Lecture 8.
Digital Demodulation

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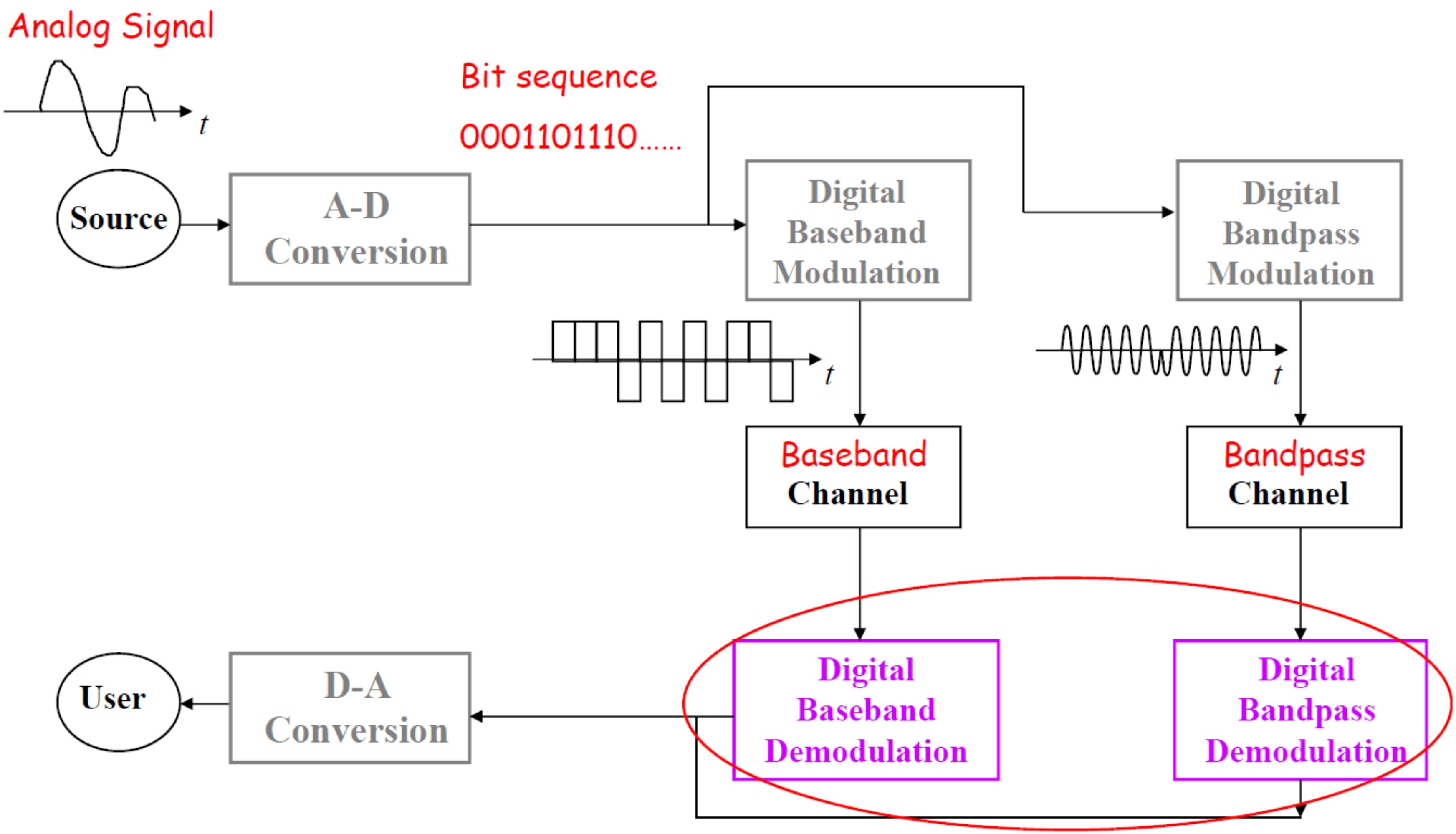
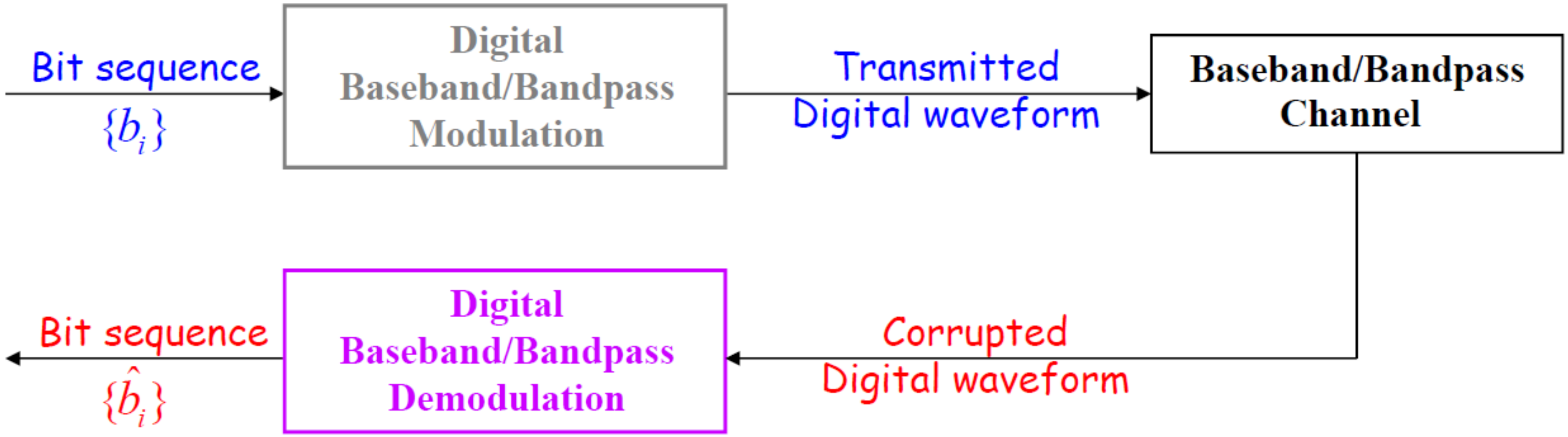
Digital Communications



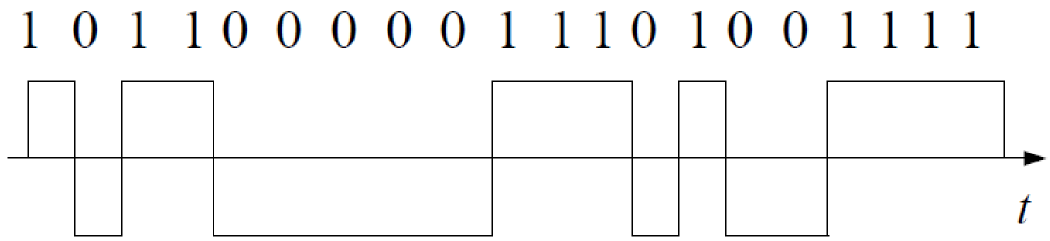
Digital Demodulation



- What are the sources of signal corruption?
- How to detect the signal (to obtain the bit sequence $\{\hat{b}_i\}$)?
- How to evaluate the fidelity performance?

Sources of Signal Corruption

Transmitted signal
$s(t)$

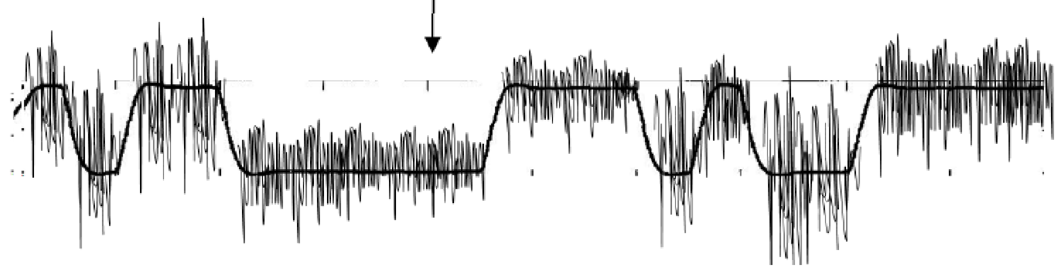


- Suppose that channel bandwidth is properly chosen such that most frequency components of the transmitted signal can pass through the channel.

Channel

- Thermal Noise: caused by the random motion of electrons within electronic devices.

Received signal
$y(t)$



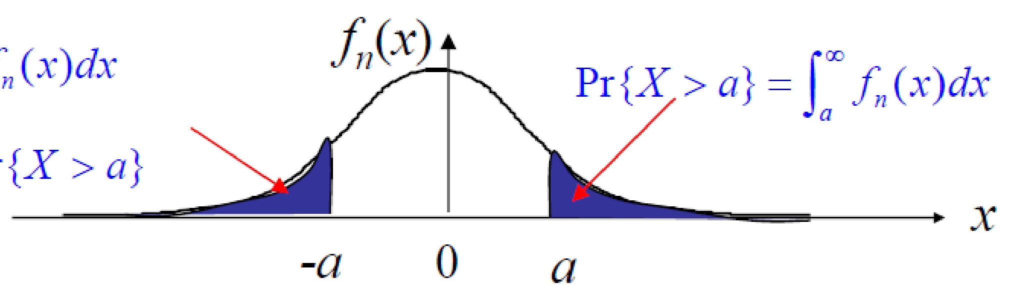
Modeling of Thermal Noise

- The thermal noise is modeled as a WSS process $n(t)$.
  - ✓ The thermal noise is superimposed (added) to the signal: $y(t)=s(t)+n(t)$
  - ✓ At each time slot t_0, $n(t_0) \sim \mathcal{N}(0, \sigma_0^2)$ (i.e., zero-mean *Gaussian* random variable with variance σ_0^2):

$$f_n(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{x^2}{2\sigma_0^2}\right)$$

$$\Pr\{X < -a\} = \int_{-\infty}^{-a} f_n(x)dx$$

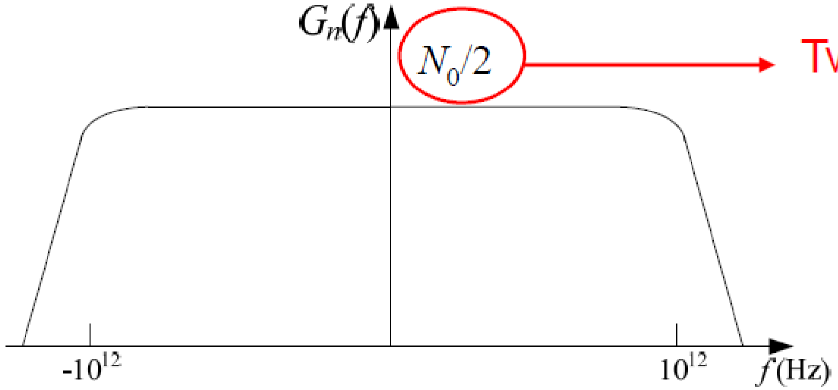
$$= \int_a^{\infty} f_n(-y)dy = \Pr\{X > a\}$$



$$\Pr\{X > a\} = \int_a^{\infty} f_n(x)dx = Q\left(\frac{a}{\sigma_0}\right)$$

Modeling of Thermal Noise

- The thermal noise has a power spectrum that is constant from dc to approximately 10^{12} Hz: $n(t)$ can be approximately regarded as a white process.

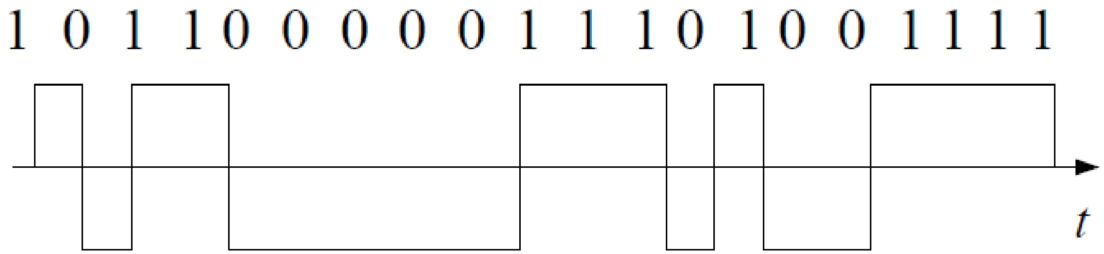


Two-sided Power Spectral Density $N_0/2$

The thermal noise is also referred to as additive white Gaussian Noise (AWGN), because it is modeled as a white Gaussian WSS process which is added to the signal.

Detection

Transmitted signal
$s(t)$



Received signal $y(t)=s(t)+n(t)$

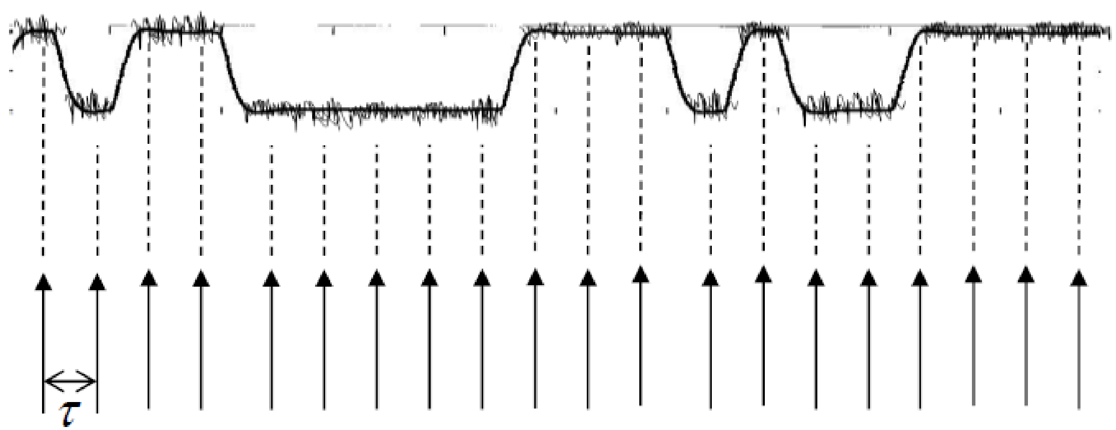
Step 1: Filtering

Step 2: Sampling

Step 3: Threshold Comparison

Sample>0 ⇨ 1

Sample<0 ⇨ 0

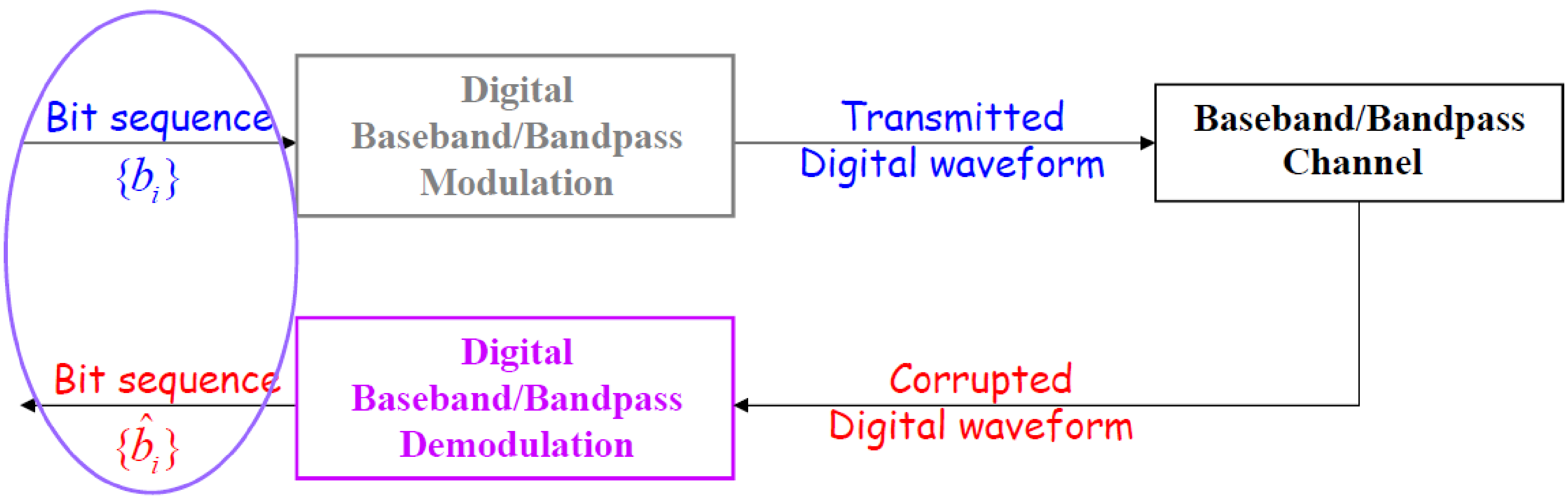


1 0 1 1 0 0 0 0 0 1 1 1 0 1 0 0 1 1 1 1

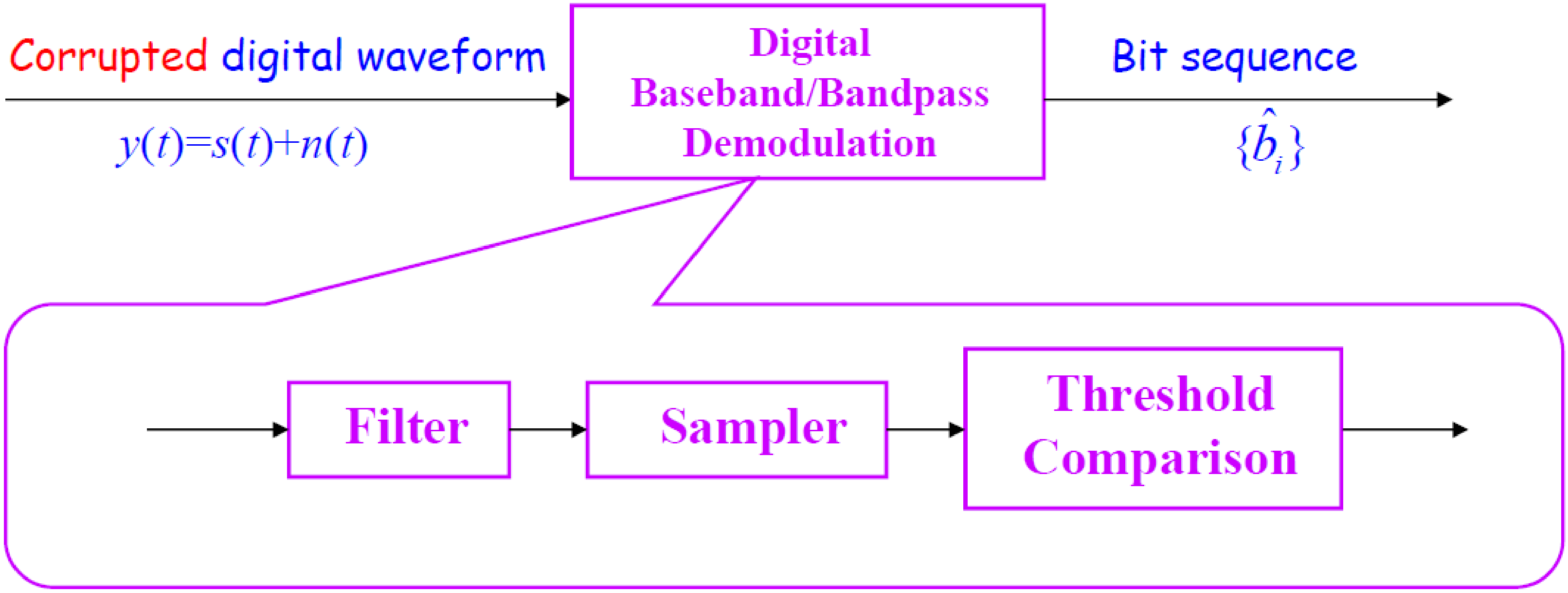
Bit Error Rate (BER)

- Bit Error: $\{\hat{b}_i \neq b_i\} = \{\hat{b}_i = 1 \text{ but } b_i = 0\} \cup \{\hat{b}_i = 0 \text{ but } b_i = 1\}$
- Probability of Bit Error (or Bit Error Rate, BER):

$$P_b = \Pr\{\hat{b}_i = 1,\, b_i = 0\} + \Pr\{\hat{b}_i = 0,\, b_i = 1\}$$



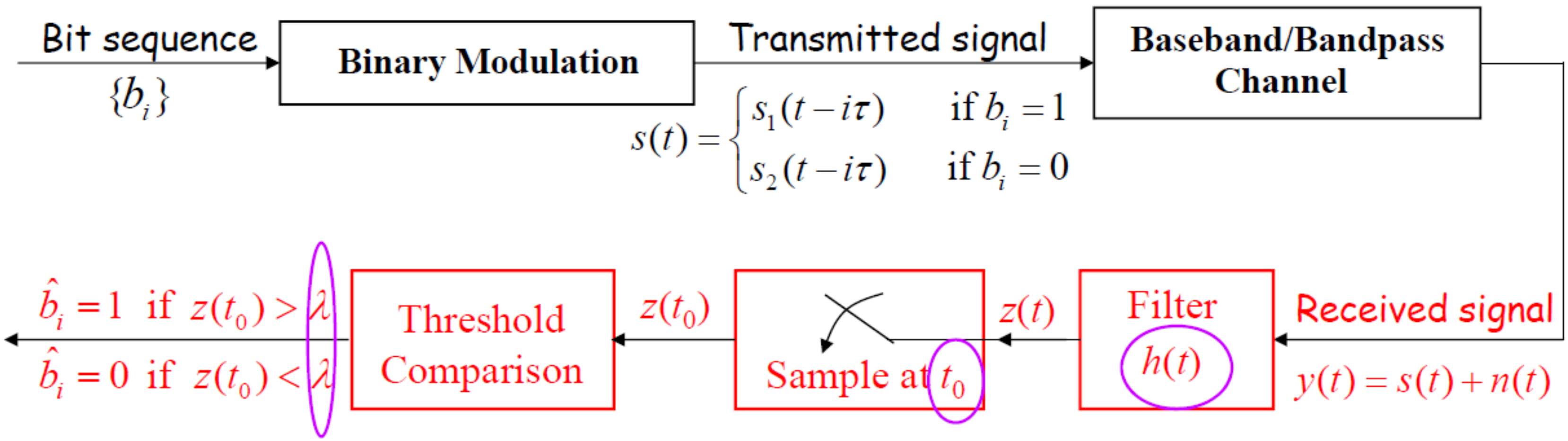
Digital Demodulation



- How to design the filter, sampler and threshold to minimize the BER?

Binary Detection

Binary Detection

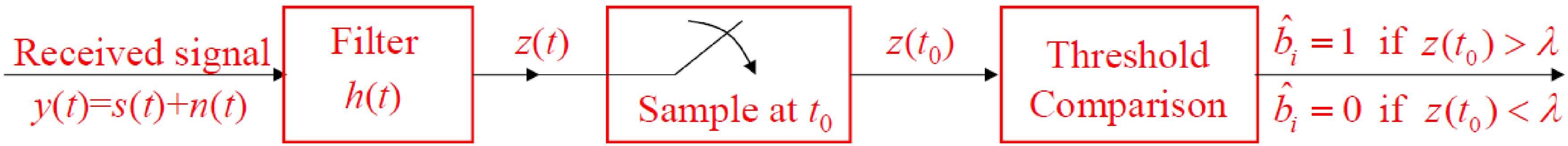


- BER: $$P_b=\Pr\{\hat{b}_i=1,\, b_i=0\}+\Pr\{\hat{b}_i=0,\, b_i=1\}$$

How to choose the threshold λ, sampling point t_0 and the filter to minimize BER?

Receiver Structure

- Transmitted signal: $s(t)=\begin{cases} s_1(t) & \text{if } b_1=1 \\ s_2(t) & \text{if } b_1=0 \end{cases} \qquad 0\le t\le \tau$



- Received signal: $y(t)=s(t)+n(t)=\begin{cases} s_1(t)+n(t) & \text{if } b_1=1 \\ s_2(t)+n(t) & \text{if } b_1=0 \end{cases}$

- Filter output: $z(t)=s_o(t)+n_o(t)=\begin{cases} s_{o,1}(t)+n_o(t) & \text{if } b_1=1 \\ s_{o,2}(t)+n_o(t) & \text{if } b_1=0 \end{cases}$

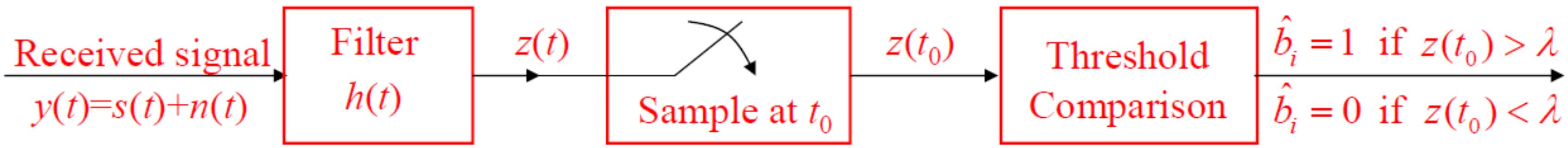
where $n_o(t)=\int_0^t n(x)h(t-x)dx,\ \ s_{o,i}(t)=\int_0^t s_i(x)h(t-x)dx,\ \ i=1,2.$

n(t) is a white process with two-sided power spectral density $N_0/2$.

Is $n_o(t)$ a white process? No!

Receiver Structure

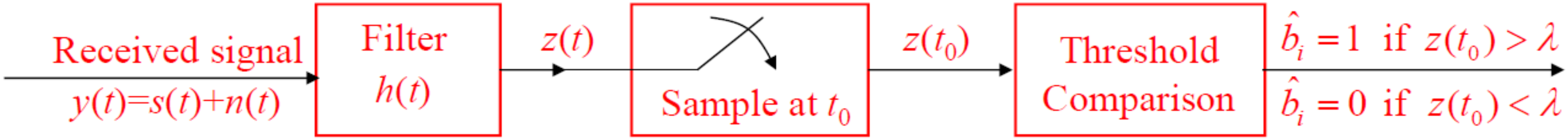
- Transmitted signal: $s(t) = \begin{cases} s_1(t) & \text{if } b_1 = 1 \\ s_2(t) & \text{if } b_1 = 0 \end{cases} \qquad 0 \le t \le \tau$



- Received signal: $y(t) = s(t) + n(t) = \begin{cases} s_1(t) + n(t) & \text{if } b_1 = 1 \\ s_2(t) + n(t) & \text{if } b_1 = 0 \end{cases}$
- Filter output: $z(t) = s_o(t) + n_o(t) = \begin{cases} s_{o,1}(t) + n_o(t) & \text{if } b_1 = 1 \\ s_{o,2}(t) + n_o(t) & \text{if } b_1 = 0 \end{cases}$
- Sampler output: $z(t_0) = s_o(t_0) + n_o(t_0) = \begin{cases} s_{o,1}(t_0) + n_o(t_0) & \text{if } b_1 = 1 \\ s_{o,2}(t_0) + n_o(t_0) & \text{if } b_1 = 0 \end{cases}$

$$n_o(t_0) \sim \mathcal{N}(0, \sigma_0^2) \quad \Rightarrow \quad \begin{array}{l} z(t_0) \mid b_1 = 1 \sim \mathcal{N}(s_{o,1}(t_0), \sigma_0^2) \\ z(t_0) \mid b_1 = 0 \sim \mathcal{N}(s_{o,2}(t_0), \sigma_0^2) \end{array}$$

BER



- **BER:**

$$P_b = \Pr\{\hat{b}_1 = 1, b_1 = 0\} + \Pr\{\hat{b}_1 = 0, b_1 = 1\} = \Pr\{z(t_0) > \lambda, b_1 = 0\} + \Pr\{z(t_0) < \lambda, b_1 = 1\}$$

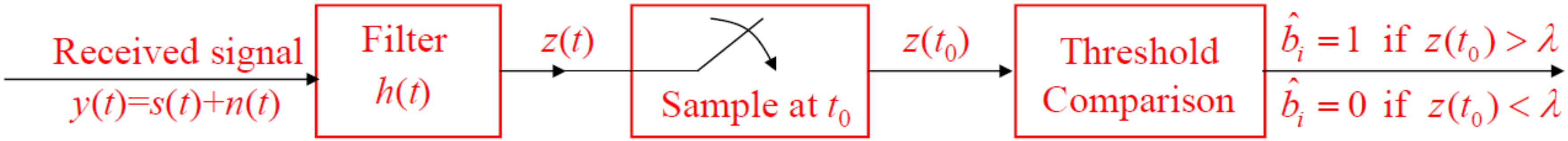
$$= \Pr\{z(t_0) > \lambda \mid b_1 = 0\}\Pr\{b_1 = 0\} + \Pr\{z(t_0) < \lambda \mid b_1 = 1\}\Pr\{b_1 = 1\}$$

$$= \tfrac{1}{2}\left[\underline{\Pr\{z(t_0) > \lambda \mid b_1 = 0\}} + \underline{\Pr\{z(t_0) < \lambda \mid b_1 = 1\}}\right] \qquad (\Pr\{b_1 = 0\} = \Pr\{b_1 = 1\} = \tfrac{1}{2})$$

Recall that $z(t_0) \mid b_1 = 0 \sim \mathcal{N}(s_{o,2}(t_0), \sigma_0^2)$ and $z(t_0) \mid b_1 = 1 \sim \mathcal{N}(s_{o,1}(t_0), \sigma_0^2)$

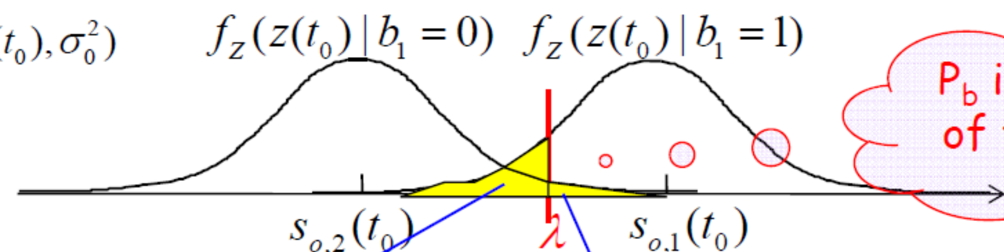
How to obtain $\Pr\{z(t_0) > \lambda \mid b_1 = 0\}$ and $\Pr\{z(t_0) < \lambda \mid b_1 = 1\}$?

BER



$z(t_0) \mid b_1 = 0 \sim \mathcal{N}(s_{o,2}(t_0), \sigma_0^2)$

$z(t_0) \mid b_1 = 1 \sim \mathcal{N}(s_{o,1}(t_0), \sigma_0^2)$



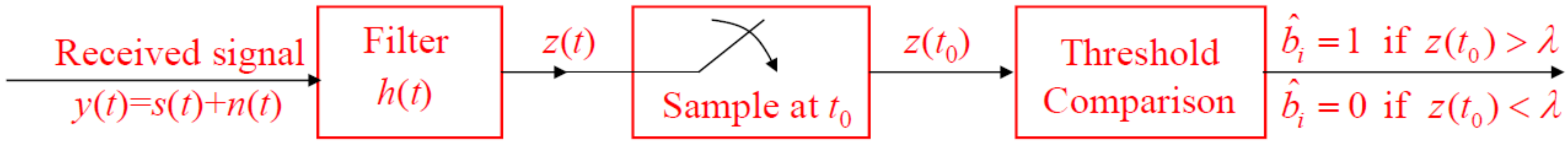
P_b is the area of the yellow zone!

- **BER:** $P_b = \frac{1}{2}\left[\underline{\Pr\{z(t_0) < \lambda \mid b_1 = 1\}} + \underline{\Pr\{z(t_0) > \lambda \mid b_1 = 0\}}\right]$

$$= \frac{1}{2}\left(Q\left(\frac{\lambda - s_{o,2}(t_0)}{\sigma_0}\right) + Q\left(\frac{s_{o,1}(t_0) - \lambda}{\sigma_0}\right)\right)$$

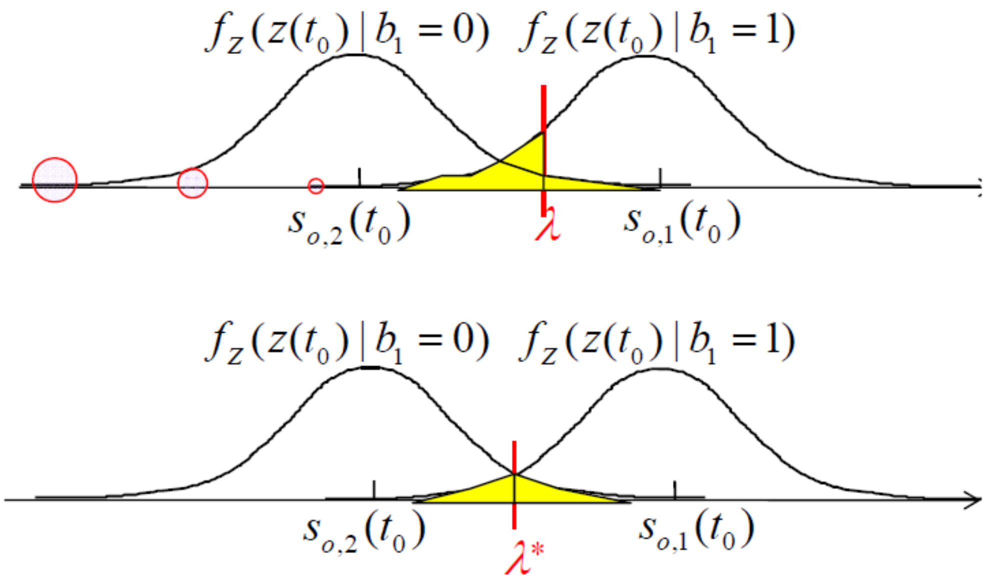
P_b is determined by the threshold λ!

Optimal Threshold to Minimize BER



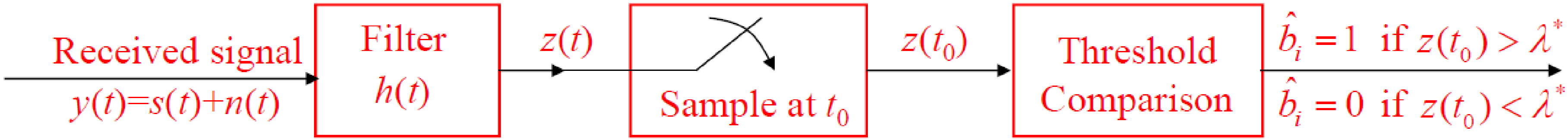
- Optimal threshold to minimize BER:

choose λ^* to minimize the yellow area !



$$\lambda^* = \frac{s_{o,1}(t_0) + s_{o,2}(t_0)}{2}$$

BER with Optimal Threshold



- BER with the optimal threshold $\lambda^* = \frac{1}{2}(s_{o,1}(t_0) + s_{o,2}(t_0))$ is

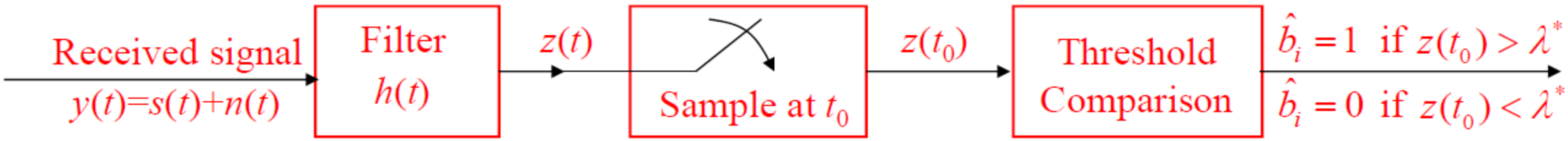
$$P_b(\lambda^*) = \frac{1}{2}\left(Q\left(\frac{\lambda^* - s_{o,2}(t_0)}{\sigma_0}\right) + Q\left(\frac{s_{o,1}(t_0) - \lambda^*}{\sigma_0}\right)\right) = Q\left(\frac{s_{o,1}(t_0) - s_{o,2}(t_0)}{2\sigma_0}\right)$$

$$= Q\left(\frac{1}{2}\sqrt{\frac{\left(\int_0^{t_0}(s_1(x) - s_2(x))h(t_0 - x)dx\right)^2}{\frac{N_0}{2}\int_0^{t_0} h^2(t_0 - x)dx}}\right)$$

$P_b(\lambda^*)$ is determined by the filter h(t) and the sampling point t_0!

where $s_{o,1}(t_0) - s_{o,2}(t_0) = \int_0^{t_0}(s_1(x) - s_2(x))h(t_0 - x)dx$, and $\sigma_0^2 = \frac{N_0}{2}\int_0^{t_0} h^2(t_0 - x)dx$. Why?

Optimal Filter to Minimize BER



- Optimal filter to minimize $P_b(\lambda^*) : \min_{h(t),\, t_0} P_b(\lambda^*) = \max_{h(t),\, t_0} \dfrac{\left[\int_0^{t_0} (s_1(x)-s_2(x))h(t_0-x)dx\right]^2}{\int_0^{t_0} \frac{N_0}{2} h^2(t_0-x)dx}$

$$\boxed{h(t)=k(s_1(\tau-t)-s_2(\tau-t)),\quad 0\le t\le \tau \quad \text{and} \quad t_0=\tau}$$

$$\frac{\left[\int_0^{t_0} (s_1(x)-s_2(x))h(t_0-x)dx\right]^2}{\int_0^{t_0} \frac{N_0}{2} h^2(t_0-x)dx} \le \frac{\int_0^{t_0} (s_1(x)-s_2(x))^2 dx \cancel{\int_0^{t_0} h^2(t_0-x)dx}}{\frac{N_0}{2}\cancel{\int_0^{t_0} h^2(t_0-x)dx}}$$

"=" holds when $h(t)=k(s_1(t_0-t)-s_2(t_0-t))$

$$= \frac{\int_0^{t_0} (s_1(x)-s_2(x))^2 dx}{N_0/2} \le \frac{\int_0^{\tau} (s_1(x)-s_2(x))^2 dx}{N_0/2}$$

"=" holds when $t_0=\tau$

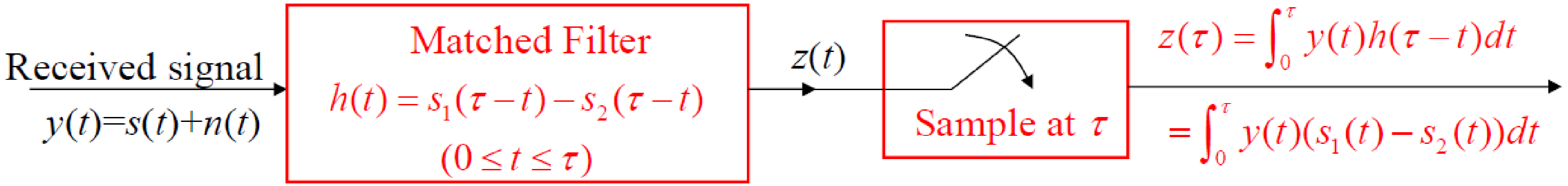
Matched Filter

- Optimal filter: $h(t) = k(s_1(\tau - t) - s_2(\tau - t)) \quad (0 \le t \le \tau)$

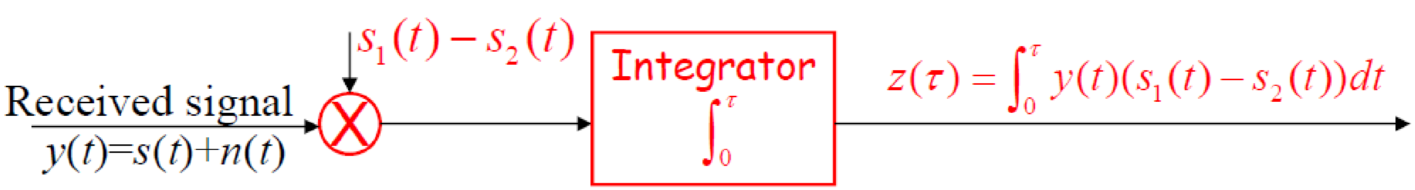
$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = k\int_0^{\tau}(s_1(\tau - t) - s_2(\tau - t))e^{-j2\pi ft}dt = k(S_1^*(f) - S_2^*(f))e^{-j2\pi f\tau}$$

The optimal filter is called matched filter, as it has a shape matched to the shape of the input signal.

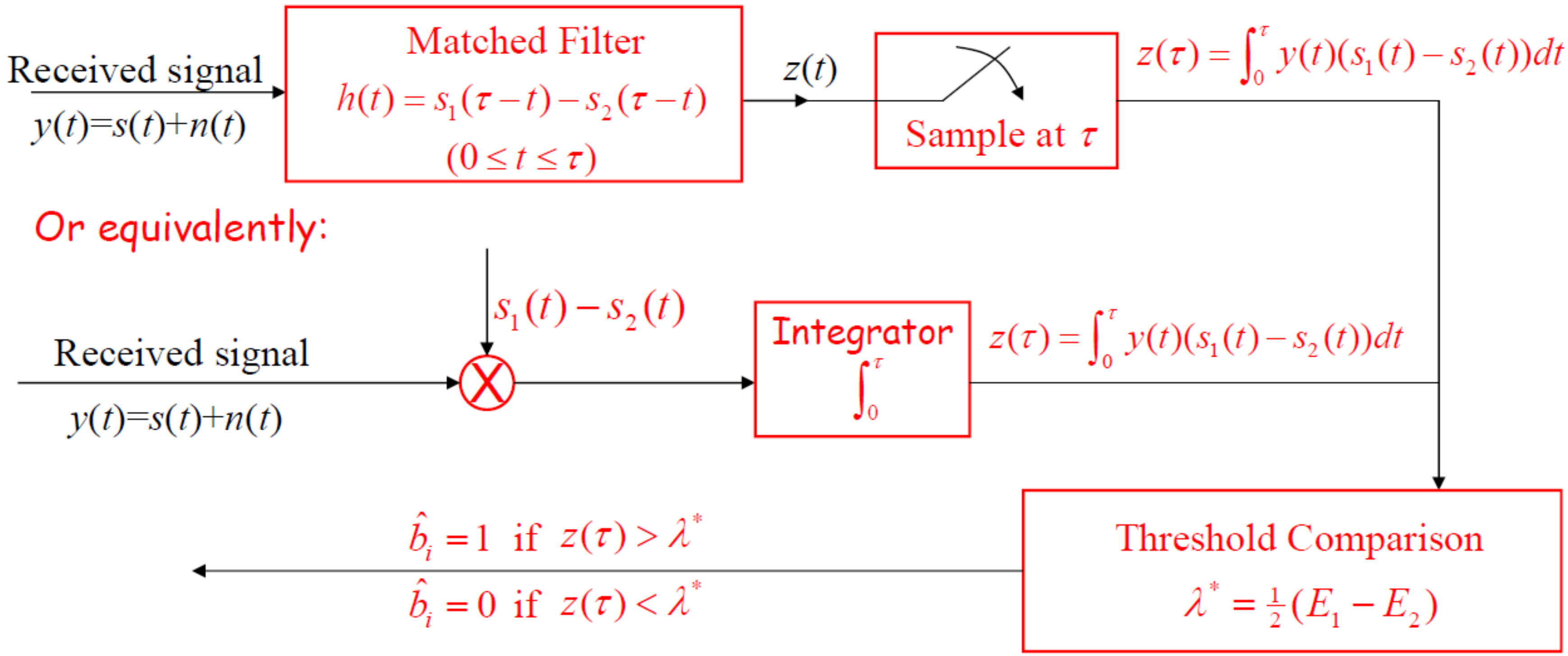
- Output of Matched Filter:



- Correlation realization of Matched Filter:



Optimal Binary Detector



$$\lambda^*=\tfrac{1}{2}(s_{o,1}(\tau)+s_{o,2}(\tau))=\tfrac{1}{2}\int_0^\tau(s_1(x)+s_2(x))h(\tau-x)dx=\tfrac{1}{2}\Big(\underbrace{\int_0^\tau s_1^2(t)dt}_{E_1}-\underbrace{\int_0^\tau s_2^2(t)dt}_{E_2}\Big)$$

$$=\tfrac{1}{2}(E_1-E_2)$$

Energy of $s_i(t)$: E_1 E_2

BER of Optimal Binary Detector

- BER with the optimal threshold: $P_b(\lambda^*) = Q\left(\frac{1}{2}\sqrt{\frac{\left(\int_0^{t_0}(s_1(x)-s_2(x))h(t_0-x)dx\right)^2}{\frac{N_0}{2}\int_0^{t_0}h^2(t_0-x)dx}}\right)$
- Impulse response of matched filter: $h(t) = s_1(\tau - t) - s_2(\tau - t) \qquad (0 \le t \le \tau)$
- Optimal sampling point: $t_0 = \tau$

$$\Downarrow$$

- BER of the Optimal Binary Detector:

$$P_b^* = Q\left(\frac{1}{2}\sqrt{\frac{\left(\int_0^{\tau}(s_1(x)-s_2(x))(s_1(x)-s_2(x))dx\right)^2}{\frac{N_0}{2}\int_0^{\tau}(s_1(x)-s_2(x))^2dx}}\right) = Q\left(\sqrt{\frac{\int_0^{\tau}(s_1(t)-s_2(t))^2dt}{2N_0}}\right)$$

Energy per Bit E_b and Energy Difference per Bit E_d

- Energy per Bit: $E_b = \frac{1}{2}(E_1 + E_2) = \frac{1}{2}\int_0^{\tau}(s_1^2(t) + s_2^2(t))dt$

- Energy difference per Bit: $E_d = \int_0^{\tau}\left(s_1(t) - s_2(t)\right)^2 dt$

  - E_d can be further written as

$$E_d = \underbrace{\int_0^{\tau} s_1^2(t)dt + \int_0^{\tau} s_2^2(t)dt}_{2E_b} - 2\underbrace{\int_0^{\tau} s_1(t)s_2(t)dt}_{\rho = \frac{1}{E_b}\int_0^{\tau} s_1(t)s_2(t)dt} = 2(1-\rho)E_b$$

> Cross-correlation coefficient $-1 \le \rho \le 1$ is a measure of similarity between two signals $s_1(t)$ and $s_2(t)$.

BER of Optimal Binary Detector

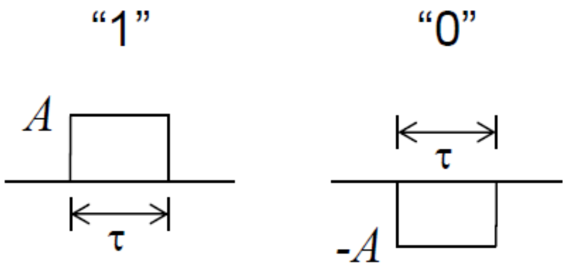
- BER of the Optimal Binary Detector:

$$\boxed{P_b^* = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)}$$

  - The BER performance is determined by 1) E_b/N_0 and 2) Cross-correlation coefficient ρ.
  - P_b^* decreases as E_b/N_0 increases.
  - P_b^* is minimized when cross-correlation coefficient ρ=-1.

BER of Binary Signaling

BER of Binary PAM



$s_1(t) = A$, $0 \le t \le \tau$

$s_2(t) = -A$, $0 \le t \le \tau$

- Energy of $s_i(t)$: $E_1 = E_2 = \int_0^\tau A^2 dt = A^2\tau$
- Energy per bit: $E_{b,BPAM} = \frac{1}{2}(E_1 + E_2) = A^2\tau$
- Cross-correlation coefficient:

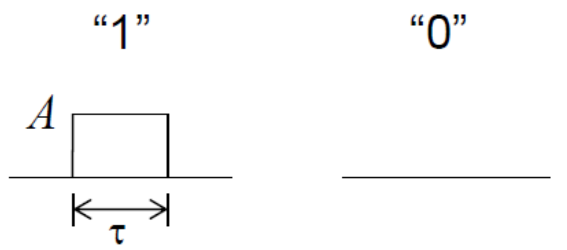
$$\rho_{BPAM} = \frac{1}{E_b}\int_0^\tau s_1(t)s_2(t)dt = -\frac{1}{E_b}\int_0^\tau A^2 dt = -1$$

- Power: $\mathcal{P}_{BPAM} = A^2$

- Optimal BER: $P^*_{b,BPAM} = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2\tau}{N_0}}\right) = Q\left(\sqrt{\frac{2\mathcal{P}_{BPAM}}{R_{b,BPAM}N_0}}\right)$

BER of Binary OOK

- Energy of $s_i(t)$: $E_1 = A^2\tau,\ \ E_2 = 0.$



$s_1(t) = A$

$0 \le t \le \tau$

$s_2(t) = 0$

- Energy per bit: $E_{b,BOOK} = \frac{1}{2}(E_1 + E_2) = \frac{1}{2}A^2\tau$

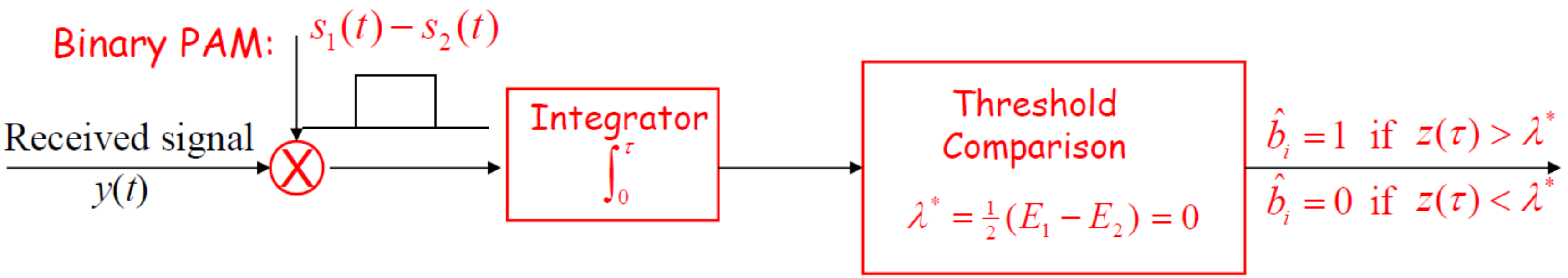
- Cross-correlation coefficient:

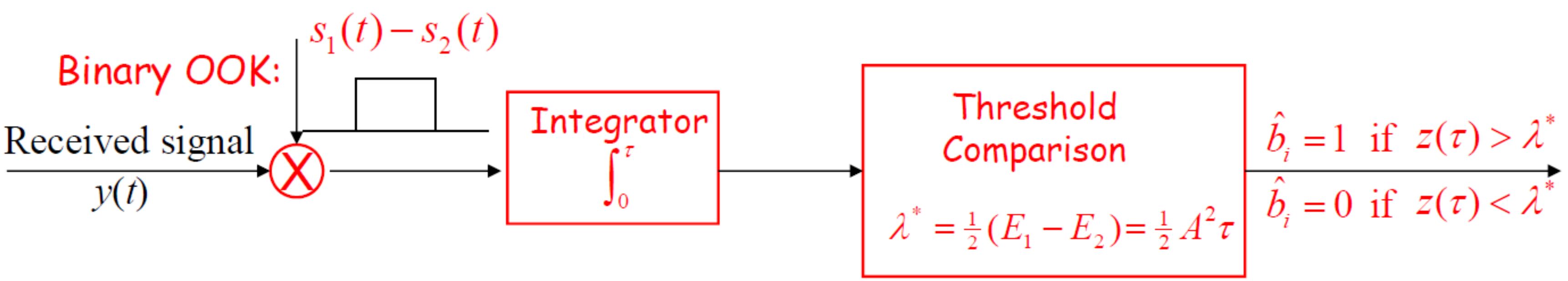
$$\rho_{BOOK} = \frac{1}{E_b}\int_0^{\tau} s_1(t)s_2(t)dt = 0$$

- Power: $\mathcal{P}_{BOOK} = A^2/2$

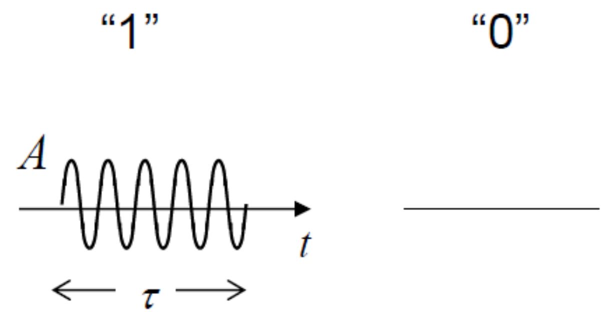
- Optimal BER: $P^*_{b,BOOK} = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2\tau}{2N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{P}_{BOOK}}{R_{b,BOOK}N_0}}\right)$

Optimal Receiver of Binary PAM and OOK





BER of Binary ASK



$$s_1(t) = A\cos(2\pi f_c t) \qquad s_2(t) = 0$$
$$0 \le t \le \tau$$

(τ is an integer number of $1/f_c$)

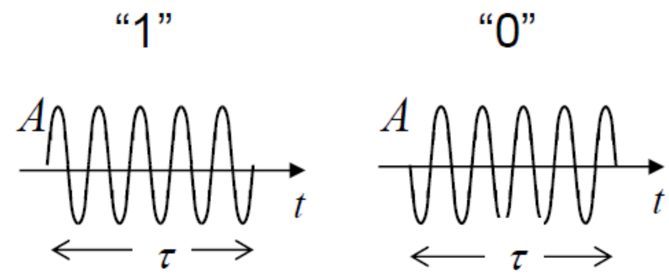
- Energy of $s_i(t)$: $E_1 = \frac{1}{2}A^2\tau,\ \ E_2 = 0.$
- Energy per bit: $E_{b,BASK} = \frac{1}{2}(E_1 + E_2) = \frac{1}{4}A^2\tau$
- Cross-correlation coefficient:

$$\rho_{BASK} = \frac{1}{E_b}\int_0^\tau s_1(t)s_2(t)dt = 0$$

- Power: $\mathcal{P}_{BASK} = A^2/4$

- Optimal BER: $P^*_{b,BASK} = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2\tau}{4N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{P}_{BASK}}{R_{b,BASK}N_0}}\right)$

BER of Binary PSK



$s_1(t) = A\cos(2\pi f_c t)$ $\quad 0 \le t \le \tau$

$s_2(t) = A\cos(2\pi f_c t + \pi)$ $\quad 0 \le t \le \tau$

(τ is an integer number of $1/f_c$)

- Energy of $s_i(t)$: $E_1 = E_2 = \frac{1}{2} A^2 \tau$

- Energy per bit: $E_{b,BPSK} = \frac{1}{2}(E_1 + E_2) = \frac{1}{2} A^2 \tau$

- Cross-correlation coefficient:

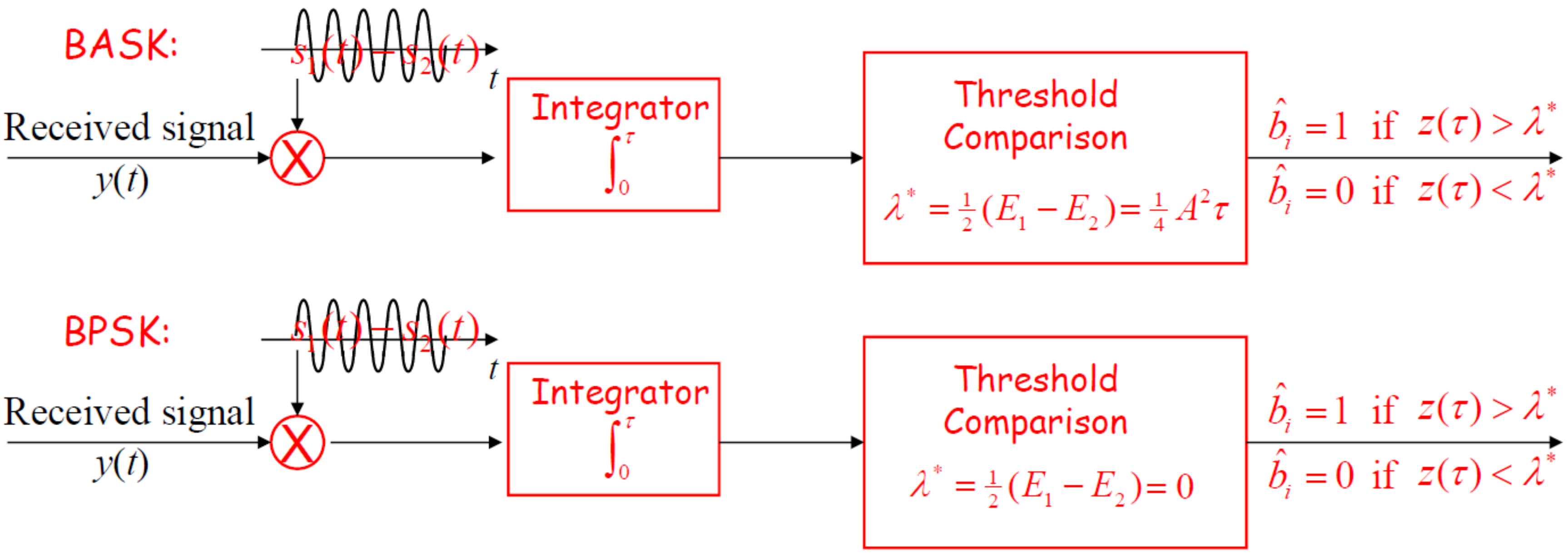
$$\rho_{BPSK} = \frac{1}{E_b}\int_0^{\tau} s_1(t) s_2(t)\,dt = -\frac{1}{E_b}\int_0^{\tau} s_1^2(t)\,dt = -1$$

$$s_2(t) = s_1(t+\pi) = -s_1(t)$$

- Power: $\mathcal{P}_{BPSK} = A^2/2$

- Optimal BER: $P^*_{b,BPSK} = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2\tau}{N_0}}\right) = Q\left(\sqrt{\frac{2\mathcal{P}_{BPSK}}{R_{b,BPSK} N_0}}\right)$

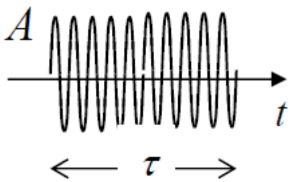
Coherent Receiver of BASK and BPSK



The optimal receiver is also called "coherent receiver" because it must be capable of internally producing a reference signal which is in exact phase and frequency synchronization with the carrier signal $\cos(2\pi f_c t)$.

BER of Binary FSK

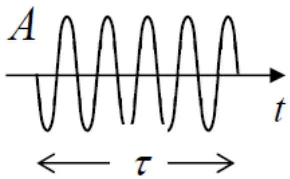
"1"



$s_1(t) = A\cos(2\pi(f_c+\Delta f)t)$

$0 \le t \le \tau$

"0"



$s_2(t) = A\cos(2\pi(f_c-\Delta f)t)$

$0 \le t \le \tau$

(τ is an integer number of $1/(f_c \pm \Delta f)$)

- Energy of $s_i(t)$: $E_1 = E_2 = \frac{1}{2}A^2\tau$
- Energy per bit:

$$E_{b,BFSK} = \tfrac{1}{2}(E_1+E_2) = \tfrac{1}{2}A^2\tau$$

- Cross-correlation coefficient:

$$\rho_{BFSK} = \frac{1}{E_b}\int_0^\tau s_1(t)s_2(t)dt = \frac{2}{\tau}\int_0^\tau \cos(2\pi(f_c+\Delta f)t)\cos(2\pi(f_c-\Delta f)t)dt$$

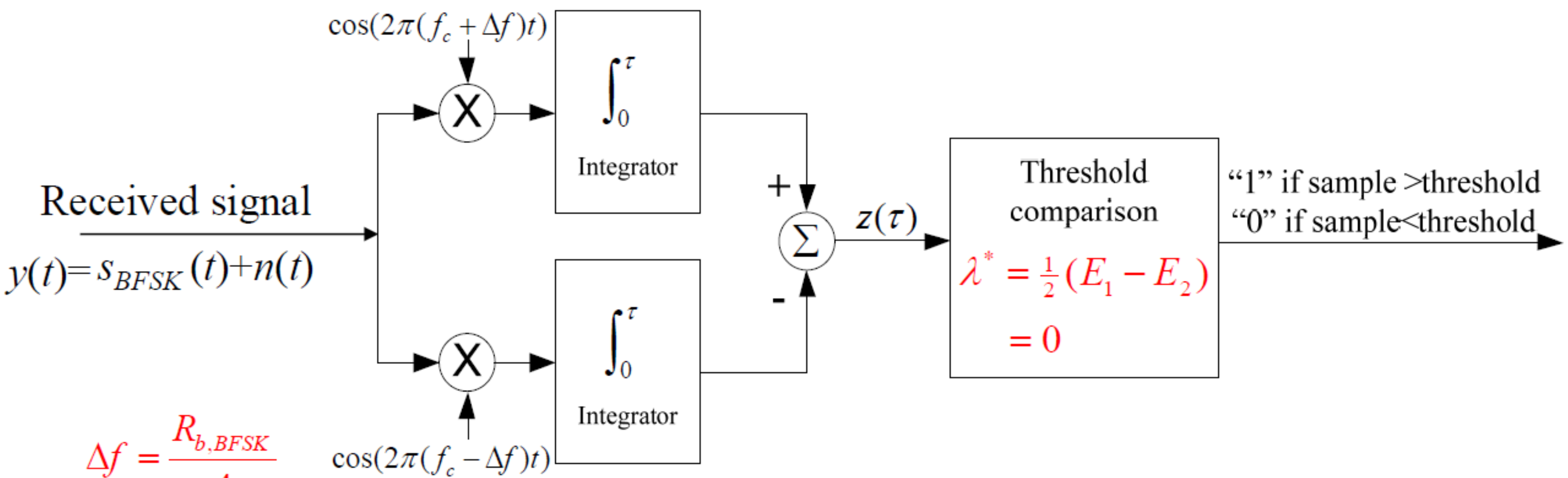
$$= \frac{1}{\tau}\left(\int_0^\tau \cos(4\pi\Delta f t)dt \cancel{+\int_0^\tau \cos(4\pi f_c t)dt}\right) = \frac{1}{\tau}\int_0^\tau \cos(4\pi\Delta f t)dt = \frac{1}{4\pi\Delta f\tau}\sin(4\pi\Delta f\tau)$$

✓ What is the minimum Δf to achieve $\rho_{BFSK} = 0$?

$$\boxed{\min \Delta f = \frac{1}{4\tau} = \frac{R_{b,BFSK}}{4}}$$

Coherent BFSK Receiver

- Power: $\mathcal{P}_{BFSK} = A^2/2$
- Optimal BER: $P^*_{b,BFSK} = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2\tau}{2N_0}}\right) = Q\left(\sqrt{\frac{\mathcal{P}_{BFSK}}{R_{b,BFSK}N_0}}\right)$



$\Delta f = \frac{R_{b,BFSK}}{4}$

Bandwidth Efficiency of Coherent BFSK

With $\Delta f = \frac{R_{b,BFSK}}{4}$:

- The required channel bandwidth for 90% in-band power:

$$B_{h\_90\%} = 2\Delta f + 2R_{b,BFSK} = 2.5R_{b,BFSK}$$

- Bandwidth efficiency of coherent BFSK:

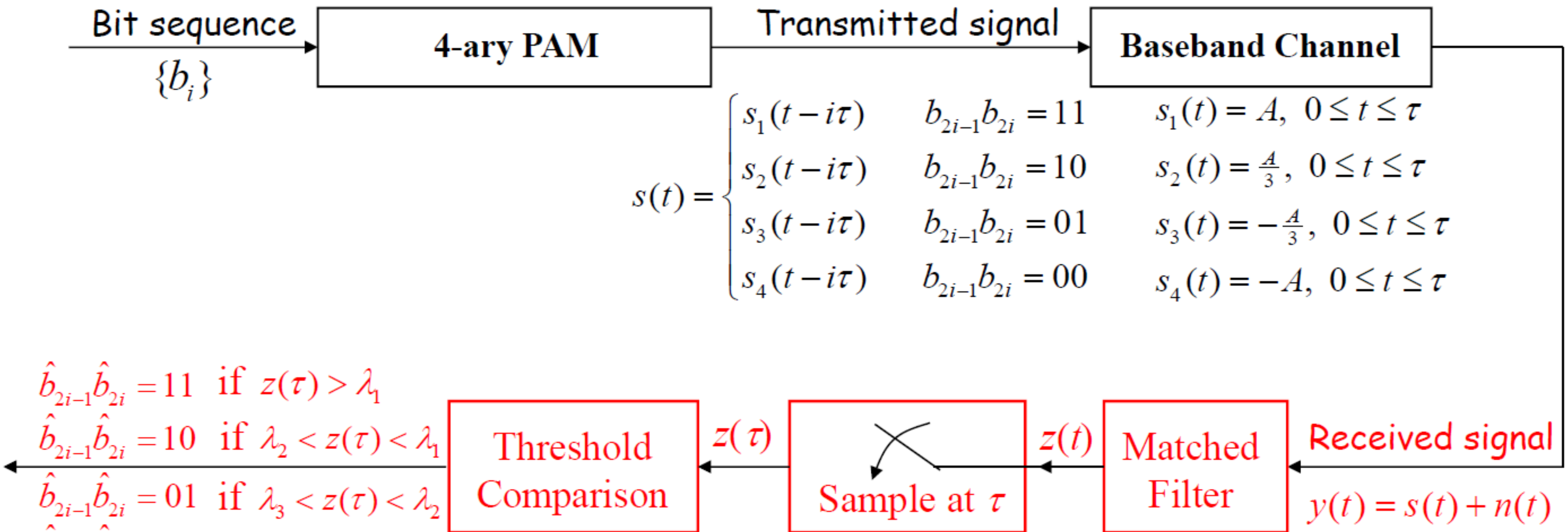
$$\gamma_{BFSK} = \frac{R_{b,BFSK}}{B_{h\_90\%}} = 0.4$$

Summary I: Binary Modulation and Demodulation

	Bandwidth Efficiency	BER
Binary PAM	1 (90% in-band power)	$Q\left(\sqrt{\frac{2E_{b,BPAM}}{N_0}}\right)$
Binary OOK	1 (90% in-band power)	$Q\left(\sqrt{\frac{E_{b,BOOK}}{N_0}}\right)$
Coherent Binary ASK	0.5 (90% in-band power)	$Q\left(\sqrt{\frac{E_{b,BASK}}{N_0}}\right)$
Coherent Binary PSK	0.5 (90% in-band power)	$Q\left(\sqrt{\frac{2E_{b,BPSK}}{N_0}}\right)$
Coherent Binary FSK	0.4 (90% in-band power)	$Q\left(\sqrt{\frac{E_{b,BFSK}}{N_0}}\right)$

M-ary Detection

Detection of 4-ary PAM



$$s(t) = \begin{cases} s_1(t - i\tau) & b_{2i-1}b_{2i} = 11 \\ s_2(t - i\tau) & b_{2i-1}b_{2i} = 10 \\ s_3(t - i\tau) & b_{2i-1}b_{2i} = 01 \\ s_4(t - i\tau) & b_{2i-1}b_{2i} = 00 \end{cases}$$

$$s_1(t) = A,\ 0 \le t \le \tau$$
$$s_2(t) = \tfrac{A}{3},\ 0 \le t \le \tau$$
$$s_3(t) = -\tfrac{A}{3},\ 0 \le t \le \tau$$
$$s_4(t) = -A,\ 0 \le t \le \tau$$

$$\hat{b}_{2i-1}\hat{b}_{2i} = 11 \text{ if } z(\tau) > \lambda_1$$
$$\hat{b}_{2i-1}\hat{b}_{2i} = 10 \text{ if } \lambda_2 < z(\tau) < \lambda_1$$
$$\hat{b}_{2i-1}\hat{b}_{2i} = 01 \text{ if } \lambda_3 < z(\tau) < \lambda_2$$
$$\hat{b}_{2i-1}\hat{b}_{2i} = 00 \text{ if } z(\tau) < \lambda_3$$

- Symbol Error: $\{\hat{b}_{2i-1}\hat{b}_{2i} \ne b_{2i-1}b_{2i}\}$

$$= \{\hat{b}_{2i-1}\hat{b}_{2i} \ne 11 \text{ but } b_{2i-1}b_{2i} = 11\} \cup \{\hat{b}_{2i-1}\hat{b}_{2i} \ne 10 \text{ but } b_{2i-1}b_{2i} = 10\}$$
$$\cup \{\hat{b}_{2i-1}\hat{b}_{2i} \ne 01 \text{ but } b_{2i-1}b_{2i} = 01\} \cup \{\hat{b}_{2i-1}\hat{b}_{2i} \ne 00 \text{ but } b_{2i-1}b_{2i} = 00\}$$

SER

- Probability of Symbol Error (or Symbol Error Rate, SER):

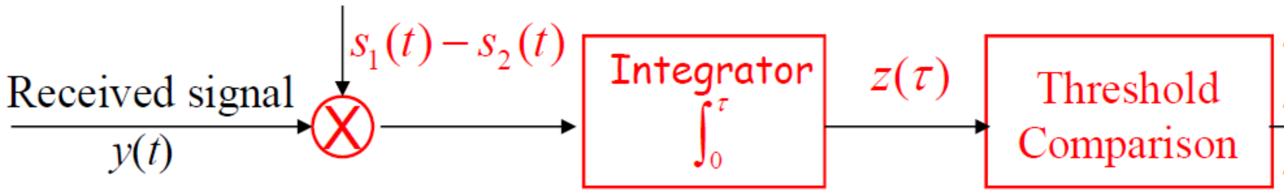
$$P_s = \Pr\{\hat{b}_{2i-1}\hat{b}_{2i} \neq 11,\ b_{2i-1}b_{2i} = 11\} + \Pr\{\hat{b}_{2i-1}\hat{b}_{2i} \neq 10,\ b_{2i-1}b_{2i} = 10\}$$
$$+ \Pr\{\hat{b}_{2i-1}\hat{b}_{2i} \neq 01,\ b_{2i-1}b_{2i} = 01\} + \Pr\{\hat{b}_{2i-1}\hat{b}_{2i} \neq 00,\ b_{2i-1}b_{2i} = 00\}$$

- SER vs. BER:

$$\begin{aligned} P_s &= \Pr\{b_{2i-1} \text{ is received in error or } b_{2i} \text{ is received in error}\} \\ &= 1 - \Pr\{b_{2i-1} \text{ is received correctly and } b_{2i} \text{ is received correctly}\} \\ &= 1 - \Pr\{b_{2i-1} \text{ is received correctly}\} \cdot \Pr\{b_{2i} \text{ is received correctly}\} \\ &= 1 - (1 - P_b)^2 = 2P_b - P_b^2 \approx 2P_b \quad \text{for small } P_b \end{aligned}$$

What is the minimum SER of 4-ary PAM and how to achieve it?

SER of 4-ary PAM Receiver



$$
\begin{aligned}
\hat{b}_{2i-1}\hat{b}_{2i} &= 11 \quad \text{if } z(\tau) > \lambda_1 \\
\hat{b}_{2i-1}\hat{b}_{2i} &= 10 \quad \text{if } \lambda_2 < z(\tau) < \lambda_1 \\
\hat{b}_{2i-1}\hat{b}_{2i} &= 01 \quad \text{if } \lambda_3 < z(\tau) < \lambda_2 \\
\hat{b}_{2i-1}\hat{b}_{2i} &= 00 \quad \text{if } z(\tau) < \lambda_3
\end{aligned}
$$

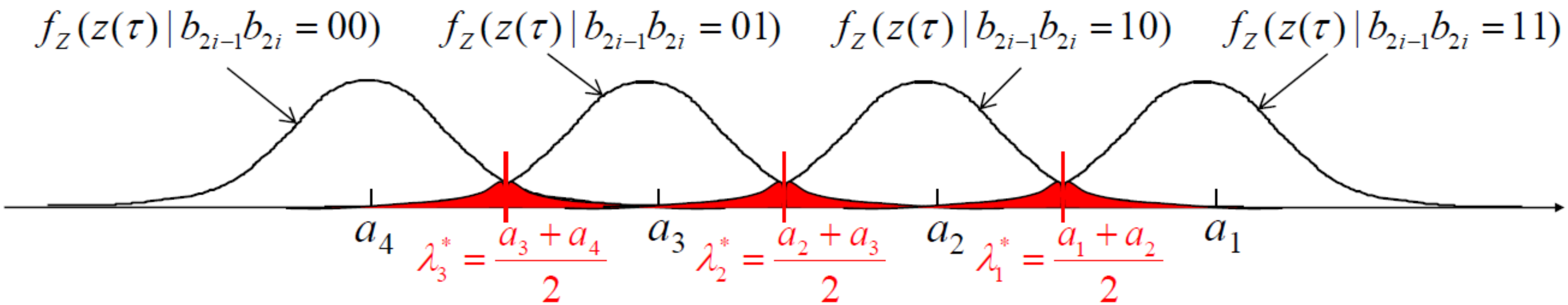
- SER:

$$
\begin{aligned}
P_s = {} & \Pr\{z(\tau) < \lambda_1, b_{2i-1}b_{2i} = 11\} && (\hat{b}_{2i-1}\hat{b}_{2i} \neq 11) \\
& + \Pr\{z(\tau) < \lambda_2, b_{2i-1}b_{2i} = 10\} + \Pr\{z(\tau) > \lambda_1, b_{2i-1}b_{2i} = 10\} && (\hat{b}_{2i-1}\hat{b}_{2i} \neq 10) \\
& + \Pr\{z(\tau) < \lambda_3, b_{2i-1}b_{2i} = 01\} + \Pr\{z(\tau) > \lambda_2, b_{2i-1}b_{2i} = 01\} && (\hat{b}_{2i-1}\hat{b}_{2i} \neq 01) \\
& + \Pr\{z(\tau) > \lambda_3, b_{2i-1}b_{2i} = 00\} && (\hat{b}_{2i-1}\hat{b}_{2i} \neq 00)
\end{aligned}
$$

$$
z(\tau) = \begin{cases}
\underbrace{\int_0^\tau s_1(t)\big(s_1(t) - s_2(t)\big)dt}_{a_1} + n_o(\tau) & \text{if } b_{2i\text{-}1}b_{2i}=11 \\
\underbrace{\int_0^\tau s_2(t)\big(s_1(t) - s_2(t)\big)dt}_{a_2} + n_o(\tau) & \text{if } b_{2i\text{-}1}b_{2i}=10 \\
\underbrace{\int_0^\tau s_3(t)\big(s_1(t) - s_2(t)\big)dt}_{a_3} + n_o(\tau) & \text{if } b_{2i\text{-}1}b_{2i}=01 \\
\underbrace{\int_0^\tau s_4(t)\big(s_1(t) - s_2(t)\big)dt}_{a_4} + n_o(\tau) & \text{if } b_{2i\text{-}1}b_{2i}=00
\end{cases}
$$

$$
\begin{aligned}
z(\tau)\,|_{b_{2i-1}b_{2i}=11} &\sim \mathcal{N}(a_1, \sigma_0^2) \\
z(\tau)\,|_{b_{2i-1}b_{2i}=10} &\sim \mathcal{N}(a_2, \sigma_0^2) \\
z(\tau)\,|_{b_{2i-1}b_{2i}=01} &\sim \mathcal{N}(a_3, \sigma_0^2) \\
z(\tau)\,|_{b_{2i-1}b_{2i}=00} &\sim \mathcal{N}(a_4, \sigma_0^2)
\end{aligned}
$$

Optimal Thresholds

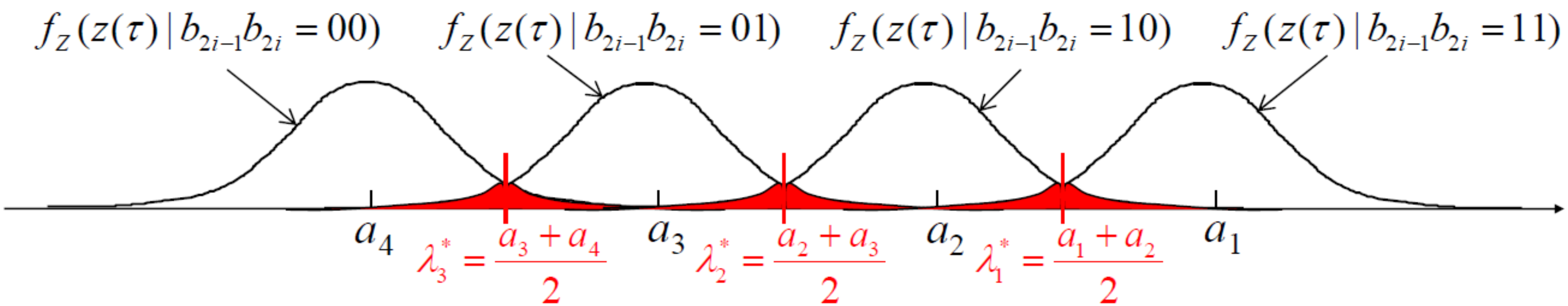


- SER of the optimal 4-ary PAM receiver:

$$
\begin{aligned}
P_s^* &= \Pr\{z(\tau)<\lambda_1^*, b_{2i-1}b_{2i}=11\} + \Pr\{z(\tau)>\lambda_3^*, b_{2i-1}b_{2i}=00\} \\
&\quad + \Pr\{z(\tau)<\lambda_2^*, b_{2i-1}b_{2i}=10\} + \Pr\{z(\tau)>\lambda_1^*, b_{2i-1}b_{2i}=10\} \\
&\quad + \Pr\{z(\tau)<\lambda_3^*, b_{2i-1}b_{2i}=01\} + \Pr\{z(\tau)>\lambda_2^*, b_{2i-1}b_{2i}=01\} \\
&= \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=00} > \tfrac{1}{2}(a_3+a_4)\right\}\cdot \Pr\{b_{2i-1}b_{2i}=00\} \\
&\quad + \left(\Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=01} < \tfrac{1}{2}(a_3+a_4)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=01} > \tfrac{1}{2}(a_2+a_3)\right\}\right)\cdot \Pr\{b_{2i-1}b_{2i}=01\} \\
&\quad + \left(\Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=10} < \tfrac{1}{2}(a_2+a_3)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=10} > \tfrac{1}{2}(a_1+a_2)\right\}\right)\cdot \Pr\{b_{2i-1}b_{2i}=10\} \\
&\quad + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=11} < \tfrac{1}{2}(a_1+a_2)\right\}\cdot \Pr\{b_{2i-1}b_{2i}=11\}
\end{aligned}
$$

1/4

SER of Optimal 4-ary PAM Receiver



- SER of the optimal 4-ary PAM receiver:

$$P_s^* = \frac{1}{4}\Big(\Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=00} > \tfrac{1}{2}(a_3+a_4)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=01} < \tfrac{1}{2}(a_3+a_4)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=01} > \tfrac{1}{2}(a_2+a_3)\right\}$$

$$+\Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=10} < \tfrac{1}{2}(a_2+a_3)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=10} > \tfrac{1}{2}(a_1+a_2)\right\} + \Pr\left\{z(\tau)|_{b_{2i-1}b_{2i}=11} < \tfrac{1}{2}(a_1+a_2)\right\}\Big)$$

$$= \frac{1}{4}\left(2Q\left(\frac{a_3-a_4}{2\sigma_0}\right) + 2Q\left(\frac{a_2-a_3}{2\sigma_0}\right) + 2Q\left(\frac{a_1-a_2}{2\sigma_0}\right)\right) = \frac{6}{4}Q\left(\frac{a_1-a_2}{2\sigma_0}\right)$$

$$\left.\begin{aligned} a_i &= \int_0^\tau s_i(t)\big(s_1(t)-s_2(t)\big)\,dt \\ \sigma_0^2 &= \tfrac{N_0}{2}\int_0^\tau \big(s_1(t)-s_2(t)\big)^2\,dt \end{aligned}\right\} \quad P_s^* = \frac{3}{2}Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

SER and BER of Optimal 4-ary PAM Receiver

- SER: $$P^*_{s,4PAM} = \frac{3}{2}Q\left(\sqrt{\frac{E_{d,4PAM}}{2N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{0.4E_{s,4PAM}}{N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{0.8E_{b,4PAM}}{N_0}}\right)$$

- BER: $$P^*_{b,4PAM} \approx \frac{1}{2}P^*_{s,4PAM} = \frac{3}{4}Q\left(\sqrt{\frac{E_{d,4PAM}}{2N_0}}\right) = \frac{3}{4}Q\left(\sqrt{\frac{0.8E_{b,4PAM}}{N_0}}\right)$$

Energy difference E_d:

$$E_{d,4PAM} = \int_0^\tau (s_1(t)-s_2(t))^2\,dt = \int_0^\tau (s_2(t)-s_3(t))^2\,dt = \int_0^\tau (s_3(t)-s_4(t))^2\,dt = \frac{4}{9}A^2\tau = 0.8E_S$$

Energy per symbol E_s:

$$E_{s,4PAM} = \frac{1}{4}\int_0^\tau s_1^2(t)dt + \frac{1}{4}\int_0^\tau s_2^2(t)dt + \frac{1}{4}\int_0^\tau s_3^2(t)dt + \frac{1}{4}\int_0^\tau s_4^2(t)dt = \frac{5}{9}A^2\tau$$

Energy per bit E_b: $E_{b,4PAM} = \frac{1}{2}E_{s,4PAM}$

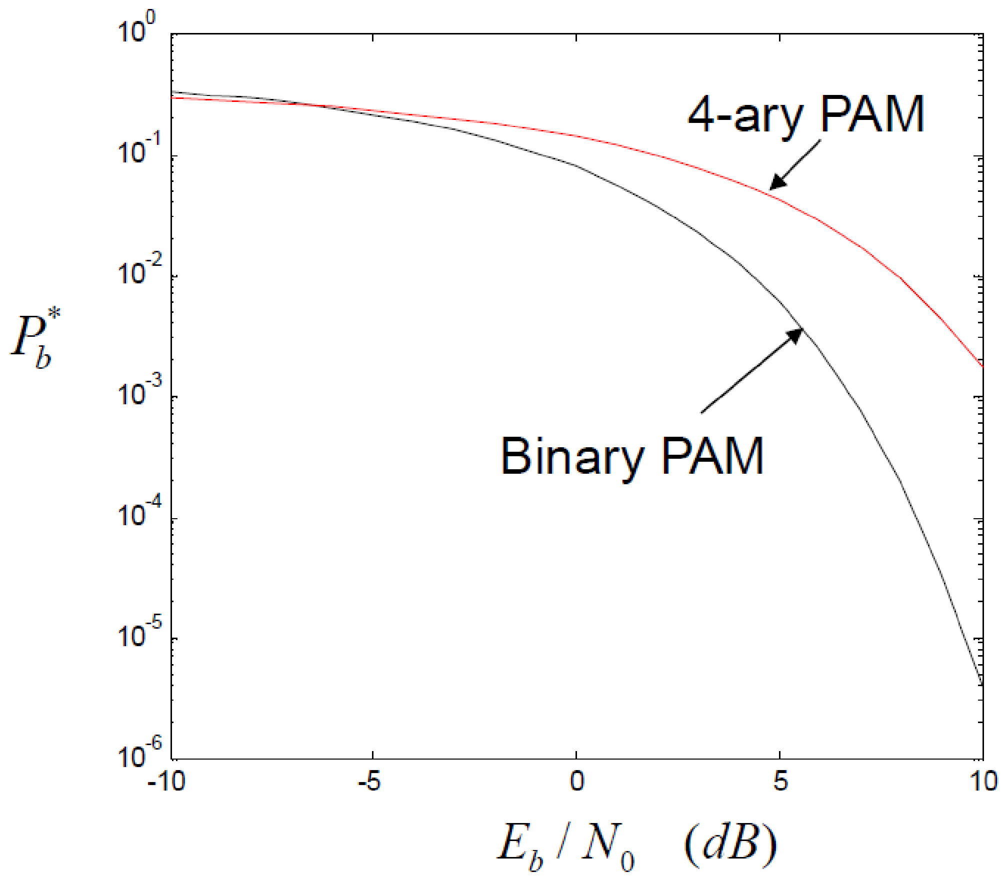
Performance Comparison of Binary PAM and 4-ary PAM

	BER (optimal receiver)	Bandwidth Efficiency (90% in-band power)
Binary PAM	$Q\left(\sqrt{\frac{2E_{b,BPAM}}{N_0}}\right)$	1
4-ary PAM	$\frac{3}{4}Q\left(\sqrt{\frac{0.8E_{b,4PAM}}{N_0}}\right)$	2

- 4-ary PAM is more bandwidth-efficient, but more susceptible to noise.

BER Comparison of Binary PAM and 4-ary PAM

- Suppose $E_{b,BPAM} = E_{b,4PAM} = E_b$

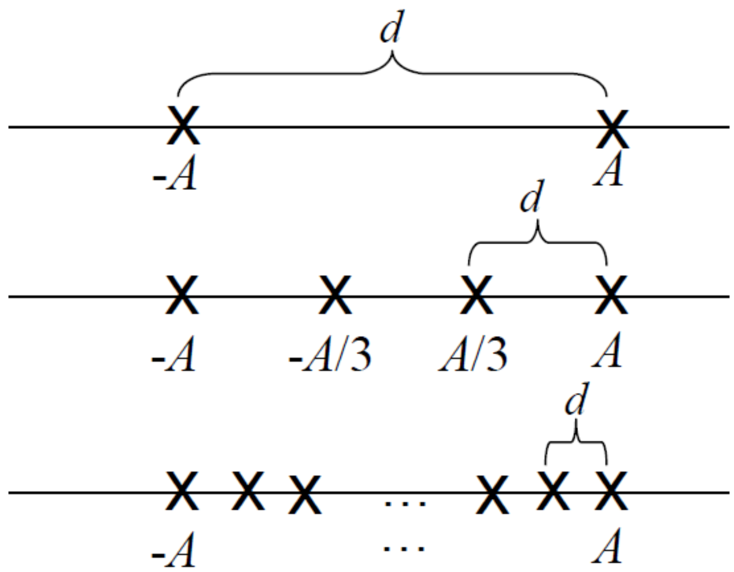


Constellation Representation of M-ary PAM

Binary PAM:

4-ary PAM:

M-ary PAM:



$$d = \frac{A-(-A)}{M-1}$$

$$s_i(t) = A-(i-1)\cdot d$$
$$i=1,\ldots,M,\ 0\le t\le\tau$$

- Energy per symbol:

$$E_s = \frac{1}{M}\sum_{i=1}^{M}\int_0^{\tau} s_i^2(t)dt = \frac{\tau}{M}\sum_{i=1}^{M}(A-(i-1)\cdot d)^2 = \frac{M+1}{3(M-1)}A^2\tau$$

- Energy difference:

$$E_d = \int_0^{\tau}\left(s_1(t)-s_2(t)\right)^2 dt = \tau\cdot d^2 = \frac{4A^2\tau}{(M-1)^2} = \frac{12E_S}{(M+1)(M-1)}$$

Given E_s, E_d decreases as M increases!

SER of M-ary PAM

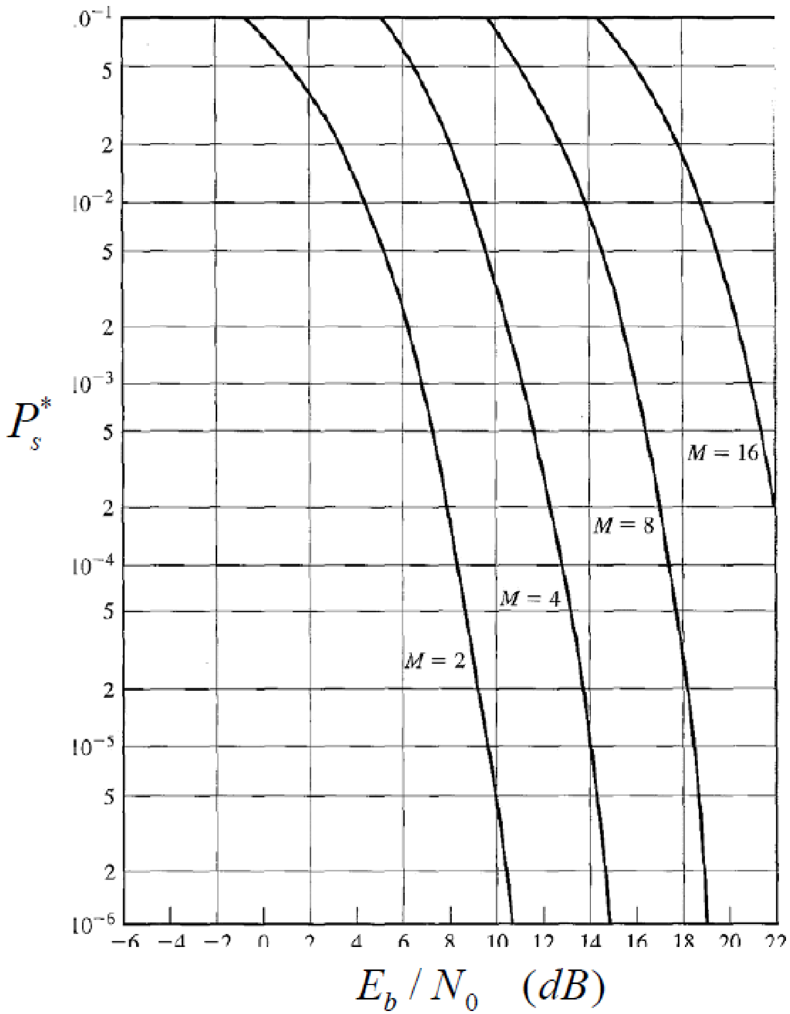
- SER of M-ary PAM:

$$\left.\begin{aligned} P_s^* &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{E_d}{2N_0}}\right) \\ E_d &= \frac{12}{M^2-1} E_s \\ E_s &= E_b \log_2 M \end{aligned}\right\} \quad \begin{aligned} P_s^* &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_b \log_2 M}{N_0(M^2-1)}}\right) \\ E_d &= \frac{12 \log_2 M}{M^2-1} E_b \end{aligned}$$

- Given E_b,
  - ✓ E_d decreases as M increases;
  - ✓ P_s^* increases as M increases.

A larger M leads to a smaller energy difference ---- a higher SER
(As two symbols become closer in amplitude, distinguishing them becomes harder.)

Performance of M-ary PAM



- Fidelity performance of M-ary PAM:

$$P_s^* = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_b \log_2 M}{N_0(M^2-1)}}\right)$$

- Bandwidth Efficiency of M-ary PAM:

$$\gamma_{MPAM} = k = \log_2 M$$

(with 90% in-band power)

With an increase of M, M-ary PAM becomes:

1) more bandwidth-efficient;

2) more susceptible to noise.

M-ary PSK

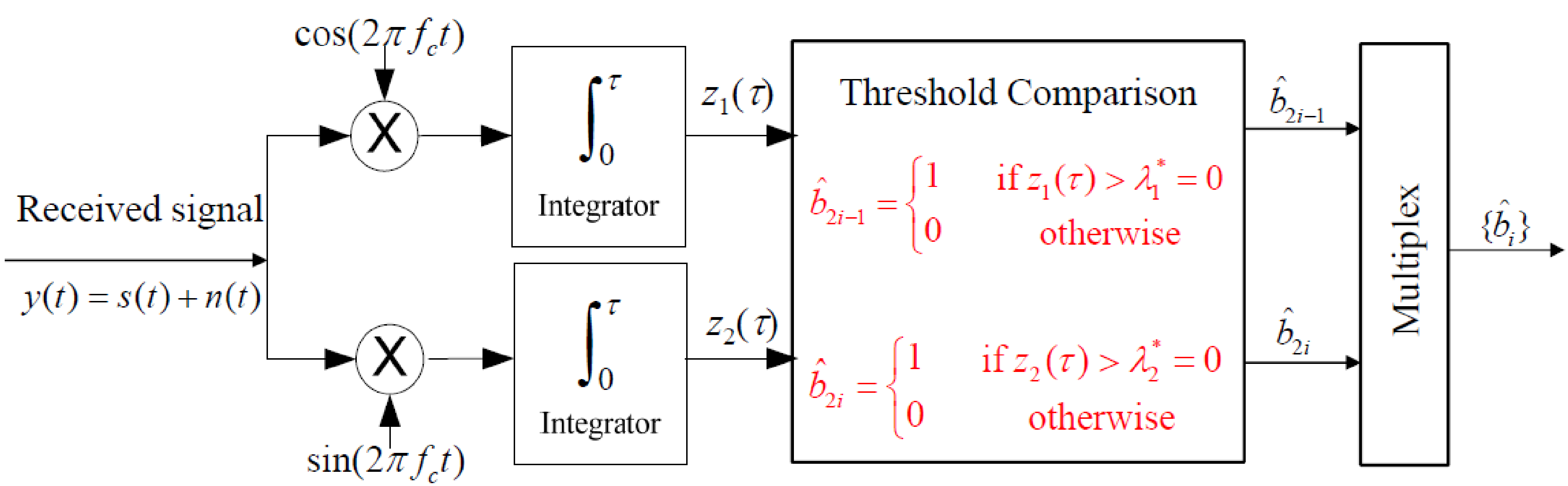
Coherent Demodulator of QPSK

- QPSK $s_{QPSK}(t) = d_I \frac{A}{\sqrt{2}} \cos(2\pi f_c t) + d_Q \frac{A}{\sqrt{2}} \sin(2\pi f_c t)$

a combination of two orthogonal BPSK signals

$$d_I = \begin{cases} 1 & \text{if } b_{2i-1} = 1 \\ -1 & \text{if } b_{2i-1} = 0 \end{cases} \qquad d_Q = \begin{cases} 1 & \text{if } b_{2i} = 1 \\ -1 & \text{if } b_{2i} = 0 \end{cases}$$

- Coherent Demodulator of QPSK



BER of Coherent QPSK

- BER of Coherent QPSK:

$$P^*_{b,QPSK} = Q\left(\sqrt{\frac{2E_{b,QPSK}}{N_0}}\right)$$

QPSK has the same BER performance as BPSK if $E_{b,QPSK}=E_{b,BPSK}$, but is more bandwidth-efficient!

Energy per bit: $E_{b,QPSK} = \frac{1}{2}E_{s,QPSK} = \frac{A^2\tau}{4} = \frac{A^2}{2R_{b,QPSK}}$

Energy per symbol: $E_{s,QPSK} = \frac{A^2\tau}{2} = \frac{A^2}{2R_{s,QPSK}} = \frac{A^2}{R_{b,QPSK}}$

Performance Comparison of BPSK and QPSK

	BER (coherent demodulation)	Bandwidth Efficiency (90% in-band power)
BPSK	$Q\left(\sqrt{\frac{2E_{b,BPSK}}{N_0}}\right)$	0.5
QPSK	$Q\left(\sqrt{\frac{2E_{b,QPSK}}{N_0}}\right)$	1

- What if the two signals, QPSK and BPSK, are transmitted with the same amplitude A and the same bit rate?

  Equally accurate!
  (BPSK requires a larger bandwidth)

- What if the two signals, QPSK and BPSK, are transmitted with the same amplitude A and over the same channel bandwidth? BPSK is more accurate!
  (but lower bit rate)

M-ary PSK

- M-ary PAM: transmitting pulses with M possible different amplitudes, and allowing each pulse to represent $\log_2 M$ bits.

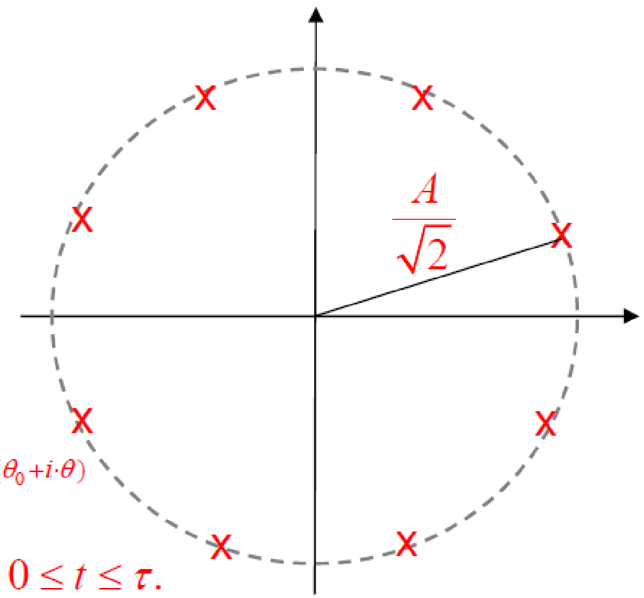
- M-ary PSK: transmitting pulses with M possible different carrier phases, and allowing each pulse to represent $\log_2 M$ bits.

$$s_i(t) = A\cos(2\pi f_c t + \theta_0 + i\cdot\theta)$$

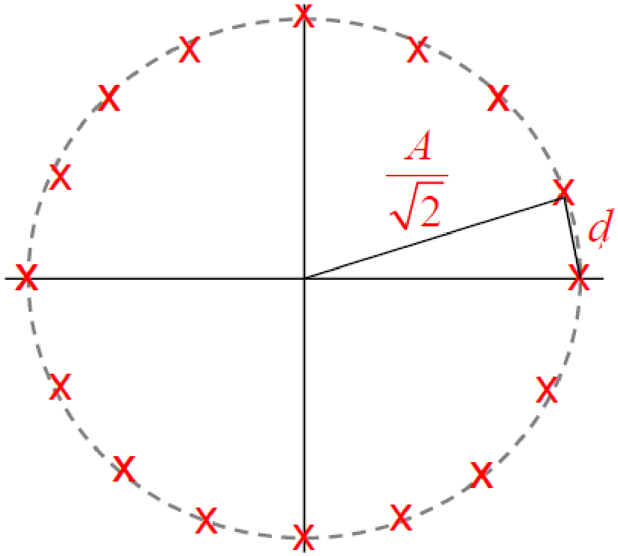
$$i = 0, \ldots, M-1, \; 0 \le t \le \tau. \quad \theta = \frac{2\pi}{M}$$

$$\Rightarrow \quad s_i(t) = \frac{A}{\sqrt{2}} e^{j(\theta_0 + i\cdot\theta)}$$

$$i=0, \ldots, M\text{-}1, \; 0 \le t \le \tau.$$



SER of M-ary PSK



$$= \sqrt{2}A\sin\frac{\pi}{M}$$

- What is the minimum phase difference between symbols?

$$2\pi / M$$

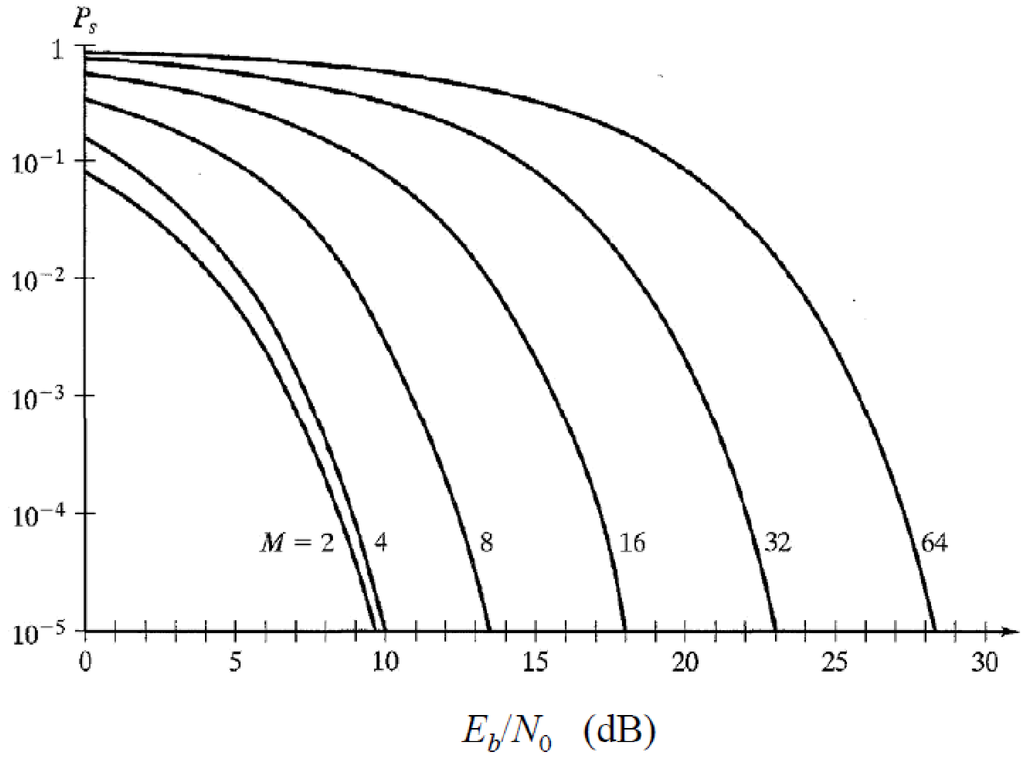
- What is the energy difference between two adjacent symbols?

$$E_d = \tau \cdot d^2 = 2A^2\tau\sin^2\frac{\pi}{M} = 4E_S\sin^2\frac{\pi}{M}$$

- What is the SER with optimal receiver?

$$P_s^* \approx 2Q\left(\sqrt{\frac{2E_S}{N_0}}\sin\frac{\pi}{M}\right) \quad \text{with a large M}$$

SER of M-ary PSK



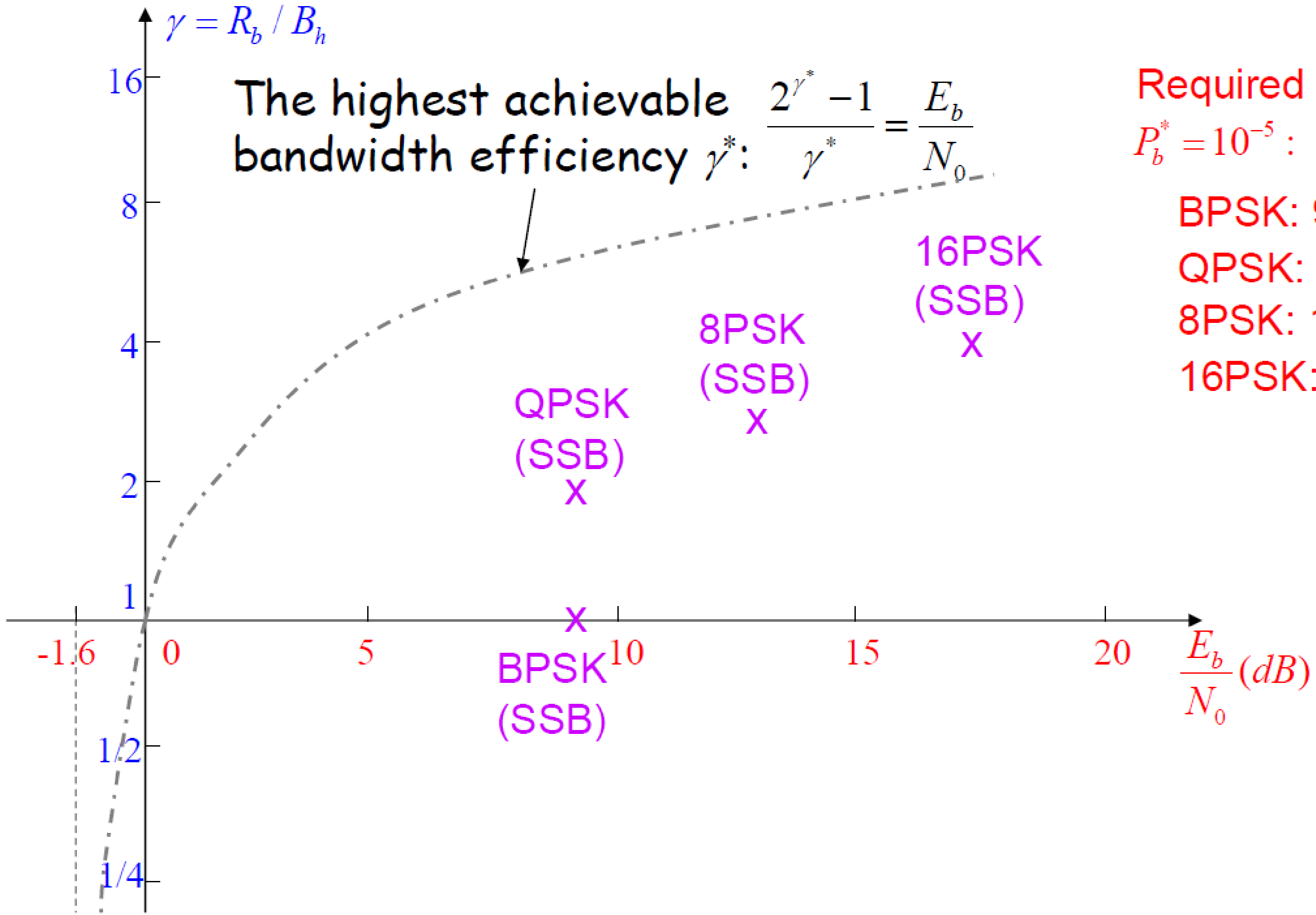
- A larger M leads to a smaller energy difference ---- a higher SER

(As two symbols become closer in phase, distinguishing them becomes harder.)

Summary II: M-ary Modulation and Demodulation

	Bandwidth Efficiency	BER
Binary PAM	1 (90% in-band power)	$Q\left(\sqrt{\frac{2E_{b,BPAM}}{N_0}}\right)$
4-ary PAM	2 (90% in-band power)	$\frac{3}{4}Q\left(\sqrt{\frac{0.8E_{b,4PAM}}{N_0}}\right)$
M-ary PAM (M>4)	$\log_2 M$ (90% in-band power)	$\approx \frac{2}{\log_2 M}Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\cdot\frac{E_{b,MPAM}}{N_0}}\right)$
Binary PSK	0.5 (90% in-band power)	$Q\left(\sqrt{\frac{2E_{b,BPSK}}{N_0}}\right)$
QPSK	1 (90% in-band power)	$Q\left(\sqrt{\frac{2E_{b,QPSK}}{N_0}}\right)$
M-ary PSK (M>4)	$\frac{1}{2}\log_2 M$ (90% in-band power)	$\approx \frac{2}{\log_2 M}Q\left(\sqrt{2\sin^2\frac{\pi}{M}\log_2 M\cdot\frac{E_{b,MPSK}}{N_0}}\right)$

Performance Comparison of Digital Modulation Schemes



Required E_b/N_0 when $P_b^* = 10^{-5}$:

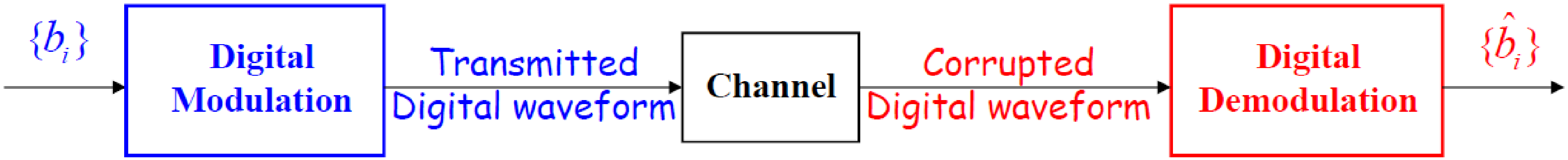
BPSK: 9.6dB

QPSK: 9.6dB

8PSK: 12.9dB

16PSK: 17.4dB

Digital Communications Systems



- **Bandwidth Efficiency**

$$\gamma \triangleq \frac{\text{Information Bit Rate } R_b}{\text{Required Channel Bandwidth } B_h}$$

- **BER (Fidelity Performance)**

Binary: $P_b = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$

- What is the highest bandwidth efficiency for given E_b/N_0?

  Information theory -- AWGN channel capacity

- How to achieve the highest bandwidth efficiency?

  Channel coding theory

- What if the channel is not an LTI system? Wireless communication theory