Lecture 11.

MFSK Error Analysis

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Lecture Outline

- **Have previously considered**
  - Optimum detector structure.
  - The optimum receiver (in the sense of minimizing P_e) for general M-ary signaling in the presence of AWGN
  - Graphical interpretation of decision regions

» **Will now consider**
  - » Probability of error expressions
    - » Case study: Orthogonal signaling
  - » Union bound on Pe for generic M-ary modulations
  - » Orthogonal signaling & its variations

Prelude

Performance evaluation of M-ary modulation:

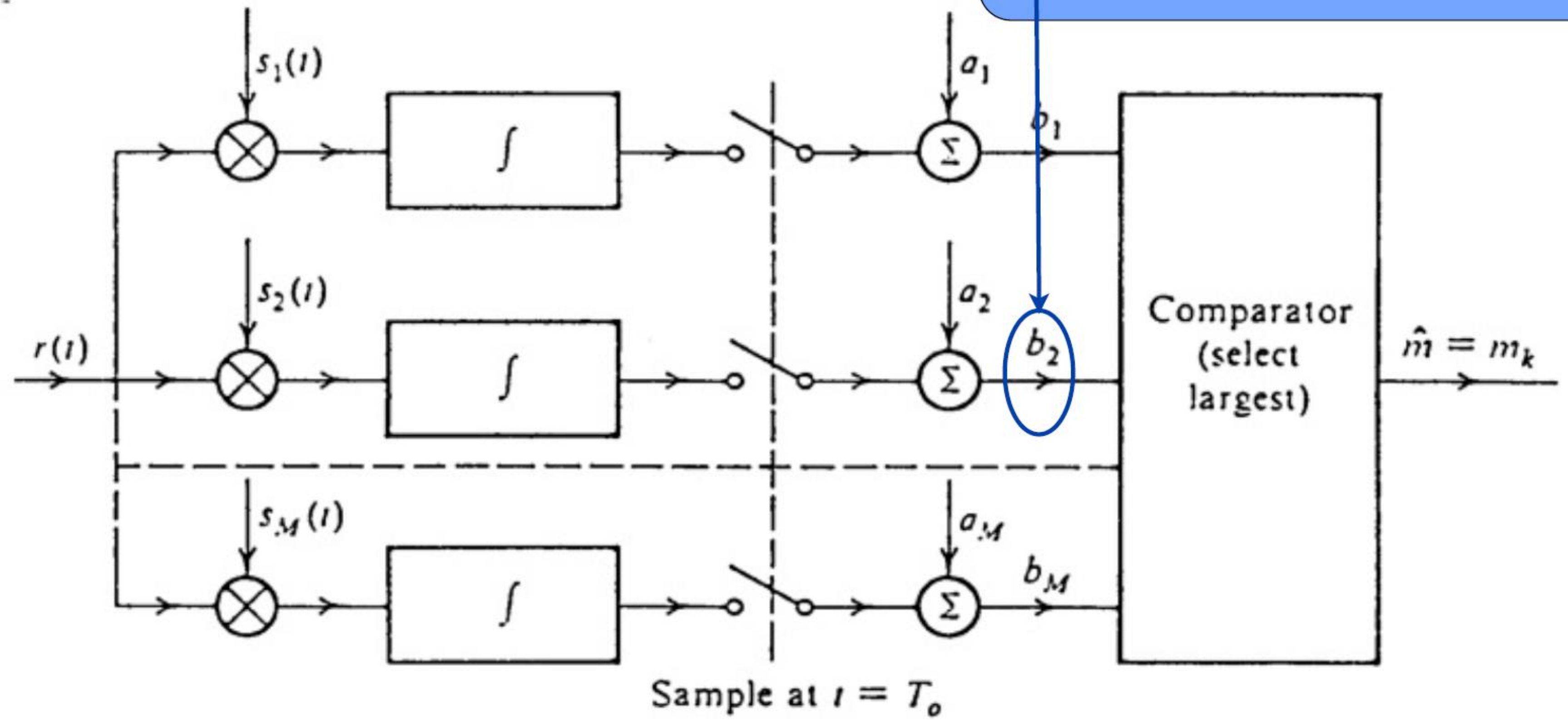
- Exact error probability computation is quite complicated
- Engineers typically look for good approximation that makes system analysis and design less complicated

Union Bound will be developed

- A common tool used by communication engineers
- Very easy to derive
- Can give accurate probability estimates

Recall the Optimum M-ary Receiver

$$b_k = \int_0^{T_s} r(t)s_k(t)dt - E_k/2$$



General Expression for P_e

If m_1 is sent, ⇒ correct decision is made only if $b_1 > b_2, b_3, \ldots, b_M$

$$P(c/m_1) = P(b_1 > b_2, b_3, \ldots, b_M / m_1)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{b_1}\cdots\int_{-\infty}^{b_1} f(b_1, b_2, \ldots, b_M / m_1)\, db_M\, db_{M-1} \ldots db_1$$

b_1 .... b_M

⇒

$$P(c) = \sum_{j=1}^{M} P(c/m_j)P(m_j)$$

and $$P_e = P_{eM} = 1 - P(c)$$

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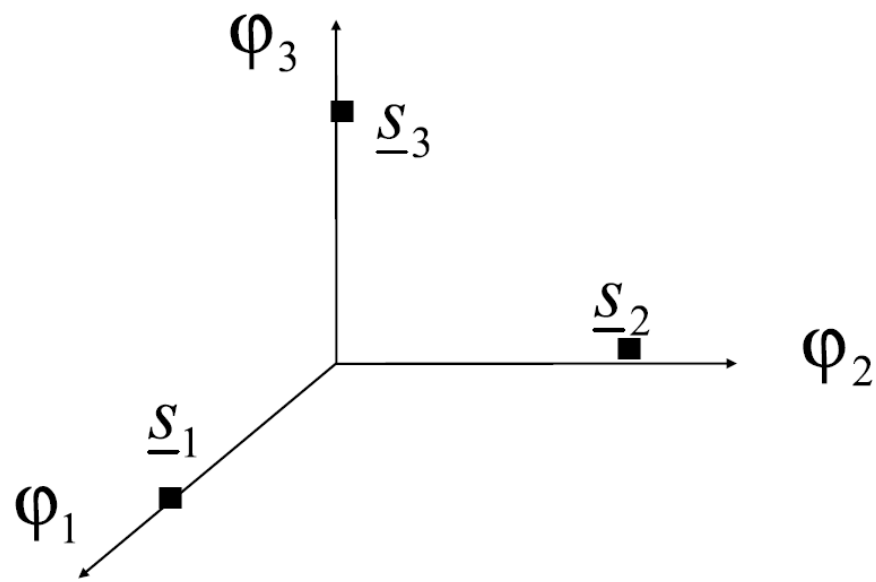
P_e for Orthogonal Signals

$$s_k(t) = \sqrt{E}\varphi_k(t) \qquad \underline{s}_j \cdot \underline{s}_k = \begin{cases} 0 & j \neq k \\ E & j = k \end{cases}$$

Assume equiprobable messages

$$P(m_k) = \frac{1}{M} \qquad \forall k = 1,2,\ldots,M$$



P_e for Orthogonal Signals

Now, since $\{s_k\}$ is an orthogonal set

$$b_k = \begin{Bmatrix} E + n_1 & \text{if k=1} \\ n_k & \text{otherwise} \end{Bmatrix} \quad \text{where } n_k = \int_0^{T_s} n(t) s_k(t) dt$$

$\{n_k\}$ are i.i.d. Gaussian r.v.'s $\quad \overline{n}_k = 0 \quad E[n_k^2] = EN_0/2$

$$\Rightarrow f(b_1, b_2, \ldots, b_M / m_1) = \frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_1 - E)^2}{N_0 E}} \prod_{k=2}^{M} \left[\frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_k)^2}{N_0 E}} \right]$$

$$\Rightarrow P(c / m_1) = \frac{1}{\sqrt{\pi N_0 E}} \int_{-\infty}^{\infty} e^{-\frac{(b_1 - E)^2}{N_0 E}} \left\{ \prod_{k=2}^{M} \int_{-\infty}^{b_1} \frac{1}{\sqrt{\pi N_0 E}} e^{-\frac{(b_k)^2}{N_0 E}} db_k \right\} db_1$$

P_e for Orthogonal Signals

Now note that since signal set is geometrically symmetric:

$$P(c/m_1) = P(c/m_2) = \ldots = P(c/m_M)$$

$$P(c) = P(c/m_1)$$

and

$$P_e = P_{eM} = 1 - P(c)$$

P_e for Orthogonal Signals

Let $\lambda = \dfrac{E}{N_0}$ and $y = x + \sqrt{\dfrac{2E}{N_0}}$

$$\Rightarrow \quad P_{eM} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{[1 - Q[y]]\}^{M-1} e^{-\frac{(y-\sqrt{2\lambda})^2}{2}} dy$$

Overall Probability of Symbol Error.

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P_e for Orthogonal Signals

Let $$x = \frac{b_1 - E - a}{\sqrt{\frac{N_0 E}{2}}}$$ ⇒

$$P(c/m_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ 1 - Q\left[x + \sqrt{\frac{2E}{N_0}} \right] \right\}^{M-1} e^{-\frac{x^2}{2}} dx$$

Symbol energy vs bit energy

Next recall that

- # bits that can be represented by a set of M signals:

$$k = log_2(M)$$

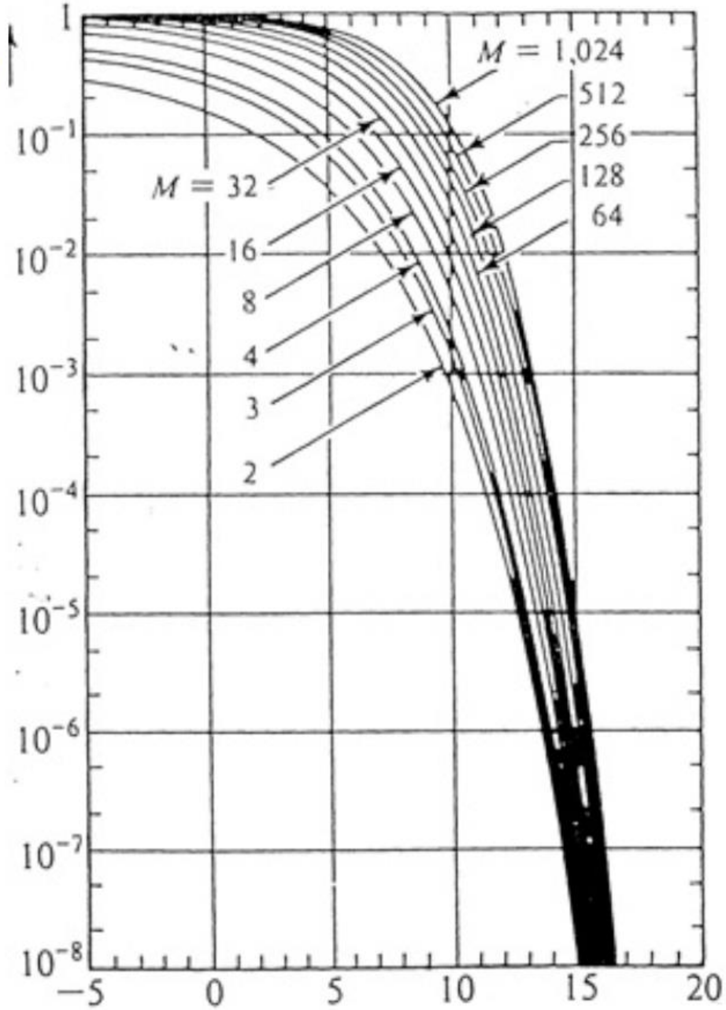
- **Energy per bit**

$$E_b = \frac{E}{\log_2(M)} = \frac{E}{k}$$

$$\Rightarrow \quad \frac{2E}{N_0} = \frac{2kE_b}{N_0}$$

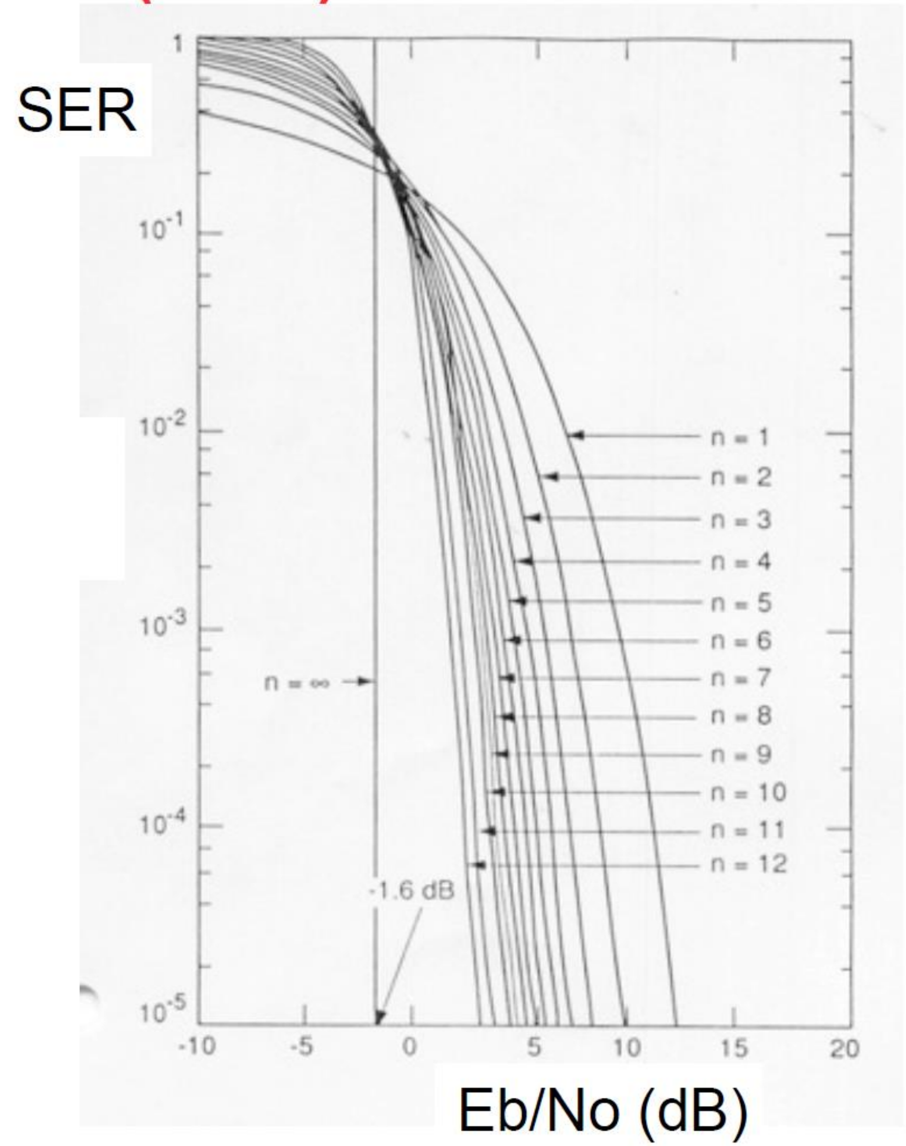
Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Es/No

SER



Es/No (dB)

Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No



Symbol Errors to Bit Errors

Symbol errors are different from bit errors.

When a symbol error occurs all $k=\log_2(M)$ bits could be in error

For orthogonal modulation when an error occurs anyone of the other symbols may result equally likely

On average therefore half the bits will be incorrect

That is k/2 bits in error every k bits will on average be in error when there is a symbol error

Therefore for a particular bit the probability of error is half the symbol error

$$P_e \cong \frac{1}{2} P_{eM}$$

Exact derivation

In orthogonal modulation when there is an error it will lead to any one of the other $M-1=2^k-1$ possible symbols equally. That is, when there is an error event the probability of a particular symbol getting that error is $P_{eM}/(M-1)$

For this given symbol error assume there are n bits in error

There are (k,n) combinations in which this may happen and therefore (k,n) symbols in total with a possible n bit errors.

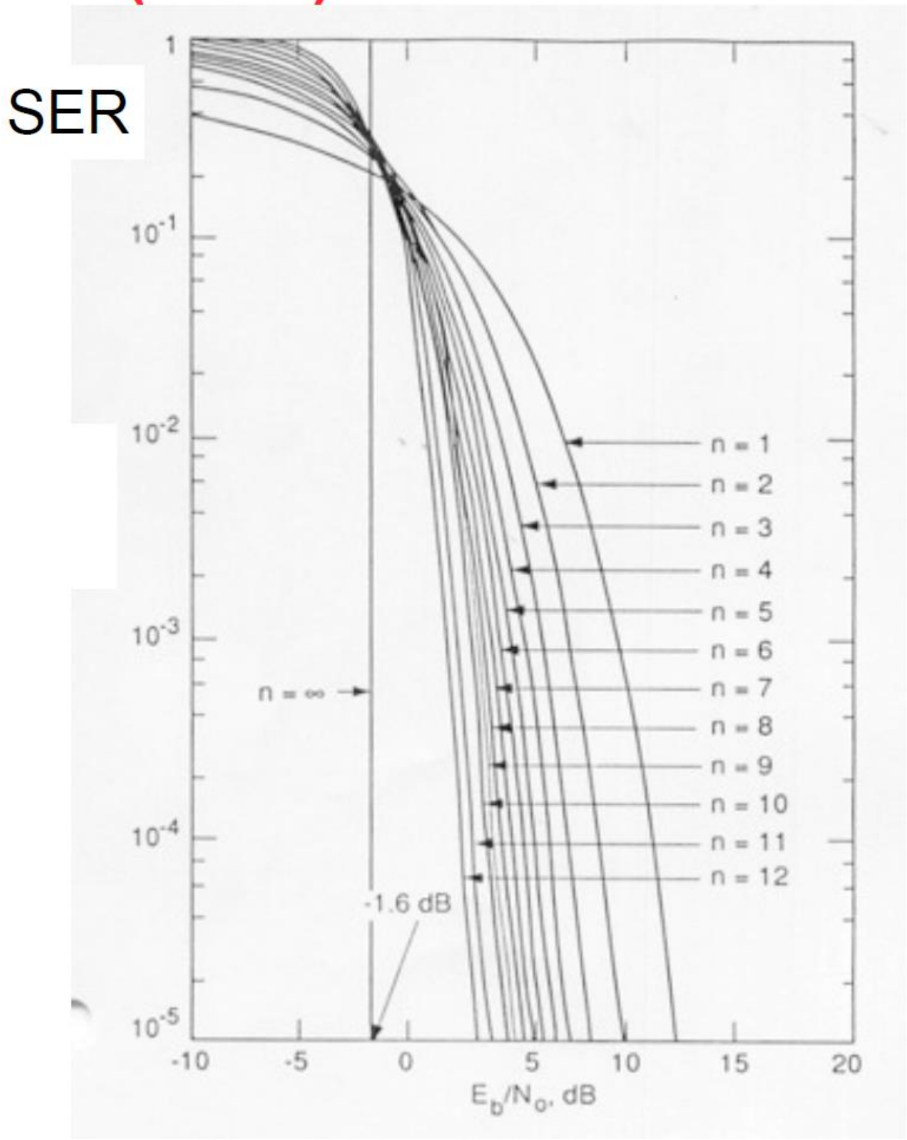
Therefore the probability of a n bit errors occurring is $\binom{k}{n}\frac{P_{eM}}{(M-1)}$

Thus for every k bits there will be on average $\sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{(M-1)}$ bit errors

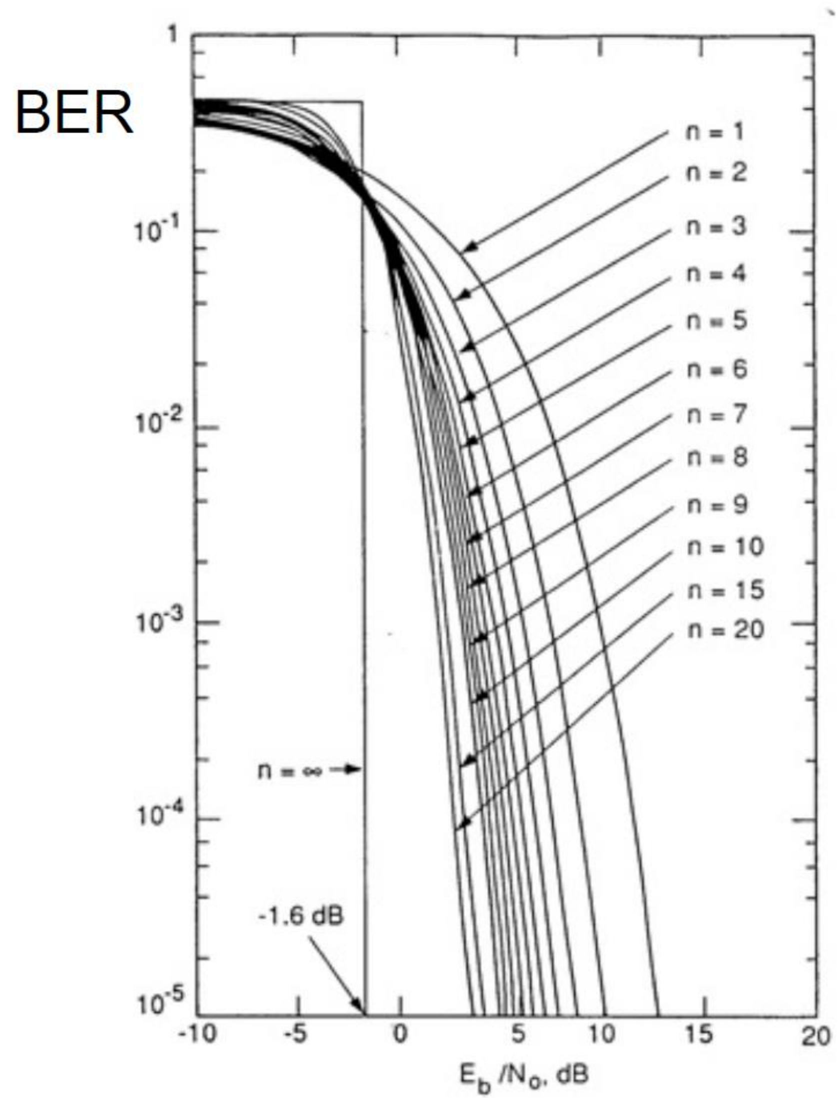
Therefore $$P_b = P_e = \frac{1}{k}\sum_{n=1}^{k} n\binom{k}{n}\frac{P_{eM}}{(M-1)} = \frac{M}{2(M-1)}P_{eM}$$ and for large M

$$P_e \cong \frac{1}{2}P_{eM}$$

Symbol error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No



Bit error probability for M orthogonal signals ($M=2^n$) as a function of Eb/No



M-ary Communication
Performance Evaluation

So far, we have considered the performance evaluation of M-ary digital modulation.

- Typically involve the computation of error probability.
- Such computation is often very complicated.
- Derived the symbol error probability of orthogonal signals.
- Derived the bit error probability of orthogonal signals.

Bit error probability and **Symbol** error probability **ARE DIFFERENT**.

Bit errors

Signal errors

Union Bound

Multi-dimension integral and quite difficult to evaluate

$$P(error/m_j) = 1 - P(\bigcap(b_j > b_k / m_j \quad \forall k \neq j))$$

OR

$$P(error/m_j) = P(\bigcup(b_j \leq b_k / m_j \quad \forall k \neq j))$$

Now we simplify Pe calculation using an ***approximation*** known as the **union upper bound**

But note that

$$P(error/m_j) \leq \sum_{\substack{k=1 \\ k \neq j}}^{M} P(b_j \leq b_k / m_j)$$

$$P(\bigcup_i A_i) \leq \sum_i P(A_i)$$

Union Bound

The key approximation is that there may be several pairwise comparisons that imply the same symbol error

The union bound does not subtract out this intersecting possibility- therefore it is an upper bound

Now for equally likely symbols,

$$P_{eM} = \frac{1}{M} \sum_{j=1}^{M} P(error / m_j)$$

$$P_{eM} \leq \frac{1}{M} \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k \neq j}}^{M} P(b_j \leq b_k / m_j)$$

Pairwise error probability

Union Bound

Let $P_e(j,k)$ = Pairwise error probability for signals j and k

$$P_e(j,k) = Q\left[\sqrt{\frac{d_{kj}^2}{2N_0}}\right]$$

where $$d_{ij}^2 = \int_0^T \left[s_i(t) - s_j(t)\right]^2 dt = \left\| \underline{s}_i - \underline{s}_j \right\|^2 = 2E$$

$$\Rightarrow \quad P_{eM} \leq (M-1)Q\left[\sqrt{\frac{E}{N_0}}\right]$$

$$P_b = \frac{M}{2(M-1)} P_{eM} \quad \Rightarrow \quad \boxed{P_b \leq \frac{M}{2} Q\left[\sqrt{\frac{E}{N_0}}\right]}$$

Union Bound for Orthogonal signals

Also we can upper bound the Q function with (different from previous approximation)

$$Q[x] \le \frac{1}{2} e^{-\frac{x^2}{2}}$$ → Quite accurate for $x \ge 3$

$$P_b \le \frac{M}{4} e^{-\frac{E}{2N_0}}$$ → Widely used

where

$$k = log_2(M)$$

$$E_b = \frac{E}{\log_2(M)} = \frac{E}{k}$$

$$P_b \approx \frac{M}{4} e^{-\frac{kE_b}{2N_0}}$$

Comparison of union bound with the exact P_e

Comparison of union bound with exact result for orthogonal signals

	$E/N_0 = 18.2$	
M	Exact $P(\varepsilon)$	Union bound
2	10^{-5}	10^{-5}
4	2.9×10^{-5}	3×10^{-5}
8	6.9×10^{-5}	7×10^{-5}
16	1.45×10^{-4}	1.5×10^{-4}
32	2.70×10^{-4}	3.1×10^{-4}
64	5.1×10^{-4}	6.3×10^{-4}
128	1.1×10^{-3}	1.27×10^{-3}

Union Bound

- Have shown that the Union bound is a nice and good approximation.
- Must keep in mind that one does not know a priori (before hand) when such approximation is accurate or not.
- In general, SIMULATION is the practical alternative.
- Typical bit error rate is small. Hence, simulation can also take a very long time.