Lecture 3.
Random Signal Analysis

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Discrete Random Variables

- A discrete random variable takes on a countable number of possible values.

Suppose that a discrete random variable X takes on one of the values $x_1, \ldots, x_n$.

✓ Distribution functions:

$$\boxed{\sum_{i=1}^{n} p(x_i) = 1}$$

Probability Mass Function: $p(x_i) = \Pr\{X = x_i\}$

Cumulative Distribution Function: $F(a) = \Pr\{X \le a\} = \sum_{x_i \le a} p(x_i)$

✓ Moments:

Expected Value, or Mean: $\mu_X = E[X] = \sum_{i=1}^{n} x_i p(x_i)$

The m-th Moment: $E[X^m] = \sum_{i=1}^{n} x_i^m p(x_i), \ m = 1, 2, \ldots$

Continuous Random Variables

- A continuous random variable has an uncountable set of possible values.

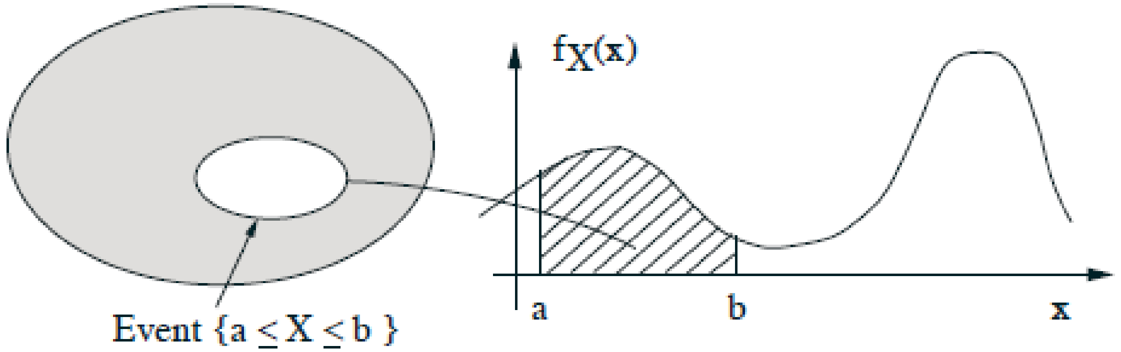
  X is a continuous random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers, $\Pr\{X \in B\} = \int_B f(x)dx$.

  ✓ f is called the **probability density function (pdf)** of X, denoted as: $f_X(x)$

$$\boxed{\int_{-\infty}^{\infty} f_X(x)dx = 1}$$

Continuous r.v.'s and pdf's

- A continuous r.v. is described by a probability density function f_X



$$\mathrm{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \qquad \mathrm{P}(X \in B) = \int_B f_X(x)\,dx, \quad \text{for “nice” sets } B$$

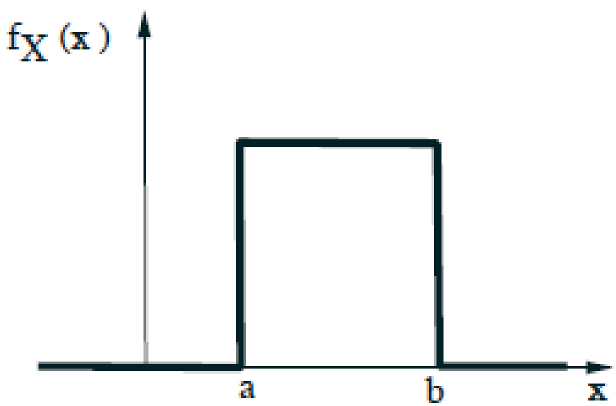
Means and variances

- $\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

- $\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

- $\mathrm{var}(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \mathrm{E}[X])^2 f_X(x)\, dx$

- **Continuous Uniform r.v.**



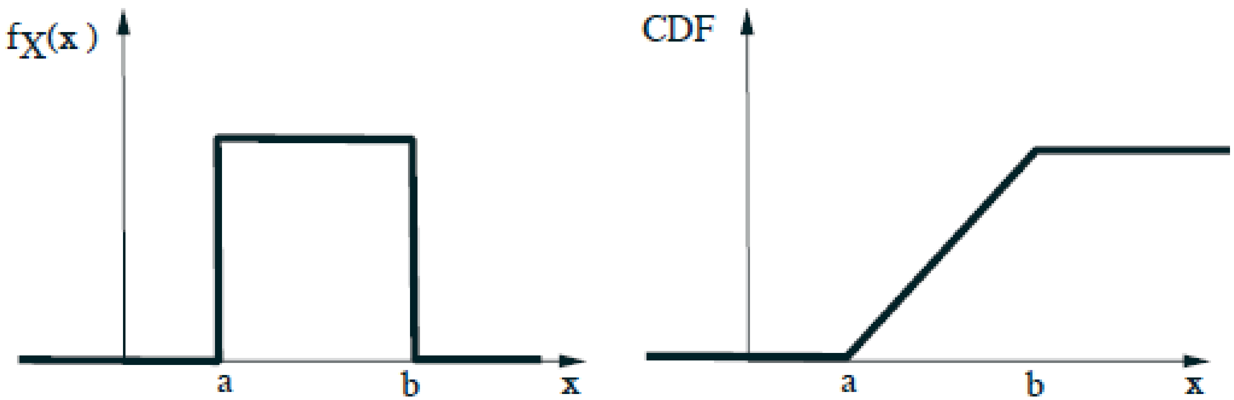
- $f_X(x) = \qquad\qquad a \leq x \leq b$

- $\mathrm{E}[X] =$

- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\, dx = \frac{(b-a)^2}{12}$

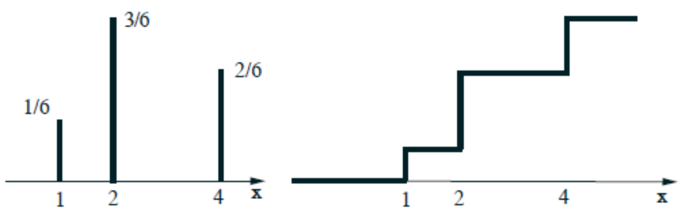
Cumulative distribution function (CDF)

$$F_X(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\, dt$$



- Also for discrete r.v.'s:

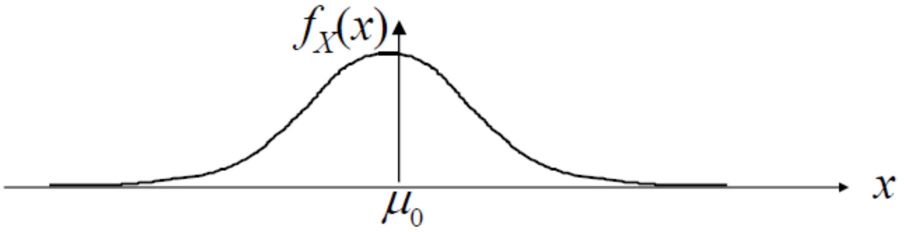
$$F_X(x) = \mathrm{P}(X \le x) = \sum_{k \le x} p_X(k)$$



Gaussian (Normal) Distribution

- X is a Gaussian random variable with parameters μ_0 and σ_0^2 if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2} \right)$$



- X is denoted as $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$

✓ Mean: $\mu_X = \mu_0$

✓ Variance: $\sigma_X^2 = \sigma_0^2$

✓ cdf: $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = 1 - \int_{a}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{ -\frac{(x-\mu_0)^2}{2\sigma_0^2} \right\} dx$

$$\overset{z=\frac{x-\mu_0}{\sigma_0}}{=} 1 - \int_{\frac{a-\mu_0}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{z^2}{2} \right\} dz = 1 - Q\left(\frac{a-\mu}{\sigma_0} \right)$$

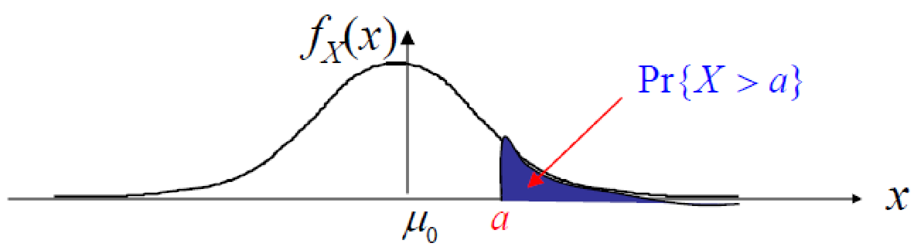
$$\boxed{Q(\alpha) \triangleq \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\} dx}$$

More about Q Function

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

- $Q(\alpha)$ is a decreasing function of α.
- For $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$,

$$\Pr\{X > a\} = \int_a^{\infty} f_X(x)dx = \int_a^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a-\mu_0}{\sigma_0}\right)$$



$$Q(x) \le \frac{1}{2} e^{-x^2/2}, \qquad \forall x \ge 0$$

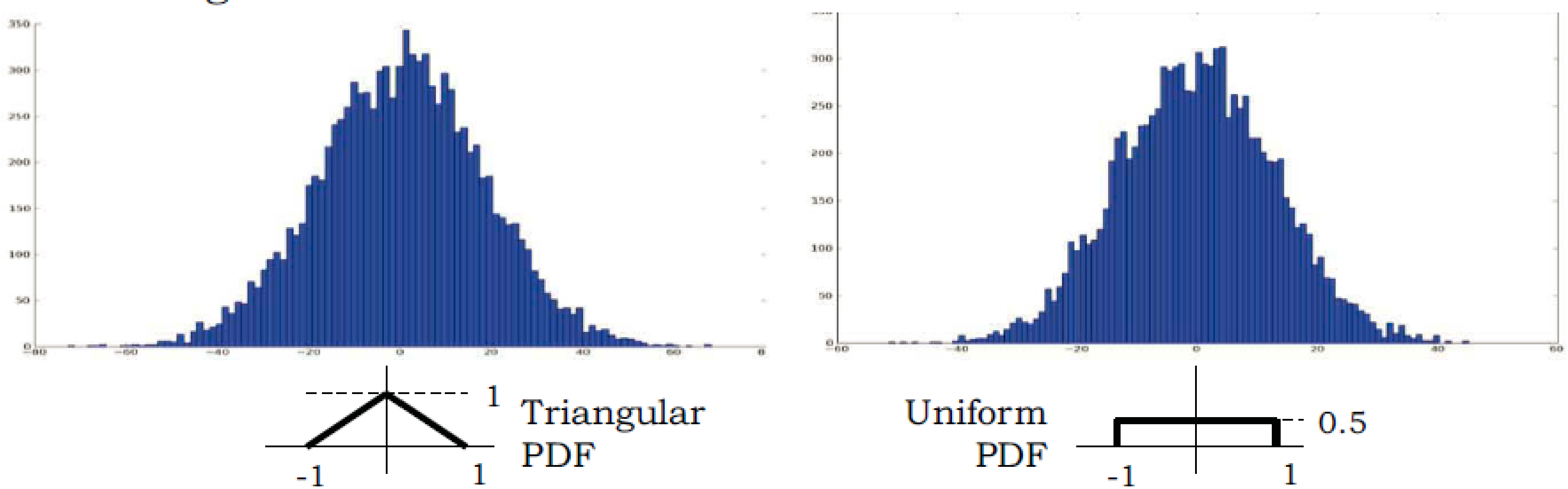
$$Q(x) < \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}, \qquad \forall x \ge 0$$

$$Q(x) > \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2}, \qquad \forall x \ge 0$$

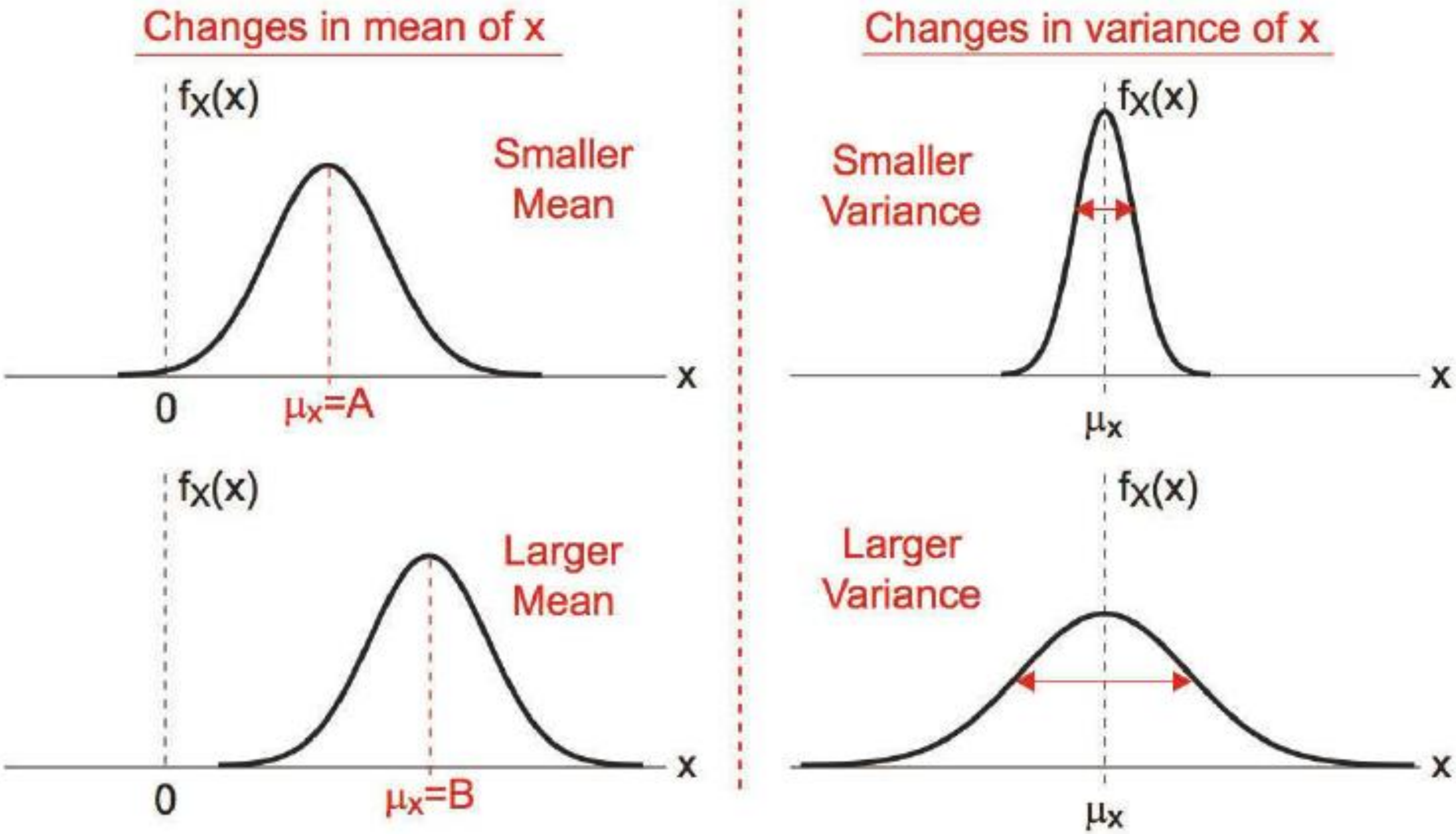
The Ubiquity of Gaussian Noise

The net noise observed at the receiver is often the sum of many small, independent random contributions from many factors. If these independent random variables have finite mean and variance, the Central Limit Theorem says their sum will be a *Gaussian*.

The figure below shows the histograms of the results of 10,000 trials of summing 100 random samples drawn from [-1,1] using two different distributions.



Visualizing Mean and Variance



Changes in mean shift the center of mass of PDF

Changes in variance narrow or broaden the PDF (but area is always equal to 1)

Joint PDF $f_{X,Y}(x,y)$

$$\mathrm{P}((X,Y) \in S) = \int\int_S f_{X,Y}(x,y)\,dx\,dy$$

- Interpretation:

$$\mathrm{P}(x \le X \le x+\delta,\ y \le Y \le y+\delta) \approx f_{X,Y}(x,y)\cdot\delta^2$$

- Expectations:

$$\mathrm{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- From the joint to the marginal:

$$f_X(x)\cdot\delta \approx \mathrm{P}(x \le X \le x+\delta) =$$

- X and Y are called independent if

Sum of i.i.d. rvs?

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \qquad \text{for all } x,y$$

Conditioning

- Recall

$$P(x \le X \le x + \delta) \approx f_X(x) \cdot \delta$$

- By analogy, would like:

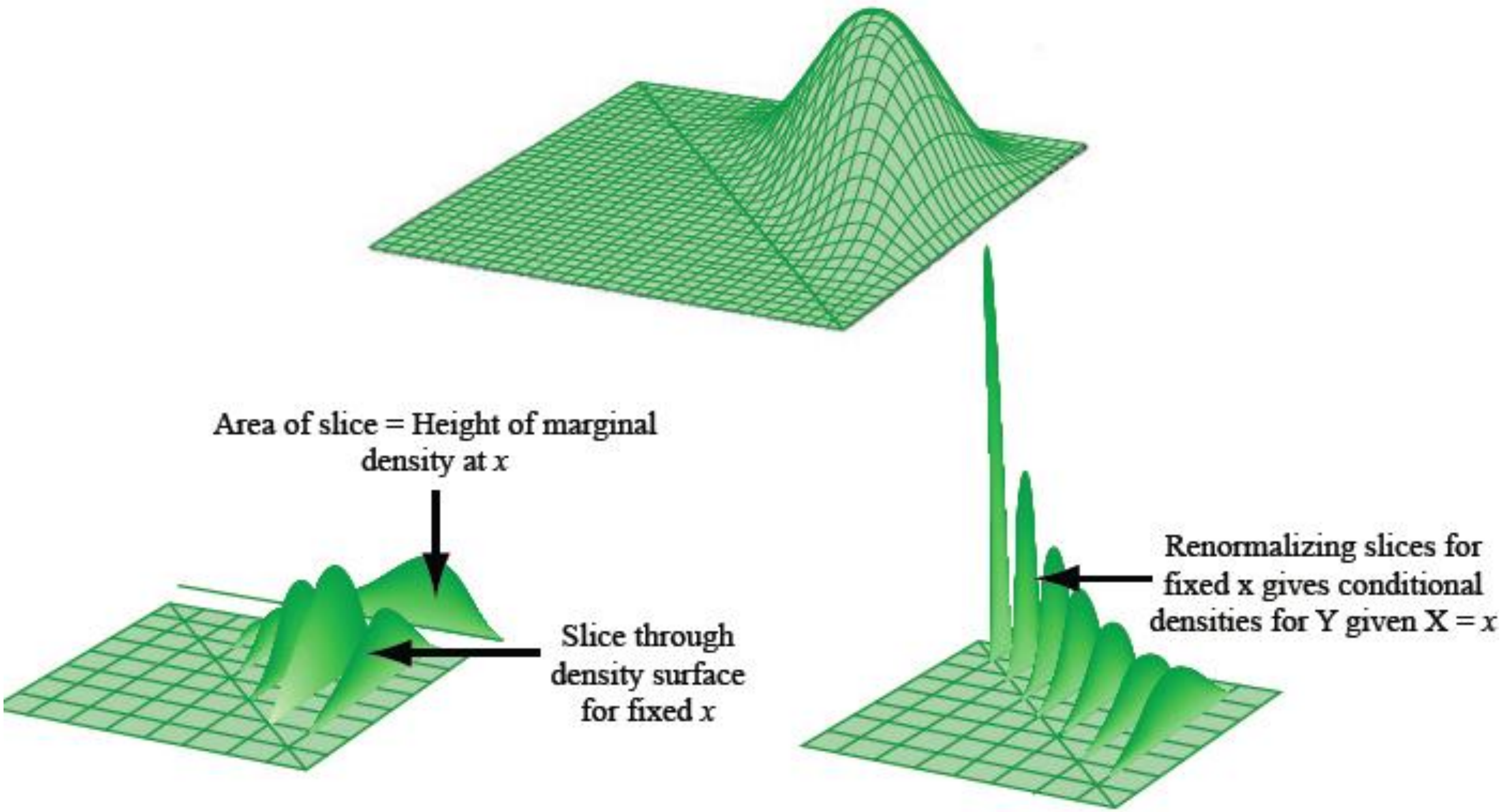
$$P(x \le X \le x + \delta \mid Y \approx y) \approx f_{X|Y}(x \mid y) \cdot \delta$$

- This leads us to the **definition**:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if } f_Y(y) > 0$$

- For given y, conditional PDF is a (normalized) "section" of the joint PDF

Joint, Marginal and Conditional Densities



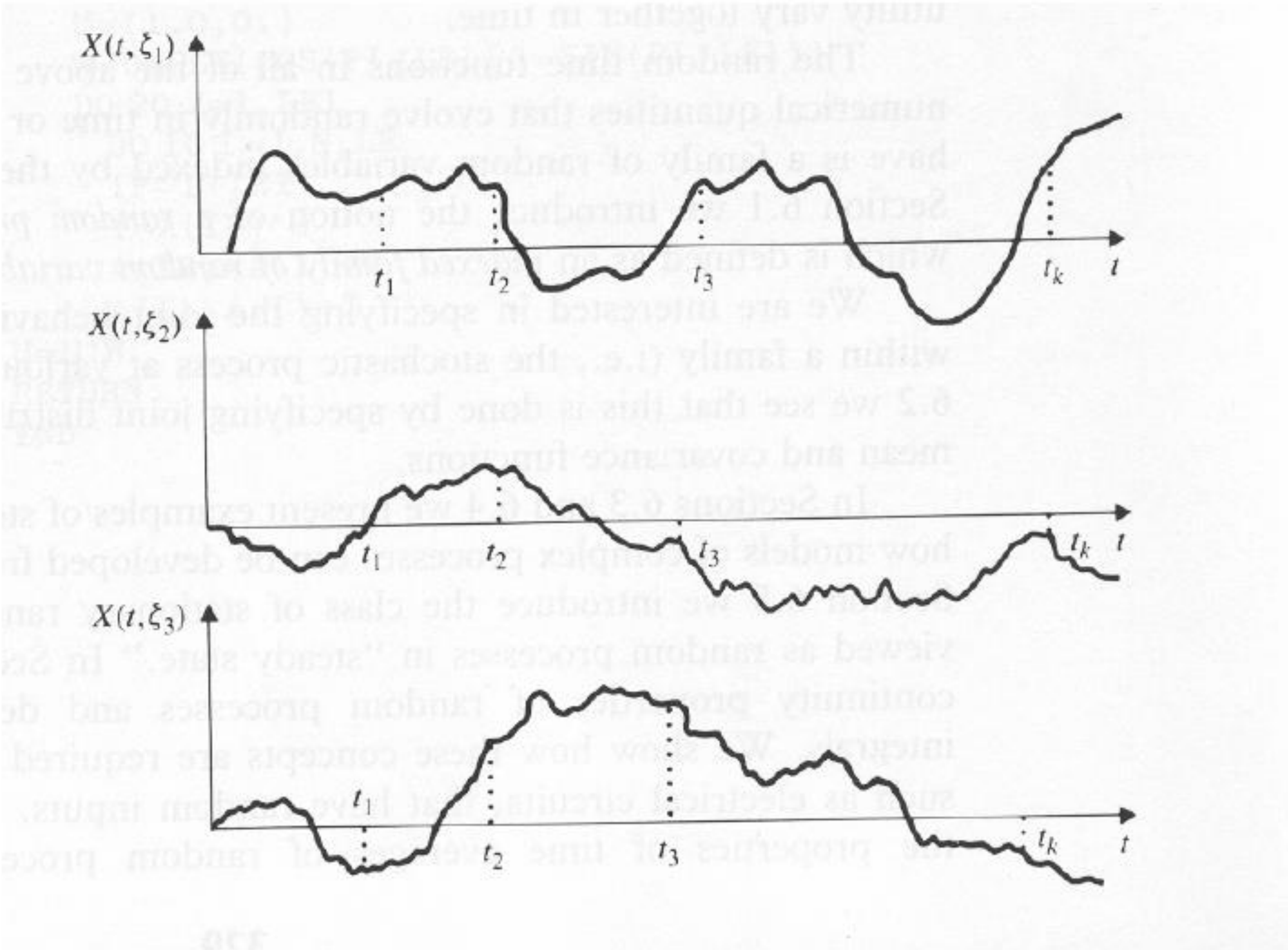
Definition of a Random Process

- Random experiment with sample space S.
- To every outcome $\zeta \in S$, we assign a function of time according to some rule:

$$X(t, \zeta) \qquad t \in I.$$

- For fixed ζ, the graph of the function $X(t, \zeta)$ versus t is a sample function of the random process.
- For each fixed t_k from the index set I, $X(t_k, \zeta)$ is a random variable.

- The indexed family of random variables $\{X(t, \zeta), t \in I\}$ is called a **random process** or **stochastic process**.
- A stochastic process is said to be **discrete-time** if the index set I is a countable set.
- A **continuous-time** stochastic process is one in which I is continuous.

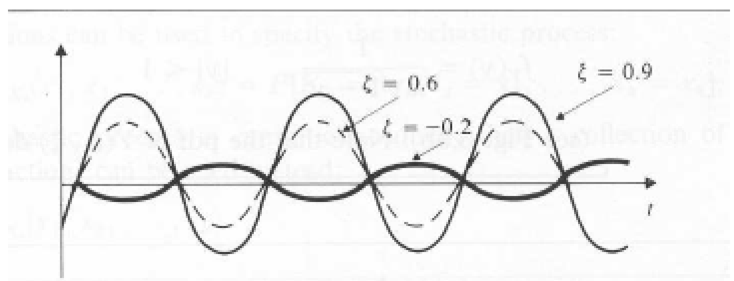


$X(t,\xi_1)$
t_1
t_2
t_3
t_k
t
$X(t,\xi_2)$
t_1
t_2
t_3
t_k
t
$X(t,\xi_3)$
t_1
t_2
t_3
t_k
t

Example:

1. Let $\zeta \in S = [-1, +1]$ be selected at random. Define the continuous-time random process $X(t, \zeta)$ by

$$X(t, \zeta) = \zeta \cos(2\pi t) \qquad -\infty < t < \infty.$$



Specifying a Random Process

Joint Distributions of Time Samples

- Let $X_1, X_2, \ldots, X_k$ be the k random variables obtained by sampling the random process $X(t, \zeta)$ at the time $t_1, t_2, \ldots, t_k$:

$$X_1 = X(t_1, \zeta), \quad X_2 = X(t_2, \zeta), \ldots, \quad X_k = X(t_k, \zeta).$$

- The joint behavior of the random process at these k time instants is specified by the joint cdf of $(X_1, X_2, \ldots, X_k)$.

A stochastic process is specified by the collection of kth-order joint cumulative distribution functions:

$$F_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = P[X_1 \le x_1,\ldots,X_k \le x_k]$$

for any k and any choice of sampling instants $t_1,\ldots,t_k$.

- If the stochastic process is discrete-valued, then a collection of probability mass functions can be used to specify the stochastic process

$$p_{X_1,\ldots,X_k}(x_1,\ldots,x_k) = P[X_1 = x_1,\ldots,X_k = x_k].$$

- If the stochastic process is continuous-valued, then a collection of probability density functions can be used instead:

$$f_{X_1,\ldots,X_k}(x_1,\ldots,x_k).$$

The Mean, Autocorrelation, and Autocovariance Functions

- The **mean** $m_X(t)$ of a random process $X(t)$ is defined by

$$m_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f_{X(t)}(x) dx,$$

where $f_{X(t)}(x)$ is the pdf of $X(t)$.

- $m_X(t)$ is a function of time.

- The **autocorrelation** $R_X(t_1, t_2)$ of a random process $X(t)$ is defined as

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{X(t_1),X(t_2)}(x, y) dx dy. \end{aligned}$$

- In general, the autocorrelation is a function of t_1 and t_2.

Stationary Random Processes

- We now consider those random processes that Randomness in the processes does not change with time, that is, they have the same behaviors between an observation in (t_0, t_1) and $(t_0 + \tau, t_1 + \tau)$.

- A discrete-time or continuous-time random process $X(t)$ is **stationary** if the joint distribution of any set of samples does not depend on the placement of the time origin. That is,

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k)$$

for all time shift τ, all k, and all choices of sample times $t_1, \ldots, t_k$.

- The first-order cdf of a stationary random process must be independent of time, i.e.,

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \quad \text{for all } t \text{ and } \tau;$$
$$m_X(t) = E[X(t)] = m \qquad \text{for all } t;$$
$$\text{VAR}[X(t)] = E[(X(t)-m)^2] = \sigma^2 \qquad \text{for all } t.$$

- The second-order cdf of a stationary random process is with

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \quad \text{for all } t_1,\ t_2.$$

- The autocorrelation and autocovariance of stationary random process $X(t)$ depend only on $t_2 - t_1$:

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \quad \text{for all } t_1, t_2;$$
$$C_X(t_1, t_2) = C_X(t_2 - t_1) \quad \text{for all } t_1, t_2.$$

Wide-Sense Stationary Random Processes

- A discrete-time or continuous-time random process $X(t)$ is **wide-sense stationary** (WSS) if it satisfies

$$m_X(t) = m \quad \text{for all } t \text{ and}$$

$$C_X(t_1, t_2) = C_X(t_1 - t_2) \quad \text{for all } t_1 \text{ and } t_2.$$

- When $X(t)$ is wide-sense stationary, we have

$$C_X(t_1, t_2) = C_X(\tau) \text{ and } R_X(t_1, t_2) = R_X(\tau),$$

where $\tau = t_1 - t_2$.

- Stationary random process $\longrightarrow$ wide-sense stationary process

- Assume that $X(t)$ is a wide-sense stationary process.
- The **average power** of $X(t)$ is given by

$$E[X(t)^2] = R_X(0) \quad \text{for all } t.$$

- The autocorrelation function of $X(t)$ is an even function since

$$R_X(\tau) = E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] = R_X(-\tau).$$

- The autocorrelation function is a measure of the rate of change of a random process.

Power Spectral Density

- The *power spectral density* (psd) of a WSS random process $X(t)$, is the Fourier Transform of $R_X(\tau)$,

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi\tau f} d\tau$$

- For a discrete time process X_n, the power spectral density is the discrete Fourier Transform (DFT) of the sequence $R_X(n)$,

$$S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n) e^{-j2\pi n f}, \text{ for } |f| < \frac{1}{2}$$

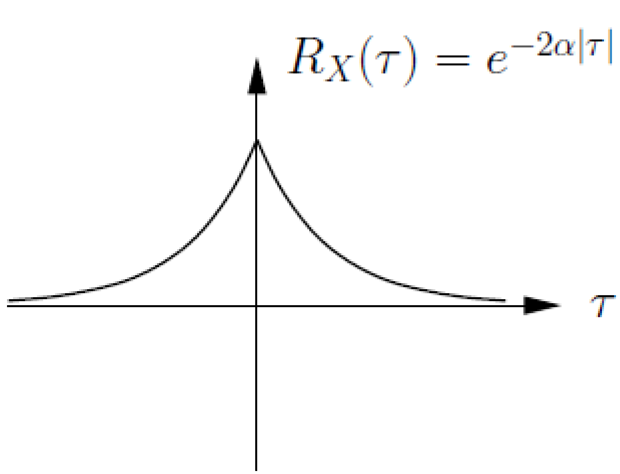
- $R_X(\tau)$ (or $R_X(n)$) can be recovered from $S_X(f)$ by taking the inverse Fourier Transform, *i.e.*,

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi\tau f} df, \text{ and inverse DFT,}$$

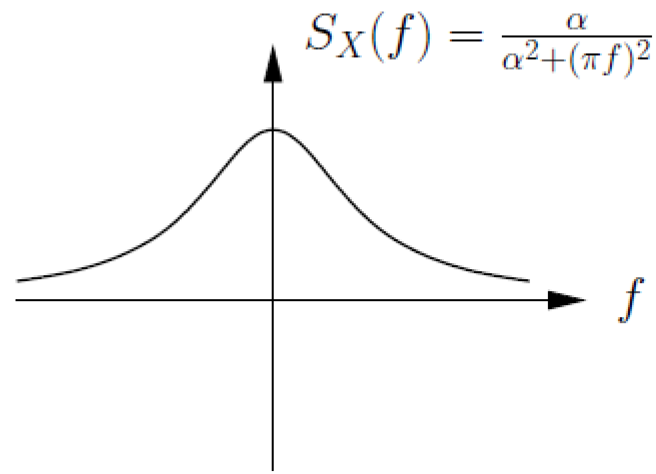
$$R_X(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{j2\pi n f} df$$

Examples

1.

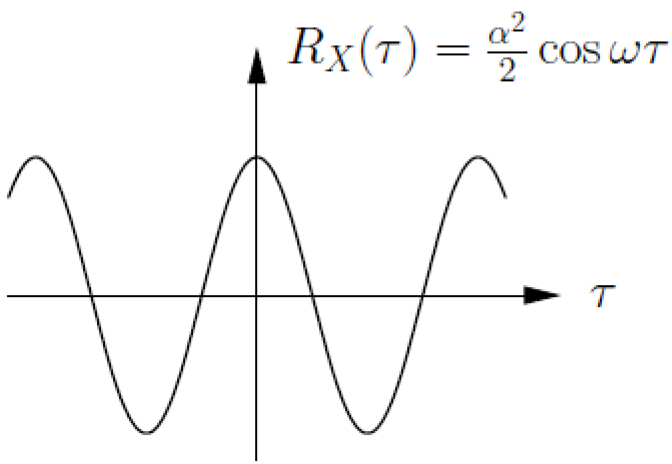


$R_X(\tau) = e^{-2\alpha|\tau|}$
τ

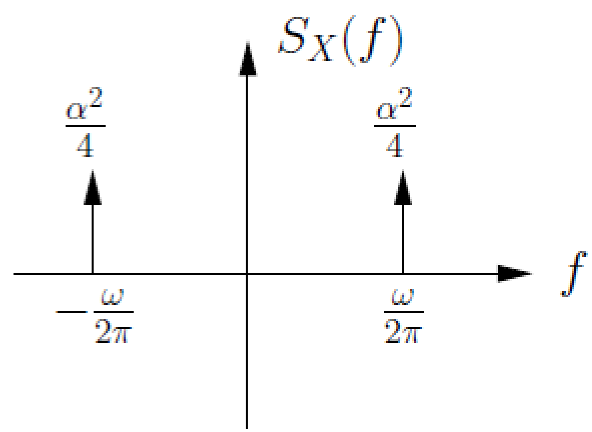


$S_X(f) = \frac{\alpha}{\alpha^2+(\pi f)^2}$
f

2.

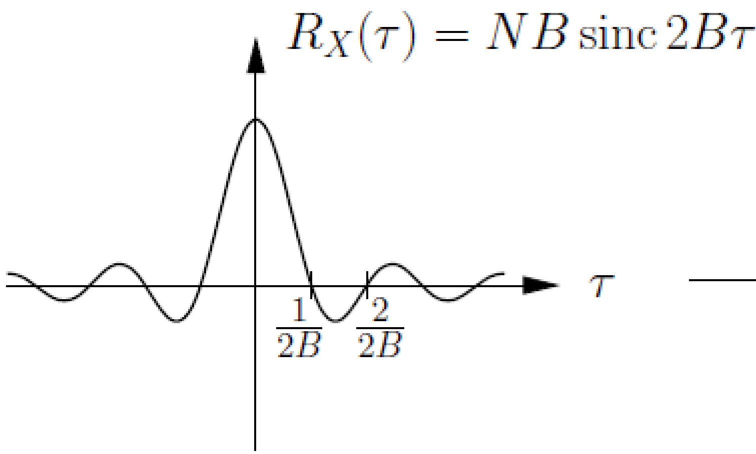


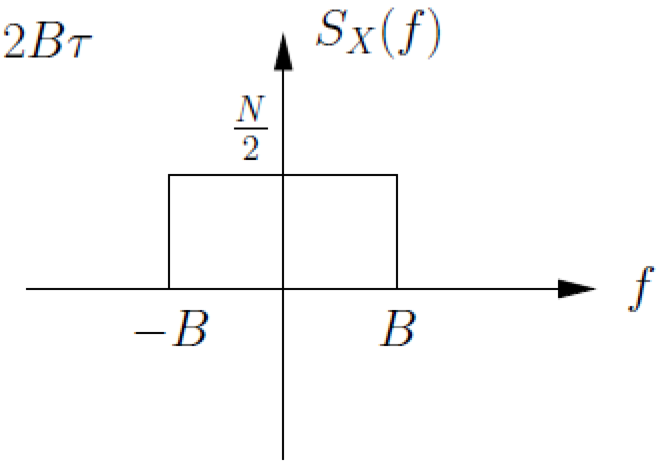
$R_X(\tau) = \frac{\alpha^2}{2}\cos\omega\tau$
τ



$S_X(f)$
$\frac{\alpha^2}{4}$
$\frac{\alpha^2}{4}$
f
$-\frac{\omega}{2\pi}$
$\frac{\omega}{2\pi}$

5. Bandlimited white noise process: WSS zero mean process $X(t)$ wit





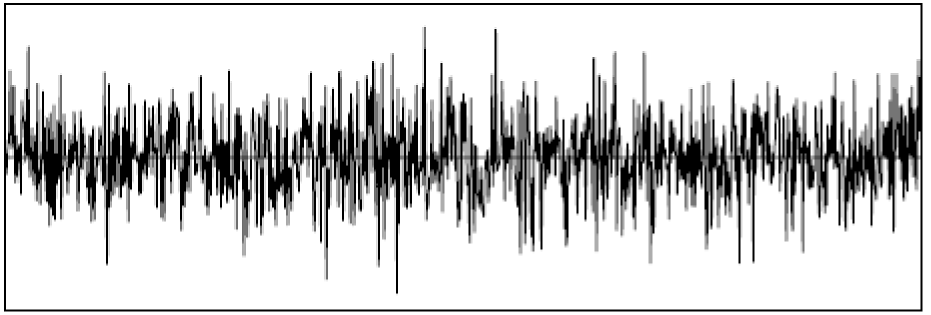
For any t, the samples $X(t \pm \frac{n}{2B})$, for $n = 0, 1, 2, \ldots$, are uncorrelated

6. White noise process: Now if we let $B \to \infty$ in the previous example, we get a *white noise process*, which has

$$S_X(f) = \frac{N}{2}, \text{ for all } f, \text{ and}$$

$$R_X(\tau) = \frac{N}{2}\delta(\tau)$$

— If, in addition, $X(t)$ is a GRP, then we get the famous white gaussia.. noise (WGN) process



- Remarks on white noise:
  - For a white noise process all samples are uncorrelated
  - The process is not physically realizable, since it has infinite power
  - However, it plays a similar role in random processes to the role of a point mass in physics and delta function in EE
  - Thermal and shot noise are well modelled as white gaussian noise, since they have very flat psd over very wide band (GHzs)

Time Averages of Random Processes and Ergodic Theorems

- We consider the measurement of repeated random experiments.
- We want to take arithmetic average of the quantities of interest.
- To estimate the mean $m_X(t)$ of a random process $X(t,\zeta)$ we have

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t,\zeta_i),$$

where N is the number of repetitions of the experiment.

- **Time average** of a single realization is given by

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t,\zeta)dt.$$

- **Ergodic theorem** states conditions under which a time average converges as the observation interval becomes large.

- We are interested in ergodic theorems that state when time average converge to the ensemble average.

- The strong law of large numbers given as

$$P\left[\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} X_i = m\right] = 1$$

is one of the most important ergodic theorems, where X_n is an iid discrete-time random process with finite mean $E[X_i] = m$.

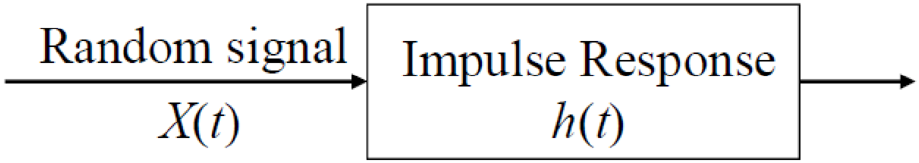
Example: Let $X(t) = A$ for all t, where A is a zero mean, unit-variance random variable. Find the limit value of the time average.

The mean of the process $m_X(t) = E[X(t)] = E[A] = 0$. The time average gives

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} A dt = A.$$

The time average does not converge to $m_X(t) = 0$. $\rightarrow$ Stationary processes need not be ergodic.

Random Process Through a LTI System



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau$$

- If a WSS random process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will be also WSS with mean

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt = \mu_X H(0)$$

autocorrelation

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

and power spectrum

$$G_Y(f) = G_X(f)\,|\,H(f)\,|^2$$

Gaussian Processes

- A random process $X(t)$ is a *Gaussian* process if for all n and all $(t_1, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian pdf.

✓ For Gaussian processes, knowledge of the mean and auto-correlation gives a complete statistical description of the process.

✓ If a Gaussian process $X(t)$ is passed through an LTI system, the output process $Y(t)$ will also be a Gaussian process.

Properties of Gaussian Processes

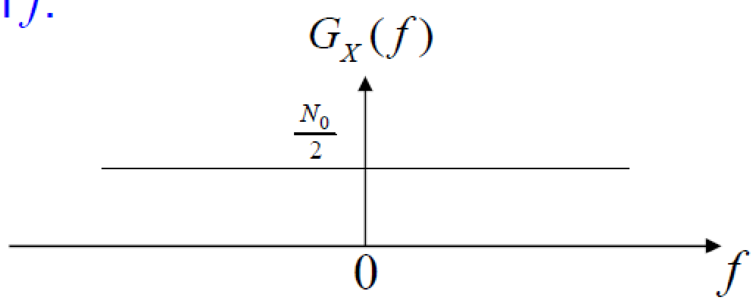
The properties of Gaussian processes are as follows:

1. The N-dimensional Gaussian pdf is completely specified by the first and second order moments (means, variances and covariances).
2. Since $x(t_i)$ are jointly Gaussian then $x(t_i)$ are individually Gaussian.
3. The Gaussian random variables are independent when they are uncorrelated.
4. A linear transformation of a set of Gaussian random variables produces another set of Gaussian random variables.
5. A WSS Gaussian random process is also SSS.

➢ If the input to a linear system is a Gaussian random process, the system output is also a Gaussian random process.

White Processes

- A random process $X(t)$ is called a white process if it has a flat spectral density, i.e., if $G_X(f)$ is a constant for all f.



✓ Power: $P_X = \int_{-\infty}^{\infty} G_X(f)df = \infty$

✓ Autocorrelation:

$$G_X(f) \Leftrightarrow R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

$\frac{N_0}{2}$: two-sided power spectral density

✓ If a white process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will not be white any more.

Power spectrum of $Y(t)$: $G_Y(f) = \frac{N_0}{2} |H(f)|^2$

Power of $Y(t)$: $P_Y = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt$