6.632 Solution to Problem Set 2

Solution P2.1

(a) The penetration depth is 2 cm.
(b) At 60 Hz, loss tangent is 1.5×10^7 and penetration depth 32 m.
At 10 MHz, loss tangent is 90 and penetration depth 8 cm.
(c) $|\overline{E}|$ is 1 volt/m just beneath surface and 0.019 volt/m at 100 m depth. Power density is $25\,\text{watt/m}^2$ just beneath surface and $8.39\,\text{mW/m}^2$ at 100 m depth.

Solution P2.2

(b) $N = 7 \times 10^{28}\,\text{m}^{-3}$

$$d_p = \sqrt{\frac{9.1 \times 10^{-31}}{7 \times 10^{28} \times (1.6 \times 10^{-19})^2 \times 4\pi \times 10^{-7}}} \simeq 2 \times 10^{-8}\,\text{m}$$

(c) The skin depth of a good conductor $\delta \propto 1/\sqrt{f}$. When the frequency is very low, δ becomes large; hence a field can penetrate it. For superconductors, however, the penetration depth is independent of frequency. This suggests that even slowly-varying fields (E and H) cannot penetrate the superconductor. Note that a steady current can exist within the superconductor without the support of an electric field.

Solution P2.3

When the wave frequency is much greater than the electron collision frequency, the collision effect can be neglected and the plasma can be treated as a lossless medium. Using Lorentz force law and Newton's law, we have

$$m\frac{d\overline{v}}{dt} = q(\overline{E} + \overline{v} \times \overline{B})$$

Under the time-harmonic excitation and with $\overline{v}$ much smaller than c, we find

$$-i\omega m\overline{v} = q(\overline{E} + \overline{v} \times \overline{B}_0)$$

Defining a vector $\overline{\omega}_c = q\overline{B}_0/m$, we write

$$\overline{\omega}_c \times \overline{v} = \frac{q}{m}\overline{E} + i\omega\overline{v}$$

we cross-multiply and dot-multiply the above equation by $\overline{\omega}_c$.

$$-i\omega\overline{\omega}_c \times \overline{v} = \frac{q}{m}\overline{\omega}_c \times \overline{E} + \omega_c^2\overline{v} - \overline{\omega}_c(\overline{\omega}_c \cdot \overline{v})$$
$$-i\omega\overline{\omega}_c \cdot \overline{v} = \frac{q}{m}\overline{\omega}_c \cdot \overline{E}$$

Identifying $\overline{J} = Nq\overline{v}$ and $\omega_p^2 = Nq^2/m\epsilon_0$, we obtain

$$-i\omega\epsilon_o\omega_p^2\overline{E} + \omega^2\overline{J} = \epsilon_o\omega_p^2\overline{\omega}_c \times \overline{E} + \omega_c^2\overline{J} - i\epsilon_o\frac{\omega_p^2}{\omega}\overline{\omega}_c(\overline{\omega}_c \cdot \overline{E})$$

we thus have

$$\overline{J} = \frac{-i\omega\epsilon_0}{\omega^2 - \omega_c^2}\left\{i\frac{\omega_p^2}{\omega}\overline{\omega}_c \times \overline{E} - \omega_p^2\overline{E} + \frac{\omega_p^2}{\omega^2}\overline{\omega}_c\overline{\omega}_c \cdot \overline{E}\right\}$$

From Ampere's law, we find, since $\overline{\omega}_c = qB_0/m$,

$$\nabla \times$$

H = -i $\omega\epsilon_0\overline{E} + \overline{J} = i\omega \left\{ i\epsilon_g \times \overline{E} + \epsilon\overline{E} + \frac{\omega_p^2\omega_c^2\epsilon_0}{\omega^2(\omega^2-\omega_c^2)} \cdot \overline{E} \right\} = -i\omega\overline{\overline{\epsilon}} \cdot \overline{E}$
where ϵ_g and ϵ are as defined. Writing $\overline{\overline{\epsilon}}$ in the matrix form, we find

$$\epsilon_z = \epsilon_0 \left[1 - \frac{\omega_p^2/\omega^2}{1-\omega_c^2/\omega^2} + \frac{\omega_p^2\omega_c^2/\omega^4}{1-\omega_c^2/\omega^2} \right] = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2} \right]$$

Carrying on the inverse of $\overline{\overline{\epsilon}}$, we find for $\overline{\overline{\kappa}} = \overline{\overline{\epsilon}}^{-1}$

$$\begin{aligned}
\kappa &= \frac{\epsilon}{\epsilon^2 - \epsilon_g^2} = \frac{1}{\epsilon_0} \left[\frac{1 - \omega_p^2/\omega^2 - \omega_c^2/\omega^2}{(1-\omega_p^2/\omega^2)^2 - \omega_c^2/\omega^2} \right] \\
\kappa_g &= \frac{-\epsilon_g}{\epsilon^2 - \epsilon_g^2} = \frac{1}{\epsilon_0} \left[\frac{\omega_c\omega_p^2/\omega^3}{(1-\omega_p^2/\omega^2)^2 - \omega_c^2/\omega^2} \right] \\
\kappa_z &= \frac{1}{\epsilon_z} = \frac{1}{\epsilon_0} \left[\frac{1}{1-\omega_p^2/\omega^2} \right]
\end{aligned}$$

It is easily shown that $\overline{\overline{\epsilon}} \cdot \overline{\overline{\kappa}} = \overline{\overline{I}}$.

In the case of an infinitely strong magnetic field, $\omega_c \to \infty$ and we have

$$\begin{aligned}
\epsilon &= \epsilon_0 & \kappa &= \frac{1}{\epsilon_0} \\
\epsilon_g &= 0 & \kappa_g &= 0 \\
\epsilon_z &= \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) & \kappa_z &= \frac{1}{\epsilon_0} \left[\frac{1}{1-\omega_p^2/\omega^2} \right]
\end{aligned}$$

and the medium becomes a uniaxial plasma.

Solution P2.4

$$\bar{E} = \hat{x}e^{ikz} = \frac{1}{2} \left[(\hat{x} + i\hat{y}) + (\hat{x} - i\hat{y}) \right] e^{ikz}$$