Homework 4.

Question 1

1) Obviously, the orthonormal basis functions are:

$$\phi_1(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}, \quad \phi_2(t) = \begin{cases} 1, & 1 \leq t < 2 \\ 0, & \text{otherwise} \end{cases}, \quad \phi_3(t) = \begin{cases} 1, & 2 \leq t \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

2) According to the basis functions in 1), the dimension of the space is:

$n\text{-dim} = 3.$

3) $s_1(t) = \phi_1(t) + \phi_2(t) + \cancel{\phi_3(t)}.$

$s_2(t) = 2\phi_2(t).$

$s_3(t) = \phi_1(t) + \phi_2(t) + \phi_3(t)$

$s_4(t) = -\phi_3(t)$

4) According to Gram-Schmidt procedure:

① $\varphi_1(t) = \dfrac{s_1(t)}{\|s_1(t)\|} = \dfrac{s_1(t)}{\sqrt{\langle s_1(t), s_1(t)\rangle}} = \begin{cases} \frac{1}{\sqrt{2}}, & 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$

② $\nu_2(t) = s_2(t) - \langle s_2(t), \varphi_1(t)\rangle \cdot \varphi_1(t) = s_2(t) - \sqrt{2}\,\varphi_1(t) = \cancel{\begin{cases} -\sqrt{2}, & 0 \le t < 1 \\ 2-\sqrt{2}, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}} \quad \begin{cases} -1, & 0 \le t < 1 \\ 1, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$

$\therefore \varphi_2(t) = \dfrac{\nu_2(t)}{\|\nu_2(t)\|} = \dfrac{1}{\sqrt{2}}\nu_2(t) = \begin{cases} -\frac{1}{\sqrt{2}}, & 0 \le t < 1 \\ \frac{1}{\sqrt{2}}, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases}$

③ $\nu_3(t) = s_3(t) - \sum_{i=1}^{2} \langle s_3(t), \varphi_i(t)\rangle \cdot \varphi_i(t) = s_3(t) - \sqrt{2}\,\varphi_1(t) - 0\cdot\varphi_2(t) = \begin{cases} 1, & 2 < t \le 3 \\ 0, & \text{otherwise.} \end{cases}$

$\therefore \varphi_3(t) = \dfrac{\nu_3(t)}{\|\nu_3(t)\|} = \begin{cases} 1, & 2 < t \le 3 \\ 0, & \text{otherwise.} \end{cases}$

④ $\nu_4(t) = s_4(t) - \sum_{i=1}^{3} \langle s_4(t), \varphi_i(t)\rangle \cdot \varphi_i(t) = s_4(t) - 0\cdot\varphi_1(t) - 0\cdot\varphi_2(t) + \varphi_3(t) = 0.$

$\therefore$ the orthonormal basis functions are:

$\varphi_1(t) = \begin{cases} \frac{1}{\sqrt{2}}, & 0 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases} \qquad \varphi_2(t) = \begin{cases} -\frac{1}{\sqrt{2}}, & 0 \le t < 1 \\ \frac{1}{\sqrt{2}}, & 1 \le t \le 2 \\ 0, & \text{otherwise.} \end{cases} \qquad \varphi_3(t) = \begin{cases} 1, & 2 < t \le 3 \\ 0, & \text{otherwise.} \end{cases}$

Question 2.

1) $\because s_1(t) = 2\cos(wt + \frac{\pi}{6}) + \sin(wt), \quad 0 \le t \le \frac{2\pi}{w}.$

$\therefore E_1 = \int_0^{\frac{2\pi}{w}} |s_1(t)|^2\, dt = \int_0^{\frac{2\pi}{w}} 4\cos^2(wt+\frac{\pi}{6}) + \sin^2(wt) + 4\cos(wt+\frac{\pi}{6})\sin(wt)\, dt$

$= \frac{4\pi}{w} + \frac{\pi}{w} + (-\frac{2\pi}{w}) = \frac{3\pi}{w}.$

$\because s_2(t) = 3\sin(wt+\frac{\pi}{4}),\ 0 \le t \le \frac{2\pi}{w}.$

$\therefore E_2 = \int_0^{\frac{2\pi}{w}} |s_2(t)|^2 dt = \int_0^{\frac{2\pi}{w}} 9\sin^2(wt+\frac{\pi}{4})dt = \frac{9}{2}\cdot\frac{2\pi}{w} = \frac{9\pi}{w}.$

2) According to Gram-Schmidt procedure:

① $\phi_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \frac{1}{\sqrt{E_1}} s_1(t) = \sqrt{\frac{w}{3\pi}}\cdot\sqrt{3}\cos(wt) = \sqrt{\frac{w}{\pi}}\cos(wt),\quad 0 \le t \le \frac{2\pi}{w}$

② $v_2(t) = s_2(t) - \langle s_2(t), \phi_1(t)\rangle \cdot \phi_1(t) = s_2(t) - \frac{3}{\sqrt{2}}\sqrt{\frac{\pi}{w}}\phi_1(t)$

$= \frac{3}{2}\sqrt{2}\,[\sin(wt)+\cos(wt)] - \frac{3}{2}\sqrt{2}\cos(wt) = \frac{3}{2}\sqrt{2}\sin(wt),\quad 0 \le t \le \frac{2\pi}{w}.$

$\therefore \phi_2(t) = \frac{v_2(t)}{\|v_2(t)\|} = \sqrt{\frac{2w}{9\pi}} \times \frac{3}{2}\sqrt{2}\sin(wt) = \sqrt{\frac{w}{\pi}}\sin(wt).\quad 0 \le t \le \frac{2\pi}{w}.$

$\therefore$ the basis functions are:

$\phi_1(t) = \sqrt{\frac{w}{\pi}}\cos(wt),\quad 0 \le t \le \frac{2\pi}{w}.$

$\phi_2(t) = \sqrt{\frac{w}{\pi}}\sin(wt),\quad 0 \le t \le \frac{2\pi}{w}.$

and: $s_1(t) = \sqrt{\frac{3\pi}{w}}\phi_1(t),\quad s_2(t) = \sqrt{\frac{9\pi}{2w}}\phi_1(t) + \sqrt{\frac{9\pi}{2w}}\phi_2(t)$

3) From 2), we can know that:

$\vec{s_1} = (\sqrt{\frac{3\pi}{w}}, 0),\quad \vec{s_2} = (\sqrt{\frac{9\pi}{2w}}, \sqrt{\frac{9\pi}{2w}}).$

$\therefore E_1 = \|\vec{s_1}\|^2 = \frac{3\pi}{w}.$

$E_2 = \|\vec{s_2}\|^2 = \frac{9\pi}{2w} \times 2 = \frac{9\pi}{w}.$

4) The inner product of the signals ~~or~~ is:

$\langle \vec{s_1}, \vec{s_2}\rangle = \vec{s_1}\cdot\vec{s_2} = \sqrt{\frac{3\pi}{w}}\cdot\sqrt{\frac{9\pi}{2w}} + 0 = \frac{3\sqrt{6}}{2}(\sqrt{\frac{\pi}{w}})^2 = \frac{3\sqrt{6}}{2}\frac{\pi}{w}$

Question 3

a) Obviously, the dimension of 8-FSK is: $n\text{-}dim = 8$.

Using the union bound, we can know that:

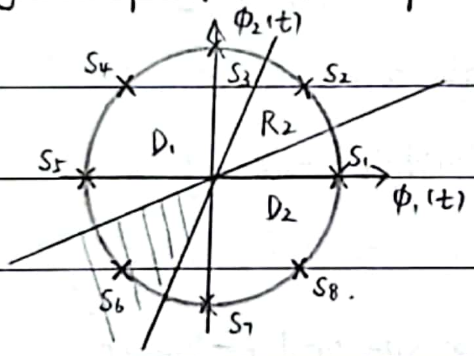
$P_b \le \frac{M}{2} Q\left(\sqrt{\frac{E_s}{N_0}}\right)$

$\because Q(x) \le \frac{1}{2} e^{-\frac{x^2}{2}}$

$\therefore P_b \le \frac{M}{2} Q\left(\sqrt{\frac{E_s}{N_0}}\right) \le \frac{M}{4} e^{-\frac{E_s}{2N_0}}$

$\therefore P_b \approx \frac{M}{4} e^{-\frac{E_s}{2N_0}}$

b) The constellation diagram of 8PSK is as follows:



And the probability of error can be overbounded by noting that the total area represented by the two half planes D_1 and D_2 is greater than the area except R_2

$\therefore$ the probability of symbol error is overbounded by the probability the received data point lies in either half plane.

Consider a single signal point, the minimum distance it is away from the boundary is: $d = \sqrt{E_s}\sin(\frac{\pi}{M})$.

Consider the noise component $N_\perp$, which contributes to the error of received data. $N_\perp$ has zero mean and a variance $\frac{N_0}{2}$.

$\therefore P_b < Pr(z \in D_1 \text{ or } D_2) = 2Pr(z \in D_1) = 2Pr(d + N_\perp < 0) = 2Pr(\overset{N_\perp}{\cancel{d}} < -\overset{d}{\cancel{N_\perp}})$

$\because N_\perp \sim N(0, \frac{N_0}{2})$. $d = \sqrt{E_s}\sin(\frac{\pi}{M})$.

$\therefore P_b < 2Q(\sqrt{\frac{2E_s}{N_0}}\sin(\frac{\pi}{M}))$, that is, $P_b \approx 2Q(\sqrt{\frac{2E_s}{N_0}}\sin(\frac{\pi}{M}))$.

c) $\because$ The dimension of 8PSK is $2 < 8$.

$\therefore$ 8FSK requires more correlators.

~~8FSK~~: MFSK:

advantages: has a higher power efficiency and a better performance in long distance transmission.

disadvantages: poor bandwidth efficiency.

MPSK:

advantages: has a better spectral efficiency.

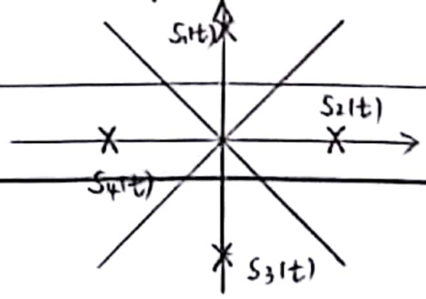
disadvantages: poor power efficiency.

d) I will choose MFSK for its better power efficiency.

Question 4.

(a) Apparently, the dimension of the signal space is 2.

(b) The regions are shown as follows:



(c) QPSK can be regarded as ~~tw~~ a combination of two BPSK.

$\because$ when the symbol received is correct, it means the two bits are right.

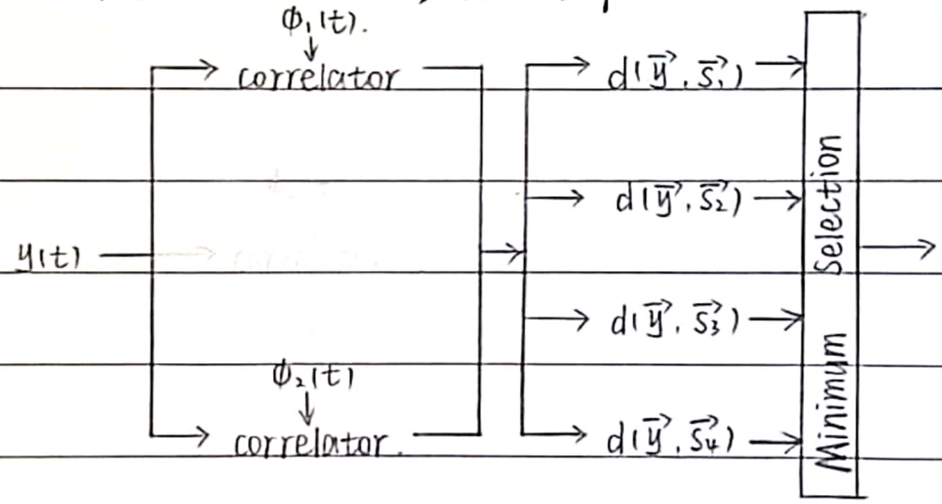
$\therefore P(C|s_1) = (1-P_{e,BPSK})^2 = [1-Q(\sqrt{\frac{2E_b}{N_0}})]^2$.

(d) According to the title, $p(s_k) = \frac{1}{4}$ for all k.

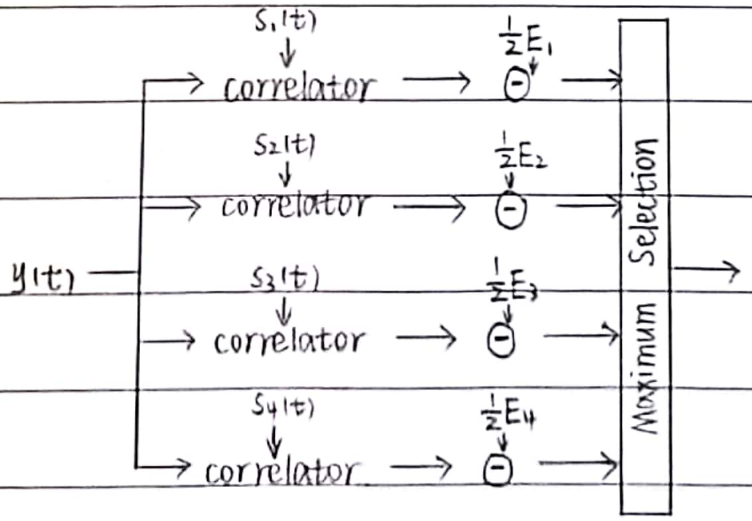
$P_{eM} = 1 - P(c|s_i) = 1-1-[Q(\sqrt{\frac{2E_b}{N_0}})]^2 + 2Q(\sqrt{\frac{2E_b}{N_0}}) \approx 2Q(\sqrt{\frac{2E_b}{N_0}})$

(e) The optimal receiver structures are as follows:

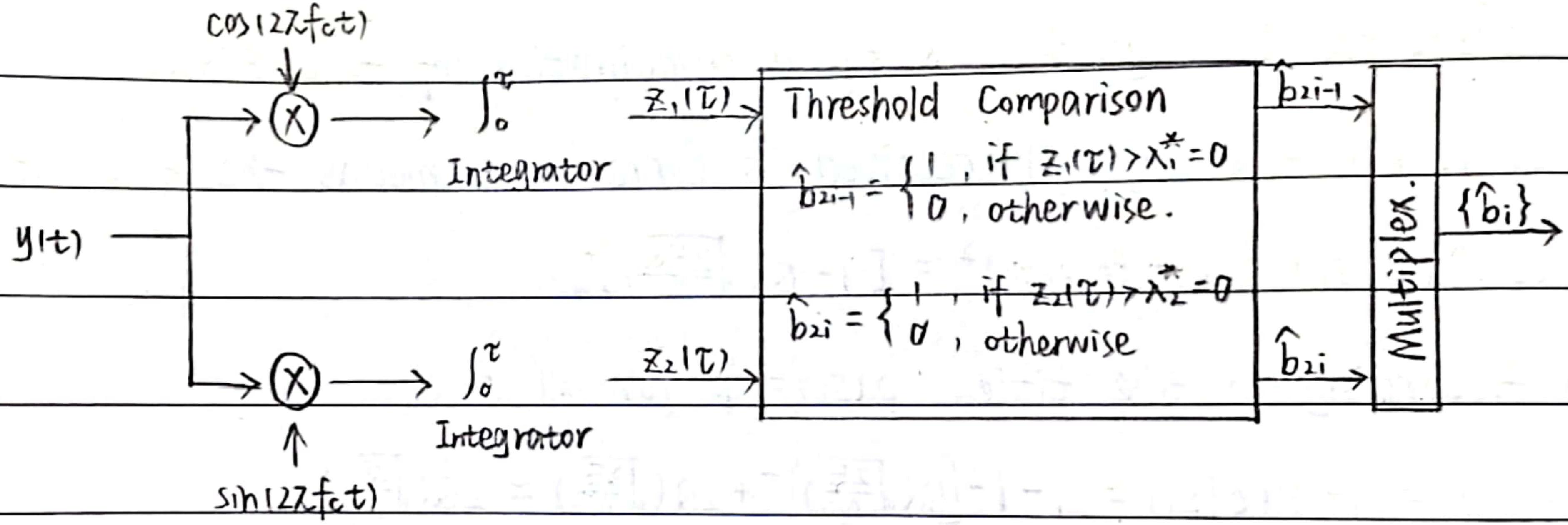
①



②



(f) The optimal receiver using matched filter is as follows:



cos(2πfct)
y(t)
Integrator
Integrator
sin(2πfct)
z1(τ)
z2(τ)
Threshold Comparison
b̂2i-1 = { 1, if z1(τ) > λ1* = 0 ; 0, otherwise.
b̂2i = { 1, if z2(τ) > λ2* = 0 ; 0, otherwise
b̂2i-1
b̂2i
Multiplex.
{b̂i}