Lecture 7.

Digital Modulation

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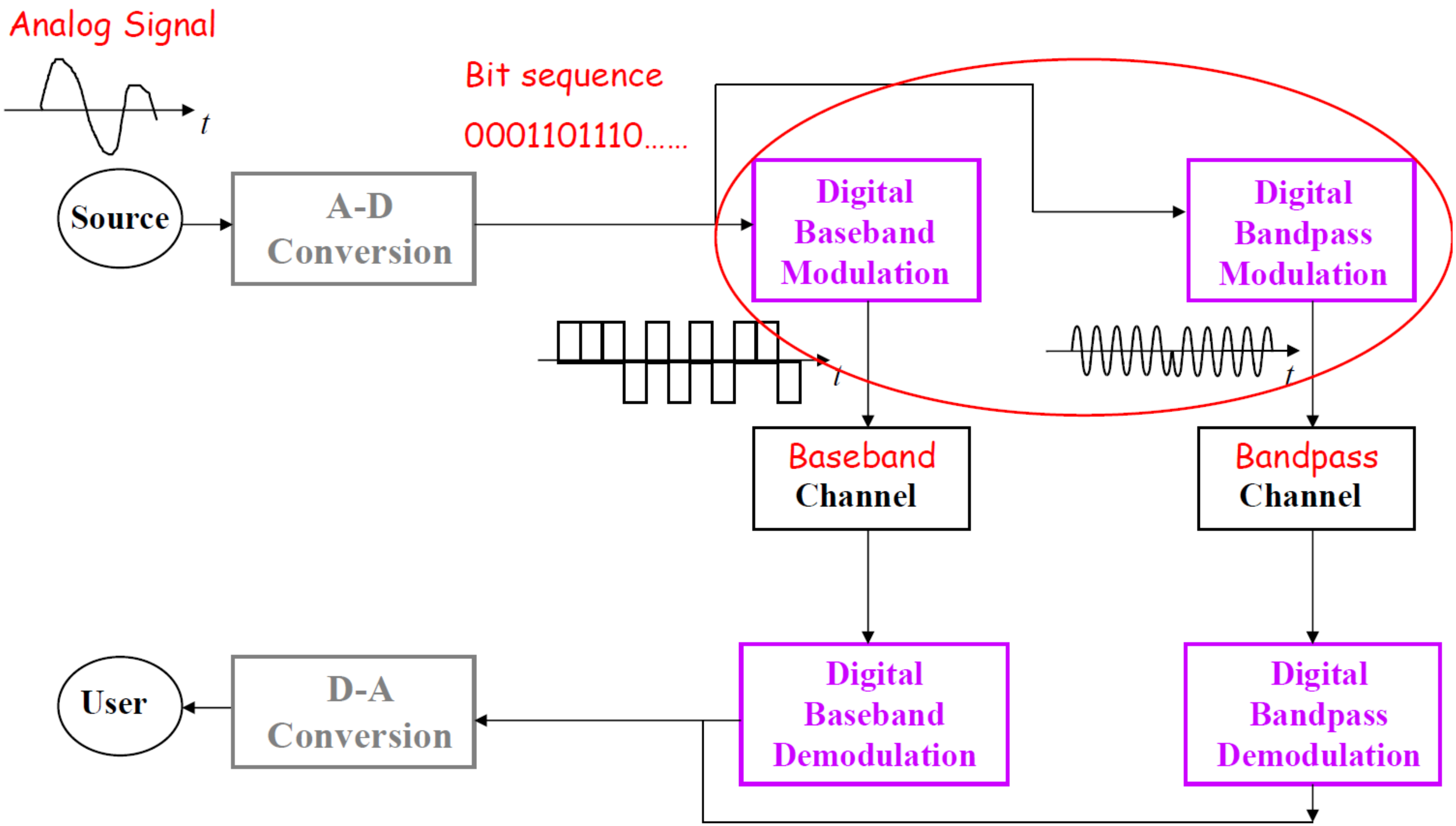
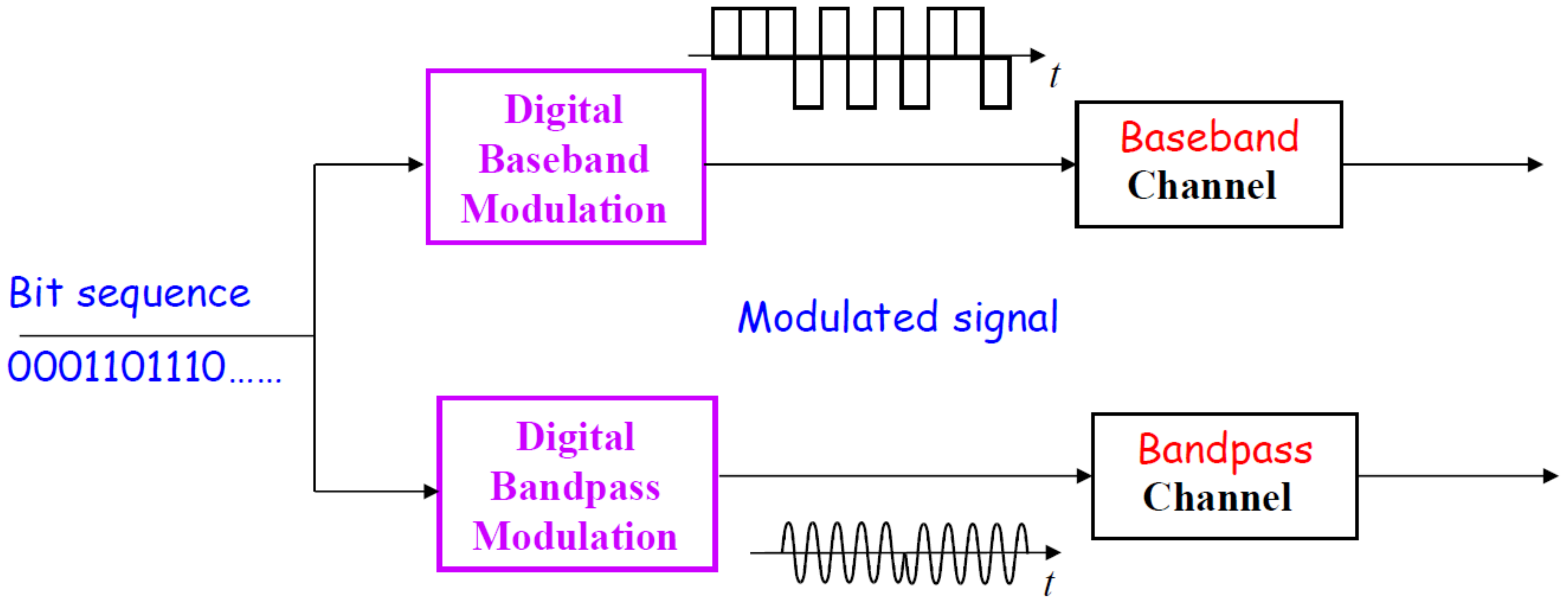
Digital Communications

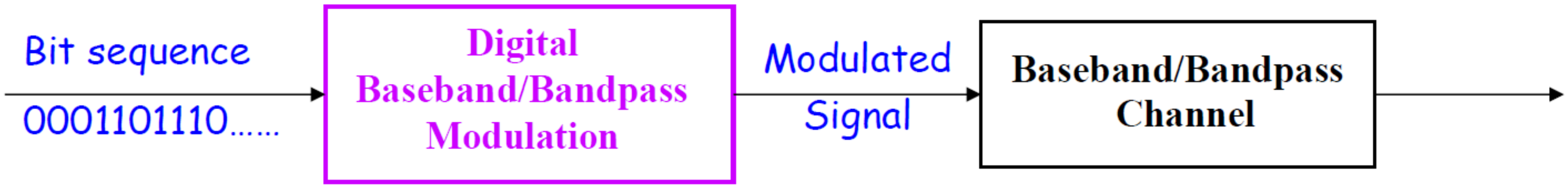


Digital Modulation



- How to choose proper digital waveforms to “carry” the digits?

Digital Modulation



- Bit Rate: number of bits transmitted in unit time
- Required channel bandwidth: determined by the bandwidth of the modulated signal.
- Bandwidth Efficiency:

$$\boxed{\gamma \triangleq \frac{\text{Information Bit Rate } R_b}{\text{Required Channel Bandwidth } B_h}}$$

Digital Baseband Modulation

Digital Baseband Modulation

- Choose baseband signals to carry the digits.
  - Each baseband signal can carry multiple bits.

Binary

- Each baseband signal carries 1 bit.
- Bit Rate: $R_b = 1/\tau$
- Totally 2 baseband signals are required.

M-ary

- Each baseband signal carries a symbol (with $\log_2 M$ bits).
- Symbol Rate: $R_s = 1/\tau$ Bit Rate: $R_b = (\log_2 M)/\tau$
- Totally M baseband signals are required.

Digital Baseband Modulation

- Focus on “amplitude modulation”
  - The baseband signals have the same shape, but different amplitudes.
  - Time-domain representation of the modulated signal:

$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

where Z_n is a discrete random variable with $\Pr\{Z_n = a_i\} = 1/M,\ i = 1,\ldots,M,$

$v(t)$ is a unit baseband signal.

  - Power spectrum of the modulated signal:

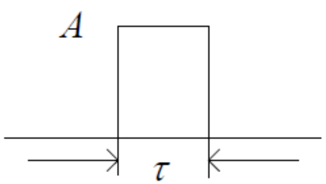
$$\boxed{G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau}\sum_{m=-\infty}^{\infty}\delta\left(f - \frac{m}{\tau}\right)\right)}$$

Read the supplemental material for details.

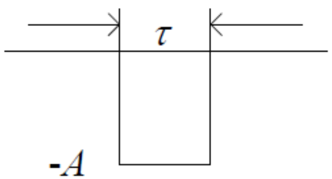
Pulse Amplitude Modulation (PAM)

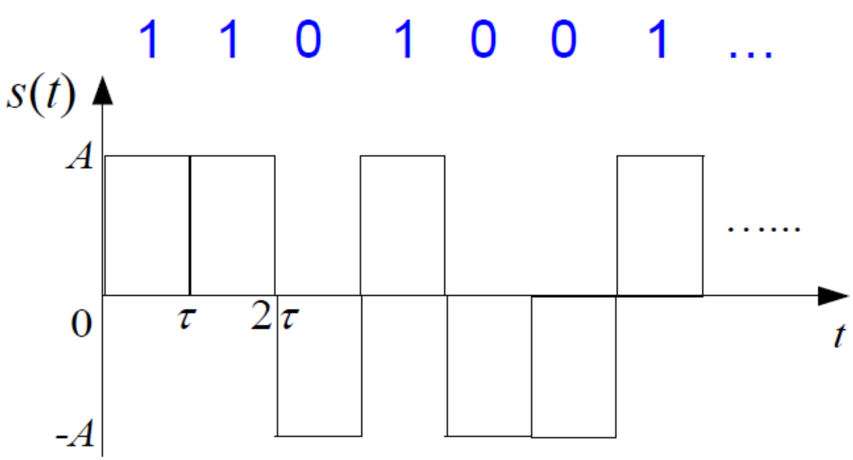
Binary PAM

1: a positive rectangular pulse with amplitude A and width τ



0: a negative rectangular pulse with amplitude $-A$ and width τ





$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

✓ $\Pr\{Z_n = \pm 1\} = 1/2$

✓ $v(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & \text{otherwise} \end{cases}$

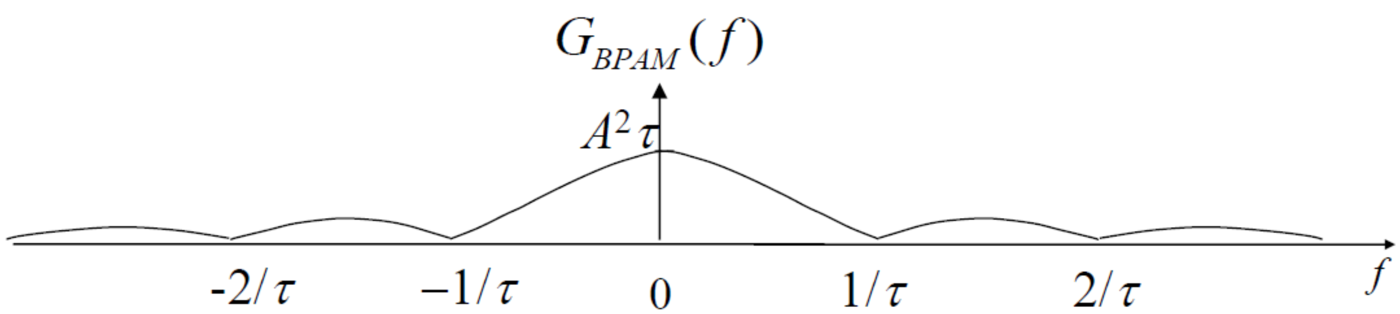
Power Spectrum of Binary PAM

$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau} \right) \right)$$

With Binary PAM: $V(f) = A\tau \mathrm{sinc}(f\tau)$

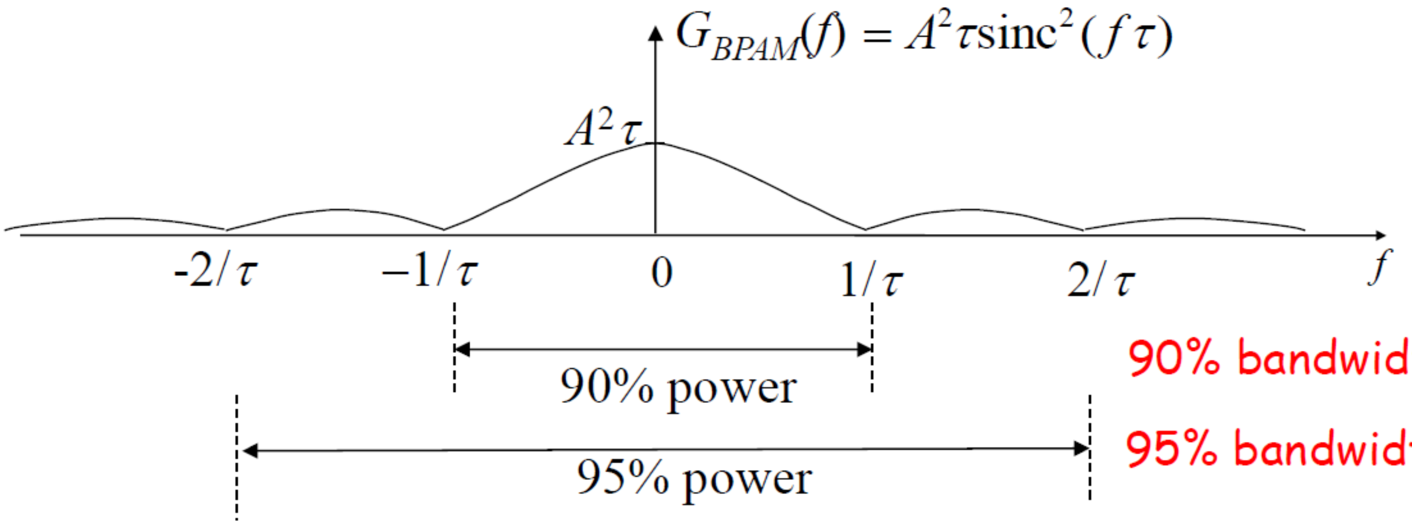
$\mu_Z = 0,\ \sigma_Z^2 = 1$

$$G_{BPAM}(f) = A^2 \tau \mathrm{sinc}^2(f\tau)$$



See Textbook (Sec. 3.2) or Reference [Proakis & Salehi] (Sec. 8.2) for more details.

Effective Bandwidth of Binary PAM



90% bandwidth: $1/\tau$

95% bandwidth: $2/\tau$

- Suppose 90% of signal power must pass through the channel (90% in-band power):

Required Channel Bandwidth: $B_{h\_90\%} = 1/\tau$

Bit rate: $R_b = 1/\tau$

$$\left.\begin{array}{l} B_{h\_90\%} = 1/\tau \\ R_b = 1/\tau \end{array}\right\} B_{h\_90\%} = R_b$$

- Suppose 95% of signal power must pass through the channel (95% in-band power):

Required Channel Bandwidth: $B_{h\_95\%} = 2/\tau = 2R_b$

Bandwidth Efficiency of Binary PAM

- Bandwidth Efficiency : $\gamma = \dfrac{\text{Information Bit Rate } R_b}{\text{Required Channel Bandwidth } B_h}$

- Bandwidth Efficiency of Binary PAM:

$$R_b = 1/\tau$$

$$B_{h\_90\%} = 1/\tau$$

$$B_{h\_95\%} = 2/\tau$$

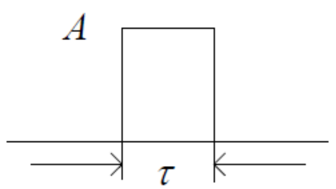
$\gamma_{BPAM} = 1$ with 90% in-band power

$\gamma_{BPAM} = 0.5$ with 95% in-band power

What if the two pulses have unsymmetrical amplitudes?

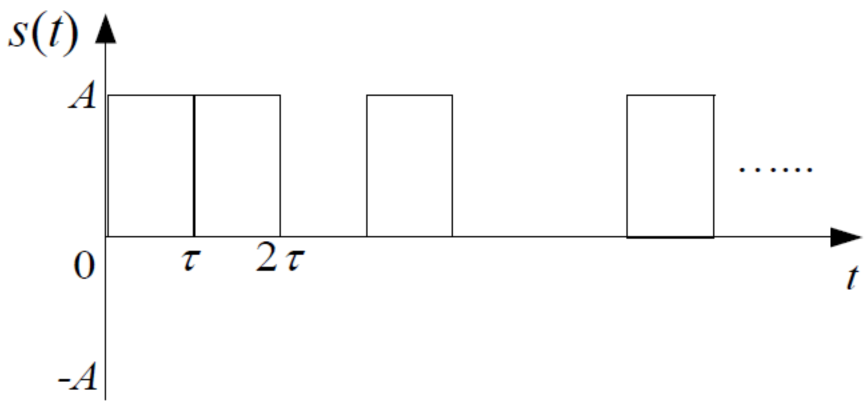
Binary On-Off Keying (OOK)

1: a positive rectangular pulse with amplitude A and width τ



0: nothing (can be regarded as a pulse with amplitude 0)

1 1 0 1 0 0 1 …



$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

✓ $\Pr\{Z_n = 1\} = \Pr\{Z_n = 0\} = 1/2$

✓ $$v(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0, & \text{otherwise} \end{cases}$$

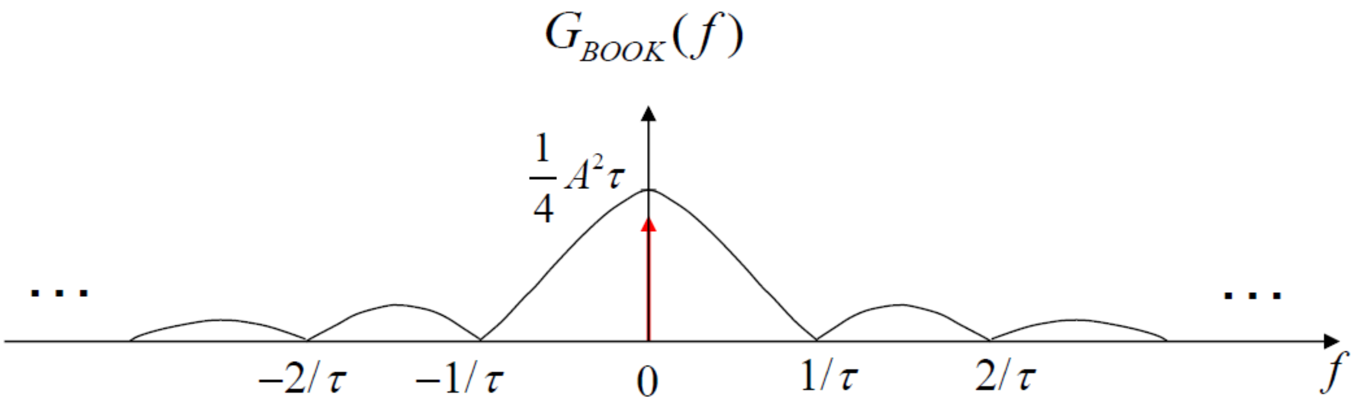
Power Spectrum of Binary OOK

$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau} \right) \right)$$

With Binary OOK: $V(f) = A\tau \operatorname{sinc}(f\tau)$

$\mu_Z = 1/2, \ \ \sigma_Z^2 = 1/4$

$$G_{BOOK}(f) = \frac{1}{\tau}\left(A\tau \operatorname{sinc}(f\tau) \right)^2 \cdot \left(\frac{1}{4} + \frac{1}{4\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau} \right) \right)$$



Bandwidth Efficiency of Binary OOK

- Bandwidth Efficiency : $\gamma = \dfrac{\text{Information Bit Rate } R_b}{\text{Required Channel Bandwidth } B_h}$

- Bandwidth Efficiency of Binary OOK:

$$R_b = 1/\tau$$

$$B_{h\_90\%} = 1/\tau$$

$$B_{h\_95\%} = 2/\tau$$

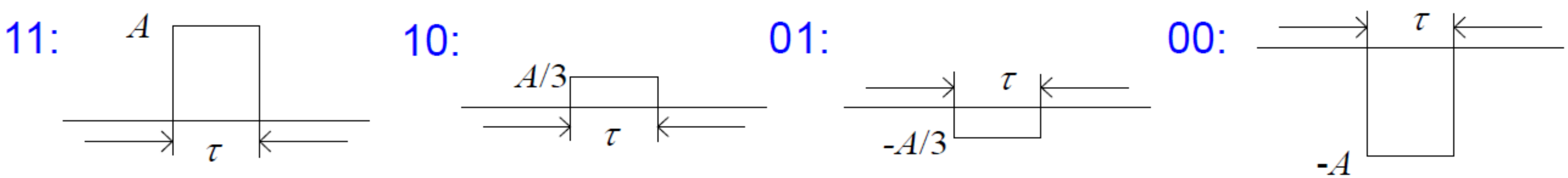
$\Rightarrow$

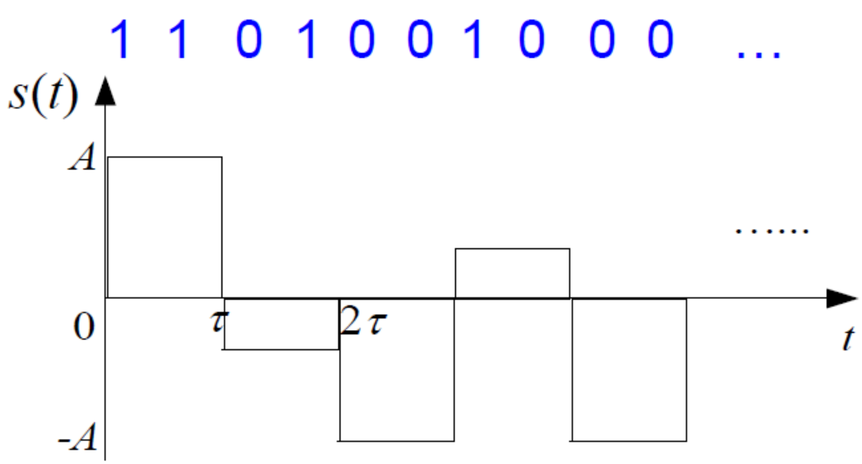
$\gamma_{BOOK} = 1$	with 90% in-band power
$\gamma_{BOOK} = 0.5$	with 95% in-band power

Can we improve the bandwidth efficiency without sacrificing the in-band power?

4-ary PAM

- 4-ary PAM: Each waveform carries 2-bit information.





$$s(t)=\sum_{n=-\infty}^{\infty} Z_n \cdot v(t-n\tau)$$

✓ $\Pr\{Z_n=1\}=\Pr\{Z_n=1/3\}$
$=\Pr\{Z_n=-1\}=\Pr\{Z_n=-1/3\}$
$=1/4$

✓ $v(t)=\begin{cases} A, & 0\le t\le \tau \\ 0, & \text{otherwise} \end{cases}$

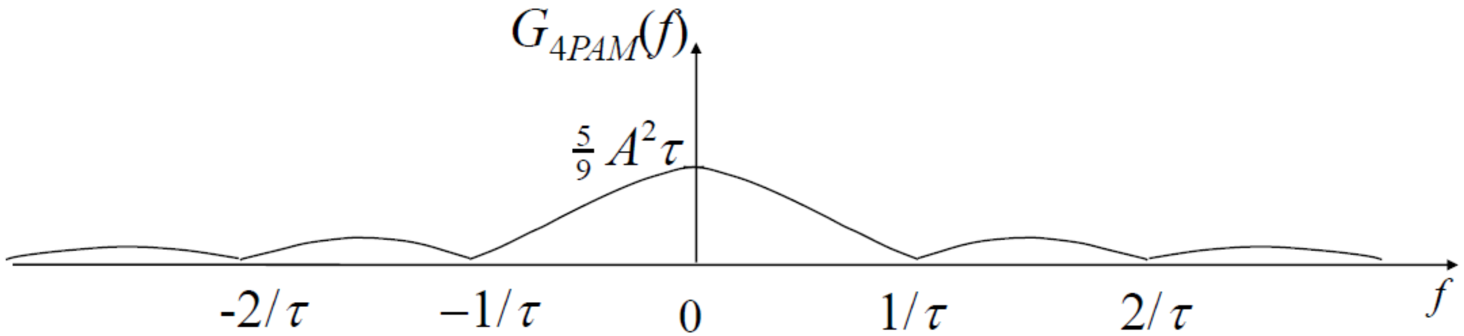
Power Spectrum of 4-ary PAM

$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau} \right) \right)$$

With 4-ary PAM: $V(f) = A\tau \mathrm{sinc}(f\tau)$

$\mu_Z = 0, \ \sigma_Z^2 = 5/9$

$$\Bigg\} \; G_{4PAM}(f) = \tfrac{5}{9} A^2 \tau \mathrm{sinc}^2(f\tau)$$



- Required channel bandwidth with 90% in-band power: $B_{h\_90\%} = 1/\tau$
- Required channel bandwidth with 95% in-band power: $B_{h\_95\%} = 2/\tau$

Bandwidth Efficiency of 4-ary PAM

- Symbol rate: $R_S = 1/\tau$
- Bit rate: $R_b = 2 \cdot R_S = 2/\tau$
- Require channel bandwidth:

with 90% in-band power: $B_{h\_90\%} = 1/\tau = R_S = \frac{1}{2} R_b$

with 95% in-band power: $B_{h\_95\%} = 2/\tau = 2R_S = R_b$

$\Rightarrow$

$\gamma_{4PAM} = 2$ with 90% in-band power

$\gamma_{4PAM} = 1$ with 95% in-band power

4-ary PAM achieves higher bandwidth efficiency than binary PAM!

Bandwidth Efficiency of M-ary PAM

- Suppose there are totally M distinct amplitude (power) levels.
- How many bits are carried by each symbol?

$$M = 2^k \Rightarrow k = \log_2 M$$

- What is the relationship between symbol rate R_S and bit rate R_b?

$$R_S = R_b / k \quad \text{or} \quad R_b = kR_S$$

- What is the required channel bandwidth with 90% in-band power?

$$B_{h\_90\%} = R_S = R_b / k$$

- Bandwidth Efficiency of M-ary PAM

Tradeoff between bandwidth efficiency and fidelity performance

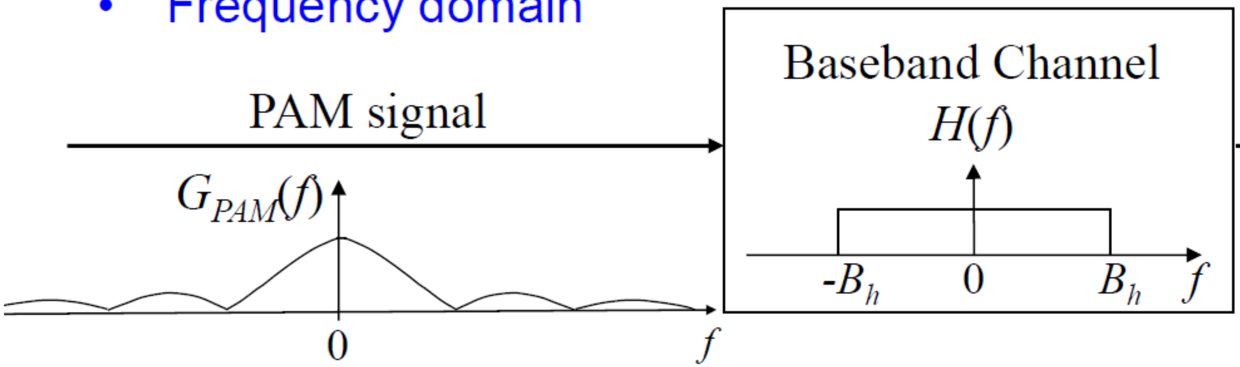
$$\boxed{\gamma_{MPAM} = k = \log_2 M \quad \text{with 90\% in-band power}}$$

- A larger M also leads to a smaller minimal amplitude difference – higher error probability (to be discussed).

Pulse Shaping

Transmission over Bandlimited Channel

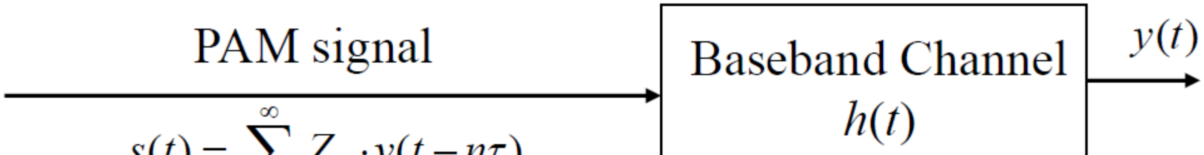
- Frequency domain



$$G_Y(f) = G_{PAM}(f)\,|H(f)|^2$$

The signal distortion incurred by channel is always non-zero!!

- Time domain



$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

$$y(t) = s(t) * h(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot x(t - n\tau)$$

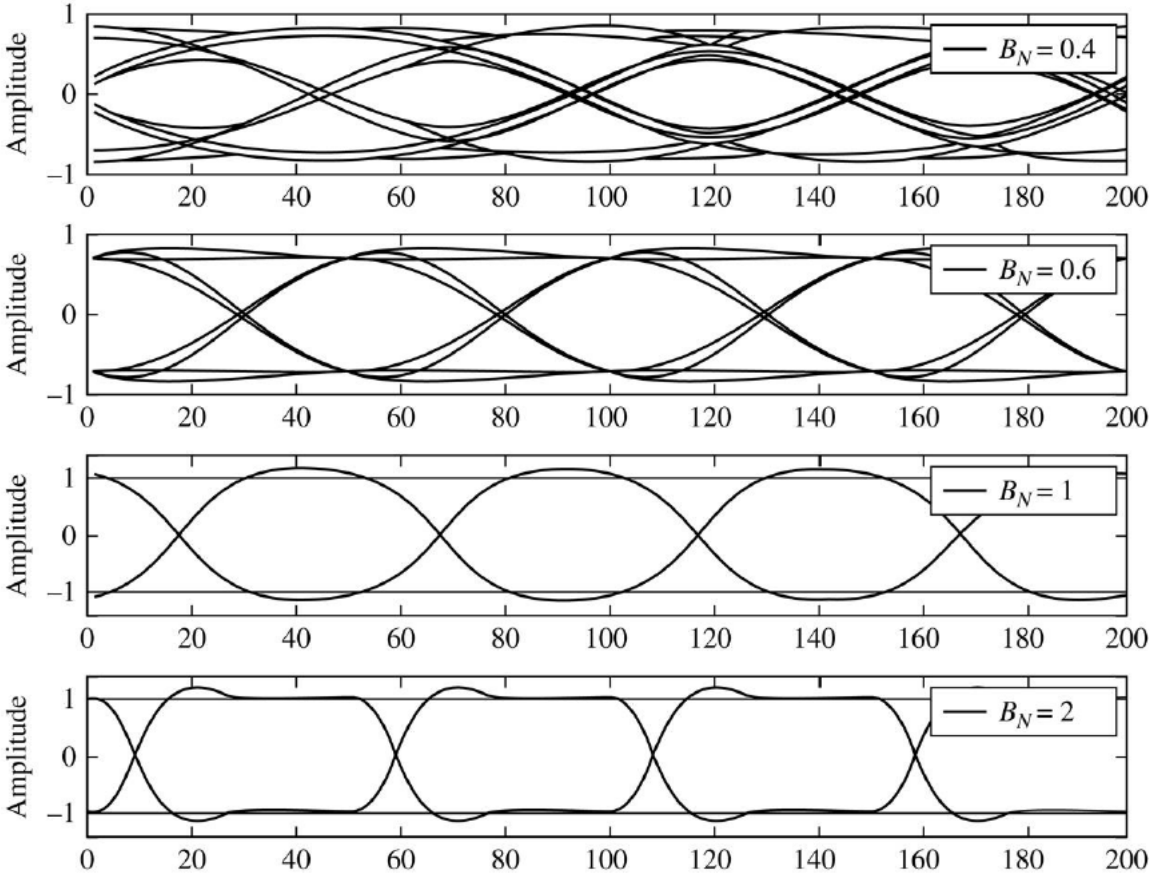
$$x(t) = v(t) * h(t)$$

Sample $y(t)$ at $m\tau$, $m=1,2,\ldots,$ we have

$$y(m\tau) = \sum_{n=-\infty}^{\infty} Z_n \cdot x(m\tau - n\tau) = Z_m \cdot x(0) + \sum_{n \neq m} Z_n \cdot x(m\tau - n\tau)$$

Inter-symbol Interference (ISI)!

ISI and Eye Diagram



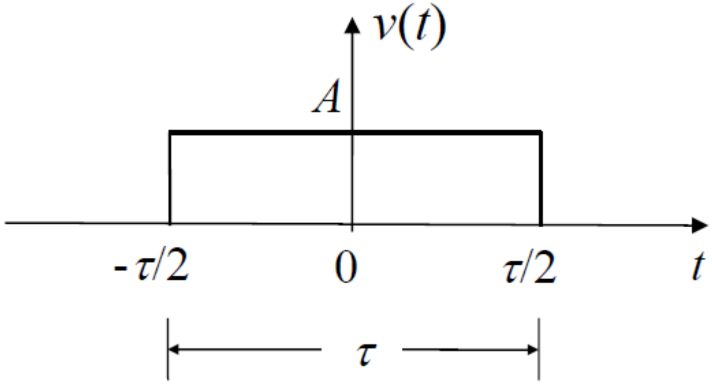
- An eye diagram is constructed by plotting overlapping k-symbol segments of a baseband signal.
- An eye diagram can be displayed on an oscillo-scope by triggering the time sweep of the oscilloscope.

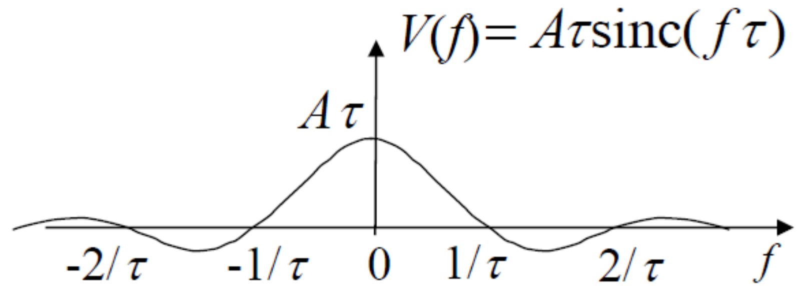
See Reference [Ziemer & Tranter] (Sec. 4.6) for more details about eye diagram.

- ISI is caused by insufficient channel bandwidth.
- Any better choice than rectangular pulse?

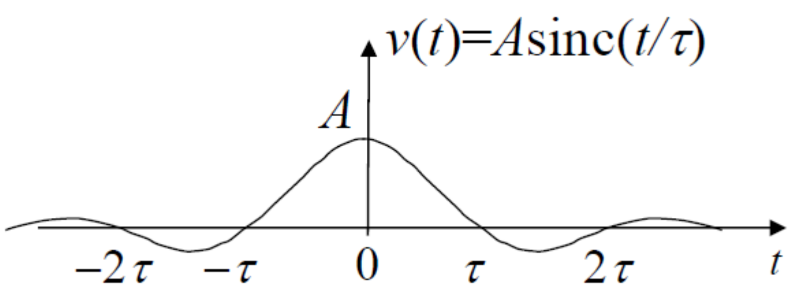
Sinc-Shaped pulse

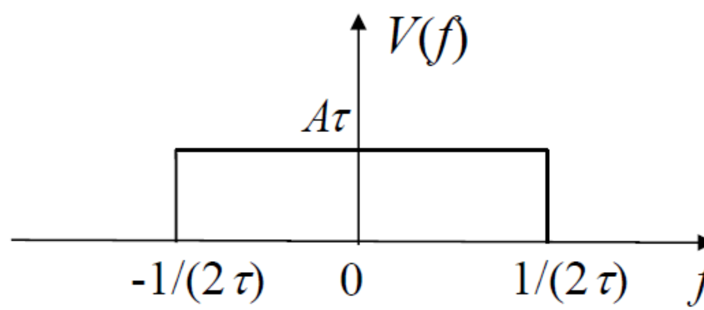
Sinc-Shaped Pulse





Rectangular Pulse

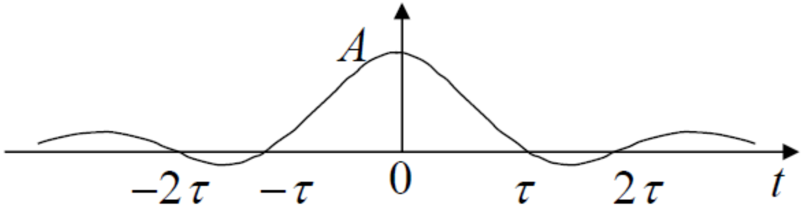




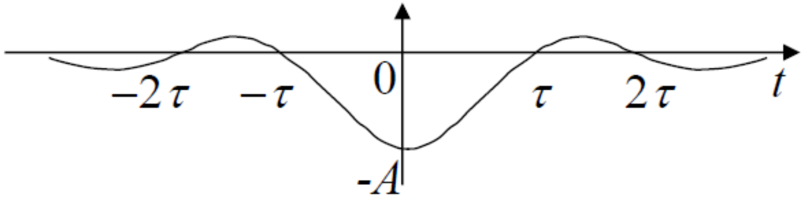
Sinc-Shaped Pulse

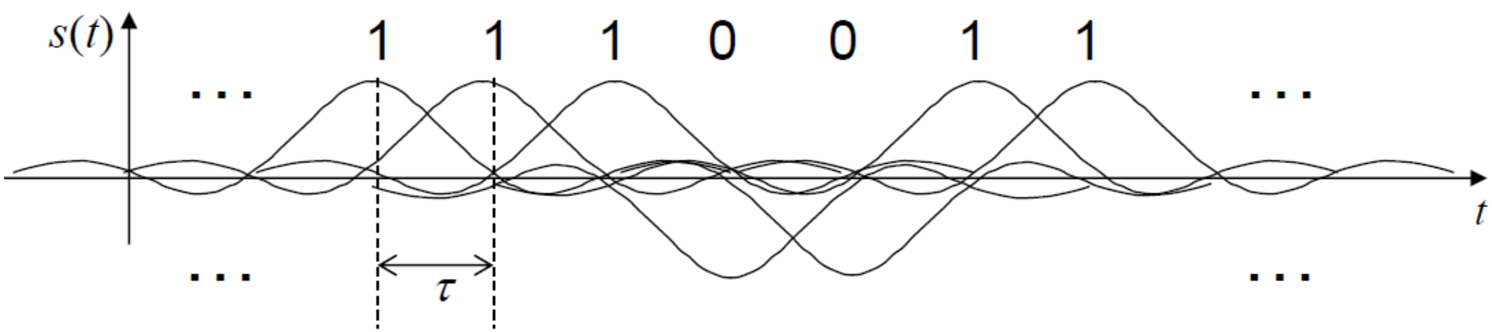
Binary Sinc-Shaped-Pulse Modulated Signal

1: a positive sinc-shaped pulse with amplitude A and first crossing-zero point $\pm\tau$



0: a negative sinc-shaped pulse with amplitude -A and first crossing-zero point $\pm\tau$





$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau)$$

✓ $\Pr\{Z_n = \pm 1\} = 1/2$

✓ $v(t) = A\operatorname{sinc}(t/\tau)$

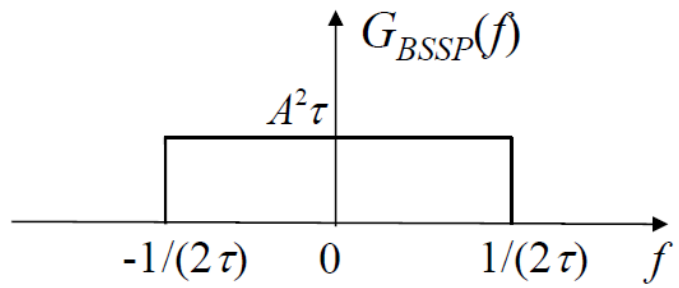
Power Spectrum of Sinc-Shaped-Pulse Modulated Signal

$$G_s(f) = \frac{1}{\tau}|V(f)|^2 \cdot \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{\tau}\right) \right)$$

With Binary Sinc-Shaped-Pulse Modulated Signal: $\mu_Z = 0,\ \sigma_Z^2 = 1$

$V(f) = A\tau, \quad |f| \le \frac{1}{2\tau}$

$G_{BSSP}(f) = A^2\tau, \quad |f| \le \frac{1}{2\tau}$



Bit Rate: $R_b = 1/\tau$

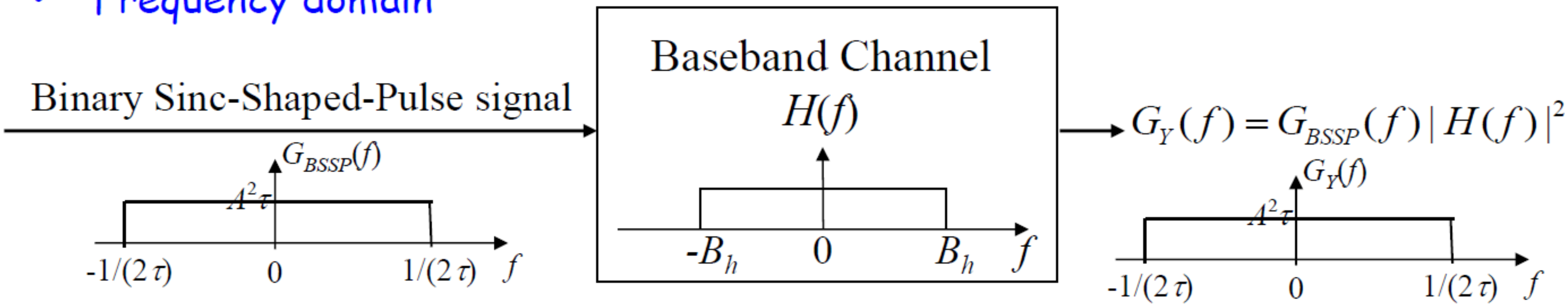
Required channel bandwidth: $B_h = 1/(2\tau) = R_b/2$

$\gamma_{BSSP} = 2$

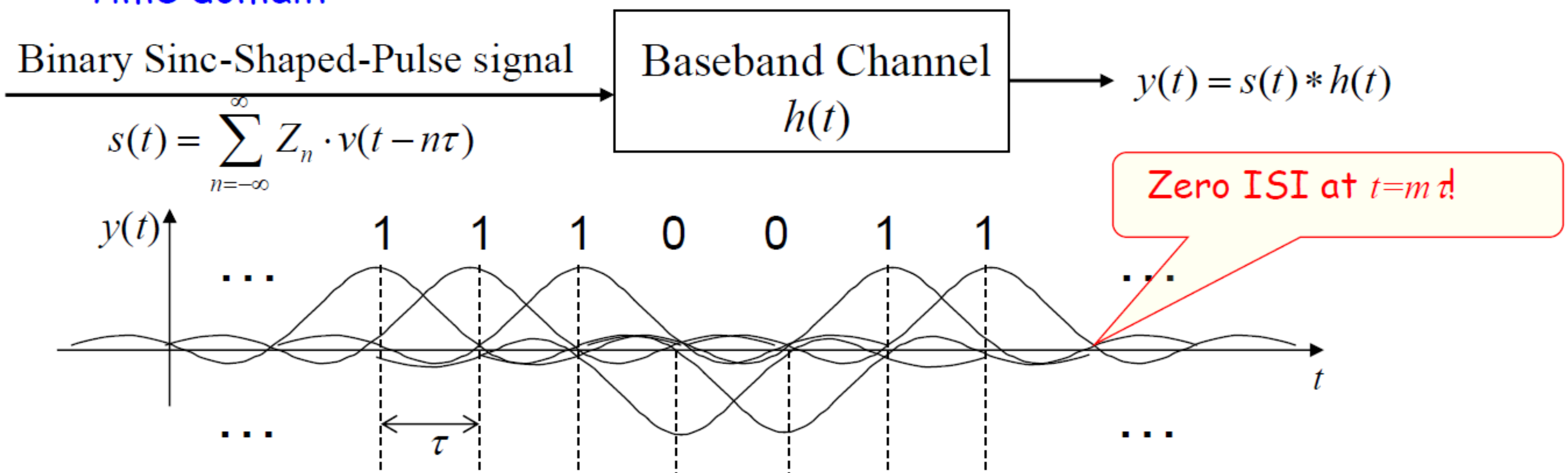
(with 100% in-band power)

Sinc-Shaped-Pulse Modulated Signal over Bandlimited Channel

- Frequency domain



- Time domain



Are there any other (better) choices to achieve zero ISI?

Nyquist Pulse-Shaping Criterion for Zero ISI

Nyquist pulse-shaping criterion for zero ISI

A necessary and sufficient condition for pulse $v(t)$ to satisfy

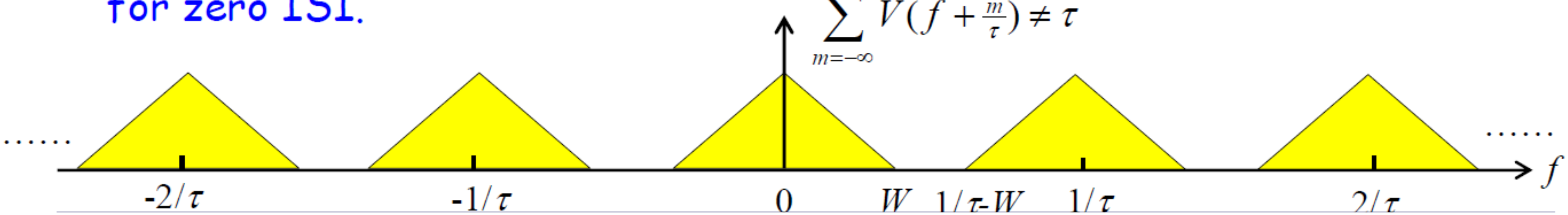
$$v(n\tau) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

is that its Fourier transform $V(f)$ satisfies $\sum_{m=-\infty}^{\infty} V(f + \frac{m}{\tau}) = \tau$.

Suppose that the bandwidth of unit pulse v(t) is W, which is also the required channel bandwidth. To pass the digital modulated signal with symbol rate $1/\tau$ through the channel:

- If $1/\tau$-W>W, there is no way to satisfy the Nyquist pulse-shaping criterion for zero ISI.

$$\sum_{m=-\infty}^{\infty} V(f + \tfrac{m}{\tau}) \neq \tau$$



Nyquist Pulse-Shaping Criterion for Zero ISI

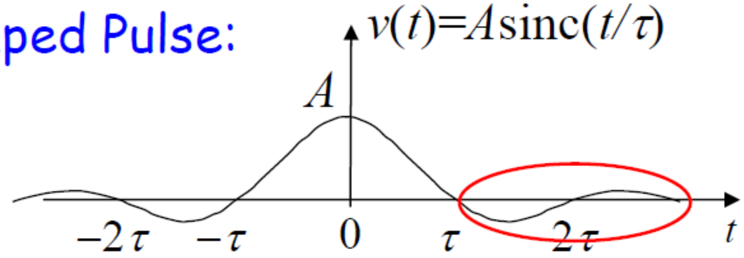
According to Nyquist pulse-shaping criterion for zero ISI:

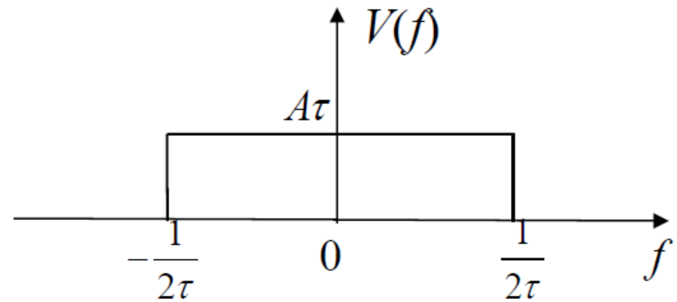
- ✓ If the symbol rate $1/\tau > 2W$, there is no way that we can design a system with zero ISI.
- ✓ If the symbol rate $1/\tau = 2W$, we must have $V(f) = \begin{cases} \tau, & |f| < W \\ 0, & \text{otherwise} \end{cases}$
  - The maximum symbol rate for zero ISI is 2W.
  - In the binary case, the highest bandwidth efficiency for zero-ISI is 2, which is achieved by the binary sinc-shaped-pulse modulated signal.
- ✓ If the symbol rate $1/\tau < 2W$, we have numerous choices. One of them is called Raised-Cosine Pulse.

See Textbook (Sec. 4.4) for more details.

Raised-Cosine Pulse: Tradeoff between Bandwidth Efficiency and Robustness

Sinc-Shaped Pulse:

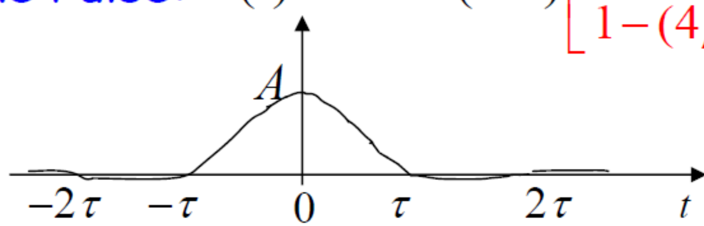


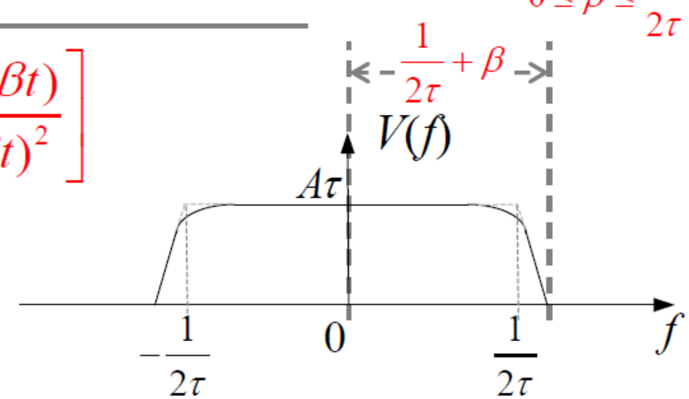


- Strong ISI at $t \neq n\tau$.
- Perfect synchronization is required at the receiver side.

Raised-Cosine Pulse: $v(t) = A\mathrm{sinc}(t/\tau)\left[\dfrac{\cos(2\pi\beta t)}{1-(4\beta t)^2}\right]$

$0 \le \beta \le \frac{1}{2\tau}$





- Larger β → More robust
- Larger β → Larger bandwidth

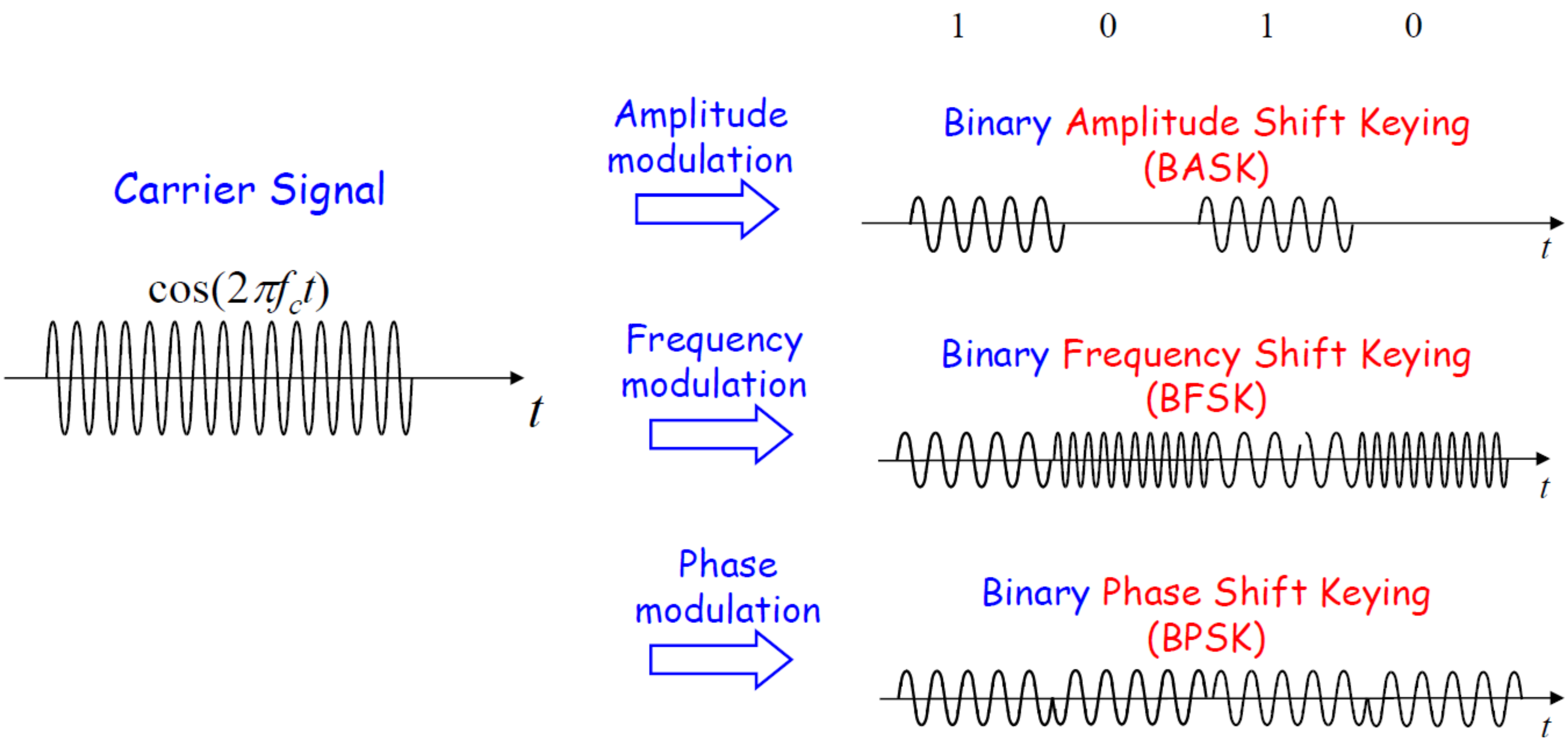
Summary I: Digital Baseband Modulation

		Complexity	Bandwidth Efficiency
PAM	Binary PAM	Low	1 (90% in-band power)
	4-ary PAM	Low	2 (90% in-band power)
Binary Sinc-Shaped-Pulse Modulation		High (Susceptible to timing jitter)	2 (100% in-band power)
Binary Raised-Cosine-Pulse Modulation		Moderate	$1 < \frac{R_b}{\frac{1}{2}R_b + \beta} < 2$ (100% in-band power)

Digital Bandpass Modulation

Digital Bandpass Modulation

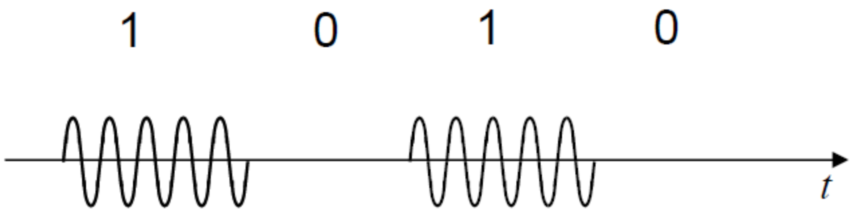
- How to transmit a baseband signal over a bandpass channel?



Binary Amplitude Shift Keying (ASK)

- Generate a binary ASK signal:
  - Send the carrier signal if the information bit is "1";
  - Send 0 volts if the information bit is "0".

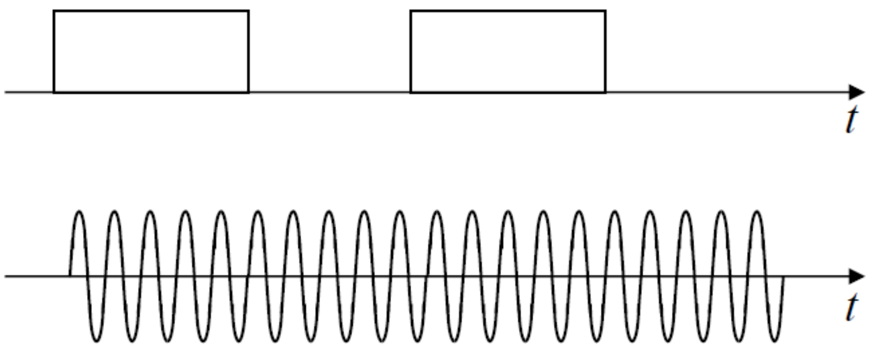
$$s_{BASK}(t) = s_{BOOK}(t)\cos(2\pi f_c t)$$



Binary On-Off Keying $s_{BOOK}(t)$

X

$\cos(2\pi f_c t)$



Power Spectrum of BASK

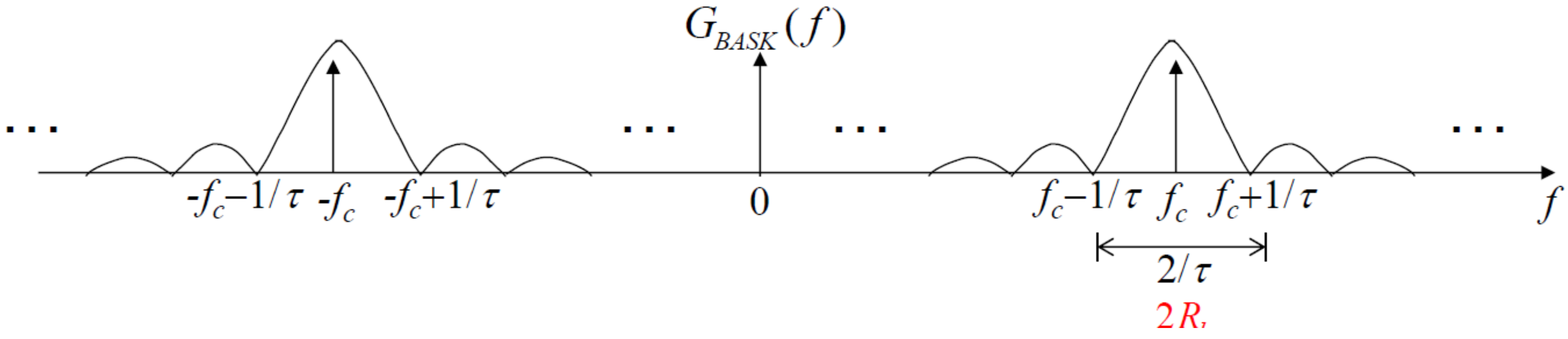
- Power spectrum of Binary OOK:

$$G_{BOOK}(f) = \frac{1}{\tau}\left(A\tau\operatorname{sinc}(f\tau)\right)^2 \cdot \left(\frac{1}{4} + \frac{1}{4\tau}\sum_{m=-\infty}^{\infty}\delta\left(f - \frac{m}{\tau}\right)\right)$$

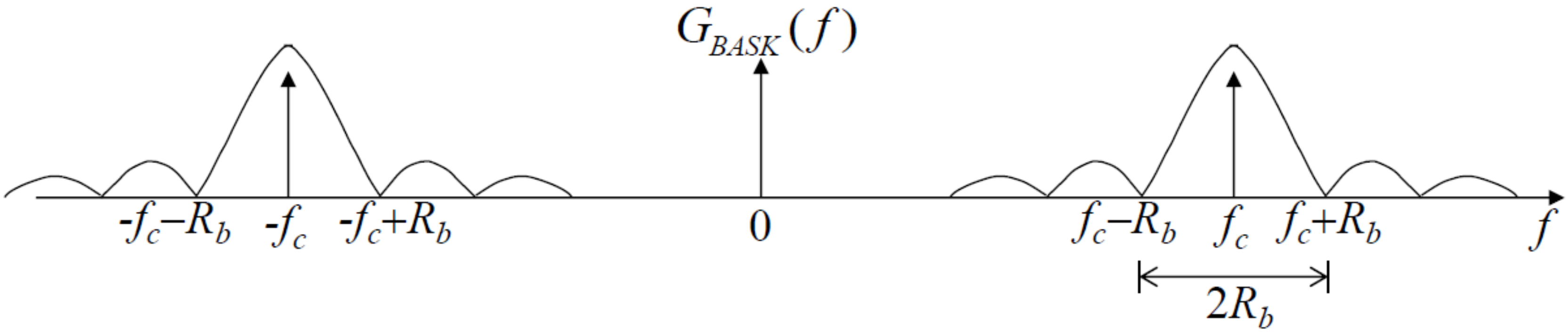
- Power spectrum of Binary ASK:

$$G_{BASK}(f) = \tfrac{1}{4}[G_{BOOK}(f - f_c) + G_{BOOK}(f + f_c)]$$

Read the supplemental material for details.



Bandwidth Efficiency of BASK



The bandwidth of BASK signal is twice of that of its baseband signal (binary On-Off Keying)!

- The required channel bandwidth for 90% in-band power:

$$B_{h\_90\%} = 2R_b$$

- Bandwidth Efficiency of BASK:

$$\boxed{\begin{array}{l} \gamma_{BASK} = 0.5 \text{ with 90\% in-band power} \\ \gamma_{BASK} = 0.25 \text{ with 95\% in-band power} \end{array}}$$

Binary Frequency Shift Keying (BFSK)

- Generate a binary FSK signal:

  Frequency offset

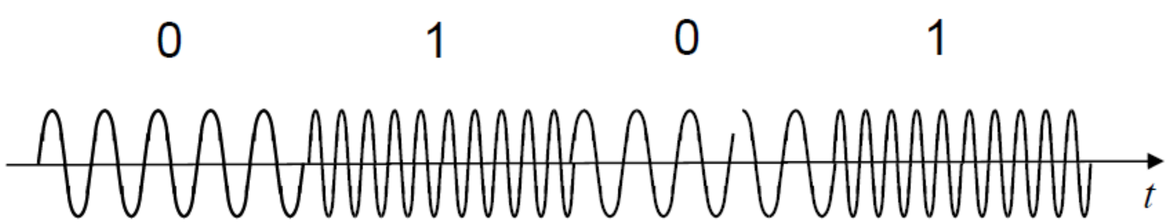
  - Send the signal $A\cos(2\pi(f_c+\Delta f)t)$ if the information bit is "1";
  - Send the signal $A\cos(2\pi(f_c-\Delta f)t)$ if the information bit is "0".

$$s_{BFSK}(t) = s_{b1,BFSK}(t)\cos(2\pi(f_c+\Delta f)t) + s_{b2,BFSK}(t)\cos(2\pi(f_c-\Delta f)t)$$

$$s_{b1,BFSK}(t) = \begin{cases} A & b_i = 1 \\ 0 & b_i = 0 \end{cases} \qquad s_{b2,BFSK}(t) = \begin{cases} 0 & b_i = 1 \\ A & b_i = 0 \end{cases}$$

Binary On-Off Keying

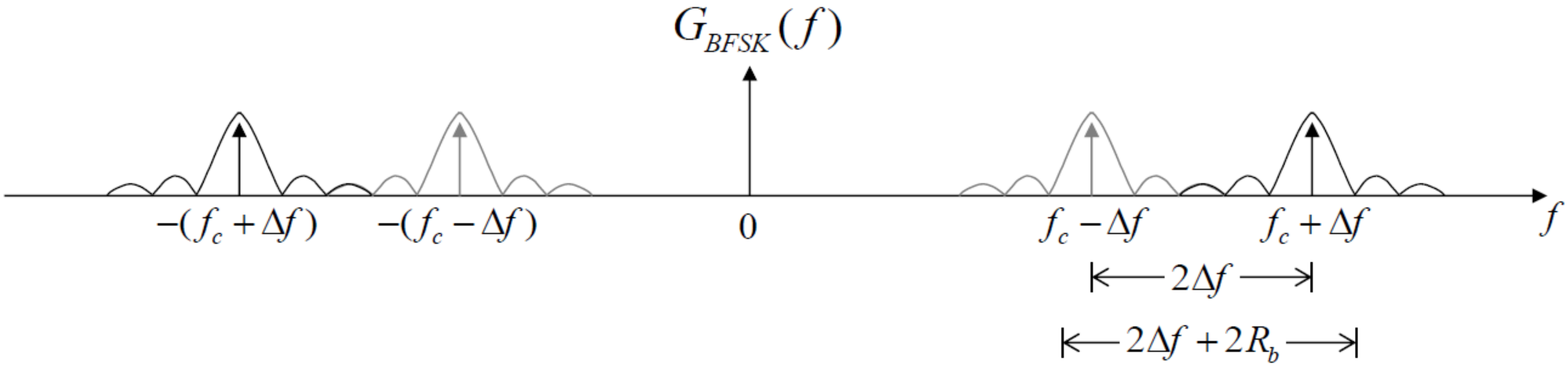
Binary On-Off Keying



Bandwidth Efficiency of BFSK

$$G_{BFSK}(f) = \tfrac{1}{4}[G_{b1,BFSK}(f-(f_c+\Delta f)) + G_{b1,BFSK}(f+(f_c+\Delta f))]$$

$$+\tfrac{1}{4}[G_{b2,BFSK}(f-(f_c-\Delta f)) + G_{b2,BFSK}(f+(f_c-\Delta f))]$$



- The required channel bandwidth for 90% in-band power:

$$B_{h\_90\%} = 2\Delta f + 2R_b$$

- Bandwidth efficiency of BFSK: (with 90% in-band power)

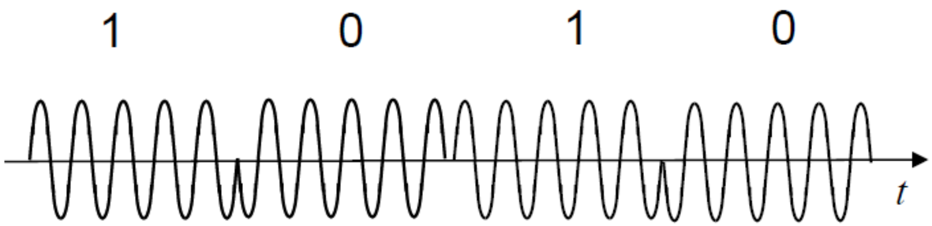
$$\gamma_{BFSK} = 0.5\cdot\frac{1}{1+\Delta f/R_b} < 0.5 = \gamma_{BASK}$$

The bandwidth efficiency of BFSK signal is lower than that of BASK signal!

Binary Phase Shift Keying (BPSK)

- Generate a binary PSK signal:
  - Send the signal $A\cos(2\pi f_c t)$ if the information bit is “1”;
  - Send the signal $A\cos(2\pi f_c t+\pi)$ if the information bit is “0”.
    $=-A\cos(2\pi f_c t)$

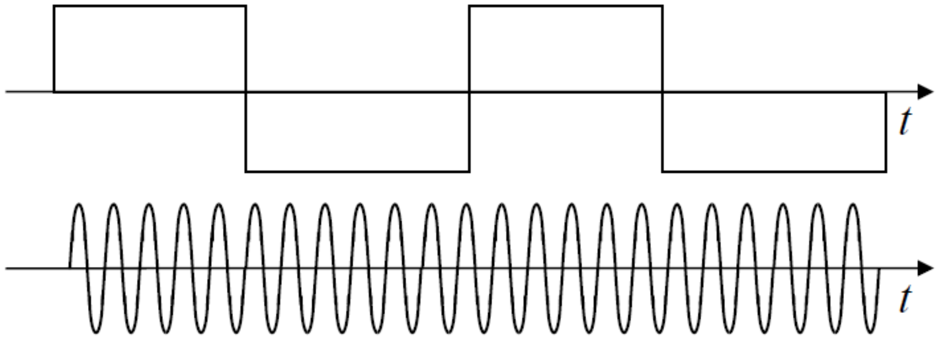
$s_{BPSK}(t) = s_{BPAM}(t)\cos(2\pi f_c t)$



Binary PAM $s_{BPAM}(t)$

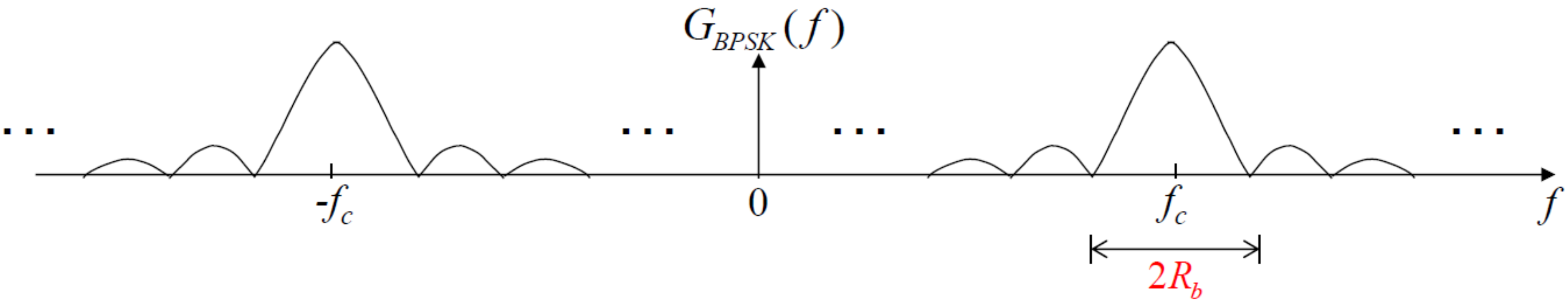
X

$\cos(2\pi f_c t)$



Bandwidth Efficiency of BPSK

$$G_{BPSK}(f) = \tfrac{1}{4}[G_{BPAM}(f - f_c) + G_{BPAM}(f + f_c)]$$



- The required channel bandwidth for 90% in-band power:

$$B_{h\_90\%} = 2R_b$$

- Bandwidth Efficiency of BPSK:

$$\boxed{\begin{array}{l} \gamma_{BPSK} = 0.5 \text{ with 90\% in-band power} \\ \gamma_{BPSK} = 0.25 \text{ with 95\% in-band power} \end{array}}$$

The bandwidth efficiency of BPSK signal is the same as that of BASK signal!

M-ary PSK

- M-ary PSK: transmitting pulses with M possible different carrier phases, and allowing each pulse to represent $\log_2 M$ bits.

✓ Binary PSK:

"1" $s_1(t) = A\cos(2\pi f_c t)$

"0" $s_2(t) = A\cos(2\pi f_c t + \pi)$

✓ Quaternary PSK: (QPSK)

"11" $s_1(t) = A\cos(2\pi f_c t + (-\pi / 4))$

"10" $s_2(t) = A\cos(2\pi f_c t + \pi / 4)$

"00" $s_3(t) = A\cos(2\pi f_c t + 3\pi / 4)$

"01" $s_4(t) = A\cos(2\pi f_c t + 5\pi / 4)$

QPSK

"1 1" $s_1(t) = A\cos(2\pi f_c t - \pi/4) = +\frac{A}{\sqrt{2}}\cos(2\pi f_c t) + \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$

"1 0" $s_2(t) = A\cos(2\pi f_c t + \pi/4) = +\frac{A}{\sqrt{2}}\cos(2\pi f_c t) - \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$

"0 0" $s_3(t) = A\cos(2\pi f_c t + 3\pi/4) = -\frac{A}{\sqrt{2}}\cos(2\pi f_c t) - \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$

"0 1" $s_4(t) = A\cos(2\pi f_c t + 5\pi/4) = -\frac{A}{\sqrt{2}}\cos(2\pi f_c t) + \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$

A QPSK signal can be decomposed into the sum of two PSK signals: an in-phase component and a quadrature component.

$$s_{QPSK}(t) = d_I \frac{A}{\sqrt{2}}\cos(2\pi f_c t) + d_Q \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$$

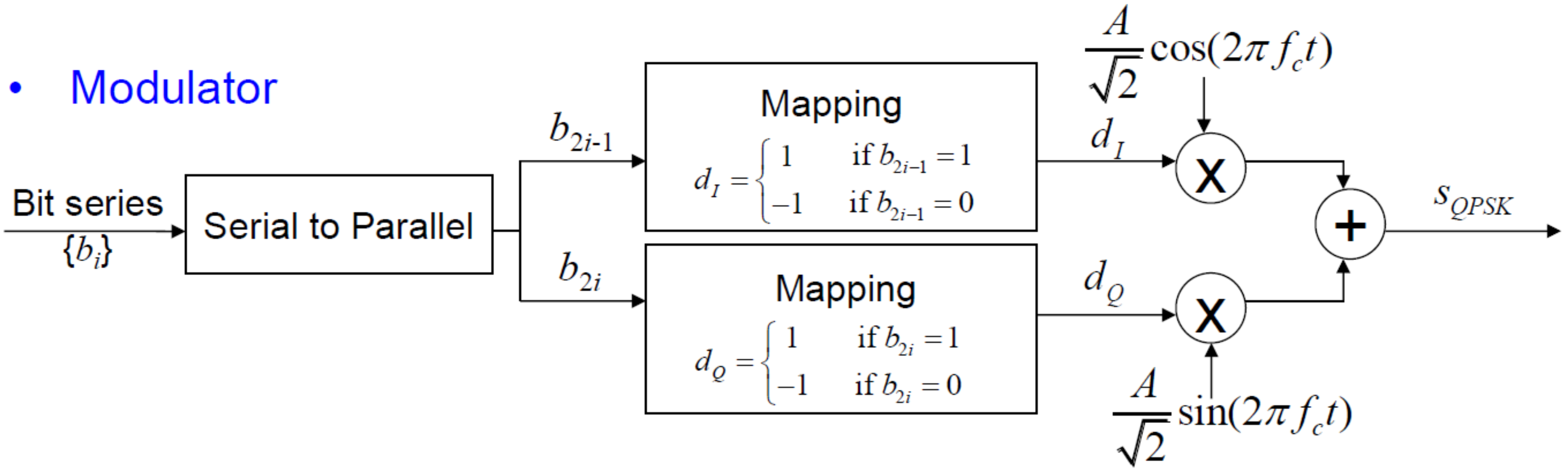
$$d_I = \begin{cases} 1 & \text{if } b_{2i-1} = 1 \\ -1 & \text{if } b_{2i-1} = 0 \end{cases} \qquad d_Q = \begin{cases} 1 & \text{if } b_{2i} = 1 \\ -1 & \text{if } b_{2i} = 0 \end{cases}$$

QPSK Modulator

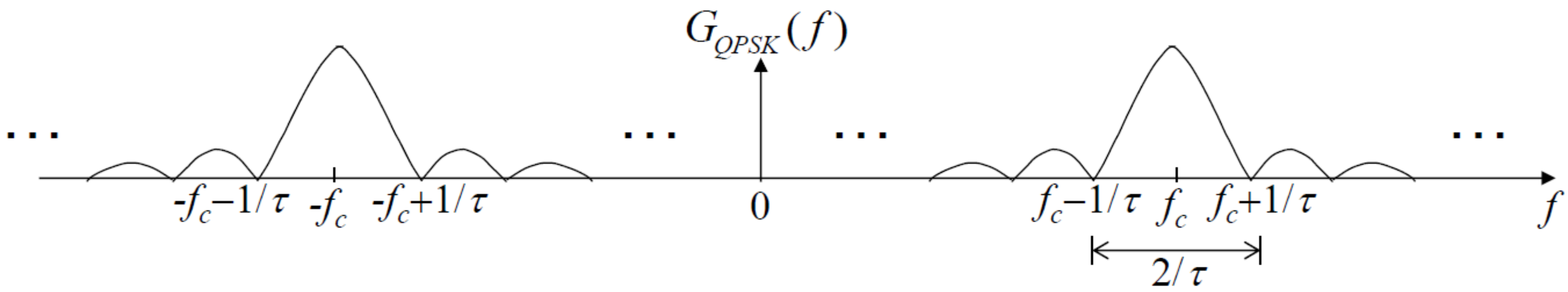
$$s_{QPSK}(t) = d_I \frac{A}{\sqrt{2}}\cos(2\pi f_c t) + d_Q \frac{A}{\sqrt{2}}\sin(2\pi f_c t)$$

$$d_I = \begin{cases} 1 & \text{if } b_{2i-1} = 1 \\ -1 & \text{if } b_{2i-1} = 0 \end{cases} \qquad d_Q = \begin{cases} 1 & \text{if } b_{2i} = 1 \\ -1 & \text{if } b_{2i} = 0 \end{cases}$$

- Modulator



Bandwidth Efficiency of QPSK



- Symbol rate: $R_{S,QPSK} = 1/\tau$
- Bit rate: $R_{b,QPSK} = 2R_{S,QPSK} = 2/\tau$
- Required Channel Bandwidth: $B_{h\_90\%} = 2R_{S,QPSK} = R_{b,QPSK}$
  $B_{h\_95\%} = 4R_{S,QPSK} = 2R_{b,QPSK}$
- Bandwidth Efficiency:

$\gamma_{QPSK} = 1$ with 90% in-band power

$\gamma_{QPSK} = 0.5$ with 95% in-band power

QPSK achieves higher bandwidth efficiency than BPSK!

Summary II: Digital Bandpass Modulation

	Bandwidth Efficiency (90% in-band power)
Binary ASK	0.5
Binary FSK	$0.5 \cdot \frac{1}{1+\Delta f / R_b}$
Binary PSK	0.5
QPSK	1