Lecture 2. Deterministic Signal Analysis

- Fourier Transform
- Energy Spectrum, Power Spectrum and Signal Bandwidth
- Signal Transmission through a Linear System

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Signals in Time Domain

- A signal is a set of data or information, which can be represented as a function of time: $s(t)$
- Deterministic signal is a signal whose physical description is known completely, either in a mathematical form or a graphical form.
  - Signal Energy: $E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt$
  - Signal Power: $P_s = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt$
- Signal Classification
  - Continuous-time vs. Discrete-time signal
  - Periodic signal vs. Aperiodic signal

Signals in Frequency Domain

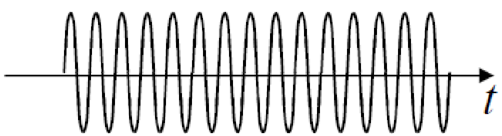
Time domain

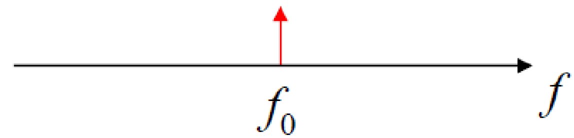
$$s(t)$$

Frequency domain

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt$$

- $\cos(2\pi f_0 t)$

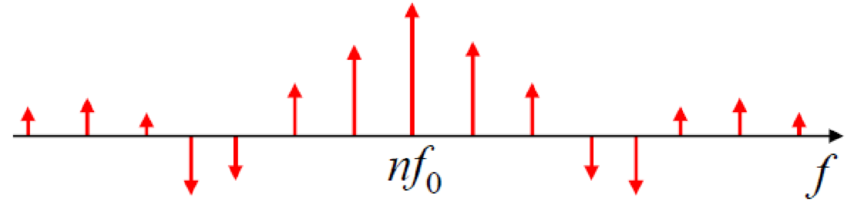




- Periodic signal with period $1/f_0$:

$$s_n = f_0 \int_0^{1/f_0} s(t) e^{-j2\pi n f_0 t}\, dt$$

$$\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$$



Fourier Transform

Fourier Transform

Given a time domain signal $s(t)$, its Fourier transform is defined as follows.

Fourier transform: $$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$

The time domain signal $s(t)$ can be expressed by $S(f)$ using an inverse transform.

Inverse Fourier transform: $$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

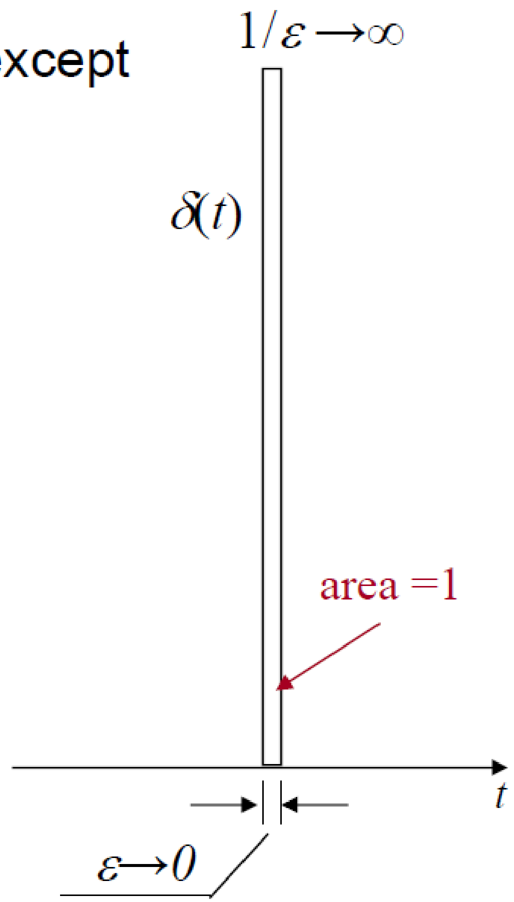
- (Fourier) spectrum of $s(t)$: $S(f)$
- Magnitude spectrum of $s(t)$: $|S(f)|$

$$s(t) \Leftrightarrow S(f)$$

Example 1: Spectrum of Unit Impulse

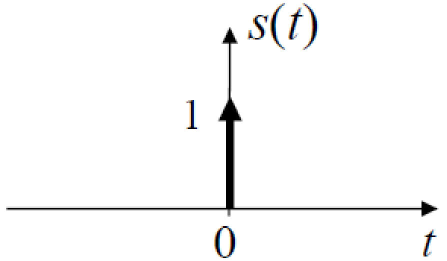
- $\delta(t)$ is a unit impulse, which is zero everywhere except at $t=0$, and has unit area.

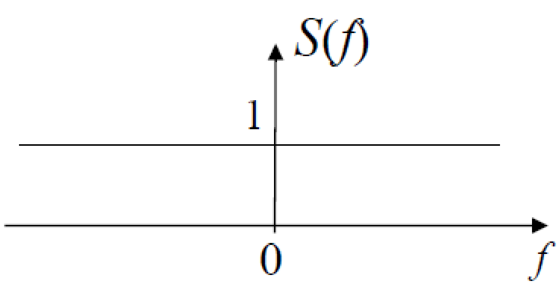
$$\delta(t)=\begin{cases}\infty & t=0\\ 0 & t\neq 0\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)dt=1$$



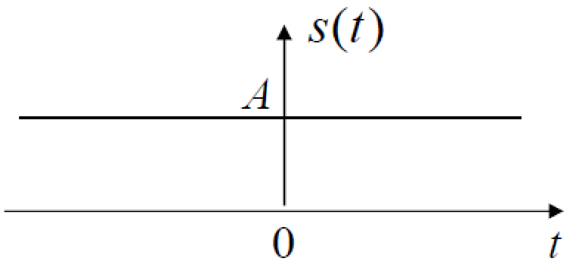
$$s(t)=\delta(t) \quad \Leftrightarrow \quad S(f)=\int_{-\infty}^{\infty}\delta(t)e^{-j2\pi ft}dt$$

$$=1$$

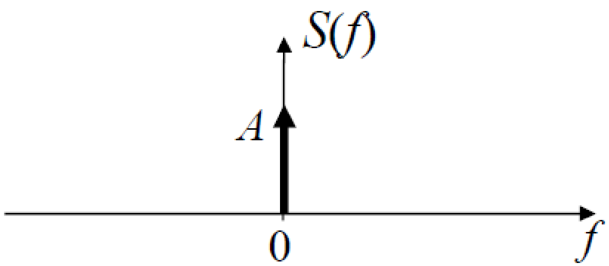




Example 2: Spectrum of Constant Signal



$$s(t) = A$$



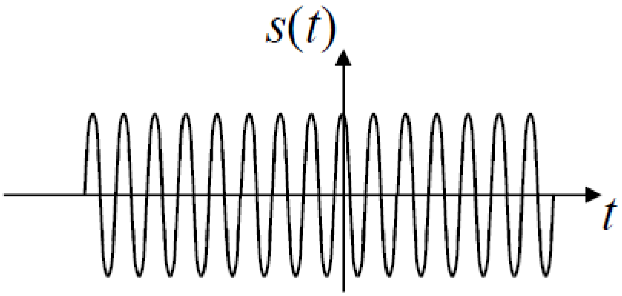
$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}\,dt$$

$$S(0) = A\int_{-\infty}^{\infty} e^{-j2\pi 0t}\,dt = A\int_{-\infty}^{\infty} 1dt = \infty$$

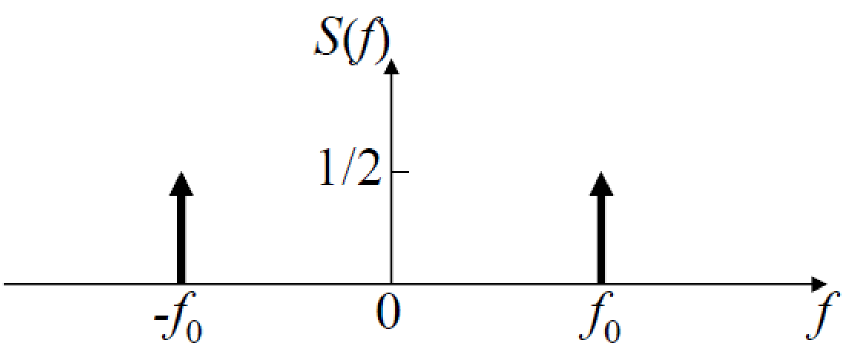
$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}\,dt = 0 \quad \text{for} \quad f \neq 0$$

$$\boxed{S(f) = A\delta(f)}$$

Example 3: Spectrum of Sinusoidal Signal



$$s(t) = \cos(2\pi f_0 t)$$



$$S(f) = \int_{-\infty}^{\infty} \cos 2\pi f_0 t \cdot e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} \tfrac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \cdot e^{-j2\pi ft}\, dt$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t}\, dt + \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t}\, dt$$

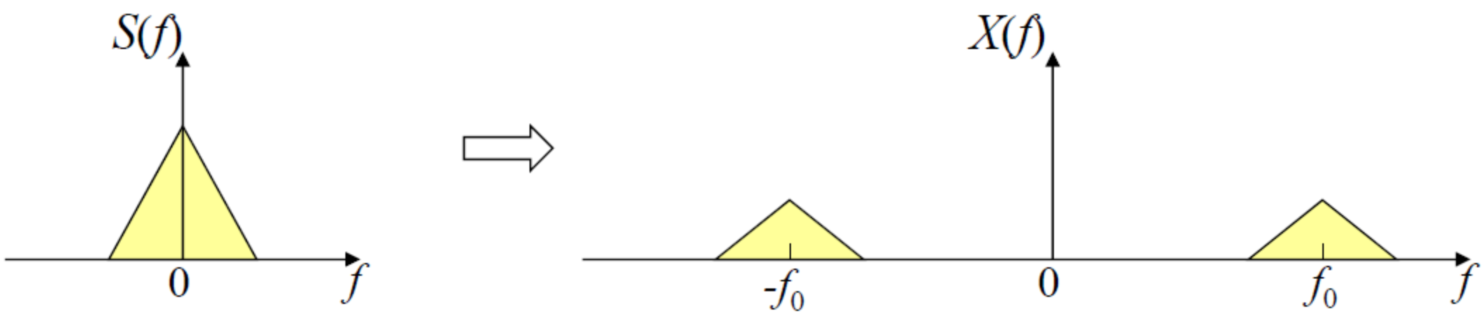
$$= \tfrac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

Example 4: Spectrum of $s(t)\cos(2\pi f_0 t)$

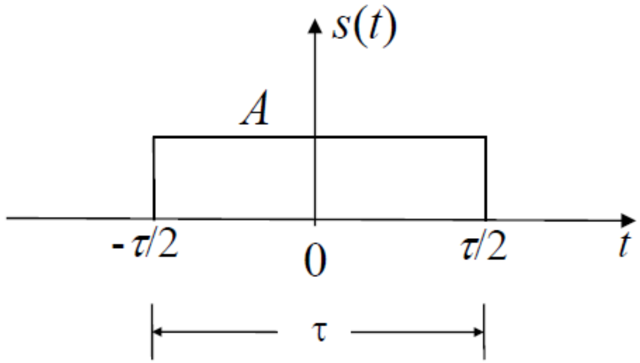
$$x(t) = s(t)\cos(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{\infty} s(t)\cos(2\pi f_0 t)\cdot e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} s(t)\cdot \tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\cdot e^{-j2\pi ft}dt$$

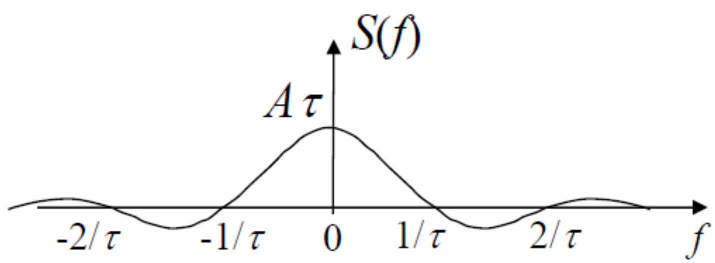
$$= \tfrac{1}{2}\int_{-\infty}^{\infty} s(t)\cdot e^{-j2\pi(f-f_0)t}dt + \tfrac{1}{2}\int_{-\infty}^{\infty} s(t)\cdot e^{-j2\pi(f+f_0)t}dt = \tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$$



Example 5: Spectrum of Single Rectangular Pulse



$$s(t)=\begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



$$S(f)=A\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt \;=A\cdot\frac{e^{-j\pi f\tau}-e^{+j\pi f\tau}}{-j2\pi f}$$

$$=A\tau\cdot\frac{\sin(\pi f\tau)}{\pi f\tau}=A\tau\,\mathrm{sinc}(f\tau)$$

$$\mathrm{sinc}(x)\doteq\frac{\sin(\pi x)}{\pi x}$$

- sinc function is an even, oscillating function with a decreasing magnitude.
- It has unit peak at x=0, and zero crossing points at x= non-zero integers.

Properties of Fourier Transform

$\alpha s_1(t)+\beta s_2(t)$	$\Leftrightarrow$	$\alpha S_1(f)+\beta S_2(f)$	Linearity
$s_1(t)s_2(t)$	$\Leftrightarrow$	$S_1(f)*S_2(f)$	Convolution
$S(t)$	$\Leftrightarrow$	$s(-f)$	Duality
$s(t-\tau)$	$\Leftrightarrow$	$S(f)e^{-j2\pi f\tau}$	Time shift
$s(t)e^{-j2\pi f_0 t}$	$\Leftrightarrow$	$S(f+f_0)$	Frequency shift
$s(t)\cos(2\pi f_0 t)$	$\Leftrightarrow$	$\frac{1}{2}[S(f-f_0)+S(f+f_0)]$	Modulation
$s(at)$ (for any real $a \neq 0$)	$\Leftrightarrow$	$\frac{1}{\lvert a\rvert}S\left(\frac{f}{a}\right)$	Time scale

Review Examples 2 & 4

$s(t)=\delta(t) \;\Leftrightarrow\; S(f)=1$

$s(t)=1 \;\Leftrightarrow\; S(f)=\delta(f)$

Duality: $S(t) \Leftrightarrow s(-f)$

$$x(t)=s(t)\cos(2\pi f_0 t)$$

$$\Leftrightarrow$$

$$X(f)=S(f)*\left[\tfrac{1}{2}(\delta(f-f_0)+\delta(f+f_0))\right]$$
$$=\tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$$

Modulation:

$s_1(t)\cos(2\pi f_0 t)$

$\Leftrightarrow$

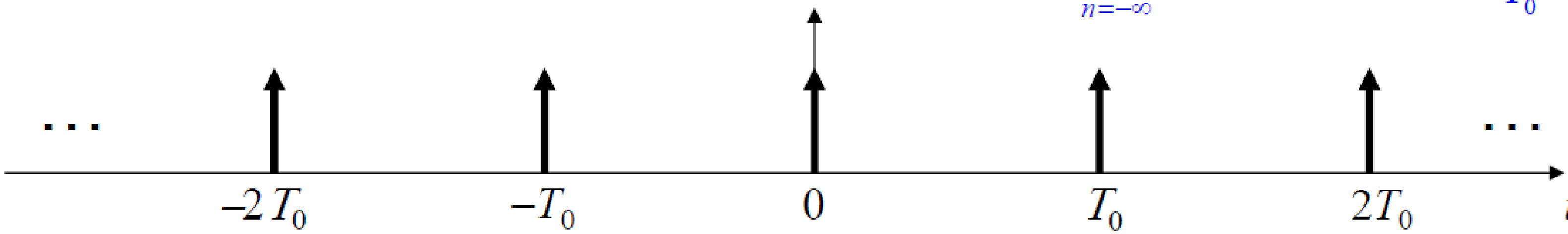
$\tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$

Convolution:

$s_1(t)s_2(t) \Leftrightarrow S_1(f)*S_2(f)$

Example 6: Spectrum of Impulse Train

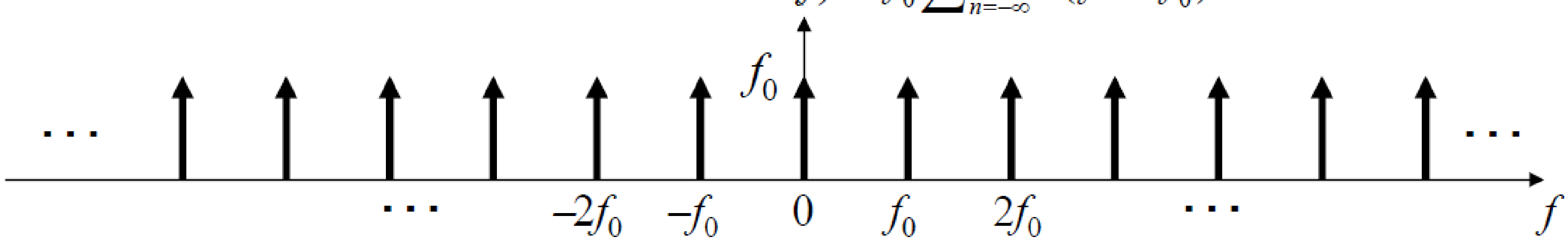
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t} \qquad f_0 = \frac{1}{T_0}$$



$$S(f) = \sum_{n=-\infty}^{\infty} s_n \delta(f - nf_0)$$

$$s_n = \frac{1}{T_0}\int_0^{T_0} s(t)e^{-j2\pi n f_0 t}dt = f_0 \int_0^{1/f_0} \delta(t)e^{-j2\pi n f_0 t}dt = f_0$$

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$



Example 7: Spectrum of Periodic Signal

- For periodic signal $s(t)$ with period T_0, define $s_{T_0}(t)$ as

$$s_{T_0}(t) = \begin{cases} s(t) & -T_0/2 < t < T_0/2 \\ 0 & otherwise \end{cases}$$

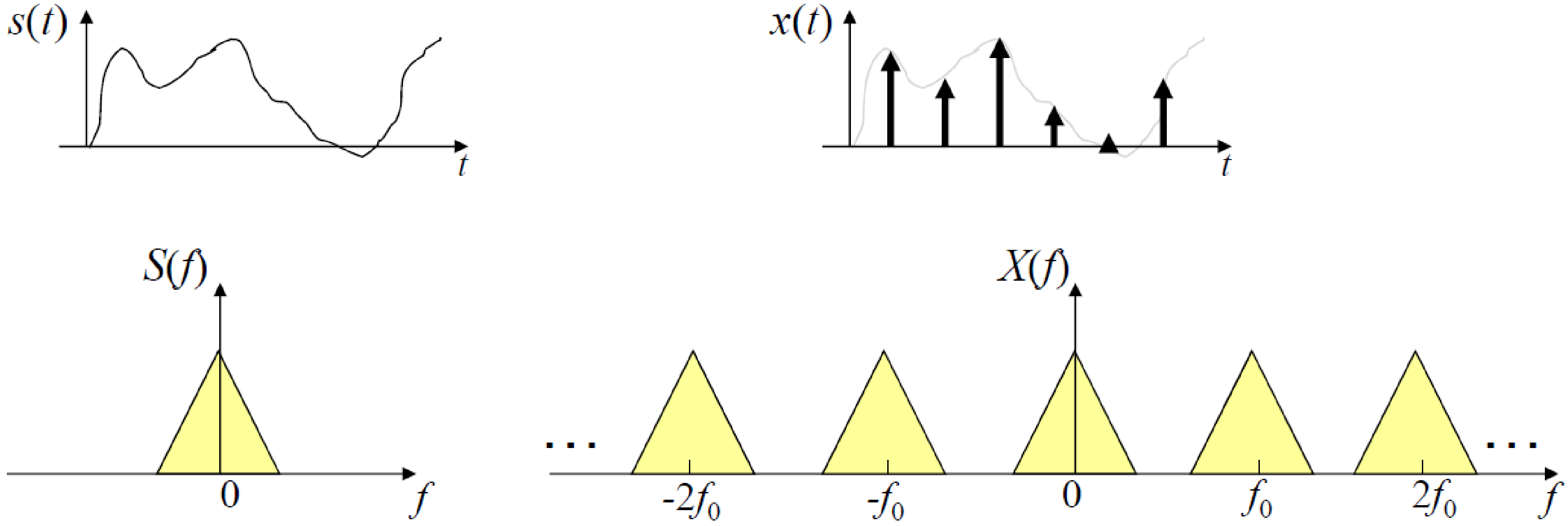
- $s(t) = \sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0) \ = s_{T_0}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

- $S(f) = S_{T_0}(f) \cdot f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}(nf_0) \delta(f - nf_0)$

Example 8: Spectrum of Sampled Signal

$$x(t) = s(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$X(f) = S(f) * f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S(f - nf_0)$$



Energy Spectrum, Power Spectrum and Signal Bandwidth

Energy-type Signal and Power-type Signal

- Energy-type Signal: A signal is an energy-type signal if and only if its energy is positive and finite.
  - ✓ s(t) is an energy-type signal if and only if $0 < E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt < \infty.$

- Power-type Signal: A signal is a power-type signal if and only if its power is positive and finite.
  - ✓ s(t) is a power-type signal if and only if $0 < P_s = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt < \infty.$

How to determine if a signal is an energy-type signal or a power-type signal from the frequency domain?

Energy and Energy Spectrum

- Energy of energy-type signal s(t):

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} s(t)s^*(t)dt = \int_{-\infty}^{\infty} s(t)\left[\int_{-\infty}^{\infty} S^*(f)e^{-j2\pi ft}df\right]dt$$

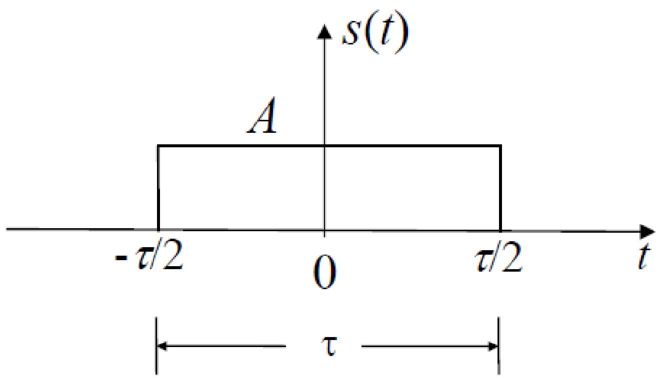
$$= \int_{-\infty}^{\infty} S^*(f)\left[\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt\right]df = \int_{-\infty}^{\infty} S^*(f)S(f)df = \int_{-\infty}^{\infty} |S(f)|^2\, df$$

$$= \int_{-\infty}^{\infty} U_s(f)df$$

Parseval's Theorem: $\boxed{E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} |S(f)|^2\, df}$

- Energy spectrum: $U_s(f) \triangleq |S(f)|^2$

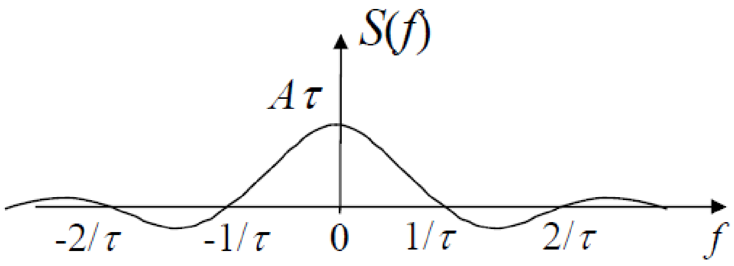
Example 9: Energy Spectrum of Single Rectangular Pulse



$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

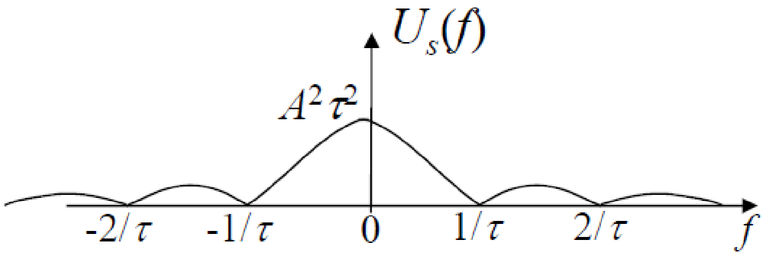
- Fourier spectrum:

  $$S(f) = A\tau \operatorname{sinc}(f\tau)$$



- Energy spectrum:

  $$U_s(f) = |S(f)|^2 = A^2\tau^2 \operatorname{sinc}^2(f\tau)$$



Power and Power Spectrum

- Power of power-type signal s(t):

$$s_T(t) \triangleq \begin{cases} s(t) & -T/2 < t < T/2 \\ 0 & otherwise \end{cases}$$

$$P_s = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|s(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|s_T(t)|^2\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|S_T(f)|^2\,df = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}|S_T(f)|^2\,df = \int_{-\infty}^{\infty}G_s(f)df$$

- Power spectrum:

$$G_s(f) \triangleq \lim_{T\to\infty}\frac{1}{T}|S_T(f)|^2$$

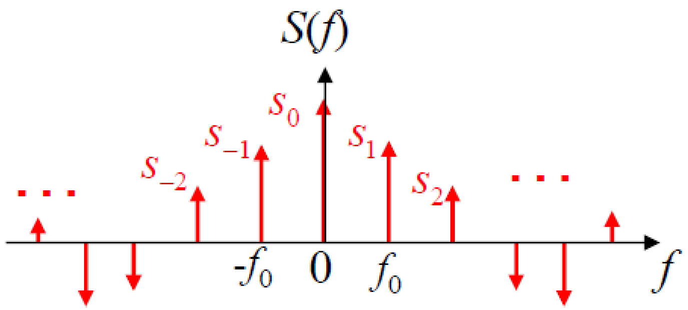
$$G_s(f) \Leftrightarrow \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}s(t+\tau)s^*(t)dt$$

Example 10: Power Spectrum of Periodic Signal

For periodic signal $s(t)$ with period T_0: $s(t)=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$ $\qquad f_0 = 1/T_0$

- Fourier spectrum:

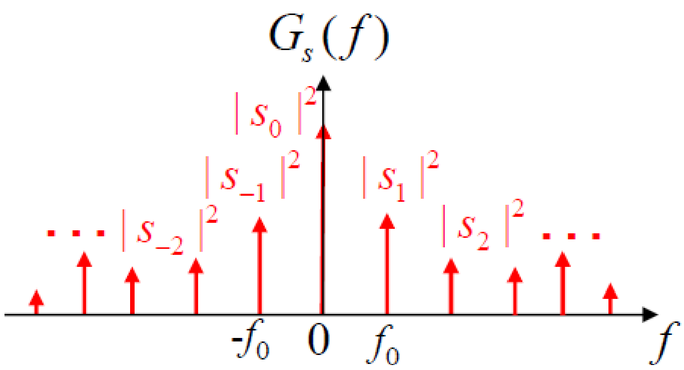
$$S(f)=\sum_{n=-\infty}^{\infty} s_n \delta(f-nf_0)$$



- Power spectrum:

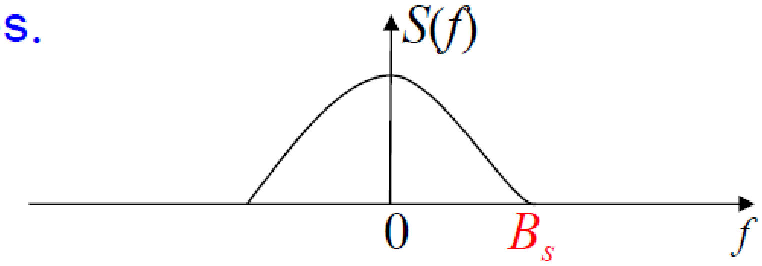
$$G_s(f) \Leftrightarrow \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} s(t+\tau)s^*(t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t+\tau)s^*(t)dt = \sum_{n=-\infty}^{\infty}|s_n|^2 e^{j2\pi n f_0 \tau}$$

$$G_s(f)=\sum_{n=-\infty}^{\infty}|s_n|^2 \delta(f-nf_0)$$

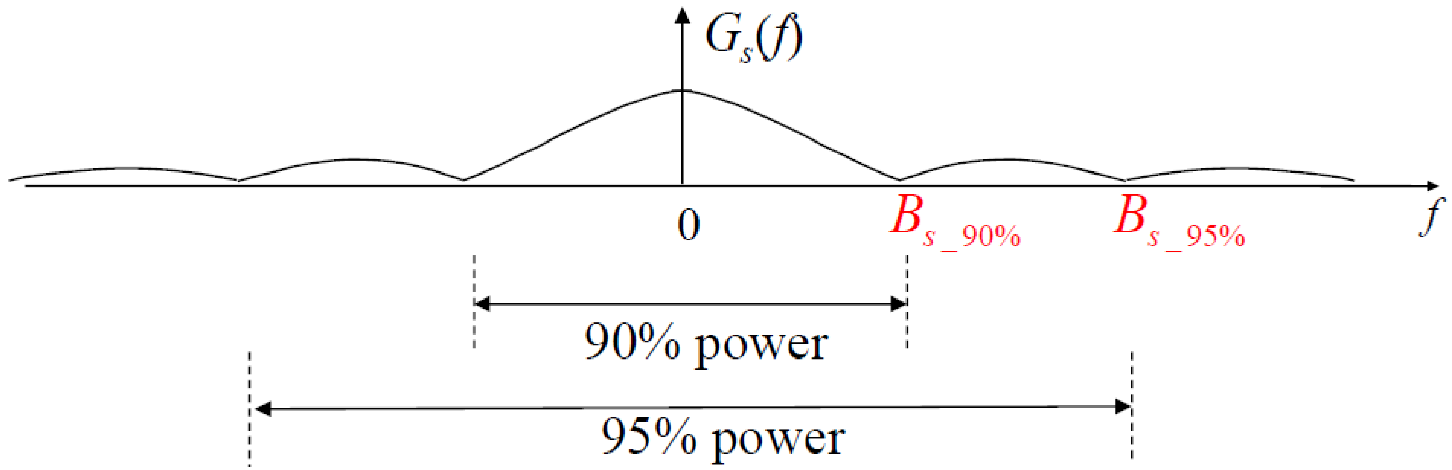


Signal Bandwidth

- Bandwidth of signal $s(t)$: the amount of **positive** frequency spectrum that signal $s(t)$ occupies.



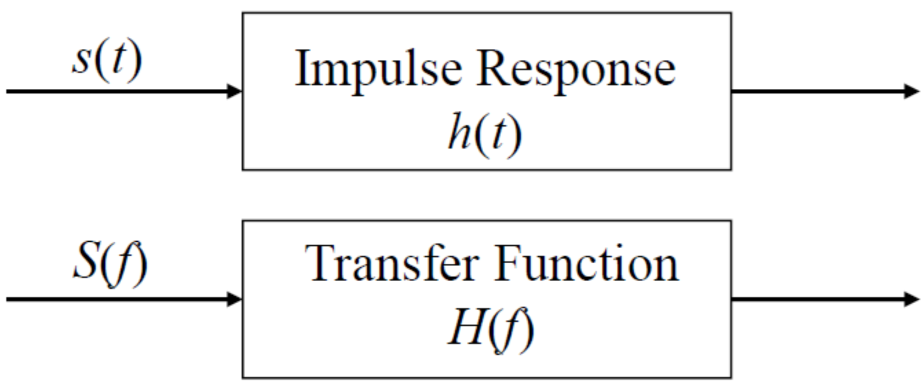
- Effective Bandwidth: $x\%$ of the signal's power (energy) are included.



Signal Transmission through a Linear System

Linear Time Invariant (LTI) System

- Linear system: in the time domain, a linear system is described in terms of its *impulse response* (the response of the system to a unit impulse $\delta(t)$).
- Linear Time Invariant (LTI) system: the shape of the impulse response is the same no matter when the unit impulse $\delta(t)$ is applied to the system.

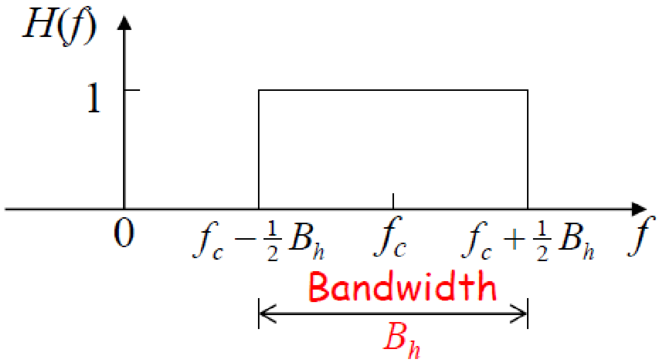


$$y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

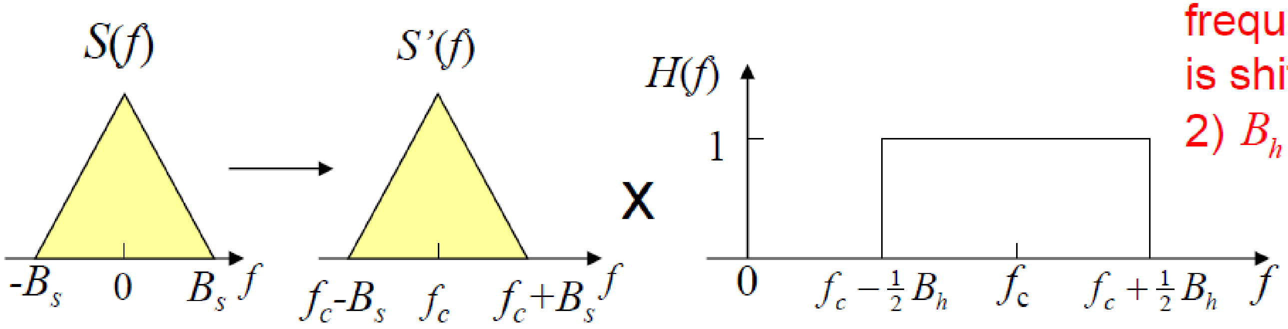
$$Y(f) = S(f) \cdot H(f)$$

Ideal Bandpass System

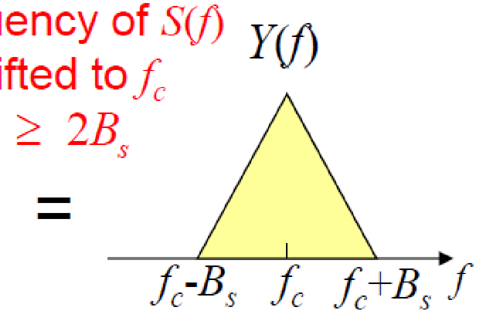
- Transfer Function $H(f)$ of an ideal bandpass system:



- For a baseband input signal with bandwidth B_s:

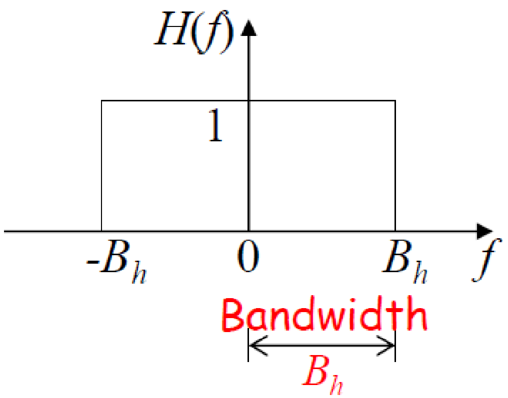


If: 1) the center frequency of $S(f)$ is shifted to f_c
2) $B_h \geq 2B_s$

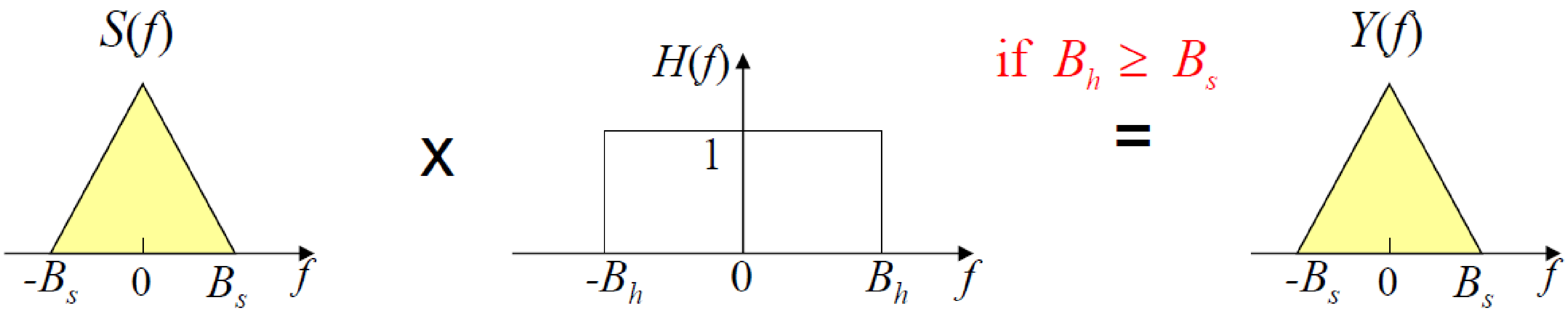


Ideal Lowpass System

- Transfer Function $H(f)$ of an ideal lowpass system:



- For a baseband input signal with bandwidth B_s:



Baseband Channel and Bandpass Channel

- Baseband channel
  - A baseband channel efficiently passes frequency components from dc (zero) to the cutoff frequency B_h Hz.
  - Examples: coaxial cable

- Bandpass channel
  - A bandpass channel efficiently passes frequency components within a certain band, say, between $f_c - \frac{1}{2}B_h$ and $f_c + \frac{1}{2}B_h$ Hz.
  - Examples: EM wave, fibre

In this course, a baseband channel and a bandpass channel are modeled as an ideal low-pass LTI system and an ideal bandpass LTI system, respectively.