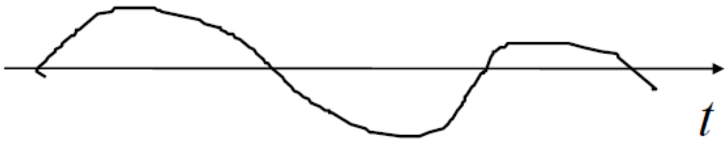
Lecture 6.

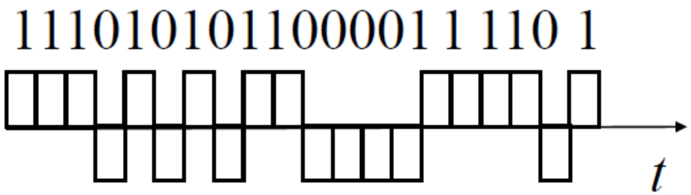
Analog-to-Digital (A-D) and Digital-to-Analog (D-A) Conversion

Prof. An Liu
College of ISEE, Zhejiang University

Analog Signal and Digital Signal



Analog signal
(continuous amplitude)

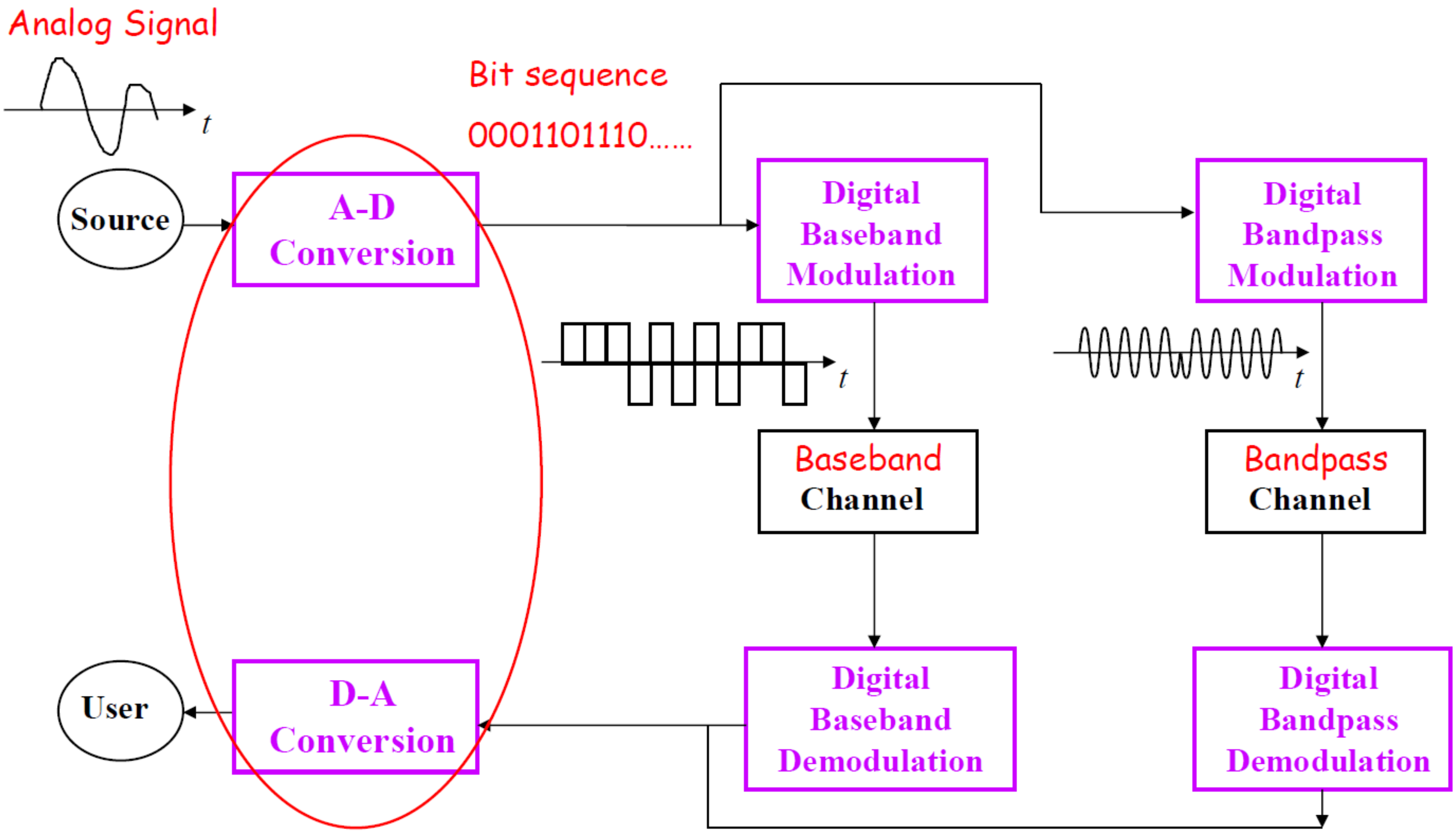


Digital signal
(discrete amplitude)

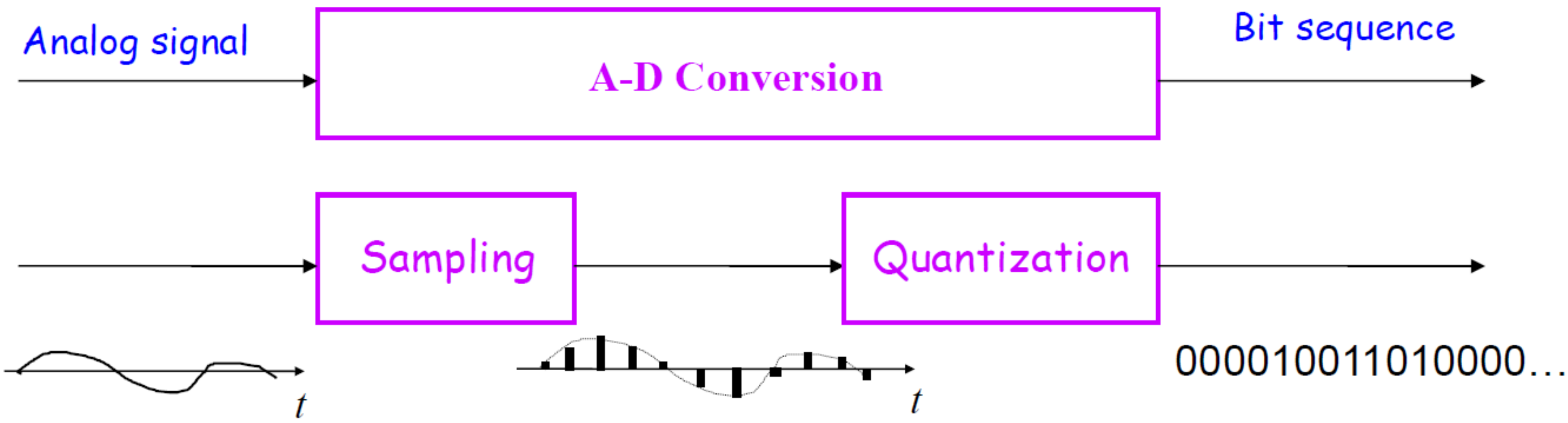
Why digital?

- easier to be regenerated
- complete theory
- flexible hardware implementation and efficient storage

Digital Communications



A-D Conversion



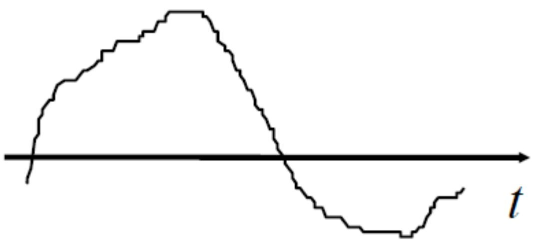
- Sampling
  - A continuous-time analog signal is transformed into a discrete-time signal.
- Quantization
  - A discrete-time signal is transformed into a series of binary bits.

Sampling

Sampling

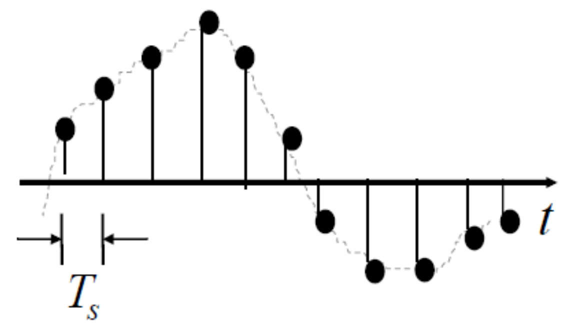
Time Domain

$x(t)$

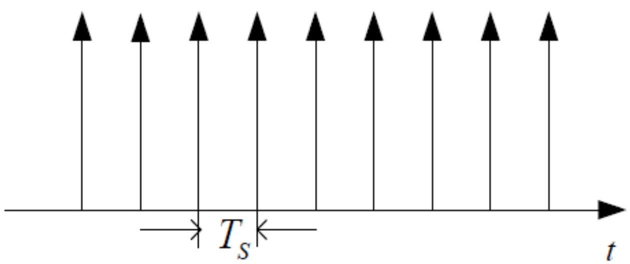


Sampling

$$x_S(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_S)$$



$$\sum_{n=-\infty}^{\infty}\delta(t - nT_S)$$



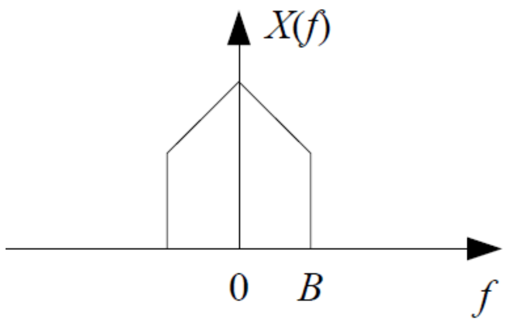
T_S: Sampling period

$f_S = 1/T_S$: Sampling rate

Sampling

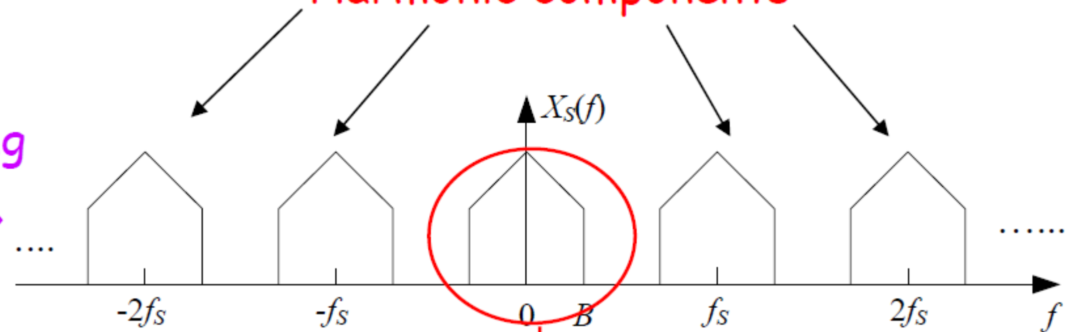
Frequency Domain

$$X_S(f) = X(f) * \left(f_S \sum_{n=-\infty}^{\infty} \delta(f - nf_S)\right) = f_S \sum_{n=-\infty}^{\infty} X(f - nf_S)$$

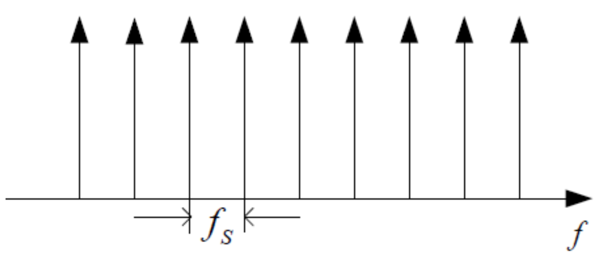


Sampling

Harmonic components



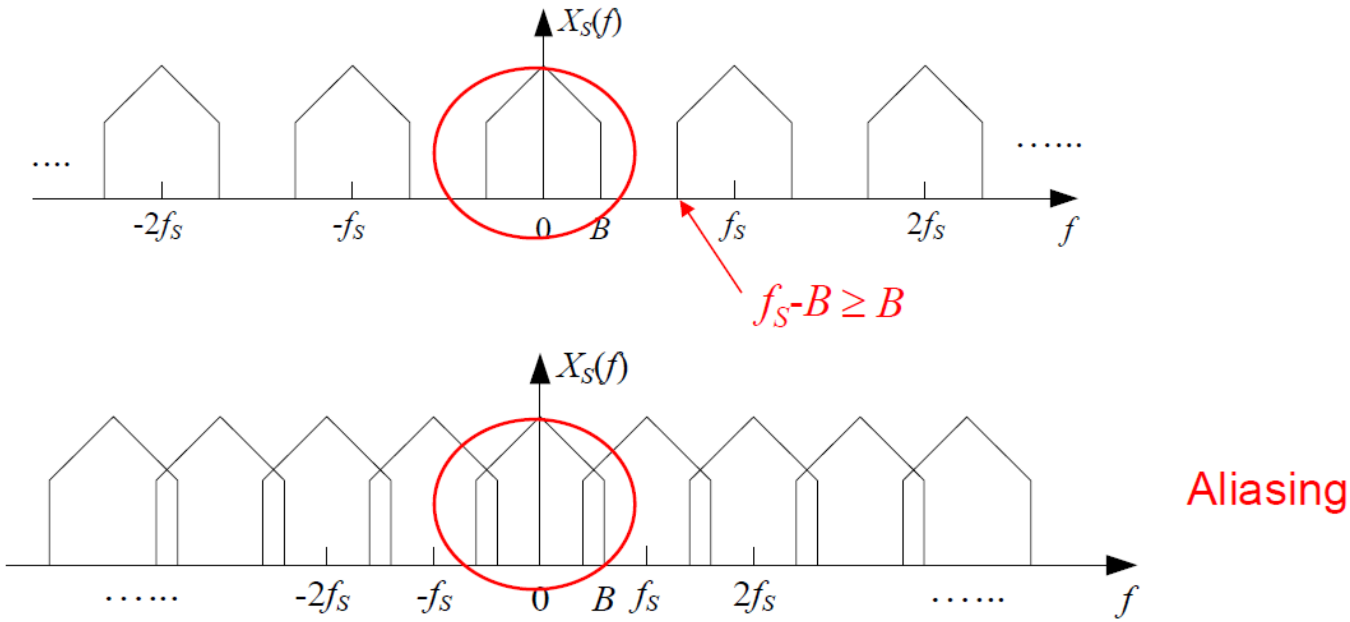
$$f_S \sum_{n=-\infty}^{\infty} \delta(f - nf_S)$$



- How to recover the original signal $x(t)$?

Lowpass filtering

Aliasing



Distortion will be incurred if there's aliasing.

- What is the requirement on the sampling rate to avoid aliasing?

$$f_S \geq 2B$$

Nyquist Sampling Theorem

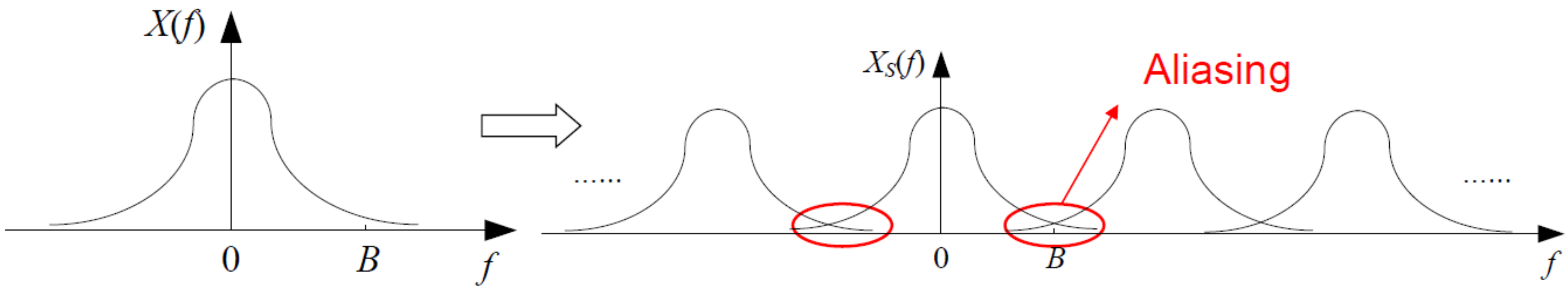
Nyquist Sampling Criteria : A baseband signal with limited bandwidth B can be uniquely determined by its values at uniformly spaced intervals if and only if the sampling rate $f_S \geq 2B$.

$f_S = 2B$ is called the Nyquist sampling rate, representing the minimum requirement without introducing distortion.

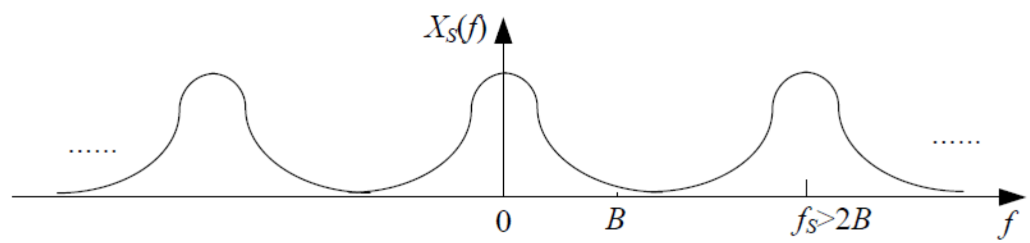
- A signal with a larger bandwidth requires a higher sampling rate.

What about a signal which is not strictly bandlimited to B?

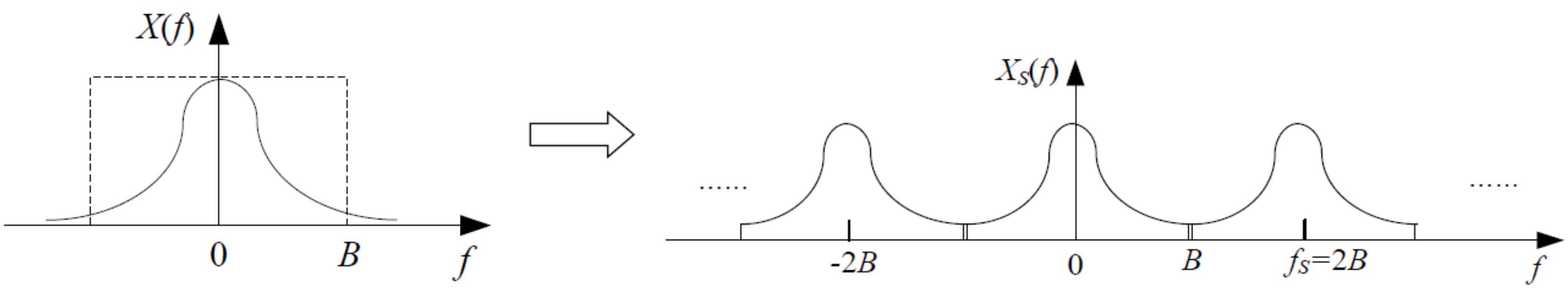
Practical Considerations on Sampling



- Use a sampling rate $f_S > 2B$

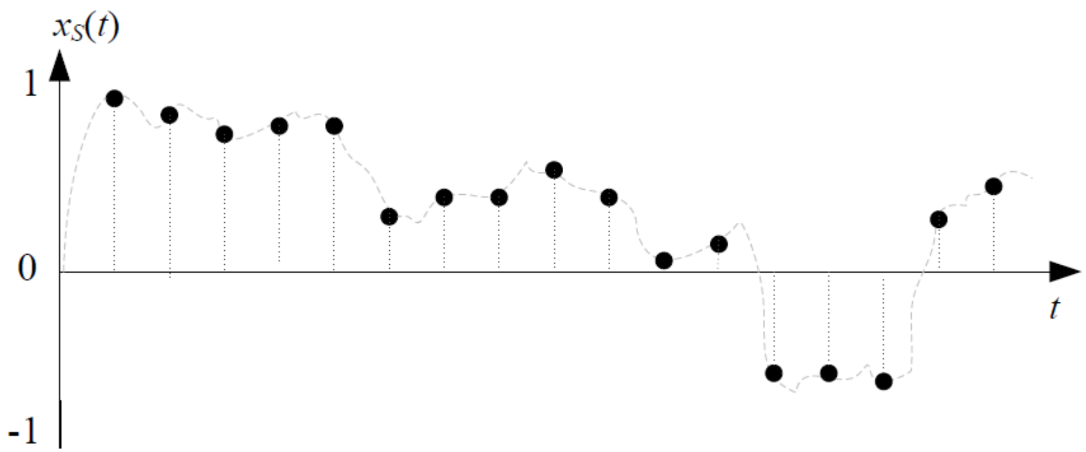


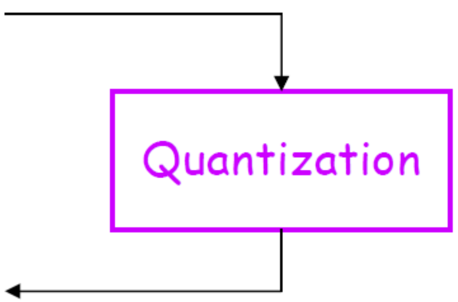
- Pre-filtering



Quantization

Quantization

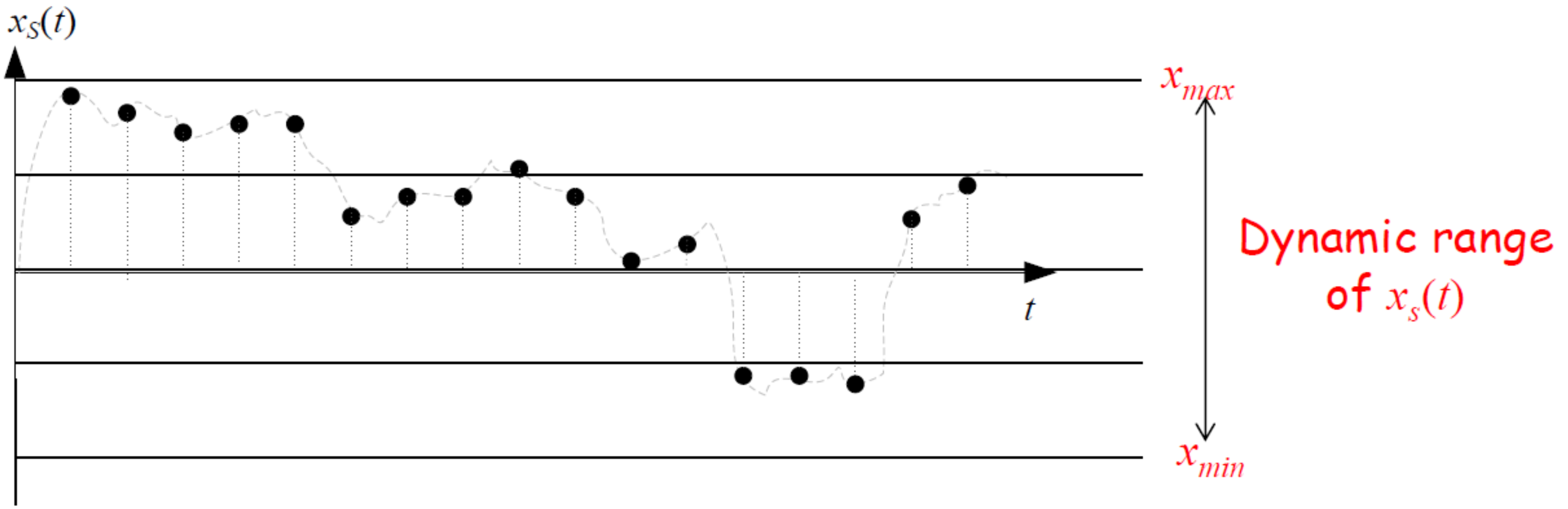




11 11 11 11 11 10 10 10 11 10 10 10 00 00 00 10 10…

- Quantization: to transform a discrete-time signal into a series of binary bits.
  - Step 1: Use a finite number of values to represent the amplitude;
  - Step 2: Assign binary codes to different quantization levels.

Uniform Quantization

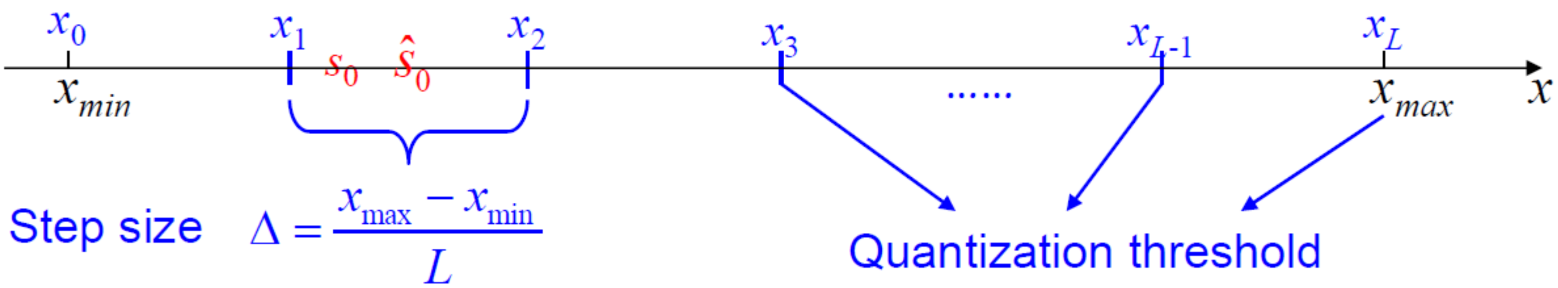


- Suppose the sampled signal amplitude x varies from x_{min} to x_{max}.
- Suppose the number of quantization levels $L = 2^b$ (use b bits to represent a sampling symbol)

Uniform Quantization: the dynamic range is divided into L equal-width quantization regions.

Uniform Quantization

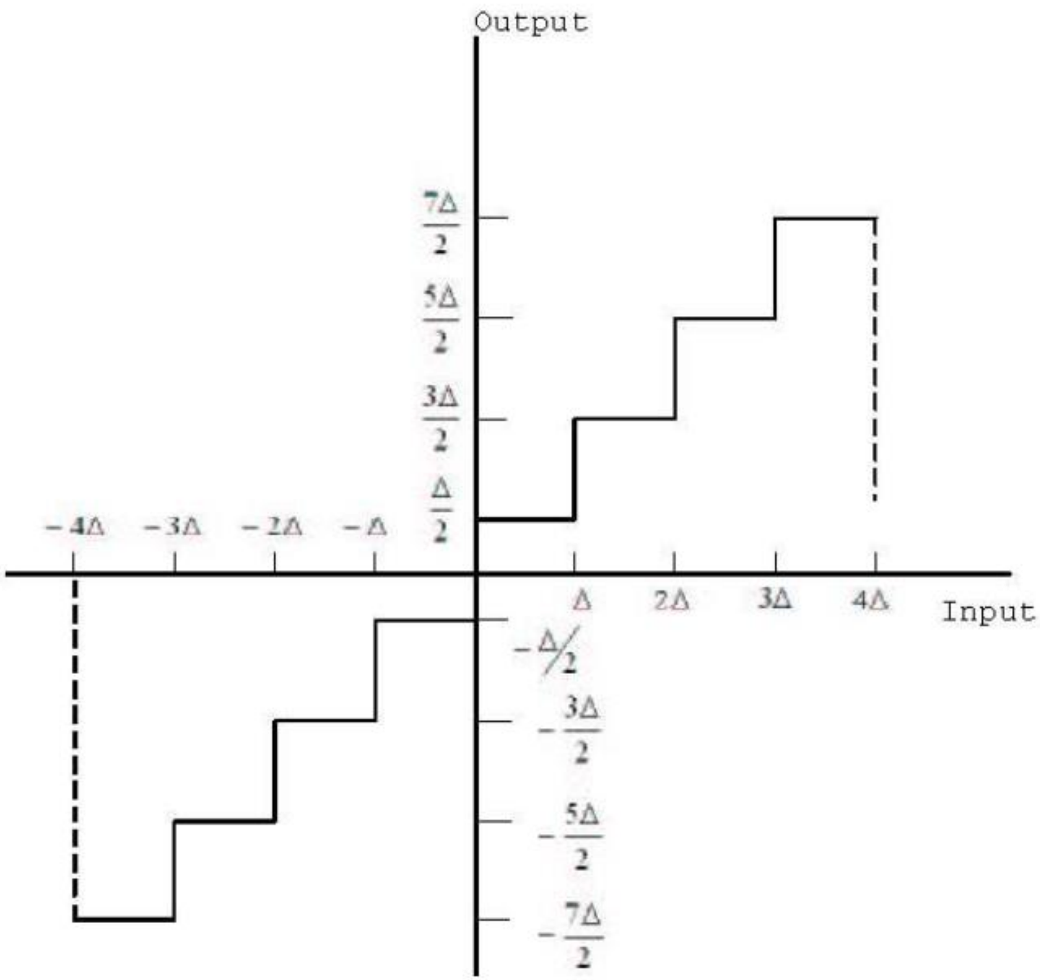
- Uniform Quantization: the dynamic range is divided into L equal-width quantization regions.



Let $\hat{s}_0$ represent the quantized value of the input value s_0.

Midriser: $\hat{s}_0 = x_i + \Delta/2$ if $x_i < s_0 \leq x_{i+1}$.

Transfer Function of Midriser



Example 1: Midriser

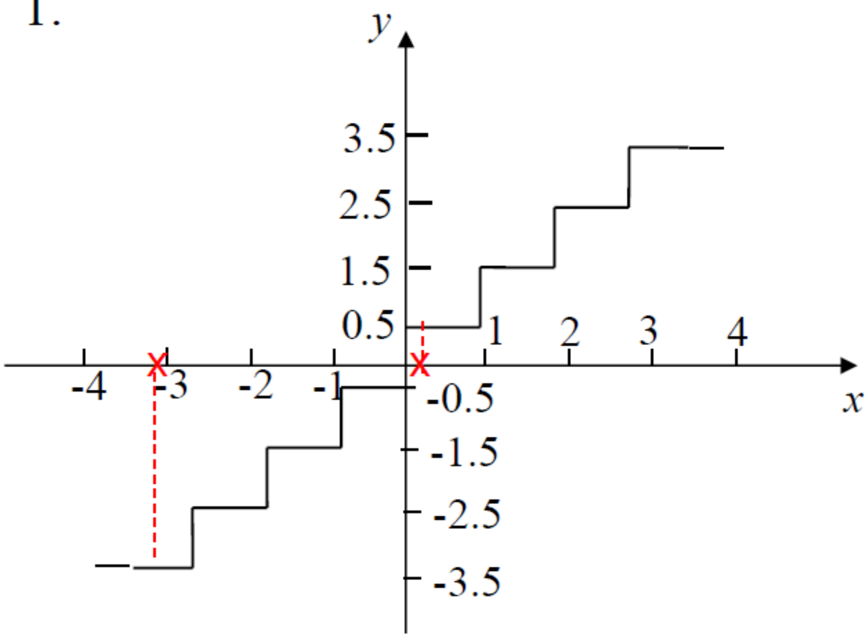
Suppose that a signal $x(t)$ with dynamic range (-4,4) volts is applied to a 3-bit midriser.

1. Plot the transfer function;
2. Determine the quantized values of 0.15V and -3.1V.

Solution

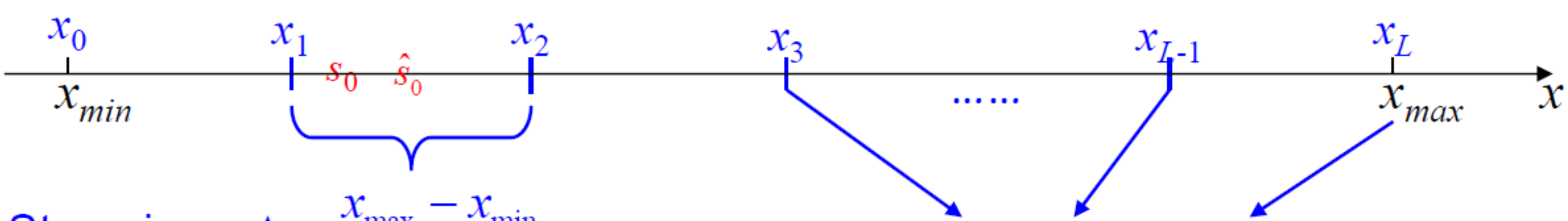
The quantization step size at the output is $\Delta = (4-(-4))/2^3 = 1$

1.



2. Quantized values of 0.15V and -3.1V are 0.5V and -3.5V, respectively.

Quantization Error



Step size $\Delta = \dfrac{x_{\max} - x_{\min}}{L}$

Quantization threshold

- What is the maximal difference between the input s_0 and its quantized value $\hat{s}_0$?

$$\Delta / 2$$

Let $e = \hat{s}_0 - s_0$. $\quad -\Delta/2 \le e \le \Delta/2$

Quantization Error (noise)

e can be regarded as a uniformly distributed random variable with pdf

$$f_e(x) = \begin{cases} 1/\Delta & -\Delta/2 \le x \le \Delta/2 \\ 0 & otherwise \end{cases} \quad \Rightarrow \quad \mu_e = 0, \quad \sigma_e^2 = \Delta^2/12$$

Signal-to-Quantization-Noise-Ratio (SQNR)

- Quantization error power = $\sigma_e^2 = \Delta^2/12 = \dfrac{(x_{\max} - x_{\min})^2}{12L^2} = \dfrac{(x_{\max} - x_{\min})^2}{12 \cdot 2^{2b}}$

  Step size $\Delta = \dfrac{x_{\max} - x_{\min}}{L}$

  Number of quantization levels $L = 2^b$

  ✓ A larger b leads to a lower quantization error power.
  (We can use more bits to improve the quantization precision.)

- SQNR:

$$SQNR = P_x / \sigma_e^2$$

P_x and σ_e^2 are the power of the signal and the quantization error, respectively.

Example 2: SQNR

A sinusoidal signal with peak amplitude A_m is applied to a uniform quantizer with a dynamic range of $(-A_m, A_m)$. Determine the output SQNR.

We have $P_x = A_m^2/2$ and $\sigma_e^2 = \Delta^2/12 = \dfrac{(x_{\max} - x_{\min})^2}{12 \cdot 2^{2b}} = \dfrac{(2A_m)^2}{12 \cdot 2^{2b}}$

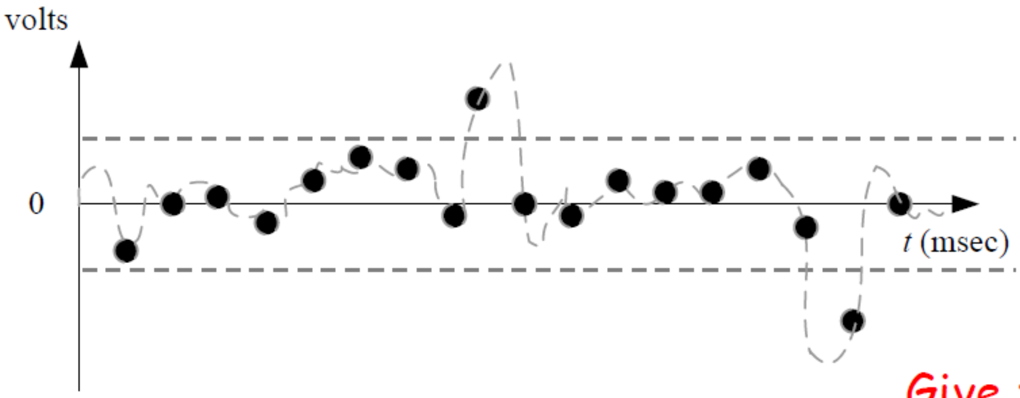
Therefore, $SQNR = \dfrac{P_x}{\sigma_e^2} = 1.5 \cdot 2^{2b}$

Write it in the form of decibels:

$$10\log_{10}(SQNR) = 10\log_{10}1.5 + 2b \cdot 10\log_{10}2 \approx 1.8 + 6b \text{ (dB)}$$

Non-Uniform Quantization

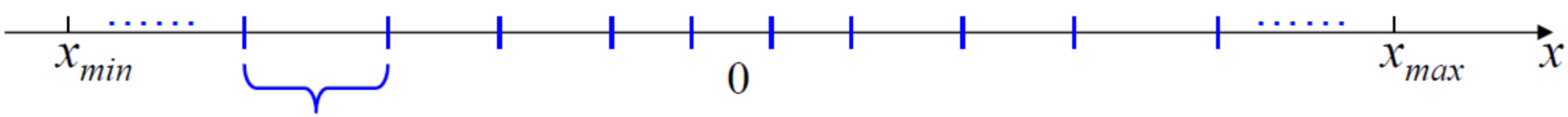
Human speech:



- Small samples have a larger percentage;
- Small samples are more susceptive to noise.

Give the small samples more priority! (more bits!)

Use small intervals for signals with small amplitudes and large intervals for signals with large amplitudes:



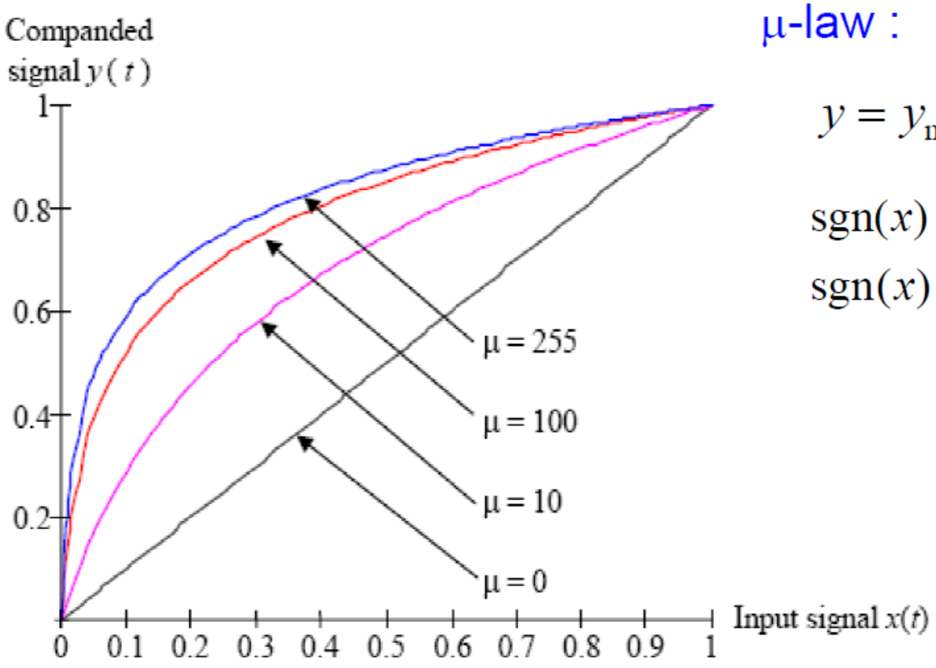
Step size Δ increases with the signal magnitude.

Non-Uniform Quantization

How to perform non-uniform quantization?

- non-linear compression on the analog signal;
- uniform quantization.

Compress the original signal such that its amplitude roughly follows a uniform distribution!



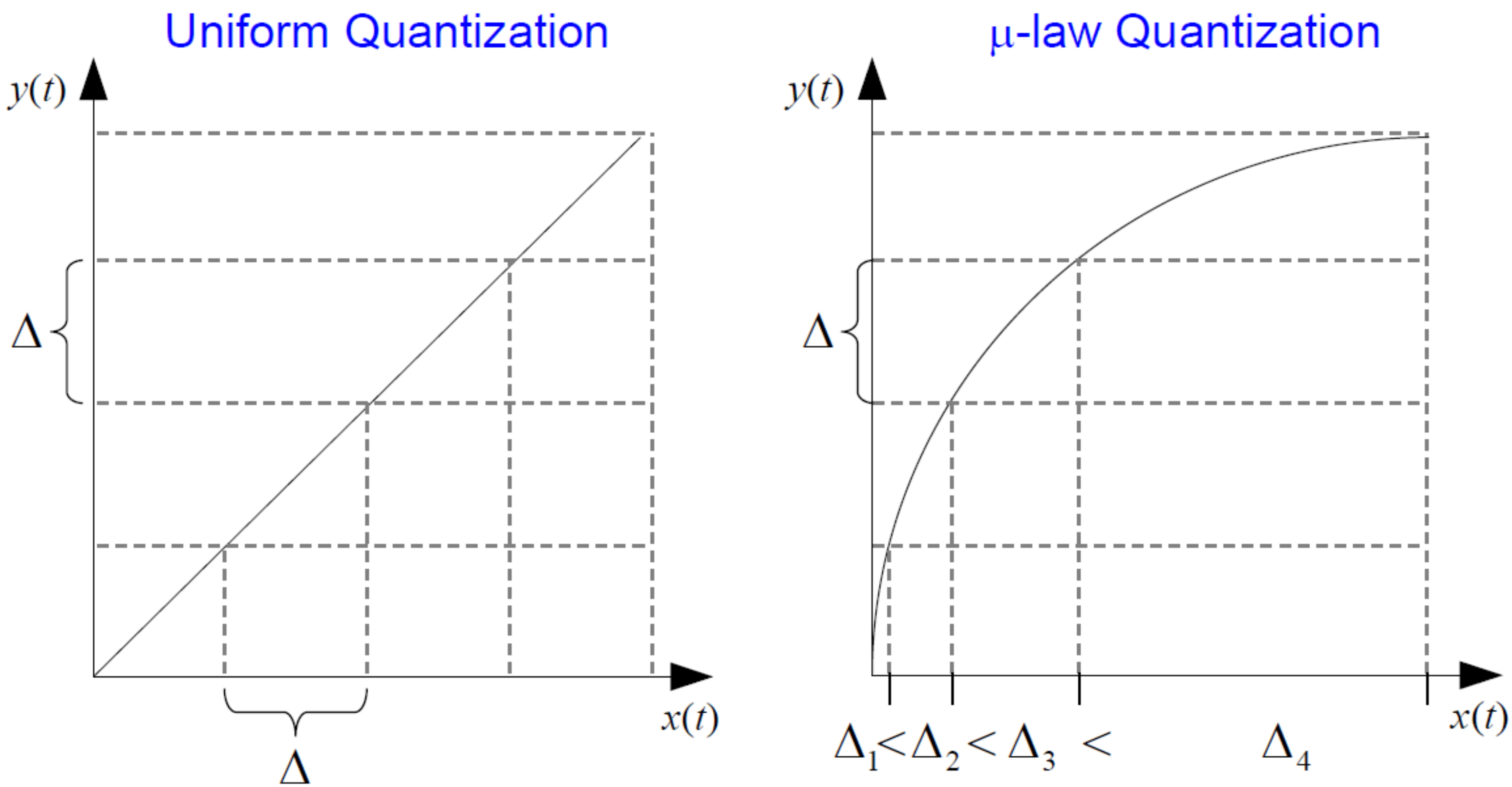
μ-law :

$$y = y_{\max} \frac{\ln\left[1 + \mu(|x| / x_{\max})\right]}{\ln(1+\mu)} \operatorname{sgn}(x)$$

$$\operatorname{sgn}(x) = 1 \quad \text{if} \quad x > 0$$

$$\operatorname{sgn}(x) = -1 \quad \text{if} \quad x < 0$$

SQNR of μ-Law Quantization



- For weak signals (with low power), μ-law has better SQNR as $\Delta_{\mu-law} < \Delta_{uniform}$
- For strong signals (with large power), μ-law has lower SQNR as $\Delta_{\mu-law} > \Delta_{uniform}$

Example 3: μ-Law Quantizer

Suppose that a signal $x(t)$ with dynamic range (-4,4) volts is applied to a 3-bit μ-law quantizer. Determine the quantization thresholds at both input and output sides.

For simplicity, we assume that $y_{\max}=x_{\max}$ and $y_{\min}=x_{\min}$, and $\mu=255$.

Solution

The quantization step size at the output is $\Delta = (4-(-4))/2^3 = 1$

The quantization thresholds at the output are then given by

$$y_{th} = [-4, -3, -2, -1, 0, 1, 2, 3, 4]$$

According to $y = y_{\max} \dfrac{\ln\left[1+\mu(|x|/x_{\max})\right]}{\ln(1+\mu)} \operatorname{sgn}(x)$

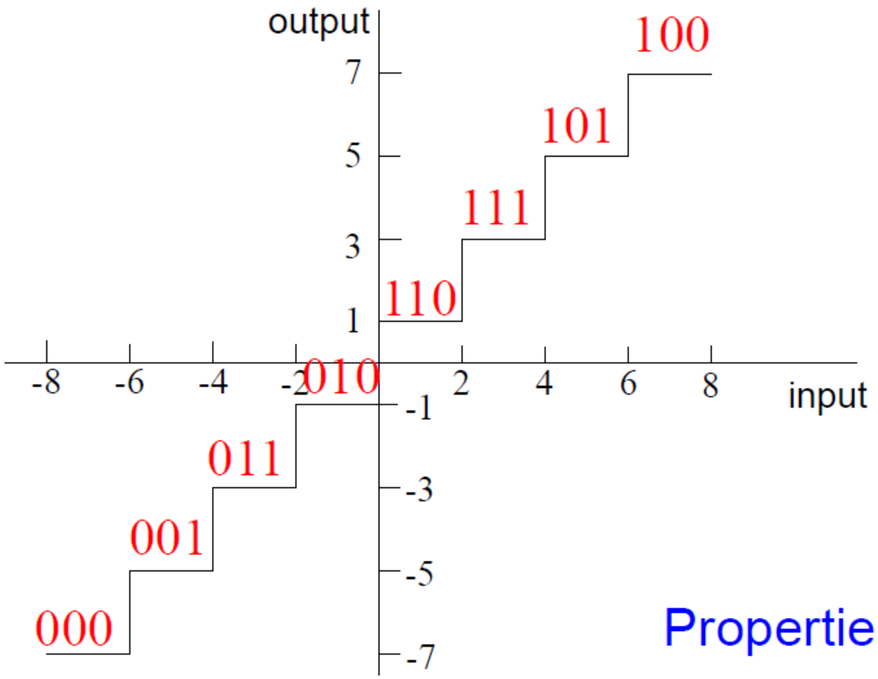
We have $x = (x_{\max}/\mu)\left[(1+\mu)^{|y|/y_{\max}} - 1\right] \cdot \operatorname{sgn}(y)$

Finally, according to $x_{th} = (x_{\max}/\mu)\left[(1+\mu)^{|y_{th}|/x_{\max}} - 1\right] \operatorname{sgn}(y_{th})$

The quantization thresholds at the input are then given by

$$x_{th} = [-4, -0.9882, -0.2353, -0.0471, 0, 0.0471, 0.2353, 0.9882, 4]$$

Code Assignment



Gray Code

000→000   001→001
010→011   011→010 ……

Properties:

1. The first bit is the sign bit;
2. Adjacent words differ only by one bit;
3. Except for the sign bit, the codewords are mirror symmetrical about the horizontal axis.

Gray Code Generation

0
1

(a)

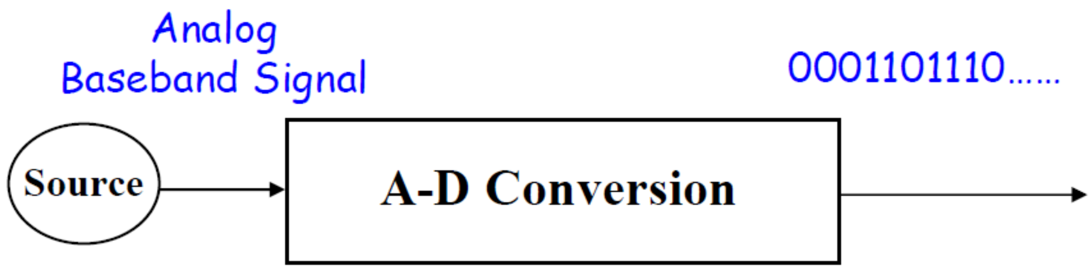
0	0	0 0
1	1	0 1
	1	1 1
	0	1 0

(b)

0 0	0 0	0 0 0
0 1	0 1	0 0 1
1 1	1 1	0 1 1
1 0	1 0	0 1 0
	1 0	1 1 0
	1 1	1 1 1
	0 1	1 0 1
	0 0	1 0 0

(c)

A-D and D-A



- Sampling
- Quantization

- Decoding (0001101110… → 0.1 -0.3 0.2 …)
- Generate a pulse the amplitude of which is the quantized value (repeat)
- Lowpass filtering