Basic Tools

- Random Process
  - A random process X(t) is a collection of random variables at different sample time $\{X(t_1),....,X(t_n)\}$
  - To completely specify a general random process X(t), we need joint pdf.
  - Yet, for Gaussian random process, we can completely specify it by the
    - 1st order moment: $m_X(t) = E[X(t)]$
    - 2nd order moment: $R_X(t_1,t_2) = E\left[X(t_1)X^*(t_2)\right]$
  - For stationary zero mean Gaussian random processs, we only need 2nd order moment
    - Autocorrelation $R_X(t_1,t_2) = E\left[X(t_1)X^*(t_2)\right] = R_X(|t_1 - t_2|)$
    - OR Equivalently, power spectral density $R_x(\tau) \xrightarrow{\mathcal{F}} S_X(f)$

- Random Process
  - White Gaussian random process
    - Power spectral density $S_X(f)$ is a constant.
  - LTI filtering of stationary gaussian random process
    - Input stationary Gaussian r.p. ➔ output stationary Gaussian r.p.
    - Output psd: $S_Y(f) = S_X(f)\left|H(f)\right|^2$

$$X(t) \longrightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ h(t)\text{, or}\\ \text{Transfer function}\\ H(f)\end{array}} \longrightarrow Y(t)$$

$R_X(\tau) \leftrightarrow S_X(f)$

$R_Y(\tau) \leftrightarrow S_Y(f)$

(m2) Binary Modulation

- Considered a special binary modulator s(t) = {+A, -A} with symbol duration T.
  - Integrator Detection + 0 threshold
    - Detection Scheme $V = \int_0^T \left[s(t) + n(t) \right] dt \quad = \begin{cases} AT + N & \text{if "1" is sent} \\ -AT + N & \text{if "0" is sent} \end{cases}$
    - Error probability analysis $P_e = Q\left[\sqrt{\frac{A^2T^2}{\sigma^2}} \right] = Q\left[\sqrt{\frac{2E_b}{N_0}} \right]$
- For general binary modulator with s(t) = {s_0(t), s_1(t)} with symbol duration T.
  - Optimal Detector
    - Integrator → replaced by an LTI filter {h(t)} ➔ MF or correlator {h(t) = s1(T-t) – s0(T-t)}
    - 0 Threshold → general threshold V_T $V_T = (s_{o1} + s_{o0})/2$
    - Error probability analysis

$$Pe = Q\left[\sqrt{\frac{\left(s_{o1} - s_{o0}\right)^2}{4\sigma^2}} \right] = Q\left[\sqrt{\frac{E_0 + E_1 - 2\rho_{12}\sqrt{E_1E_2}}{4\sigma^2}} \right]$$

(m3) Signal Space

Signal Space

- Same as vector space by considering "signals" (with finite duration) as "vectors"
- Dot product for "signals" are defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \langle x(t), y(t) \rangle \triangleq \int_0^{T_s} x(t) y^*(t) dt$
- Similar to the convectional vector space, to define a "signal space", we need to first define the "axis" or "basis functions" ("axis-equivalent" for signal space).
- A signal space is D-dimensional if it has D has D orthonormal basis $\{\phi_1(t), \phi_2(t), .., \phi_D(t)\}$
  - "Ortho" means $\langle \varphi_j(t), \varphi_k(t) \rangle = \int_{-\infty}^{\infty} \varphi_j(t) \varphi_k(t) dt = 0 \quad \text{for } j \neq k$
  - "Normal" means "length" = 1:

$$\left\| \varphi_j(t) \right\|^2 = \langle \varphi_j(t), \varphi_j(t) \rangle = \int_{-\infty}^{\infty} \left| \varphi_j(t) \right|^2 dt = 1$$

(m3) Signal Space

- Procedure for representing M given "signals" by M "vectors"
  - Step 1: Given a set of M signals, $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ define a D-dim signal space with basis $\{\phi_1(t), \phi_2(t), .., \phi_D(t)\}$.
    - <Follow GS procedure>
  - Step 2: Find out the coordinates of each signals by:

$$s_i(t) \to \vec{s}_i = (s_{i,1}, s_{i,2}, .., s_{i,D}) \qquad s_{ij} = \left\langle s_i(t), \phi_j(t) \right\rangle = \int_0^{T_s} s_i(t) \phi_j^*(t) dt$$

- **Important Connection between time domain ←→ Geometric domain
  - Via dot product
  - E.g. "Length of vector in a signal space" = "energy of signal"

(m4) M-ary Modulation

- Generalize BPSK to other types
  - Represent signals in basis functions
  - Signal dimension
  - Signals as points on a constellation
  - Orthogonal signals

(m4) M-ary Modulation

- Modulation symbols
  - signal pulses of finite duration T_s .
  - For example, binary modulator takes in 1 bit {0,1} and output one modulation symbol from the set $\{s_1(t), s_2(t)\}$
- Baud Rate (Symbol Rate)

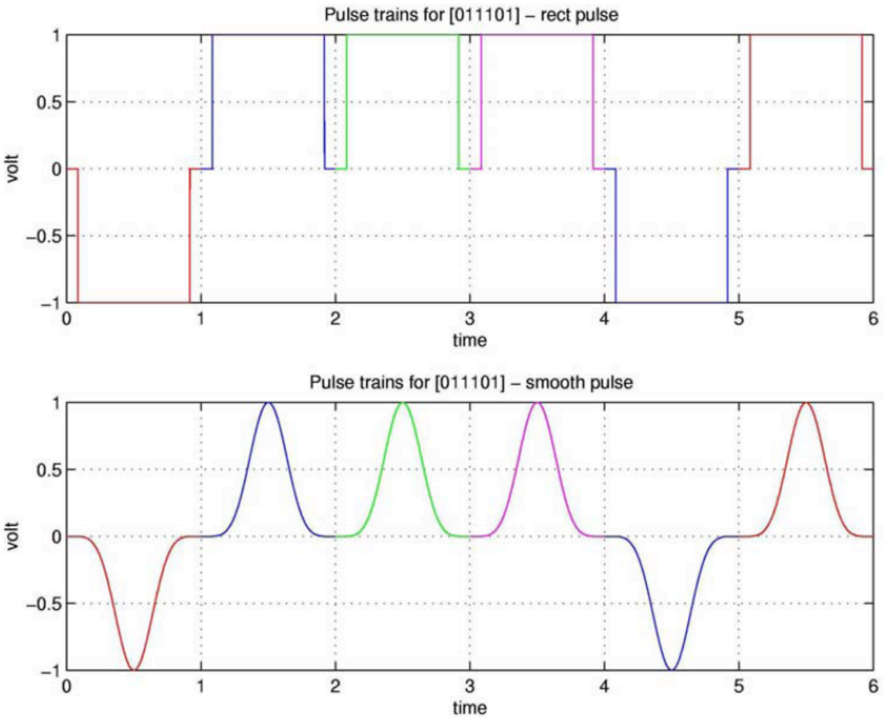
$$R_s = \frac{1}{T_s} symbol/s$$

- Bit Rate (bps) =

$$R_b = \left(\frac{1}{T_s}\right) \times \left(\frac{bits}{symbol}\right) = \left(\frac{1}{T_s}\right) \times \left(\log_2 M\right) b/s$$

- Transmission BW

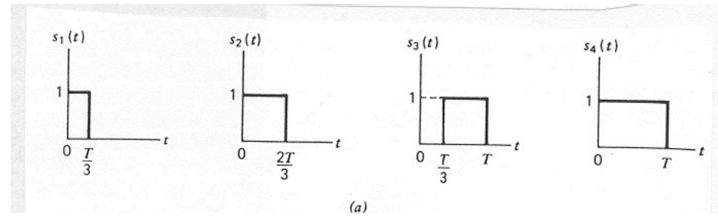
$$W_{tx} = BW\left\{\sum_{m=1}^{M} \left|S_m(f)\right|^2\right\} \approx \left(\frac{1}{T_s}\right)(1+\alpha)$$

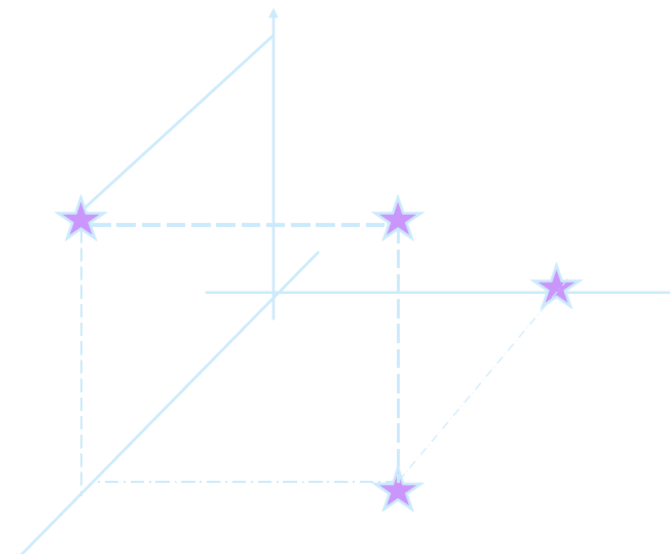


- M-ary Modulator
  - One out of M symbols $\{s_1(t), s_2(t), \ldots, s_M(t)\}$

(m4) M-ary Modulation

– Constellation
  - The set of M modulation signals could be represented geometrically as points in the signal space.
  - Hence, M-ary modulator could be represented by M-points in a D-dim signal space. This is called the constellation of the modulator.





– Average Energy out of modulator

(equivalent to average c… re of all the c… points from origin)

$$E = \frac{1}{M}\sum_{i=1}^{M} |\vec{s}_i|^2$$

(m4) Optimal Detection

- Received signal through AWGN channel
  - Time domain: $$r(t) = \overbrace{s(t)}^{\substack{\text{transmitted}\\ \text{signal}}} + \overbrace{n(t)}^{\substack{\text{random}\\ \text{noise}}}$$
  - Geometric domain:
    - Let the signal s(t) is contained in a D-dim signal space
    - The projected receive vector is given by:

$$\underbrace{\vec{z}}_{\substack{\text{projected}\\ \text{receive}\\ \text{vector}}} = \underbrace{\vec{s}}_{\substack{\text{transmitted}\\ \text{vector}}} + \underbrace{\vec{n}}_{\substack{\text{projected}\\ \text{noise vector}}}$$

  - Task of Demodulator is
    - Given the received point $(\vec{\mathbf{z}})$, decode or find out the transmitted message m from one of the M possible messages $\{m_1, m_2, \ldots, m_M\}$

(m4) Optimal Detection

- Two important Messages
  - Message 1: For AWGN channel and equiprobable messages, optimal detection is "min distance detection"

$$\text{Choose } \vec{s} \in \left\{\vec{s}_1, \ldots, \vec{s}_M\right\} \text{ such that } \left\|\vec{z}\text{-}\vec{s}\right\| \text{ is minimized}$$

$$\text{OR (using mathematical language), } \hat{\vec{s}} = \underbrace{\arg\min}_{\vec{s}\in\left\{\vec{s}_1,\ldots,\vec{s}_M\right\}} \left\|\vec{z}\text{-}\vec{s}\right\|$$

  - Why??
    - For general channel, equiprobable messages, the optimal detection rule (optimal = minimize SER) is by "maximum likelihood"

$$\text{Choose } \vec{s} \in \left\{\vec{s}_1, \ldots, \vec{s}_M\right\} \text{ such that } \underbrace{f\left(\vec{z}\middle|\vec{s}\right)}_{\text{likelihood function}} \text{ is maximized OR (using mathematical language), } \hat{\vec{s}} = \underbrace{\arg\max}_{\vec{s}\in\left\{\vec{s}_1,\ldots,\vec{s}_M\right\}} f\left(\vec{z}\middle|\vec{s}\right)$$

    - For AWGN, it turns out that the "likelihood function" is of the form

$$f\left(\vec{z}\middle|\vec{s}\right) \sim K \exp\left(-\frac{\left\|\vec{z}-\vec{s}\right\|^2}{2\sigma_n^2}\right) \Rightarrow \text{ maximizing likelihood function is the same as minimizing distance}$$

(m4) Optimal Detection

Two Important messages

- Message 2:
  - For AWGN channel, equi-probable messages, the symbol error probability is upper bounded by:

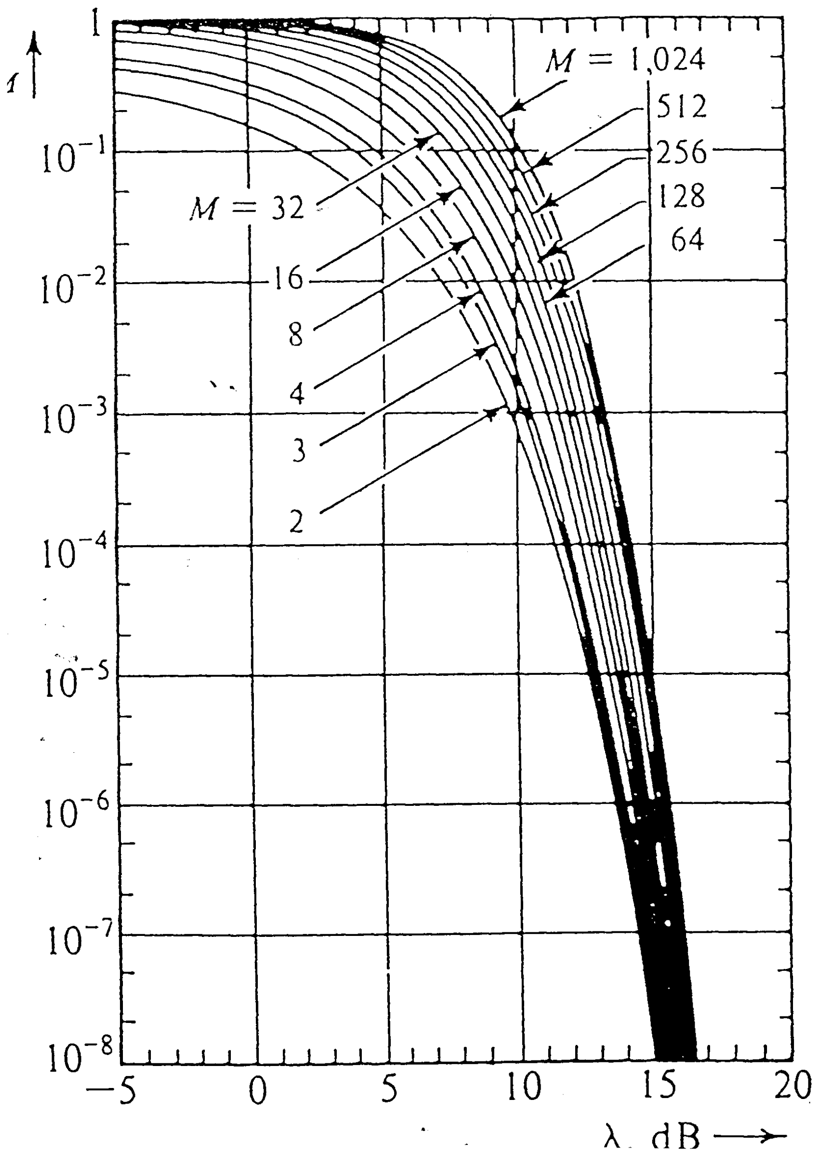
$$P_e(i) = \Pr\left[\text{detection error} \mid \mathrm{s}_\mathrm{i} \text{ is transmitted}\right] \le \sum_{j \ne i} P_j(i) = \sum_{j \ne i} Q\left(\sqrt{\frac{\left\|\mathbf{s}_j - \mathbf{s}_i\right\|^2}{2N_0}}\right)$$

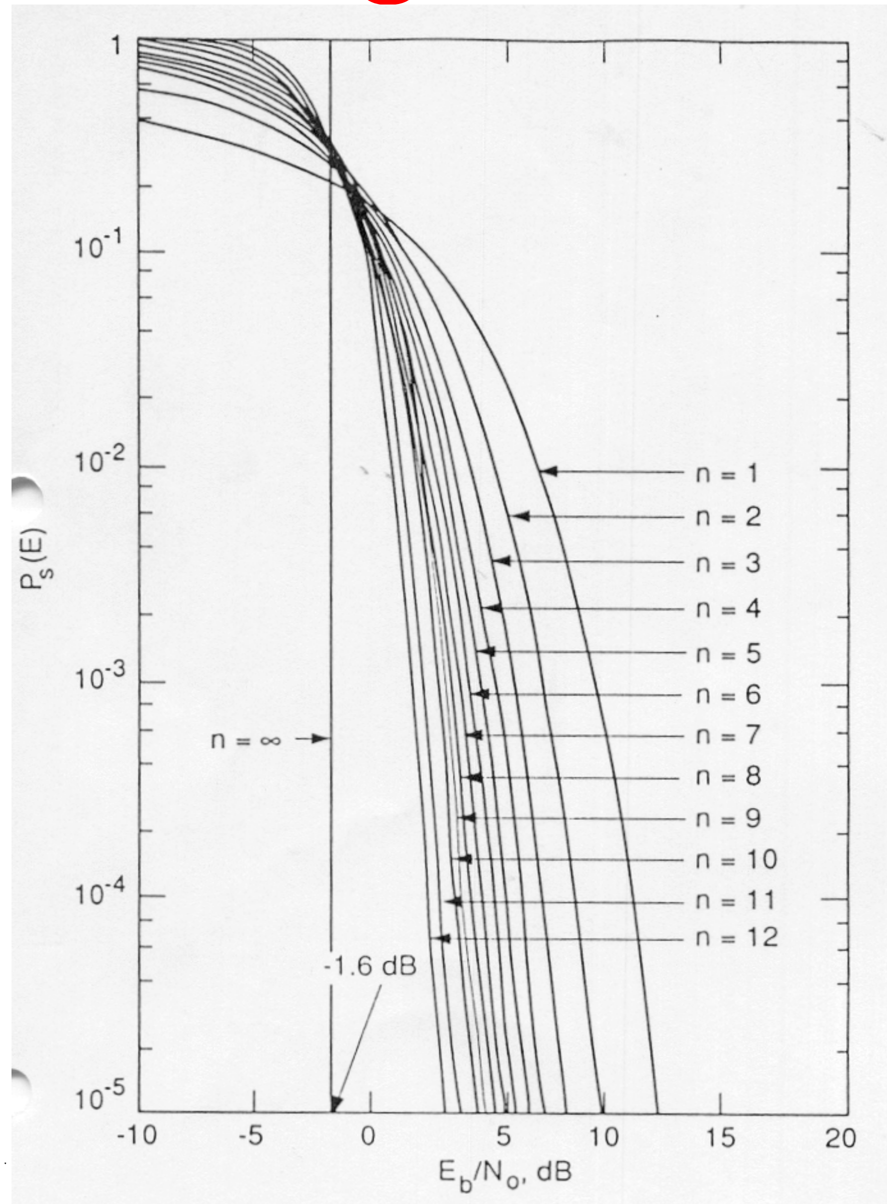
  - Hence, the error probability is limited by the smallest distance between ANY TWO constellation points
  - If for a given constellation, there are K points closest to the transmitted point (i), the average symbol error probability is:

$$P_e(i) = \Pr\left[\text{detection error} \mid \mathrm{s}_\mathrm{i} \text{ is transmitted}\right] \le \sum_{j \ne i} Q\left(\sqrt{\frac{\left\|\mathbf{s}_j - \mathbf{s}_i\right\|^2}{2N_0}}\right) \le KQ\left(\sqrt{\frac{\min_{j \ne i}\left\|\mathbf{s}_j - \mathbf{s}_i\right\|^2}{2N_0}}\right)$$

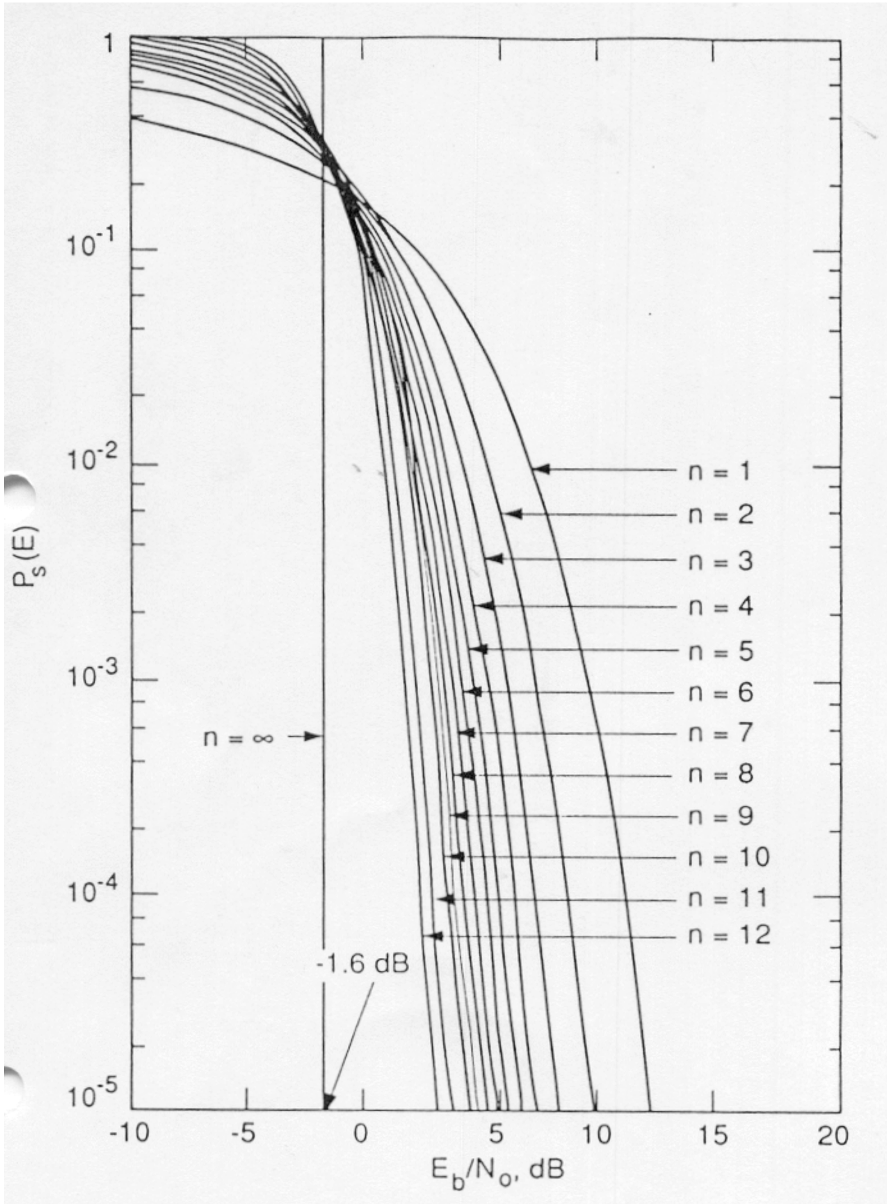
    - E.g. For MPSK, there are always 2 closest points about each transmitted point → K=2 for MPSK
    - E.g. For MFSK, there are always M-1 closest points next to each transmitted points→ K=M-1 for MFSK

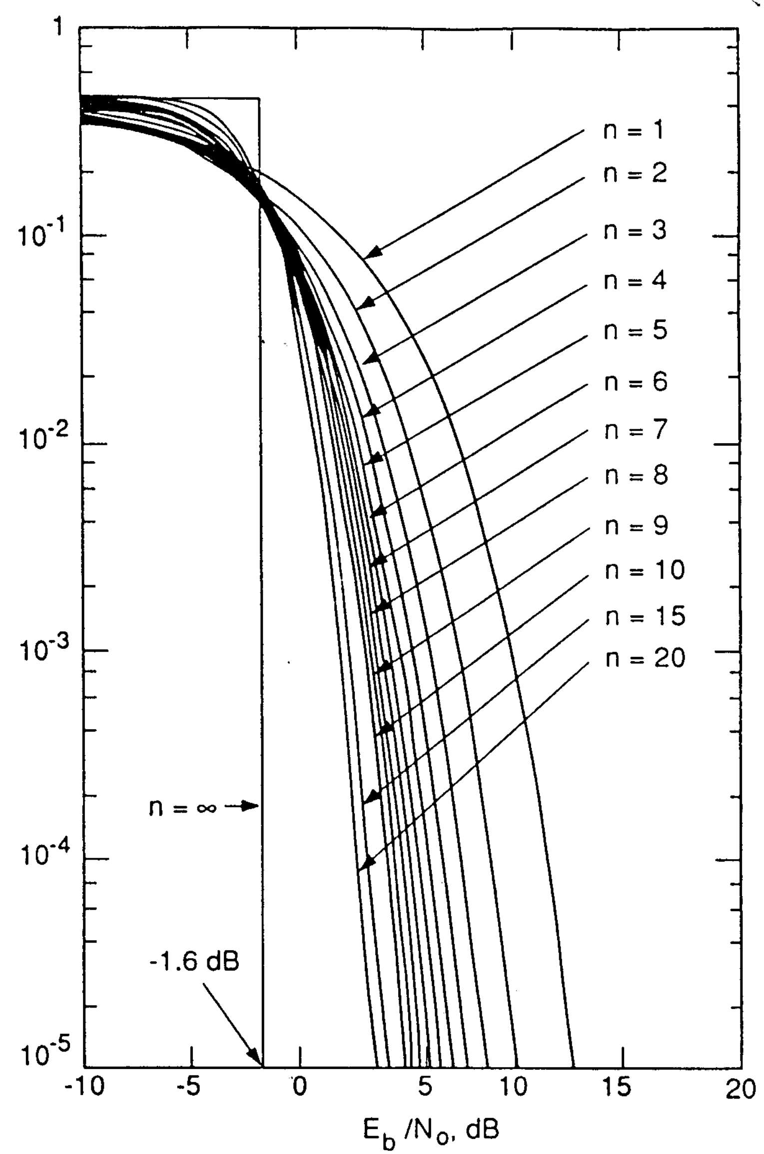
(m6) Orthogonal Signals





(m6) Orthogonal Signals





- Union Bound
  - Approximation to exact error probabilities
  - Based on $P(\bigcup_i A_i) \le \sum_i P(A_i)$
  - Need to know pairwise or binary probabilities of error between symbols

$$P_{eM} \le \frac{1}{M} \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k \ne j}}^{M} P(b_j \le b_k / m_j)$$

Binary or pair comparisons

$$= \frac{1}{M} \sum_{j=1}^{M} \sum_{\substack{k=1 \\ k \ne j}}^{M} Q\left[\sqrt{\frac{d_{kj}^2}{2N_0}}\right]$$

(m6-m7)Basic Signal Types

- Power Efficient Schemes
  - Orthogonal
  - Bi-orthogonal
  - Simplex
- Bandwidth Efficient Schemes
  - MPSK
  - MASK
  - MQAM

(m7) Tradeoff in Digital Communication Systems

- Performance Measure
  - Bit Rate (b/s)

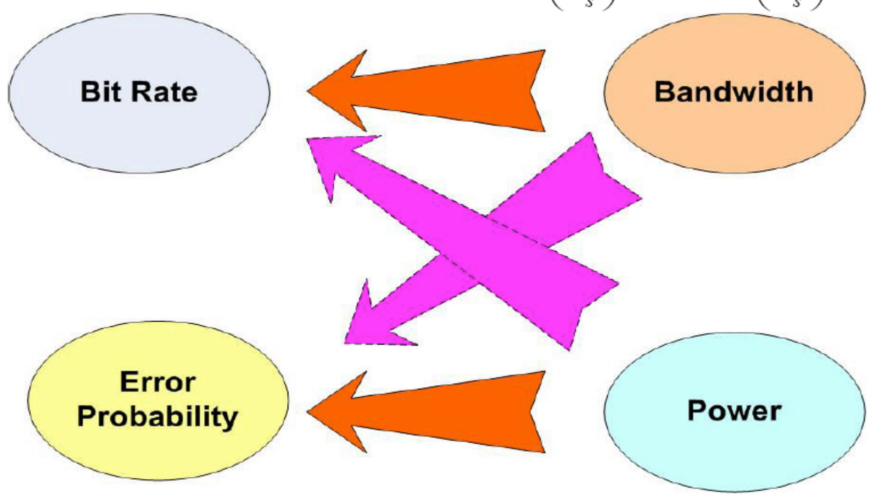
    $$R_b = \log_2 M \times baudrate(symbolrate) = (\log_2 M)\left(\frac{1}{T_s}\right)$$
  - BER (Bit Error Rate)

    $$BER \approx Q\left(\sqrt{\frac{\left\|\mathbf{s}_i - \mathbf{s}_j\right\|^2}{2N_0}}\right) = Q\left(\sqrt{\frac{\int_0^{T_s}\left(s_i(t) - s_j(t)\right)^2 dt}{2N_0}}\right)$$
- Resource Measure
  - Tx Power

    $$P_{tx} = \frac{1}{MT_s}\sum_{m=1}^{M}\left\|s_m\right\|^2 = \frac{1}{MT_s}\sum_{m=1}^{M}\int_0^{T_s} s_m^2(t)dt$$
  - Tx BW

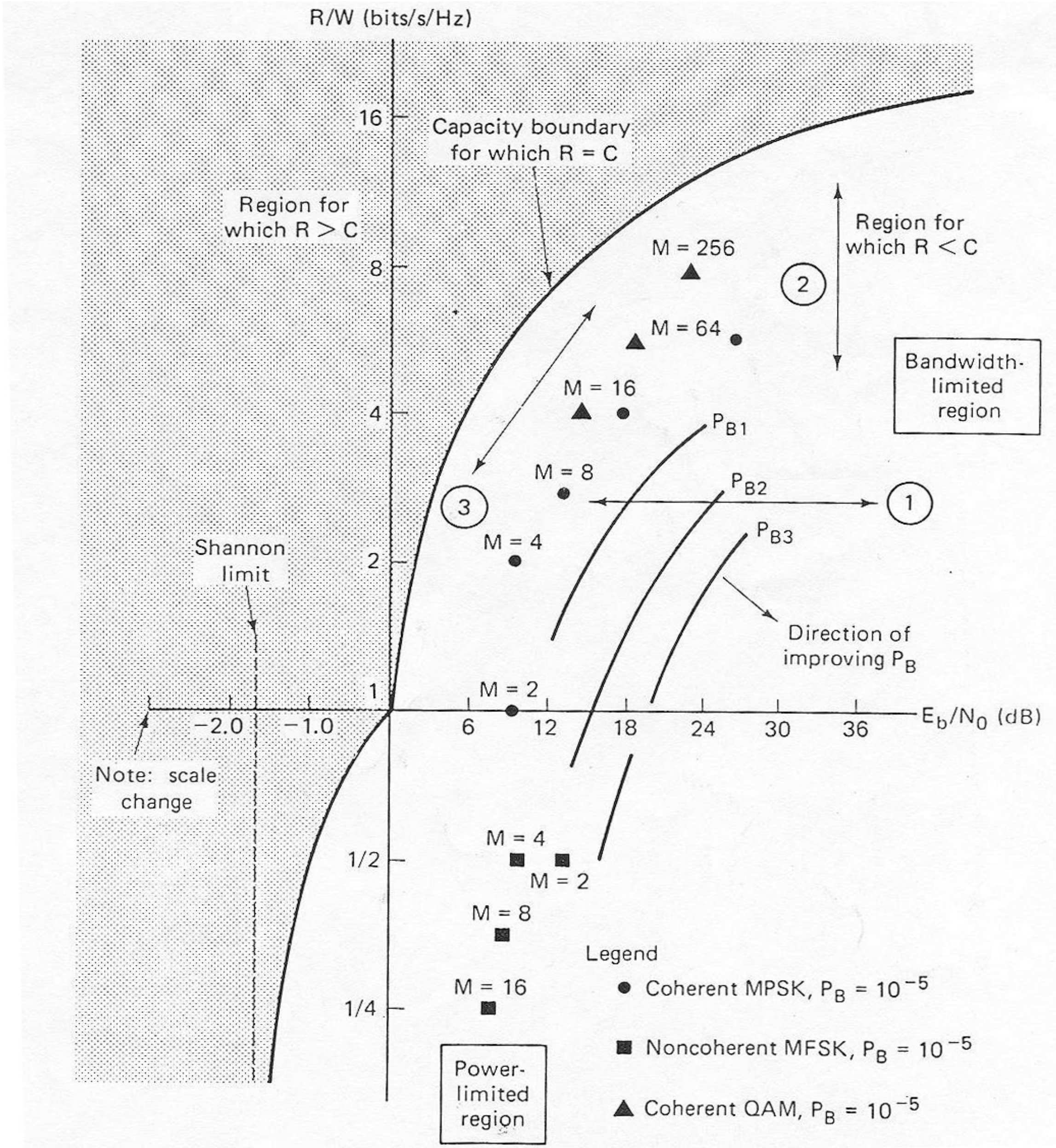
    $$W_{BP} = \left(\frac{1}{T_s}\right)(1+\alpha) \approx \left(\frac{1}{T_s}\right)$$

Tradeoff



(m7) Summary of M-ary Modulation Schemes

	M-FSK	M-PSK	M-QAM
Bit Rate	$R_b = \frac{1}{T_s}\log_2(M)$	$R_b = \frac{1}{T_s}\log_2(M)$	$R_b = \frac{1}{T_s}\log_2(M)$
BW (Bandpass)	$BW = \frac{M+1}{2T_s}$	$BW = \frac{1}{T_s}$	$BW = \frac{1}{T_s}$
Average Transmit Power	$\frac{E_s}{T_s}$	$\frac{E_s}{T_s}$	$\frac{4\alpha^2}{\sqrt{M}T_s}\sum_{i=1}^{\log_2 M/2}(2^i-1)^2$
Average Symbol Error Probability (SER)	$P_e \leq (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	$P_e \leq 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin(\pi/M)\right)$	$P_M \approx 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\frac{3}{M-1}\frac{E_s}{N_0}\right)$
Remarks	• Orthogonal Signaling Schemes (Equi-energy points & Mutually orthogonal signals) • Enhance Energy Efficiency at the expense of extra BW	• Equi-energy constellation (information carried by phase values only) • Dimension of the signal set is always 2 (I-Q modulator) • Enhance spectral efficiency at the expense of extra power	• Points are NOT equi-energy • Information is carried by both amplitude and phase • Enhance spectral efficiency at the expense of extra power • Better than M-PSK for M>4.


R/W (bits/s/Hz)
16
8
4
2
1
1/2
1/4
Capacity boundary for which R = C
Region for which R > C
Region for which R < C
M = 256
M = 64
M = 16
M = 8
M = 4
M = 2
M = 4
M = 2
M = 8
M = 16
2
3
1
Bandwidth-limited region
P_{B1}
P_{B2}
P_{B3}
Shannon limit
Direction of improving P_B
−2.0
−1.0
6
12
18
24
30
36
E_b/N_0 (dB)
Note: scale change
Legend
Coherent MPSK, $P_B = 10^{-5}$
Noncoherent MFSK, $P_B = 10^{-5}$
Coherent QAM, $P_B = 10^{-5}$
Power-limited region