Lecture 12.
MQAM Error Analysis

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Update/Outline

- Previously we have discussed
  - Probability of Error for Orthogonal M-ary modulation
  - Union Bound
  - Some signal types
- We will now consider
  - M-ary Modulation Types
  - MQAM, MPSK
  - Tradeoffs

M-ary Modulation Types

- We have seen how to:
  - Design an optimum M-ary Receiver
  - Calculating the probability of bit and symbol errors
  - Considered some general transmission signal types
- Now we wish to discuss specific but popular modulation formats and determine their properties and when to use which one

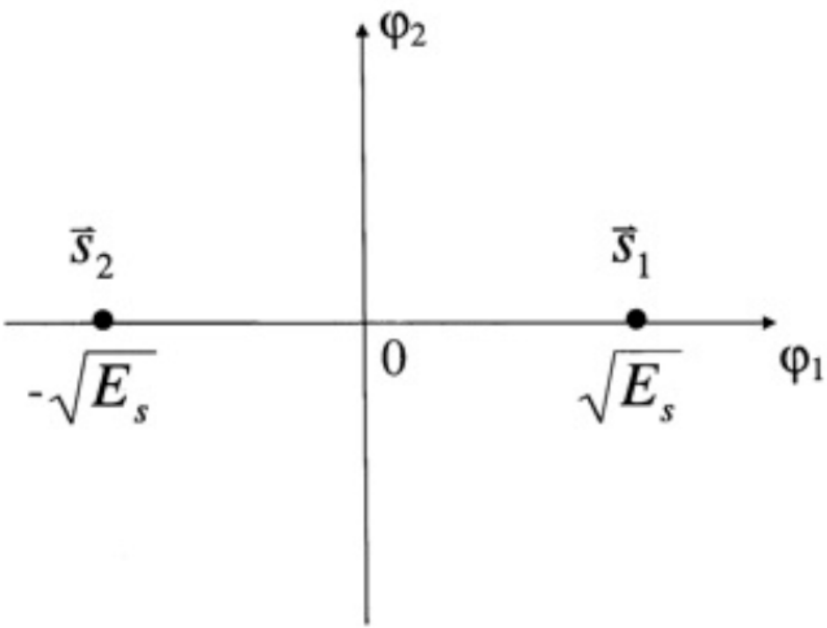
M-ary Phase-Shift Keying (MPSK)

$$s_k(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left[\omega_c t + \underbrace{\frac{2\pi(k-1)}{M}}_{\theta_k}\right] \quad 0 \le t \le T_s, k = 1,2,\ldots, M$$

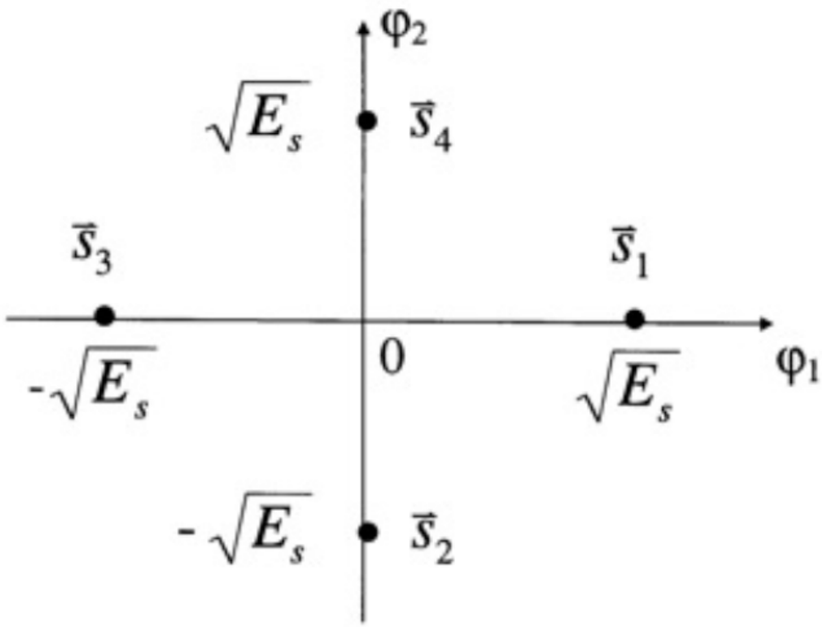
- Assuming that $\omega_c = \dfrac{\text{integer} \times 2\pi}{T_s}$

$$\Rightarrow \left\| \begin{aligned} \varphi_1(t) &= \sqrt{\frac{2}{T_s}} \cos(\omega_c t) \\ \varphi_2(t) &= \sqrt{\frac{2}{T_s}} \sin(\omega_c t) \end{aligned} \right.$$

Examples of MPSK constellation



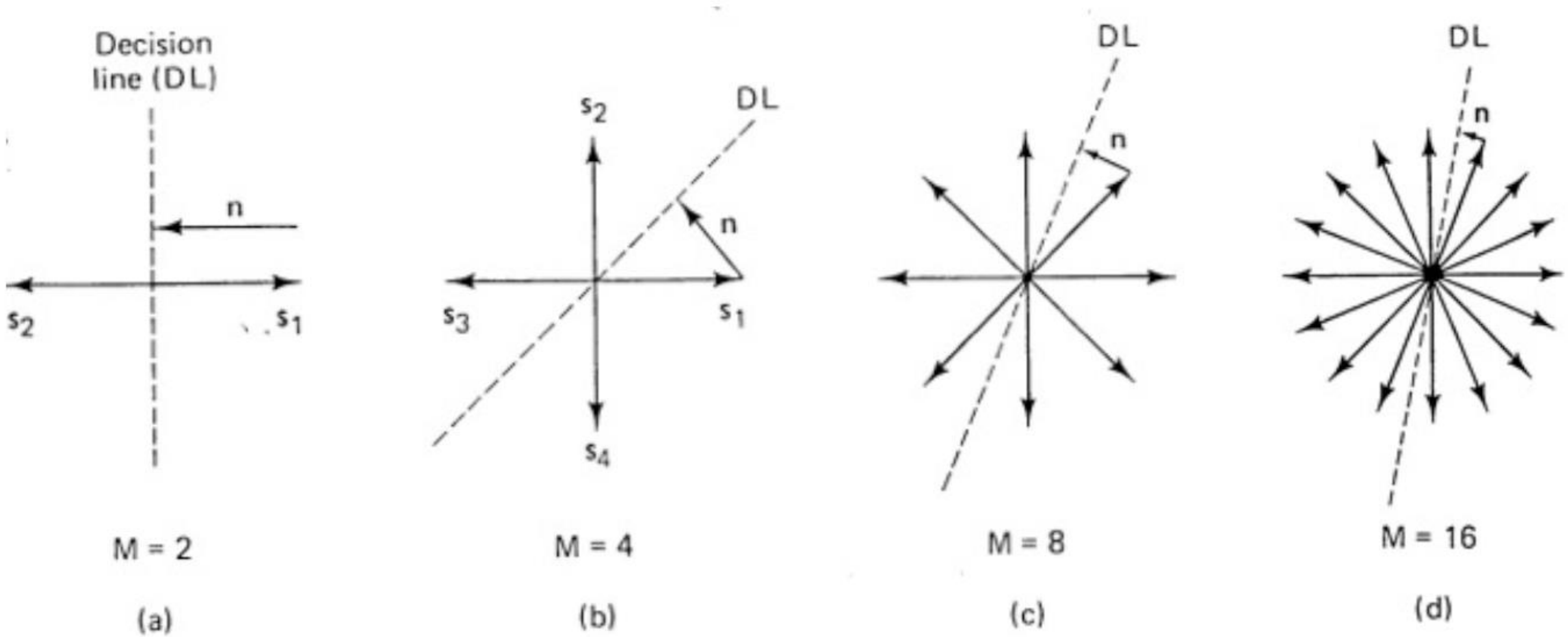
M = 2 (2-PSK ≡ BPSK)



M = 4 (4-PSK)

Also called **QPSK**

MPSK signal sets for M=2,4,8,16



Error Performance of MPSK

- It can be shown that

$$P_{eM} = \frac{1}{\pi}\int_0^{\pi(1-1/M)} \exp\left[\frac{(E_S/N_0)\sin^2(\pi/M)}{\sin^2\phi}\right] d\phi$$

i.e. Eqn.(4-98) in Ziemer and Peterson

Except for M = 2 and 4, numerical integration is needed.

Alternatively, we can use the tight upper and lower bounds below.

$$\boxed{Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi/M)\right] \le P_{e,M} \le 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi/M)\right]}$$

SER vs Eb/No for MPSK

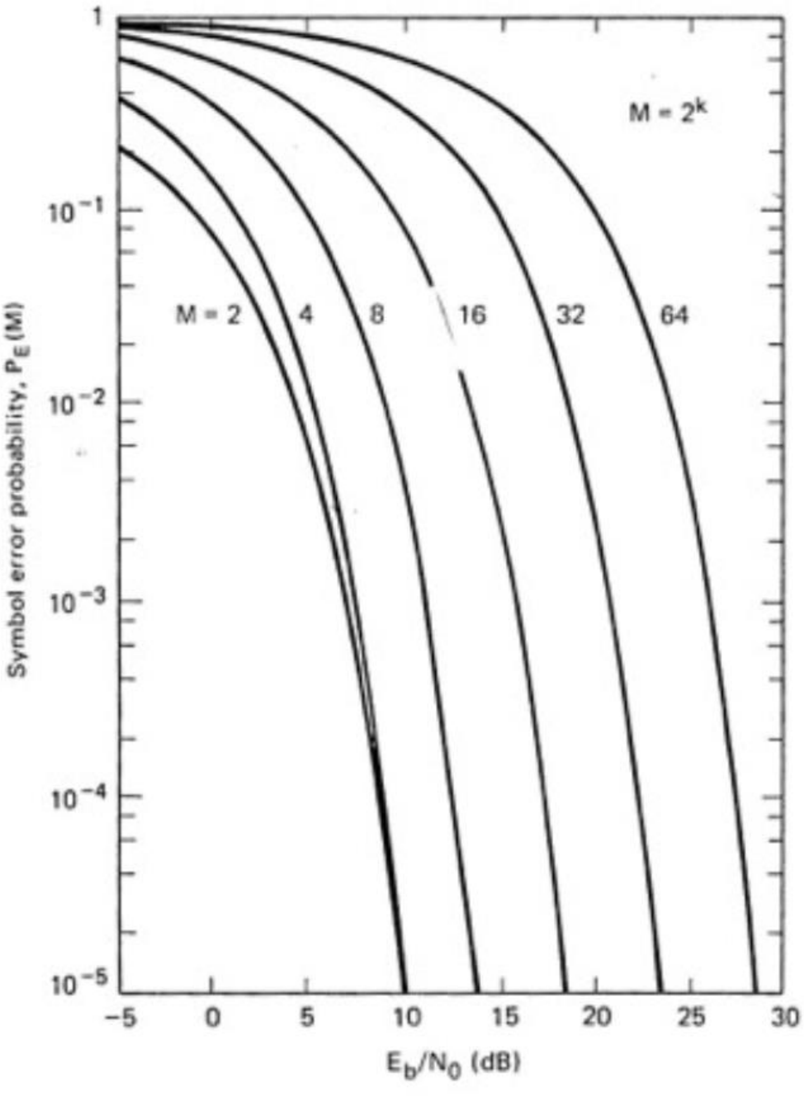


Figure 3.32 Symbol error probability for coherently detected multiple phase signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

For **large** Es/No:

$$P_{e,M} \approx 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi / M)\right]$$

Very tight for **fixed M** as **E_s/N_0 increases.**

How about Bit error prob?
Different from M-ary orthogonal, it depends on the bit labeling scheme used.

Typically **Gray Coding** is used for MPSK.

Binary Code

Gray Code

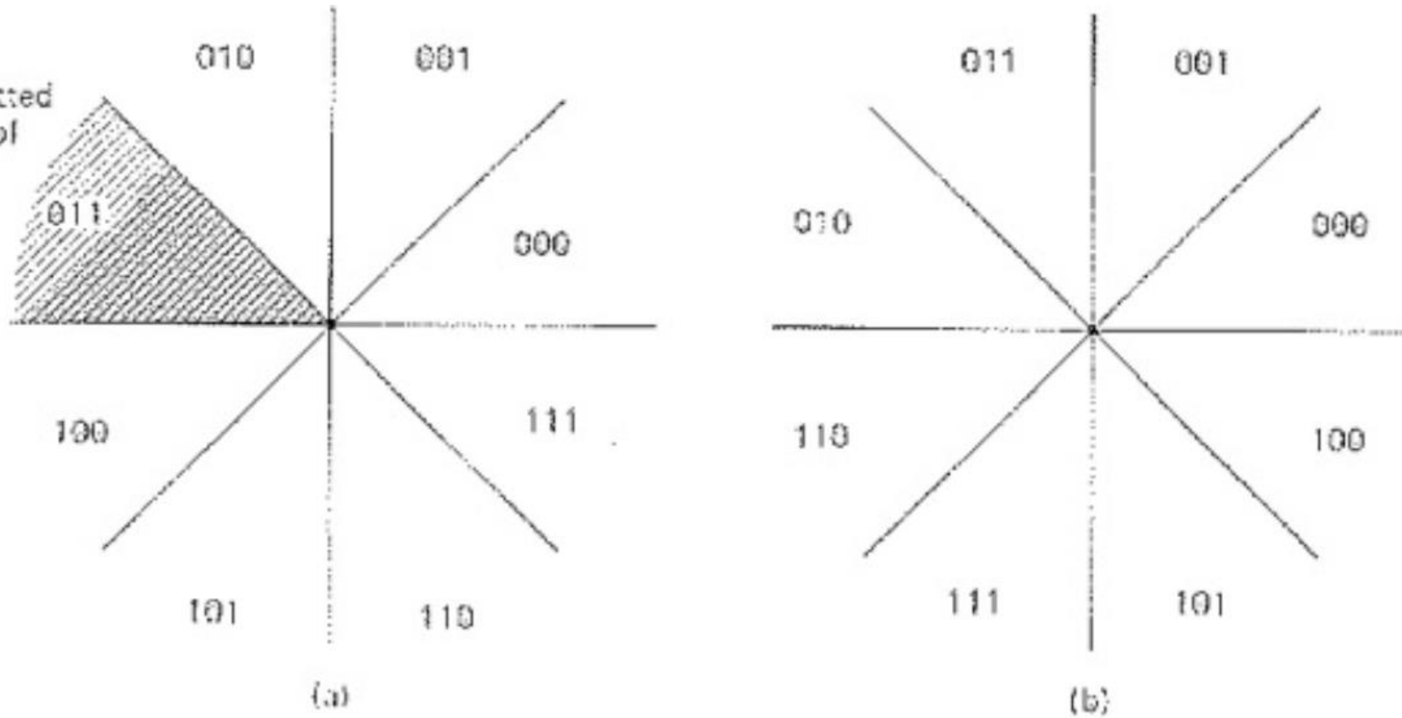


Figure 3.36 Binary-coded versus Gray-coded decision regions in an MPSK signal space. (a) Binary coded. (b) Gray coded.

Gray Coding

- Allows representation of symbols or **bit-to-symbol mapping**
- In going from one symbol to an **adjacent** symbol, **only one bit** out of the k (or n bits in text) bits **changes**.
- An adjacent symbol error (i.e. the most likely symbol error) will therefore be accompanied by one and only one bit error.
- Thus, the **bit error** probability of Gray-coded MPSK can be well approximated by

$$P_b \cong \frac{P_{e,M}}{\log_2 M}$$

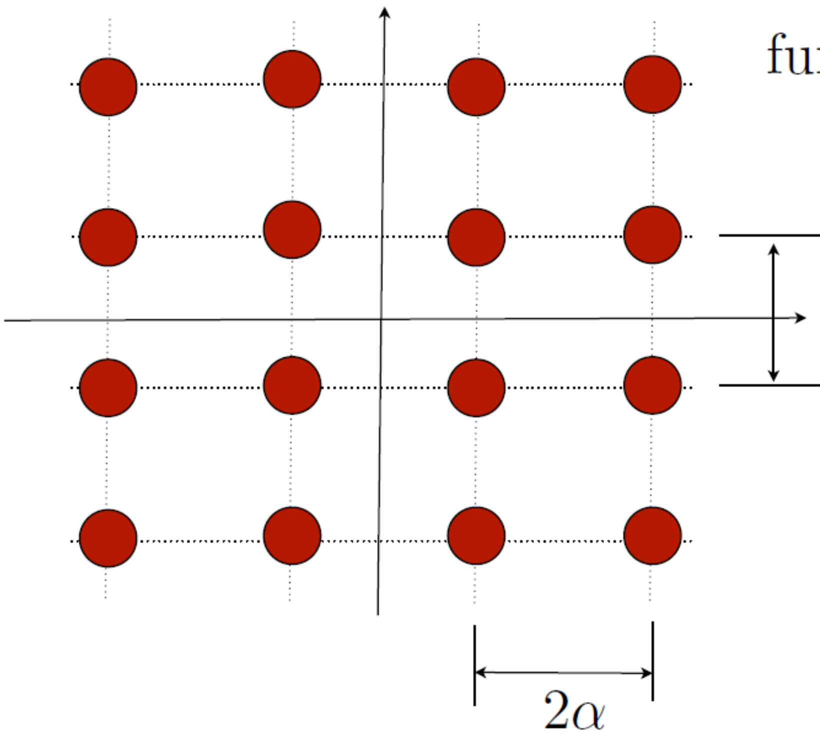
M-QAM Modulation

Time Domain Description

$$s_k(t) = a_k \cos(\omega_c t) + b_k \sin(\omega_c t) \text{ for } t \in [0, T_s], k = \{1, 2, ..., M\}$$

where $a_k, b_k \in \{\pm\alpha, \pm 3\alpha, ..., \pm(\sqrt{M} - 1)\alpha\}$ and $M = \{4, 16, 64, 256, ...\}$

Geometric Domain Description



2-dim signal space with basis functions $\{\cos(\omega_c t), \sin(\omega_c t)\}$

Rectangular M-QAM signals can be generated easily using I-Q modulator

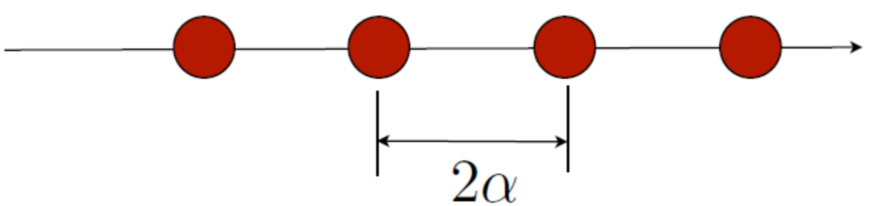
MQAM Performance

- Bit Rate: $R_b = \frac{1}{T_s} \log_2 M$
- Average Symbol Energy: $E_s = \frac{2}{\sqrt{M}} \left(2\alpha^2 + 2(3\alpha)^2 + ...\right)$
  - e.g. for 16QAM, $E_s = \frac{2}{4} \left(2\alpha^2 + 2(3\alpha)^2\right) = 10\alpha^2$
- Average Transmit Power: $P_s = \frac{E_s}{T_s}$
- Remarks:
  - Not all M points have the same energy.
  - Information can be visualized as carried by the amplitude and phase information of the points

MQAM Performance

- Symbol Error Probability (SER) Analysis:
  - M-QAM can be regarded as two independent M-PAM, each having $\sqrt{M} = 2^{k/2}$ points
  - Probability of correct decision for M-QAM is given by:

$P_c = (1 - P_{\sqrt{M}})^2$ where $P_{\sqrt{M}}$ is the probability of error of a $\sqrt{M}$-PAM.



M-QAM Performance

- It can be shown that

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

- Hence,

$$P_M = 1 - (1 - P_{\sqrt{M}})^2 \approx 2P_{\sqrt{M}} = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

- Recall that the MPSK SER is given by:

$$P_M \approx 2Q\left(\sin\frac{\pi}{M}\sqrt{\frac{2E_s}{N_0}}\right)$$

- Compare M-QAM with M-PSK SER:
  - The gain of M-QAM over M-PSK is given by:

$$gain = \frac{3/(M-1)}{2\sin^2(\pi/M)}$$

M-QAM Union Bound

- Instead of working out the exact Pe, the union bound is simpler and is asymptotically accurate for high SNR.

$$P_e \leq 4Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) \approx 4Q\left(\sqrt{\frac{(2\alpha)^2}{2N_0}}\right) \approx 4Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

On average 4 nearest neighbors

- Compare with the exact Pe, they are very close especially for large Es/No.

Choices of M-ary Modulations

- There are a number of factors that one needs to consider in order to pick a modulation scheme. For example:
  - **Power efficiency** (inversely propotional to **Eb/No)**
  - **Bandwidth efficiency (bit rate/bandwidth)**
- M-ary modulation allows us to **trade** in **power**, **bit rate** and **bandwidth**

Error Probability Performance Curves

- Allow us to design and set an operating point for a system
- Consider **MFSK**
  - **Increasing M can provide an improvement in P_b,or reduction in the E_b/N_0 required, at the cost of increased bandwidth**
- Consider **MPSK**
  - **Increasing M can provide a reduction in bandwidth requirement, at the cost of degraded P_b, or increase in the E_b/N_0 requirement**

Theoretical Limits on Performance

- **Channel Capacity** is the theoretical upper bound for the maximum rate at which information could be transmitted without error *(Shannon 1948)*
- For a bandlimited channel that is corrupted by AWGN $\left(S_n(f) = \frac{N_0}{2}\right)$ the maximum rate achievable is given by

$$R = B\log_2(1+SNR) = B\log_2(1+\frac{P_s}{N_0 B})$$

Spectral Efficiency

Shannon Limit

- $N = N_0B$, hence

$$\frac{R}{B} = \log_2\left(1 + \frac{P}{N_0B}\right) = \log_2\left(1 + \frac{P}{N}\right)$$

- Next note that

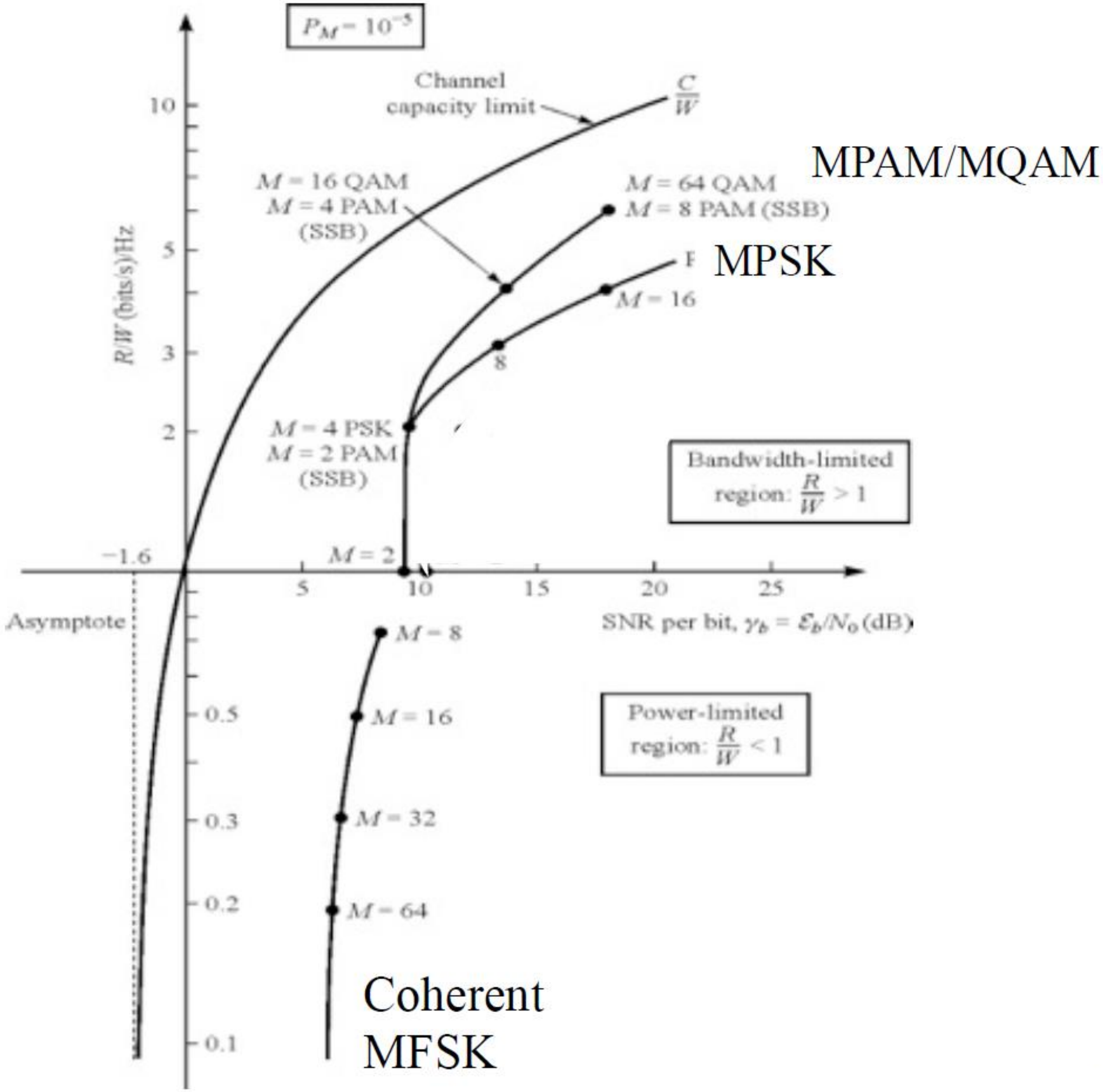
Energy Efficiency

$$\frac{E_b}{N_0} = \frac{PT}{N_0} = \frac{P}{RN_0} = \frac{PB}{RN_0B} = \frac{P}{N}\frac{B}{R}$$

- Then

$$\frac{E_b}{N_0} = \frac{B}{R}\left(2^{R/B} - 1\right)$$

Bandwidth- efficiency plane



Trade-Offs

- ***Power-Limited Systems***: Power scarce but bandwidth available
  - Improved P_b by expanding bandwidth (for a given E_b/N_0) or required E_b/N_0 can be reduced by expanding bandwidth (for a given P_b)
- ***Bandwidth-Limited Systems***: bandwidth scarce
  - Maximize R over the bandlimited channel at the expense of E_b/N_0 (for a given P_b)

Shannon Limit

- In the limit as R/B goes to 0, we get

$$\frac{E_b}{N_0} = \frac{1}{\log_2 e} = 0.693 = -1.59dB$$

This value is called the Shannon Limit (what is the relationship with k going to infinity)

Received Eb/N0 must be > –1.6dB for reliable communications to be possible

Summary of M-ary Modulation Scheme

	M-FSK	M-PSK	M-QAM
Bit Rate	$R_b = \frac{1}{T_s}\log_2(M)$	$R_b = \frac{1}{T_s}\log_2(M)$	$R_b = \frac{1}{T_s}\log_2(M)$
BW (Bandpass)	$BW = \frac{M+1}{2T_s}$	$BW = \frac{1}{T_s}$	$BW = \frac{1}{T_s}$
Average Transmit Power	$\frac{E_s}{T_s}$	$\frac{E_s}{T_s}$	$\frac{4\alpha^2}{\sqrt{M}T_s}\sum_{i=1}^{\log_2 M/2}(2^i-1)^2$
Average Symbol Error Probability (SER)	$P_e \le (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$	$P_e \le 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin(\pi/M)\right)$	$P_M \approx 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\frac{3}{M-1}\frac{E_s}{N_0}\right)$
Remarks	• Orthogonal Signaling Schemes (Equi-energy points & Mutually orthogonal signals) • Enhance Energy Efficiency at the expense of extra BW	• Equi-energy constellation (information carried by phase values only) • Dimension of the signal set is always 2 (I-Q modulator) • Enhance spectral efficiency at the expense of extra power	• Points are NOT equi-energy • Information is carried by both amplitude and phase • Enhance spectral efficiency at the expense of extra power • Better than M-PSK for M>4.