

数值分析方法

作业3

Problem 1

解: a. $\because \mathbf{x} = (0, -7, 5)^t, \tilde{\mathbf{x}} = (-0.2, -7.5, 5.4)^t$

$$\therefore \mathbf{x} - \tilde{\mathbf{x}} = (0.2, 0.5, -0.4)^t$$

$$\therefore \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty} = 0.5$$

将 $\tilde{\mathbf{x}}$ 代入方程组, 可得:

$$A\tilde{\mathbf{x}} - \mathbf{b} = \begin{pmatrix} 0 \\ -0.3 \\ -0.2 \end{pmatrix}$$

$$\therefore \|A\tilde{\mathbf{x}} - \mathbf{b}\|_{\infty} = 0.3$$

b. $\because \mathbf{x} = (0, -7, 5)^t, \tilde{\mathbf{x}} = (-0.33, -7.9, 5.8)^t$

$$\therefore \mathbf{x} - \tilde{\mathbf{x}} = (0.33, 0.9, -0.8)^t$$

$$\therefore \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\infty} = 0.9$$

将 $\tilde{\mathbf{x}}$ 代入方程组, 可得:

$$A\tilde{\mathbf{x}} - \mathbf{b} = \begin{pmatrix} 0.27 \\ -0.16 \\ 0.21 \end{pmatrix}$$

$$\therefore \|A\tilde{\mathbf{x}} - \mathbf{b}\|_{\infty} = 0.27$$

Problem 2

解: $\because \mathbf{A}$ 是对称矩阵, 则 $\mathbf{A}^T = \mathbf{A}$

$$\text{又 } \|\mathbf{A}\|_2 = \sqrt{\rho(\mathbf{A}\mathbf{A}^T)}$$

$$\therefore \|\mathbf{A}\|_2 = \sqrt{\rho(\mathbf{A}^2)}$$

由定义可知, $\rho(\mathbf{A}) = \max |\lambda| > 0$, 其中 λ 为 \mathbf{A} 的特征根

$$\therefore \rho(\mathbf{A}^2) = \max |\lambda^2| = (\max |\lambda|)^2 = \rho^2(\mathbf{A})$$

$$\therefore \|\mathbf{A}\|_2 = \sqrt{\rho^2(\mathbf{A})} = \rho(\mathbf{A})$$

证毕.

Problem 3

解: 代码如下:

```

1  #include<stdio.h>
2  #include<stdlib.h>
3  #include<math.h>
4  int main()
5  {
6      int n;
7      scanf("%d",&n);
8      double a[n][n+1];
9      double s[n],m[n][n];
10     double max,sum;
11     double x[n];
12     int i=0,j=0,k,p;
13     int row[n],temp;
14     while (i<n*(n+1)){
15         scanf("%lf",&a[0][i]);
16         i++;
17     }
18     for (i=0;i<n;i++){
19         j=0;
20         s[i]=fabs(a[i][0]);
21         while (j<n){
22             if (fabs(a[i][j])>s[i]){
23                 s[i]=fabs(a[i][j]);
24             }
25             if (s[i]==0){
26                 printf("Algorithm \
27 failed");
28                 exit(0);
29             }
30             j++;
31         }
32         row[i]=i;
33     }
34     for (i=0;i<n-1;i++){
35         max=a[row[i]][i]/s[row[i]];
36         for (j=i;j<n;j++){
37             if (max<a[row[j]][i]/\
38                 s[row[j]]){
39                 max=a[row[j]][i]/\
40                 s[row[j]];
41                 p=j;

```

```

42         }
43     }
44     if (a[row[p]][i]==0){
45         printf("Algorithm failed");
46         exit(0);
47     }
48     if (row[i]!=row[p]){
49         temp=row[i];
50         row[i]=row[p];
51         row[p]=temp;
52     }
53     for (j=i+1;j<n;j++){
54         m[row[j]][i]=a[row[j]][i]/\
55         a[row[i]][i];
56         for (k=0;k<=n;k++){
57             a[row[j]][k]-=m[row\
58             [j]][i]*a[row[i]][k];
59         }
60     }
61 }
62 if (a[row[n-1]][n-1]==0){
63     printf("Algorithm failed");
64     exit(0);
65 }
66 x[n-1]=a[row[n-1]][n]/a[row[n-1]][n-1];
67 for (i=n-2;i>=0;i--){
68     sum=0;
69     for (j=i+1;j<n;j++){
70         sum+=x[j]*a[row[i]][j];
71     }
72     x[i]=(a[row[i]][n]-sum)/a[row[i]][i];
73 }
74 for (i=0;i<n;i++){
75     printf("x[%d]=%.8lf\n",i+1,x[i]);
76 }
77 }

```

a. 由题意: 增广矩阵为

$$\tilde{A} = \begin{pmatrix} 0.03 & 58.9 & 59.2 \\ 5.31 & -6.10 & 47.0 \end{pmatrix}$$

代入程序可得: $\tilde{\mathbf{x}} = (10.00000000, 1.00000000)^T$.

b. 由题意: 增广矩阵为

$$\tilde{\mathbf{A}} = \begin{pmatrix} 3.03 & -12.1 & 14 & -119 \\ -3.03 & 12.1 & -7 & 120 \\ 3.11 & -14.2 & 21 & -139 \end{pmatrix}$$

代入程序可得: $\tilde{\mathbf{x}} = (0.00000000, 10.00000000, 0.14285714)^T$.

Problem 4

解: 代码如下:

```
1  #include<stdio.h>
2  #include<stdlib.h>
3  #include<math.h>
4  int main()
5  {
6      int n;
7      scanf("%d",&n);
8      double a[n][n];
9      double b[n];
10     double x0[n],x[n];
11     double sum=0;
12     int i,j,k=0;
13     for (i=0;i<n*n;i++){
14         scanf("%lf",&a[0][i]);
15     }
16     for (i=0;i<n;i++){
17         scanf("%lf",&b[i]);
18     }
19     for (i=0;i<n;i++){
20         scanf("%lf",&x0[i]);
21     }
22     while (k<3){
23         for (i=0;i<n;i++){
24             sum=0;
25             for (j=0;j<n;j++){
26                 if (j!=i){
27                     sum+=a[i][j]*x0[j];
28                 }
29             }
```

```

30         x[i]=(b[i]-sum)/a[i][i];
31     }
32     for (i=0;i<n;i++){
33         x0[i]=x[i];
34     }
35     k++;
36     printf("n = %d,x = ( ",k-1);
37     for (i=0;i<n-1;i++){
38         printf("%10.8lf,",x[i]);
39     }
40     printf("%10.8lf )^T\n",x[n-1]);
41 }
42 }

```

a. 由题意:

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -1 \\ -1 & 3 & 1 \\ 2 & 2 & 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$$

$$\because \mathbf{x}^{(0)} = \mathbf{0}$$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	1.25000000	-1.33333333	0.20000000
2	1.63333333	-0.98333333	0.23333333
3	1.55416667	-0.86666667	-0.06000000

故近似解为 $\tilde{\mathbf{x}} = (1.55416667, -0.86666667, -0.06000000)^T$.

b. 由题意:

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0.5 \\ 1 & -2 & -0.5 \\ 0 & 1 & 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$$

$$\because \mathbf{x}^{(0)} = \mathbf{0}$$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	-2.00000000	2.00000000	0.00000000
2	-1.00000000	1.00000000	-1.00000000
3	-1.75000000	1.75000000	-0.50000000

故近似解为 $\tilde{\mathbf{x}} = (-1.75000000, 1.75000000, -0.50000000)^T$.

Problem 5

解：当使用雅可比迭代时,代码如下：

```
1  #include<stdio.h>
2  #include<stdlib.h>
3  #include<math.h>
4  int main()
5  {
6      int n;
7      scanf("%d",&n);
8      double a[n][n];
9      double b[n];
10     double x0[n],x[n];
11     double sum=0;
12     double norm[n],norm1,tol;
13     int i,j,k=1;
14     for (i=0;i<n*n;i++){
15         scanf("%lf",&a[0][i]);
16     }
17     for (i=0;i<n;i++){
18         scanf("%lf",&b[i]);
19     }
20     for (i=0;i<n;i++){
21         scanf("%lf",&x0[i]);
22     }
23     scanf("%lf",&tol);
24     while (k){
25         for (i=0;i<n;i++){
26             sum=0;
27             for (j=0;j<n;j++){
28                 if (j!=i){
29                     sum+=a[i]\
30                     [j]*x0[j];
31                 }
32             }
33             x[i]=(b[i]-sum)/a[i][i];
34             norm[i]=fabs(x[i]-x0[i]);
35         }
36         for (i=0;i<n;i++){
37             x0[i]=x[i];
38         }
39         k++;
```

```

40     printf("n = %d, x = ( ", k-1);
41     for (i=0; i<n-1; i++){
42         printf(" %10.8lf ", x[i]);
43     }
44     printf(" %10.8lf )", x[n-1]);
45     //Output Ax-b
46     printf("Ax-b = ( ");
47     for (i=0; i<n-1; i++){
48         sum=0;
49         for (j=0; j<n; j++){
50             sum+=a[i][j]*x[j];
51         }
52         sum-=b[i];
53         printf(" %10.8lf ", sum);
54     }
55     sum=0;
56     for (j=0; j<n; j++){
57         sum+=a[i][j]*x[j];
58     }
59     sum-=b[i];
60     printf(" %10.8lf )^T\n", sum);
61     norml=norm[0];
62     for (i=0; i<n; i++){
63         if (norm[i]>norml){
64             norml=norm[i];
65         }
66     }
67     if (norml<=tol){
68         break;
69     }
70 }
71 }

```

a. 由题意:

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

令 $\mathbf{x}^{(0)} = \mathbf{0}$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	0.33333333	0.00000000	0.57142857
2	0.14285714	-0.35714286	0.42857143
3	0.07142857	-0.21428571	0.66326531
4	0.04081633	-0.25680272	0.63265306
5	0.03684807	-0.23129252	0.66399417
6	0.03490444	-0.23975543	0.65476190
7	0.03516089	-0.23570619	0.65922185
8	0.03502399	-0.23732106	0.65737656
9	0.03510079	-0.23663751	0.65812732

故近似解为 $\tilde{\mathbf{x}} = (0.03510079, -0.23663751, 0.65812732)^T$.

b. 由题意:

$$\mathbf{A} = \begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix}$$

令 $\mathbf{x}^{(0)} = \mathbf{0}$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	0.90000000	0.70000000	0.60000000
2	0.97000000	0.91000000	0.74000000
3	0.99100000	0.94500000	0.78200000
4	0.99450000	0.95550000	0.78900000
5	0.99555000	0.95725000	0.79110000
6	0.99572500	0.95777500	0.79145000

故近似解为 $\tilde{\mathbf{x}} = (0.99572500, 0.95777500, 0.79145000)^T$.

当使用高斯-赛德尔迭代时,代码如下:

```

1  #include<stdio.h>
2  #include<stdlib.h>
3  #include<math.h>
4  int main()
5  {
6      int n;
7      scanf("%d",&n);
8      double a[n][n];
9      double b[n];
10     double x0[n],x[n];
11     double sum1=0,sum2=0,sum;
12     double norm[n],norml,tol;

```



```

13     int i,j,k=1;
14     for (i=0;i<n*n;i++){
15         scanf("%lf",&a[0][i]);
16     }
17     for (i=0;i<n;i++){
18         scanf("%lf",&b[i]);
19     }
20     for (i=0;i<n;i++){
21         scanf("%lf",&x0[i]);
22     }
23     scanf("%lf",&tol);
24     while (k){
25         for (i=0;i<n;i++){
26             sum1=0;
27             sum2=0;
28             for (j=0;j<i;j++){
29                 sum1+=a[i][j]*x[j];
30             }
31             for (j=i+1;j<n;j++){
32                 sum2+=a[i][j]*x0[j];
33             }
34             x[i]=(b[i]-sum1-sum2)/a[i]\
35                 [i];
36             norm[i]=fabs(x[i]-x0[i]);
37         }
38         for (i=0;i<n;i++){
39             x0[i]=x[i];
40         }
41         k++;
42         printf("n = %d,x = ( ",k-1);
43         for (i=0;i<n-1;i++){
44             printf("%10.8lf,",x[i]);
45         }
46         printf("%10.8lf )",x[n-1]);
47         //Output Ax-b
48         printf("Ax-b = ( ");
49         for (i=0;i<n-1;i++){
50             sum=0;
51             for (j=0;j<n;j++){
52                 sum+=a[i][j]*x[j];
53             }

```

```

54         sum-=b[i];
55         printf("%10.8lf",sum);
56     }
57     sum=0;
58     for (j=0;j<n;j++){
59         sum+=a[i][j]*x[j];
60     }
61     sum-=b[i];
62     printf("%10.8lf )\n",sum);
63     norml=norm[0];
64     for (i=0;i<n;i++){
65         if (norm[i]>norml){
66             norml=norm[i];
67         }
68     }
69     if (norml<=tol){
70         break;
71     }
72 }
73 }

```

a. 由题意:

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

令 $\mathbf{x}^{(0)} = \mathbf{0}$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	0.33333333	-0.16666667	0.50000000
2	0.11111111	-0.22222222	0.61904762
3	0.05291005	-0.23280423	0.64852608
4	0.03955656	-0.23595364	0.65559875
5	0.03614920	-0.23660752	0.65733928
6	0.03535107	-0.23678863	0.65775895

故近似解为 $\hat{\mathbf{x}} = (0.03535107, -0.23678863, 0.65775895)^T$.

b. 由题意:

$$\mathbf{A} = \begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix}$$

令 $\mathbf{x}^{(0)} = \mathbf{0}$

可得:

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
1	0.90000000	0.79000000	0.75800000
2	0.97900000	0.94950000	0.78990000
3	0.99495000	0.95747500	0.79149500
4	0.99574750	0.95787375	0.79157475

故近似解为 $\tilde{\mathbf{x}} = (0.99574750, 0.95787375, 0.79157475)^T$.

Problem 6

解: 由数学归纳法:

当 $n = 1$ 时, $\because \chi_1 \neq \mathbf{0}$

则 χ_1 线性无关

假设 $n = k - 1$ 时, $\chi_1, \chi_2, \dots, \chi_{k-1}$ 线性无关

当 $n = k$ 时, 设有

$$c_1\chi_1 + c_2\chi_2 + \dots + c_{k-1}\chi_{k-1} + c_k\chi_k = \mathbf{0} \quad (1)$$

(1) 式两边同乘 ρ_k , 得:

$$c_1\rho_k\chi_1 + c_2\rho_k\chi_2 + \dots + c_{k-1}\rho_k\chi_{k-1} + c_k\rho_k\chi_k = \mathbf{0} \quad (2)$$

$$\because \mathbf{A}\chi_i = \rho_i\chi_i$$

(1) 式两边同乘 \mathbf{A} , 则:

$$c_1\rho_1\chi_1 + c_2\rho_2\chi_2 + \dots + c_{k-1}\rho_{k-1}\chi_{k-1} + c_k\rho_k\chi_k = \mathbf{0} \quad (3)$$

用 (2) 减去 (3) 式, 得:

$$c_1(\rho_k - \rho_1)\chi_1 + c_2(\rho_k - \rho_2)\chi_2 + \dots + c_{k-1}(\rho_k - \rho_{k-1})\chi_{k-1} = \mathbf{0} \quad (4)$$

$\because \chi_1, \chi_2, \dots, \chi_{k-1}$ 线性无关, 且特征根两两互异

$$\therefore c_1 = c_2 = \dots = c_{k-1} = 0$$

代入 (3) 式得: $c_k = 0$

故 $\chi_1, \chi_2, \dots, \chi_{k-1}, \chi_k$ 线性无关, 证毕.

Problem 7

解: 假设严格对角占优矩阵 \mathbf{A} 不可逆, 则 $\mathbf{A}\mathbf{x} = \mathbf{0}$ 存在无穷多个解

记 \mathbf{A} 中某一行 $\{a_{kj}\}, 1 \leq k \leq n, j = 1, 2, \dots, n$

设其一解为 $\mathbf{x} = \{x_1, x_2, \dots, x_n\}^T$, 且满足 $|x_k| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$, 则:

$$\sum_{j=1}^n a_{kj}x_j = 0$$

于是:

$$\sum_{j \neq k}^n a_{kj} x_j = -a_{kk} x_k$$

$$\therefore |a_{kk} x_k| = \left| \sum_{j \neq k}^n a_{kj} x_j \right|$$

又

$$|a_{kk} x_k| = |a_{kk}| |x_k| > \left| \sum_{j \neq k}^n a_{kj} \right| |x_k| \geq \sum_{j \neq k}^n |a_{kj}| |x_j| \geq \left| \sum_{j \neq k}^n a_{kj} x_j \right|$$

矛盾, 假设不成立.

故严格对角占优矩阵必可逆.