Lecture 9.
Signal Space

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Teaching Schedule

» **Consider M-ary Digital Transmission**
  - Introduction
  - Signal Space Concepts
  - Basis Vectors/functions
  - Determination of an orthogonal basis set (Gram-Schmidt Orthogonalization)

Signal Space Concepts and Signal Representation

It turns out that the key to analyzing and understanding the performance of digital transmission is the realization that signals used in communications can be expressed and visualized graphically.

Thus

We need to understand signal space concepts as applied to digital communications

Overall Objectives/Goals

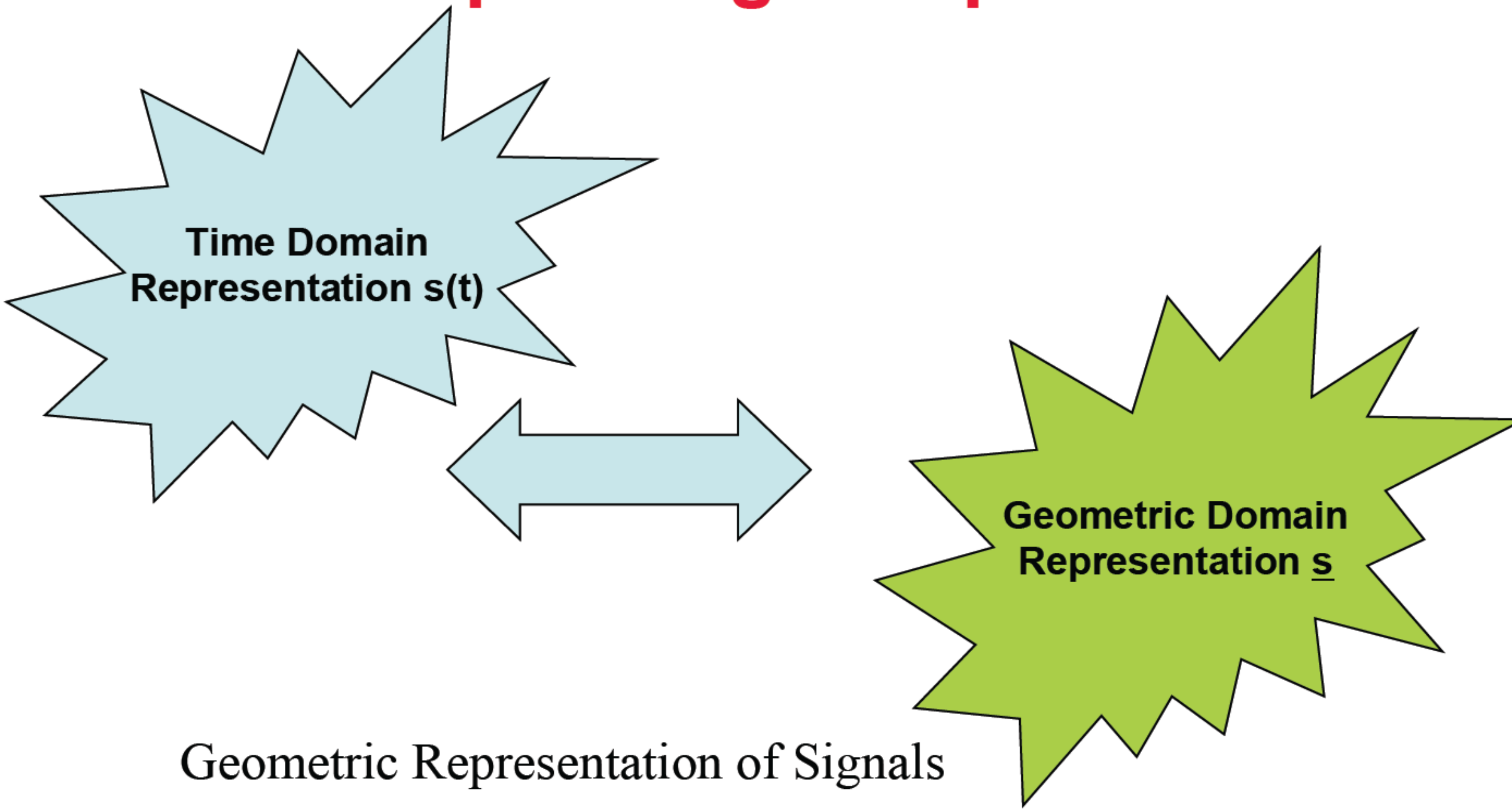
- **To analyze the problem of digital signal detection *from a fundamental point of view*.**

- **To understand the digital modulation and demodulation from a geometric perspective**
  - **Easy to understand**
  - **Useful design insights can be obtained without too much math**
  - **Concept of Signal Space**

Signal Space Concepts

- Signal space concepts will allow a more general way of looking at modulation schemes.
- By choosing an appropriate set of axis for our signal constellation, one can:
  - Design modulation types which have desirable properties
  - Construct optimal receivers for a given modulation technique
  - Analyze the performance of modulation schemes using very general techniques.

Concept of Signal Space



Geometric Representation of Signals

Representation of Signals

- (1) Time Domain:

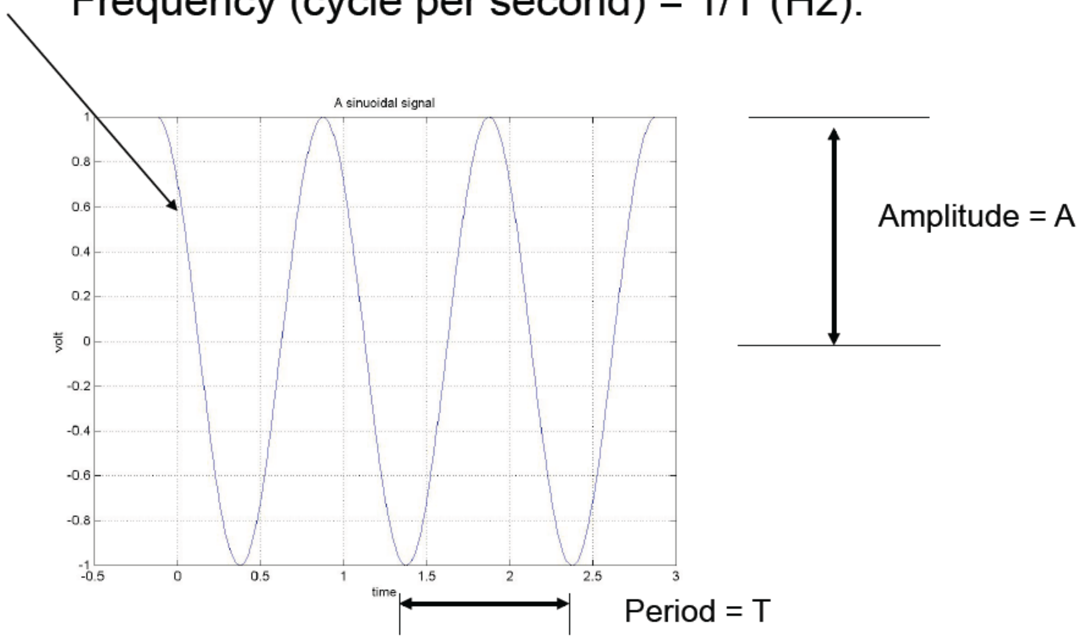
  Signal is represented by a function in time, s(t).

  Waveform (the shape of the function) could be observed.

  Periodic Signal

  S(t + T) = s(t) for all t. T = period

  Frequency (cycle per second) = 1/T (Hz).



Representation of Signals

- Frequency Domain:
  Signal could be represented by a function of frequency S(f) as well.
  For some aperiodic signals,
  could be decomposed into components of "sin" and "cos".
  Each component has different (amplitude, frequency, phase).

Fourier Transform - Transform Equation:

$$F(f) = \int_{-\infty}^{\infty} f(t) \exp(-j2\pi ft) dt$$

Fourier Transform - Analysis Equation:

$$f(t) = \int_{-\infty}^{\infty} F(f) \exp(j2\pi ft) df$$

Representation of Signals

- **Geometric Domain (Signal space)**
  - **Signal s(t) is represented as a "vector" s (with coordinates)**
  - **For a vector to be meaningful, we need to define the space first**
    - » **What is the "frame-of-reference"?**
    - » **The "frame-of-reference" is defined by "x-axis", "y-axis",.....**

Define the "**frame-of-reference**"

- Geometric Domain (Signal Space)
  - Signal could be represented by a point in a space.
  - Step 1: Given a set of M signals, $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ define a D-dim signal space with basis $\{\phi_1(t), \phi_2(t), \ldots, \phi_D(t)\}$.
  - Step 2: Find out the coordinates of each signals by: $s_i(t) \rightarrow \vec{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,D})$

$$s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

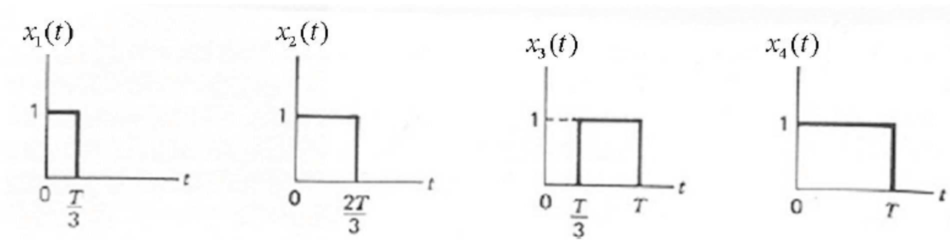
Geometric Representation of Signals

- **Time Domain (x(t)), Frequency Domain (X(f)), Geometric Domain (x) are just different views looking at the same coin.**
  - **The physical characterization of the coin will be the same no matter you are computing from which domains**
  -

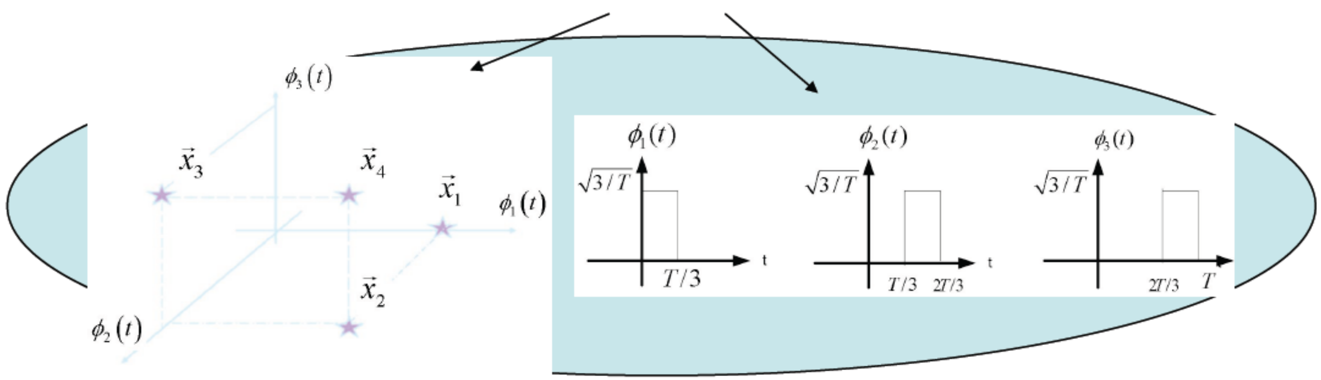
$$E = \int_0^T |x(t)|^2 dt \text{ (Time Domain Energy)}$$
$$E = \int_{-\infty}^{\infty} |H(f)|^2 df \text{ (Frequency Domain Energy)}$$
$$E = < \vec{x}, \vec{x} > = \|\vec{x}\|^2 \text{ (Geometric Domain Energy)}$$

Example 1

- **Consider 4 signals**

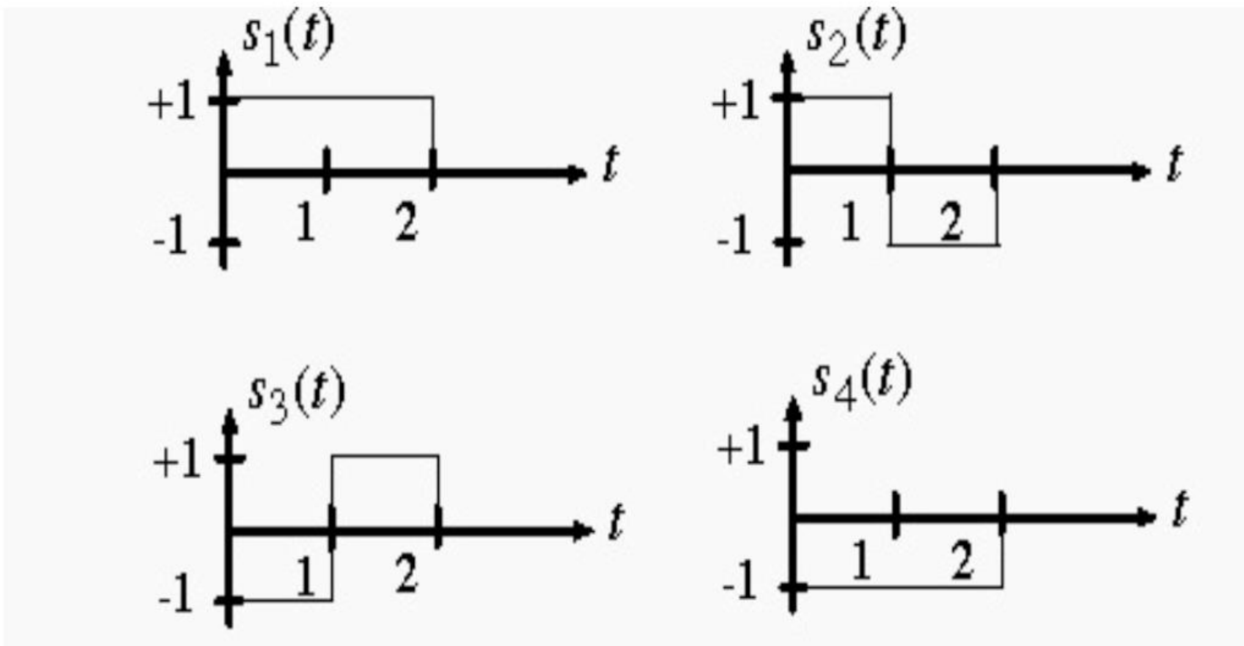


Find the orthonormal basis functions (orthonormal axis) of the Signal Space that contains the 4 signals.



Example 2

Consider the following signal set:



Basis Functions

- By inspection, the signals can be expressed in terms of the following functions:

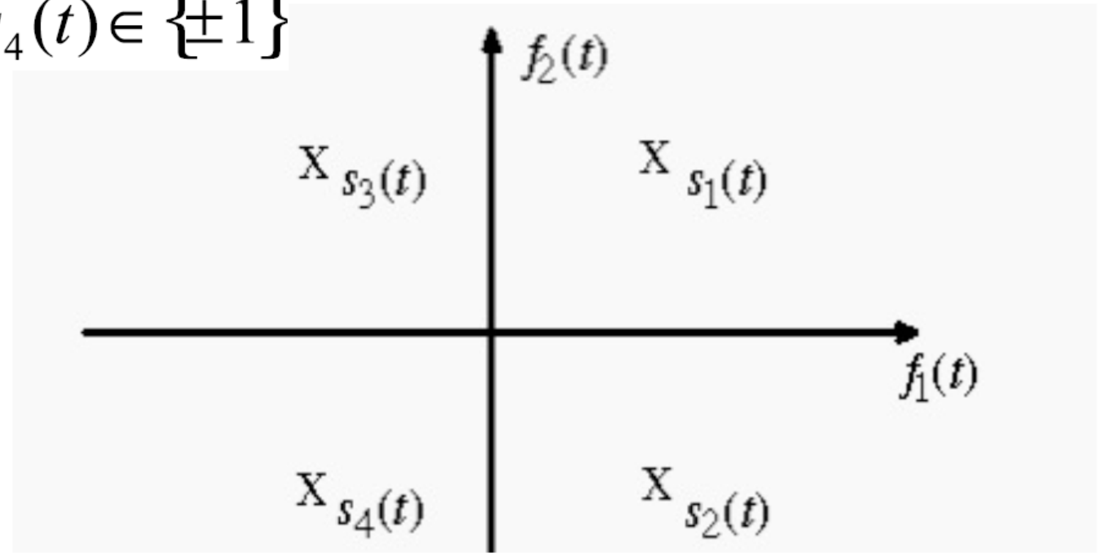
$$f_1(t) = rect(t - 0.5)$$

$$f_2(t) = rect(t - 3/2)$$

- These are known as basis functions.

Constellation Diagram

$s_1(t),\ldots,s_4(t) \in \{\pm 1\}$



Signal Space and Basis Functions

- Two entirely different signal sets can have the same geometric representation.
- The underlying geometry will determine the performance and the receiver structure for a signal set.
- In the previous examples, we were able to guess the correct basis functions.
- In general, is there any method which allows us to find a complete orthonormal basis for an arbitrary signal set?
  - Gram-Schmidt Orthogonalization (GSO) Procedure

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Vector Space

- **A vector space V over a field F is a set of "abstract objects" called "vectors".**
  - **The elements of V are called "Vectors".**
  - **The elements of F are called "Scalars".**
  - **Two basic "binary operations" (1) Vector additions; (2) Scalar Multiplications that satisfy the following AXIOMS**
    - » **Associativity of Addition: u + (v + w) = (u + v) + w**
    - » **Commutativity of Addition: u + v = v + u**
    - » **Identity Elements of Addition: There exists 0 \in V s.t. 0 + u = u for all u \in V.**
    - » **Inverse Elements of Addition: For every v\in V, there exists -v \in V s.t. v + (-v) = 0**
    - » **Distributivity of Scalar Multiplication (w.r.t. Vector Addition):** a(**u+v) =** au **+** av
    - » **Distributivity of Scalar Multiplication (w.r.t. Field Addition): (**a + b)**u =** au **+** b**u.**
    - » **Compatibility of scalar multiplication:** a(b**v) =** (ab)**v**
    - » **Identity element of scalar multiplication: there exists** 1 \in F s.t. 1**v = v for all v \in V.**

Vector Space Example

- **Coordinate Space over Real elements:-**
  - **V = {**$(a_1, a_2, ..., a_n)$: $a_i \in R$} a vector space can be composed of n-tuples of real numbers. (Field = R)
- **Coordinate Space over Complex elements:-**
  - **V = {**$(a_1, a_2, ..., a_n)$: $a_i \in C$} a vector space can be composed of n-tuples of complex numbers. (Field = C)
- **Function Space (Signal Space):-**
  - V = Functions from any fixed domain to F also forms a vector space.
  - e.g. Functions of time --> R (signal space) is a vector space.

Inner Product Space

- A vector space (V,F) does not have notion of geometry (or topology)
  - Notion of distance? (Two vectors are close or far away from each other)
  - Notion of topology? (open set, closed set, limits)
  - Notion of geometry? (Circle??)
  - All these requires “norm”
  - Notion of angle? (angle between two vectors)
  - All these requires “inner product”
- A vector space (V,F) with an “inner product” is called “inner product space”
  - Inner Product is a mapping <u,v>: V x V --> F that satisfy the following axioms
    - » $\langle u,v\rangle = \langle v,u\rangle^*$
    - » $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$
    - » $\langle au,v\rangle = a\langle u,v\rangle$
    - » $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u = 0$.

Geometric Concepts in Inner Product Space

- **Length of a vector:**
  - $||v||^2 = \langle v,v \rangle$
- **Distance between two vectors:**
  - $||v-w||^2 = \langle (v-w),(v-w) \rangle$
- **Angle between two vectors:**

$$\cos\theta = \frac{<\mathbf{v},\mathbf{w}>}{||\mathbf{v}||\,||\mathbf{w}||}$$

- **Orthogonal vectors: $\langle v,w \rangle = 0$**
- **Circle (x_c, r): $||x - x_c|| = r$**
- **Limit of a sequence:**

$$\lim_{n\to 0} \mathbf{v}_n = \mathbf{v}$$

For any $\epsilon > 0$, there exists n_0 such that for all $n > n_0$, $||\mathbf{v}_n - \mathbf{v}|| < \epsilon$

Vectors and Space Concepts

- An n-dimensional space S is defined by a set of n basis vectors (e_1, $e_{2,}$ … e_n);
  - S = span (e_1, $e_{2,}$ … e_n);

$\Rightarrow$ Any vector $\underline{a}$ can be written as

$$\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$$

n = dimension = maximum number of *linearly independent vectors* in the vector space

Definitions and Properties in Vector Space

- Notation:

Coordinate Representation of vector a.

$$\underline{\mathbf{a}} = \sum_{i=1}^{n} a_i \underline{e}_i \Leftrightarrow \underline{\mathbf{a}} = (a_1, a_2, \ldots, a_n)$$

- Definitions:

1) **Inner Product** : $\langle \underline{\mathrm{a}}, \underline{\mathrm{b}} \rangle = \underline{\mathrm{a}} \cdot \underline{\mathrm{b}} = \sum_{i=1}^{n} a_i b_i$

2) $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$ are **Orthogonal** ($\perp$) if

$$\underline{\mathrm{a}} \cdot \underline{\mathrm{b}} = 0$$

3) $\|\underline{\mathrm{a}}\| = \sqrt{\langle \underline{\mathrm{a}}, \underline{\mathrm{a}} \rangle} = \sqrt{\sum_{i=1}^{n} a_i^2}$

$=$ **Norm of $\underline{\mathbf{a}}$**

Definitions and Properties in Vector Space

4) A set of vectors are orthonormal if they are mutually $\perp$ and all have unity norm.

So if $(\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n) \sim$ Orthonormal basis

$$\Rightarrow a_i = \underline{a} \cdot \underline{e}_i \quad \text{or} \quad \underline{a} = \sum_{i=1}^{n} (\underline{a} \cdot \underline{e}_i) \underline{e}_i$$

- **5) A transformation h(·) is said to be Linear if**

$$h(\alpha \underline{a} + \beta \underline{b}) = \alpha h(\underline{a}) + \beta h(\underline{b})$$

$\forall \alpha, \beta \in$ IR and $\forall$a and b.

Definitions and Properties in Vector Space

6) $\underline{a}_1$, $\underline{a}_2$, …, $\underline{a}_n$ are independent if none of these vectors can be written as a linear combination of the others.

7) Triangular Inequality:

For any vectors $\underline{a}$ and $\underline{b}$,

$$\|\underline{a}+\underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$$

With equality iff

$$\underline{a} = k\underline{b} \qquad \text{for some } k \in \mathbb{R}$$

Definitions and Properties in Vector Space

8) Cauchy – Schwartz Inequality**:**

$$\left|\underline{a}\cdot\underline{b}\right| \leq \left\|\underline{a}\right\| \cdot \left\|\underline{b}\right\|$$

with equality if a = kb

9) Pythagorean Relation

if a and b are $\perp$

$\Rightarrow$

$$\left\|\underline{a}+\underline{b}\right\|^2 = \left\|\underline{a}\right\|^2 + \left\|\underline{b}\right\|^2$$

Basis Vectors

- **Let ($\underline{a}_1$, $\underline{a}_2$, …, $\underline{a}_n$) be a set of n vectors. These vectors are** independent **if it is impossible to find constants α_1, α_2, …α_n (<u>not</u>** all zero) **such that**

$$\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \cdots + \alpha_n \underline{a}_n = 0$$

- **In an** n-dim **space, we can have at most n independent vectors**

Signal Space Concepts

- Basic Idea: **Any entity that can be represented by n-tuple is an n-dim Vector** $\Rightarrow$ If a finite-duration signal (Ts) can be represented by n-tuple, then it is a vector.

- **Let $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ be n finite duration signals (Ts)**

- **Consider a finite-duration signal x(t) and suppose that**

$$x(t) = \sum_{i=1}^{n} x_i \varphi_i(t)$$

- **If every signal can be written as above** $\Rightarrow$ **$\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\}$ ~** basis and have n-dim space

$x(t) \Leftrightarrow \mathbf{x} = (x_1, \ldots, x_n)$ with respect to basis $\{\varphi_1(t), \ldots \varphi_n(t)\}$

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- **Define "dot-product" as** $< x(t), y(t) >= \int_0^{T_s} x(t)y^*(t)d$

- **Basis set $\{\varphi_k(t)\}^n$ is an** orthogonal **set if**

$$\int_{-\infty}^{\infty} \varphi_j(t)\varphi_k(t)dt = \begin{cases} 0 & j \neq k \\ k_j & j = k \end{cases}$$

- **If k_j=1 $\forall j \Rightarrow \{\varphi_k(t)\}$ is an** orthonormal **set. In this case,**

$$x_k = \int_{-\infty}^{\infty} x(t)\varphi_k(t)dt$$

$$x(t) = \sum_{i=1}^{n} x_i \varphi_i(t)$$

$$\underline{x} = (x_1, x_2, \ldots, x_n)$$

Key Property

Given a signal space $\mathcal{S} = span\{\varphi_1(t), ...\varphi_n(t)\}$ and a finite duration signal $x(t) \in \mathcal{S}$

(1) Computing Dot-Product

Let $x(t), y(t) \in \mathcal{S}$, $x(t) \Leftrightarrow \mathbf{x} = (x_1, .., x_n)$, $y(t) \Leftrightarrow \mathbf{y} = (y_1, .., y_n)$. For orthonormal basis, $< x(t), y(t) >= \sum_{i=1}^{n} x_i y_i$

(2) Energy of x(t)

$E_s = \int_0^{T_s} |x(t)|^2 dt$ (Time Domain Method)

$E_s = \int_{-\infty}^{\infty} |X(f)|^2 df$ (Frequency Domain Method)

$E_s = \|\mathbf{x}\|^2 =< x(t), x(t) >$ (Geometric Domain Method)

Geometric Domain Representation

- Geometric Domain (Signal Space)
  - Signal could be represented by a point in a space.
  - Step 1: Given a set of M signals, $\{s_1(t), s_2(t), \ldots, s_M(t)\}$ define a D-dim signal space with basis $\{\phi_1(t), \phi_2(t), .., \phi_D(t)\}$.
  - Step 2: Find out the coordinates of each signals by: $s_i(t) \rightarrow \vec{s}_i = (s_{i,1}, s_{i,2}, .., s_{i,D})$

$$s_{ij} = \int_0^{T_s} s_i(t)\phi_j(t)dt$$

- **Question 1) How to find the signal space (basis signals) that contains $\{s_1(t), \ldots, s_M(t)\}$**
- **Question 2) How to find the coordinate of each signal?**

Step 1) Gram-Schmidt Orthogonalization for Vectors

- Given a set of M vectors $\{\vec{x}_1, \ldots, \vec{x}_M\}$, the G-S procedure allows one to find out the "orthonormal basis" $\{\vec{\phi}_1, \ldots, \vec{\phi}_M\}$ of the signal space (with the minimum dimension) to contain all the M vectors.
  - **Step 1:** $\vec{\phi}_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|}$
  - **Step 2:** $\vec{v}_2 = \vec{x}_2 - \langle \vec{\phi}_1, \vec{x}_2 \rangle \vec{\phi}_1, \vec{\phi}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$
  - **Step m:** $\vec{v}_m = \vec{x}_m - \sum_{i=1}^{m-1} \langle \vec{\phi}_i, \vec{x}_m \rangle \vec{\phi}_i, \vec{\phi}_m = \dfrac{\vec{v}_m}{\|\vec{v}_m\|}$

Projection of $\vec{x_m}$ on the current vector space spanned by $\{\vec{\phi}_1, \ldots, \vec{\phi}_{m-1}\}$

The process continues until m=M or $\vec{\phi}_m = \vec{0}$ for some $m \in [1, M]$

- Similarly, for signal space, vector = signal.
  - Given a set of M "signals" (vectors), we can use the same GS procedure to find out the "orthogonal basis" (basis signals) of the signal space (with min dimension) to contain all the M signals.
  - Use the same procedure except with the understanding that $\langle x(t), y(t) \rangle = \int_0^{T_s} x(t) y^*(t) dt$

Summary of GSO

- 1st basis function is a normalized version of 1st signal.
- Remaining basis functions are found by removing portions of signals which are correlated to previous basis functions, and normalizing the result.
- This procedure is repeated until all basis functions are found.

Step 2) Computing the Coordinates

Given the orthonormal basis $\{\phi_1(t), ..., \phi_D(t)\}$ that contains the M finite duration signals $\{s_1(t), .., s_M(t)\}$,

$$s_i(t) \Leftrightarrow \mathbf{s_i} = (s_{i1}, ..., s_{iD})$$

$$\Downarrow$$

$$s_{ij} =< s_i(t), \phi_j(t) >= \int_0^{T_s} s_i(t)\phi_j^*(t)dt$$

Example

- Consider the following two signals that are defined on $[0,T)$

$$s_0(t) = A\cos(2\pi f_c t) \qquad s_1(t) = A\sin(2\pi f_c t)$$

where fc=n/T with n being an integer.

- Find an orthonomal basis set for these two signals.

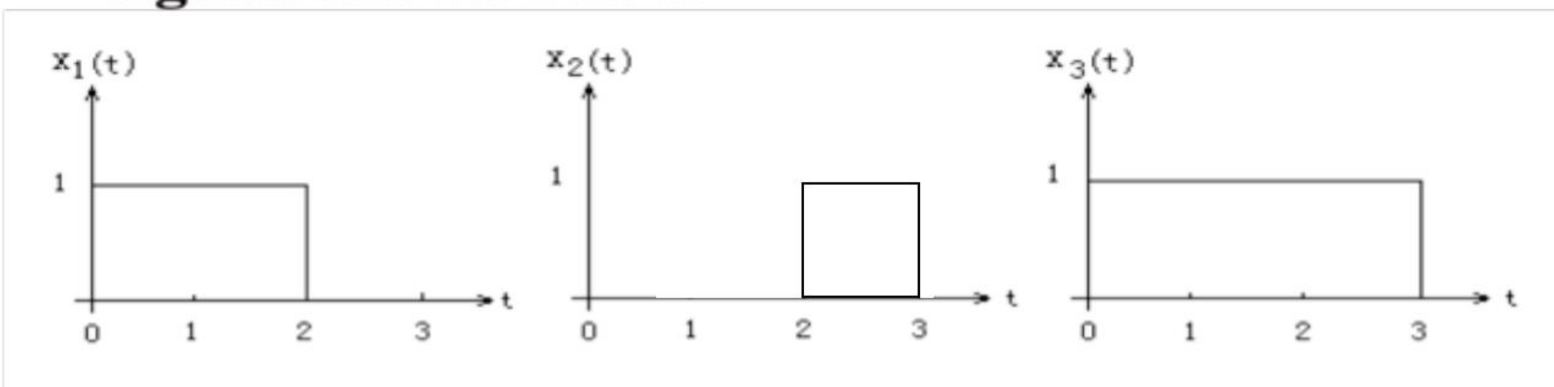
- Repeat the above problem if we now have M-ary signals where

$$s_i(t) = A\cos\left(\omega_c t + \frac{2\pi(m-1)}{M}\right), \quad m = 1,2,\ldots,M$$

- What is the dimension of the resulting signal space ?
- Express $s_i(t)$ in terms of these basis functions and the signal energy E_s

Example

a. Use the Gram-Schmidt procedure to find a set of orthonormal basis functions corresponding to the signals shown below.



b. Express x1, x2, and x3 in terms of the orthonormal basis functions found in Part a.

c. Draw the constellation diagram for the signals