数值分析方法

作业1

Problem 1

解: a. $\because f \in C[a,b]$,且$x_1, x_2 \in [a,b]$,不妨令$x_1 < x_2, f(x_1) \le f(x_2)$,则: $f \in C[x_1, x_2]$

$$\because f \in C[x_1,x_2], f(x_1) \le f(x_2)$$
$$\therefore \frac{f(x_1)+f(x_2)}{2} \in \big[f(x_1), f(x_2)\big]$$

由介值定理可知,$\exists \xi \in [x_1, x_2]$,使得

$$f(\xi) = \frac{f(x_1)+f(x_2)}{2} = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2)$$

成立;
证毕.

b. 同理，不妨令$x_1 < x_2, f(x_1) \le f(x_2)$,有$f \in C[x_1, x_2]$

$$\because \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} - f(x_1) = \frac{c_2}{c_1+c_2}\big(f(x_2) - f(x_1)\big) \ge 0$$
$$\therefore \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} \ge f(x_1)$$

同理，

$$\because \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} - f(x_2) = \frac{c_1}{c_1+c_2}\big(f(x_1) - f(x_2)\big) \le 0$$
$$\therefore \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} \le f(x_2)$$

所以有：

$$f(x_1) \le \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} \le f(x_2)$$

由介值定理可得,$\exists \xi \in [x_1, x_2]$,使得

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2}$$

成立;
证毕.

c. 令$c_1 = 2, c_2 = -1$,则b中:

$$\frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} - f(x_1) = \frac{c_2}{c_1+c_2}\big(f(x_2) - f(x_1)\big) = f(x_1) - f(x_2) \le 0$$

$$\frac{c_1 f(x_1) + c_2 f(x_2)}{c_1+c_2} - f(x_2) = \frac{c_1}{c_1+c_2}\big(f(x_1) - f(x_2)\big) = 2\big(f(x_1) - f(x_2)\big) \le 0$$

不满足介值定理条件.

若进一步,令$f(x)=x^2, x_1=0, x_2=1$,则

$$\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}=2f(x_1)-f(x_2)=-1$$

显然,由于$f(x)=x^2$在$[0,1]$恒为正,即不存在ξ,使得$f(\xi)=-1$,所以当c_1, c_2异号且$c_1+c_2\neq 0$时,b中结论不一定成立.

Problem 2

解: a. 由题意: $\widetilde{f}(x_0)=f(x_0+\epsilon)$

不妨设$\epsilon>0$,则:

$\because f\in C[x_0, x_0+\epsilon], f\in D(x_0, x_0+\epsilon)$

$\therefore$由中值定理:

$$f(x_0)-\widetilde{f}(x_0)=f(x_0)-f(x_0+\epsilon)=f'(\xi)\big(x_0-(x_0+\epsilon)\big)=-f'(\xi)\epsilon$$

其中ξ介于x_0与$x_0+\epsilon$之间.

$\therefore$ 绝对误差

$$|f(x_0)-\widetilde{f}(x_0)|=|f'(\xi)\epsilon|=\epsilon|f'(\xi)|\le\epsilon\max_{x_0<x<x_0+\epsilon}|f'(x)|$$

$\therefore$ 相对误差

$$\frac{|f(x_0)-\widetilde{f}(x_0)|}{|f(x_0)|}=\frac{|f'(\xi)\epsilon|}{|f(x_0)|}\le\frac{\epsilon\max\limits_{x_0<x<x_0+\epsilon}|f'(x)|}{|f(x_0)|}$$

b. $\because \epsilon=5\times10^{-6}, x_0=1$,由a可得:

i. 若$f(x)=e^x$,则$f'(x)=e^x>0$

$$\therefore\max_{x_0<x<x_0+\epsilon}|f'(x)|=f'(x_0+\epsilon)=e^{1+5\times10^{-6}}$$

$\therefore$ 绝对误差:

$$|f(x_0)-\widetilde{f}(x_0)|\le 5\times10^{-6}\times e^{1+5\times10^{-6}}=1.36\times10^{-5}$$

同理,相对误差:

$$\frac{|f(x_0)-\widetilde{f}(x_0)|}{|f(x_0)|}\le\frac{5\times10^{-6}\times e^{1+5\times10^{-6}}}{e}=5.00\times10^{-6}$$

ii. 若$f(x)=\sin x$,则$f'(x)=\cos x$

又$|\cos x|$在$(0,\frac{\pi}{2})$上单减

$$\therefore\max_{x_0<x<x_0+\epsilon}|f'(x)|=|f'(x_0)|=\cos 1$$

$\therefore$ 绝对误差:

$$|f(x_0)-\widetilde{f}(x_0)|\le 5\times10^{-6}\times\cos 1=2.70\times10^{-6}$$

同理,相对误差:

$$\frac{|f(x_0)-\widetilde{f}(x_0)|}{|f(x_0)|}\le\frac{5\times10^{-6}\times\cos 1}{\sin 1}=3.21\times10^{-6}$$

c. 当$\epsilon = (5\times10^{-6})x_0, x_0 = 10$时,有$\epsilon = 5\times10^{-5}$,由b得:

i. 若$f(x) = e^x$,则$f'(x) = e^x > 0$

$$\therefore \max_{x_0<x<x_0+\epsilon} |f'(x)| = f'(x_0+\epsilon) = e^{10+5\times10^{-5}}$$

$\therefore$ 绝对误差:

$$|f(x_0) - \widetilde{f}(x_0)| \le 5\times10^{-5}\times e^{10+5\times10^{-5}} = 1.10$$

同理,相对误差:

$$\frac{|f(x_0) - \widetilde{f}(x_0)|}{|f(x_0)|} \le \frac{5\times10^{-5}\times e^{10+5\times10^{-5}}}{e^{10}} = 5.00\times10^{-5}$$

ii. 若$f(x) = \sin x$,则$f'(x) = \cos x$

又$|\cos x|$在$(3\pi, \frac{7\pi}{2})$上单减

$$\therefore \max_{x_0<x<x_0+\epsilon} |f'(x)| = |f'(x_0)| = -\cos 10$$

$\therefore$ 绝对误差:

$$|f(x_0) - \widetilde{f}(x_0)| \le 5\times10^{-5}\times(-\cos 10) = 4.20\times10^{-5}$$

同理,相对误差:

$$\frac{|f(x_0) - \widetilde{f}(x_0)|}{|f(x_0)|} \le \frac{5\times10^{-5}\times(-\cos 10)}{-\sin 10} = 7.71\times10^{-5}$$

Problem 3

解: a. (i)

$$\frac{4}{5} + \frac{1}{3} = \frac{17}{15}$$

(ii) $\because \frac{4}{5} = 0.8, \frac{1}{3} = 0.\dot{3}$

$\therefore$ 当考虑three-digit chopping时,有:

$$fl(\frac{4}{5}) = 0.800\times10^0 = 0.800$$

$$fl(\frac{1}{3}) = 0.333\times10^0 = 0.333$$

于是

$$fl(\frac{4}{5} + \frac{1}{3}) = 1.13$$

(iii) 由(ii)可得,$\frac{4}{5}$与$\frac{1}{3}$展开的第四位均小于5，故直接截断

$\therefore$ 当考虑three-digit rounding时,有:

$$fl(\frac{4}{5} + \frac{1}{3}) = 1.13$$

(iv) $\because \frac{17}{15} = 1.1\dot{3}$

$\therefore$ 使用three-digit chopping和rounding时的相对误差均为

$$\frac{|1.1\dot{3}-1.13|}{|1.1\dot{3}|}=3\times10^{-3}$$

b. (i)

$$(\frac{1}{3}+\frac{3}{11})-\frac{3}{20}=\frac{301}{660}$$

(ii) $\because \frac{1}{3}=0.3\dot{3}, \frac{3}{11}=0.\dot{2}\dot{7}, \frac{3}{20}=0.15$

$\therefore$ 当考虑three-digit chopping时,有:

$$fl(\frac{1}{3})=0.333$$

$$fl(\frac{3}{11})=0.272$$

$$fl(\frac{3}{20})=0.150$$

于是

$$fl\Big[(\frac{1}{3}+\frac{3}{11})-\frac{3}{20}\Big]=0.455$$

(iii) 由(ii)可得,$\frac{1}{3}$与$\frac{3}{20}$展开的第四位均小于5,故不进位;

$\frac{3}{11}$展开的第四位大于5,故进位,即

$$fl(\frac{3}{11})=(0.272+0.001)\times10^{0}=0.273$$

$\therefore$ 当考虑three-digit rounding时,有:

$$fl\Big[(\frac{1}{3}+\frac{3}{11})-\frac{3}{20}\Big]=0.456$$

(iv) $\because \frac{301}{660}=0.456\dot{0}\dot{6}$

由上述可知,使用three-digit chopping时,相对误差为

$$\frac{|0.456\dot{0}\dot{6}-0.455|}{0.456\dot{0}\dot{6}}=2.33\times10^{-3}$$

使用three-digit chopping时,相对误差为

$$\frac{|0.456\dot{0}\dot{6}-0.456|}{0.456\dot{0}\dot{6}}=1.33\times10^{-4}$$

Problem 4

解: a. $\because$ 当 $x\to0$时,$F_1(x)=L_1+O(x^{\alpha}), F_2(x)=L_2+O(x^{\beta})$

$\therefore \lim\limits_{x\to0}F_1(x)=L_1$,且当$x$充分小时,有$|F_1(x)-L_1|\le k_1|x^{\alpha}|$

$\lim\limits_{x\to0}F_2(x)=L_2$,且当$x$充分小时,有$|F_2(x)-L_2|\le k_2|x^{\beta}|$

其中k_1,k_2均为常数.

又$F(x)=c_1F_1(x)+c_2F_2(x)$

$\therefore \lim\limits_{x\to0}F(x)=c_1L_1+c_2L_2$

$\because |F(x)-c_1L_1-c_2L_2|=|c_1F_1(x)+c_2F_2(x)-c_1L_1-c_2L_2|$

$$
\begin{aligned}
&\le \left|c_1\big(F_1(x)-L_1\big)\right| + \left|c_2\big(F_2(x)-L_2\big)\right| \\
&= c_1|F_1(x)-L_1| + c_2|F_2(x)-L_2| \\
&\le c_1k_1|x^\alpha| + c_2k_2|x^\beta|
\end{aligned}
$$

x充分小.

又$\gamma=\min\{\alpha,\beta\}$

$\therefore$ 当x充分小时,有$|x^\alpha|\le|x^\gamma|,|x^\beta|\le|x^\gamma|$

$\therefore |F(x)-c_1L_1-c_2L_2|\le c_1k_1|x^\alpha|+c_2k_2|x^\beta|\le(c_1k_1+c_2k_2)|x^\gamma|$

由定义可知,当$x\to0$时,有

$$F(x)=c_1L_1+c_2L_2+O(x^\gamma)$$

证毕.

b.由a得:

$$
\begin{aligned}
&\lim_{x\to0}F_1(x)=L_1,\text{且当}x\text{充分小时,有}|F_1(x)-L_1|\le k_1|x^\alpha| \\
&\lim_{x\to0}F_2(x)=L_2,\text{且当}x\text{充分小时,有}|F_2(x)-L_2|\le k_2|x^\beta|
\end{aligned}
$$

其中k_1,k_2均为常数.

$\therefore$ 当x充分小时,有

$$|F_1(c_1x)-L_1|\le k_1\left|(c_1x)^\alpha\right|=k_1c_1^\alpha|x^\alpha|$$

$$|F_2(c_2x)-L_2|\le k_2\left|(c_2x)^\beta\right|=k_2c_2^\beta|x^\beta|$$

$\because G(x)=F_1(c_1x)+F_2(c_2x)$

$\therefore \lim_{x\to0}G(x)=L_1+L_2$

$$
\begin{aligned}
\because |G(x)-L_1-L_2| &= |F_1(c_1x)+F_2(c_2x)-L_1-L_2| \\
&\le |F_1(c_1x)-L_1|+|F_2(c_2x)-L_2| \\
&\le k_1c_1^\alpha|x^\alpha|+k_2c_2^\beta|x^\beta|
\end{aligned}
$$

x充分小.

又$\gamma=\min\{\alpha,\beta\}$

$\therefore$ 当x充分小时,有$|x^\alpha|\le|x^\gamma|,|x^\beta|\le|x^\gamma|$

$\therefore |G(x)-L_1-L_2|\le k_1c_1^\alpha|x^\alpha|+k_2c_2^\beta|x^\beta|\le(k_1c_1^\alpha+k_2c_2^\beta)|x^\gamma|$

由定义可知,当$x\to0$时,有

$$G(x)=L_1+L_2+O(x^\gamma)$$

证毕.

Problem 5

解: a. 代码如下:

```
#include<stdio.h>
#include<math.h>

```

```c
#define e 2.718281828459

int main()
{
        int i=0;
        double a,b,p;
        double TOL;
        scanf("%lf,%lf,%lf",&a,&b,&TOL);
        double fa,fp;
        fa=pow(e,a)-a*a+3*a-2;
        while(b-a>TOL){
                p=a+(b-a)/2;
                fp=pow(e,p)-p*p+3*p-2;
                if (fp==0){
           printf("n = %2d, The zero of the \
function is %.8lf ",++i,p);
                        break;
                }
                if (fp*fa>0){
                        a=p;
                }
                else {
                        b=p;
                }
                printf("n = %2d, the midpoint = \
%.8lf, a-b = %.8lf, f(p) = %12.8lf\n",++i,p,a-b,fp);
        }
 }
```

所得中点为:

n	midpoint(p_n)	$a_n - b_n$	$f(p_n)$
1	0.50000000	-0.50000000	0.89872127
2	0.25000000	-0.25000000	-0.02847458
3	0.37500000	-0.12500000	0.43936641
4	0.31250000	-0.06250000	0.20668169
5	0.28125000	-0.03125000	0.08943320
6	0.26562500	-0.01562500	0.03056423
7	0.25781250	-0.00781250	0.00106637
8	0.25390625	-0.00390625	-0.01369868
9	0.25585938	-0.00195313	-0.00631481
10	0.25683594	-0.00097656	-0.00262388
11	0.25732422	-0.00048828	-0.00077867
12	0.25756836	-0.00024414	0.00014387
13	0.25744629	-0.00012207	-0.00031740
14	0.25750732	-0.00006104	-0.00008676
15	0.25753784	-0.00003052	0.00002855
16	0.25752258	-0.00001526	-0.00002910
17	0.25753021	-0.00000763	-0.00000028

$\therefore$ 近似解为 $p_{17} = 0.25753021$.

b. 同理,代码如下:

```
#include<stdio.h>
#include<math.h>

int main()
{
        int i=0;
        double a,b,p;
        double TOL;
        scanf("%lf,%lf,%lf",&a,&b,&TOL);
        double fa,fp;
        fa=a*cos(a)-2*a*a+3*a-1;
        while(b-a>TOL){
                p=a+(b-a)/2;
                fp=p*cos(p)-2*p*p+3*p-1;
                if (fp==0){
            printf("n = %2d, The zero of the \
function is %.8lf ",++i,p);
                        break;
```

```
                }
                if (fp*fa>0){
                        a=p;
                }
                else {
                        b=p;
                }
                printf("n = %2d, the midpoint = \
%.8lf, a-b = %.8lf, f(p) = %12.8lf\n",++i,p,a-b,fp);
        }
}
```

当$x \in [0.2, 0.3]$时,所得中点为:

n	midpoint(p_n)	$a_n - b_n$	$f(p_n)$
1	0.25000000	-0.05000000	-0.13277189
2	0.27500000	-0.02500000	-0.06158307
3	0.28750000	-0.01250000	-0.02711272
4	0.29375000	-0.00625000	-0.01016096
5	0.29687500	-0.00312500	-0.00175623
6	0.29843750	-0.00156250	0.00242831
7	0.29765625	-0.00078125	0.00033752
8	0.29726562	-0.00039063	-0.00070898
9	0.29746094	-0.00019531	-0.00018564
10	0.29755859	-0.00009766	0.00007597
11	0.29750977	-0.00004883	-0.00005483
12	0.29753418	-0.00002441	0.00001057
13	0.29752197	-0.00001221	-0.00002213
14	0.29752808	-0.00000610	-0.00000578

$\therefore$ 近似解为 $p_{14} = 0.29752808$.

当$x \in [1.2, 1.3]$时,所得中点为:

n	midpoint(p_n)	$a_n - b_n$	$f(p_n)$
1	1.25000000	-0.05000000	0.01915295
2	1.27500000	-0.02500000	-0.05458535
3	1.26250000	-0.01250000	-0.01722489
4	1.25625000	-0.00625000	0.00108689
5	1.25937500	-0.00312500	-0.00803829
6	1.25781250	-0.00156250	-0.00346802
7	1.25703125	-0.00078125	-0.00118864
8	1.25664063	-0.00039063	-0.00005040
9	1.25644531	-0.00019531	0.00051837
10	1.25654297	-0.00009766	0.00023402
11	1.25659180	-0.00004883	0.00009182
12	1.25661621	-0.00002441	0.00002071
13	1.25662842	-0.00001221	-0.00001484
14	1.25662231	-0.00000610	0.00000294

$\therefore$ 近似解为 $p_{14} = 1.25662231$.

Problem 6

解:　a. 代码如下:

```c
#include<stdio.h>
#include<math.h>

#define pi 3.1415926535898

int main()
{
        int i=0;
        double p,p0,delta=1;
        double TOL;
        scanf("%lf,%lf",&p0,&TOL);
        while(delta>TOL){
                p=(2*sin(pi*p0)+4*p0)/3;
                delta=fabs(p-p0);
                printf("n = %2d, p = %10.8lf, \
|p-p0| = %.8lf\n",++i,p,delta);
                p0=p;
        }
 }
```

$\because p_0 = 1$,可得:

n	p_n	$\|p_n - p_{n-1}\|$
1	1.33333333	0.33333333
2	1.20042751	0.13290582
3	1.20798916	0.00756165

$\therefore$ 近似解为 $p_3 = 1.20798916$.

b. 易知该函数在$(0,1)$上存在一零点,令

$$g(x) = \sqrt{\frac{e^x}{3}}$$

则当$x \in (0,1)$时,其将自身映射到$(0,1)$,且

$$|g'(x)| = \left|\frac{\sqrt{3e^x}}{6}\right| < 1$$

满足收敛条件.

代码如下:

```c
#include<stdio.h>
#include<math.h>

#define e 2.718281828459

int main()
{
        int i=0;
        double p,p0,delta=1;
        double TOL;
        scanf("%lf,%lf",&p0,&TOL);
        while(delta>TOL){
                p=sqrt(pow(e,p0)/3);
                delta=fabs(p-p0);
                printf("n = %2d, p = %.8lf, |p-p0|\
 = %.8lf\n",++i,p,delta);
                p0=p;
        }
}
```

令$p_0 = 1$,可得:

n	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.95188967	0.04811033
2	0.92926502	0.02262465
3	0.91881210	0.01045291
4	0.91402250	0.00478960

$\therefore$ 近似解为 $p_4 = 0.91402250$.

同理,在$(3,4)$上,令

$$g(x) = x - \frac{3x^2 - e^x}{6x - e^x}$$

则代码:

```c
#include<stdio.h>
#include<math.h>

#define e 2.718281828459

int main()
{
        int i=0;
        double p,p0,delta=1;
        double TOL;
        scanf("%lf,%lf",&p0,&TOL);
        while(delta>TOL){
                p=p0-(3*p0*p0-pow(e,p0))/(6*p0-\
                  pow(e,p0));
                delta=fabs(p-p0);
                printf("n = %2d, p = %.8lf, |p-p0|\
 = %.8lf\n",++i,p,delta);
                p0=p;
        }
 }
```

令$p_0 = 4$,可得:

n	p_n	$\lvert p_n - p_{n-1}\rvert$
1	3.78436115	0.21563885
2	3.73537938	0.04898177
3	3.73308390	0.00229548

$\therefore$ 近似解为 $p_3 = 3.73308390$.

综上,该方程的两近似解分别为0.91402250、3.73308390.

Problem 7

解: $\because g \in C^1[a,b]$,则: $g' \in C^1[a,b]$

又$p \in (a,b)$

$\therefore g'$在$x=p$点连续

$\because |g'(p)| > 1$

由连续的性质可知: $\exists \delta > 0$,当$0 < |x-p| < \delta$时,有$|g'(x)| > 1$

设p_0是迭代的初始点, $p_0 \neq p$,且$0 < |p_0 - p| < \delta$, $p_1 = g(p_0)$

$\therefore$由中值定理:

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|$$

其中ξ介于p_0与p之间,即$0 < |\xi - p| < \delta$.

$\therefore |g'(\xi)| > 1$

$\therefore |p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$

由此可见,当$p_0 \neq p$时,后续迭代得到的p_1会远离p点,不动点迭代将不会收敛.