Confidence interval for a mean

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Lemma

Let $X_1, X_2, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$, independent, with **known variance σ^2**. The following interval

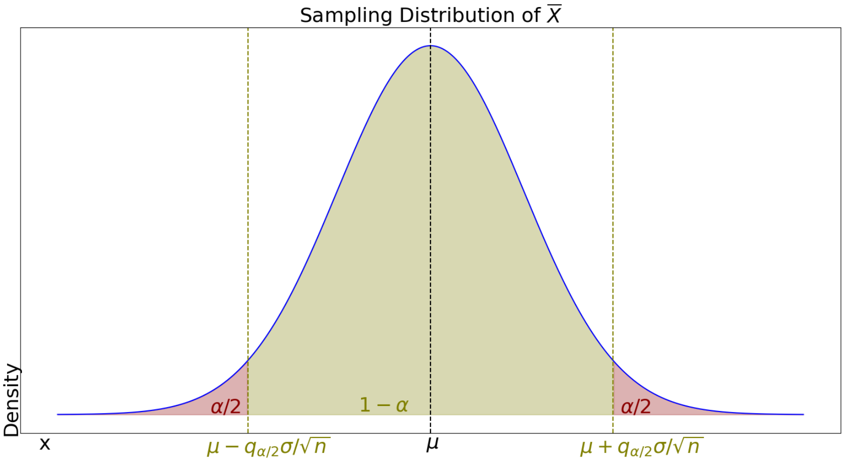
$$\left[\overline{X} - \frac{q_{\alpha/2}\sigma}{\sqrt{n}},\ \overline{X} + \frac{q_{\alpha/2}\sigma}{\sqrt{n}}\right]$$

where $0 < \alpha < 1$, and $q_{\alpha/2}$ is the $(1-\alpha/2)$-**quantile** of the standard normal distribution, is a $(1-\alpha)$-confidence interval for the **unknown mean μ**.

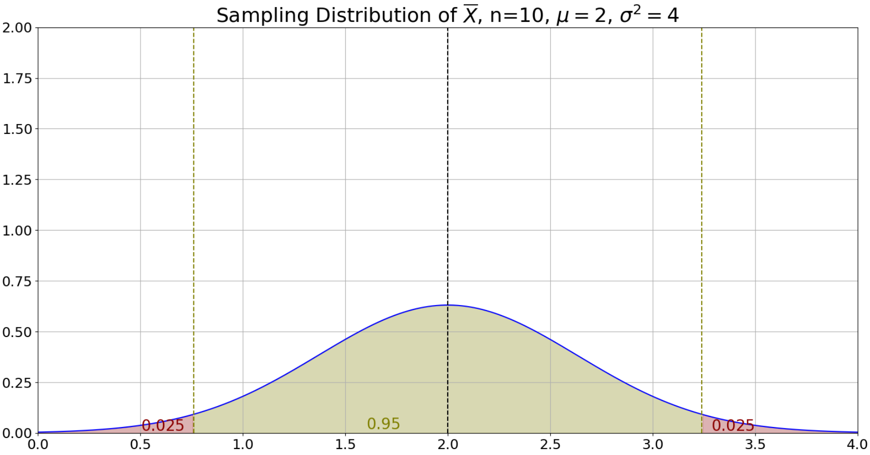
Recall: $q_{\alpha/2}$ is defined by the equation:

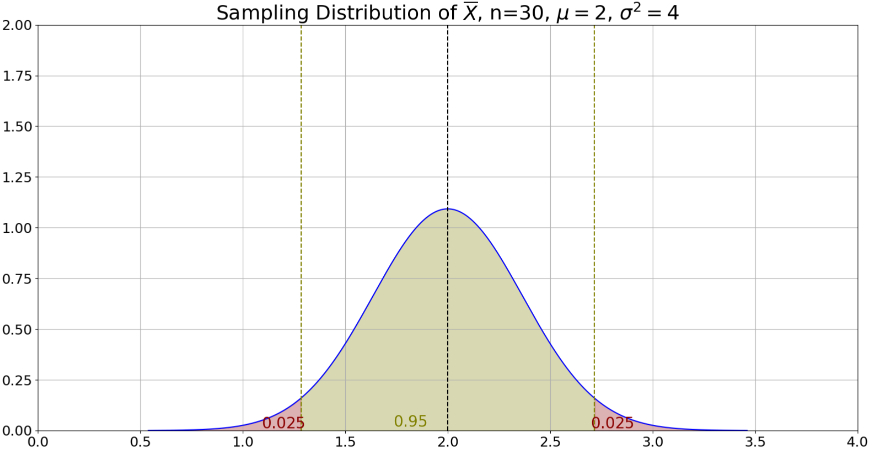
$$\mathbb{P}(Z \leq q_{\alpha/2}) = 1 - \frac{\alpha}{2}$$

where $Z \sim \mathcal{N}(0,1)$.

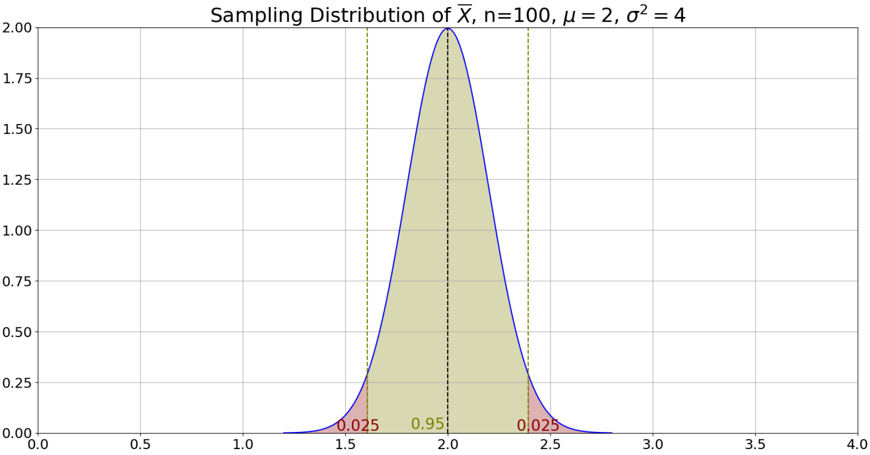


Sampling Distribution of $\overline{X}$
Density
x
$\alpha/2$
$1-\alpha$
$\alpha/2$
$\mu - q_{\alpha/2}\sigma/\sqrt{n}$
μ
$\mu + q_{\alpha/2}\sigma/\sqrt{n}$


Sampling Distribution of $\overline{X}$, n=10, $\mu = 2$, $\sigma^2 = 4$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.025
0.95
0.025



Sampling Distribution of $\overline{X}$, n=30, $\mu = 2$, $\sigma^2 = 4$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.025
0.95
0.025


Sampling Distribution of $\overline{X}$, n=100, $\mu = 2$, $\sigma^2 = 4$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
0.025
0.95
0.025

The Pivotal Quantity Method in a Nutshell

The **Pivotal Quantity Method** consists of finding a function of the observed data and the unknown parameter to estimate, which has a **known distribution** independent of the parameter.

Combination of Independent Normal Random Variables

Proposition: For $X_1, X_2, \ldots, X_n$ independent normal variables and deterministic variables $a_1, a_2, \ldots, a_n$:

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), \quad i = 1, \ldots, n,$$

then:

$$\sum_{i=1}^{n} a_i X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

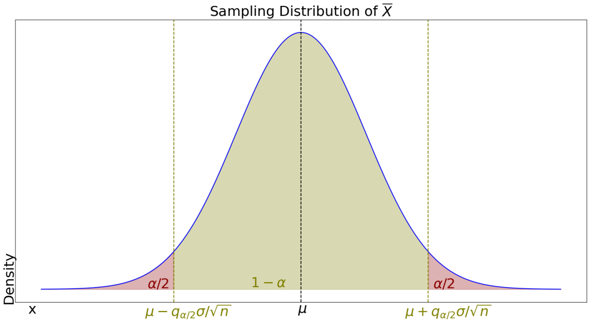
Remark: A deterministic variable c can be viewed as:

$$c \sim \mathcal{N}(c, 0).$$

Finding a Pivot

- **Sample Mean:**

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$



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$$\overline{X} - \mu \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)$$ **(It is a pivot.)**

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- **Normalisation:**

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$ **(It's also a pivot!)**

Construction of the Confidence Interval

We want to construct an interval that contains the real mean μ with probability $1 - \alpha$. Since the normal distribution is symmetric, we may construct an interval centered on $\overline{X}$.

$$\mathbb{P}\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \in [-q_{\alpha/2}, q_{\alpha/2}]\right) = 1 - \alpha \quad \textbf{(distribution of the pivot)}$$

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$$\Leftrightarrow \mathbb{P}\left(\frac{\mu}{\sigma/\sqrt{n}} \in \left[\frac{\overline{X}}{\sigma/\sqrt{n}} - q_{\alpha/2}, \frac{\overline{X}}{\sigma/\sqrt{n}} + q_{\alpha/2}\right]\right) = 1-\alpha$$

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$$\Leftrightarrow \mathbb{P}\left(\mu \in \left[\overline{\mathbf{X}} - \frac{\mathbf{q}_{\alpha/2}\sigma}{\sqrt{\mathbf{n}}}, \overline{\mathbf{X}} + \frac{\mathbf{q}_{\alpha/2}\sigma}{\sqrt{\mathbf{n}}}\right]\right) = 1-\alpha$$

Numerical Example

We have observed $n = 100$ independent realisations, $x_1, x_2, \cdots, x_{100}$ of a $\mathcal{N}(\mu, 4)$ distribution, and the sample mean is $\overline{x} = 1.72$. Give a 95%-confidence interval for the true mean μ.

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- Finding the quantile $q_{\alpha/2}$ in the $\mathcal{N}(0, 1)$ table:

Z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.8	0.964	0.965	0.966	0.966	0.967	0.968	0.969	0.969	0.970	0.971
1.9	0.971	0.972	0.973	0.973	0.974	0.974	**0.975**	0.976	0.976	0.977
2.0	0.977	0.978	0.978	0.979	0.979	0.980	0.980	0.981	0.981	0.982
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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- $\alpha = 1 - 0.95 \Leftrightarrow 1 - \alpha/2 = 1 - 0.05/2 = 0.975$
- $q_{\alpha/2} = 1.96$
- Conclusion:

$$\left[\overline{x} - \frac{q_{\alpha/2}\sigma}{\sqrt{n}},\ \overline{x} + \frac{q_{\alpha/2}\sigma}{\sqrt{n}}\right] = \left[1.72 - 1.96\frac{2}{10}, 1.72 + 1.96\frac{2}{10}\right]$$
$$= [1.328, 2.112]$$