A Small Theorem on the Interval Approach Criterion for Systems of Linear Equations.

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1 Motivation

The development of Unums encourages reformulation of computational mathematics to approach numerical problems, keeping in mind traditional performance metrics, such as speed and energy consumption, while still yielding mathematically accurate solutions. Traditional numerical systems, such as IEEE floating points, abandon accuracy for performance. Symbolic systems incur high calculation costs and still may encounter situations without closed form solutions. Interval arithmetic can have exploding uncertainty under certain operations without clear guidance on how to resolve this. Unums, however, provide a natural solution to the exploding uncertainty, by way of methods such as the Ubox method.

For solving systems of linear equations, the naive solution is to first perform a gauss-jordan elimination and then whittle down the solution set using Uboxes. This is inefficient and cannot distinguish between a unit cell that contains a solution and a unit cell that merely happens to be at the intersection of all of the hyperplanes. Moreover, the search time proceeds with time O(something large)

2 The Interval Approach (uSlice) method

To address this issue, I developed a strategy called the interval approach method. The procedure is as follows: Select a dimension. Partition a superset of the solved interval in this dimension with equal width (simplest to choose a power of two), called "uslices". Apply the uslices as values to the system of linear equations and measure the maximum error from the constant vector of the linear equations. Because the equations are linear, conceptually the "outside" slices should have greater error than the "inside" slices, with a few exceptions that are otherwise easily solved (e.g. an irrelevant dimension that trivially reduces to a smaller system of linear equations).

In practice, however, this interval approach method fails in the event that a bounding interval is *asymmetric* about the solution. This document describes the derivation of an "Interval Approach Criterion" which guarantees that the uslices with lowest error bound the solution in the exact case and must contain the solution in the inexact case.

3 Derivation of the Interval Approach Criterion: 2-dimensions, positive matrix, exact solution

Given a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with constant vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or written as $\left(\begin{array}{cc|c} A & B & v_1 \\ C & D & v_2 \end{array}\right)$. The solution $s = \begin{pmatrix} x \\ y \end{pmatrix}$ and without loss of generality over variable order y is bounded by an asymmetric open interval $y \in (y - \delta_l, y + \delta_h)$, and we examine the following intervals for x: $v_{far} = (x - 2\epsilon, x - \epsilon)$ and $v_{near} = (x - \epsilon, x)$

The first equation evaluates as follows:

$(M \cdot v_{far})_1 = A(x - 2\epsilon, x - \epsilon) + B(y - \delta_l, y + \delta_h)$

And:

$(M \cdot v_{near})_1 = A(x - \epsilon, x) + B(y - \delta_l, y + \delta_h)$

Subtracting $v_1 = Ax + By$ from these intervals results in the following error intervals:
$(-2\epsilon A - B\delta_l, -\epsilon A + B\delta_h)$
And:
$(-\epsilon A - B\delta_l, B\delta_h)$

Since the outer values of the intervals are necessarily extrema, the "maximal" error for these uSlices is attained at the absolute value. The assertion of interval approach is thus satisfied when $max(2\epsilon A + B\delta_l, |-\epsilon A + B\delta_h|) > max(\epsilon A + B\delta_l, B\delta_h)$. Since always $\epsilon A + B\delta_l < 2\epsilon A + B\delta_l$, this value is not in contention. We must only check that $B\delta_h$ is not greater than either critical value. By simple rearrangement,

$$\epsilon > \frac{(B\delta_h - B\delta_l)}{2A}$$

satisfies this condition.