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1. What is logic? Differentiate propositional logic and predicate logic with suitable example.

⇒ Logic is the tool or language for reasoning about the truth and false of statement or simply it is the generation of idea for solving problem.

The differents between propositional logic and predicate logic are :-

Propositional Logic	Predicate logic
i) Logic that deals with propositions are called propositional logic.	i) Predicate logic are those which are defined using some predicate.
ii) It is associated with finite models.	ii) It is related to both finite and infinite.
iii) It is decidable	iii) It is undecidable
iv) In propositional logic, statements are expressed as a whole or as a combination of statements.	iv) In predicate logic, statements have a specific format comprising predicates and terms.
v) E.g:- - 2+10=12 (T) - Pokhara lies in Kathmandu district (F)	v) E.g:- Let E(x,y) denote "x=y" Let x(a,b,c) denote "a+b+c=0"

2. What do you mean by logical equivalence? Show that implification and its contrapositive are logically equivalent.

⇒ If the both compound proposition p and q have the identical values in the truth table, then the proposition are said to be logically equivalent.

Implification and its contrapositive are logically equivalent are shown below:-

p	q	$\neg p$	$\neg q$	$p \to q$	~~q~~ $\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

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3. Differentiate between universally quantified and existentially quantified statements. What is the truth value of the statement, $x^2-1>0$ for every real number x.

⇒ The different between universally quantified and existentially quantified statements are :-

Universally Quantified statements	Existentially quantified statement
i) Universal quantifier states that the statements within its scope are true for every value of the specific variable.	i) Existential quantifier states that the statements within its scope are true for some values of the specific variable.
ii) It is denoted by the symbol $\forall$	ii) It is denoted by the symbol $\exists$.
iii) $\forall x P(x)$ is read as for every value of x, $P(x)$ is true.	iii) $\exists x P(x)$ is read as for some values of x. $P(x)$ is true.
E.g:- "Man is mortal" can be transformed into the propositional form $\forall x P(x)$ where $P(x)$ is the predicate which denotes x is mortal and the universe of discourse is all men.	E.g:- "Some people are dishonest" can be transformed into the propositional form $\exists x P(x)$ where $P(x)$ is predicate which denotes x is dishonest and the universe of discourse is some people.

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Sol^n.

$x^2 - 1 > 0$ for $x \in R$

$P(x): x^2 - 1 > 0$,

$x = 1$, $0 > 0$ (false)

$x = 2$, $2^2 - 1 > 0$

$3 > 0$ (True)

$x = 3$, $3^2 - 1 > 0$

$8 > 0$ (True)

The given statement is not true for all the values i.e $x = 1$ it is false.

So, $\forall x P(x)$ is false.

→ Here, $x = 1$ is a counter example that makes the universally quantified statement false.

4. Define the term tautology. Show that $p \leftrightarrow q$ is logically equivalent to $(p \wedge q) \vee (\neg p \wedge \neg q)$

⇒

⇒ The compound statement is said to be tautology if all the values of the truth table is true, whatever the constituent proposition may holds.

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p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

Hence, $p \leftrightarrow q$ is logically equivalent to $(p \wedge q) \vee (\neg p \wedge \neg q)$.

5. State the rules of inference for propositional logic and quantified statements.

⇒ Rules of inference for propositional logic are:-

(i) Modus Ponens :-

wherever the proposition p and $p \to q$ are true, then we confirm that q is true.

i.e.
$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

(ii) Modus Tollens :-

when two proposition $p \to q$ and $\neg q$ is true then $\neg p$ is also true.

i.e.
$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

(iii) Hypothetical syllogism

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

(iv) Disjunctive syllogism

$$\begin{array}{c} p \lor q \\ \neg p \\ \hline \therefore q \end{array} \qquad \begin{array}{c} p \lor q \\ \neg q \\ \hline \therefore p \end{array}$$

(v) Additive rule

$$\begin{array}{c} p \\ \hline \therefore p \lor q \end{array}$$

(vi) simplification

$$\begin{array}{c} p \land q \\ \hline \therefore p, q \end{array}$$

(vii) Conjunction

$$\begin{array}{c} p \\ q \\ \hline p \land q \end{array}$$

(viii) Resolution.

$$\begin{array}{c} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

Rules of inference for Quantified statement

① Universal Instantiation

$\forall x P(x)$

$\therefore P(d)$

where 'd' is the value from universe of discourse.

E.g:- All birds can fly.

(ii) Universal Generalization

P(d) is true where 'd' is the universe of discourse, then.

$\forall x P(x)$ is true.

(iii) Existential Instantiation

$\exists x P(x)$

$\therefore P(d)$ is true.

(iv) Existential Generalization

$P(d) \in D$, D is universe of discourse.

$\therefore \exists x P(x)$

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6. Give an argument using the rules of inference to show that the conclusion follows from the hypothensis.

a. Everyone loves either Microsoft or Apple. Lynn does not love Microsoft. show that the conclusion. Lynn loves apple follows from the hypothesis.

b. If clinton does not live in France, then he doesn't speak french. clinton does not drive a Datsun. If clinton lives in France, then he rides a motorcycle. Either clinton speaks french or he drives a Datsun. leads to the conclusion clinton ride a motorcycle.

c. Everyone in the class has a graphic calculator. Everyone who has a graphic calculator understands the trignometric functions. leads to conclusion Ralphie, who is in the class, understands the trignometric functions.

$\Rightarrow$

a) soln

Defining prediction

$M(x)$: x loves Microsoft

$A(x)$: x loves Apple

Hypothesis

(i) $\forall x[M(x) \vee A(x)]$

(ii) $\neg M(Lynn)$

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conclusion
A(Lynn)

Proof

steps	Reasons
(i) $\forall x\,[M(x) \vee A(x)]$	Given hypothesis
(ii) $M(Lynn) \vee A(Lynn)$	Using universal instantiation hypothesis (i) $P(d) \in D$, d: lynn
(iii) $\neg M(Lynn)$	Given hypothesis
(iv) $A(Lynn)$	Applying disjunctive syllogism on hypothesis (ii) & (iii)

b) soln

Identifying the individual sentences.

a. Clinton live in france.

b. He speak french

c. Clinton drive a Datsun

d. He rides a motorcycle

P.T.O

$p \vee q$
$\neg p$
$\therefore q$

Writing the given statements into propositional logic statements.

Hypothesis

(i) $\neg a \to \neg b$ $\qquad b \to a$
(ii) $\neg c$
(iii) $a \to d$ $\qquad b \vee c$
(iv) $b \vee c$ $\qquad \neg c$
$\qquad b$

Conclusion

d

Proof :-

Steps	Reasons
(i) $\neg a \to \neg b$	(i) Given hypothesis
(ii) $b \to a$	(ii) Applying contrapositive in (i)
(iii) $b \vee c$	(iii) Given hypothesis
(iv) $\neg c$	(iv) Given hypothesis
(v) b	(v) Using Disjunctive Syllogism in (iv)
(vi) $a \to d$	(vi) Given hypothesis
(vii) $b \to d$	(vii) Using hypothetical in (ii) and (vi)
(viii) d	(viii) Using modus ponens in (v) and (vii)

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c. Sol

~~Identifying the individual~~ sentences
Defining prediction
$S(x)$: x is student of this class
$G(x)$: x has graphing calculator
$T(x)$: x understands trignometric function.

Hypothesis
$\forall x\, S(x) \rightarrow G(x)$
$\forall x\, G(x) \rightarrow T(x)$.
~~S(Ralphie)~~ S(Ralphie)
Conclusion
~~Proo~~ T(Ralphie)

Proof:-

Steps	Reasons
(i) $\forall x\, S(x) \rightarrow G(x)$	Given hypothesis
(ii) S(Ralphie) → G(Ralphie)	Using universal instantiation on hypothesis (i)
(iii) $\forall x\, G(x) \rightarrow T(x)$	Given hypothesis
(iv) G(Ralphie) → T(Ralphie)	Using universal instantiation on hypothesis (iii)
(v) S(Ralphie) → T(Ralphie)	Using hypothetical on (ii) & (iv)
(vi) S(Ralphie)	Given hypothesis
(vii) T(Ralphie)	Using modus Ponens on (v) & (vi)

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7. Prove the validity of the following argument "If I get the job and work hard. Then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get job or I will not work hard.

⇒ Soln

Let P: I get the job
Q: I work hard
R: I will get promoted
S: I will be happy

Then the given premises are

(i) $(P \wedge Q) \rightarrow R$
(ii) $R \rightarrow S$
(iii) $\neg S$

Conclusion

$\neg P \vee \neg Q$

Proof

steps	Reasons
(i) $(P \wedge Q) \rightarrow R$	(i) Given hypothesis
(ii) $R \rightarrow S$	(ii) Given hypothesis
(iii) $(P \wedge Q) \rightarrow S$	(iii) Using hypothetical syllogism on (i) & (ii)
(iv) $\neg S \rightarrow \neg(P \wedge Q)$	(iv) Applying contrapositive on hypothesis (iii)

P.T.O

(v) $\neg s$ (v) Given hypothesis

(vi) $\neg(P \wedge Q)$ (vi) Using modus ponens on hypo (iv) & (v)

(vii) $\neg P \vee \neg Q$ (vii) Using De-Morgan's theorem in (vi)

8. Show that $n^2 > 2n+1$ for $n >= 3$ by mathematical induction.

Sol^n

Basic step

For $n=3$

$P(n) = n^2 > 2n+1$

$3^2 > 2\times3+1$

$9 > 7$

$P(3)$ is true

For inductive step

We assume that $P(k)$ is true for any arbitary k

$P(k)$ is true

i.e $k^2 > 2k+1$

Now, we have to proves for $k+1$

$P(k+1) = (k+1)^2 > 2(k+1)+1$

$k^2+\cancel{2k}+1 > \cancel{2k}+2+1$

$k^2+1 > 3$

$k^2 > 2$ (which is true, for $k>=3$)

9. Using mathematical induction prove that the statement $6.7^n - 2.3^n$ is divisible by 4 for $n = 1, 2, 3 ..$

Sol^n

Basic step, for $n=1$

$P(n) = 6.7^n - 2.3^n$

$6.7^1 - 2.3^1$

$= 36$ which is divisible by 4

$P(1)$ is true.

For inductive step

We assume that $P(K)$ is true for any arbitrary K

$P(K)$ is true

i.e. $6.7^K - 2.3^K$

Now, we try to proves for $K+1$.

$P(K+1) = 6.7^{K+1} - 2.3^{K+1}$

$= 6.7^K.7 - 2.3^K.3$

$= 6.7^K(4+3) - 2.3^K.3$

$= 6.4.7^K + 3.6.7^K - 2.3^K.3$

$= 4.6.7^K + 3(6.7^K - 2.3^K)$

Since $4.6.7^K$ is divisible by 4, from our assumption $(6.7^K - 2.3^K)$ is also divisible by 4

Two individual number is also divisible by 4.

So, we can say that $P(n) = 6.7^n - 2.3^n$ is divisible by 4 using mathematical induction.

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10. Use mathematical induction to show that if $r \neq 1$ then $a + ar^1 + ar^2 \dots + ar^n = a(r^{n+1} - 1)/r-1$

<u>soln</u>

<u>Basic step</u>

for $n = 1$

$L.H.S = a$

$R.H.S = a\dfrac{(r^1 - 1)}{r-1} = a$

$\therefore$ P(1) is true.

<u>For Inductive step</u>

we assume that P(K) is true for any arbitrary K.

$$P(K) = a + ar + ar^2 + \dots + ar^{K-1} = \frac{a(r^K - 1)}{r-1}$$

Now, we try to proves for K+1

$$P(K+1) = a + ar + ar^2 + \dots + ar^{K-1} + ar^K = \frac{a(r^{K+1} - 1)}{r-1}$$

$$= \frac{a(r^K - 1)}{r-1} + ar^K$$

$$= \frac{\cancel{a(r^K - 1) + ar^{K+1}}(r-1)}{r-1}$$

$$= \cancel{a[r^K - 1 + r^{K+2} - r^{K+1}]}$$

$$= \frac{a[r^K - 1 + r^K(r-1)]}{r-1}$$

$$= \frac{a[r^K - 1 + r^{K+1} - r^K]}{r-1}$$

$$= \frac{a(r^{K+1} - 1)}{r-1}$$

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Therefore, the statement if $r \neq 1$ then $a + ar^1 + ar^2 + ar^n = a(r^n - 1)/r-1$ is valid for mathematical induction.

11. Use mathematical induction to show that for all $n >= 1$

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Sol^n

Basic step

for $n = 1$

$L.H.S = 1(1+1) = 2$

$R.H.S = \frac{1(1+1)(1+2)}{3} = 2$

$P(1)$ is true

For inductive step

We assume that $P(K)$ is true for any arbitary K

$P(K)$ is true.

$\therefore P(K) = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + K(K+1) = \frac{K(K+1)(K+2)}{3}$

Now, we try to proves for $K+1$

$P(K+1) = 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots K(K+1) + (K+1)\{(K+1)+1\} = \frac{(K+1)\{(K+1)+1\}\{(K+1)+\ldots}{3}$

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$$\Rightarrow \frac{K(K+1)(K+2)}{3} + (K+1)\{(K+1)+1\}$$

$$= \frac{K(K+1)(K+2) + 3(K+1)\{K+2)}{3}$$

$$= \frac{(K+1)(K+2)\{K+3)}{3}$$

$$= \frac{(K+1)\{(K+1)+1\}\{(K+1)+2)}{3}$$

Hence, $1\times2 + 2\times3 + 3\times4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

is valid for mathematical induction

12. Describe direct proof and indirect proof techniques. Prove that the product of two odd numbers is an even number.

⇒ A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if p then q" or "p implies q" which we can write as $p \to q$. The method of the proof is to takes an original statement p, which we assume to be true, and use it to show directly that another statement q is true. So, a direct proof has following step

→ Assume the statement is true

→ Use what we know about p and other facts

P.T.O

as necessary to deduce that another statement q is true. ~~so, a direct proof ho that~~ is show $p \rightarrow q$ is true.

Indirect proof

For certain case, the direct proof may not be appropriate. For e.g if $(3n+2)$ is odd, then n is odd. Here using the direct proof we may not reach the conclusion such problem is considered as "dead end" of proof. To overcome such problem, we can use indirect proof.

Two types of indirect proof are:

a) proof by contradiction
b) proof by contrapositive

When we use an indirect proof to prove a theory, we follow three steps.

i) start by assuming that the theory is false.
ii) Next, we go about our proof and eventually run into a contradiction, that is something that doesn't make sense.
iii) The contradiction from step 2 proves our assumption of the theory being false not to be the case, so the theory must be true

proof:-

Let n and m be two odd ~~int~~ number.

By definition of odd, we have

$n = 2a+1$ and

$m = 2b+1$

Now,

$$nm = (2a+1)(2b+1)$$
$$= 4ab + 2a + 2b + 1$$
$$= 2(2ab + a + b) + 1$$
$$= 2K + 1$$

where, $K = 2ab + a + b$ is an integer.

Therefore, by definition of odd, we have shown that the product of two odd number is odd.

14. Give a direct proof of the theorem " For all m and n. if m is odd and n is even then m+n is odd."

⇒ Since m is odd and n is even

∴ By definition of odd,

$m = 2a+1$

By definition of even,

$n = 2a$

Now,

$$m+n = 2a+1+2a$$
$$m+n = 4a+1 \quad \text{(which is odd)}$$

Hence, we have shown that if m is odd and n is even then m+n is odd.

13. Differentiate between proof by contradiction and proof by contrapositive with suitable example.

⇒ The different between proof by contradiction and proof by contrapositive are :-

Proof by contradiction	Proof by contrapositive
i) A proof by contradiction is a method of proving a statement by assuming the hypothesis to be true and conclusion to be false and then deriving a contradiction	i) A proof by contraposition is based on the fact that the statement 'If A, then B is logically equivalent to the statement 'if not B, then not A'. In other words, we have, $(A \to B) \leftrightarrow (\neg B \to \neg A)$
ii) In proof by contradiction we show that $A \wedge \neg B \to C \wedge \neg C$, for some C (C may not necessary be A)	ii) In proof by contrapositive we show that $\neg B \to \neg A$.

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15. Give a proof by contradiction that if 40 coins are distributed among nine bags so that each bag contains at least coin, at least two bags contain same number of coins.

⇒ Suppose that the conclusion is false, that is that no two bags contains the same number of coins. Suppose that we rearrange the bags in increasing order of the number of coins that they contain. Then the first bag contains at least one ~~con~~ coin, the second bag contains at least two coins and so on Thus, the total number of coins is at least $1+2+3+\cdots+9=45$. This contradicts the hypothesis that there are 40 coins. Hence, the given hypothesis is true.

16. Using indirect proof show that if $3n+2$ is odd then n is odd.

⇒ soln

If $3n+2$ is odd then n is odd

p : $3n+2$ is odd

q : n is odd

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

we assume that negation of conclusion is true i.e. n is even

Now, we try to prove $\neg p$ is also true
i.e. $(3n+2)$ is ~~true~~ even.

By definition of even number
$n = 2k$

Now,
$3(2k) + 2$
$= (6k+2)$ which is even

$\therefore (3n+2)$ is even.
i.e negation of hypothesis is also true.
Here, the negation of ~~hypothesis is als~~ conclusion leads to negation of hypothesis. So, we conclude that when $(3n+2)$ is odd then 'n' is odd by using proof by contrapositive.

17. Rewrite the following arguments using quantifier, variable and predicate symbol.

a. All birds can fly

$\Rightarrow$

$B(x)$: x is bird

$F(x)$: x can fly

$\forall x\, [B(x) \rightarrow F(x)]$

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b. No all birds can fly.

⇒

$B(x)$: x is bird

$F(x)$: x can fly

$\exists x\,[B(x) \wedge F(x)]$

c. Some men are genius.

⇒

$M(x)$: x are men

$F(x)$: x are genius

$\exists x\,[M(x) \wedge F(x)]$

d. Some numbers are not rational.

⇒

$R(x)$: x ~~numbers~~ are ~~rational~~ numbers

$F(x)$: x are rational

$\exists x\,[R(x) \wedge F(x)]$

e. Some real numbers have square three.

⇒

$R(x)$: x is reals numbers

$S(x)$: x have square three

$\exists x\,[R(x) \wedge S(x)]$

f. Every student either can speak English or know programming Java.

~~x S(x) : x is a student~~
E(x) : x student can speak English
J(x) : x knows Java

$\therefore \forall x [E(x) \vee J(x)]$

g. There is a student who likes MFCS but not Applied Mechanics.

~~x S(x) : x is a student~~
M(x) : x likes MFCS
A(x) : x like applied mechanics

$\therefore$

$\exists x [M(x) \wedge \neg A(x)]$.

Tutorial 2

1. What are recurrence relations. Model any real-life problem using recurrence relation.

⇒ Let $a_0, a_1, a_2, \ldots, a_n$ be the n terms of a sequence a_n. then a_n is said ~~is~~ to be recurrence relation if a_n can be expressed by an equation in terms of it's previous elements.

For e.g:- $a_n = 5a_{n-1} + 6a_{n-2}$ is a recurrence relation.

Let's take an example,

A person invests \$1000 at 12 percent interest compound annually. If A_n represents the amount ~~the~~ at the end of n years, find a recurrence relation and initial conditions that define the sequence $\{A_n\}$

Soln

The recurrence relation & initial condition are followed.

$$A_n = A_{n-1} + 12\% * A_{n-1}$$

$$A_n = 1.12 \times A_{n-1}$$

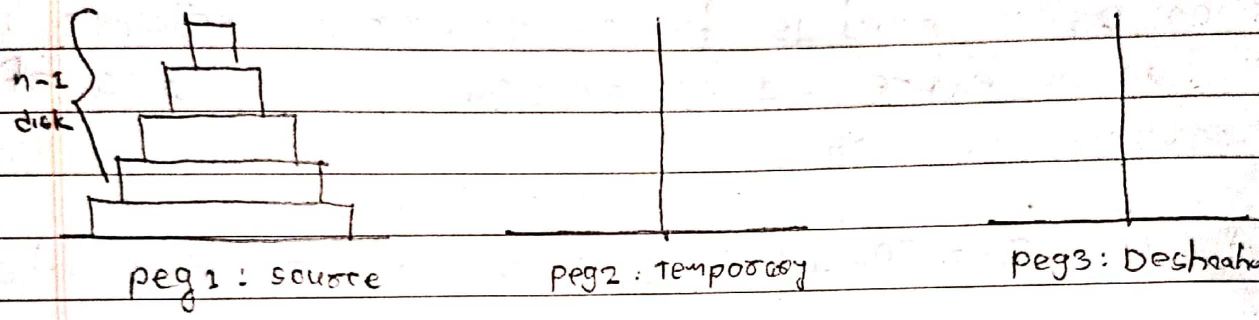
$$A_0 = 1000$$

The explicit formula after solving the recurrence relation is followed

$A_n = r^n \times A_0$ where r is the sum of 1 and interest rate.

2. Model the Tower of Hanoi problem using recurrence relation.

⇒



Let H_n be the moves required to move 'n' disk from source (peg 1) to destination (peg 3)

→ first of all with the help of peg 2 and peg 3 (n-1) disks from source is arranged to temporary (peg 2). This requires H_{n-1} moves.

→ Then the largest disk from peg-1 is moved to peg-3, which requires '1' move.

→ Finally (n-1) disk from peg 2 are moved to peg 3 with the help of peg 1 & peg 2 which further requires H_{n-1} moves.

→ Hence we can define the recurrence relation as :

$$H_n = H_{n-1} + 1 + H_{n-1}$$

$$\boxed{H_n = 2H_{n-1} + 1}$$

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which is required recurrence relation.

Now,

$$H_n = 2H_{n-1} + 1$$
$$2(2H_{n-2}+1) + 1$$
$$= 2(2(2H_{n-3}+1)+1) + 1$$
$$= 2(4(2H_{n-4}+1)+2)+1$$
$$= 2(8H_{n-4}+4+2)+1$$
$$= 2(2^3 H_{n-4} + 2^2 + 2^1 + 2^0)$$

For n disks

$$2(2^{n-1}.1 + 2^{n-2} + \dots + 2^3 + 2^2 + 2^1 + 2^0)$$

$$\text{common relation} = \frac{2^1}{2^0} = 2$$

This relation satisfies the geometric series with common ratio 2.

$$\text{sum of G.S} = \frac{a(r^n - 1)}{r-1}$$
$$= \frac{1(2^n - 1)}{2-1}$$
$$= 2^n - 1$$

3. Assume that the deer population of Rushc country is 1000 at time $n=0$ and the increase from $n-1$ to time n by 10% of the size at time $n-1$. Write a recurrence relation and an initial condition then define the deer population at time n and then solve the recurrence relation.

Sol^n

Let d_n denote the deer population at time n.
We have the initial condition.

$$d_0 = 10000$$

The recurrence relation

$$d_n - d_{n-1} = 0.1\, d_{n-1}$$

The recurrence relation may be solved by iteration.

$$d_n = 1.1\, d_{n-1} = 1.1\,(1.1\, d_{n-2}) = \dots = (1.1)^n d_0$$
$$= (1.1)^n \times 10000$$

4. Suppose the number of virus in a colony triples every hour. Find recurrence relation for the number of virus after n hours have ~~elt~~ elapsed. If 100 viruses were there in a colony in the beginning. how many viruses will be there after 12 hours.

P.T.O

Sol^n

Let a_n represents the number of virus after n hours have ellopsed.

Every hour, the number of virus triples. Thus the number of virus is the number of virus at an hour ~~ego~~ ago multiplied by 3

$$a_n = 3a_{n-1}$$

we have given,

At the beginning, no. of viruses = 100

$$\therefore a_0 = 100$$

We successively apply the recurrence relation

$$a_n = 3a_{n-1} = 3^1 a_{n-1}$$
$$= 3(3a_{n-2}) = 3^2 a_{n-2}$$
$$= 3^2(3a_{n-3}) = 3^3 a_{n-3}$$
$$= 3^3(3a_{n-4}) = 3^4 a_{n-4}$$
$$\vdots$$
$$= 3^n a_{n-n}$$
$$= 3^n \cdot a_0$$
$$= 100 \cdot 3^n$$

Evaluate the found expression at $n = 12$

$$a_{12} = 100 \cdot 3^{12} \simeq 53144100$$

Thus, there are 53144100 viruses after 12 hours.

5. Derive explicit formula for fibonacci series

$\Rightarrow$

$$fib(n) = \begin{cases} n & \text{if } n<2 \\ fib(n-1) + fib(n-2) \end{cases}$$

The reccurrence relation of the fibonacci series

$a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$ $a_1 = 1$

The corresponding characteristics equation is

$r^2 - r - 1 = 0$

On solving

$r = \frac{1+\sqrt{5}}{2}$, $r = \frac{1-\sqrt{5}}{2}$

when roots are not same, the sol^n in the root

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{— (A)}$$

when $n=0$

$$a_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$0 = \alpha_1 + \alpha_2 \quad \text{— (1)}$$

$$\alpha_1 = -\alpha_2$$

when $n=1$

$$a_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)$$

$2 = \alpha_1 + \alpha_1 \cdot \sqrt{5} + \alpha_2 - \sqrt{5} \cdot \alpha_2$

~~$2 = (1+\sqrt{5})\alpha_1 + (1-\sqrt{5}$~~

$2 = (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2$

$2 = (1+\sqrt{5})(-\alpha_2) + (1-\sqrt{5})\alpha_2$ from C

$2 = -\cancel{\alpha_2} - \sqrt{5}\alpha_2 + \cancel{\alpha_2} - \sqrt{5}\alpha_2$

or, $\cancel{2} = -\cancel{2}\sqrt{5}\alpha_2$

$\alpha_2 = -\frac{1}{\sqrt{5}}$

$\alpha_1 = \frac{1}{\sqrt{5}}$

putting the value in A

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

6. solve the recurrence relation.

a. $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 5$ and $a_1 = 15$

Sol^n

The characteristics eq^n is

$r^2 - 7r + 10 = 0$

or, $r^2 - 5r - 2r + 10 = 0$

or, $r(r-5) - 2(r-5) = 0$

$(r-5)(r-2) = 0$

$r = 2, 5$

P.T.O

The general sol^n for the distinct root of the form

$$a_n = \alpha_1 . 2^n + \alpha_2 . 5^n \quad \text{--- (i)}$$

$\because$ Given,

$$a_0 = 5$$

eq^n (i) becomes

$\therefore \quad a_0 = \alpha_1 . 2^0 + \alpha_2 . 5^0$

$$\alpha_1 + \alpha_2 = 5 \quad \text{--- (a)}$$

Also, $\quad a_1 = 15$

$$a_1 = \alpha_1 . 2^1 + \alpha_2 . 5^1$$

$$2\alpha_1 + 5\alpha_2 = 15 \quad \text{--- (b)}$$

solving eq^n (a) and (b), we get,

$$\alpha_1 = 10/3$$

$$\alpha_2 = 5/3$$

$\therefore$ Hence,

$$a_n = \frac{10}{3} . 2^n + \frac{5}{3} . 5^n$$

d. $a_n = 7a_{n-1} - 10a_{n-2} + 16n$

sol^n

Associated part.

$$a_n = 7a_{n-1} - 10a_{n-2} \cancel{+0}$$

The characteristics eq^n is

$$r^2 - 7r + 10 = 0$$

$$\text{or, } r^2 - 5r - 2r + 10 = 0$$

$$\text{or, } (r-5)(r-2) = 0$$

$$r = 2, 5$$

The general sol^n for the distinct root is of the form

$$a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 5^n \quad \text{— (a)}$$

The non-homogenerous part is

$$a_n = 16n$$

The particular sol^n is of the form

$$a_n = Cn + D$$

$$Cn + D = 7(n-1) - 10(n-2) + 16n$$

$$Cn + D = 7n - 7 - 10n + 20 + 16n$$

$$Cn + D = 15n + 13$$

$$\therefore C = 15$$

$$D = 13$$

$$a_n = 15n + 13$$

$$a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 5^n + 15n + 13$$

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e. $2a_n = 7a_{n-1} - 3a_{n-2} + 2^n$ with $a_0 = 1, a_1 = 5$

Soln

Assosciate part :

$2a_n = 7a_{n-1} - 3a_{n-2}$

The characteristics eqn is

$2r^2 - 7r + 3 = 0$

or, $2r^2 - 6r - r + 3 = 0$

or, $2r(r-3) - 1(r-3) = 0$

or, $(r-3)(2r-1) = 0$

$\therefore r = 3$

$r = 1/2$

The general soln for the distinct root is of the form

$a_n = \alpha_1 . 3^n + \alpha_2 . \left(\frac{1}{2}\right)^n$

The non-homogeneous part is

$a_n = 2^n$

the soln of non-homogeneous part is of

$a_n = c . 2^n$ where c is constant

$2 . c . 2^n = 7c . 2^{n-1} - 3c . 2^{n-2} + 2^n$

Dividing by 2^n

$2c = 7c . \frac{1}{2} - 3c . \frac{1}{4} + 1$

$2c = \frac{7c}{2} - \frac{3c}{4} + 1$

or, $2c = \frac{14c - 3c + 4}{4}$

or, $8c = 11c + 4$

or, $-3c = 4$

$c = -4/3$

The particular sol^n is

$2a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot \left(\frac{1}{2}\right)^n - \frac{4}{3} 2^n$

Here,

$a_0 = 1$

$\therefore 2a_0 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot \left(\frac{1}{2}\right)^0 - \frac{4}{3} \cdot 2^0$

$2 \times 1 = \alpha_1 + \alpha_2 - \frac{4}{3}$

$\alpha_1 + \alpha_2 = 2 + 4/3 =$

$\alpha_1 + \alpha_2 = 10/3$ — (a)

also,

$a_1 = 5$

$2a_1 = \alpha_1 \cdot 3^1 + \alpha_2 \cdot \left(\frac{1}{2}\right)^1 - \frac{4}{3} \cdot 2^1$

$2 \times 5 = 3\alpha_1 + \frac{1}{2}\alpha_2 - \frac{8}{3}$

$3\alpha_1 + \frac{1}{2}\alpha_2 = 10 + 8/3$

$3\alpha_1 + \frac{1}{2}\alpha_2 = 38/3$ — (b)

P.T.O

solving eqⁿ (a) and (b)

$\alpha_1 = \frac{22}{5}$

$\alpha_2 = \frac{-16}{15}$

$\therefore \quad 2a_7 = \frac{22}{5} \cdot 3^7 + \frac{16}{15} \cdot \left(\frac{1}{2}\right)^7 - \frac{4}{3} 2^7$ Ans

7. What is graph theory? List out the application of graph theory with suitable example of one of them.

⇒ Many situation that occur in computer science, physical science, chemical science, economics and many ~~more~~ other area can be analyzed big using techniques found in a relatively new area of mathematics called as graph theory.

Application of Graph theory

Graph theory has its applications in diverse fields of engineering :-

P.T.O

* Electrical Engineering :-
The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named ~~are~~ as topologies. Some examples for topologies are star, bridge, series and parallel topologies.

* Computer science :-
Graph theory is used for the study of algorithms. For example,
→ Kruskal's Algorithm
→ Prim's Algorithm
→ Dijkstra's Algorithm

* Computer Network :-
The relationships among interconnected computers in the network follows the principles of graph theory.

* Science :-
The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc are represented by graph.

* Linguistics :-
The parsing tree of a language and

grammar of a language uses graph.

* General :-

Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.

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Q. Define the terms: Multigraph, Pseudo graph, complete graph, bi-p

⇒ The terms are defined below:-

Multigraph:-
A graph G(V,E) is said to be multigraph such that some of the edges are parallel.

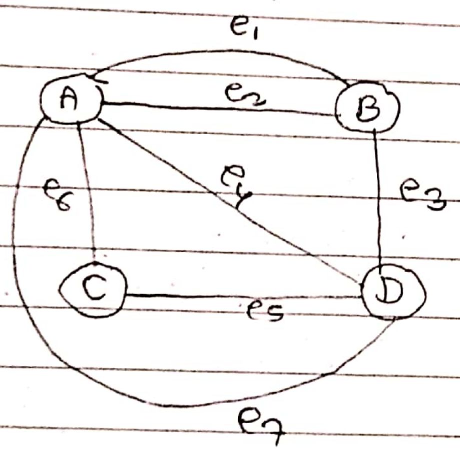

fig:- Multigraph

Pseudo graph:-
A graph G(V,E) is said to be pseudo graph if G has both loops and multi edges or loops only.

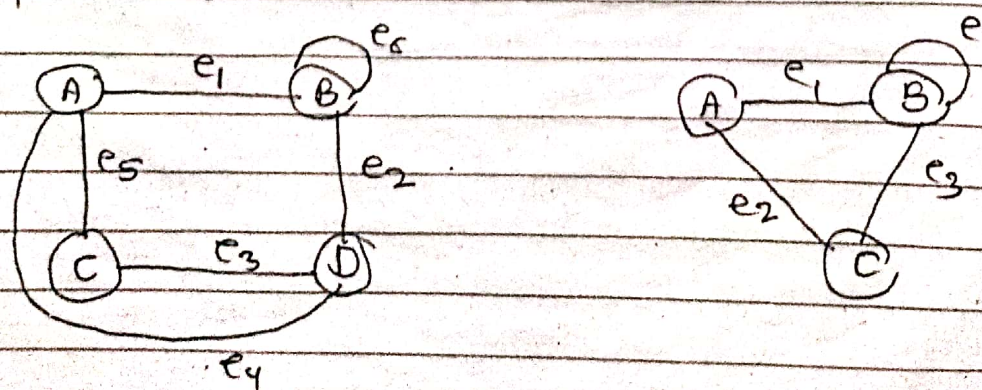

fig:- Pseudo graph

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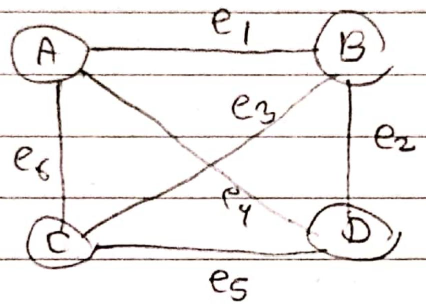
<u>complete graph</u> :-

A graph where each vertex are connected to each ~~vo~~ other is called complete graph.

A complete graph with 'n' verhces is denoted by k_n.

E.g k_4



<u>Bi-partile graph</u>

A graph is said to be partile if is verhces are divided into two parts such that the verhces of first part are connected to verhces of second part but verhces of some part are not connected.

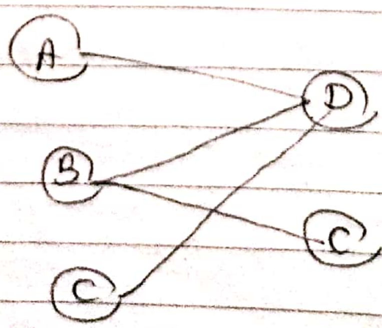


fig:- Bi-parhle graph

Regular graph:
If every vertex of a graph consists same degree, then such graphs are called regular graph.

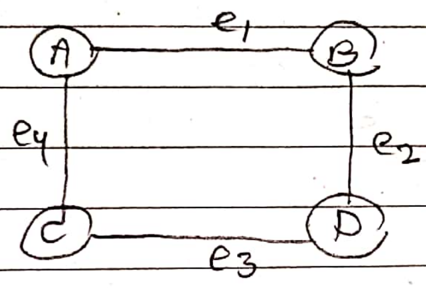


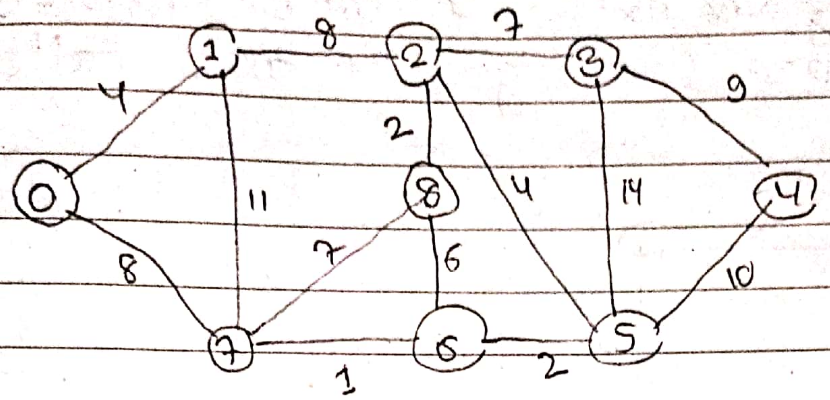
fig:- Regular graph.

9. What is minimum spanning tree? Find the minimum spanning tree of the graph using Kruskal algorithm and Prim's algorithm.

⇒ In minimum spanning tree, the tree with the minimum cost is constructed. The cost of tree is computed by adding all the weight of the edge included in the spanning tree.

P.T.O

Using Prim's Algorithm



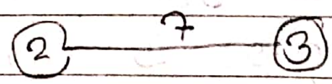
We start with any arbitrary vertex. Let us choose V3. Now, the adjacent vertex pair of V3 are

$(3, 4) = 9$

$(3, 5) = 14$

$(3, 2) = 7$

We choose the vertex pair with minimum weight i.e $(3, 2) = 7$ and add it to our tree to be constructed.



Now, the adjacent of V2 and remaining vertex ~~of~~ pair of previous set.

$(2, 1) = 8$

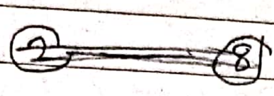
$(2, 8) = 2$

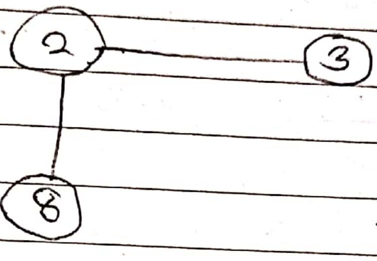
$(2, 5) = 4$

$(3, 4) = 9$

$(3, 5) = 14$

→ $(2,8) = 2$ is the vertex pair with minimum weight so ~~were~~ we add it to our tree





→ The adjacent of v_8 are

$(8,7) = 7$

$(8,6) = 6$

$(2,1) = 8$

$(2,5) = 4$ -

The adjacent of v_5 are

$(5,6) = \cancel{6}\ 2$ -

$(8,7) = 7$

$(8,6) = 6$

~~$(2,1) = 8$~~

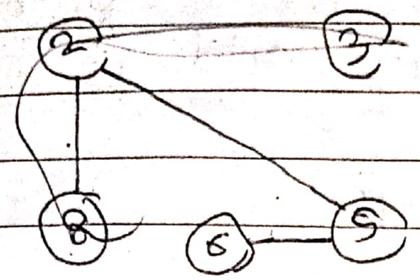
The adjacent of v_6 are

$(6,7) = 1$

The adjacent of v_7 are

$(7,0) = 8$ -

$(7,1) = 11$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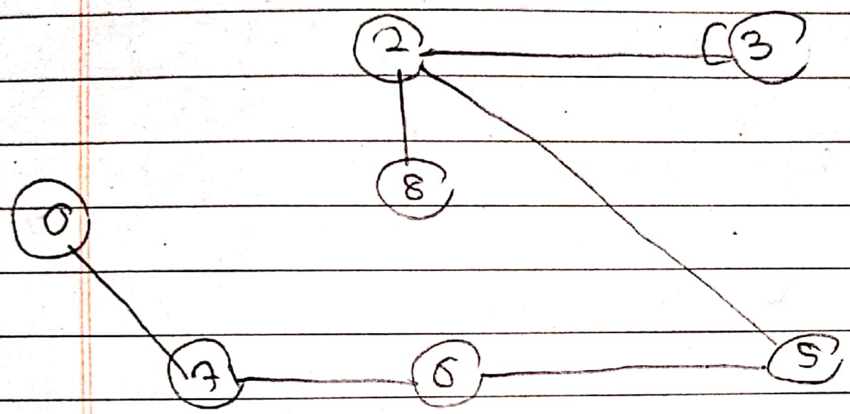
2
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10. Explain Dijkstra's algorithm with an example

=> Dijkstra's Algorithm is used to find the shortest distance between source and destination node.

→ For this we first assign the source node distance by zero and other nodes by infinity. So that to indicate distance from source to other nodes yet to be calculated.

→ Then we calculate the distance of all the adjacent nodes of source node and select the adjacent node with least distance.

→ Let this adjacent node be 'v' such that its distance from source node is $L(v)$.

→ After that we find the adjacent node of 'v' and choose the node with the least distance. Let that node be 'w' with distance from 'v' as $L(v,w)$. The distance of 'w' from starting node is

$$L(v) + L(v,w)$$

→ For the node 'w' there may be previously calculated distance. Let that distance be $L(w)$.

Scanned with CamScanner

→ Now, we compare the previously calculated distance with newly calculated distance and if the new calculated distance is smallest one we discard the previously calculated distance.

i.e.

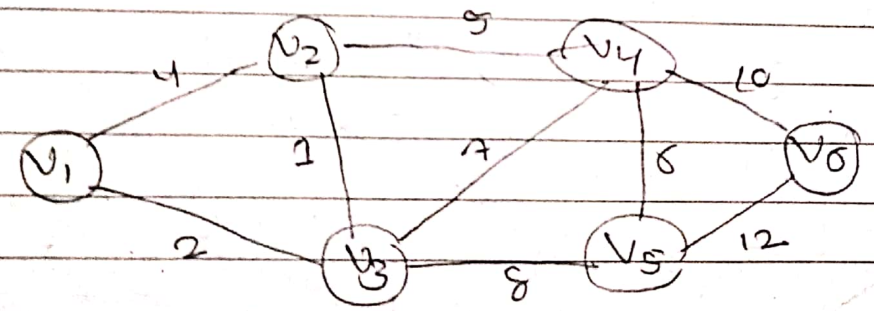
$L(w) > L(v) + L(v,w)$

then

$L(w) = L(v) + L(v,w)$

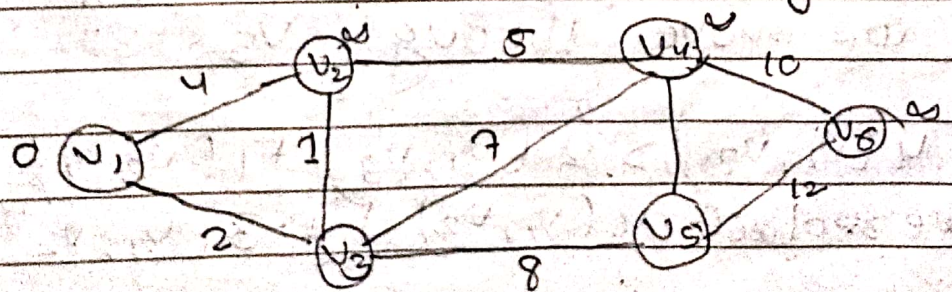
→ This process is continued until all paths for the destination nodes are calculated. The final gives path given by the algorithm is the shortest path.

E.g:-

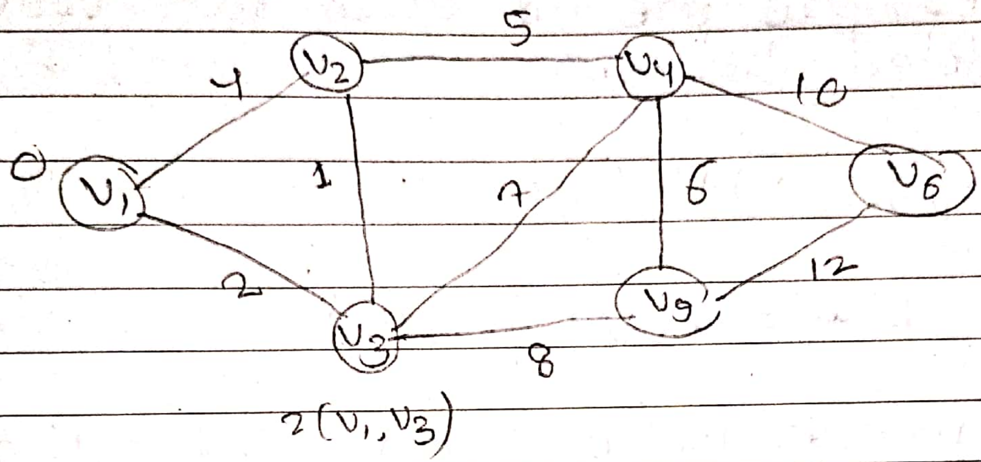


sol

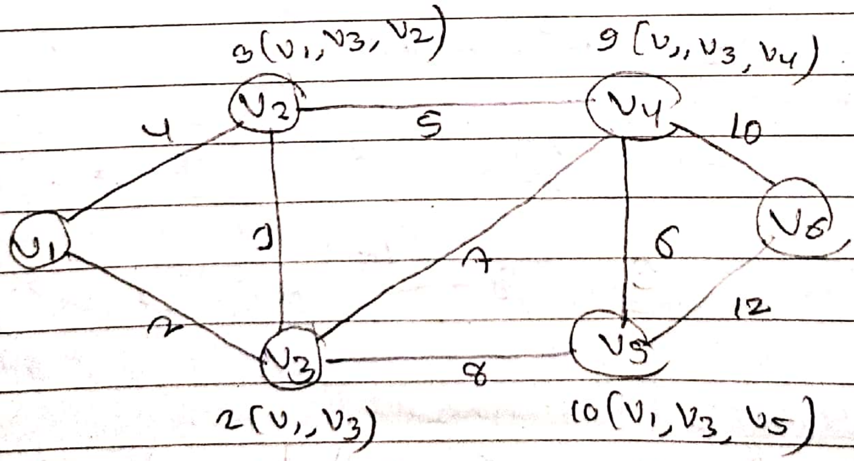
step1 :- Assign 0 (zero) to starting node i.e. V_1 and infinity to other ~~side~~ node



The adjacent of v_1 is v_2 and v_3 then we explore these two vertices from v_1 with their weight 4 (v_1, v_2).



Here, the distance of v_3 is least so we proceed through v_3. The adjacent of v_3 are v_2, v_4 and v_5

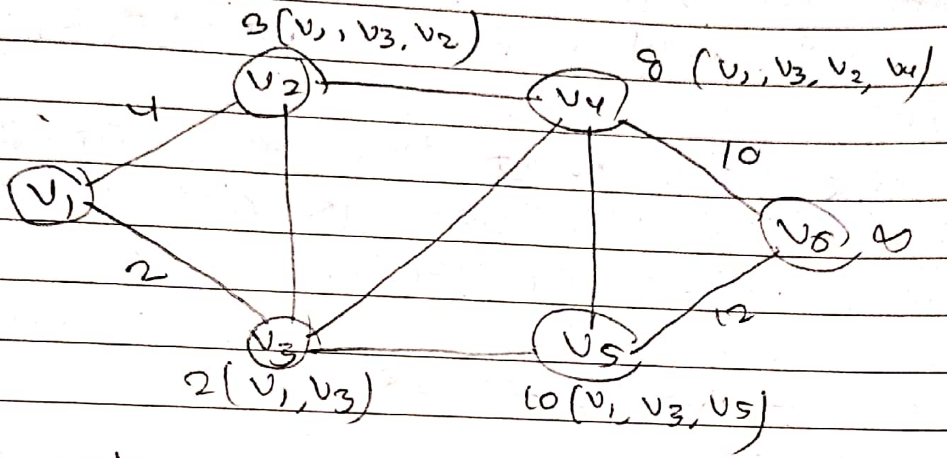


Here, the distance of v_2 from v_1 is short when we move through v_3.

i.e. $4 (v_1, v_2) > 2 (v_1, v_3) + 1 (v_3, v_2)$

so we replace $4 (v_1, v_2)$ by $3 (v_1, v_3, v_2)$

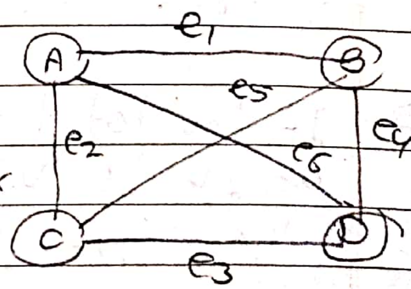
Here, v_2 has the least distance so we proceed through v_2. The adjacent of v_2 is v_4.



Here to reach v_4, 8 (v_1, v_3, v_2, v_4)

12. Show that for a complete graph with n vertices the number of edges is given by $n(n-1)/2$.

Sol^n



Let us consider a graph with n vertices since the graph is complete each vertex are connected to each other by distinct edges

So, the total number of degree for each vertex is $(n-1)$

The sum of all the degree of a graph is given by

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$deg(v_1) + deg(v_2) + \dots + \cancel{fd}\ degree(v_n)$
$(n-1) + (n-1) + \dots + (n-1)$
for n-vertices
$\therefore n(n-1)$ is total degree.

Again,
We know that the sum of all the degree of vertices is equal to twice the number of edges.

i.e. $$\sum_{i=1}^{n} deg(v_i) = 2e$$

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2}$$ proved//

13. State and prove Euler's formula for planar graph.

⇒ If the total number of vertices of a planar graph is 'V' edges is 'E' and faces is F then $V - E + F = 2$.

Proof:-
We use the mathematical inductor to prove the formula

we have,

$V - E + F = 2$

Basic step

$e = 0$

$V = 1, \quad e = 0 \quad F = 1$

So,

$V - E + F =$

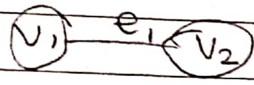
$1 - 0 + 1$

$= 2$

which is true

when $E = 1$

case 1



$V = 2 \quad E = 1 \quad F = 1$

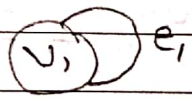
$V - E + F$

$2 - 1 + 1$

$= 2$

case II

$V = 1, \; E = 1, \; F = 2$



$V - E + F.$

$1 - 1 + 2$

$= 2$

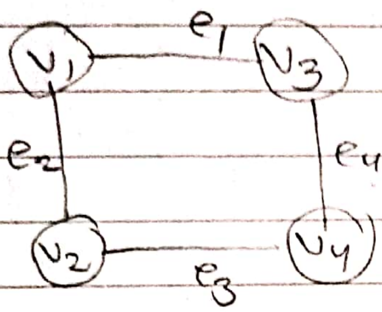
which is true

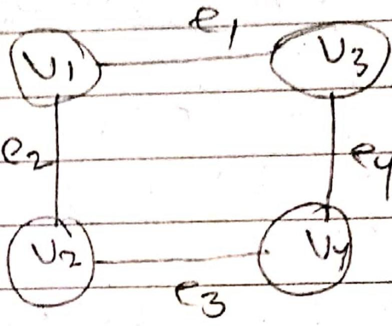
Induction step

We assume that the formula is true for 'E' number of edges and try to prove it for $(e+1)$ number of edges.

case I

By adding the edges the number of vertices remarks the same but number of faces increases.





Let v', e' and f' be the vertices, edges and faces of the newly formed graph respectively.

Now,

$$v' - e' + f' = 2$$

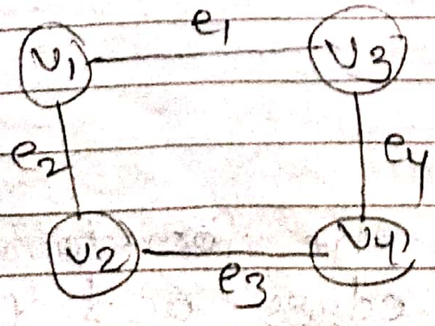
$$v - (e+1) + (f+1) = 2$$

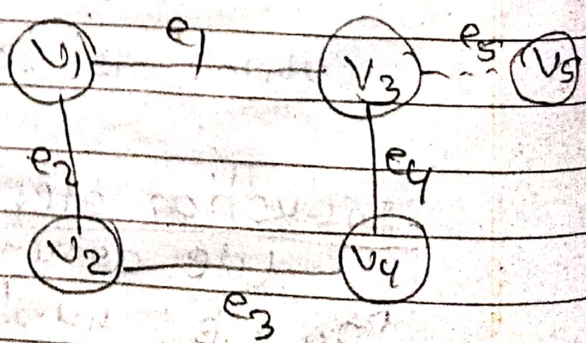
$$v - e - \cancel{1} + f + \cancel{1} = 2$$

or, $\boxed{v - e + f = 2}$

case II

By adding the edges the number of vertices increases but faces remains the same.





Let v', e' and f' be the total number of vertices and faces of newly formed graph respectively

i.e. $v' = v+1$, $e' = e+1$, $f' = f$

Now,

$$v' - e' + f = 2$$

$$(v+1) - (e+1) + f = 2$$

$$\boxed{v - e + f = 2}$$

which is true.

Hence, the euler's formula for planar graph i.e. $\boxed{v - e + f = 2}$ is valid using mathematical induction.

11. Define a planar graph. Show that in any simple planar graph, and $v >= 3$, $e <= 3v - 6$.

⇒ A graph drawn in the plane where no edges intersects to each other is called planar graph.

Proof:

The sum of the degrees of the faces is equal to twice the number of edges. But each face must have degree ≥ 3 so we have

$3f \leq 2e$

Euler's formula says that $v - e + f = 2$

so, $f = e - v + 2$ and thus $3f = 3e - 3v + 6$.

combining this with $3f \leq 2e$, we get
$3e - 3v + 6 \leq 2e$. so, $e \leq 3v - 6$.

14. State and prove Euler's Theorem.

⇒ A graph 'G' consists of Euler's cycle if and only if every vertex of G has even degree.

P: A Graph 'G' consists of Euler's cycle
Q: Every vertex of 'G' has even degree

$P \leftrightarrow Q$

1st part

If a graph G contains euler cycle then every vertex of 'G' has even degree.

Proof:-

We know that Euler's cycle is a continuous path that start with any arbitary vertex and ends at that vertex from where we started to move. Furthermore, it is formed by visiting every incoming and one-out going edges for every vertex. This contributes degree 2 for each vertex which is even.

→ If a vertex is repeated or visited twice, it provides further degree 2 to that

vertex which is also even.
→ It is also true for starting vertex since when the continuous path starts it odds degree '1' to that vertex and when it ends to that vertex it adds further degree '1' making the degree of vertex '2' which is even

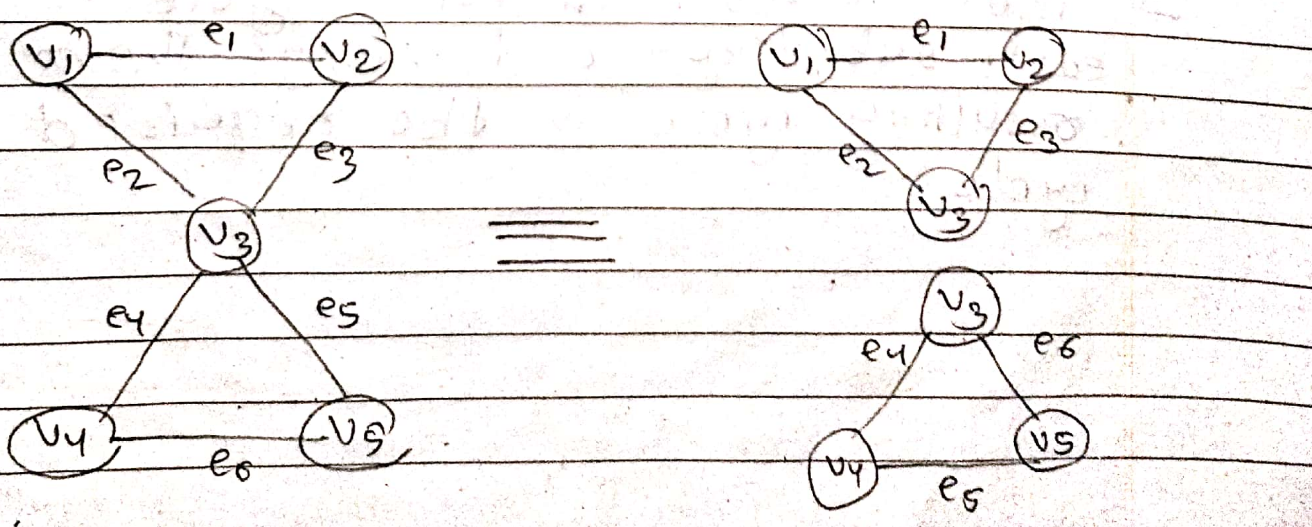
→ Hence, we can say that every vertex of G' has even degree.

<u>second part</u>

If every vertex of 'G' has even degree then it contains Euler's cycle.

<u>Proof:-</u>

To prove the statement we need to search for the continuous path that is formed by visiting every edges exactly once and ends in the vertex from where we started.



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→ Let us start with any arbitary vertex, we know that its minimum degree is 2, we can say that there may exists an outgoing edges from the vertex to its adjacent vertex.

→ This case may exist for all the vertices of a graph.

→ By ~~proqe~~cessing this we find a path in which every edge is included and ends in the arbitary vertex from where we started, thus, we can say that the euler's cycle exists.

→ But there may be the case, where cycle may form in part of the graph. In this case we split the given graph into multiple sub-graph by taking reference of comman vertex.

→ Then we form different cycle in ~~which~~ each sub-graph and merge them and resulting cycle is the required euler's cycle.

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15. Define Alphabet and Language in finite state Automata. Describe the working principle of DFA.

⇒ Alphabet is the collection of input symbols. It is denoted by Σ

E.g:- binary alphabet = $\{0, 1\}$

Language :- The collection of all possible strings over some given alphabet. It is denoted by L.

E.g:- $L = \{0, 1, 11, 001, 00110, \ldots\}$

DFA (Deterministic finite Automata) consists of 5 tuples $\{Q, \Sigma, q, F, \delta\}$

Q : set of all states

Σ : set of input symbols. (symbols which machine takes as input)

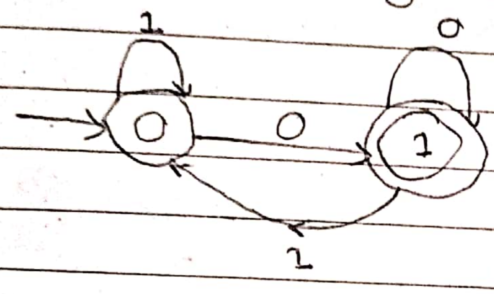
q : initial state. (starting state of a machine)

F : set of final state.

δ : Transition function, defined as
$\delta : Q \times \Sigma \rightarrow Q$

In a DFA, for a particular input character the machine goes to one state only. A transition function is defined on every state for every input symbol. Also in DFA null (or ε) move is not allowed, i.e. DFA cannot change state without any input ~~or~~ character.

For example, below DFA with $\Sigma = \{0, 1\}$ accepts all strings ending with 0.



There can be many possible DFAs for a pattern. A DFA with minimum number of states is generally preferred.

17. Draw the transition diagram of a finite state automata that accepts the string starts with baa over {a, b}

Soln

$M = \{Q, \Sigma, \delta, q_0, F\}$

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{a, b\}$

$q_0 \rightarrow$ starting point $= \{q_0\}$

$F = \{q_3\}$

δ consist of:

$\delta(q_0, a) \rightarrow \phi$

$\delta(q_0, b) \rightarrow q_1$

$\delta(q_1, a) \rightarrow q_2$

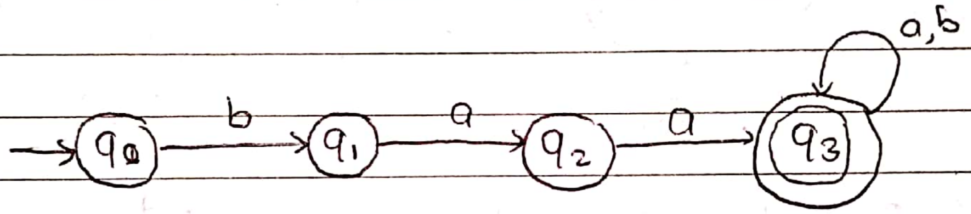
$\delta(q_1, b) \rightarrow \phi$

$\delta(q_2, a) \rightarrow q_3$

$\delta(q_2, b) \rightarrow \phi$

$\delta(q_3, a) \rightarrow q_3$

$\delta(q_3, b) \rightarrow q_3$



18. Define finite state machine. what are the difference between DFA and NFA. Constr the FA using the following transition table

Q/Σ	a	b	c
q_0	q_1	q_0	q_2
q_1	q_0	q_3	q_0
q_2	q_3	q_2	q_0
q_3	q_1	q_0	q_1

⇒ A finite state machine (FSM) is defined mathematically by 5 tuple.

$M = (Q, \Sigma, O, F, G)$
where,

Q = finite set of states
Σ = finite set of inputs
O = finite set of outputs
F = transition function
G = Output relation.

The difference between DFA and NDFA are:-

DFA	NDFA
i) The transition from a state is to a single particular next state for each input symbol. Hence it is called deterministic.	i) The transition from a state can be to multiple next states for each input symbol. Hence it is called non-deterministic.
ii) ~~Each~~ empty string transitions are not seen in DFA	ii) NDFA permits empty string transitions.
iii) Backtracking is allowed in DFA	iii) In NDFA, backtracking is not always possible.
iv) Requires more space	iv) Requires less space.
v) A string is accepted by a DFA, if it transits to a final state.	v) A string is accepted by a NDFA, if at least one of all possible transitions ends in a final state.