Factoring and Manipulating Algebraic Equations

25 January 2025

§1 Factoring

Here are some basic identities that will be used in this section:

- **Difference of Powers**:

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$$

- **Sum of Odd Powers**

$$x^{2n+1} + y^{2n+1} = (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots - xy^{2n-1} + y^{2n})$$

Although we will primary focus on and work with difference of squares, difference of cubes and sum of cubes, it is still important to know the general forms of these identities.

Example 1.1

Factor the following expressions

a. $2x^2 + 4xy + 4xz + 2y^2 + 4yz + 2z^2$

b. $x^2 - y^2 + 6x - 12y - 27$

Solution. (a) Rearranging the equation gives us

$$2x^2 + 2y^2 + 2z^2 + 4xy + 4xz + 4yz$$

We can factor the 2 out

$$2(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz)$$

Then, we can factor it was

$$2(x+y+z)^2$$

b) We can complete the square in this expression to get

$$x^2 - y^2 + 6x - 12y - 27 = (x+3)^2 - (y+6)^2$$

Now we can use difference of squares to factor it to get

$$((x+3) + (y+6))((x+3) - (y+6)) = (x+y+9)(x-y-3)$$

□

Example 1.2 (2023 AMC 12A)
What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

Solution. Since we have differences of cubes, we should use our difference of cubes identity. We can rewrite the expression as

$$(2-1)(2^2+2(1)+1^2)+(4-3)(4^2+4(3)+3^2)+\cdots+(18-17)(18^2+18(17)+17^2)$$
$$=(2^2+2(1)+1^2)+(4^2+4(3)+3^2)+\cdots+(18^2+18(17)+17^2)$$
$$=1^2+2^2+3^2+\cdots+18^2+1(2)+3(4)+\cdots+17(18)$$

We can split the equation as the sum of consecutive squares and the sum of product of consecutive numbers. For the sum of consecutive squares we can use the formula to get

$$1^2+2^2+3^2+\cdots+18^2 = \frac{18(18+1)(2(18)+1)}{6} = 2109$$

We can then write the other part as

$$\sum_{k=1}^{9}(2k)(2k-1) = \sum_{k=1}^{9}(4k^2-2k) = 4\sum_{k=1}^{9}(k^2) - 2\sum_{k=1}^{9}(k) = 1050$$

$$2109 + 1050 = \boxed{3159}$$

□

Exercise 1.3. Find all integer solutions (x, y, z) to

$$x^2 + y^2 = -z^2 - 2$$

$$4x + 8y + 16z = -167$$

Exercise 1.4. Factor

$$x^6 - 50x^4 + 18x^3 + 200x^2 - 119$$

Exercise 1.5. Factor

$$x^4 + 2x^3 + 2x^2 + x$$

§2 Finding Solutions to Equations

When trying to find integer solutions to an equation of a product of numbers that equals to a constant integer term is easy. We just need to know the prime factorization of that number. Simon's Favorite Factoring Trick is a powerful tool that allow us to factor an equation of the form

$$kxy + qx + ry + c$$

for some k, q, r, c.

Example 2.1
Find real numbers x and y such that $xy + 6x + 4y + 25 = 0$ in the form (x, y).

Solution. We can factor the equation as

$$xy + 6x + 4y + 25 = (x + 4)(y + 6) + 1 = 0$$

Moving the one to the other side, we get

$$(x + 4)(y + 6) = -1$$

We get the solutions to the equation when one of the variables are equal to -1 and the other is 1. This gives us $(-3, -7)$ and $(-5, -5)$ □

Example 2.2 (Math League HS 2005-2006)
What are both pairs of integers (x, y) for which $4^y - 615 = x^2$

Solution. We start by moving the constant to the right side of the equation and simplifying.

$$4^y - x^2 = 615$$

$$2^{2y} - x^2 = 615$$

We notice this is a difference of squares so we apply out identity.

$$(2^y - x)(2^y + x) = 615$$

We see that the prime factorization of 615 is $3 * 5 * 41$ and that $(2^y + x) \geq (2^y - x)$ If $(2^y + x) = p$ and $(2^y - x) = q$ for some p and q, then

$$(2^y - x) + (2^y + x) = 2^{y+1} = p + q$$

We can see that p and q sum to a power of 2. The possible ordered pairs of (p, q) are $(41, 15)$, $(123, 5)$, $(205, 3)$, $(615, 1)$. Checking we see that $123 + 5 = 128 = 2^7$ so $y = 6$ and $x = 59$. We can verify this by plugging it back into the equation.

Hence, $(59, 6)$ is answer. But the question says that there are two solutions so what went wrong? Well, we forgot to account for negative numbers. Considering the cases $(-15, -41)$, $(-5, -123)$, $(-3, -205)$ and $(-1, -615)$. We can also find $(-59, 6)$.

Therefore, the solutions are $\boxed{\text{(59, 6)}}$ and $\boxed{\text{(-59, 6)}}$. □

Remark 2.3. Differences and sums of powers are perfect for trying to find integer solutions!

Exercise 2.4. How many integer solutions (x, y) are there to the equation

$$5x^2y + 5xy^2 + 35x + 35y = 4$$

Exercise 2.5. Find real solutions to

$$16^x + 16^y + 2(4^{x+y}) = 64$$

§3 The Binomial Theorem

Theorem 3.1 (The Binomial Theorem)

Let x, y be any real or complex number and n be any non-negative number. The Binomial Theorem states that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Example 3.2 (2008 iTest)

Find the sum of the 2007 roots of $(x-1)^{2007} + 2(x-2)^{2006} + 3(x-3)^{2005} + \cdots + 2006(x-2006)^2 + 2007(x-2007)$

Solution. To find the sum of the roots of a polynomial, we know by Vieta's formula that it is equal to the negative coefficient of the x^{2007} term over the coefficient of the x^{2008} term. Looking at $(x-1)^{2007}$ and applying the Binomial Theorem, we get

$$x^{2007} - \binom{2007}{2006} x^{2006} + \cdots - 1$$

Looking at the other term, $2(x-2)^{2006}$ we can see that the coefficient of x^{2006} is 2;

$$-\frac{-\binom{2007}{2006} + 2}{1} = \boxed{2005}$$

□

§4 Working w/ Exponents, Radicals and Logarithms

Problems involving exponents, radicals and/or logarithms are usually great problems to to use techniques such as factoring and substitution. Below is a review on the basic log identities you should know.

- $\log_b xy = \log_b x + \log_b y$
- $\log_b \frac{x}{y} = \log_b x - \log_b y$
- $\log_b a = \frac{\log_c a}{\log_c b}$
- $\log_{b^y} a^x = \frac{x}{y} \log_b a$

Lets start with two simple problems first.

Example 4.1 (2022 BMT Algebra)

The equation

$$4^x - 5 \cdot 2^{x+1} + 16 = 0$$

has two integer solutions for x. Find their sum.

Solution. Substituting $y = 2^x$, we get

$$y^2 - 10y + 16$$

This is just a simple quadratic. Factoring we get

$$(y-8)(y-2) = 0$$

Therefore, $2^x = 8$ and $2^x = 2$ and $x = 3$ and $x = 1$.

$$3 + 1 = \boxed{4}$$

$\square$

Example 4.2 (2024 China HS Math League)

Real number $m > 1$ satisfies $\log_9 \log_8 m = 2024$. Find the value of $\log_3 \log_2 m$.

Solution. This problem can be solved entirely using log identities.

$$\log_9 \log_8 m = 2024$$

$$9^{2024} = \log_8 m$$

$$8^{9^{2024}} = m$$

$$2^{3*3^{2*2024}} = m$$

$$2^{3^{4049}} = m$$

$$\boxed{4049} = \log_3 \log_2 m$$

$\square$

Now a more challenging problem

Example 4.3 (2016 PUMaC Algebra A)

For positive real numbers x and y, let $f(x, y) = x^{\log_2 y}$. The sum of the solutions to the equation

$$4096 f(f(x, x), x) = x^{13}$$

can be written in simplest form as $\frac{m}{n}$. Compute $m + n$.

Solution. We should first simplify the equation by repeatedly using the function and moving all the exponents to one side.

$$4096 f(f(x, x), x) = x^{13}$$

$$4096 x^{(\log_2 x)^2} = x^{13}$$

$$4096 x^{(\log_2 x)^2 - 13} = 1$$

How should we proceed from here? We have a 1 on one side of the equation, maybe we can equate the power to 0 if we manipulate this further.

$$2^{12} x^{(\log_2 x)^2 - 13} = 1$$

Since $2^{\log_2 x} = x$, we can combine the left side

$$2^{12} 2^{(\log_2 x)((\log_2 x)^2 - 13)} = 1$$

$$2^{(\log_2 x)((\log_2 x)^2 - 13) + 12} = 1$$

We can now simplify further

$$(\log_2 x)((\log_2 x)^2 - 13) + 12 = 0$$

$$(\log_2 x)^3 - 13 \log_2 x + 12 = 0$$

The obvious solution is $x = 2$

$$(\log_2 x - 1)((\log_2 x)^2 + \log_2 x - 12) = 0$$

$$(\log_2 x - 1)(\log_2 x + 4)(\log_2 x - 3) = 0$$

We get the solutions 2, 8 and $\frac{1}{16}$.

$$2 + 8 + \frac{1}{16} = \boxed{\frac{161}{16}}$$

□

Solution. □

There aren't much to learn about exponents, radicals and logarithms except memorizing log identities. To really learn to manipulate them, you need to practice!

§5 Lesser Used Identities

Below are some lesser used factoring identities but are still interesting and are sometimes useful.

- **Sophie Germain Identity**:

$$a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$$

- **Sum of Three Cubes**:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz)$$

- **Common Trinomial**:

$$x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$$

§6 Problems

Problem 6.1 (2018 Duke Math Meet)**.** Find the sum of the unique prime factors of $(2018^2 - 121) \cdot (2018^2 - 9)$.

Problem 6.2 (SMT 2012 Algebra)**.** Find all real values of x such that

$$(\frac{1}{5}(x^2 - 10x + 26))^{x^2-6x+5} = 1$$

Problem 6.3 (2012 SMT Algebra)**.** For real numbers (x, y, z) satisfying the following equations, find all possible values of $x + y + z$

$$x^2y + y^2z + z^2x = -1$$

$$xy^2 + yz^2 + zx^2 = 5$$

$$xyz = -2$$

Problem 6.4 (2016 CMIMC Algebra Round)**.** A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$, respectively. What is the smallest possible value of $a + b$?

Problem 6.5 (2016 CMIMC Algebra Round)**.** Suppose a, b, c, and d are positive real numbers that satisfy the system of equations

$$\begin{aligned}(a + b)(c + d) &= 143,\\ (a + c)(b + d) &= 150,\\ (a + d)(b + c) &= 169.\end{aligned}$$

Compute the smallest possible value of $a^2 + b^2 + c^2 + d^2$.

Problem 6.6 (2007 Purple Comet HS)**.** Find the maximum possible value of $8 \cdot 27^{\log_6 x} + 27 \cdot 8^{\log_6 x} - x^3$ as x varies over the positive real numbers.

Problem 6.7 (Dutch MO 2011)**.** Determine all pairs of positive real numbers (a, b) with $a > b$ that satisfy the following equations: $a\sqrt{a} + b\sqrt{b} = 134$ and $a\sqrt{b} + b\sqrt{a} = 126$.

Problem 6.8 (2024 AMC 12A)**.** Integers a, b, and c satisfy $ab + c = 100$, $bc + a = 87$, and $ca + b = 60$. What is $ab + bc + ca$?

Problem 6.9 (2019 BMT Algebra)**.** Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2+3y^2+3z^2+2xy+2xz-4yz < 5$.

Problem 6.10 (2014 USAMO)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Problem 6.11 (2019 Purple Comet HS)**.** There are positive integers m and n such that $m^2-n = 32$ and $\sqrt[5]{m + \sqrt{n}}+\sqrt[5]{m - \sqrt{n}}$ is a real root of the polynomial $x^5-10x^3+20x-40$. Find $m + n$.

Problem 6.12 (2019 BMT Algebra)**.** Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.

Problem 6.13 (2020 BMT Algebra Round)**.** Let a, b, and c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=5$. The value of

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Problem 6.14 (1984 USAMO)**.** In the polynomial $x^4-18x^3+kx^2+200x-1984=0$, the product of 2 of its roots is -32. Find k.

§7 Some Harder Problems

Problem 7.1 (2015 PUMaC Algebra)**.** There are real numbers a,b,c,d such that for all (x,y) satisfying $6y^2=2x^3+3x^2+x$, if $x_1=ax+b$ and $y_1=cy+d$, then $y_1^2=x_1^3-36x_1$. What is $a+b+c+d$?

Problem 7.2 (2007 Alabama ARML TST)**.** If $w^{2007}=1$ and $w\neq 1$, then evaluate

$$\frac{1}{1+w}+\frac{1}{1+w^2}+\frac{1}{1+w^3}+\cdots+\frac{1}{1+w^{2007}}.$$

Express your answer as a fraction in lowest terms.

Problem 7.3 (2005 Purple Comet HS)**.** Find the number of quadruples (a,b,c,d) of integers which satisfy both

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{2}\qquad\text{and}$$

$$2(a+b+c+d)=ab+cd+(a+b)(c+d)+1.$$

Problem 7.4 (2008 Canadian MO)**.** Let a, b, c be positive real numbers for which $a+b+c=1$. Prove that

$$\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab}\leq\frac{3}{2}.$$

Problem 7.5 (2016 CMIMC Algebra Round)**.** Let ℓ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2}=\frac{13}{37}$. Then

$$\frac{(1+\ell)^3}{1+\ell^3}=\frac{m}{n},$$

where m and n are positive coprime integers. Find $m+n$.

Problem 7.6 (2019 Irish MO)**.** Three non-zero real numbers a,b,c satisfy $a+b+c=0$ and $a^4+b^4+c^4=128$. Determine all possible values of $ab+bc+ca$.

Problem 7.7 (2000 Pan African MO)**.** Define the polynomials $P_0,P_1,P_2\cdots$ by:

$$P_0(x)=x^3+213x^2-67x-2000$$

$$P_n(x)=P_{n-1}(x-n), n\in N$$

Find the coefficient of x in $P_{21}(x)$.

Problem 7.8 (2005 British MO Round 2)**.** The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$

Prove that N is a perfect square.

Problem 7.9 (2006 Canadian MO)**.** Let $f(n, k)$ be the number of ways distributing k candies to n children so that each child receives at most two candies. For example, $f(3, 7) = 0$, $f(3, 6) = 1$, and $f(3, 4) = 6$. Evaluate $f(2006, 1) + f(2006, 4) + f(2006, 7) + \cdots + f(2006, 1003)$.

§8 Challenge Problems

These problems are really difficult! Don't get discouraged if you aren't able to solve these!

Problem 8.1 (2015 AIME II)**.** Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Problem 8.2 (2017 CMIMC Algebra Round)**.** Let a, b, and c be complex numbers satisfying the system of equations

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= 9, \\ \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} &= 32, \\ \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} &= 122. \end{aligned}$$

Find abc.

Problem 8.3 (1974 USAMO)**.** Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Problem 8.4 (2022 AIME I)**.** Let x, y, and z be positive real numbers satisfying the system of equations:

$$\begin{aligned} \sqrt{2x - xy} + \sqrt{2y - xy} &= 1 \\ \sqrt{2y - yz} + \sqrt{2z - yz} &= \sqrt{2} \\ \sqrt{2z - zx} + \sqrt{2x - zx} &= \sqrt{3}. \end{aligned}$$

Then $[(1 - x)(1 - y)(1 - z)]^2$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 8.5 (1971 IMO Shortlist)**.** Knowing that the system

$$x + y + z = 3,$$

$$x^3 + y^3 + z^3 = 15,$$

$$x^4 + y^4 + z^4 = 35,$$

has a real solution x, y, z for which $x^2 + y^2 + z^2 < 10$, find the value of $x^5 + y^5 + z^5$ for that solution.

Problem 8.6 (2021 Kazakhstan MO)**.** Let a be a positive integer. Prove that for any pair (x, y) of integer solutions of equation

$$x(y^2 - 2x^2) + x + y + a = 0$$

we have:

$$|x| \leqslant a + \sqrt{2a^2 + 2}$$

Problem 8.7 (2024 Russia MO)**.** Call a triple (a, b, c) of positive numbers mysterious if

$$\sqrt{a^2 + \frac{1}{a^2c^2} + 2ab} + \sqrt{b^2 + \frac{1}{b^2a^2} + 2bc} + \sqrt{c^2 + \frac{1}{c^2b^2} + 2ca} = 2(a + b + c).$$

Prove that if the triple (a, b, c) is mysterious, then so is the triple (c, b, a).

§9 Solutions to Exercises

1.3 Find all integer solutions (x, y, z) to

$$x^2 + y^2 = -z^2 - 2$$

$$4x + 8y + 16z = -167$$

Solution: Moving all the terms in both equations to one side yields

$$x^2 + y^2 + z^2 + 2 = 0$$

$$4x + 8y + 16z + 167 = 0$$

Doubling the second equation and adding it to the first yields

$$x^2 + 8x + y^2 + 16y + z^2 + 32z + 336 = 0$$

Completing the square for the three variables yields

$$(x + 4)^2 + (y + 8)^2 + (z + 16)^2 = 0$$

By the trivial inequality $a^2 \geq 0$, the only solution is $\boxed{\text{(-4, -8, -16)}}$

1.4 Factor

$$x^6 - 50x^4 + 18x^3 + 200x^2 - 119$$

Solution: Rearranging, we get

$$x^6 + 18x^3 - 50x^4 + 200x^2 - 119$$

We can complete the square to get

$$(x^3 + 9)^2 - 50x^4 + 200x^2 - 200$$

After testing integers, it doesn't look possible to factor it but trying square roots allow for us to factor

$$(x^3 + 9)^2 - (5\sqrt{2}x^2 + 10\sqrt{2})^2$$

Differences of squares allow us to get $\boxed{(x^3 + 9 + 5\sqrt{2}x + 10\sqrt{2})(x^3 + 9 - 5\sqrt{2}x - 10\sqrt{2})}$

1.5 Factor

$$x^4 + 2x^3 + 2x^2 + x$$

Solution: After playing around with the equation and looking at it, we might notice that this looks similar to $(x^2 + x + 1)^2$ but with a few missing terms. It turns out it just factor nicely into

$$(x^2 + x + 1)^2 - (x^2 + x + 1)$$

Therefore we can factor it into $\boxed{(x^2 + x + 1)(x^2 + x)}$

2.4 How many integer solutions (x, y) are there to the equation

$$5x^2y + 5xy^2 + 35x + 35y = 4$$

Solution: We can first factor out the 5 and then factor by splitting the equation into two

$$5(x^2y + xy^2 + 7x + 7y)$$

$$5(xy(x + y) + y(x + y))$$

$$5(xy + y)(x + y) = 4$$

$$(xy + y)(x + y) = \frac{4}{5}$$

Since the product of integers cannot equal a fraction, there are no solutions. (You can also notice from the initial equation that the sum of multiples of 5 cannot equal 4.)

2.5 Find real solutions to

$$16^x + 16^y + 2(4^{x+y}) = 64$$

Solution: We should first simplify the powers

$$2^4x + 2^4y + 2(2^{2x+2y}) = 2^6$$

We notice that there are a lot of 2^x and 2^y terms. Therefore we should substitute $a = 2^x$ and $b = 2^y$.

$$a^4 + b^4 + 2(a^2b^2) = 2^6$$

We should notice the sum of squares immediately and factor accordingly

$$(a^2 + b^2)^2 = 2^6$$

$$a^2 + b^2 = \pm 2^3$$

Sums of squares are always greater or equal to 0 so we just need to consider when they are equal to 2^3. a and b can also not be negative because they are 2 to the power of something so the only solution is when $a = 2$ and $b = 2$. This gives us the solution $\boxed{(\frac{1}{2}, \frac{1}{2})}$