Discrete Update for 2D Damped Wave Equation

AudioRipple Project

Continuous PDE

$$\frac{\partial^2 Z}{\partial t^2} = c^2 \nabla^2 Z - \gamma \frac{\partial Z}{\partial t}, \tag{1}$$

where $\nabla^2 Z$ is the Laplacian operator defined as

$$\nabla^2 Z = \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}. \tag{2}$$

Approximate Time Derivatives

Second time derivative

$$\frac{\partial^2 Z}{\partial t^2} \approx \frac{Z_{i,j}^{n+1} - 2Z_{i,j}^{n} + Z_{i,j}^{n-1}}{\Delta t^2}. \tag{3}$$

First time derivative (damping)

$$\frac{\partial Z}{\partial t} \approx \frac{Z_{i,j}^{n} - Z_{i,j}^{n-1}}{\Delta t}. \tag{4}$$

Approximate Spatial Derivatives (Laplacian)

In x

$$\frac{\partial^2 Z}{\partial x^2} \approx \frac{Z_{i+1,j} - 2Z_{i,j} + Z_{i-1,j}}{(\Delta x)^2}. \tag{5}$$

In y

$$\frac{\partial^2 Z}{\partial y^2} \approx \frac{Z_{i,j+1} - 2Z_{i,j} + Z_{i,j-1}}{(\Delta y)^2}. \tag{6}$$

Combined 2D Laplacian (five-point stencil)

$$\nabla^2 Z_{i,j} \approx \frac{Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1} - 4Z_{i,j}}{(\Delta x)^2}. \tag{7}$$

Solve for Next Time Value

$$\frac{Z_{i,j}^{n+1} - 2Z_{i,j}^{n} + Z_{i,j}^{n-1}}{\Delta t^2} = c^2 \nabla^2 Z_{i,j}^{n} - \gamma \frac{Z_{i,j}^{n} - Z_{i,j}^{n-1}}{\Delta t}. \tag{8}$$

$$\Longrightarrow$$

$$\begin{aligned} Z_{i,j}^{n+1} = 2Z_{i,j}^{n} - Z_{i,j}^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \big(Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1} - 4Z_{i,j}\big) \\ - \gamma \Delta t \, (Z_{i,j}^{n} - Z_{i,j}^{n-1}). \end{aligned} \tag{9}$$

Code Expression (Final Form)

$$\begin{aligned} Z_{\text{new}} &= \underbrace{2Z - Z_{\text{old}}}_{\text{leap-frog}} + \underbrace{c2\_dt2 \cdot \text{laplacian}(Z)}_{\text{curvature}} - \underbrace{(1 - \text{damping}) \cdot \Delta t \cdot (Z - Z_{\text{old}})}_{\text{damping correction}}, \\ c2\_dt2 &= \left(\frac{c\Delta t}{\Delta x}\right)^2. \end{aligned} \tag{10}$$

Stability Condition

$$\frac{c\Delta t}{\Delta x} \leq \frac{1}{\sqrt{2}}. \tag{11}$$

Graph Laplacian Viewpoint

The discrete Laplacian matrix L on a regular 2D grid with 4-connected neighbours corresponds to

$$LZ = -4Z_{i,j} + Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1}. \tag{12}$$

Which in matrix form can be expressed as

$$L = \begin{bmatrix} -4 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 1 & 0 & \cdots \\ 0 & 1 & -4 & 1 & \cdots \\ 0 & 0 & 1 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{13}$$

This is exactly the five-point stencil.