Discrete Update for 2D Damped Wave Equation

AudioRipple Project

Continuous PDE

$$\frac{\partial^2 Z}{\partial t^2} = c^2 \nabla^2 Z - \gamma \frac{\partial Z}{\partial t}, \tag{1}$$

where $\nabla^2 Z$ is the Laplacian operator defined as

$$\nabla^2 Z = \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}. \tag{2}$$

Approximate Time Derivatives

Second time derivative

$$\frac{\partial^2 Z}{\partial t^2} \approx \frac{Z_{i,j}^{n+1} - 2Z_{i,j}^{n} + Z_{i,j}^{n-1}}{\Delta t^2}. \tag{3}$$

First time derivative (damping)

$$\frac{\partial Z}{\partial t} \approx \frac{Z_{i,j}^{n} - Z_{i,j}^{n-1}}{\Delta t}. \tag{4}$$

Approximate Spatial Derivatives (Laplacian)

In x

$$\frac{\partial^2 Z}{\partial x^2} \approx \frac{Z_{i+1,j} - 2Z_{i,j} + Z_{i-1,j}}{(\Delta x)^2}. \tag{5}$$

In y

$$\frac{\partial^2 Z}{\partial y^2} \approx \frac{Z_{i,j+1} - 2Z_{i,j} + Z_{i,j-1}}{(\Delta y)^2}. \tag{6}$$

Combined 2D Laplacian (five-point stencil)

$$\nabla^2 Z_{i,j} \approx \frac{Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1} - 4Z_{i,j}}{(\Delta x)^2}. \tag{7}$$

Solve for Next Time Value

$$\frac{Z_{i,j}^{n+1} - 2Z_{i,j}^{n} + Z_{i,j}^{n-1}}{\Delta t^2} = c^2 \nabla^2 Z_{i,j}^{n} - \gamma \frac{Z_{i,j}^{n} - Z_{i,j}^{n-1}}{\Delta t}. \tag{8}$$

$$\Longrightarrow$$

$$\begin{aligned} Z_{i,j}^{n+1} = 2Z_{i,j}^{n} - Z_{i,j}^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \big(Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1} - 4Z_{i,j}\big) \\ - \gamma \Delta t \,(Z_{i,j}^{n} - Z_{i,j}^{n-1}). \end{aligned} \tag{9}$$

Code Expression (Final Form)

$$\begin{aligned} Z_{\text{new}} &= \underbrace{2Z - Z_{\text{old}}}_{\text{leap-frog}} + \underbrace{c2\_dt2 \cdot \text{laplacian}(Z)}_{\text{curvature}} - \underbrace{(1 - \text{damping}) \cdot \Delta t \cdot (Z - Z_{\text{old}})}_{\text{damping correction}}, \\ c2\_dt2 &= \left(\frac{c\Delta t}{\Delta x}\right)^2. \end{aligned} \tag{10}$$

Stability Condition

$$\frac{c\Delta t}{\Delta x} \leq \frac{1}{\sqrt{2}}. \tag{11}$$

Boundary Conditions

Finite-difference solvers need a rule for every "missing neighbour" that lies outside the computational grid. Four families cover most practical situations:

1. Periodic (wrap-around)

Neighbour indices are taken modulo the grid size, so waves leaving one edge instantly re-enter from the opposite edge. Energy is conserved and no reflection occurs.

2. Dirichlet (fixed value)

We require

$$Z = Z_{\text{bnd}} \quad (\text{often } Z_{\text{bnd}} = 0).$$

Physically this models a string clamped at its end or an acoustic pressure-release surface. Waves reflect with *unit amplitude* and a 180° *phase flip*. No energy is absorbed unless additional damping terms are added.

3. Neumann (zero-gradient)

We impose

$$\partial_{\mathbf{n}} Z = 0,$$

i.e. the normal derivative vanishes. This represents a perfectly rigid wall (hard boundary) or a free string end. Waves reflect with unit amplitude and *no* phase flip.

4. Absorbing / radiative

To let energy leave the domain, we add a spatially varying damping term $-\sigma(\mathbf{x})\,Z$, often chosen to increase smoothly toward the grid edge ("sponge layer") or derived from a perfectly matched layer (PML). σ can itself be frequency-dependent to model materials that absorb high frequencies more strongly than low.

Unified view. All four cases can be expressed with the mask-based Laplacian of Section : set edge weights and (optionally) diagonal damping so the discrete operator already *knows* the boundary behaviour—no per-step if-statements are needed.

Internal Reflective (and Absorbing) Interfaces

Suppose we embed an arbitrary interface (e.g. a circle) inside a rectangular grid. A binary mask

$$M_{ij} = \begin{cases} 1, & \text{cell lies on the interface,} \\ 0, & \text{otherwise} \end{cases}$$

marks those locations.

1. Continuous boundary statements

- **Dirichlet:** $Z = 0$ at the interface (soft wall, pressure release). Reflection coefficient $R = -1$ (full magnitude, phase inversion).
- **Neumann:** $\partial_{\mathbf{n}} Z = 0$ (rigid wall). Reflection coefficient $R = +1$.

2. Why the ghost values are $-Z$ or $+Z$

Consider a one-dimensional grid with points $\ldots, i-1, i, i+1, \ldots$ where i is just *inside* the interface.

1. The second derivative at i needs Z_{i+1}, which we do not store. We invent a *ghost value* Z^*_{i+1}.
2. For a Dirichlet wall we want the physical displacement to be zero *exactly halfway* between i and $i+1$. Extending the function *oddly* ($Z(-x) = -Z(x)$) guarantees that midpoint is zero. Hence
$$Z^*_{i+1} = -\,Z_i.$$
3. For a Neumann wall we want zero slope, so we extend the function *evenly* ($Z(-x) = Z(x)$), giving
$$Z^*_{i+1} = Z_i.$$

Stencil check (1-D)

$$\frac{Z^*_{i+1} - 2Z_i + Z_{i-1}}{(\Delta x)^2} = \begin{cases} \dfrac{-3Z_i + Z_{i-1}}{(\Delta x)^2}, & \text{Dirichlet,} \\ \dfrac{-Z_i + Z_{i-1}}{(\Delta x)^2}, & \text{Neumann.} \end{cases}$$

Both choices recover the correct reflective behaviour in leap-frog time stepping.

3. Masked Laplacian (graph view)

Define edge indicators $\eta^{\uparrow}_{ij}, \eta^{\downarrow}_{ij}, \eta^{\leftarrow}_{ij}, \eta^{\rightarrow}_{ij} \in \{0,1\}$ that equal 1 when the neighbour is *not* masked. Let $d_{ij} = \sum \eta$. Then

$$\nabla^2_{\text{mask}} Z_{ij} = \big(\eta^{\uparrow} Z_{i-1,j} + \eta^{\downarrow} Z_{i+1,j} + \eta^{\leftarrow} Z_{i,j-1} + \eta^{\rightarrow} Z_{i,j+1}\big) - \begin{cases} d_{ij}\, Z_{ij}, & \text{Neumann,} \\ 4\, Z_{ij}, & \text{Dirichlet.} \end{cases}$$

Setting the diagonal to 4 in the Dirichlet case mimics the $Z^* = -Z$ rule while keeping the operator symmetric.

4. Partial transmission (hybrid interface)

We can interpolate continuously between full reflection and full transmission by scaling each edge that crosses the interface:

$$w \in [0,1], \quad R = \frac{1-w}{1+w}, \qquad w = \frac{1-R}{1+R}.$$

- $w = 0 \;\Rightarrow\; R = 1$: perfect mirror.

- $w = 1 \Rightarrow R = 0$: transparent membrane.
- $0 < w < 1$: partial reflection $\big(|R| < 1\big)$.

Implementationally we multiply the adjacency entries that cross the mask by w.

5. Absorbing (damped) interface

If a portion of the wave should disappear inside the interface (e.g., porous absorber), add a local damping term $-\sigma_{ij}\, Z_{ij}$ with $\sigma_{ij} > 0$. Frequency-dependent absorption can be modelled by letting σ (or w) depend on frequency—a topic beyond this primer.

Stability. Because the modified Laplacian remains symmetric positive semi-definite (and damping is non-negative), the standard CFL limit

$$c\,\Delta t/\Delta x \;\le\; 1/\sqrt{2}$$

still guarantees stability.

Graph Laplacian Viewpoint

The discrete Laplacian matrix L on a regular 2D grid with 4-connected neighbours corresponds to

$$LZ = -4Z_{i,j} + Z_{i+1,j} + Z_{i-1,j} + Z_{i,j+1} + Z_{i,j-1}. \tag{12}$$

Which in matrix form can be expressed as

$$L = \begin{bmatrix} -4 & 1 & 0 & 0 & \cdots \\ 1 & -4 & 1 & 0 & \cdots \\ 0 & 1 & -4 & 1 & \cdots \\ 0 & 0 & 1 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{13}$$

This is exactly the five-point stencil.