SCALAR-MASSIVE VECTOR MODEL WITH GRAVITY

DAMIAN SOWINSKI

Let's begin with the following action:

$$S = \int d^4x \; \sqrt{|g|} \left(\frac{R - 2\Lambda}{16\pi G} + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2(\phi) A_\mu A^\mu \right) \tag{1}$$

The first term is the Einstein-Hilbert Lagrangian with a cosmological constant. This term will lead to the Einstein equation that couples gravity to the matter fields. The next two terms correspond to the scalar field, its kinetic term followed by its potential. The last two terms are the kinetic term for the vector field and a coupling between it and the scalar field.

We will combine all the scalar and vector terms into a single matter lag range density, and then vary this action with respect to the metric tensor:

$$\delta S = \int d^4x \sqrt{|g|} \left(\frac{G_{\mu\nu} + \Lambda g_{\mu\nu}}{16\pi G} + \frac{\delta(\mathcal{L}_{matter} \sqrt{|g|})}{\sqrt{|g|}\delta g^{\mu\nu}} \right) \delta g^{\mu\nu} \tag{2}$$

Requiring that it vanish for arbitrary variations leads us to the Einstein equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3}$$

if we identify he stress energy tensor as

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|}\mathcal{L}_{matter})}{\delta g^{\mu\nu}} \\ &= -2\frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{matter} \\ &= \left(-\partial_\mu \phi \partial_\nu \phi + \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi + g_{\mu\nu} V(\phi) \right) + \\ &\qquad \left(-F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) + m^2(\phi) A_\mu A_\nu \end{aligned} \tag{4}$$

Here the first term is the stress energy of the scalar field and the second term is the stress energy of the vector field. The final term, which is an interaction between the scalar and vector fields, can be interpreted as giving an effective mass to the vector field.

We can find the equations of motion by applying the Euler Lagrange equations with respect to each of the fields. For the scalar field we find:

$$\nabla_\mu \nabla^\mu \phi + \frac{\partial V}{\partial \phi} + m \frac{dm}{d\phi} A_\mu A^\mu = 0 \tag{5}$$

and for the vector field we find:

$$\nabla_\mu F^{\mu\nu} - m^2(\phi) A^\nu = 0 \tag{6}$$

Together equations (3), (5), and (6) describe the dynamics of this model.