STRESS ENERGY OF A GAS OF PARTICLES

DAMIAN SOWINSKI

Consider the stress energy tensor of a collection of N particles with 4-momenta p_n^μ and mass m:

$$\begin{aligned} T^{\mu\nu} &= \sum_{n=0}^{N} \int d\tau \; p_n^\mu u_n^\nu \frac{\delta^4(x^\alpha - x_n^\alpha(\tau))}{\sqrt{|g|}} \\ &= m \sum_{n=0}^{N} \int \gamma_n dt \; v_n^\mu v_n^\nu \frac{\delta(t-t_n)\delta^3(\vec{x}-\vec{x}_n(t))}{\sqrt{|g|}} \end{aligned} \tag{1}$$

We can get a better idea as to the meaning of this expression by contracting it with the 4-velocity of some stationary observer to measure the total momentum of this gas as seen by the observer. Note that $u_\mu v^\mu = 1$. We have:

$$\begin{aligned} p^\mu_{gas} &= u_\nu T^{\mu\nu} \\ &= m \sum_{n=0}^{N} \gamma_n v_n^\mu(t_n, \vec{x}) \frac{\delta^3(\vec{x}-\vec{x}_n(t))}{\sqrt{|g|}} \end{aligned} \tag{2}$$

This is simply the sum of the 4-momenta of all the particles, so makes sense. Now let's consider the continuum limit by allowing $N \to \infty$. This allows us to define an average density of particles:

$$\begin{aligned} T^{\mu\nu} &\to \lim_{N\to\infty} T^{\mu\nu} \\ &= \int dt \; \lim_{N\to\infty} \sum_{n=0}^{N} m \frac{\delta^3(\vec{x}-\vec{x}_n(t))}{\sqrt{|g|}} \gamma_n \; v_n^\mu v_n^\nu \delta(t-t_n) \\ &= \int dt \int d^3\vec{x}' \rho(\vec{x}', t') v^\mu(\vec{x}', t') v^\nu(\vec{x}', t') \delta^3(\vec{x}-\vec{x}')\delta(t-t') \\ &= \rho(\vec{x}, t) v^\mu(\vec{x}, t) v^\nu(\vec{x}, t) \end{aligned} \tag{3}$$

Next we will assume this gas is isotropic, and homogenous. This means the velocities and densities will not depend on position, and that we can take a directional average of all the velocities. We have then:

$$\begin{aligned} v_x = v\sin\theta\cos\phi \qquad v_y &= v\sin\theta\sin\phi \qquad v_z = v\cos\theta \\ \langle T^{\mu\nu} \rangle &= \frac{1}{4\pi} \int d\Omega \; T^{\mu\nu} \\ &= \rho \; diag(1, \frac{v^2}{3}, \frac{v^2}{3}, \frac{v^2}{3}) \end{aligned} \tag{4}$$

Under these assumptions we see then that this gas of particles can be interpreted as a perfect fluid with

$$P = \frac{1}{3}\rho v^2 \tag{5}$$

Furthermore, taking the classical limit where $v \to 0$ we see that the gas becomes pressureless.