EVOLUTION OF SCALE FACTOR FOR DIFFERENT EOS

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Consider an FRW spacetime filled with a fluid satisfying the equation of state

$$P = \omega\rho \tag{1}$$

Let's examine the Friedmann equations for the scale factor when we plug in (1):

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{3}\rho \tag{2}$$

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + P)\\ &= -\frac{4\pi G}{3}\rho(1+3\omega) \end{aligned} \tag{3}$$

Combining (2) and (3) we find a differential equation for the scale factor:

$$\begin{aligned} a\ddot{a} &= -k\dot{a}^2 \\ k &= \frac{1+3\omega}{2} \end{aligned} \tag{4}$$

By noting that there are the same number of derivatives on each side, we assume an ansatz of a power law, for the case $\omega \neq -1$:

$$a(t) \propto t^n \tag{5}$$

$$\Rightarrow n(n-1) = -kn^2$$

$$\Rightarrow n = \frac{1}{1+k} = \frac{2}{3(1+\omega)}$$

$$a(t) \propto t^{\frac{2}{3(1+\omega)}} \tag{6}$$

For $\omega = -1$ we have instead from (2):

$$a\ddot{a} - \dot{a}^2 = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\dot{a}}{a}\right)^2 = 0$$

$$a(t) \propto e^{Ct} \tag{7}$$