Introducing features of agency into computational models of infectious disease

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Outline

1 Introduction to Computational Disease Modelling
- Graph Theory
- Games on Graphs

2 Extending existing graph models to account for agency
- Attributes of Agency
- Protection Rating Allocation and Defence Strategies

3 Using Compartmental Frameworks
- *SIR* graph model
- Adding a new state

4 Describing Compartmental Graph Models Exactly
- Required Equations
- Closures

5 Conclusion

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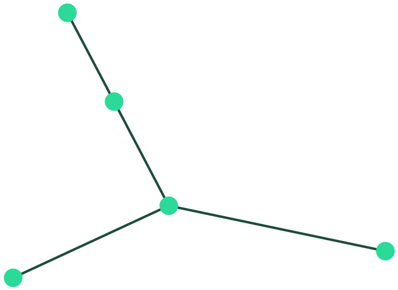
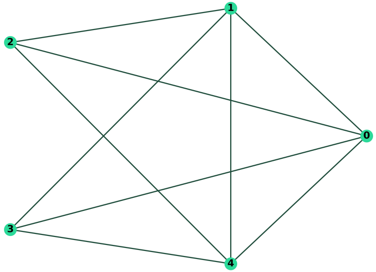


Examples of Graphs



Example: Erdős-Rényi

Random graph with a given number of vertices (in this case, 5) and a probability with which an edge exists between any two vertices.

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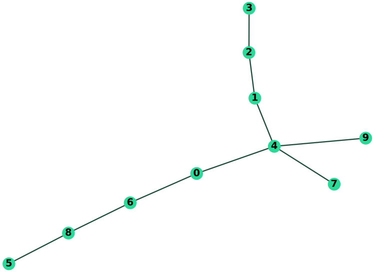

Example: Tree

A tree is a special type of graph where *any two vertices are connected by exactly one path.*

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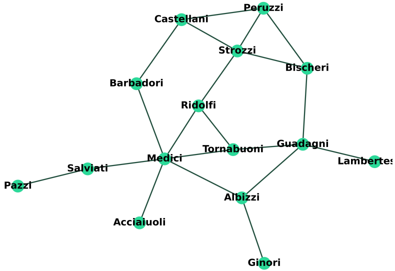

Example: Florentine Families Graph

Depicts the marital alliances between Renaissance Florentine families [4].

Using Graphs to Model Disease Spread

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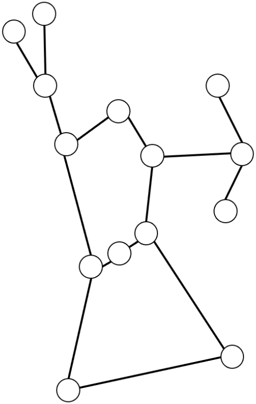
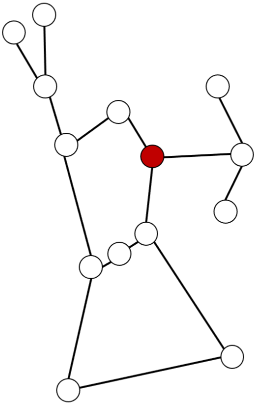
- Firefighter then ‘protects’ some other vertex.
- Fire spreads to any adjacent vertices neither protected nor burnt.
- Firefighter protects another vertex, the fire spreads again and so on.

Example



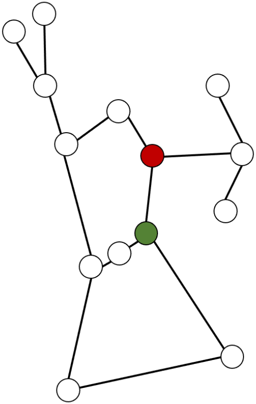
Original Graph

Example



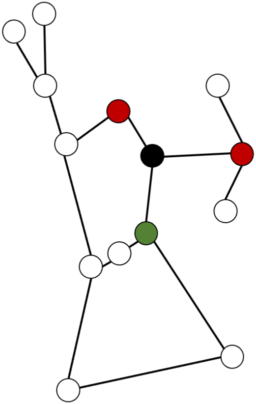
Outbreak

Example



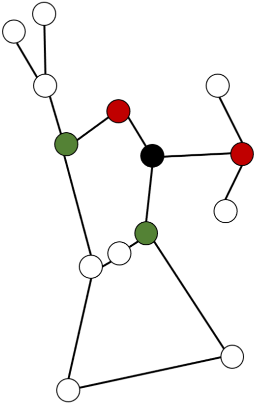
Defence

Example



Fire Spreads

Example



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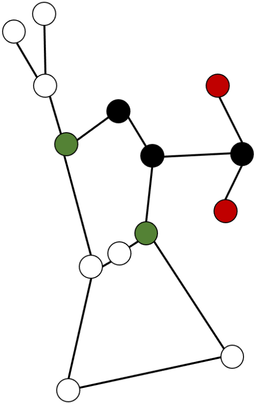
Example



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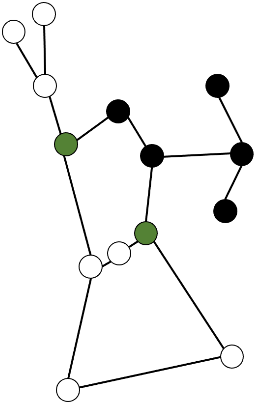

Fire contained

Firefighter as a Model for Disease Spread

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Limitations:

- Fairly rudimentary model for disease spread but already NP-hard.
- Defence and infection are discrete but epidemic propagation is a stochastic process.
- Only interventions in halting disease spread are *external*, no way for individuals to avoid contraction personally.

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Attributes of Agency

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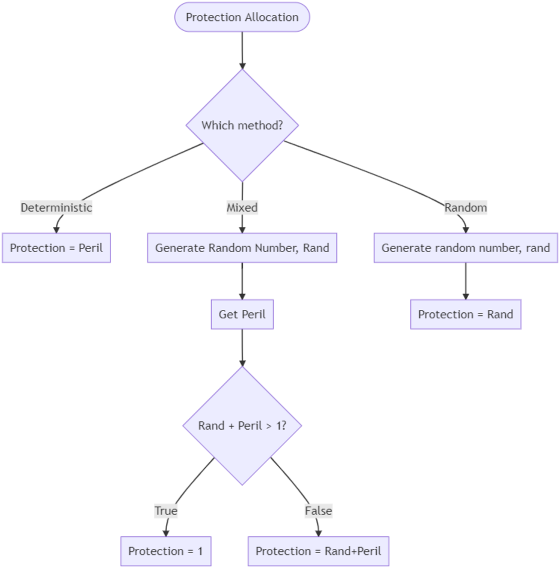
- Wearing PPE correctly
- Hand hygiene
- Strict physical distancing


Protection Allocation
Which method?
Deterministic
Mixed
Random
Protection = Peril
Generate Random Number, Rand
Generate random number, rand
Get Peril
Protection = Rand
Rand + Peril > 1?
True
False
Protection = 1
Protection = Rand+Peril

Defence Strategies

We should now ask what these amendments mean for how the game is played. In the usual formulation, general rule of thumb: *for sparse graphs, defend based on proximity to fire (breaking ties on degree); for dense graphs, defend based on degree (breaking ties on proximity)*.

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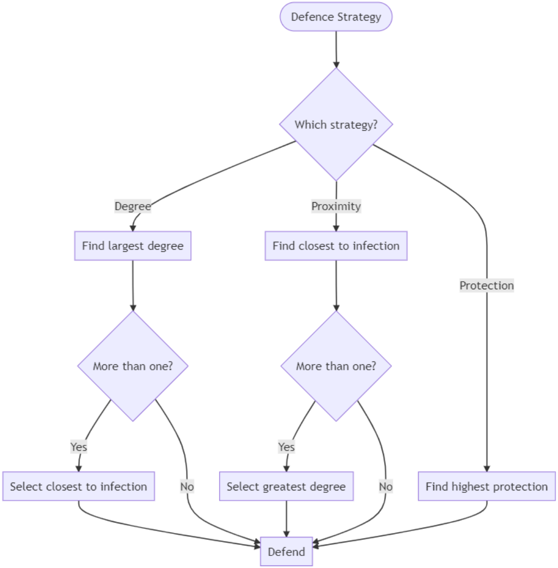
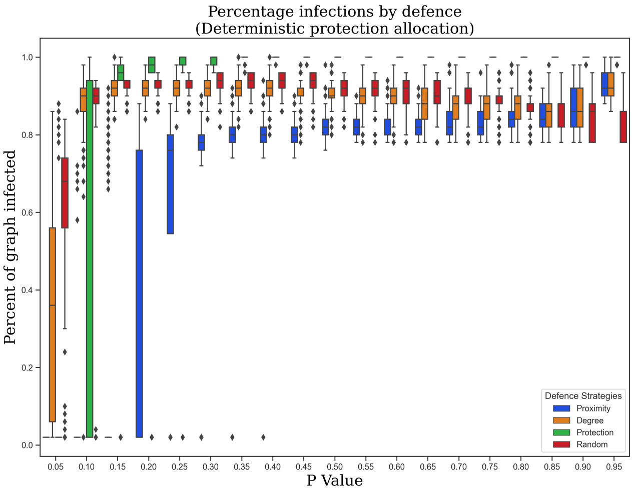
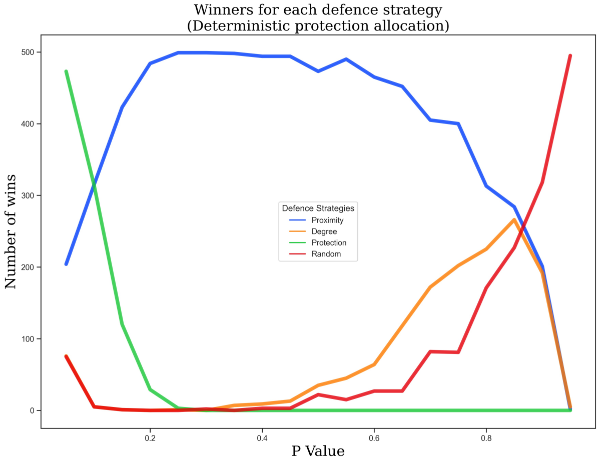
However, in our adjusted formulation we have more candidates for defence strategies. One such novel strategy is to defend based on highest agent protection rating.


Defence Strategy
Which strategy?
Degree
Proximity
Protection
Find largest degree
Find closest to infection
More than one?
More than one?
Yes
No
Yes
No
Select closest to infection
Select greatest degree
Find highest protection
Defend

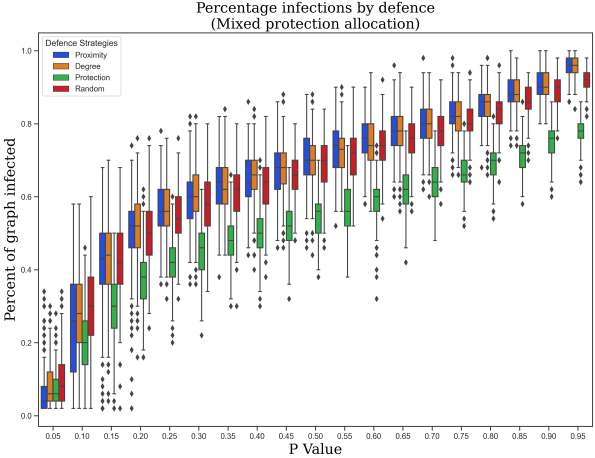
Deterministic Protection



Deterministic Protection



Mixed Protection



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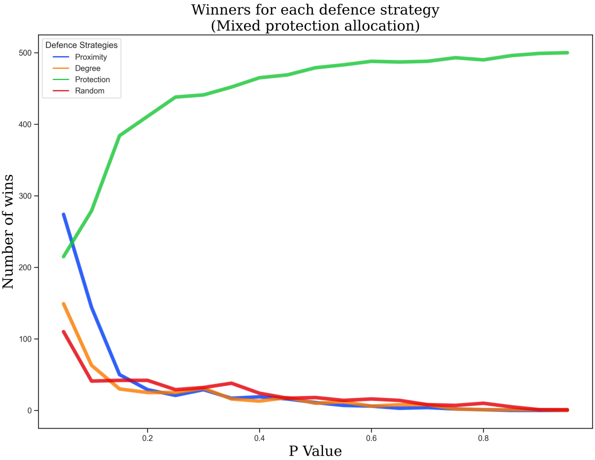
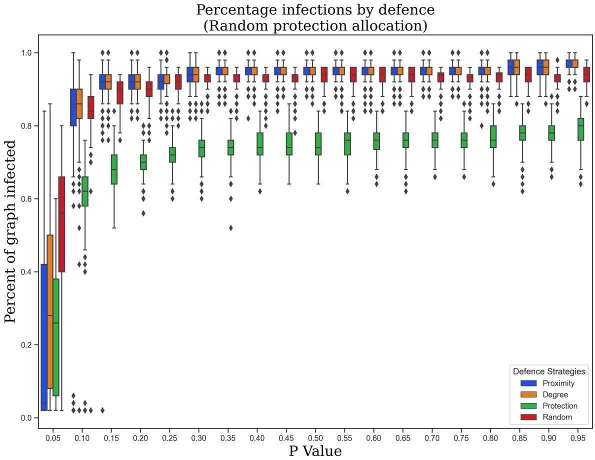


Random Protection



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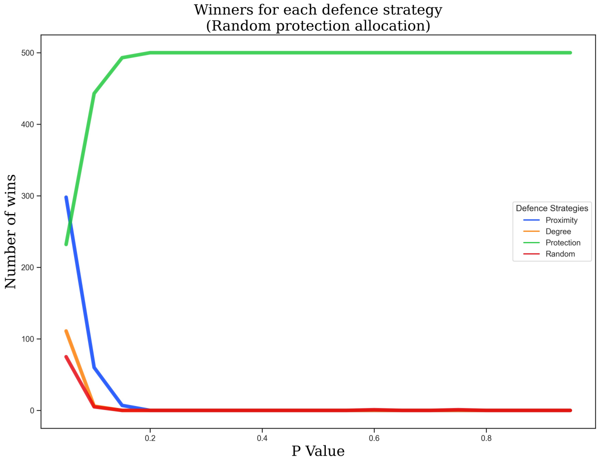

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- For transmission matrix T, assign $T_{ij} = \beta_i$ if there is a route of infection between i and j ($T_{ij} = 0$ otherwise).
- Often, we will consider unweighted and undirected graphs, but in general T_{ij} may not equal T_{ji}.

We can replace β_i with a term involving a transmission matrix of a network to begin extending the usual *SIR* model into a network realm.

$$
\begin{aligned}
\langle \dot{S}_i \rangle &= -\sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle \\
\langle \dot{I}_i \rangle &= \sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle - \gamma_i \langle I \rangle \\
\langle \dot{R}_i \rangle &= \gamma_i \langle I \rangle,
\end{aligned}
$$

which are the evolution equations given in [3].

Protected State

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Using these rates of protection and effectiveness, for fixed population size the differential equations become:

$$\langle \dot{S}_i \rangle = \alpha_i \langle P_i \rangle - \sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle - \zeta_i \langle S_i \rangle \tag{1}$$

$$\langle \dot{I} \rangle = \sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle - \gamma_i \langle I \rangle \tag{2}$$

$$\langle \dot{R}_i \rangle = \gamma_i \langle I \rangle \tag{3}$$

$$\langle \dot{P}_i \rangle = \zeta_i \langle S_i \rangle - \alpha_i \langle P_i \rangle . \tag{4}$$

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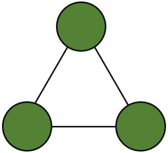
Numbers of Equations

Consider the triangle graph. The equations required to precisely express the system *SIR* dynamics of this network are as below [3].



$$\text{6 singles: } \langle \dot{S_1} \rangle, \langle \dot{S_2} \rangle, \langle \dot{S_3} \rangle, \langle \dot{I_1} \rangle, \langle \dot{I_2} \rangle, \langle \dot{I_3} \rangle. \tag{5}$$

$$\text{6 doubles: } \langle \dot{S_1 I_2} \rangle, \langle \dot{I_1 S_2} \rangle, \langle \dot{S_1 I_3} \rangle, \langle \dot{I_1 S_3} \rangle, \langle \dot{S_2 I_3} \rangle, \langle \dot{I_2 S_3} \rangle. \tag{6}$$

$$\text{6 triples: } \langle \dot{S_1 I_2 I_3} \rangle, \langle \dot{S_1 I_2 S_3} \rangle, \langle \dot{S_1 S_2 I_3} \rangle, \langle \dot{I_1 S_2 S_3} \rangle, \langle \dot{I_1 I_2 S_3} \rangle, \langle \dot{I_1 S_2 I_3} \rangle. \tag{7}$$

Using the equations for the *SIRP* model, we have the following equation requirements:

$$
\begin{aligned}
\text{9 singles: (5) and } & \langle \dot{P}_1 \rangle, \langle \dot{P}_2 \rangle, \langle \dot{P}_3 \rangle. \\
\text{18 doubles: (6) and } & \langle S_1 \dot{} P_2 \rangle, \langle P_1 \dot{} S_2 \rangle, \langle I_1 \dot{} P_2 \rangle, \langle P_1 \dot{} I_2 \rangle, \langle S_1 \dot{} P_3 \rangle, \langle P_1 \dot{} S_3 \rangle, \\
& \langle I_1 \dot{} P_3 \rangle, \langle P_1 \dot{} I_3 \rangle, \langle S_2 \dot{} P_3 \rangle, \langle P_2 \dot{} S_3 \rangle, \langle I_2 \dot{} P_3 \rangle, \langle P_2 \dot{} I_3 \rangle. \\
\text{24 triples: (7) and } & \langle S_1 \dot{S}_2 P_3 \rangle, \langle S_1 \dot{P}_2 S_3 \rangle, \langle S_1 \dot{I}_2 P_3 \rangle, \langle S_1 \dot{P}_2 I_3 \rangle, \langle S_1 \dot{P}_2 P_3 \rangle, \\
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& \langle P_1 \dot{S}_2 S_3 \rangle, \langle P_1 \dot{S}_2 I_3 \rangle, \langle P_1 \dot{I}_2 S_3 \rangle, \langle P_1 \dot{I}_2 I_3 \rangle, \langle P_1 \dot{P}_2 I_3 \rangle, \\
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The number of other equations will usually increase when we require closures, as closure equations will sometimes require equations that we don't usually consider as they are dynamically uninteresting, e.g. $\langle S_i S_j \rangle$ where i and j are adjacent.

Consider the equations below for calculating singles and pairs:

$$
\begin{aligned}
\langle \dot{S_i} \rangle &= -\sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle, \\
\langle \dot{I_i} \rangle &= \sum_{j=1}^{N} T_{ij} \langle S_i I_j \rangle - \gamma_i \langle I_i \rangle, \\
\langle \dot{S_i I_j} \rangle &= \sum_{k=1, k \neq i}^{N} T_{jk} \langle S_i S_j I_k \rangle - \sum_{k=1, k \neq j}^{N} T_i k \langle I_k S_i I_j \rangle \\
&\quad - T_{ij} \langle S_i I_j \rangle - \gamma_i \langle S_i I_j \rangle, \\
\langle \dot{S_i S_j} \rangle &= -\sum_{k=1, k \neq j}^{N} T_{ik} \langle I_k S_i S_j \rangle - \sum_{k=1, k \neq i}^{N} T_{jk} \langle S_i S_j I_k \rangle,
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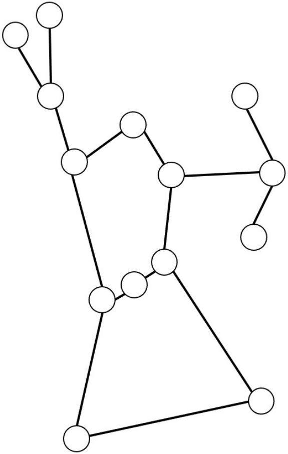
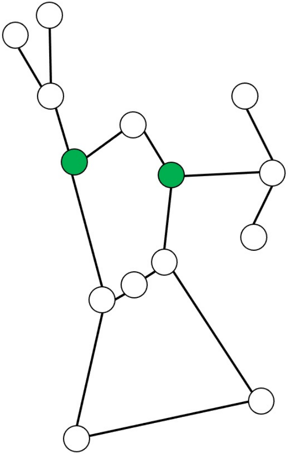
&\quad - T_{ij} \langle S_i I_j \rangle - \gamma_i \langle S_i I_j \rangle, \\
\langle \dot{S_i S_j} \rangle &= -\sum_{k=1, k \neq j}^{N} T_{ik} \langle I_k S_i S_j \rangle - \sum_{k=1, k \neq i}^{N} T_{jk} \langle S_i S_j I_k \rangle,
\end{aligned}
$$

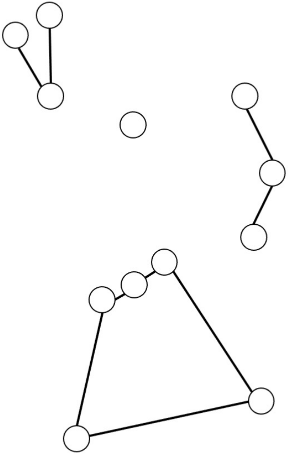
This is *not* a closed system - we require equations for triples.



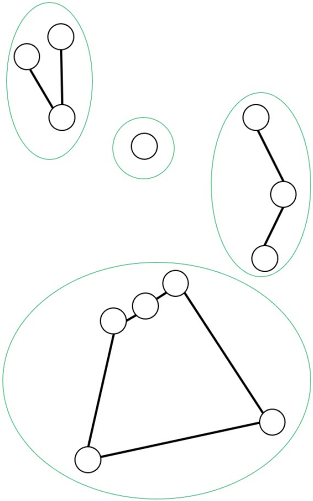
Original Graph



Identified Cut Vertices



Removed Cut Vertices



4 Connected Components

Calculating Closures from Cut-Vertex Sets

Let $G = \{V, E\}$ be a graph on N vertices.

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Let $G = \{V, E\}$ be a graph on N vertices. Consider a connected subset of vertices $F = v_1, v_2, \ldots, v_k \subset V$ and assume $\exists v_{i^*} \in F$ a cut-vertex in G, such that $F \setminus \{v_{i^*}\}$ is partitioned into at least two disjointed components with vertices $F_1 = v_1, v_2, \ldots, v_{i-1}$ and $F_2 = v_{i+1}, v_{i+2}, \ldots, v_k$ belonging to any such two, distinct and disjointed components or subnetworks.

Calculating Closures from Cut-Vertex Sets

Let $G = \{V, E\}$ be a graph on N vertices. Consider a connected subset of vertices $F = v_1, v_2, \ldots, v_k \subset V$ and assume $\exists v_{i^*} \in F$ a cut-vertex in G, such that $F \setminus \{v_{i^*}\}$ is partitioned into at least two disjointed components with vertices $F_1 = v_1, v_2, \ldots, v_{i-1}$ and $F_2 = v_{i+1}, v_{i+2}, \ldots, v_k$ belonging to any such two, distinct and disjointed components or subnetworks. Then the following equation holds:

$$\langle Z_{v_1} Z_{v_2}, \ldots Z_{v_{i-1}} S_{v_i^*} Z_{v_{i+1}} \ldots Z_{v_k} \rangle(t) = \frac{\langle Z_{v_1} Z_{v_2} \ldots Z_{v_{i-1}} S_{v_i^*} \rangle(t) \langle S_{v_i^*} Z_{v_{i+1}} \ldots Z_{v_k} \rangle}{\langle S_{v_i^*} \rangle(t)}$$

where $Z_{v_i} \in \{S, I\}$ $\forall v_i \neq v_{i^*}$. [3]

Bounding

Denote by $\mathrm{Ind}(v_{i_j})$ for $j = 1, 2, \ldots, L$ the number of subgraph v_{i_j} belongs to.

Upper bound:

$$N_{EQ}(G) = \sum_{i=1}^{P} m_i f_i - 2\sum_{j=1}^{L}(\mathrm{Ind}(v_{i_j}) - 1).$$

We take a sum across number of equations for all subgraphs and subtract unnecessary multiplications from cut-vertices being replaced into produced sub-graphs. [3]

Bounding

Upper bound:

$$N_{EQ}(G) = \sum_{i=1}^{P} m_i f_i - 2\sum_{j=1}^{L}(\mathrm{Ind}(v_{i_j}) - 1).$$

Note: m_i denotes the number of equations required for subnetwork i, e.g.

- An edge requires 7 equations (4 for singles, 3 for pairs)
- A triangle requires 22 equations
- A cycle on 4 vertices requires 45 equations

We can in fact bound these multipliers. From the requirements of a subgraph (connected, no cut vertices), for a number of vertices n our worst-case scenario is the complete graph (everything is connected to everything else) and our best-case scenario is the cycle on n vertices.

Outline

1 Introduction to Computational Disease Modelling
- Graph Theory
- Games on Graphs

2 Extending existing graph models to account for agency
- Attributes of Agency
- Protection Rating Allocation and Defence Strategies

3 Using Compartmental Frameworks
- *SIR* graph model
- Adding a new state

4 Describing Compartmental Graph Models Exactly
- Required Equations
- Closures

5 Conclusion

We discussed:

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- Existing approaches to computational models of disease spread

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- Existing approaches to computational models of disease spread
- How to extend existing approaches to better account for agency
- How using a compartmental framework can help introduce agency formally
- Feasibility of describing compartmental graph models exactly

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Questions?