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Level-4 Honours project

Mathematics of the Pipe Organ

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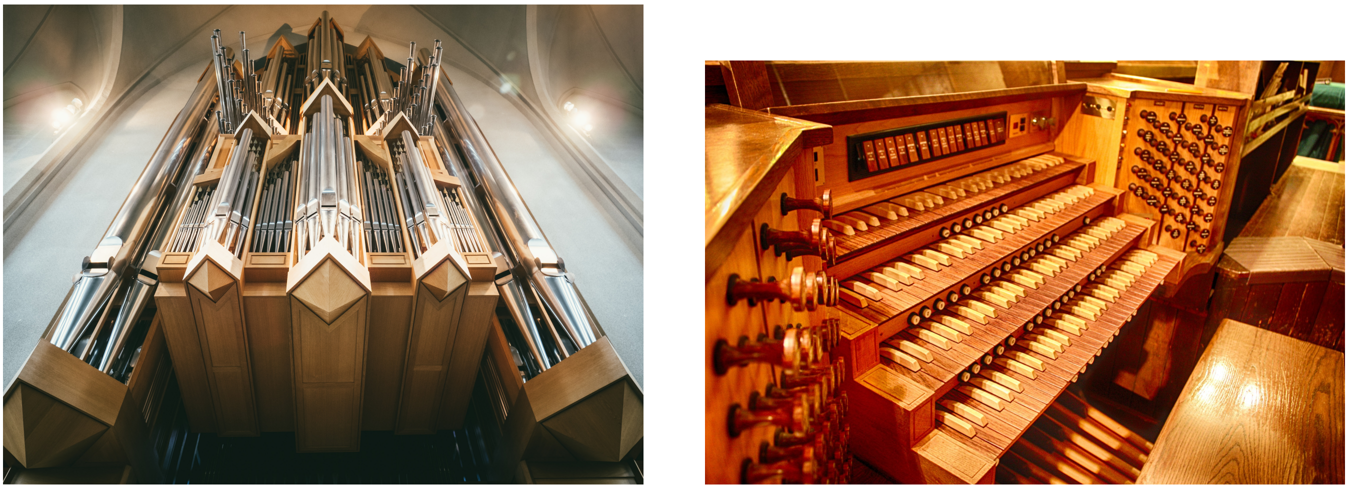
Abstract

In this paper, we investigate the mathematics behind the pipe organ. To do this, we define equations related to resonance in pipes such as the Acoustic Wave Equation, Associated Legendre Equation and the Bessel Equation. We later use these results to show how the sound of a 32-foot organ pipe can be recreated by using combinations of shorter pipes, which is clearly financially preferable for many situations. Further, we solve the Wave Equation in cylindrical and spherical polar coordinates to find solutions in cylindrical and conical pipes respectively. These are used to plot the pressure waveforms inside of organ pipes. Our results will be used to confirm and explain common phenomena known of the pipe organ, such as the sounding of even harmonics only by open ended pipes. Only a brief discussion of reed pipes is included, along with an explanation as to why they are difficult to consider and model mathematically in similar ways as flue pipes. We will also provide the theoretical groundwork for possible experimental work on spatial geometric and sound pressure level spectral research, which we hope will be taken up in future in order to determine methods for pipe design for admittance of high-resonance harmonics.

Contents

1 Introduction

Acoustics as a discipline deals with any and all mechanical waves, not only sound waves propagating through air as our discussion will focus upon. One could use the same and similar results as in this work to study waves propagated through solids, liquids and other gases, and these waves can be vibrations and other forms of sound, such as ultra- or infra-sound. Acousticians develop methods to improve many technological marvels, most notably in the audiological industry with speakers and other sound production equipment. Indeed, the entirety of human civilisation is founded upon our ability to communicate, and so (with the exception of visual languages such as Sign Languages, used by and for those with hearing impairments) hearing is a vital component of our cultures, even our survival. Hence, the study of acoustics more generally reaches far into medicine, to improve the hearing of those with impairments thereof; architecture, to study how sound will travel within spaces and whether it is necessary to impede or amplify waves depending on the function(s) of the space; and many other aspects of industry. For the purposes of this research, we will limit the remit of our discussion to the domain of sound production in pipes composing a pipe organ in order to explain many of the phenomena associated with the instrument and some aspects of the musicality of the instrument.



(a) An example of a pipe organ

(b) A particularly attractive example of an organ console in the baroque style

Figure 1: Aspects of the pipe organs - see Appendix D for a note on the use of images in this paper.

The pipe organ as we know it today is the product of 4,000 years of innovation and engineering. Today, they are imposing and powerful constructions, hence the term the 'King of Instruments.' From the water organ of Ancient Greece to the present day, the organ reflects both developments in scientific understanding and the tastes of the given time, and has enjoyed uses such as entertainment (with theatre organs such as the Wurlitzer), religious worship and even therapeutic purposes in hospitals [7]. Throughout its history, a thorough understanding of the mathematical reality of its workings has been sought after; through the use of various forms of analysis, we will aim to detail the current understanding of the mathematics behind this instrument by systematically examining the various forms of organ pipes and the mechanisms by which they produce sound.

The pipes of an organ, organised inside of the *wind chest*, are actuated by manual(s) composing the console of the organ. The manuals are the keyboard(s) on the console (as seen on Figure 1b, which has four manuals) - they are typically a good deal smaller than the keyboard of a standard piano. In order to select a set of pipes, the organist will pull out the *stop* corresponding to it. In Figure 1b, they are small, circular pistons to the left and right of the manuals. Pulling out a stop releases a barrier, which otherwise blocks wind from the blowers from entering the pipes, allowing air into the pipe of a particular pitch when the corresponding key is depressed. There are a great many technicalities involved in this process, which vary enormously between instruments. Combinations of stops are drawn by the organist to give a wide variety of sounds, effects and timbres.[1] [30]

The keen eye may also note the small, rectangular controls above the uppermost manual: these are known as rocking tabs, and are most commonly *couplers.* They give the organist the ability to play stops from more than one manual at a time by 'linking' them. That is, playing one keyboard will actuate the notes corresponding to the same keys being played on both the keyboard the organist is currently playing at but also the one to which it has been coupled. For example, coupling the Great Organ to the Swell Organ will present the organist with the option to control the volume of the stops belonging to the Swell since they are enclosed inside a box with flaps on the front that can be opened and closed using expression pedals underneath the manuals, operated by the feet. On the topic of controlling aspects of the instrument with feet, one may also note the pedal board underneath the manuals also visible in Figure 1b. These are keys to be operated by the feet and provide the very lowest bass sounds. There are two different types of pipes: flue and reed pipes. A flue pipe sounds due to the vibrations in the air inside it: air from the flue is pushed across an opening against the *Labium* - the sharp ridge at the bottom of the aperture in a flue pipe over which air is forced at pressure. This gives rise to a region of low pressure below the opening. Then, when this pressure is low enough, the flow of air is directed underneath the lip of the *Labium*, and then another region of low pressure forms over the *Labium* and the process repeats there, leading to a constant switching between high and low pressure, giving rise to a standing wave and therefore sound - see Figure 8 [3], [16].

Reed pipes, on the other hand, sound due to the presence of moving parts: a small piece of brass is acted upon by air under pressure and vibrates against a resonator at the desired frequency. Some reed stops are mounted horizontally to the wind chest (and can be see sticking out from the instrument) in order to direct their sound towards the listener and are known as *en chamades*: if this is not possible, then an organ builder may opt for a hooded pipe, which includes a right angle turn in the vertical resonator. [24, Chapter 8]

In the following sections, we will delve more into the mathematics explaining the results known of pipe organs. Firstly we provide in Section (2) many concepts utilised later on in the work so that they may be used without extensive commentary during the later computations and manipulations. Further, in Section (3) we will consider two main notions, beginning with the cylindrical flue pipe. To describe and discuss this, we consider the wave equation in cylindrical polar coordinates and the separable solutions thereof. In a similar manner, we then consider the wave equation in spherical polar coordinates in order to discuss conical organ pipes. In both cases, we will show that the solutions obtained give rise to pictures coherent with our understanding of standing waves and sound production in organ pipes. We end this section by discussing the difficulties in comprehensive and satisfactory mathematical derivations of the same kind with the second main notion, reed pipes. Following this, in Section (4), we discuss and lay the framework for possible future (more experimentally-oriented research) by detailing a simple mechanical model with which to discuss the Röhrflute organ pipe and the results others have found in similar research. Finally, in Section (5), we will conclude this work with a discussion of what has been found in this research paper and interweave these reflections with speculation on possible further research, particularly experimentally.

[1] Anecdotally, the author once met an organist who - given a large enough selection of stops on an organ - could use a combination of stop selection and manual technique to produce an incredibly convincing imitation of church bells. The author has never since been able to recreate this effect!

2 Mathematical Basis of Sound

2.1 Resonance in Pipes

2.1.1 Helmholtz Resonators

An organ pipe can be thought of as a Helmholtz resonator: a volume with a small opening containing a fluid in which resonance takes place. To see this, note first that resonance is the ability of a system to take energy from a given frequency and vibrate sympathetically to it, and the resonant frequency of such a resonator is approximately [18]

$$f = \frac{v}{2\pi}\sqrt{\frac{A}{lV}}. \tag{2.1}$$

Here, v is the speed of sound in ms^{-1}, A is the area of the cross-section of the neck, V is the volume of the resonance chamber and l is the length of the neck - see Figure 2. However, this is only an approximation owing to the effect of air molecules around the opening. This is called 'end correction,' and will be discussed later in this paper.

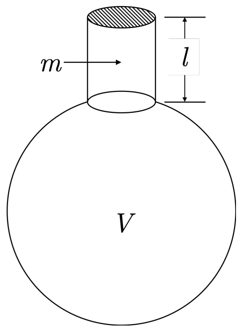


Figure 2: A Helmholtz Resonator

The Helmholtz equation is a special, time-independent case of the Acoustic Wave Equation:

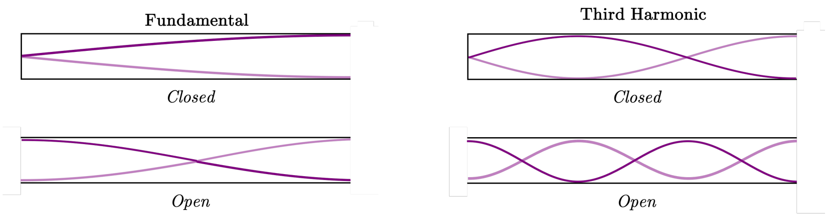
$$\nabla^2 f = -k^2 f \tag{2.2}$$

which can be solved using a separation of variables. [32, p. 4]
Because Helmholtz resonators produce a standing wave, we can consider the normal mode,[2] where the wave moves with constant frequency and phase relation everywhere along it. This is a characteristic of *all* Helmholtz resonators, and the property of absorbing external frequencies and internally resonating possessed by Helmholtz resonators is a direct consequence of the normal mode of a standing wave being produced inside of it. Specifically, this means normal modes form an important part of the fingerprint or characteristics of an organ pipe.

2.1.2 Modes of Stopped Organ Pipes

We notice in Figures 3 and 4 that pipes open at both ends have an antinode at both ends and pipes closed at one end have a node at the closed end and an antinode at the open end.[3] This is true in general: *open ends produce antinodes and closed ends produce nodes.* We can see in Figure 3 that the wave of the harmonic inside the closed pipe is one quarter of a sine wave, and so $\lambda_1 = 4l$; that of the

[2] Also called an 'eigenmode.'
[3] As before, we neglect to consider end correction for simplicity as these equations are purely illustrative.



(a) Fundamental Harmonic **(b)** Third Harmonic

Figure 3: Fundamental and Third Harmonic in Open and Closed Pipes

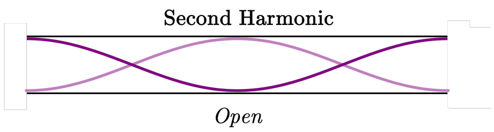


Figure 4: Second Harmonic in an Open Pipe

third harmonic is three quarters of a sine wave, so $\lambda_2 = \frac{4l}{3}$ and so on. We find that the wavelength is given by, for $n \in \mathbb{N}$:

$$\lambda_n = \frac{4l}{2n-1}.$$

This confirms that a stopped pipe will sound only odd harmonics. Similarly, we can deduce the expression for open pipes which do sound even harmonics: for $m \in \mathbb{N}$ we find

$$\lambda_m = \frac{4l}{2m} \implies \lambda_m = \frac{2l}{m}.$$

In Table 1, we see the intervals represented by each harmonic up to the fifth harmonic. This is determined using that each harmonic is an integer multiple of the fundamental frequency: for example, if the fundamental frequency is 10Hz, then the second harmonic is $2 \cdot 10 = 20$Hz, the third is $3 \cdot 10 = 30$Hz and so on. While theoretical harmonic frequencies generally deviate from "actual" tuning, this example is simply an illustration of how to determine the harmonics generated by a given organ pipe [12].

Harmonic	Interval	Example on C4	Theoretical Freq., Hz
First	Fundamental	C4	262
Second	Octave	C5	524
Third	Fifth	G5	1048
Fourth	Fourth	C6	2096
Fifth	(Major) Third	E6	4192

Table 1: Table of harmonics and the musical intervals they represent, relative to the fundamental frequency

2.2 End correction

As previously discussed, the antinode at the open end of a pipe lies slightly beyond the end of the internal length.[4] That is, when a wave forms in an organ pipe, the end of the wave lies outside of the

[4] That is, the length of the resonator. This is also applicable to reed pipes: we consider the length of the pipe in order to produce the best quality sound, but it is not an essential component as it is for flue pipes.

end of the pipe due to the effects of air particles outside of the pipe. Hence, we need to determine a theoretical *end correction* to model these pipes, a value we add on to the physical internal length of the pipe to determine harmonic frequency. It is necessary to consider end corrections for both ends of an open pipe, since both have antinodes that lie slightly further from the actual end of the theoretical length of pipe. End correction is approximately $0.6r$, [13] meaning it is exclusively dependent on the radius of the pipe, r. Using $\lambda = v/f$, the end correction, denoted δ, is

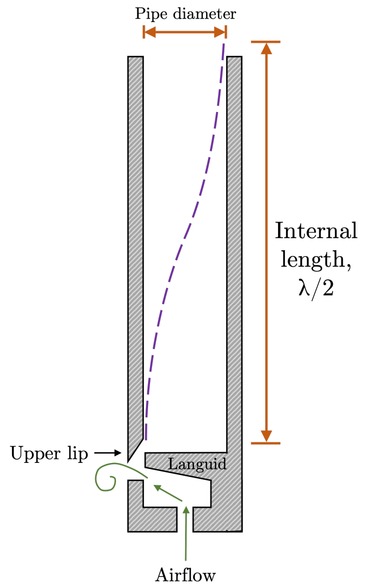
$$\text{For an open pipe:} \quad l + 2\delta = \frac{\lambda}{2}n,$$
$$\text{For a closed pipe:} \quad l + \delta = \frac{\lambda}{4}(2n-1).$$

2.3 The (Acoustic) Wave Equation

Acoustic waves are a form of energy transmission involving adiabatic compression and decompression, which can only occur through a medium. The air jets which cause a standing wave to be produced in an organ pipe are typically characterised by vortices, a motion we describe as turbulent. To this end, organ builders often make small cuts in the languid (a horizontal piece of material, usually metal, which blocks most of the air and forces it through a much smaller aperture to increase the pressure - see Figure 8) in order to ensure fully turbulent motion. This is depicted in Figure (5) [9]. Such waves are described by the Acoustic Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) \tag{2.3}$$

Figure 5: A side, cross-sectional view of the organ flue pipe depicting air flow and vortex generation.



2.4 The Bessel Equation and Functions

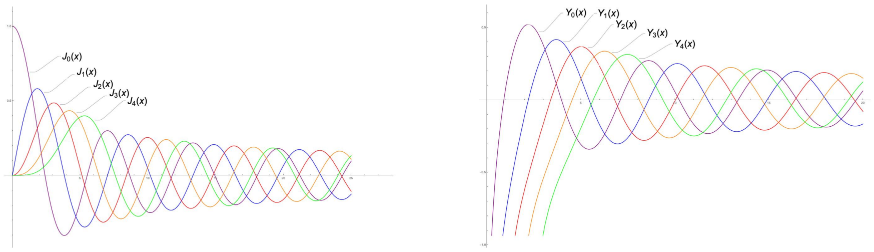
Solutions of the Bessel Equation are well-known for having uses in fields as diverse as Quantum Mechanics, the study of DNA and Probability Theory. We will see that solutions to the Bessel equation can be used to solve the wave equation in cylindrical and spherical polar coordinates. Indeed, if we express the Helmholtz equation (2.2) in polar coordinates, we find that the spatial derivative is such that the radial variation can be solved by Bessel functions. The Bessel equation is a linear second order ordinary differential equation

$$x^2 y'' + x y' + (x^2 - v^2) y = 0, \tag{2.4}$$

where v is the *order* of the Bessel equation. [1, p. 247] The solutions to the Bessel equation are known as the Bessel functions.We first note that - since the Bessel equation is a linear, second-order differential equation - we know that it must have two linearly independent solutions, so the general solution can be written as

$$y = C_1 J_v(x) + C_2 Y_v(x), \tag{2.5}$$

where C_1 and C_2 are arbitrary constants and J_v and Y_v are the Bessel functions of the first and second kind respectively, which we will now discuss.



(a) Bessel Functions of the First Kind

(b) Bessel Functions of the Second Kind

Figure 6: The first four Bessel functions of the first and second kind

2.4.1 Bessel functions of the first kind

The Bessel function of the first kind can be represented using the Gamma function, the generalisation of the factorial function from the integers to the real numbers, which is given as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \tag{2.6}$$

Using (2.6), we can write $J_v(x)$ from our general solution (2.5) as

$$J_v(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{\Gamma(p+1)\Gamma(p+v+1)} \frac{x}{2}^{2p+v}$$

which we can see, from Figure 6(a), are oscillating, real-valued functions [1].

2.4.2 Bessel functions of the second kind

The Bessel function of the second kind is related to that of the first kind in the following way:

$$Y_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}.$$

Note that, for $v \in \mathbb{Z}$, the functions J_v and J_{-v} are *not* linearly independent, and we therefore have:

$$J_{-v}(x) = (-1)^v J_v(x)$$
$$Y_{-v}(x) = (-1)^v Y_v(x),$$

but for $v \notin \mathbb{Z}$, J_v and J_{-v} *are* linearly independent [1]. We will not use these functions, and so this will end our discussion of Bessel functions of the second kind.

2.5 Associated Legendre Equations

Another concept we will utilise is the Associated Legendre Polynomial. These are the solutions to the Associated Legendre Equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_l^m(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0. \tag{2.7}$$

We define the associated Legendre functions P_l^m for $-l \leq m \leq l$ as [4]

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{n+m}(x^2-1)^n,$$

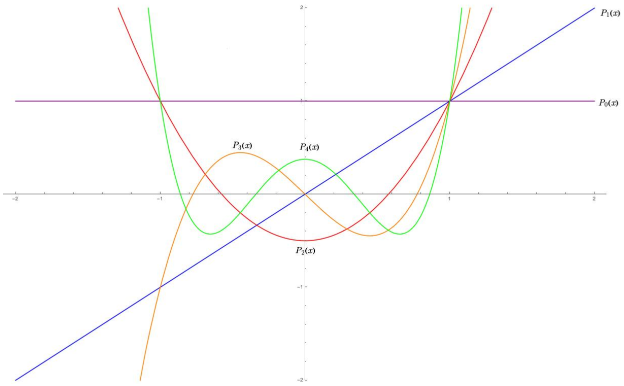


Figure 7: The first 5 (Unassociated) Legendre Polynomials

Where Rodrigues' Formula for the *unassociated* Legendre polynomials of the first kind is given by[5] [15]

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Moreover, we can see that the functions P_n^m are either polynomials of degree n or $\sqrt{1-x^2}$ multiplied by a polynomial of degree $(n-1)$ for m even or odd respectively. Hence, the parity of P_n^m is $(-1)^{n+m}$ and we find, for $m \geq 0$

$$\sqrt{1-x^2}P_n^m(x) = \frac{1}{2n+1}\left(P_{n-1}^{m+1} - P_{n+1}^{m+1}\right).$$

Further, for $m < 0$ we have that the associated Legendre polynomials of the first kind are given by

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!}P_n^m(x).$$

Note that, for $m = 1$ the Legendre polynomials are called *unassociated.* We calculate a handful associated Legendre polynomials as follows:

$$\begin{aligned} P_0^0(x) &= 1 & P_2^0(x) &= \frac{1}{2}(3x^2-1) \\ P_1^0(x) &= x & P_2^1(x) &= -3x(1-x^2)^{1/2} \text{[c]} \\ P_1^1(x) &= -(1-x^2)^{1/2} & P_2^2(x) &= 3(1-x^2) \end{aligned}$$

[5]Note that $m = 0 \implies P_n^m = P_n$.

3 Mathematics of the Pipe Organ

3.1 Harmonics and Resultant Stops

As we have seen, the sound of a rank of pipes is largely characterised by its harmonics, meaning a pipe can be described as having an harmonic 'fingerprint' of sorts. One interesting use of this is the resultant stop: for space or budget constraints, it may not be possible to have a $32'$ rank in a given organ, so the organ builder may use a combination of smaller pipes in order to create the illusion of a $32'$ pipe, known as the *resultant.* To show this, consider the equation

$$f_n = n\left(\frac{v}{2l}\right) \tag{3.1}$$

where v is the speed of sound, $n \in \mathbb{N}$ is the number of the harmonic and l is the length of the organ pipe. We can obtain Table 2.[6] For example, to calculate the first harmonic of a $32'$ pipe we use Equation 3.1 with $n = 1$,

$$(3.1) \implies f_1 = \frac{v}{2l} \implies f_1 = \frac{343.2}{2 \times 9.753} \approx 17.59.$$

Following this process yields Table 2. We can see that an open $16'$ pipe combined with either an open $10\frac{2}{3}'$ or a closed $5\frac{1}{3}'$ pipe fills out harmonics 2, 3, 4, 6, 8 and 9. What of the other harmonics? For the fundamental of the "true" $32'$ pipe, this is at the lower bound of human hearing (around 16Hz) so the musical character of the notes is the same, the only difference is a missing 'bass' feel [29].

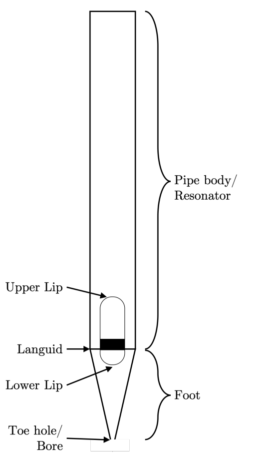
Harmonic	1	2	3	4	5	6	7	8	9
$32'$	17.6	35.2	52.8	70.4	88.0	105.6	123.2	140.8	158.4
Open $16'$	-	35.2	-	70.4	-	105.6	-	140.8	-
Open $10.66'$	-	-	52.8	-	-	105.6	-	-	158.4
Closed $5.33'$	-	-	52.8	-	-	-	-	-	158.4

Table 2: Table of Harmonics in Hz [29]

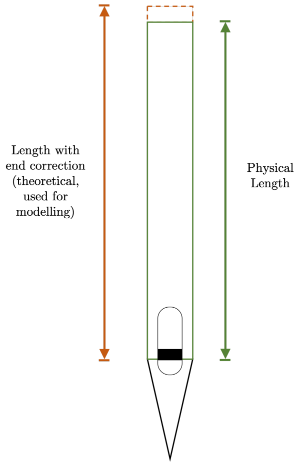
3.2 Cylindrical Flue Pipes

Flue pipes usually come from one of three basic tonal families: diapasons, flutes and strings. Organ builders construct flue pipes to "imitate" the sounds of the instruments they take their names from. The flute is larger in diameter, so the lower harmonics (especially the fundamental) are more prevalent, giving a full and rich quality. The diapason (or principal) pipe has a smaller diameter, with a more even distribution of harmonics that gives rise to the "classic" organ sound. String pipes are the narrowest flue pipes and sometimes have a wooden rod running through the centre which has a similar effect to dragging a rosined bow across a string. Having the smallest diameters of the flue pipes, they produce more upper partials and favour the fundamental frequencies far less, giving a bright sound [30]. It is also common for organs to include undulating stops, ranks of pipes slightly out of tune with the organ, for example the *voix céleste* and *unda maris* that are tuned slightly sharp and flat respectively, providing colour from alternating constructive and destructive interference. [24, Chapter 8]

[6] This idea was described by Bennett, [29] who describes the use of an open $16'$ pipe with a closed $5.33'$ pipe to create a "phantom" $32'$ pipe, although the values differ slightly - owing to either differing degrees of accuracy or end correction he has neglected to discuss!



(a) Diagram of a Flue Pipe



(b) Actual and theoretical lengths of a flue pipe

Figure 8: The Flue Pipe and a representation of end correction

Figure 9: The facade of pipe organs is composed of flue pipes, since reed pipes are usually unsightly

3.2.1 The Wave Equation in Cylindrical Polar Coordinates

In order to study standing waves inside of a cylindrical organ pipe, we will consider solutions to the wave equation in three spacial dimensions, which is given as[7]

$$\nabla^2 U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}.$$

Using the definition of the Laplacian operator, we can express this in cylindrical polar coordinates (ρ, ϕ, z). Now, consider $f(\rho, \phi, z)$. We have, by the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}.$$

[7]See [22] for a similar discussion.

Taking another derivative with respect to x gives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \rho^2}\left(\frac{\partial \rho^2}{\partial x}\right) + \frac{\partial^2 f}{\partial \phi \partial \rho}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \rho}\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 f}{\partial \rho \partial \phi}\frac{\partial \rho}{\partial x}\frac{\partial \phi}{\partial x} \tag{3.2}$$

$$+ \frac{\partial^2 f}{\partial \phi^2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial f}{\partial \phi}\frac{\partial^2 \phi}{\partial x^2}. \tag{3.3}$$

We eventually find the first term of the Laplacian is

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} =& \frac{\partial^2 f}{\partial \rho^2}\cos^2(\phi) - \frac{\partial^2 f}{\partial \phi \partial \rho}\frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial f}{\partial \rho}\frac{\sin^2(\phi)}{\rho} \\ &- \frac{\partial^2 f}{\partial \rho \partial \phi}\frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial^2 f}{\partial \phi^2}\frac{\sin^2(\phi)}{\rho^2} + \frac{\partial f}{\partial \phi}\frac{2\sin(\phi)\cos(\phi)}{\rho^2}.\end{aligned}$$

Similarly, we find the second term to be

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} =& \frac{\partial^2 f}{\partial \rho^2}\sin^2(\phi) + \frac{\partial^2 f}{\partial \phi \partial \rho}\frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial f}{\partial \rho}\frac{\cos^2(\phi)}{\rho} \\ &+ \frac{\partial^2 f}{\partial \rho \partial \phi}\frac{\sin(\phi)\cos(\phi)}{\rho} + \frac{\partial^2 f}{\partial \phi^2}\frac{\cos^2(\phi)}{\rho^2} - \frac{\partial f}{\partial \phi}\frac{2\sin(\phi)\cos(\phi)}{\rho^2}.\end{aligned}$$

This gives us the Laplacian in cylindrical polar coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Hence, the wave equation (2.3) in cylindrical polar coordinates is given by

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}. \tag{3.4}$$

Then, let $U = U(\rho, \phi, z, t) = R(\rho)\Phi(\phi)Z(z)T(t)$ be a product solution to equation (3.4). Then this can be written as

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}\left(R'' + \frac{1}{\rho}\right) + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z}. \tag{3.5}$$

3.2.2 Time (t) dependence

Considering equation (3.5), we see that since the right hand side (RHS) is independent of t, the left hand side (LHS) must be equal to some separation constant $-a^2$, hence

$$\frac{1}{c^2}\frac{T''}{T} = -a^2 \iff T'' = -c^2 a^2 T.$$

This is the harmonic oscillator equation, which has solution

$$T_a(t) = T_+ e^{iact} + T_- e^{-iact}.$$

We require a real solution, so we take this to be, for some constant $\alpha \in \mathbb{R}$,

$$T_a(t) = T_0 \sin(act + \alpha),$$

3.2.3 Height (z) dependence

Again, consider equation (3.5). We now have

$$-a^2 - \frac{Z''}{Z} = \left(R'' + \frac{1}{\rho}\right)\frac{1}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi}. \tag{3.6}$$

Noting the RHS is independent of z implies the LHS must equal some separation constant $-b^2$, so we have

$$-a^2 - \frac{Z''}{Z} = -b^2 \implies Z'' + (a^2 - b^2)Z = 0.$$

Again, the harmonic oscillator equation. We consider the three possibilities for a and b:

- Solutions oscillate when $a^2 < b^2 \implies Z = Z_0 e^{\pm\sqrt{a^2-b^2}z}$.
- Solutions grow exponentially when $a^2 > b^2 \implies Z = Z_0 e^{\pm\sqrt{a^2-b^2}z}$ We require the wave to be constricted to the internal volume of the cylinder, so we discount this solution.
- Solutions are constant when $a^2 = b^2$, that is $a = b \implies Z^{\pm}_{a,a} = Z_0 e^{\pm\sqrt{a^2-a^2}z} = Z_0 e^0 = Z_0$. We will also discount this solution, as we are interested only in oscillatory solutions.

We therefore impose the constraint $a^2 > b^2$. For simplicity, define $\kappa^2 = a^2 - b^2$. We therefore obtain the solution

$$Z_\kappa = Z_+ e^{i\kappa z} + Z_- e^{-i\kappa z}.$$

Once again, to consider only the real case we take this solution to be, for $A, B \in \mathbb{R}$,

$$Z_\kappa = A\cos(\kappa z) + B\sin(\kappa z).$$

We must now consider two cases: the open and closed organ pipe. As before, we have an antinode at open ends and a node at closed ends. Hence, our constraints are determined by taking a wave in the open pipe as maximal at both ends and the closed pipe as being zero at the closed end, which yields:

For an open pipe: $Z^O_\kappa = A\cos(\kappa z)$, and for a closed pipe: $Z^C_\kappa = B\sin(\kappa z)$

Let the length of the pipe be $Z = L$, a constant. Then, the above results give, for $m \in \mathbb{N}$

$$\text{Open pipe: } Z^O_\kappa(L) = A\cos(m\pi) \qquad \implies \kappa_O = \frac{m\pi}{L} \tag{3.7}$$

$$\text{Closed pipe: } Z^C_\kappa(L) = B\sin\left[\frac{\left(m\pi + \frac{\pi}{2}\right)}{L}\right] \qquad \implies \kappa_C = \frac{\pi(2m-1)}{2L} \tag{3.8}$$

Reassuringly, the plots in Figures (10), (11) and (12) confirm the previous diagrams of standing waves in pipes in Figures (3) and (4). We again confirm that open ends give rise to antinodes and closed ends give rise to nodes, and we see that inputting $m = 2$ for the closed pipe actually gives the third harmonic, confirming the result earlier that closed pipes do not sound even numbered harmonics. To simplify, we index by 3 for the third harmonic, rather than 2 in this case as the choice of m would dictate.

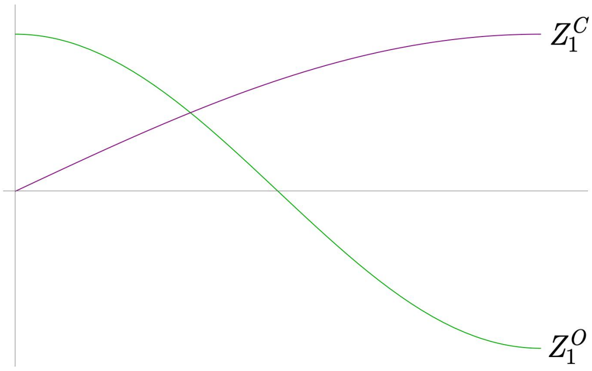


Figure 10: Plot of $Z(z)$ for $m = 1$ over $[0, L]$

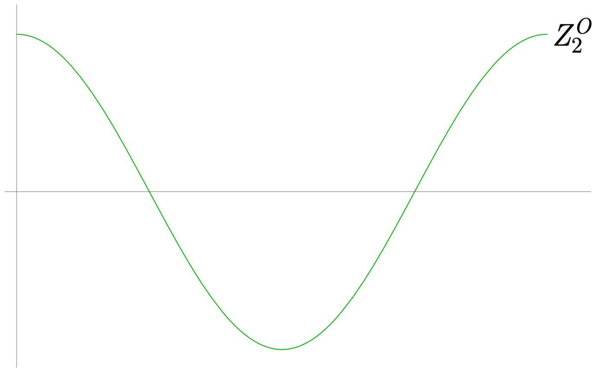


Figure 11: Plot of $Z(z)$ for $m = 2$ over $[0, L]$

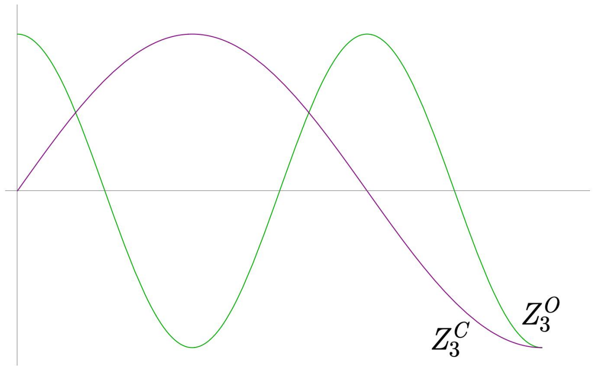


Figure 12: Plot of $Z(z)$ for $m = 3$ over $[0, L]$

3.2.4 Angular (ϕ) dependence

Now, we can replace the LHS of Equation 3.6 with $-b^2$, yielding

$$-b^2 = \left(R'' + \frac{1}{\rho}R'\right)\frac{1}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi}.$$

This can be rewritten as

$$-\frac{\Phi''}{\Phi} = \left(R'' + \frac{1}{\rho}R'\right)\frac{\rho^2}{R} + \rho^2 b^2.$$

We can see that both the LHS and RHS are dependent on one variable each so we define our third separation constant, n^2 and we have

$$\Phi'' + n^2\Phi = 0. \tag{3.9}$$

Once again, we have the harmonic oscillator equation, which has solutions of the form

$$\Phi_n = \Phi_+ e^{in\phi} + \Phi_- e^{-in\phi},$$

and similarly to the previous cases we specify a real solution for a constant $\beta \in \mathbb{R}$,

$$\Phi_n = D\sin(n\phi + \beta).$$

Since we expect solutions to equation (3.9) to be the same for $\phi = 0$ and $\phi = 2\pi$, we find that

$$\Phi(0) = \Phi(2\pi) \implies D\sin(\beta) = D\sin(2n\pi + \beta).$$

Clearly, this can only be the case when $n \in \mathbb{Z}$, so we have our ϕ solution:

$$\Phi_n = D\sin(n\phi + \beta),\ n \in \mathbb{Z}.$$

3.2.5 Radial coordinate (ρ) dependence

We now set the RHS of equation (3.2.4) equal to n^2, yielding

$$n^2 = \left(R'' + \frac{1}{\rho}R'\right)\frac{\rho^2}{R} + \rho^2 b^2.$$

Rearranging gives:

$$\rho^2 R'' + \rho R' + (b^2\rho^2 - n^2)R = 0 \tag{3.10}$$

In order to transform this equation into a standard form, we will use the substitution $s = b\rho$, we have

$$\implies s^2 R''(s) + sR'(s) + (s^2 - n^2)R(s) = 0.$$

This is Bessel's Equation, discussed in section 2.4, which has linearly independent solutions $J_n(s)$ and $Y_n(s)$, the Bessel functions of the first and second kind respectively, where n is the order of the solution. Given that $s = b\rho$, the solutions to equation (3.10) are $J_n(b\rho)$ and $Y_n(b\rho)$, so we have

$$R = c_1 J_n(b\rho) + c_2 Y_n(b\rho)$$

for arbitrary constants c_1 and c_2.

We note, however that Bessel Functions of the second kind, Y_n, tend to infinity as n tends to zero, hence we disregard this solution for our purposes. We now let $c_1 = R_0$, so we are left with

$$R = R_0 J_n(b\rho).$$

Our constraint here will be that, for r the constant radius of the pipe, $0 \leq \rho \leq r$ and that, when $\rho = r$, $R_{b,n} = 0$.

3.2.6 Combining the separable solutions

Note that, from our preceding separation of variables, we have the relation

$$\kappa^2 = a^2 - b^2 = \begin{cases} \dfrac{m\pi}{L}, & \text{for an open pipe,} \\ \dfrac{\pi(2m-1)}{2L}, & \text{for a closed pipe.} \end{cases}$$

This shows that $\kappa^2 = a^2 - b^2$ can be written in terms of m (and L, a constant we can choose to be, say $16'$ or some other suitable pipe length) and further $R_{b,n}$ has been chosen so that, for r the radius of the organ pipe, $R_{n,b}(r) = 0$ i.e. $J_n(r) = 0$. Hence, the combined product solution is

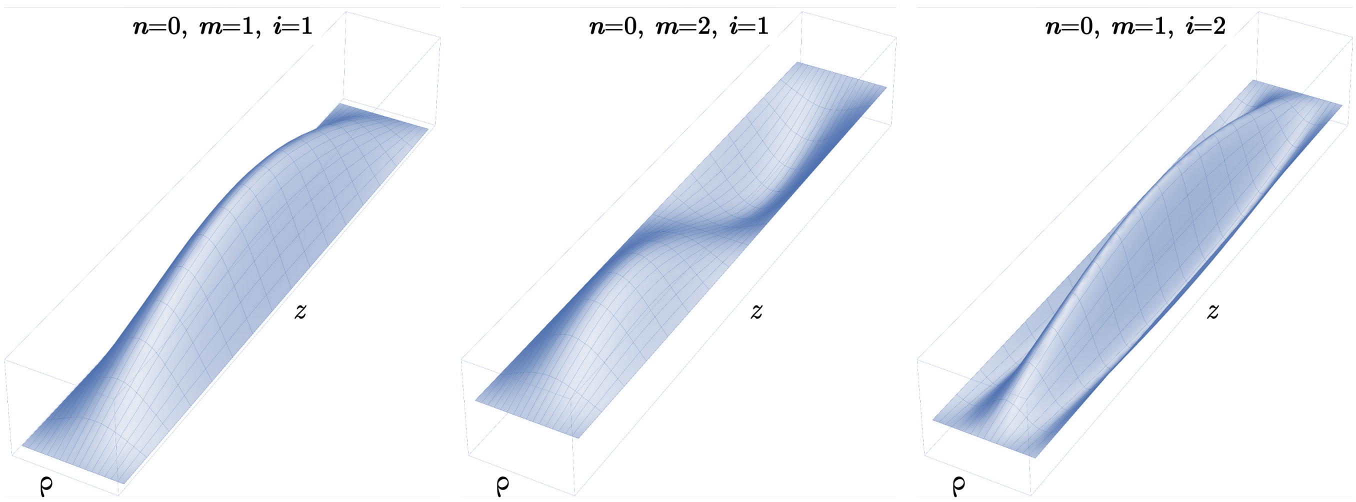
$$U = U_0(n,m)J_n(b\rho)\sin(act + \alpha_m)\sin(\kappa z)\sin(n\phi + \beta_n),$$

where U_0 is the combination of the other preceding constant multiplier terms. Since κ, a and b can all be written in terms of m and n and the equation is linear, the superposition principle holds. Hence, we obtain the general solution

$$U = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_0(n,m)J_n(n_{n,m}\rho)\sin(a_{n,m}ct + \alpha_m)\sin(\kappa_m z)\sin(n\phi + \beta_n). \tag{3.11}$$

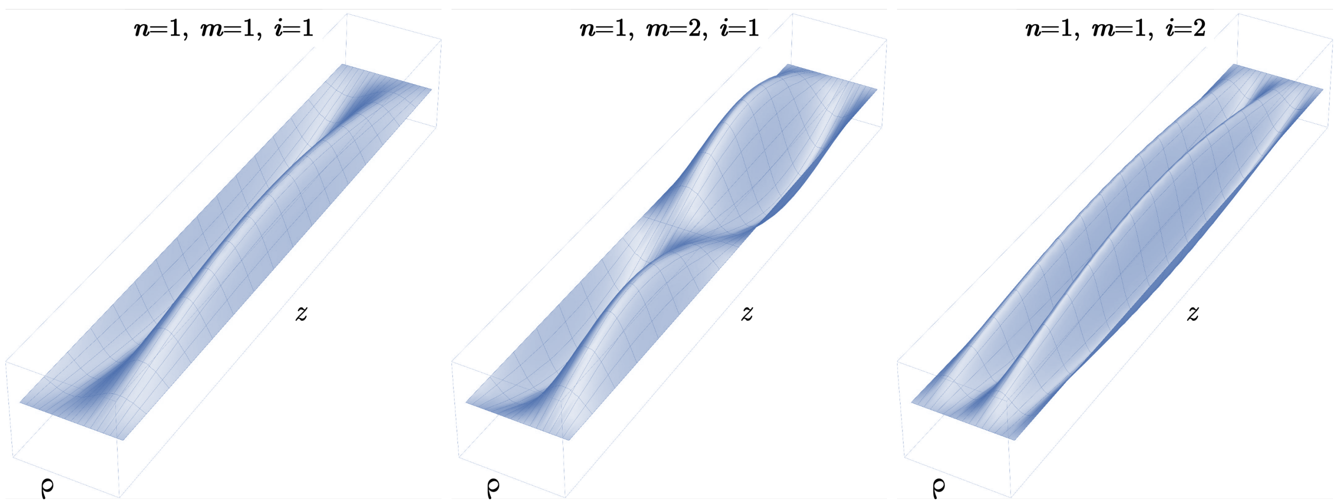
3.2.7 Plotting the separable solution

In order to plot these solutions, we will consider two separate cases: U dependent upon ρ and θ, and U dependent upon ρ and z. First, we will consider $U(\rho,\theta)$: let $s_{n,i}$ denote the i^{th} zero of the Bessel function J_n, that is $J_n(s_{n,i}) = 0$. Then, we take $J_n(b_{n,i}r) = 0 \implies b_{n,i} = \frac{s_{n,i}}{r}$, where r is the radius of the organ pipe. This gives us the plots in Figures (13), (14), (15) and (16).



(a) $U(\rho, z)$ for $n = 0, m = 1, i = 1$. **(b)** $U(\rho, z)$ for $n = 0, m = 2, i = 1$. **(c)** $U(\rho, z)$ for $n = 0, m = 1, i = 2$.

Figure 13: Plots of $U(\rho, z)$ for various m, i with $n = 0$ for an open-ended organ pipe.



(a) $U(\rho, z)$ for $n = 1, m = 1, i = 1$. **(b)** $U(\rho, z)$ for $n = 1, m = 2, i = 1$. **(c)** $U(\rho, z)$ for $n = 1, m = 1, i = 2$.

Figure 14: Plots of $U(\rho, z)$ for various m, i with $n = 1$ for an open-ended organ pipe.

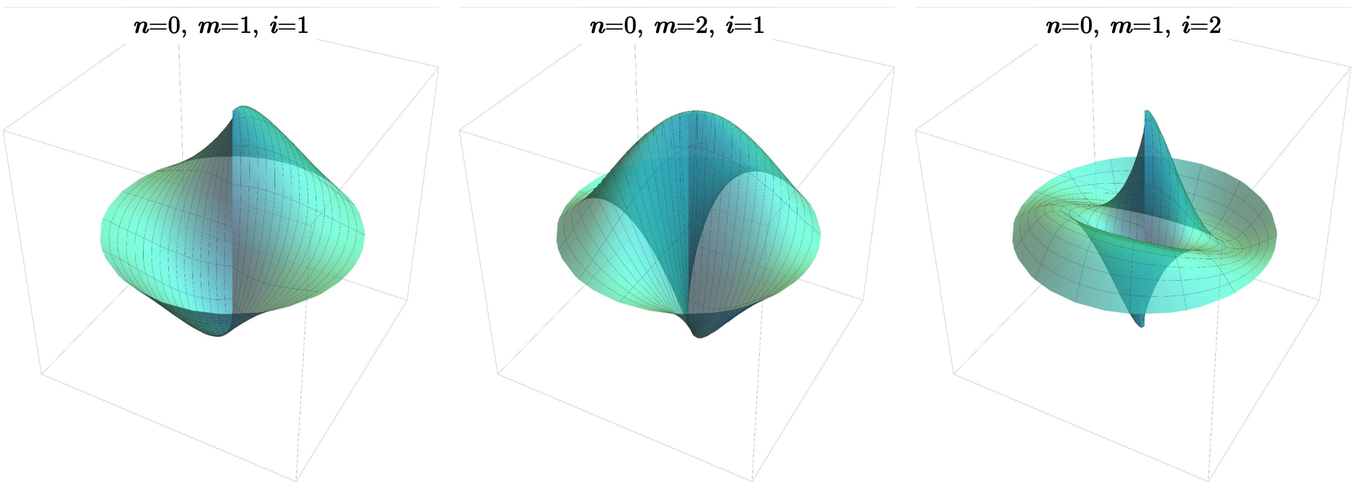
3.3 Conical Pipes

Conical pipes are rarely discussed in the literature surrounding the physical principles and mathematics of the pipe organ: their most common application is as the resonator of a reed pipe (see section 3.4) but flue pipes (as discussed in section 3.2) can also be conical. In this section, we will first give the wave equation (2.3) in polar coordinates and, as in sections 3.2.1 to 3.2.5, we will separate the variables in different ways and then discuss the significance and meaning of these solutions. [8]

3.3.1 The Wave Equation in Spherical Polar Coordinates

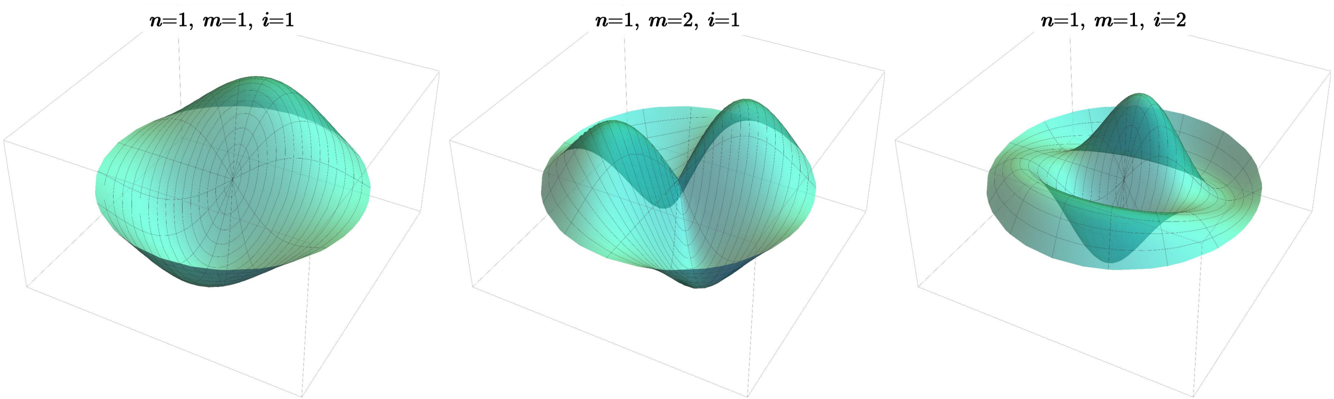
We can model a conical pipe as having a pressure node at the tip (rather than pressure reducing to zero at this point, as seems perhaps intuitive) and using this to derive our boundary conditions in the following computations.

[8]See [5] for a similar discussion of this separation of variables.



(a) $U(\rho,\theta)$ for $n=0, m=1, i=1$. **(b)** $U(\rho,\theta)$ for $n=0, m=2, i=1$. **(c)** $U(\rho,\theta)$ for $n=0, m=1, i=2$.

Figure 15: Plots of $U(\rho,\theta)$ for various m, i with $n=0$.



(a) $U(\rho,\theta)$ for $n=1, m=1, i=1$. **(b)** $U(\rho,\theta)$ for $n=1, m=2, i=1$. **(c)** $U(\rho,\theta)$ for $n=1, m=1, i=2$.

Figure 16: Plots of $U(\rho,\theta)$ for various m, i with $n=1$.

The wave equation is

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{3.12}$$

Using the definition of the Laplacian operator in spherical polar coordinates, we can express Equation (3.12) in spherical polar coordinates as

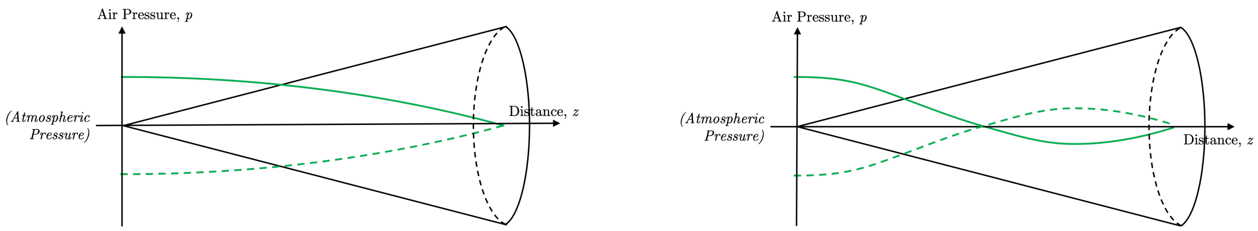
$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \tag{3.13}$$

where the first two terms are the radial part and the second two are the angular portion. In a similar manner to the cylindrical case, we will assume a product solution of the form

$$u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t).$$

Substituting this into the wave equation yields

$$\begin{aligned}\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} + \frac{1}{r^2}\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}& \\ = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2}&.\end{aligned}$$



(a) Fundamental Harmonic in a Conical Pipe **(b)** Second Harmonic in a Conical Pipe

Figure 17: Harmonic Resonance in Conical Pipes

3.3.2 Time (t) dependence

We have that

$$\mathrm{LHS}(r,\theta,\phi) = \mathrm{RHS}(t) = \text{constant} = -k^2.$$

Hence,

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2 \tag{3.14}$$

which is of simple harmonic form, and so gives sinusoids as solutions:

$$\frac{d^2T}{dt^2} = -c^2k^2T \equiv -\omega_k^2 T\ ,$$

for $\omega_k = c_k$.

3.3.3 Radial (r) dependence

Multiply the LHS equation by r^2 and rearrange:

$$-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + k^2r^2. \tag{3.15}$$

We choose the separation constant to be λ and so multiplying the RHS equation by R/r^2 gives the R equation:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{\lambda}{r^2}\right]R = 0.$$

Multiplying through by r^2 gives

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left[k^2r^2 - \lambda\right]R = 0,$$

which is the Bessel Equation.

3.3.4 Azimuthal angle (ϕ) dependence

Multiplying LHS of Equation (3.15) by $\sin^2\theta$ and rearranging gives

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

and we choose the separation constant to be m^2. The RHS equation easily gives the Φ equation

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi\ . \tag{3.16}$$

We find

$$\Phi(\phi + 2\pi) = \Phi(\phi) \qquad \Rightarrow \qquad e^{i2\pi m} = 1 \qquad \Rightarrow \qquad m \text{ is an integer.}$$

Using a change of variables $w = \cos\theta$ and after manipulation, we have

$$\left(\frac{d}{dw}(1 - w^2)\frac{d}{dw} + \lambda - \frac{m^2}{1 - w^2} \right) \Theta(w) = 0 \; .$$

This is known as the *Associated Legendre Equation.* Solutions of the Associated Legendre Equation are the *Associated Legendre Polynomials* [5]. Further information on these polynomials can be found in Section 2.5.

3.3.5 Consequences of Solutions

The significance of finding these solutions is that we want to consider the specific case of pressure $P = u$, after the removal of time t-dependence, satisfying the equation

$$\nabla^2 P = \frac{-\omega^2}{v_{sound}^2} P.$$

Using the boundary conditions $P = 0$ at the opening and $dP/dr = 0$ at the apex $r = 0$, we work with the apex of the cone as the centre of our spherical axes. Then, the pressure will depend only on r and the solutions come from section 3.3.3, since we are considering an increasing cross-sectional area. These solutions are spherical Bessel functions. In particular, pressure is therefore given by

$$P(r) = P_0 \frac{\sin(kr)}{r}, \; k = \frac{2\pi}{\lambda} \tag{3.17}$$

which has solutions to $P(r) = 0$ exactly when the sine function is zero, with the exception of at the origin [5]. This solution has been plotted in Figure 18 for $\lambda = 1, 2, 3$. This confirms our intuitions and pictures of standing waves in a conical pipe (such as in Figures 17a and 17b): it is clear that pressure is greatest at the apex, and that it tends to zero as r goes to infinity.

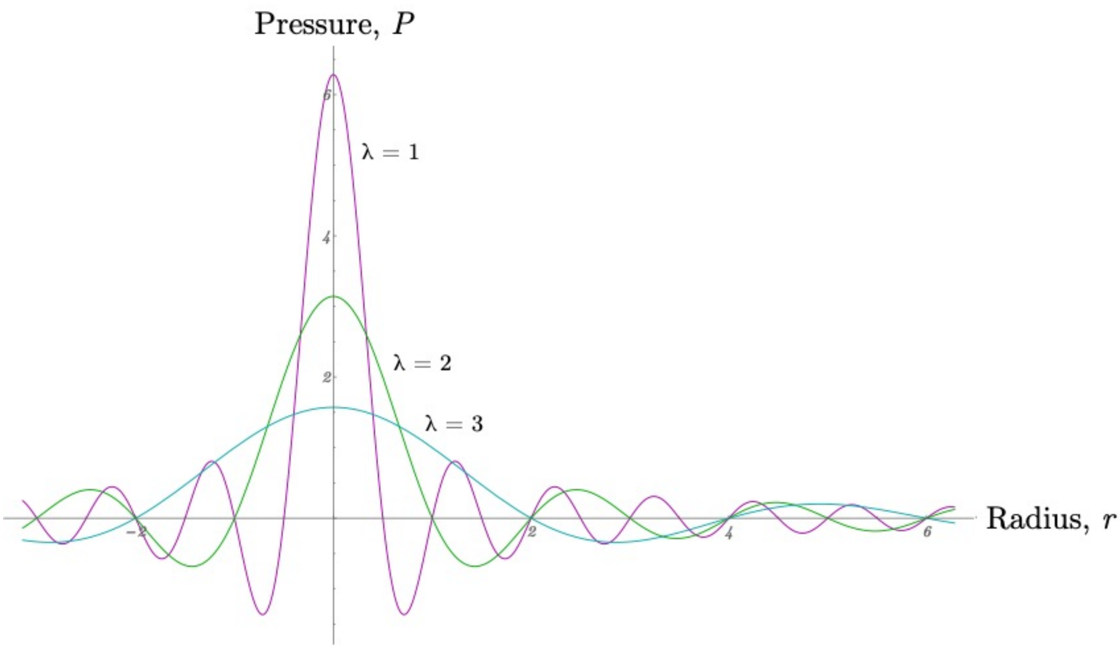


Figure 18: Cartesian Plot of Equation (3.17) for $\lambda = 1, 2, 3$.

3.4 Reed Pipes

Like flue pipes, seen in section 3.2, a reed pipe is broadly grouped into a boot (or generator) and a resonator: the reed, a brass (traditionally, although wood, aluminium and bakelite have been used also) vibrates against the shallot, also (typically) made of brass with an aperture cut into it, which is how the reed pipe speaks. This vibration gives rise to a waveform of air pressure that is then amplified by the resonator. The tuning wire can be taken up or down, which shortens or lengthens the reed in order to produce the desired pitch in the organ pipe, which is how the reed pipe is tuned. The reed and shallot have a hardwood wedge in between them to separate them.

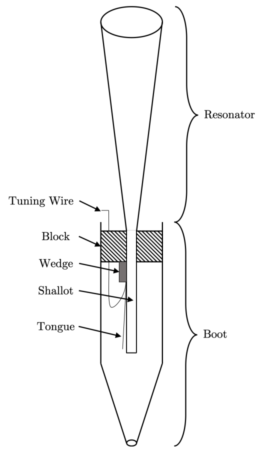


Figure 19: A Common Reed Pipe

The other main part of the pipe, the resonator, can (to a more limited extent than the mechanisms within the boot) affect the sound produced by the pipe. Whereas in the flue pipe where the generator and resonator are inseparable (in the sense that neither will produce sound on their own), the generator of a reed pipe can generate sound with or without its resonator - that section of piping is there more for practicality and timbral considerations. The resonator functions as an acoustic transformer of sorts: it takes highly-resisted acoustic energy and converts it into much less-resisted acoustic energy. This level of resistance is known as the 'acoustic impedance,' and so in these terms the resonator transforms the high impedance output to low impedance output. This process is called impedance matching, and is the main function of the resonator. [31] See figure 19 for a detailed diagram of the reed pipe. There are also such pipes as 'free reeds,' but they are very uncommon and will not be discussed here: the only significant use of note is that of the harmonium, which sounds exclusively free reeds. [21, p. 3462]

Not a great deal is written about reed pipes: a great many authors refuse to cover them in their discussions of the mathematical and physical characteristics of pipe organs simply because these characteristics of reed pipes are so complex. When they are discussed, especially on their own, they often centre on experimental data with theoretical explanations of the results obtained.[9] Indeed, considerations for determining a comprehensive physical and mathematical understanding of this type of pipe include: dimensions of the internal mechanics of the boot (such as material, size and shape of the shallot, the size and shape of the opening in the shallot through which the vibrations travel and the material, curvature and length of the reed); the geometry of the resonator, which varies greatly between different reed pipes; variations in wind pressure, and so on.
Having said that, there is some research done on the sounding of reed pipes. For instance, if one were to pluck the reed in order to induce sound production, we may model this as a cantilever with either clamped or free ends depending on the particular situation [20] and therefore consider the differential equation for a cantilever (which is energy-lossless) [33]

$$\frac{\partial \xi(z,t)}{\partial t^2} blh\rho + \frac{Eblh^3}{12}\frac{\partial^4 \xi(z,t)}{\partial z^4} = F(z,t) \tag{3.18}$$

where ρ is Young's modulus, $b, l,$ and h are width, length and breadth respectively, ξ is the displacement of the reed we are actuating and F, a function of the spatial variable and time, is the function

[9]For an example of this, see [20] - an accessible yet somewhat complex paper detailing several experimental situations for actuating reed pipes and the observations made before going on to discuss theoretical considerations and how this compares to their results.

acting upon the reed [20]. Setting $F = 0$ allows us to solve for the resonant frequency of the reed. We, once again assume a product solution: for the displacement, we assume

$$\xi(z,t) = T(t)Z(z)$$

. Our boundary conditions for an unclamped cantilever are, for $l \in \mathbb{R}$ the length of the reed, are

$$Z(l) = 1, \quad Z(0) = 0, \quad \dot{Z}(0) = 0, \quad \ddot{Z}(l) = 0 \text{ and } \dddot{Z}(l) = 0.$$

Such a function may be

$$Z(z) = \frac{1}{2}\left[\cosh\left(\alpha\frac{z}{l}\right) - \cos\left(\alpha\frac{z}{l}\right)\right] - 0.37\left[\sinh\left(\alpha\frac{z}{l}\right) - \sin\left(\alpha\frac{z}{l}\right)\right]$$

found in [20]. Further, we substitute our product solution into equation 3.18 and assume t-dependence in order to find the following:

$$\frac{d^2T}{dt^2} + \frac{Eh^2}{12\rho}\frac{\alpha^4}{l^4}T = 0.$$

By further manipulation, one can see that we will arrive at a nonlinear differential equation, which can be solved only numerically, in order to determine the time-dependent displacement of the free end of the reed [20]. Hence, the closest that we may come to this form of analysis would be through experimentation in order to confirm or deny various iterative results, for instance for the angular frequency, which can be obtained using the resulting nonlinear differential equation, and so ends our discussion of the sounding of the reed pipe. Unfortunately, once again the mathematical and physical realities of the reed pipe and the mechanisms contributing to its sounding are remarkably complex, meaning that a comprehensive yet concise discussion such as that of the flue pipe is simply not possible for our purposes.

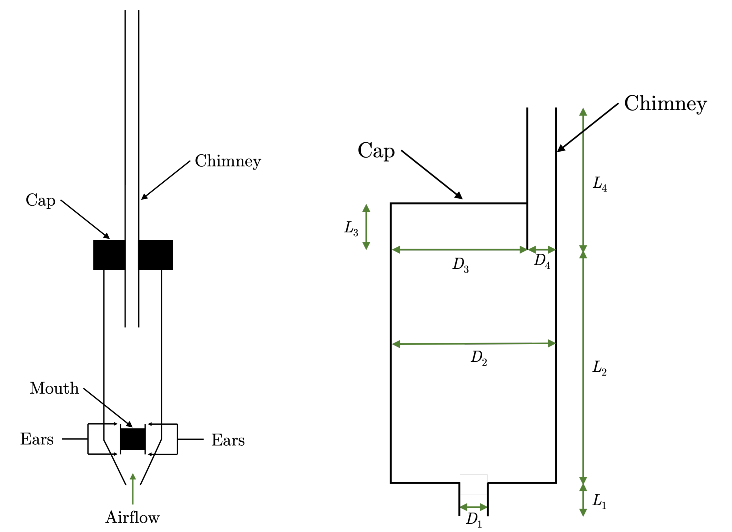
4 Further Applications

In this section, we provide a model as a framework for experimental evidence to build upon. Whilst we believe the current work would have been greatly enriched by accumulating physical, experimental data to verify results we have obtained mathematically, several factors has meant that this is unfortunately not possible. Hence, we provide this as one notable example of an area experimental evidence would greatly help.

4.1 Spatial Geometries and Sound Pressure Level Spectra

We begin by considering the amplitudes of the various harmonic partials produced by a given pipe. These amplitudes depend upon factors such as the material of the pipe, the dimensions of the pipe, the geometry of the environment surrounding the pipe (for instance how close or far away walls or surrounding pipes are) and auto-synchronisation with the pipe's own sound signal [10]. A discussion of all of these aspects would require a dedicated paper, so we will focus on the concepts related to pipe geometries since they are the most researched and the most influential to the overall timbre [2].

For instance, music requiring an emphasis of the major third, such as classical and romantic-era music can utilise pipes which amplify the fifth harmonic; we apply the above ideas to determine the geometries of a pipe which would produce this effect. Others have experimentally determined that the *röhrflute* (or chimney pipe) has this quality [14]. Using a simple mechanical model, we can determine the exact dimensions of the pipe to maximise the fifth harmonic production that we desire in our example: however, the number of variables to consider means this model can only take us so far theoretically before we need to supplement with experimental evidence to narrow down the possible solutions to ones that would suit specific needs.



(a) Typical Röhrflute Pipe

(b) The Röhrflure pipe model we consider

Figure 20: A labelled diagram of a Röhrflure pipe and the one-dimensional acoustical model of a Röhrflure pipe that we consider [14]

Let us assume plane-wave propagation inside the the röhrflute, depicted in Figure 20, which we will model as in Figure 20(b). Let the length of the chimney of the röhrflute be l_4, the cross section be $S_4 = \frac{\pi D^2}{4}$ inside of the main section of the pipe which is of length $l_2 + l_3$, where $l_4 \geq l_3 \geq 0$. A cap is placed at the end of l_3 and has cross section $S_3 = S_2 - S_4$ where $S_2 = \frac{\pi D_2^2}{4}$ is the cross-section of the lower pipe. The mouth pipe is a segment of length l_1 and the cross-section of which is equal to the cross-section of the mouth, which we will call S_1. We can find experimentally that L_1 is determined by contributions of sound radiation, effects of the small diameter of the mouth of the pipe, and the positioning of the "ears" around the mouth of the röhrflute [14]. Helmholtz proposed several design constraints in order to take into account all of these effects [26]. By noting that the fundamental resonance of the chimney should be equal to the frequency of the fifth harmonic of the main pipe, which oscillates at frequency $\lambda_0 = \frac{c_0}{f_0}$, then the length of the chimney is given by [14]

$$2L_4 = \frac{\lambda_0}{5}.$$

This gives us an expression for the length, but where should the chimney pipe be positioned? Whilst Figure 20 has the chimney sitting partially inside of the main pipe body, most röhrflutes have the chimney either fully inside or outside of the pipe [9]. Experimentally, it can be found that the best position for the pipe in order to optimise the amplification effect on the fifth harmonic is at an intermediate position - half of the chimney pipe inside of the main pipe and the other half outside [2]. We can explain this by considering the standing waves created: we have standing waves in both the main pipe body and the chimney, and so if we place the chimney fully outside of the main pipe we impose a pressure node at the closed end, which destructively interferes with the standing wave pattern of the stopped pipe in the fifth harmonic - as discussed in section 2.1.2. Instead of this, and contrary to Helmholtz's initial thinking, we conjecture that the chimney of a röhrflute ought not to be thought of as a resonating device, but rather an instrument capable of admitting high resonance in its own right. [14, p. 50]

In light of this, we see that the chimney of the röhrflute plays a very particular role (if so desired): it can be thought of as a separate pipe to amplify and match with high admittance the fifth harmonic. We find that the best positioning of the chimney is in the intermediate (half-in, half-out) and that this leaves a great deal of room for variation in the possible lengths of the pipe [14]. Further analysis must, however be left to those willing to do so experimentally and use this to build on the findings of this model we have provided. It is our hope that this simple mechanical model can be used to optimise further the röhrflute design and even to generate future, similar models which may give high admittance to different harmonics as different organs, organists and compositions require.

5 Discussion

5.1 Concluding Remarks and Speculation on Possible Future Research and Applications

One notion used throughout this work to model organ pipes is the Helmholtz Resonator. Whilst this is the fundamental model for an organ pipe and gives us a highly accurate idea of the physical principles governing sound production in the instrument, this also proves useful elsewhere. When a Helmholtz resonator is tuned to a particular frequency, it absorbs the energy producing this sound to resonate inside itself instead and, if the opening is small enough, effectively removes that particular frequency. We propose this could be useful for studying the makeup of a particular sound, as one could study each harmonic produced by a given organ pipe by systematically removing all harmonics with resonators and reintroducing each one individually. This would allow us to study individual amplitudes, giving us a 'fingerprint' for a given type of pipe. This would be a natural extension of the current work, to confirm or indeed deny the theoretical results obtained with experimental data using this approach. Another use outside of this discussion is in architectural acoustics, where creating a Helmholtz Resonator with the resonant frequency of a troublesome, unwanted frequency produced by, say electronics or machinery nearby will allow us to absorb the sound energy into the resonator, largely eliminating it from the environment and thereby reducing (hopefully to a level below human auditory levels) the given sound.

Furthermore, much of the work in this paper depended upon the Wave Equation. From understanding earthquakes to modelling light waves, this equation proves its use in many other mathematical and physical applications. Interestingly, it is a direct consequence of Newton's Second Law (the rate of change of momentum of a particle is proportional to the force applied). In 1746, Jean Le Rond modelled a vibrating violin string as a collection of point masses and applied this principle; through some manipulation, he derived the Wave Equation [25]. Using this equation in cylindrical polar coordinates, we showed - through separation of variables - that ρ-dependence gives rise to the Bessel Equation, and the other dependences gave harmonic oscillatory solutions. We then combined the solutions and discussed the implications of these results for cylindrical pipes, including generating diagrams. We confirmed our original pictures of standing waves in pipes, and produced some three dimensional plots of the physical situation occurring inside of an organ pipe. In a similar way, we used spherical polar coordinates in order to discuss standing waves in conical pipes, which are much rarer but this was nonetheless a non-trivial manipulation and required discussion of the Associated Legendre Equation and Polynomials.

After these examinations of flue pipes, we went on to briefly discuss reed pipes, beginning with the reason it is so infrequently mentioned in the literature: the complexity of the mathematics involved. We presented some notions that may be used in order to consider its sound generation as an unclamped cantilever and determine a nonlinear differential equation using some mechanical assumptions and boundary conditions relating to this situation. However, it is not within the remit of the current work in order to go about solving this equation numerically, and so this ended our discussion of reed pipes. Finally, we went on to discuss a model that may be used in order to produce some very interesting explanations of experimental results, which this work would have benefitted greatly from if conditions had allowed. However, this has been presented so that others may use this model to determine the optimal dimensions and chimney positions of the röhrflute for various purposes and discuss the problems that some have experienced in creating such pipes, perhaps such as practical construction issues. It is our hope that this model may prove useful to other researchers with the capacity to experimentally confirm or deny the theoretical results produced and briefly discussed in the previous section.

Appendix A Derivation of Helmholtz Frequency Equation

The equation for resonant angular frequency is

$$\omega_H = \sqrt{\gamma \frac{A^2}{m} \frac{P_0}{V_0}}$$

where γ is the heat capacity ratio, A is the area of the cross-section of the neck, m is the mass in the neck, P_0 is the static pressure in the cavity and V_0 is the static volume of the cavity. Given that, for cylindrical or rectangular necks, we have

$$A = \frac{V_n}{l},$$

where l is the equivalent length with end correction accounted for (i.e. $l = l_n + 0.3D$, for l_n the physical length and D the hydraulic diameter of the neck), we have

$$\omega_H = \sqrt{\gamma \frac{P_0}{\rho} \frac{A}{V_0 l}}.$$

Given that mass density ρ and resonant frequency f_H are given by

$$\rho = \frac{m}{V_n} \text{ and } f_H = \frac{\omega_H}{2\pi},$$

and that the speed of sound in a gas is given by the equation

$$v = \sqrt{\gamma \frac{P_0}{\rho}}$$

we have that the Helmholtz frequency is given by

$$f = \frac{v}{2\pi} \sqrt{\frac{A}{lV}}.$$

which is exactly Equation (2.1).[10] □

[10] See [19] for a similar discussion

Appendix B List of Figures

Appendix C List of Tables

Appendix D Use of Images

We would like to offer our thanks to the originators of the images used in this paper, which were sourced from royalty-free image websites `www.unsplash.com` and `www.pixabay.com`. Although these sites do not require that credit is provided for the use of images, as a sign of our gratitude we have credited the artists below.

- Figure 1a: From Unsplash. Taken by John Salvino (@jsalvino), a pipe organ in Hallgrimskirkja, Reykjavík, Iceland.
- Figure 1b: From Pixabay. Taken by James Smith, a particularly attractive console in the baroque style.
- Figure 9: From Unsplash. Taken by Denny Müller (@redaquamedia), a pipe organ facade in an Evangelical church, Germany.

All diagrams, plots and models have been created by the author and are, as such, our own intellectual property - any work upon which they may rely in part or entirely has been referenced accordingly.

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