SI140 Probability & Mathematical Statistics

Homework 7

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Problem 5.8

Solution:

(a)

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x 12t^2(1-t)\,\mathrm{d}t = 4x^3 - 3x^4, \quad \text{for } 0 < x < 1$$

(b)

$$P(0 < X < 1/2) = F(1/2) = \frac{5}{16}$$

(c)

$$E(X) = \int_0^1 xf(x)\,\mathrm{d}x = \int_0^1 12x^3(1-x)\,\mathrm{d}x = \frac{3}{5}$$

To get variance, we have:

$$E(X^2) = E(X) = \int_0^1 x^2 f(x)\,\mathrm{d}x = \frac{2}{5}$$

So that

$$V(X) = E(X^2) - E^2(X) = \frac{1}{25}$$

Problem 5.14

Solution:

$$F(x) = P(X \leqslant x) = P(U_1 \leqslant x, U_2 \leqslant x, \dots, U_n \leqslant x) = P(U_1 \leqslant x)P(U_2 \leqslant x)\dots P(U_n \leqslant x) = x^n$$

So the PDF of X is:

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} = nx^{n-1}$$

And we get:

$$EX = \int_0^1 xf(x)\,\mathrm{d}x = \frac{n}{n+1}$$

Problem 5.31

Solution:

(a) For $y < 0$, since $Y \geqslant 0$ so $F(y) = 0$ in this case. For $y \geqslant 0$, $P(Y \leqslant y) = P(-y \leqslant x \leqslant y) = \Phi(\frac{y-\mu}{\sigma}) - \Phi(\frac{-y-\mu}{\sigma})$, so that:

$$F(y) = \begin{cases} \Phi(\frac{y-\mu}{\sigma}) - \Phi(\frac{-y-\mu}{\sigma}) & y \geqslant 0 \\ 0 & y < 0 \end{cases}$$

(b) For $y < 0, f(y) = 0$; for $y > 0, f(y) = \frac{\mathrm{d}F(y)}{\mathrm{d}y} = \varphi(\frac{y-\mu}{\sigma})\frac{1}{\sigma} + \varphi(\frac{-y-\mu}{\sigma})\frac{1}{\sigma}$; But since $f(y)$ has no limit at 0, we can't define the value here, let's just say that $f(0) = c$ where c can be any real number. So that:

$$f(y) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \cdot (e^{-\frac{(y-\mu)^2}{2\sigma^2}} + e^{-\frac{(y+\mu)^2}{2\sigma^2}}) & y \geqslant 0 \\ c & y = 0 \\ 0 & y < 0 \end{cases}$$

(c) Not continuous, since the left limit does not equal to the right limit. There is no problem to using it to find probabilities because the values of PDF at finite number of ys won't affect area under the graph of PDF.

Problem 5.45

Solution:

(a) It is easy to notice that if the number of failures is 1, then $G = 1$ and $T = \Delta t$, if 2 then $G = 2$ and $T = 2\Delta t \ldots$, we found that $\Delta t \cdot G = T$.

(b)

$$F(t) = P(T \leqslant t) = 1 - P(T > t) = 1 - P(G > \frac{t}{\Delta t}) = 1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}}$$

(c) Using the compound interest limit, we have

$$\lim_{\Delta t \to 0} \left[1 - (1 - \lambda\Delta t)^{\frac{t}{\Delta t}}\right] = 1 - e^{-\lambda t}, \quad \text{for all fixed } t > 0$$

That is right as the Expo(λ) CDF.

Problem 5.55

Solution:

$$\begin{aligned}
\mathrm{MSE}(T) - \mathrm{Var}(T) &= E(T^2 + \theta^2 - 2T\theta) - (E(T^2) - E^2(T)) \\
&= E(T^2) + E(\theta^2) - E(2T\theta) - E(T^2) + E^2(T) \\
&= E(T^2) + \theta^2 - 2\theta E(T) - E(T^2) + E^2(T) \\
&= E^2(T) - 2\theta E(T) + \theta^2 \\
&= (b(T))^2
\end{aligned}$$

That is $\mathrm{MSE}(T) = \mathrm{Var}(T) + (b(T))^2$.

Problem 6.25

Solution:

(a)

$$\begin{aligned}
P(Y > s+t | Y > s) &= \frac{P(Y > s+t,\, Y > s)}{P(Y > s)} = \frac{P(Y > s+t)}{P(Y > s)} \\
&= \frac{P(X > \sqrt[3]{s+t})}{P(X > \sqrt[3]{s})} = \frac{1 - P(X \leqslant \sqrt[3]{s+t})}{1 - P(X \leqslant \sqrt[3]{s})} = \frac{e^{-\sqrt[3]{s+t}}}{e^{-\sqrt[3]{s}}} \neq e^{-\sqrt[3]{t}}
\end{aligned}$$

So Y does **not** have the memoryless property.

(b) Since $F(y) = P(Y \leqslant y) = P(X \leqslant \sqrt[3]{y})$, we have:

$$f(y) = \frac{\mathrm{d}F(y)}{\mathrm{d}y} = \frac{\mathrm{d}(1 - e^{-y^{1/3}})}{\mathrm{d}y} = \frac{y^{-2/3} \cdot e^{-y^{1/3}}}{3}$$

$$\begin{aligned}
E(Y^n) &= \int_0^{+\infty} (y^n \cdot \frac{y^{-2/3} \cdot e^{-y^{1/3}}}{3})\,\mathrm{d}y = \frac{1}{3}\int_0^{+\infty} (y^{n-2/3} \cdot e^{-y^{1/3}})\,\mathrm{d}y \\
&= \frac{1}{3}\int_0^{+\infty} (u^{3n-2} \cdot e^{-u})\,\mathrm{d}u^3 = \int_0^{+\infty} (u^{3n} \cdot e^{-u})\,\mathrm{d}u = \Gamma(3n+1) = (3n)!
\end{aligned}$$

So we have $E(Y) = 3! = 6, \quad E(Y^2) = (3 \cdot 2)! = 720, \quad V(Y) = E(Y^2) - E^2(Y) = 684$

(c)

$$M(t) = \sum_{n=0}^{\infty} M^{(n)}(0)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E(Y^n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(3n)! \cdot t^n}{n!}$$

It does not converge at any non-zero point since $t^n = \omega(\frac{n!}{(3n)!})$ when $t > 0$ using Stirling's formula. So the MGF does **not** exist.