上海科技大学
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SI140 Probability & Mathematical Statistics

Homework 8

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Problem 7.9

Solution:

(a)

$$\begin{aligned} p_{X,Y,N}(x,y,n) &= P(X=x, Y=y, N=n) \\ &= P(N=n|X=x,Y=y)P(X=x|Y=y)P(Y=y) \\ &= 1 \cdot P(X=x) \cdot P(Y=y) \\ &= (1-p)^{x+y}p^2 \end{aligned}$$

(b)

$$\begin{aligned} p_{X,N}(x,n) &= P(X=x, N=n) \\ &= P(N=n|X=x)P(X=x) \\ &= P(Y=n-x) \cdot P(X=x) \\ &= (1-p)^n p^2 \end{aligned}$$

(c)

$$\begin{aligned} P(N=n) &= \sum_{x=0}^{n} P(N=n|X=x)P(X=x) \\ &= \sum_{x=0}^{n} P(Y=n-x)P(X=x) \\ &= (n+1)(1-p)^n p^2 \end{aligned}$$

$$p_{X|N} = P(X=x|N=n) = \frac{P(N=n|X=x)P(X=x)}{P(N=n)} = \frac{(1-p)^n p^2}{(n+1)(1-p)^n p^2} = \frac{1}{n+1}$$

Intuitively, if we have $n+2$ trials with the $(n+2)^{\text{th}}$ trial is success, and the first $n+1$ trials contains 1 success, that is $N=n$. Since we know where the success apears is completely with the same probability having $n+1$ choices, so the $p_{X|N} = \frac{1}{n+1}$.

Problem 7.10

Solution:

(a)

$$F_T(t|X=x) = P(T \leqslant t|X=x) = P(Y \leqslant t-x) = 1 - e^{-\lambda(t-x)}, \quad t > x$$
$$F_T(t|X=x) = 0, \quad t \leqslant x$$

(b)

$$f_{T|X}(t|x) = \frac{\mathrm{d}F_T(t|X=x)}{\mathrm{d}t} = \lambda e^{-\lambda(t-x)}, \quad t > x$$
$$f_{T|X}(t|x) = 0, \quad t \leqslant x$$

It is valid 'cause for any $\lambda > 0$, $f_{T|X}(t|x) \geqslant 0$ for $t \in \mathbb{R}$. And we have

$$\int_{-\infty}^{\infty} f_{T|X}(t|x)\,\mathrm{d}t = \int_0^{\infty} \lambda e^{-\lambda u}\,\mathrm{d}u = \lambda \cdot \frac{-1}{\lambda}(\lim_{u\to\infty} e^{-\lambda u} - 1) = 1$$

(c)

$$f_{X|T}(x|t) = \frac{f_{T|X}(t|x)f_X(x)}{f_T(t)} = \frac{\lambda^2 e^{-\lambda t}}{f_T(t)}$$

Since when $x \leqslant 0$ and $x \geqslant t$, the PDF is always zero, to make it valid, we get

$$1 = \int_0^t \frac{\lambda^2 e^{-\lambda t}}{f_T(t)}\,\mathrm{d}x = \frac{\lambda^2 e^{-\lambda t} t}{f_T(t)}$$

So that $f_{X|T}(x|t) = \frac{1}{t}$ for a given positive constant t. Since t is positive and as we show above we know that the integral of $f_{X|T}(x|t)$ in $\mathbb{R}$ is 1, we get this is valid PDF.

(d) From (c) we know that

$$1 = \int_0^t \frac{\lambda^2 e^{-\lambda t}}{f_T(t)}\,\mathrm{d}x = \frac{\lambda^2 e^{-\lambda t} t}{f_T(t)}$$

So we have: $f_T(t) = \lambda^2 t e^{-\lambda t}$

Problem 7.17

Solution:

(a)

$$1 = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_x^1 cxy\,\mathrm{d}y = \frac{c}{8}$$

$$\Rightarrow c = 8$$

(b)

$$f_X(x) = \int_x^1 f_{X,Y}(x,y)\,\mathrm{d}y = 4x - 4x^3, \quad x \in (0,1)$$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,\mathrm{d}x = 4y^3, \quad x \in (0,1)$$

$$\Rightarrow f_{X,Y}(x,y) = 8xy \neq (4x - 4x^3)4y^3 = f_X(x)\,f_Y(y)$$

So X and Y are not independnt.

(c) From (b), there are

$$f_X(x) = \int_x^1 f_{X,Y}(x,y)\,\mathrm{d}y = 4x - 4x^3, \quad x \in (0,1) \quad \text{otherwise } 0$$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,\mathrm{d}x = 4y^3, \quad y \in (0,1) \quad \text{otherwise } 0$$

(d)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2y}{1-x^2}, \quad 0 < x < y < 1 \quad \text{otherwise } 0$$

Problem 7.20

Solution:

(a)

$$M \leqslant m \Rightarrow U_1, U_2, U_3 \leqslant m$$

$$F_M(m) = F_{U_1}(m)F_{U_2}(m)F_{U_2}(m) = m^3, \quad (0 \leqslant m \leqslant 1 \quad \text{otherwise } 0)$$

$$f_M(m) = \frac{\mathrm{d}F_M(m)}{\mathrm{d}m} = 3m^2, \quad (0 \leqslant m \leqslant 1 \quad \text{otherwise } 0)$$

If $L \geqslant l$ and $M \leqslant m$, then $l \leqslant U_i \leqslant m$ for all $i \in \{1, 2, 3\}$. So that

$$P(L \geqslant l, M \leqslant m) = (m - l)^3$$

$$\begin{aligned}
P(M \leqslant m) &= P(L \geqslant l, M \leqslant m) + P(L \leqslant l, M \leqslant m) \\
\Rightarrow F_{L,M}(l, m) &= P(L \leqslant l, M \leqslant m) \\
&= P(M \leqslant m) - P(L \geqslant l, M \leqslant m) \\
&= m^3 - (m - l)^3, \quad (0 \leqslant m, l \leqslant 1, \quad m \geqslant l \quad \text{otherwise } 0) \\
\Rightarrow f_{L,M}(l, m) &= \frac{\mathrm{d}F_{L,M}(l, m)}{\mathrm{d}m} = 6(m - l), \quad (0 \leqslant m, l \leqslant 1, \quad m \geqslant l \quad \text{otherwise } 0)
\end{aligned}$$

(b)

$$f_L(l) = \frac{\mathrm{d}F_L(l)}{\mathrm{d}l} = \frac{\mathrm{d}(1 - l)^3)}{\mathrm{d}l} = 3(1 - l)^2, \quad (0 \leqslant l \leqslant 1, \quad \text{otherwise } 0)$$

$$f_{M|L}(m|l) = \frac{f_{M,L}(m, l)}{f_L(l)} = \frac{2(m - l)}{(1 - l)^2}, \quad (0 \leqslant m, l \leqslant 1, \quad m \geqslant l \quad \text{otherwise } 0)$$

Problem 7.24

Solution:

(a) Let $\frac{Y_1}{Y_2} = T$

$$F_T(t) = P(\frac{Y_1}{Y_2} \leqslant t) = P(Y_1 \leqslant tY_2) = \int_0^\infty \lambda_2 e^{\lambda_2 y_2}\,\mathrm{d}y_2 \int_0^{ty_2} \lambda_1 e^{-\lambda_1 y_1}\,\mathrm{d}y_1 = \frac{t\lambda_1}{t\lambda_1 + \lambda_2},$$

For $t > 0$, $\quad F_T(t) = 0$ otherwise.

$$f_T(t) = \frac{\mathrm{d}F_T(t)}{\mathrm{d}t} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

(b) When $t = 1$,

$$P(Y_1 < Y_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Problem 7.29

Solution:

(a) When $m = l \geqslant 0$,

$$p_{L,M}(l,m) = P(X = Y = l = m) = (1-p)^{2l}p^2$$

When $m > l \geqslant 0$,

$$p_{L,M}(l,m) = P(X = m, Y = l) + P(X = l, Y = m) = 2(1-p)^{l+m}p^2$$

0 otherwise. That is

$$p_{L,M}(l,m) = \begin{cases} (1-p)^{2l}p^2, & m = l \geqslant 0 \\ 2(1-p)^{l+m}p^2, & m > l \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

Since $P(M = m, L = l) = 0$ when $m < l$ while $P(M = m) \neq 0$ and $P(L = l) \neq 0$ giving $m < l$, so in this case $P(M = m, L = l) \neq P(L = l)P(M = m)$, so they are not independnt.

(b)

$$p_L(l) = \sum_{m=0}^{\infty} p_{L,M}(l,m) = \sum_{m=l}^{\infty} p_{L,M}(l,m) = (1-p)^{2l}p^2 + \sum_{m=l+1}^{\infty} p_{L,M}(l,m)$$

$$= (1-p)^{2l}p^2 + \sum_{m=l+1}^{\infty} 2(1-p)^{l+m}p^2 = (1-p)^{2l}p^2 + 2(1-p)^l p^2 \sum_{m=l+1}^{\infty} (1-p)^m$$

$$= (1-p)^{2l}p^2 + 2(1-p)^{2l+1}p = (1-p)^{2l}(2-p)p$$

Story: Performing two Geom processes α and β. Let $A =$"The α processes is in success", $B =$"The β processes is in success". $C =$"The first successful processes is in success". So that the L with the success probability that

$$p' = P(C) = P(A \cup B) = P(A) + P(B) - P(A \cup B) = 2p - p^2 = (2-p)p$$

So $L \sim Geom(p')$, $p_L(l) = (1-(2-p)p)^l(2-p)p = (1-p)^{2l}(2-p)p$

(c)

$$EM = E(X+Y) - EL = EX + EY - EL = \frac{2(1-p)}{p} - \frac{(1-p)^2}{(2-p)p} = \frac{(1-p)(3-p)}{p(2-p)}$$

(d) From Memoryless property, $p_{M-L}(k) = p(X=k) = (1-p)^k p$ whatever L is. So

$$p_{L,M-L}(l,k) = P(L=l, M-L=k) = P(M-L=k|L=l)P(L=l) = (1-p)^{2l+k}(2-p)p^2$$

For $k \geqslant 0$, $p_{L,M-L}(l,k) == 0$ otherwise. We know that from the Memoryless property, the $M-L$ is nothing to do with L, so that $p_{L,M-L}(l,k) = p_L(l)p_{M-L}(k)$. So they are independnt.