上海科技大学
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SI140 Probability & Mathematical Statistics

Homework 9

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Problem 7.38

Solution:

Since X, Y, $\max(X,Y)$ and $\min(X,Y)$ are all r.v.s, we know that X can be $\min(X,Y)$ or $\max(X,Y)$. When X is $\min(X,Y)$, Y can only take $\max(X,Y)$ ans vice versa. So the sum of them are the same that is $\max(X,Y)+\min(X,Y) = X+Y$.
However, $\text{COV}(\max(X,Y), \min(X,Y)) \neq \text{Cov}(X,Y)$ since $\text{COV}(X)$ cannot be the $\text{COV}(\max(X,Y))$ nor $\text{COV}(\min(X,Y))$ and so can't Y. And apparently, $\max(X,Y) \geqslant \min(X,Y)$ while the X and Y are not certain in who is greater. So the change-range of $X-Y$ will be sure greater than $\max(X,Y)-\min(X,Y)$.

Problem 7.48

Solution:

From the chicken-egg story, $X \sim \text{Pois}(\lambda p)$ and $X \sim \text{Pois}(\lambda q)$, X and Y are independent. So we have

$$\text{COV}(N,X) = \text{COV}(X+Y,X) = \text{COV}(X,X) + \text{COV}(Y,X) = \text{Var}(X) + 0 = \lambda p$$

So we have:

$$\text{Corr}(N,X) = \frac{\text{COV}(N,X)}{\sqrt{\text{Var}(N)\text{Var}(X)}} = \sqrt{p}$$

Problem 7.53

Solution:

Since we know that $X - Y \sim N(0,2)$. Let $X - Y = \sqrt{2}Z, \Rightarrow Z \sim N(0,1)$ and $E(|X-Y|) = \sqrt{2}E|Z|$.

$$E|Z| = \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z = 2\int_0^{\infty} z\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z = \sqrt{\frac{2}{\pi}}$$

So we have $E(M) - E(L) = E(|X-Y|) = \frac{2}{\sqrt{\pi}}$. And we have $E(M) + E(L) = E(X) + E(Y) = 0$.
So that $E(M) = -E(L) = \frac{1}{\sqrt{\pi}}$. We have:

$$\text{Cov}(M,L) = E(ML) - E(M)E(L) = E(XY) - E(M)E(L) = 0 - E(M)E(L) = \frac{1}{\pi}$$

By symmetry

$$\text{Var}(M) = \text{Var}(L) = E(M^2) - E^2(M) = E(M^2) - \frac{1}{\pi}$$

Since $E((X-Y)^2) = 0$, we have

$$\text{Var}(X-Y) = E^2(X-Y) = E^2(M-L) = E^2(M) + E^2(L) - 2E(M)E(L) = E^2(M) + E^2(L) = 2$$

So that $\text{Var}(M) = 1 - \frac{1}{\pi}$, we have

$$\text{Corr}(M,L) = \frac{\text{Cov}(M,L)}{\sqrt{\text{Var}(M)\text{Var}(L)}} = \frac{1}{\pi - 1}$$

Problem 7.55

Solution:

(a)

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(V,V) + \mathrm{Cov}(V,Z) + \mathrm{Cov}(V,W) + \mathrm{Cov}(Z,W) = \mathrm{Var}(V,V) = \lambda$$

(b) Since $\mathrm{Cov}(X,Y) \neq 0$, X and Y aree not independent.

$$\begin{aligned} P(X=x, Y=y|V=v) &= P(W=x-v, Z=y-z|V=v) \\ &= P(W=x-v, Z=y-z) \\ &= P(W=x-v)P(W=y-v) \\ &= P(X=x|V=v)P(Y=y|V=v) \end{aligned}$$

So they are conditionally independent given V.

(c) Let $L = \min(X,Y)$, we get that $V \leqslant L$

$$\begin{aligned} P(X=x, Y=y) &= \sum_{v=0}^{\infty} P(X=x, Y=y|V=v)P(V=v) \\ &= \sum_{v=0}^{L} P(X=x|V=v)P(Y=y|V=v)P(V=v) \\ &= \sum_{v=0}^{L} e^{-3\lambda} \frac{\lambda^{x-v}}{(x-v)!} \cdot \frac{\lambda^{y-v}}{(y-v)!} \cdot \frac{\lambda^{v}}{v!} \\ &= e^{-3\lambda} \sum_{v=0}^{L} \frac{\lambda^{x+y-v}}{(x-v)!(y-v)!v!} \end{aligned}$$

For $X \geqslant 0$,$Y \geqslant 0$, 0 otherwise.

Problem 7.57

Solution:

(a) Let

$$X = \sum_{i=1}^{n} x_i I_i, \quad Y = \sum_{j=1}^{n} y_j J_j$$

$$\begin{aligned}
\operatorname{Cov}(X, Y) &= \operatorname{Cov}\left(\sum_{i=1}^{n} x_i I_i, \sum_{j=1}^{n} y_j J_j\right) \\
&= \sum_{i,j=1} (x_i y_j) \operatorname{Cov}(I_i, J_j) \\
&= \sum_{i,j=1} (x_i y_j)(E(I_i J_j) - E(I_i)E(J_j)) \\
&= \sum_{i,j=1} (x_i y_j)(P(I_i = 1, J_j = 1) - P(I_i = 1)P(J_j = 1)) \\
&= \frac{n-1}{n^2} \sum_{i=j} x_i y_j - \frac{1}{n^2} \sum_{i \neq j} x_i y_j
\end{aligned}$$

$$\begin{aligned}
r &= \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\
&= \frac{1}{n} \sum_{i=1}^{n} (x_i y_i - x_i \bar{y} - y_i \bar{x} + \bar{x}\bar{y}) \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y} \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n^2} \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i \\
&= \frac{1}{n} \sum_{i=1}^{n} x_i y_i - \frac{1}{n^2} \sum_{i=j} x_i y_j - \frac{1}{n^2} \sum_{i \neq j} x_i y_j \\
&= \frac{n-1}{n^2} \sum_{i=j} x_i y_j - \frac{1}{n^2} \sum_{i \neq j} x_i y_j
\end{aligned}$$

That is $\operatorname{Cov}(X, Y) = r$.

(b) Since we get n^2 pairs of (X, Y) and $(\widetilde{X}, \widetilde{Y})$. By symmetry,

$$\begin{aligned} \textit{total signed area} &= \sum_{i,j} (y_j - y_i)(x_j - x_i) \\ &= n^2 E((X - \widetilde{X})(Y - \widetilde{Y})) \end{aligned}$$

Also,

$$\begin{aligned} \textit{total signed area} &= \sum_{i,j} (y_j - y_i)(x_j - x_i) \\ &= \sum_i \sum_j (x_j y_j - x_i y_j - x_j y_i + x_i y_j) \\ &= n \sum_i x_i y_i - n^2 \bar{x}\bar{y} - n^2 \bar{x}\bar{y} + n \sum_i x_i y_i \\ &= 2n \sum_i x_i y_i - 2n^2 \bar{x}\bar{y} = 2n^2 r \end{aligned}$$

(c) (i) The order of Height and weight doesn't matter since the area of the rectangle won't change if we exchange the first coordinate and the second coordinate. And the covariance is just a constant times of the area.

(ii) Scaling of coordinates will lead to the area scale being the product of the scaling factor of the two coordinates

(iii) Shifting a rectangle doesn't change its area.

(iv) The area when expanding one coordinate is equal to the sum of two areas of these little rectangles.

Problem 7.58

Solution:

(a)

$$E(\widehat{\theta}) = E(\omega_1\widehat{\theta}_1 + \omega_2\widehat{\theta}_2) = E(\omega_1\widehat{\theta}_1) + E(\omega_2\widehat{\theta}_2) = (\omega_1 + \omega_2)E(\widehat{\theta}) = \theta$$

(b) Since they are unbiased,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega_1}MSE(\widehat{\theta}) = \frac{\mathrm{d}}{\mathrm{d}\omega_1}\mathrm{Var}(\widehat{\theta}) &= \frac{\mathrm{d}}{\mathrm{d}\omega_1}(\omega_1^2\mathrm{Var}(\widehat{\theta}_1) + \omega_2^2\mathrm{Var}(\widehat{\theta}_2)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\omega_1}(\omega_1^2\mathrm{Var}(\widehat{\theta}_1) + (1-\omega_1)^2\mathrm{Var}(\widehat{\theta}_2)) \\
&= \omega_1\mathrm{Var}(\widehat{\theta}_1) - \omega_2\mathrm{Var}(\widehat{\theta}_2) = 0 \\
&\Rightarrow \omega_1\mathrm{Var}(\widehat{\theta}_1) = \omega_2\mathrm{Var}(\widehat{\theta}_2) \\
&\Rightarrow \omega_1 = \frac{\mathrm{Var}(\widehat{\theta}_2)}{\mathrm{Var}(\widehat{\theta}_1) + \mathrm{Var}(\widehat{\theta}_2)}, \qquad \omega_2 = \frac{\mathrm{Var}(\widehat{\theta}_1)}{\mathrm{Var}(\widehat{\theta}_1) + \mathrm{Var}(\widehat{\theta}_2)}
\end{aligned}$$

(c) Let the variance of them be σ^2.

$$\mathrm{Var}(\widehat{\theta}_1) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{\sigma^2}{n}$$

And thus

$$\mathrm{Var}(\widehat{\theta}_2) = \frac{\sigma^2}{m}$$

So we get

$$\omega_1 = \frac{\mathrm{Var}(\widehat{\theta}_2)}{\mathrm{Var}(\widehat{\theta}_1) + \mathrm{Var}(\widehat{\theta}_2)} = \frac{n}{m+n}$$

$$\omega_2 = \frac{\mathrm{Var}(\widehat{\theta}_1)}{\mathrm{Var}(\widehat{\theta}_1) + \mathrm{Var}(\widehat{\theta}_2)} = \frac{m}{m+n}$$

Finally we have

$$\begin{aligned}
\widehat{\theta} = \widehat{\theta} = \omega_1\widehat{\theta}_1 + \omega_2\widehat{\theta}_2 &= \frac{n}{m+n}\widehat{\theta}_1 + \frac{m}{m+n}\widehat{\theta}_2 \\
&= \frac{n}{m+n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + \frac{m}{m+n}\left(\frac{1}{m}\sum_{i=1}^{m} Y_i\right) \\
&= \frac{1}{m+n}\left(\sum_{i=1}^{n} X_i + \sum_{i=1}^{m} Y_i\right)
\end{aligned}$$

Thus, $\widehat{\theta}$ is the mean of the whole sample.