上海科技大学
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# SI140 Probability & Mathematical Statistics

## Homework 12


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## Problem 9.4

**Solution:**

(a)

$$\begin{aligned} E(X|X \geqslant 1)\, P(X \geqslant 1) + E(X|X = 0)\, P(X = 0) &= E(X) \\ E(X|X \geqslant 1)\,(1 - P(X = 0)) + 0 &= \lambda \\ E(X|X \geqslant 1)\,(1 - e^{-\lambda}) &= \lambda \\ E(X|X \geqslant 1) &= \frac{\lambda}{1 - e^{-\lambda}} \end{aligned}$$

(b)

$$\begin{aligned} E(X^2|X \geqslant 1)\, P(X^2 \geqslant 1) + E(X^2|X = 0)\, P(X = 0) &= E(X^2) \\ E(X^2|X \geqslant 1)\,(1 - P(X = 0)) + 0 &= \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{\lambda(e^t - 1)}\right|_{t=0} \\ E(X^2|X \geqslant 1)\,(1 - e^{-\lambda}) &= \lambda^2 + \lambda \\ E(X^2|X \geqslant 1) &= \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} \end{aligned}$$

So that

$$\mathrm{Var}(X|X \geqslant 1) = E(X^2|X \geqslant 1) - (E(X|X \geqslant 1))^2 = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} - \left(\frac{\lambda}{1 - e^{-\lambda}}\right)^2$$

**Problem 9.10**

**Solution:**

(a)

$$w_{HT}: \ \# \text{ tosses until the } HT \text{ for the first time occurs}$$

$$w_1: \ \# \text{ tosses waiting for the first } H$$

$$w_2: \ \# \text{ tosses waiting for the first } T \text{ after the first } H$$

$$w_{HT} = w_1 + w_2 \quad w_1 \sim \mathrm{Fs}(p) \quad w_2 \sim \mathrm{Fs}(1-p)$$

From the properties of Fs, we have

$$E(w_{HT}) = E(w_1 + w_2) = E(w_1) + E(w_2) = \frac{1}{p} + \frac{1}{1-p}$$

(b) By the similar definition, we can find the expecation by condition on the first toss.

$$E(w_{HH}) = E(w_{HH}|H)p + E(w_{HH}|T)(1-p) = E(w_{HH}|H)p + (1 + E(w_{HH}))(1-p)$$

By condition on the second toss, we have

$$E(w_{HH}|H) = E(w_{HH}|HH)p + E(w_{HH}|HT)(1-p) = 2p + (2 + E(w_{HH}))(1-p)$$

Solve the equation, thus

$$E(w_{HH}) = \frac{1}{p} + \frac{1}{p^2}$$

(c)

$$E\left(\frac{1}{p}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{-1}\, p^{a-1}(1-p)^{b-1}\, \mathrm{d}p = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-1)\Gamma(b)}{\Gamma(a+b-1)} = \frac{a+b-1}{a-1}$$

$$E\left(\frac{1}{1-p}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 (1-p)^{-1}\, p^{a-1}(1-p)^{b-1}\, \mathrm{d}p = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a)\Gamma(b-1)}{\Gamma(a+b-1)} = \frac{a+b-1}{b-1}$$

$$E\left(\frac{1}{p^2}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{-2}\, p^{a-1}(1-p)^{b-1}\, \mathrm{d}p = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-2)\Gamma(b)}{\Gamma(a+b-2)} = \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)}$$

By Adam's Law,

$$E(w_{HT}) = E(E(w_{HT}|p)) = E\left(\frac{1}{p}\right) + E\left(\frac{1}{1-p}\right) = \frac{a+b-1}{a-1} + \frac{a+b-1}{b-1}$$

$$E(w_{HH}) = E(E(w_{HH}|p)) = E\left(\frac{1}{p}\right) + E\left(\frac{1}{p^2}\right) = \frac{a+b-1}{a-1} + \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)}$$

## Problem 9.17

**Solution:**

From $E(Y) = 0$:

$$E(W|Z) = E(\rho X + \sqrt{1-\rho^2}\, Y|X) = E(\rho X|X) + E(\sqrt{1-\rho^2}\, Y|X) = \rho X + \sqrt{1-\rho^2}\, E(Y) = \rho X$$

From $E(XY) = E(X)E(Y) = 0$, $E(Y^2) = \text{Var}(Y) - E^2(Y) = 1$:

$$E(W^2|Z) = E(\rho^2 X^2 + (1-\rho^2)Y^2 + 2\rho\sqrt{1-\rho^2}\, XY|X) = \rho^2 X^2 + (1-\rho^2)E(Y^2) = \rho^2 X^2 + (1-\rho^2)$$

Thus

$$\text{Var}(W|Z) = E(W^2|Z) - E^2(W|Z) = 1 - \rho^2$$

## Problem 9.34

**Solution:**

(a) Consider $j$ as r.v. ranging the same as $i$.

By Adam's Law,

$$E(X_j^*) = E(E(X_j^*|j)) = E(\mu) = \mu$$

By Eve's Law,

$$\text{Var}(X_j^*) = E(\text{Var}(X_j^*|j)) + \text{Var}(E(X_j^*|j)) = E(\sigma^2) + \text{Var}(\mu) = \sigma^2$$

(b) Using the conclusions (property about independence) given by hint.

$$\begin{aligned}
E(\bar{X}^*|X_1,\dots,X_n) &= E(\frac{1}{n}(X_1^* + \cdots + X_n^*)|X_1,\dots,X_n) \\
&= \frac{1}{n}E(X_1^* + \cdots + X_n^*|X_1,\dots,X_n) \\
&= \frac{1}{n}(E(X_1^*|X_1,\dots,X_n) + \cdots + E(X_n^*|X_1,\dots,X_n)) \\
&= \frac{1}{n}(nE(X_1^*|X_1,\dots,X_n)) \\
&= E(X_1^*|X_1,\dots,X_n) \\
&= \frac{1}{n}(X_1 + \cdots + X_n)
\end{aligned}$$

$$
\begin{aligned}
\operatorname{Var}(\bar{X}^*|X_1,\dots,X_n) &= \operatorname{Var}(\frac{1}{n}(X_1^*+\cdots+X_n^*)|X_1,\dots,X_n) \\
&= \frac{1}{n^2}\operatorname{Var}(X_1^*+\cdots+X_n^*|X_1,\dots,X_n) \\
&= \frac{1}{n^2}(\operatorname{Var}(X_1^*|X_1,\dots,X_n)+\cdots+\operatorname{Var}(X_n^*|X_1,\dots,X_n)) \\
&= \frac{1}{n^2}(n\operatorname{Var}(X_1^*|X_1,\dots,X_n)) \\
&= \frac{1}{n}\operatorname{Var}(X_1^*|X_1,\dots,X_n) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2
\end{aligned}
$$

Where $\bar{X}_n = \frac{1}{n}(X_1+\cdots+X_n)$.

(c) By Adam's Law,

$$
E(\bar{X}^*) = E(E(\bar{X}^*|X_1,\dots,X_n)) = E\left(\frac{1}{n}(X_1+\cdots+X_n)\right) = \mu
$$

By Eve's Law,

$$
\begin{aligned}
\operatorname{Var}(\bar{X}^*) &= E(\operatorname{Var}(\bar{X}^*|X_1,\dots,X_n)) + \operatorname{Var}(E(\bar{X}^*|X_1,\dots,X_n)) \\
&= E\left(\frac{1}{n^2}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right) + \operatorname{Var}\left(\frac{1}{n}(X_1+\cdots+X_n)\right) \\
&= \frac{\sigma^2}{n} + \frac{1}{n^2}(\operatorname{Var}(X_1)+\cdots+\operatorname{Var}(X_n)) \\
&= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} \\
&= \frac{2\sigma^2}{n}
\end{aligned}
$$

(d) Intuitively, $X_i$s are r.v.s. contributing the variance of the mean. But $X_j^*$s are selected randomly from $X_i$s, so they have more randomicity. Reflected in the variance, it is greater.

## Problem 9.42

**Proof:**

(a) By Adam's Law,

$$E(N) = E(E(N|\lambda)) = E(\lambda) = 1$$

By Eve's Law,

$$\mathrm{Var}(N) = E(\mathrm{Var}(N|\lambda)) + \mathrm{Var}(E(N|\lambda)) = E(\lambda) + \mathrm{Var}(\lambda) = 2$$

(b) Let the dollar amount of a claim be $X$, independent of $N$. Using the properties of Log-Normal distribution.

$$E(NX) = E(N)E(X) = E(X) = e^{\mu+\frac{\sigma^2}{2}}$$

$$\mathrm{Var}(NX) = E(\mathrm{Var}(NX|N)) + \mathrm{Var}(E(NX|N)) = E(N)\mathrm{Var}(X) + \mathrm{Var}(N)E^2(X) = (e^{\sigma^2}+1)e^{2\mu+\sigma^2}$$

(c)

$$P(N=n) = \int_0^{+\infty} P(N=n|\lambda=x)f_\lambda(x)\,\mathrm{d}x = \int_0^{+\infty} \frac{x^n e^{-x}}{n!}e^{-x}\,\mathrm{d}x = \frac{1}{n!}\int_0^{+\infty} x^n e^{-2x}\,\mathrm{d}x$$
$$= \frac{1}{2^{n+1}n!}\int_0^{+\infty} u^n e^{-u}\,\mathrm{d}u = \frac{\Gamma(n+1)}{2^{n+1}n!} = \frac{1}{2^{n+1}}$$

So $N \sim \mathrm{Geom}(\frac{1}{2})$, for non-negative integer $n$.

(d)

$$f_{\lambda|N}(x|n) = \frac{P(N=n|\lambda=x)f_\lambda(x)}{P(N=n)} = \frac{\frac{x^n e^{-x}}{n!}e^{-x}}{\frac{\Gamma(n+1)}{2^{n+1}n!}} = \frac{x^n\,2^{n+1}\,e^{-2x}}{\Gamma(n+1)}$$

For $x > 0$ and non-negative integer $n$.

So $\lambda|N \sim \mathrm{Gamma}(n+1, 2)$

## Problem 9.44

**Solution:**

(a) Since they are independent, from memoryless,

$$\begin{aligned}E(X_1+X_2+X_3|X_1>1,X_2>2,X_3>3) &= E(X_1|X_1>1)+E(X_2|X_2>2)+E(X_3|X_3>3)\\ &= 1+\frac{1}{\lambda_1}+2+\frac{1}{\lambda_2}+3+\frac{1}{\lambda_3}\\ &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+6\end{aligned}$$

(b) Easy to know $P(X_1 = \min(X_1, X_2, X_3)) = P(X_1 \leqslant \min(X_2, X_3))$. (Logically Equivalence)

From the property in Expo that $\min(X_2, X_3) \sim$Expo$(\lambda_2+\lambda_3)$, we have:

$$P(X_1 = \min(X_1, X_2, X_3)) = P(X_1 \leqslant \min(X_2, X_3)) = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$$

(c) Let$M = \max(X_1, X_2, X_3)$.From the PDF of order statistic,

$$f_M(x) = 3(1-e^{-x})^2e^{-x}$$

For $x > 0$, 0 otherwise. It isn't one of the important distributions we have studied.