上海科技大学
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SI140 Probability & Mathematical Statistics

Homework 5

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Problem 2.14

Solution:

(a)

$$P(X \geqslant 1) = 1 - P(X = 0) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

$$P(X \geqslant 2) = P(X \geqslant 1) - P(X = 1) = 1 - (1 + \lambda)e^{-\lambda}$$

(b)

$$P(X = k | X \geqslant 1) = \frac{P(X \geqslant 1 | X = k)P(X = k)}{P(X \geqslant 1)} = \frac{P(X = k)}{P(X \geqslant 1)} = \frac{e^{-\lambda}\lambda^k}{(1 - e^{-\lambda})k!}, \quad k = 1, 2, \ldots$$

Problem 2.31

Solution:

(a) The number of correct guesses is X. We know that $X \sim \text{HGeom}(3, 3, 3)$. We have:

$$P(X \geqslant 2) = \frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}} + \frac{\binom{3}{3}\binom{3}{0}}{\binom{6}{3}} = \frac{1}{2}$$

(b) Let $A = \{\text{She claims that the cup was milk-first.}\}, \quad B = \{\text{The cup is milk-first.}\}$

$$\textit{posterior odds} = \frac{P(B|A)}{P(B^c|A)} = \frac{P(A|B)P(B)}{P(A|B^c)P(B^c)} = \frac{p_1}{1 - p_2}$$

Problem 2.37

Solution:

(a) When even errors occured, that is 2 and 4, it will be undetected.

$$P(\text{Errors going undetected}) = \binom{5}{2}(0.1)^2(0.9)^3 + \binom{5}{4}(0.1)^4(0.9)^1 = 0.7335$$

(b) When even errors occured, it will be undetected.

$$P(\text{Errors going undetected}) = \sum_{k \text{ is even and } 2\leqslant k\leqslant n} \binom{n}{k} p^k(1-p)^{n-k}$$

(c)

$$a + b = 1; \qquad a - b = \sum_{k=0}^{n} \binom{n}{k}(-p)^k(1-p)^{n-k} = (1-2p)^n$$

$$\Rightarrow a = \frac{1+(1-2p)^n}{2}; \qquad b = \frac{1-(1-2p)^n}{2}$$

$$\Rightarrow P(\text{Errors going undetected}) = a - \binom{n}{0}p^0(1-p)^n = \frac{1+(1-2p)^n}{2} - (1-p)^n$$

Problem 2.44

Solution:

(a)

$$P(X \oplus Y = 1) = P(X = 1|Y = 1)P(Y = 1) + P(X = 0|Y = 0)P(Y = 0) = \frac{1}{2}$$

So that

$$X \oplus Y \sim \text{Bern}(\frac{1}{2})$$

(b) Let $A = \{X \oplus Y = 1\}$; $B = \{X = 1\}$; $C = \{Y = 1\}$

$$P(A \cap B) = P(B \cap C^c) = \frac{p}{2} = P(A)P(B)$$

So that A and B are independent. That is A and B^c, A^c and B, A^c and B^c are independent. That is $X \oplus Y$ and X are independent whatever p is.

$$P(A \cap C) = P(B^c \cap C) = \frac{1-p}{2}$$

When $p = \frac{1}{2}$, $P(A \cap C) = P(A)P(C) = \frac{1}{4}$, then like the reason above, $X \oplus Y$ and Y are independent. If $p \neq \frac{1}{2}$, then $P(A \cap C) \neq P(A)P(C)$, so they are not independent.

(c) When J is of size 1, it is obvious that $Y_{J_1} \sim \text{Bern}(\frac{1}{2})$. When J is of size 2, we know that $Y_{J_2} = Y_{J_1} \oplus X_i$ for any other i. Since $X_i \sim \text{Bern}(\frac{1}{2})$, using the conclusion in (a), $Y_{J_2} \sim \text{Bern}(\frac{1}{2})$. By using induction, $Y_J \sim \text{Bern}(\frac{1}{2})$ for each nonempty subset J.

Using the explanation in Hint, we know that for Y_J and Y_K, A, B and C are independent. If $J \cap K = \emptyset$,

$$p(Y_J = 0 \cap Y_K = 0) = P(Y_J = 0)P(Y_K = 0)$$

Because they don't relate to any ame X_i, so that they are independent.

If $A \neq \emptyset$,

$$p(Y_J = 0 \cap Y_K = 0) = P(Y_J = 0 \cap Y_K = 0|A = 1)P(A = 1) + (Y_J = 0 \cap Y_K = 0|A = 0)P(A = 0)$$
$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(Y_J = 0)P(Y_K = 0)$$

So that they are independent. That is, Y_J are pairwise independent.

But since $P(Y_1 = 0 \cap Y_2 = 0 \cap Y_{\{1,2\}} = 0) = \frac{1}{4} \neq P(Y_1 = 0) \cap P(Y_2 = 0)P \cap (Y_{\{1,2\}} = 0) = \frac{1}{8}$, we know that Y_J are not independent.

Problem 2.46

Solution:

(a) If A starting with 1 dollars, and finally reaches to 0 dollars, that means A must wins twice more than failures (# failures is exactly 1 greater than # wins). So that's typically the original problem wich p_1 is the probability of the original problem.

(b) From the constrains, $p_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k+2}$ (It makes sense that the notation could be negative). The Characteristic equation is $\lambda^3 - 2\lambda + 1 = 0$. Find that:

$$\lambda_1 = 1; \quad \lambda_2 = \frac{-1-\sqrt{5}}{2}; \quad \lambda_3 = \frac{-1+\sqrt{5}}{2}$$

So

$$p_k = a + b\left(\frac{-1-\sqrt{5}}{2}\right)^k + c\left(\frac{-1+\sqrt{5}}{2}\right)^k$$

We know that

$$p_0 = 1; \quad \lim_{k\to\infty} p_k = 0$$

So that $a = b = 0, \quad c = 1$ That is

$$p_k = \left(\frac{-1+\sqrt{5}}{2}\right)^k$$

(c) The original problem is to consider the p_1. So

$$P_{\text{original}} = p_1 = \frac{-1+\sqrt{5}}{2}$$

Problem 2.47

Solution:

(a) If $n \leqslant m$, then $P = 1$; If $n > 2m$, then $P = 0$. Now consider the case that $m < n \leqslant 2m$.

$$\begin{aligned} P &= \sum_{n-m}^{m} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\ &= \sum_{k=0}^{m} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} - \sum_{k=0}^{n-m-1} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\ &= \text{pbinom}(m, n, \frac{1}{2}) - \text{pbinom}(n-m-1, n, \frac{1}{2}) \end{aligned}$$

(b) Using python to sole this problem. The result is $m = 8$ for $n = 10$, $m = 60$ for $n = 100$, $m = 531$ for $n = 1000$, $m = 5098$ for $n = 10000$. The code is as below:

```python
import random

def copyPaper(n:int):
    for m in range(int(n/2), n):
        suc = 0

        for i in range(1000):
            tray0 = tray1 = m

            for j in range(n):
                if random.randint(0, 1):
                    tray1 -= 1
                else:
                    tray0 -= 1

                if tray0 < 0 or tray1 < 0:
                    break

            else:
                suc += 1

        if (suc / 1000) >= 0.95:
            return m

n = int(input("Please enter the n: "))

print("The smallest m is: ", copyPaper(n))
```