SI140 Probability & Mathematical Statistics

Homework 4

陈昱聪

Chen Yucong ><E<>N

Student ID: 2019533079

Email: chenyc@shanghaitech.edu.cn

School of Information Science and Technology

⊚ Group#2 (TA: 曾理)

Problem 2.38

Solution:

(a) I pick the door 1. Monty open the door $2 \sim 4$.

$$M_j^i = \text{“Monty open the door } i \sim j\text{”} \qquad C_k = \text{“Car is behind the } k^{\text{th}} \text{ door”}$$

$$P(C_k) = \frac{1}{7}, \quad (k = 1, 2, \ldots, 7)$$

$$\text{LOTP:} \qquad P(M_4^2) = P(M_4^2|C_1)P(C_1) + \sum_{i=2}^{4} P(M_4^2|C_i)P(C_i) + \sum_{i=5}^{7} P(M_4^2|C_i)P(C_i)$$

$$= \frac{1}{\binom{6}{3}} \cdot \frac{1}{7} + 0 + \frac{1}{\binom{5}{3}} \cdot \frac{1}{7} \cdot 3$$

$$= \frac{1}{20}$$

$$P(C_1|M_4^2) = \frac{P(M_4^2|C_1)P(C_1)}{P(M_4^2)} = \frac{1}{7}$$

$$P(C_5|M_4^2) = P(C_6|M_4^2) = P(C_7|M_4^2) = \frac{P(M_4^2|C_5)P(C_5)}{P(M_4^2)} = \frac{2}{7} > P(C_1|M_4^2)$$

I should switch, the probability of switch to one of the remaining 3 doors is $\frac{2}{7}$.

(b) I pick the door 1. Monty open the door $2 \sim m+1$.

$$M_{m+1}^2 = \text{“Monty open the door } 2 \sim m+1\text{”} \qquad C_k = \text{“Car is behind the } k^{\text{th}} \text{ door”}$$

$$P(C_k) = \frac{1}{n}, \quad (k = 1, 2, \ldots, n)$$

$$\text{LOTP:} \qquad P(M_{m+1}^2) = P(M_{m+1}^2|C_1)P(C_1) + \sum_{i=2}^{m+1} P(M_{m+1}^2|C_i)P(C_i) + \sum_{i=m+2}^{n} P(M_{m+1}^2|C_i)P(C_i)$$

$$= \frac{1}{\binom{n-1}{m}} \cdot \frac{1}{n} + 0 + \frac{1}{\binom{n-2}{m}} \cdot \frac{1}{n} \cdot (n-m-1)$$

$$= \frac{m!}{(n-1)(n-2)\ldots(n-m)}$$

$$P(C_1|M_{m+1}^2) = \frac{P(M_{m+1}^2|C_1)P(C_1)}{P(M_{m+1}^2)} = \frac{1}{n}$$

$$P(C_i|M_{m+1}^2) = \frac{P(M_{m+1}^2|C_i)P(C_i)}{P(M_{m+1}^2)} = \frac{1}{n} \cdot \frac{n-1}{n-m-1} > P(C_1|M_{m+1}^2) \quad \text{For } i = m+2, m+3 \ldots, n$$

I should switch, the probability of switch to one of the remaining doors is $\frac{1}{n} \cdot \frac{n-1}{n-m-1}$.

Problem 2.39

Solution:

(a)

$$M_i = \text{``Monty open the door } i\text{''} \quad C_k = \text{``Car is behind the } k^{\text{th}} \text{ door''}$$

$$P(C_k) = \frac{1}{3}, \quad k = 1, 2 \text{ or } 3$$

$$P(M_2) = P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2) + P(M_2|C_3)P(C_3) = \frac{1}{3} \cdot p + 0 + \frac{1}{3} \cdot 1 = \frac{p+1}{3}$$

$$P(M_3) = 1 - P(M_2) = \frac{2-p}{3}$$

$$\begin{aligned} P(C_1|\text{After Monty chose}) &= P(C_1|M_2)P(M_2) + P(C_1|M_3)P(M_3) \\ &= P(M_2|C_1)P(C_1) + P(M_3|C_1)P(C_1) = \frac{1}{3} \end{aligned}$$

$$P(\text{Win on switching}) = 1 - P(C_1|\text{After Monty chose}) = \frac{2}{3}$$

(b)

$$P(\text{Win on switching}|M_2) = P(C_3|M_2) = \frac{P(M_2|C_3)P(C_3)}{P(M_2)} = \frac{1}{p+1}$$

(c)

$$P(\text{Win on switching}|M_3) = P(C_2|M_3) = \frac{P(M_3|C_2)P(C_2)}{P(M_3)} = \frac{1}{2-p}$$

Problem 2.42

Solution:

(a) We can say that the running total would never be negative. So we define $p_k = 0$ for $k < 0$. And it is clear that the running total reached to 0 before the very first rolling so that $p_0 = 1$. The value of p_n is that the last time reached to $n - 1$ and this time got an 1; or the last time reached to $n - 2$ and this time got a 2, and so on. So we get

$$p_n = \frac{1}{6}(p_{n-1} + p_{n-2} + p_{n-3} + p_{n-4} + p_{n-5} + p_{n-6}) \qquad n \in \mathbb{N}_+$$

(b)

$$p_0 = 1 \qquad p_1 = \frac{1}{6} \qquad p_2 = \frac{1}{6}(1 + \frac{1}{6})$$

$$p_3 = \frac{1}{6}(1 + \frac{1}{6} + \frac{1}{6}(1 + \frac{1}{6})) = \frac{1}{6}(1 + \frac{1}{6})^2$$

$$p_4 = \frac{1}{6}(1 + \frac{1}{6})^3 \qquad p_5 = \frac{1}{6}(1 + \frac{1}{6})^4 \qquad p_6 = \frac{1}{6}(1 + \frac{1}{6})^5$$

$$p_7 = \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6) = \frac{1}{6}\left((1 + \frac{1}{6})^6 - 1\right) \approx 0.2536$$

(c) On average, the number thrown is $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$. That means the growth speed of the running total is 3.5/times. That means every time we roll two times, the running total goes up over 7 numbers but we only get 2 results during the 2 rolls.

So the probability when n is big enough is $\frac{2}{7}$

Problem 2.52

Proof:

(a) **Impossible.**

If A and B are independent, then $P(A|B) = P(A) = P(A|B^c)$.

□

(b) **Impossible.**

$$\begin{aligned} P(A) = P(A|C) &= P(A|B,C)P(B) + P(A|B^c,C)P(B^c) \\ &< P(A|B^c,C)P(B) + P(A|B^c,C)P(B^c) \\ &= P(A|B^c,C) \end{aligned}$$

Similarly, we have

$$P(A) < P(A|B^c,C^c)$$

So that

$$P(A) < P(A|B^c,C)P(C) + P(A|B^c,C^c)P(C^c) = P(A|B^c)$$

Similarly, we have

$$P(A) > P(A|B)$$

So that

$$P(A|B) < P(A|B^c)$$

□

(c) **Impossible.**

$$\begin{aligned} P(A|B) &= P(A|B,C)P(C|B) + P(A|B,C^c)P(C^c|B) \\ &= P(A|B,C)P(C|B^c) + P(A|B,C^c)P(C^c|B^c) \\ &< P(A|B^c,C)P(C|B^c) + P(A|B^c,C^c)P(C^c|B^c) \\ &= P(A|B^c) \end{aligned}$$

□

Problem 2.63

(a) **Proof:**

$$A_n = \text{``Treatment } A \text{ is assigned on the } n^{th} \text{ trial''} \quad S_n = \text{``The success on the } n^{th} \text{ trial''}$$

$$p_n = P(S_n) = P(S_n|A_n)P(A_n) + P(S_n|A_n^c)P(A_n^c) = a \cdot a_n + b \cdot (1 - a_n) = (a-b)a_n + b$$

$$\begin{aligned} a_{n+1} &= P(A_{n+1}) = P(A_n \cap S_n) + P(A_n^c \cap S_n^c) = P(S_n|A_n)P(A_n) + P(S_n^c|A_n^c)P(A_n^c) \\ &= (a+b-1)a_n + a - b \end{aligned}$$

□

(b) **Proof:**

$$\begin{aligned} p_n = (a-b)a_n + b \Rightarrow a_n = \frac{p_n - b}{a-b} & \\ &\Rightarrow a_{n+1} = (a+b-1)\frac{p_n - b}{a-b} + 1 - b \\ &\Rightarrow p_{n+1} = (a-b)a_{n+1} + b = (a-b)\left[(a+b-1)\frac{p_n - b}{a-b} + 1 - b\right] + b \\ &\Rightarrow p_{n+1} = (a+b-1)p_n + a + b - 2ab \end{aligned}$$

□

(c) **Solution:**

Assume

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} p_{n+1}$$

Then we get

$$\begin{aligned} p_{n+1} &= (a+b-1)p_n + a + b - 2ab \\ \Rightarrow p &= \frac{a+b-2ab}{2-a-b} \end{aligned}$$

Problem 2.64

Solution:

(a)

$$
\begin{aligned}
P(\text{child is } AA) &= P(\text{child is } AA|\text{parents are both } AA)P(\text{parents are both } AA) \\
&\quad + P(\text{child is } AA|\text{parents are both } Aa)P(\text{parents are both } Aa) \\
&\quad + P(\text{child is } AA|\text{parents are } AA \text{ and } Aa)P(\text{parents are } AA \text{ and } Aa) \\
&= p^4 + p^2(1-p)^2 + 2p^3(1-p) = p^2
\end{aligned}
$$

Similarly,

$$P(\text{child is } Aa) = 2p(1-p) \qquad P(\text{child is } aa) = (1-p)^2$$

So that the frequencies of genotypes don't change through generations after we reaches the Hardy-Weinberg balance. So the law is stable.

(b)

$$M = \text{``child is homozygous''} \qquad N = \text{``parents are homozygous''}$$

$$
\begin{aligned}
P(M|N) &= P(\text{parents are both } AA|N) + P(\text{parents are both } aa|N) \\
&= \frac{P(N|\text{parents are both } AA)P(\text{parents are both } AA)}{P(N)} \\
&\quad + \frac{P(N|\text{parents are both } aa)P(\text{parents are both } aa)}{P(N)} \\
&= \frac{P(\text{parents are both } AA) + P(\text{parents are both } aa)}{P(N)} \\
&= \frac{p^4 + (1-p)^4}{(p^2 + (1-p)^2)^2}
\end{aligned}
$$

$$P(\text{child is } Aa|\text{parents are } Aa) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

(c)

$$
\begin{aligned}
&P(\text{child is heterozygous}|\text{no parent is } aa)\\
=&P(\text{child is } Aa|\text{no parents is } aa)\\
=&P(\text{child is } Aa|\text{parents are } AA \text{ and } Aa)P(\text{parents are } AA \text{ and } Aa)\\
+&P(\text{child is } Aa|\text{parents are both } Aa)P(\text{parents are both } Aa)\\
+&P(\text{child is } Aa|\text{parents are both } AA)P(\text{parents are both } AA)\\
=&\frac{1}{2}\cdot 2\cdot p^2\cdot 2p(1-p)+\frac{1}{2}\cdot[2p(1-p)]^2+0\\
=&2p^2(1-p)
\end{aligned}
$$

Similarly,

$$P(\text{child is not } aa|\text{no parent is } aa)=p^2(3-2p)$$

$$
\begin{aligned}
&P(\text{child is heterozygous}|\text{no parent is } aa, \text{child is not } aa)\\
=&\frac{P(\text{child is not } aa|\text{child is heterozygous, no parent is } aa)P(\text{child is heterozygous}|\text{no parent is } aa)}{P(\text{child is not } aa|\text{no parent is } aa)}\\
=&\frac{P(\text{child is heterozygous}|\text{no parent is } aa)}{P(\text{child is not } aa|\text{no parent is } aa)}\\
=&\frac{2-2p}{3-2p}
\end{aligned}
$$