上海科技大学
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SI140 Probability & Mathematical Statistics

Homework 3

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Problem 2.12

Solution:

(a) Let

$$A = \text{“Alice sends 1”} \qquad B = \text{“Bob receicves 1”}$$

Let $P(A) = P(A^c) = 0.5$, from Bayes' Rule, we have:

$$\begin{aligned} P(A|B) &= \frac{P(B|A) \cdot P(A)}{P(B)} \\ &= \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)} \\ &= \frac{0.9 \cdot 0.5}{0.9 \cdot 0.5 + 0.05 \cdot 0.5} = \frac{18}{19} \approx 0.9474 \end{aligned}$$

(b) Let

$$B_i = \text{“The } i^{th} \text{ number Bob receicved is 1”}$$

Let $P(A) = P(A^c) = 0.5$, from Bayes' Rule, we have:

$$\begin{aligned} P(A|B_1B_2B_3^c) &= \frac{P(B_1B_2B_3^c|A) \cdot P(A)}{P(A)P(B_1B_2B_3^c|A) + P(A^c)P(B_1B_2B_3^c|A^c)} \\ &= \frac{P(B_1B_2B_3^c|A) \cdot P(A)}{P(B_1B_2B_3^c|A) \cdot P(A) + P(B_1B_2B_3^c|A^c) \cdot P(A^c)} \\ &= \frac{0.9 \cdot 0.9 \cdot 0.1 \cdot 0.5}{0.9 \cdot 0.9 \cdot 0.1 \cdot 0.5 + 0.05 \cdot 0.05 \cdot 0.95 \cdot 0.5} = \frac{648}{667} \\ &\approx 0.9715 \end{aligned}$$

Problem 2.21

Solution:

(a) $A =$ "All 3 tosses landed Heads"; $B =$ "At least 2 tosses were Heads".

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(\frac{1}{2})^3}{\binom{3}{2} \cdot (\frac{1}{2})^3 + (\frac{1}{2})^3} = \frac{1}{4}$$

(b) $C =$ "Two of the slips of paper drwan show the letter H".

$$\begin{aligned} P(A|C) = \frac{P(A \cap C)}{P(C)} &= \frac{P(A \cap C)}{P(C|A)P(A) + P(C|A^c)P(A^c)} \\ &= \frac{P(A)}{P(A) + P(C \cap A^c)} \\ &= \frac{(\frac{1}{2})^3}{(\frac{1}{2})^3 + \binom{3}{2} \cdot (\frac{1}{2})^3 \cdot \frac{1}{\binom{3}{2}}} \\ &= \frac{1}{2} \end{aligned}$$

Problem 2.26

Solution:

(a)

$$P(L|M_1) = \frac{P(M_1|L)P(L)}{P(M_1)} = \frac{P(M_1|L)P(L)}{P(M_1|L)P(L) + P(M_1|L^c)P(L^c)} = \frac{0.9 \cdot 0.1}{0.9 \cdot 0.1 + 0.9 \cdot 0.1} = \frac{1}{2}$$

(b)

$$P(L|M_1 \cap M_2) = \frac{P(M_1 \cap M_2|L)P(L)}{P(M_1 \cap M_2)} = \frac{0.1 \cdot 0.9^2}{0.1 \cdot 0.9^2 + 0.1^2 \cdot 0.9} = \frac{9}{10}$$

(c)

$$\tilde{P}(L|M_2) = \frac{\tilde{P}(L \cap M_2)}{\tilde{P}(M_2)} = \frac{P(L \cap M_2|M_1)}{P(M_2|M_1)} = \frac{\frac{P((L \cap M_2) \cap M_1)}{P(M_1)}}{\frac{P(M_1 \cap M_2)}{P(M_1)}} = \frac{P(L \cap (M_2 \cap M_1))}{P(M_1 \cap M_2)} = P(L|(M_2 \cap M_1))$$

Problem 2.29

Proof:

A = "Both children are girls"; B = "At least one is a girl with characteristic C".

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Since the probability a specific child is a girl with C is $\frac{1}{2} \cdot p$, we have

$$P(B) = 1 - (1 - \frac{1}{2} \cdot p)^2$$

To be specific, we can tell that $P(A \cap B) = P(A \cap (\text{At least one child is with characteristic } C))$.
So that is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4} \cdot (1 - (1 - p)^2)}{1 - (1 - \frac{1}{2}p)^2} = \frac{2 - p}{4 - p}$$

$\square$

Problem 2.36

Solution:

(a) $A =$ "Being good at baseball"; $B =$ "Having a good math score on the test".

Without conditioning on having a good math score, the probability of being good at base ball could be:

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)}$$

Since for a student admitted to the university, he is at least with one of A and B, so use the probability of $A \cup B$ as the condition.

With the conditioning on having a good math score, the probability of being good at base ball could be:

$$P(A|B, A \cup B) = \frac{P(A \cap B|A \cup B)}{P(B|A \cup B)} = \frac{P(A)P(B)}{P(B)} = P(A) > \frac{P(A)}{P(A \cup B)}$$

.

That's because without conditioning, the probability of A would be less in student who are admitted by the colledge than in a universal student, since the demographics have changed. But with the condition, we know that the probability is the same to a universal student because the both denominators that we're picking are the same, which are both with B, in the constraint that A and B are independent. Since $P(A \cup B) < 1$, so we know they are negatively associated.

(b) As we proved in (a) as above, I'm just going to write them down again in here...

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap (A \cup B))}{P(C)} = \frac{P(A)}{P(C)}$$

But fornP$(A|B, C)$, there is

$$P(A|B, C) = \frac{P(A \cap B|C)}{P(B|C)} = \frac{\frac{P(A \cap B \cap C)}{P(C)}}{\frac{P(B \cap C)}{P(C)}} = \frac{P(A)P(B)}{P(B)} = P(A) > \frac{P(A)}{P(C)}$$

Since $P(C) < 1$.

□

Problem 2.37

Solution:

Let

$$W = W_1^c, \dots, W_{22}^c, W_{23}, W_{24}^c, \dots, W_{63}^c, W_{64}, W_{65}, W_{66}^c, \dots, W_{100}^c$$

$$X_p = (1-p_1)(1-p_2)\dots(1-p_{22})\,p_{23}\,(1-p_{24})\dots(1-p_{63})\,p_{64}\,p_{65}\,(1-p_{66})\dots(1-p_{100})$$

$$X_r = (1-r_1)(1-r_2)\dots(1-r_{22})\,r_{23}\,(1-r_{24})\dots(1-r_{63})\,r_{64}\,r_{65}\,(1-r_{66})\dots(1-r_{100})$$

From Bayes' Rule, we have

$$\begin{aligned} P(\text{spam}|W) &= \frac{P(W|\text{spam})P(\text{spam})}{P(W)} \\ &= \frac{P(W|\text{spam})P(\text{spam})}{P(W|\text{spam})P(\text{spam}) + P(W|\text{not spam})P(\text{not spam})} \\ &= \frac{p \cdot X_p}{p \cdot X_p + (1-p) \cdot X_r} \end{aligned}$$

Where X_p and X_r are defined as above.