



上海科技大学
ShanghaiTech University

SI140 Probability & Mathematical Statistics

Homework 12

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Problem 9.4

Solution:

(a)

$$E(X|X \geq 1) P(X \geq 1) + E(X|X = 0) P(X = 0) = E(X)$$

$$E(X|X \geq 1) (1 - P(X = 0)) + 0 = \lambda$$

$$E(X|X \geq 1) (1 - e^{-\lambda}) = \lambda$$

$$E(X|X \geq 1) = \frac{\lambda}{1 - e^{-\lambda}}$$

(b)

$$E(X^2|X \geq 1) P(X \geq 1) + E(X^2|X = 0) P(X = 0) = E(X^2)$$

$$E(X^2|X \geq 1) (1 - P(X = 0)) + 0 = \left. \frac{d^2}{dt^2} e^{\lambda(e^t - 1)} \right|_{t=0}$$

$$E(X^2|X \geq 1) (1 - e^{-\lambda}) = \lambda^2 + \lambda$$

$$E(X^2|X \geq 1) = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}}$$

So that

$$\text{Var}(X|X \geq 1) = E(X^2|X \geq 1) - (E(X|X \geq 1))^2 = \frac{\lambda^2 + \lambda}{1 - e^{-\lambda}} - \left(\frac{\lambda}{1 - e^{-\lambda}} \right)^2$$

Problem 9.10

Solution:

(a)

w_{HT} : # tosses until the HT for the first time occurs

w_1 : # tosses waiting for the first H

w_2 : # tosses waiting for the first T after the first H

$$w_{HT} = w_1 + w_2 \quad w_1 \sim \text{Fs}(p) \quad w_2 \sim \text{Fs}(1-p)$$

From the properties of Fs, we have

$$E(w_{HT}) = E(w_1 + w_2) = E(w_1) + E(w_2) = \frac{1}{p} + \frac{1}{1-p}$$

(b) By the similar definition, we can find the expectation by condition on the first toss.

$$E(w_{HH}) = E(w_{HH}|H)p + E(w_{HH}|T)(1-p) = E(w_{HH}|H)p + (1 + E(w_{HH}))(1-p)$$

By condition on the second toss, we have

$$E(w_{HH}|H) = E(w_{HH}|HH)p + E(w_{HH}|HT)(1-p) = 2p + (2 + E(w_{HH}))(1-p)$$

Solve the equation, thus

$$E(w_{HH}) = \frac{1}{p} + \frac{1}{p^2}$$

(c)

$$E\left(\frac{1}{p}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{-1} p^{a-1} (1-p)^{b-1} dp = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-1)\Gamma(b)}{\Gamma(a+b-1)} = \frac{a+b-1}{a-1}$$

$$E\left(\frac{1}{1-p}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 (1-p)^{-1} p^{a-1} (1-p)^{b-1} dp = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a)\Gamma(b-1)}{\Gamma(a+b-1)} = \frac{a+b-1}{b-1}$$

$$E\left(\frac{1}{p^2}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{-2} p^{a-1} (1-p)^{b-1} dp = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-2)\Gamma(b)}{\Gamma(a+b-2)} = \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)}$$

By Adam's Law,

$$E(w_{HT}) = E(E(w_{HT}|p)) = E\left(\frac{1}{p}\right) + E\left(\frac{1}{1-p}\right) = \frac{a+b-1}{a-1} + \frac{a+b-1}{b-1}$$

$$E(w_{HH}) = E(E(w_{HH}|p)) = E\left(\frac{1}{p}\right) + E\left(\frac{1}{p^2}\right) = \frac{a+b-1}{a-1} + \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)}$$

Problem 9.17

Solution:

From $E(Y) = 0$:

$$E(W|Z) = E(\rho X + \sqrt{1 - \rho^2} Y|X) = E(\rho X|X) + E(\sqrt{1 - \rho^2} Y|X) = \rho X + \sqrt{1 - \rho^2} E(Y) = \rho X$$

From $E(XY) = E(X)E(Y) = 0$, $E(Y^2) = \text{Var}(Y) - E^2(Y) = 1$:

$$E(W^2|Z) = E(\rho^2 X^2 + (1 - \rho^2)Y^2 + 2\rho\sqrt{1 - \rho^2} XY|X) = \rho^2 X^2 + (1 - \rho^2)E(Y^2) = \rho^2 X^2 + (1 - \rho^2)$$

Thus

$$\text{Var}(W|Z) = E(W^2|Z) - E^2(W|Z) = 1 - \rho^2$$

Problem 9.34

Solution:

(a) Consider j as r.v. ranging the same as i .

By Adam's Law,

$$E(X_j^*) = E(E(X_j^*|j)) = E(\mu) = \mu$$

By Eve's Law,

$$\text{Var}(X_j^*) = E(\text{Var}(X_j^*|j)) + \text{Var}(E(X_j^*|j)) = E(\sigma^2) + \text{Var}(\mu) = \sigma^2$$

(b) Using the conclusions (property about independence) given by hint.

$$\begin{aligned} E(\bar{X}^*|X_1, \dots, X_n) &= E\left(\frac{1}{n}(X_1^* + \dots + X_n^*)|X_1, \dots, X_n\right) \\ &= \frac{1}{n}E(X_1^* + \dots + X_n^*|X_1, \dots, X_n) \\ &= \frac{1}{n}(E(X_1^*|X_1, \dots, X_n) + \dots + E(X_n^*|X_1, \dots, X_n)) \\ &= \frac{1}{n}(nE(X_1^*|X_1, \dots, X_n)) \\ &= E(X_1^*|X_1, \dots, X_n) \\ &= \frac{1}{n}(X_1 + \dots + X_n) \end{aligned}$$

$$\begin{aligned}
 \text{Var}(\bar{X}^*|X_1, \dots, X_n) &= \text{Var}\left(\frac{1}{n}(X_1^* + \dots + X_n^*)|X_1, \dots, X_n\right) \\
 &= \frac{1}{n^2} \text{Var}(X_1^* + \dots + X_n^*|X_1, \dots, X_n) \\
 &= \frac{1}{n^2} (\text{Var}(X_1^*|X_1, \dots, X_n) + \dots + \text{Var}(X_n^*|X_1, \dots, X_n)) \\
 &= \frac{1}{n^2} (n \text{Var}(X_1^*|X_1, \dots, X_n)) \\
 &= \frac{1}{n} \text{Var}(X_1^*|X_1, \dots, X_n) \\
 &= \frac{1}{n^2} \sum_{i=1}^n (X_i - \bar{X}_n)^2
 \end{aligned}$$

Where $\bar{X}_n = \frac{1}{n}(X_1 + \dots + X_n)$.

(c) By Adam's Law,

$$E(\bar{X}^*) = E(E(\bar{X}^*|X_1, \dots, X_n)) = E\left(\frac{1}{n}(X_1 + \dots + X_n)\right) = \mu$$

By Eve's Law,

$$\begin{aligned}
 \text{Var}(\bar{X}^*) &= E(\text{Var}(\bar{X}^*|X_1, \dots, X_n)) + \text{Var}(E(\bar{X}^*|X_1, \dots, X_n)) \\
 &= E\left(\frac{1}{n^2} \sum_{i=1}^n (X_i - \bar{X}_n)^2\right) + \text{Var}\left(\frac{1}{n}(X_1 + \dots + X_n)\right) \\
 &= \frac{\sigma^2}{n} + \frac{1}{n^2} (\text{Var}(X_1) + \dots + \text{Var}(X_n)) \\
 &= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} \\
 &= \frac{2\sigma^2}{n}
 \end{aligned}$$

(d) Intuitively, X_i s are r.v.s. contributing the variance of the mean. But X_j^* s are selected randomly from X_i s, so they have more randomness. Reflected in the variance, it is greater.

Problem 9.42

Proof:

(a) By Adam's Law,

$$E(N) = E(E(N|\lambda)) = E(\lambda) = 1$$

By Eve's Law,

$$\text{Var}(N) = E(\text{Var}(N|\lambda)) + \text{Var}(E(N|\lambda)) = E(\lambda) + \text{Var}(\lambda) = 2$$

(b) Let the dollar amount of a claim be X , independent of N . Using the properties of Log-Normal distribution.

$$E(NX) = E(N)E(X) = E(X) = e^{\mu + \frac{\sigma^2}{2}}$$

$$\text{Var}(NX) = E(\text{Var}(NX|N)) + \text{Var}(E(NX|N)) = E(N)\text{Var}(X) + \text{Var}(N)E^2(X) = (e^{\sigma^2} + 1)e^{2\mu + \sigma^2}$$

(c)

$$\begin{aligned} P(N = n) &= \int_0^{+\infty} P(N = n|\lambda = x)f_\lambda(x) \, dx = \int_0^{+\infty} \frac{x^n e^{-x}}{n!} e^{-x} \, dx = \frac{1}{n!} \int_0^{+\infty} x^n e^{-2x} \, dx \\ &= \frac{1}{2^{n+1}n!} \int_0^{+\infty} u^n e^{-u} \, du = \frac{\Gamma(n+1)}{2^{n+1}n!} = \frac{1}{2^{n+1}} \end{aligned}$$

So $N \sim \text{Geom}(\frac{1}{2})$, for non-negative integer n .

(d)

$$f_{\lambda|N}(x|n) = \frac{P(N = n|\lambda = x)f_\lambda(x)}{P(N = n)} = \frac{\frac{x^n e^{-x}}{n!} e^{-x}}{\frac{\Gamma(n+1)}{2^{n+1}n!}} = \frac{x^n 2^{n+1} e^{-2x}}{\Gamma(n+1)}$$

For $x > 0$ and non-negative integer n .

So $\lambda|N \sim \text{Gamma}(n+1, 2)$

Problem 9.44

Solution:

(a) Since they are independent, from memoryless,

$$\begin{aligned} E(X_1 + X_2 + X_3 | X_1 > 1, X_2 > 2, X_3 > 3) &= E(X_1 | X_1 > 1) + E(X_2 | X_2 > 2) + E(X_3 | X_3 > 3) \\ &= 1 + \frac{1}{\lambda_1} + 2 + \frac{1}{\lambda_2} + 3 + \frac{1}{\lambda_3} \\ &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + 6 \end{aligned}$$

(b) Easy to know $P(X_1 = \min(X_1, X_2, X_3)) = P(X_1 \leq \min(X_2, X_3))$. (Logically Equivalence)

From the property in Expo that $\min(X_2, X_3) \sim \text{Expo}(\lambda_2 + \lambda_3)$, we have:

$$P(X_1 = \min(X_1, X_2, X_3)) = P(X_1 \leq \min(X_2, X_3)) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

(c) Let $M = \max(X_1, X_2, X_3)$. From the PDF of order statistic,

$$f_M(x) = 3(1 - e^{-x})^2 e^{-x}$$

For $x > 0$, 0 otherwise. It isn't one of the important distributions we have studied.