上海科技大学

ShanghaiTech University

SI140 Probability & Mathematical Statistics

Homework 11

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Problem 8.20

Solution:

(a) The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$$

Find the Jacobian matrix that

$$\frac{\partial(t,w)}{\partial(x,y)} = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}$$

So we have

$$f_{T,W}(t,w) = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(t,w)} \right|$$

$$= \lambda^2 e^{-\lambda(x+y)} \cdot \frac{y^2}{x+y}$$

$$= \lambda^2 e^{-\lambda t} \cdot \frac{t}{(w+1)^2}$$

For $t, w > 0$. $f_{T,W}(t,w) = 0$ otherwise. Since $f_{T,W}(t,w) = \lambda^2\, t\, e^{-\lambda t} \cdot \frac{1}{(w+1)^2}$, which factors into a function of t times a function of w, so T and W are independent.

(b)

$$f_T(t) = \int_0^{+\infty} f_{T,W}(t,w)\,\mathrm{d}w = \lambda^2\, t\, e^{-\lambda t} \int_0^{+\infty} \frac{1}{(w+1)^2}\,\mathrm{d}w = \lambda^2\, t\, e^{-\lambda t}$$

For $t > 0$. $f_T(t) = 0$ otherwise.

Since T and W are independent, we have

$$f_W(w) = f_{T,W}(t,w)/f_T(t) = \frac{1}{(w+1)^2}$$

For $w > 0$. $f_W(w) = 0$ otherwise.

Problem 8.28

Solution:

Let $T = X + Y$, we have known that

$$f_T(t) = \begin{cases} t & \text{for } 0 < t \leqslant 1 \\ 2-t & \text{for } 1 < t \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

And

$$f_Z(z) = \begin{cases} 1 & \text{for } 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $W = T + Z$,

$$f_W(w) = \int_{-\infty}^{\infty} f_T(w-z) f_Z(z) \, \mathrm{d}z$$

Find the ranges of w and z to take such that $f_W(w)$ is non-zero, we have those constrains

$$0 < z < 1$$

$$z < w \leqslant z+1 \quad \text{or} \quad z+1 < w \leqslant z+2$$

That is

1. When $0 < w \leqslant 1$, $\quad 0 < z \leqslant w$, $\quad f_W(w) = \int_0^w f_T(w-z) f_Z(z) \, \mathrm{d}z$
2. When $1 < w \leqslant 2$, $\quad 0 < z \leqslant 1$, $\quad f_W(w) = \int_{w-1}^1 f_T(w-z) f_Z(z) \, \mathrm{d}z + \int_0^{w-1} f_T(w-z) f_Z(z) \, \mathrm{d}z$
3. When $2 < w \leqslant 3$, $\quad w-2 < z \leqslant 1$, $\quad f_W(w) = \int_{w-2}^1 f_T(w-z) f_Z(z) \, \mathrm{d}z$

So we have

$$f_W(w) = \begin{cases} \dfrac{w^2}{2} & \text{for } 0 < w \leqslant 1 \\ -w^2 + 3w - \dfrac{3}{2} & \text{for } 1 < w \leqslant 2 \\ \dfrac{w^2}{2} - 3w + \dfrac{9}{2} & \text{for } 2 < w \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.30

Solution:

(a) Let $Z = X + Y$, by using convolution:

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,\mathrm{d}y \\
&= \lambda^{a+b} \frac{e^{-\lambda z}}{\Gamma(a)\,\Gamma(b)} \int_0^z y^{b-1}(z-y)^{a-1}\,\mathrm{d}y \\
&= \lambda^{a+b} \frac{e^{-\lambda z}}{\Gamma(a)\,\Gamma(b)} \int_0^1 (zu)^{b-1}(z-zu)^{a-1} z\,\mathrm{d}u \\
&= \lambda^{a+b} \frac{e^{-\lambda z}}{\Gamma(a)\,\Gamma(b)}\, z^{a+b-1} \int_0^1 u^{b-1}(1-u)^{a-1} z\,\mathrm{d}u \\
&= \lambda^{a+b} \frac{e^{-\lambda z}}{\Gamma(a)\,\Gamma(b)}\, z^{a+b-1}\, \beta(b,a) \\
&= \lambda^{a+b} \frac{e^{-\lambda z}}{\Gamma(a)\,\Gamma(b)}\, z^{a+b-1}\, \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\
&= \frac{1}{\Gamma(a+b)} (\lambda z)^{a+b}\, e^{-\lambda z} \frac{1}{z}
\end{aligned}
$$

That is, $Z \sim \text{Gamma}(a, \lambda)$.

(b) The MGF of $X + Y$ is $= M_{X+Y}(t) = M_X(t) M_Y(t) = \frac{\lambda^a}{(\lambda-t)^a} \frac{\lambda^b}{(\lambda-t)^b} = \frac{\lambda^{a+b}}{(\lambda-t)^{a+b}}$.
So $X + Y \sim \text{Gamma}(a+b, \lambda)$.

(c) Since the X is the sum of a i.i.d. r.v.s. $T \sim \text{Expo}(\lambda)$, and Y is the sum of b i.i.d. r.v.s. $T \sim \text{Expo}(\lambda)$, consider the Poisson Process, X is the time required to have the a^{th} arrival, and Y is the time to have b more arrivals after the a^{th} arrival. Since the time gaps are all i.i.d. we get X and Y are independent and they have the distributions of Gamma shown in this Question. That is, we get the time taken by $(a+b)^{\text{th}}$ arrival, $X + Y$. That is $\text{Gamma}(a+b, \lambda)$

Problem 8.36

Solution:

(a) Let $W = \frac{T_1}{T_1+T_2}$, $T = T_1 + T_2$, by the conclusion dragged by bank-post office story, the joint PDF factors into a function of t times a function of w, so T ans W are independent. Since $\frac{T_1}{T_2} = \frac{T_1}{T_1+T_2} / \frac{T_1}{T_1+T_2} = \frac{W}{1-W}$, since the expression only contains one r.v. W which is independent to T, so it is also independent to $T_1 + T_2$, so $\frac{T_1}{T_2}$ and $T_1 + T_2$ are independent.

(b) Since T_1 and T_2 are independent, we have

$$P(T_1 < T_2) = \int_0^{+\infty} \left(\int_0^{y_2} \lambda_1 e^{-\lambda_1 y_1} \, \mathrm{d}y_1 \right) \lambda_2 e^{-\lambda_2 y_2} \, \mathrm{d}y_2 = \int_0^{+\infty} (1 - e^{-\lambda_1 y_2}) \lambda_2 e^{-\lambda_2 y_2} \, \mathrm{d}y_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

In special case, if $\lambda_1 = \lambda_2$, $P(T_1 < T_2) = \frac{1}{2}$, which is intuitively known by symmetry.

(c) Divide the total time into two parts: waiting and doing. For the first part, the time taken is $L = \mathrm{Min}(T_1, T_2)$. By consider its survival function:

$$G_L(l) = P(T_1 > l, T_2 > l) = P(T_1 > l)P(T_2 > l) = G_{T_1}(l)G_{T_2}(l) = e^{-(\lambda_1+\lambda_2)l}$$

So $L \sim \mathrm{Expo}(\lambda_1 + \lambda_2)$, thus

$$\begin{aligned} E(\text{Total time}) =& E(L) + E(T_1|T_1 < T_2 \text{ (in the previous time)})P(T_1 < T_2) \\ &+ E(T_2|T_2 < T_1 \text{ (in the previous time)})P(T_2 < T_1) \\ =& \frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2} \\ =& \frac{3}{\lambda_1 + \lambda_2} \end{aligned}$$

Problem 8.43

Proof:

Using the names in the story of Bayes' billiards.
Let n r.v.s (the positions of white balls) $U_1, U_2, \ldots, U_n$ i.i.d. with distribution Unif$(0,1)$. Now we have a grey ball to be thown on to the range $(0,1)$ with the position X.

Say that we have **at least** j white balls fall to left of X, the number of it is discrete. Since each trial is binomial and the probability of success is equivalent to the position of the grey ball, then the probability is that

$$P(\textbf{At least } j \text{ white balls fall on left of } X) = \sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k} = LHS$$

That is equivalent to say, the positions of $j-1$ white balls are surely less than the X, and that is $P(U_{(j-1)} < x)$ The positions of them are continuous and we divide them into two categories with the probability t such that $0 < t < x$, we know

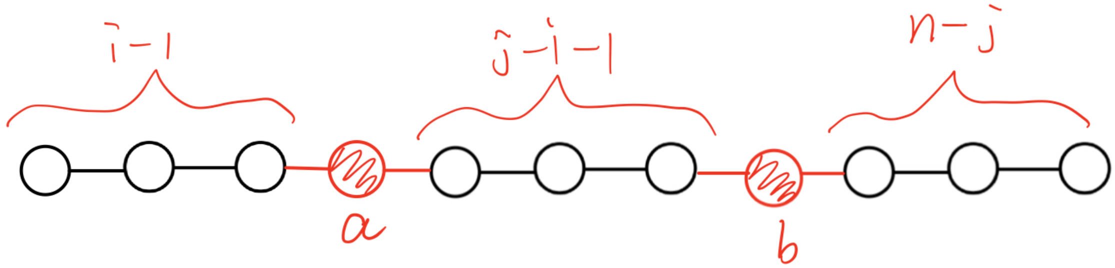
$$\begin{aligned}
&P(\text{The positions of } j-1 \text{ white balls are surely } \textbf{less than } \text{x}) \\
&= \int_0^x f_{U_{(j-1)}} \,\mathrm{d}x \\
&= \int_0^x n\binom{n-1}{j-1} t^{j-1}(1-t)^{n-j} \,\mathrm{d}x \\
&= RHS
\end{aligned}$$

Since they are equivalent, $RHS = LHS$.

□

Problem 8.48

Solution:



To have $X_{(i)}$ be in a $\varepsilon-$interval around a and $X_{(j)}$ be in a $\varepsilon-$interval around b, where $a < b$.
For all these r.v.s. there are one in the $\varepsilon-$interval around a, and then another one in the $\varepsilon-$interval around b, $i-1$ of them left of a, $j-i-1$ of them between a and b, and $n-j$ of them right to b.
So we have

$$f_{X_{(i)},X_{(j)}}(a,b) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}(F(a))^{i-1}f(a)(F(b)-F(a))^{j-i-1}f(b)(1-F(b))^{n-j}$$

For $a < b$, otherwise 0.

The coeff in front of it is given by the cases of choice. There are n possible choices for who is at a and $n-1$ possible choices for who is at b, etc. So the factor is $n(n-1)\binom{n-2}{i-1}\binom{n-i-1}{j-i-1}$. Other factors of this expression are given by their CDFs and PDFs.