Ethan Dowalter

CS 4040 - HW 2

A) $T(n) = 5T(\lfloor \frac{n}{3} \rfloor) + n$

$= n + 5(\lfloor \frac{n}{3} \rfloor + 5T(\lfloor \frac{n}{9} \rfloor))$

$= n + 5\lfloor \frac{n}{3} \rfloor + 25T(\lfloor \frac{n}{9} \rfloor)$

$= n + 5\lfloor \frac{n}{3} \rfloor + 25(\lfloor \frac{n}{9} \rfloor + 5T(\lfloor \frac{n}{27} \rfloor))$

$= n + 5\lfloor \frac{n}{3} \rfloor + 25\lfloor \frac{n}{9} \rfloor + 125T(\lfloor \frac{n}{27} \rfloor)$

$\vdots$

For i'th iteration:

$$T(n) = \sum_{j=0}^{i-1} 5^j \lfloor \frac{n}{3^j} \rfloor + 5^i T(\lfloor \frac{n}{3^i} \rfloor)$$

When $\log_3 \frac{n}{2} < i \leq \log_3 n$ then $\lfloor \frac{n}{3^i} \rfloor = 1$ and $T(1) = c$, so:

$$T(n) \leq \sum_{j=0}^{\log_3 n - 1} 5^j \lfloor \frac{n}{3^j} \rfloor + 5^{\log_3 n} c \leq \sum_{j=0}^{\log_3 n - 1} \left(\frac{5}{3}\right)^j n + 5^{\log_3 n} c$$

$$= n + \sum_{j=1}^{\log_3 n - 1} \left(\frac{5}{3}\right)^j n + 5^{\log_3 n} c$$

$$= n + \left(\frac{5n}{3}\right)\left(\frac{1 - (5/3)^{\log_3 n - 1}}{1 - 5/3}\right) + 5^{\log_3 n} c$$

$$= n + \left(\frac{5n}{3}\right)\left(-\frac{3}{2}\right)\left(1 - \left(\frac{5^{\log_3 n} 5^{-1}}{n/3}\right)\right) + 5^{\log_3 n} c$$

$$= n - \frac{5}{2}n + \left(\frac{5n}{2}\right)\left(\frac{5^{\log_3 n}}{5} \cdot \frac{3}{n}\right) + 5^{\log_3 n} c$$

$$= -\frac{3}{2}n + 5^{\log_3 n}\left(\frac{3}{2} + c\right)$$

$$= -\frac{3}{2}n + n^{\log_3 5}\left(\frac{3}{2} + c\right)$$

Therefore, $\boxed{T(n) = O(n^{\log_3 5})}$

B) $T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + 9n$

Cost

$$9n \qquad \text{Lvl } 0 = 9\lfloor n \rfloor$$

$$9\lfloor \tfrac{n}{2} \rfloor \quad 9\lfloor \tfrac{n}{2} \rfloor \quad 9\lfloor \tfrac{n}{2} \rfloor \qquad \text{Lvl } 1 = 27\lfloor \tfrac{n}{2} \rfloor$$

$$9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \; 9\lfloor \tfrac{n}{4} \rfloor \qquad \text{Lvl } 2 = 81\lfloor \tfrac{n}{4} \rfloor$$

$$\vdots$$

Base case: $\lfloor \frac{n}{2^i} \rfloor = 1$, Height $= h$ $\qquad$ Lvl $i = 3^{i+2}\lfloor \frac{n}{2^i} \rfloor$

$\Rightarrow 1 \le \frac{n}{2^h} < 2$

$2^h \le n < 2^{h+1}$

$h \le \log_2 n < h+1$

$\therefore h = \log_2 n$

$$\text{Cost} = \sum_{i=0}^{\log_2 n} 3^{i+2} \left\lfloor \frac{n}{2^i} \right\rfloor \le 9 \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i n$$

$$= 9n\left(\frac{1-(3/2)^{\log_2 n+1}}{1-3/2}\right)$$

$$= -18n\left(1 - \frac{3}{2}\left(3^{\log_2 n} \cdot \frac{1}{n}\right)\right)$$

$$= -18n + 27 \cdot 3^{\log_2 n}$$

$$= -18n + 27 \cdot n^{\log_2 3}$$

Therefore, $\boxed{\text{Cost} = O(n^{\log_2 3})}$

I.H.

Assume $T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + 9n = O(n^{\log_2 3}) \le cn^{\log_2 3} - d \quad \forall n > n_0$

I.S

Prove $T(m) = 3T(\lfloor \frac{m}{2} \rfloor) + 9m = O(m^{\log_2 3}) \le am^{\log_2 3} - b \quad \forall m > n$

$T(m) \le 3(a\lfloor \frac{m}{2} \rfloor^{\log_2 3} - b) + 9m \le am^{\log_2 3} - b$

$T(m) \le 3(a(\frac{m}{2})^{\log_2 3} - b) + 9m \le am^{\log_2 3} - b$

$T(m) \le 3(a \cdot \frac{1}{3} \cdot m^{\log_2 3} - b) + 9m \le am^{\log_2 3} - b$

$T(m) \le am^{\log_2 3} - 3b + 9m \le am^{\log_2 3} - b$

This inequality is true if $b \ge \frac{9}{2}m$, so $T(m) = O(m^{\log_2 3})$ ✓

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CS 4040 - HW 2 (cont.)

c) $T(n) = 3T(\lfloor \frac{n}{\sqrt{3}} \rfloor) + cn$

Cost

cn — Lvl $0 = cn$

$c\lfloor \frac{n}{\sqrt{3}} \rfloor \quad c\lfloor \frac{n}{\sqrt{3}} \rfloor \quad c\lfloor \frac{n}{\sqrt{3}} \rfloor$ — Lvl $1 = 3c\lfloor \frac{n}{\sqrt{3}} \rfloor$

$c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor \quad c\lfloor \frac{n}{3} \rfloor$ — Lvl 2 $= 9c\lfloor \frac{n}{3} \rfloor$

Base case: $\lfloor \frac{n}{\sqrt{3}^i} \rfloor = 1$, Height $= h$ — Lvl $i = 3^i c \lfloor \frac{n}{\sqrt{3}^i} \rfloor$

$$\Rightarrow 1 = \frac{n}{\sqrt{3}^h}$$
$$\sqrt{3}^h = n$$
$$3^{h/2} = n$$
$$\frac{h}{2} = \log_3 n$$
$$h = 2\log_3 n$$
$$\therefore \; h = 2\log_3 n$$

$$\text{Cost} = \sum_{i=0}^{2\log_3 n} 3^i c \lfloor \frac{n}{\sqrt{3}^i} \rfloor \le c \sum_{i=0}^{2\log_3 n} \sqrt{3}^i n$$
$$= cn\left(\frac{1-\sqrt{3}^{2\log_3 n+1}}{1-\sqrt{3}}\right)$$
$$= cn\left(\frac{1}{1-\sqrt{3}}\right)\left(1-\sqrt{3}(3^{1/2})^{2\log_3 n}\right)$$
$$= cn\left(\frac{1}{1-\sqrt{3}}\right)(1-\sqrt{3}\,n)$$
$$= \left(\frac{1}{1-\sqrt{3}}\right)(cn - \sqrt{3}\,cn^2)$$

Therefore, Cost $= O(n^2)$

<u>I.H.</u>

Assume $T(n) = 3T(\lfloor \frac{n}{\sqrt{3}} \rfloor) + c_1 n = O(n^2) \le c_2 n^2 - c_3 \quad \forall n > n_0$

<u>I.S.</u>

Prove $T(m) = 3T(\lfloor \frac{m}{\sqrt{3}} \rfloor) + cm = O(m^2) \le am^2 - b \quad \forall m > n$

$$T(m) \le 3(a\lfloor \tfrac{m}{\sqrt{3}} \rfloor^2 - b) + cm \le am^2 - b$$
$$T(m) \le 3(a(\tfrac{m}{\sqrt{3}})^2 - b) + cm \le am^2 - b$$
$$T(m) \le am^2 - 3b + cm \le am^2 - b$$

This inequality is true for any $b \ge \frac{cm}{2}$, so $T(m) = O(m^2)$ ✓

D) I.H.

Assume $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 13) + 2n = O(n\log n) \leq c\, n\log n - d \quad \forall n > n_0$

I.S.

Prove $T(m) = 2T(\lfloor \frac{m}{2} \rfloor + 13) + 2m = O(m\log m) \leq am\log m - b \quad \forall m > n$

$$T(m) \leq 2(a(\lfloor \tfrac{m}{2} \rfloor + 13)\log(\lfloor \tfrac{m}{2} \rfloor + 13) - b) + 2m \leq am\log m - b$$
$$\leq 2(a(\tfrac{m}{2} + 13)\log(\tfrac{m}{2} + 13) - b) + 2m \leq am\log m - b$$
$$\leq (am + \tfrac{13}{2})(\log(\tfrac{m}{2}) + \log(1 + \tfrac{26}{m})) - 2b + 2m \leq am\log m - b$$
$$\leq (am + \tfrac{13}{2})(\log m - 1 + \log(1 + \tfrac{26}{m})) - 2b + 2m \leq am\log m - b$$
$$\leq am\log m - am + am\log(1 + \tfrac{26}{m}) + \tfrac{13}{2}\log m$$
$$- \tfrac{13}{2} + \tfrac{13}{2}\log(1 + \tfrac{26}{m}) - 2b + 2m \leq am\log m - b$$

This inequality is true for any $b \geq -am + am\log(1 + \frac{26}{m}) + \frac{13}{2}\log m - \frac{13}{2} + \frac{13}{2}\log(1 + \frac{26}{m}) + 2m$, So, $T(m) = O(m\log m)$ ✓

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CS 4040 - HW 2 (cont.)

E) 1) $T(n) = 32T(\frac{n}{6}) + 5n$

Master Method: $a = 32$, $b = 6$, $f(n) = 5n$

Case 1: $f(n) \in O(n^{\log_b a - \epsilon})$ $\quad \log_6 32 \approx 1.934$

$5n \in O(n^{1.934 - \epsilon})$ ✓ $\quad$ Choose $0 < \epsilon < .934$

$\boxed{\therefore T(n) \in \Theta(n^{\log_6 32})}$

2) $T(n) = 36T(\frac{n}{6}) + 5n^2 + 4n$

Master Method: $a = 36$, $b = 6$, $f(n) = 5n^2 + 4n$

Case 1: $f(n) \in O(n^{\log_b a - \epsilon})$ $\quad \log_6 36 = 2$

$5n^2 + 4n \notin O(n^{2-\epsilon})$

Case 2: $5n^2 + 4n \in \Theta(n^2)$ ✓

$\boxed{\therefore T(n) \in \Theta(n^2 \log n)}$

3) $T(n) = 42T(\frac{n}{6}) + 25n^3 + 5n^2 + n$

Master Method: $a = 42$, $b = 6$, $f(n) = 25n^3 + 5n^2 + n$

Case 1: $f(n) \in O(n^{\log_b a - \epsilon})$ $\quad \log_6 42 \approx 2.086$

$25n^3 + 5n^2 + n \notin O(n^{2.086 - \epsilon})$

Case 2: $25n^3 + 5n^2 + n \notin \Theta(n^{2.086})$

Case 3: $25n^3 + 5n^2 + n \in \Omega(n^{2.086 + \epsilon})$ ✓ Choose $0 < \epsilon < .914$

$42[25(\frac{n}{6})^3 + 5(\frac{n}{6})^2 + \frac{n}{6}] \leq c(25n^3 + 5n^2 + n)$

$\frac{175}{36}n^3 + \frac{35}{6}n^2 + 7n \leq c(25n^3 + 5n^2 + n)$ $\quad$ Choose $c = 0.8$

$\frac{175}{36}n^3 + \frac{35}{6}n^2 + 7n \leq 20n^3 + 4n^2 + \frac{4}{5}n$ $\quad$ True $\forall n \geq 1$

$\therefore T(n) \in \Theta(25n^3 + 5n^2 + n) \Rightarrow \boxed{T(n) \in \Theta(n^3)}$

F) $T(n) = 8T\left(\frac{n}{2}\right) + 2n^2$

Master Method: $a=8$, $b=2$, $\log_b a = 3$, $f(n) = 2n^2$

Case 1: $2n^2 \in O(n^{3-\epsilon})$ ✓ Choose $0 < \epsilon < 1$

$\therefore T(n) \in \Theta(n^3)$

$T(n) = aT\left(\frac{n}{4}\right) + n^2$

Master Method: $a=a$, $b=4$, $f(n) = n^2$

Case 1: $n^2 \in O(n^{\log_4 a - \epsilon})$

$\Rightarrow 2 < \log_4 a < 3$ → in order to satisfy Case 1 and grow quicker than alg. 1

$16 < a < 64$

So, in order for algorithm 2 to be asymptotically faster than algorithm 1, a must be strictly less than 64. Therefore, the largest integer a value is $\boxed{63}$

If $a \leq 16$ then it falls down to either Case 2 or 3, in both cases that would mean alg. 2 is still faster than alg. 1.

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CS 4040 - HW 2 (cont.)

G) 1) $T(n) = 42T(\frac{n}{42}) + n^3$

M.M.: $a = 42, b = 42, \log_b a = 1, f(n) = n^3$

Case 1: $n^3 \notin O(n^{1-\epsilon})$

Case 2: $n^3 \notin \Theta(n)$

Case 3: $n^3 \in \Omega(n^{1+\epsilon})$ ✓ Choose $0 < \epsilon < 2$

$42(\frac{n}{42})^3 \leq cn^3$

$\frac{1}{1764} n^3 \leq cn^3$ ✓ Choose $\frac{1}{1764} \leq c < 1$

$\boxed{\therefore T(n) = \Theta(n^3)}$

2) $T(n) = 36T(\frac{n}{6}) + n^2$

M.M.: $a = 36, b = 6, \log_b a = 2, f(n) = n^2$

Case 1: $n^2 \notin O(n^{2-\epsilon})$

Case 2: $n^2 \in \Theta(n^2)$ ✓

$\boxed{\therefore T(n) = \Theta(n^2 \log n)}$

3) $T(n) = 7T(\frac{n}{2}) + n^{2.333}$

M.M.: $a = 7, b = 2, \log_b a \approx 2.807, f(n) = n^{2.333}$

Case 1: $n^{2.333} \in O(n^{2.807-\epsilon})$ ✓ Choose $0 < \epsilon < .474$

$\boxed{\therefore T(n) \in \Theta(n^{\log_2 7})}$

4) $T(n) = 2T(\frac{n}{16}) + \sqrt[4]{n}$

M.M.: $a = 2, b = 16, \log_b a = \frac{1}{4}, f(n) = n^{1/4}$

Case 1: $n^{1/4} \notin O(n^{1/4-\epsilon})$

Case 2: $n^{1/4} \in \Theta(n^{1/4})$ ✓

$\boxed{\therefore T(n) \in \Theta(n^{1/4} \log n)}$

5) $T(n) = T(n-2) + 3n + 1$

Iteration Method:

$T(n) = 3n + 1 + T(n-2)$

$= 3n + 1 + 3(n-2) + 1 + T(n-4)$

$= 3n + 1 + 3n - 6 + 1 + 3(n-4) + 1 + T(n-6)$

$\vdots$

For i'th iteration:

$$T(n) = \sum_{j=0}^{i-1} [3n - 6j + 1] + T(n-2i)$$

Base cases occur for T(1) and T(0), i.e. when $n-2i=0$ or $n-2i=1$. So i either equals $\frac{n}{2}$ or $\frac{n-1}{2}$, but I will choose $i=\frac{n}{2}$ because $\frac{n}{2} \geq \frac{n-1}{2}$. Also, let $T(0)=T(1)=c$

$$T(n) \leq \sum_{j=0}^{\frac{n}{2}-1} [3n-6j+1] + c$$

$$\leq \frac{n}{2}(3n) - 6\left(\frac{1}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\right) + \frac{n}{2} + c$$
$$\leq \frac{3}{2}n^2 - 3\left(\frac{n^2}{4} - \frac{3}{2}n + 2\right) + \frac{n}{2} + c$$
$$\leq \frac{3}{2}n^2 - \frac{3}{4}n^2 + \frac{9}{2}n - 6 + \frac{n}{2} + c$$
$$\leq \frac{3}{4}n^2 + 5n - 6 + c$$

Therefore, $\boxed{T(n) = O(n^2)}$

H) 1) Without considering costs of parameter passing, the worst-case running time of bsearch is on the order of $\theta(\log_2 n)$, having at worst $\lceil \log_2 n \rceil$ calls to the function. The normal recurrence relation is approx.:

$$T(n) = \theta(1) + T\left(\frac{n}{2}\right)$$

i) Pass by pointer:

$$T_i(n) = \theta(1) + T_i\left(\frac{n}{2}\right) + \theta(1) \quad \Rightarrow \theta(1)+\theta(1)=\theta(1)$$
$$= \theta(1) + T_i\left(\frac{n}{2}\right) \quad \Rightarrow \text{No change}$$
$$\boxed{\therefore T_i(n) = \theta(\log_2 n)}$$

ii) Pass by copying: (N is size of initial input size)

$$T_{ii}(n) = \theta(1) + T_{ii}\left(\frac{n}{2}\right) + \theta(N)$$
$$= cN + T_{ii}\left(\frac{n}{2}\right)$$
$$= cN + cN + T_{ii}\left(\frac{n}{4}\right)$$
$$= cN + cN + cN + T_{ii}\left(\frac{n}{8}\right)$$
$$\vdots$$

For i'th iteration:

$$T_{ii}(n) = \sum_{j=0}^{i-1} cN + T_{ii}\left(\frac{n}{2^i}\right)$$

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CS 4040 - HW 2 (cont.)

Base case occurs for T(1), i.e. when $\frac{n}{2^i} = 1$, which means $i = \log_2 n$. Also, let $T(1) = d$

$$T_{ii}(n) \leq \sum_{j=0}^{\log_2 n - 1} cN + d \leq cN\log_2 n + d$$

Therefore, $\boxed{T_{ii}(n) = \theta(n\log_2 n)}$

iii) Pass by selective copying:

$$\begin{aligned} T_{iii}(n) &= \theta(1) + T_{iii}(\tfrac{n}{2}) + \theta(\tfrac{n}{2}) \\ &= c\tfrac{n}{2} + T_{iii}(\tfrac{n}{2}) \\ &= c\tfrac{n}{2} + c\tfrac{n}{4} + T_{iii}(\tfrac{n}{4}) \\ &= c\tfrac{n}{2} + c\tfrac{n}{4} + c\tfrac{n}{8} + T_{iii}(\tfrac{n}{8}) \\ &\vdots \end{aligned}$$

For i'th iteration:

$$T_{iii}(n) = \frac{1}{2}\sum_{j=0}^{i-1} \left(\frac{1}{2}\right)^j cn + T_{iii}\left(\frac{n}{2^i}\right)$$

Base case occurs for T(1), i.e. when $\frac{n}{2^i} = 1$, which means $i = \log_2 n$. Also, let $T(1) = d$

$$\begin{aligned} T_{iii} \leq \frac{1}{2}\sum_{j=0}^{\log_2 n - 1} \left(\frac{1}{2}\right)^j cn + d &\leq \frac{1}{2}cn\left(\frac{1-(1/2)^{\log_2 n}}{1-1/2}\right) + d \\ &\leq cn\left(1 - \frac{1}{n}\right) + d \\ &\leq cn - c + d \end{aligned}$$

Therefore, $\boxed{T_{iii}(n) = O(n)}$

2) Without considering costs of parameter passing, the worst-case running time of Merge Sort is on the order of $\theta(n\log_2 n)$ and the normal recurrence relation is approximately:

$$T(n) = \theta(n) + 2T\left(\frac{n}{2}\right)$$

i) Pass by pointer:

$$T_i(n) = \Theta(n) + 2T(\tfrac{n}{2}) + \Theta(1)$$ $\Rightarrow \Theta(n)+\Theta(1)=\Theta(n)$

$$= \Theta(n) + 2T(\tfrac{n}{2})$$ $\Rightarrow$ No change

$$\boxed{\therefore T_i(n) = \Theta(n\log_2 n)}$$

ii) Pass by copying: (N is size of initial input size)

$$T_{ii}(n) = \Theta(n) + 2T(\tfrac{n}{2}) + \Theta(N)$$ $\Rightarrow \Theta(n)+\Theta(N)=\Theta(N)$

$$= cN + 2T(\tfrac{n}{2})$$ b/c $N \geq n$

M.M.: $a=2, b=2, \log_b a = 1, f(n) = cN$

Case 1: $cN \notin O(n^{1-\epsilon})$

Case 2: $cN \in \Theta(n)$ ✓

$$\boxed{\therefore T_{ii}(n) = \Theta(n\log_2 n)}$$

iii) Pass by selective copying:

$$T_{iii}(n) = \Theta(n) + 2T(\tfrac{n}{2}) + \Theta(\tfrac{n}{2})$$ $\Rightarrow \Theta(n)+\Theta(\tfrac{n}{2})=\Theta(n)$

$$= \Theta(n) + 2T(\tfrac{n}{2})$$ b/c $n \geq \frac{n}{2}$

$$\boxed{\therefore T_{iii}(n) = \Theta(n\log_2 n)}$$ Thus, no change