

第六节 空间几何.

空间曲线.

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}$$

$$\text{或} \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

点法式:

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

方向向量 $\vec{s} = (l, m, n)$ ✓

空间直线.

$$\begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt \end{cases}$$

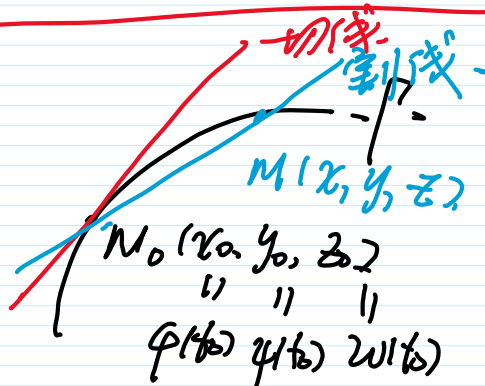
空间曲面: $F(x, y, z) = 0$ 空间平面: $Ax + By + Cz + D = 0$ 法向量: $\vec{n} = (A, B, C)$ ✓或 $z = f(x, y)$ 点法式

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

一. 空间曲线的切线与法平面

空间曲线.

$$\Gamma: \begin{cases} x = \varphi(t) \\ y = \psi(t) \\ z = \omega(t) \end{cases}$$

割线方向向量: $\overrightarrow{M_0M} = (x-x_0, y-y_0, z-z_0) \rightarrow \vec{O}$
 $(M \rightarrow M_0)$

$$\frac{1}{t-t_0} \overrightarrow{M_0M} = \left(\frac{x-x_0}{t-t_0}, \frac{y-y_0}{t-t_0}, \frac{z-z_0}{t-t_0} \right) \rightarrow \vec{T}$$

$(t \rightarrow t_0)$
 $(M \rightarrow M_0)$

定义: 切向量 \vec{T} : 即切线的方向向量

$$\vec{T} = \lim_{t \rightarrow t_0} \frac{1}{t - t_0} \overrightarrow{MM'} = (\dot{\varphi}(t_0), \dot{\psi}(t_0), \dot{\omega}(t_0)).$$

切线方程: $\frac{x - x_0}{\dot{\varphi}(t_0)} = \frac{y - y_0}{\dot{\psi}(t_0)} = \frac{z - z_0}{\dot{\omega}(t_0)}$

法平面方程: $\dot{\varphi}(t_0)(x - x_0) + \dot{\psi}(t_0)(y - y_0) + \dot{\omega}(t_0)(z - z_0) = 0$

参数形式

一般形式: $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \rightarrow \begin{cases} x = x \\ y = y(x) \\ z = z(x) \end{cases}$

在 $M_0(x_0, y_0, z_0)$ 的切线方程和法平面方程 $\vec{T} = (1, y'(x), z'(x))$

$$J = \frac{\partial(F, G)}{\partial(y, z)} = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}$$

$$y'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, z)} = -\frac{1}{J} \begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}$$

$$z'(x) = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, x)} = -\frac{1}{J} \begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}$$

$$\vec{T} = (1, -\frac{1}{J} \begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}, -\frac{1}{J} \begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}).$$

$$\vec{T} = J \vec{T} = \left(\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}, -\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \right)$$

$$\begin{vmatrix} \vec{v} & \vec{T} & \vec{k} \end{vmatrix} = 0$$

$$\vec{T} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} = \vec{\alpha} \times \vec{\beta}$$

$$\vec{\alpha} = (F_x, F_y, F_z)$$

$$\vec{\beta} = (G_x, G_y, G_z)$$

例. $\begin{cases} y = \sqrt{2}x \\ z = 1-x \end{cases}$ 在 $(2, 2, -1)$ 处的切线方程

解: $\vec{T} = \left(1, \sqrt{2} \cdot \frac{1}{2\sqrt{2}}, -1 \right) \Big|_{(2, 2, -1)}$

$$= \left(1, \frac{1}{2}, -1 \right) \quad \vec{T} = \underline{(2, 1, -2)}$$

切线方程: $\frac{x-2}{1} = \frac{y-2}{\frac{1}{2}} = \frac{z+1}{-1}$

或: $\frac{x-2}{2} = \frac{y-2}{1} = \frac{z+1}{-2}$

例. $\begin{cases} x^2 + y^2 + z^2 = 6 \\ x + y + z = 0 \end{cases}$ 在 $(1, 1, -2)$ 处的法平面方程.

解: $F(x, y, z) = x^2 + y^2 + z^2 - 6$

$$G(x, y, z) = x + y + z$$

$$F_x = 2x, \quad F_y = 2y, \quad F_z = 2z$$

$$\vec{\alpha} = (2x, 2y, 2z) \Big|_{(1, 1, -2)} = (2, 2, -4) = 2 \underline{(1, 1, -2)}$$

$$a_x=1, \quad a_y=1, \quad a_z=1$$

$$\vec{r} = (1, 1, 1) \mid (1, 1, -2) = \underline{(1, 1, 1)}$$

$$\vec{T} = \vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right)$$

$$= (3, -3, 0) = 3(1, -1, 0)$$

法平面方程: $1 \cdot (x-1) + (-1) \cdot (y-1) + 0 \cdot (z+2) = 0$

即: $x - y = 0$

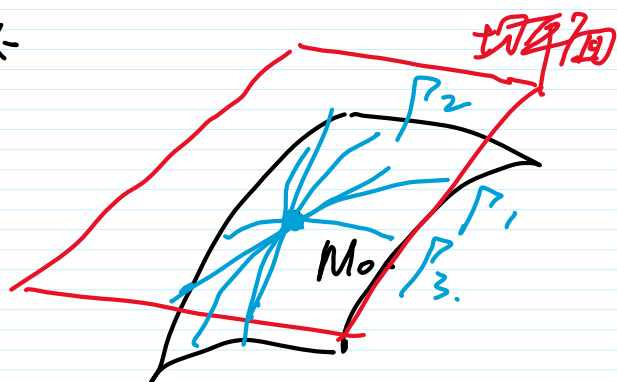
切线方程: $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z+2}{0}$

或者 $\begin{cases} x+y=2 \\ z+2=0 \end{cases}$

二. 空间曲面的切平面与法线

空间曲面: $F(x, y, z) = 0$

$M_0(x_0, y_0, z_0)$

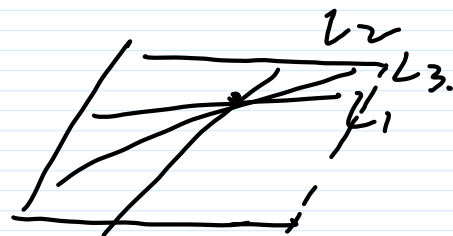


定义: 切平面是由切线共面构成的

即: 各点线相交, 去证共面.

L_1, L_2, L_3 三条直线. 相交. 去证其面.

$$L \perp L_1 \quad L \perp L_2$$



$$\Rightarrow L \perp L_1, L_2 \text{ 所张成的平面} \Rightarrow L_1, L_2, L_3 \text{ 共面} \\ L \perp L_3$$

证切线共面 \Rightarrow 证 $L \perp$ 所有切线.

$$\Leftrightarrow \vec{r} \cdot \vec{T}_i = 0$$

空间曲面内过 M_0 的任一条曲线 $\Gamma_i: \begin{cases} x = \varphi_i(t) \\ y = \psi_i(t) \\ z = w_i(t) \end{cases}$

$$x_0 = \varphi_i(t_0)$$

$$y_0 = \psi_i(t_0)$$

$$z_0 = w_i(t_0)$$

对 $\Gamma \subset (\varphi_i(t), \psi_i(t), w_i(t)) = 0$ 关于 t 求导

$$F_1' \cdot \varphi_i'(t) + F_2' \cdot \psi_i'(t) + F_3' \cdot w_i'(t) = 0$$

$$\underbrace{(F_1', F_2', F_3')}_{\vec{n}} \cdot \underbrace{(\varphi_i'(t), \psi_i'(t), w_i'(t))}_{\vec{T}_i(t)} = 0$$

定义: 法向量 $\vec{n} = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$

切平面方程: $F_x(p_0)(x-x_0) + F_y(p_0)(y-y_0) + F_z(p_0)(z-z_0) = 0$
 法线方程: $\frac{x-x_0}{F_x(p_0)} = \frac{y-y_0}{F_y(p_0)} = \frac{z-z_0}{F_z(p_0)}$

$z = f(x, y), \quad \vec{n} = (f_x, f_y, -1)|_{p_0}$
 \downarrow 或 $\vec{n} = (-f_x, -f_y, 1)|_{p_0}$
 $f(x, y) - z = 0$
 或 $z - f(x, y) = 0$

例 $z = 4 - x^2 - y^2$. 切平面平行于 $2x + 2y + z - 1 = 0$.

解: $4 - x^2 - y^2 - z = 0$ 或 $x^2 + y^2 + z - 4 = 0$
 $\pi_1 \parallel \pi_2 \Leftrightarrow \vec{n}_1 \parallel \vec{n}_2$ $2x + 2(1-y) + z - 1 = 0$
 $\vec{n}_2 = (2, -2, 1)$

设切点 (x_0, y_0, z_0)

法向量 $\vec{n}_1 = (-2x, -2y, -1)|_{(x_0, y_0, z_0)}$
 $= (-2x_0, -2y_0, -1)$

法向量 $\vec{n}_2 = (2, 2, 1)$

由 $\vec{n}_1 \parallel \vec{n}_2 \Rightarrow \frac{-2x_0}{2} = \frac{-2y_0}{2} = \frac{-1}{1} \Rightarrow x_0 = 1, y_0 = 1$

$z_0 = 4 - x_0^2 - y_0^2 = 2$

切平面方程: $2(x-1) + 2(y-1) + 1 \cdot (z-2) = 0$

$z = f(x, y)$ 的切平面方程 $(x_0, y_0, f(x_0, y_0))$

$$\vec{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$$

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1) \cdot (z - f(x_0, y_0)) = 0$$

$$z = f(x_0, y_0) + \underbrace{f_x(x_0, y_0) \overset{dx}{(x - x_0)} + f_y(x_0, y_0) \overset{dy}{(y - y_0)}}_{df(x, y)|_{(x_0, y_0)}}$$

例

直线:
$$\begin{cases} x + y + b = 0 \\ x + ay - z - 3 = 0 \end{cases} \text{ 在平面 } \pi \text{ 上}$$

且 π 与 $z = x^2 + y^2$ 相切于 $(1, -2, 5)$. 则 $a = \underline{\quad}$, $b = \underline{\quad}$.

解:
$$\vec{n} = (2x, 2y, -1)|_{(1, -2, 5)} = (2, -4, -1).$$

平面 π : $2x(x-1) + (-4)(y+2) + (-1) \cdot (z-5) = 0$

$$2x - 4y - z - 5 = 0.$$

⑤法:

$$\begin{cases} x + y + b = 0 \\ x + ay - z - 3 = 0 \\ 2x - 4y - z - 5 = 0 \end{cases}$$

$$r(A) = r(A; \eta) < 3.$$

$$r(A) = r(A; \eta) = 2.$$

$$(A; \eta) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & -b \\ 1 & a & -1 & 3 \\ 2 & -4 & -1 & 5 \end{array} \right)$$

$$a = -5, b = -2.$$

②法: 平面方程.

设 π : $\lambda(x+y+b) + \mu(x+ay-z-3) = 0$

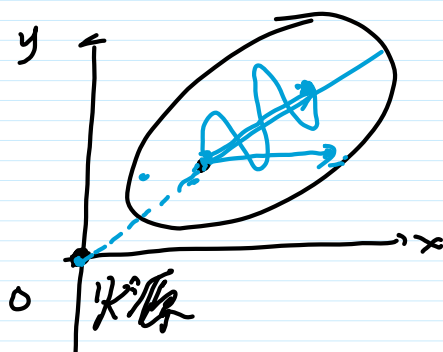
π : $2x - 4y - z - 5 = 0$

第0节 方向导数与梯度
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 实数 向量

偏导数: 偏增量与自变量增量之关系

方向导数: 全增量与自变量增量之关系

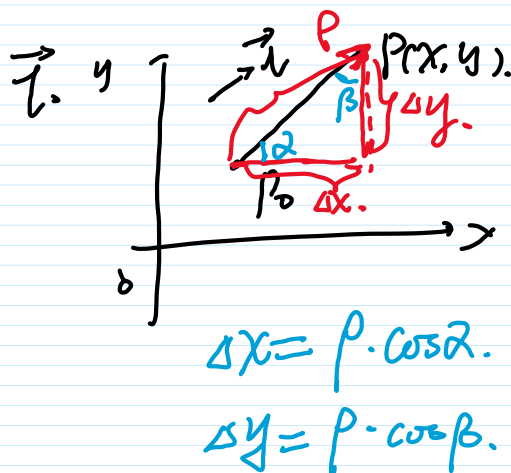
引例



定义: $z = f(x, y)$ 在 $U(P_0)$ 有定义.

沿 \vec{r} 运动到 $P(x, y)$.

全增量 $\Delta z = f(x, y) - f(x_0, y_0)$



距离 $p = |\vec{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

若 $\lim_{p \rightarrow 0^+} \frac{f(x, y) - f(x_0, y_0)}{p}$ 存在. 则称为

$z = f(x, y)$ 在 $P_0(x_0, y_0)$ 沿 \vec{r} 的方向导数记为 $\frac{\partial f}{\partial r} \Big|_{(x_0, y_0)}$

$z = f(x, y)$ 在 $P_0(x_0, y_0)$ 沿 \vec{l} 的方向导数记为 $\frac{\partial f}{\partial \vec{l}}|_{(x_0, y_0)}$

方向导数: $\frac{\partial f}{\partial \vec{l}}|_{(x_0, y_0)} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \sin \alpha) - f(x_0, y_0)}{\rho}$

单侧

$(\cos \alpha, \sin \alpha) = \frac{1}{|\vec{l}|} \vec{l}$

例: $\vec{l} = (2, -3)$. $\frac{1}{|\vec{l}|} \vec{l} = \frac{1}{\sqrt{2^2 + (-3)^2}} (2, -3)$

$= (\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$

$\cos \alpha = \frac{2}{\sqrt{13}} \quad \sin \alpha = -\frac{3}{\sqrt{13}}$

方向导数与偏导数之关系.

(1) 沿 x 轴正半轴的方向导数, $\vec{l} = (1, 0)$.

$$\begin{aligned} \frac{\partial f}{\partial x^+}|_{(x_0, y_0)} &= \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cdot 1, y_0 + \rho \cdot 0) - f(x_0, y_0)}{\rho} \\ &= \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho, y_0) - f(x_0, y_0)}{\rho} \\ &\stackrel{\rho = \Delta x}{=} \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \end{aligned}$$

$f_x(x_0, y_0)$ 存在 $\Rightarrow \frac{\partial f}{\partial x^+}|_{(x_0, y_0)} = f_x(x_0, y_0)$.

(2) 沿 x 轴负半轴, $-\vec{l} = (-1, 0)$

$\lim_{\rho \rightarrow 0^+} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{\rho}$

(2) 10 10 10 10 10 10 10 10 10 10

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 - \rho, y_0) - f(x_0, y_0)}{\rho}$$

$$\stackrel{\Delta x = -\rho}{=} \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{-\Delta x}$$

$$f_x(x_0, y_0) \text{ 存在} \Rightarrow \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = -f_x(x_0, y_0).$$

$$(3) \quad f_y(x_0, y_0) \text{ 存在} \Rightarrow \begin{cases} \frac{\partial f}{\partial y^+} \Big|_{(x_0, y_0)} = f_y(x_0, y_0) \\ \frac{\partial f}{\partial y^-} \Big|_{(x_0, y_0)} = -f_y(x_0, y_0). \end{cases}$$

推广: $u = f(x, y, z), \quad \vec{r}.$

$$\frac{\partial u}{\partial r} \Big|_{(x_0, y_0, z_0)} = \lim_{\rho \rightarrow 0^+} \frac{u(x_0 + \rho \cos \alpha, y_0 + \rho \cos \beta, z_0 + \rho \cos \gamma) - u(x_0, y_0, z_0)}{\rho}.$$

$$\text{其中 } (\cos \alpha, \cos \beta, \cos \gamma) = \frac{1}{|\vec{r}|} \vec{r}.$$

例: $z = \sqrt{x^2 + y^2}$. 在 $(0, 0)$ 连续. 偏导不存在. 不可微.

讨论 $(0, 0)$ 沿各个方向的导数是否存在.

$$\text{解: } \frac{\partial f}{\partial r} \Big|_{(0,0)} = \lim_{\rho \rightarrow 0^+} \frac{f(0 + \rho \cos \alpha, 0 + \rho \cos \beta) - f(0,0)}{\rho}$$

$$= \lim_{\rho \rightarrow 0^+} \frac{\sqrt{\rho^2 (\cos^2 \alpha + \cos^2 \beta)} - 0}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{\rho}{\rho} = 1.$$

例 1

$$\frac{\partial z}{\partial r} = \frac{x \cos \alpha + y \cos \beta}{\sqrt{x^2 + y^2}}$$

例. $f(x, y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2 + y^2}} \sin \frac{1}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

讨论在 $(0, 0)$ 处连续性、偏导数存在性、可微性及方向导数。

解: $f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0$
 $f_y(0, 0) = 0.$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{[f(\Delta x, \Delta y) - f(0, 0)] - [f_x(0, 0) \cdot \Delta x + f_y(0, 0) \cdot \Delta y]}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{(\Delta x)^2 \cdot \Delta y}{(\Delta x)^2 + (\Delta y)^2} \cdot \sin \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

夹逼、极坐标变换

$$\lim_{\rho \rightarrow 0^+} \frac{f(0 + \rho \cos \alpha, 0 + \rho \sin \alpha) - f(0, 0)}{\rho}$$

$$= \lim_{\rho \rightarrow 0^+} \frac{\rho^2 \cos^2 \alpha \cdot \rho \cos \beta}{\rho} \cdot \sin \frac{1}{\rho^2}$$

$$= \lim_{\rho \rightarrow 0^+} \rho \cos^2 \alpha \cos \beta \sin \frac{1}{\rho^2}$$

$$= 0$$

定理: 可微 \Rightarrow 方向导数存在且

$$\left. \frac{\partial f}{\partial \vec{r}} \right|_{(x_0, y_0)} = f_x(x_0, y_0) \cdot \cos \alpha + f_y(x_0, y_0) \cdot \cos \beta$$

$$\text{其中 } (\cos \alpha, \cos \beta) = \frac{1}{\sqrt{2}} \vec{e}$$

证明: $\left. \frac{\partial f}{\partial \vec{r}} \right|_{(x_0, y_0)} = \lim_{\rho \rightarrow 0^+} \frac{\Delta z}{\rho}$

直接 $\lim_{\rho \rightarrow 0^+} \frac{f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \cdot \Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})}{\rho}$

$\Delta x = \rho \cos \alpha$
 $\Delta y = \rho \cos \beta$
 $\lim_{\rho \rightarrow 0^+} \left[\frac{f_x(x_0, y_0) \cdot \rho \cos \alpha + f_y(x_0, y_0) \cdot \rho \cos \beta}{\rho} + \frac{o(\rho)}{\rho} \right]$

$$= f_x(x_0, y_0) \cos \alpha + f_y(x_0, y_0) \cos \beta$$

推广: 设 $f(x, y, z)$ 为三元函数

$$\left. \frac{\partial u}{\partial \vec{r}} \right|_{(x_0, y_0, z_0)} = f_x(x_0, y_0, z_0) \cos \alpha + f_y(x_0, y_0, z_0) \cos \beta + f_z(x_0, y_0, z_0) \cos \gamma$$