$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y). \end{cases}$$

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

公式法

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{J}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{J}\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{1}{J}\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{1}{J}\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}$$

$F(x,y)=0$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

$F(x,y,z)=0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

直接法：

对 $\begin{cases} F(x,y,u(x,y),v(x,y))=0 \\ G(x,y,u(x,y),v(x,y))=0 \end{cases}$ 两边对 x 求偏导

$$\begin{cases} F_1'\cdot 1+F_2'\cdot 0+F_3'\cdot \overset{X_1}{\boxed{\frac{\partial u}{\partial x}}}+F_4'\cdot \overset{X_2}{\boxed{\frac{\partial v}{\partial x}}}=0 \\ G_1'\cdot 1+G_2'\cdot 0+G_3'\cdot \overset{X_1}{\boxed{\frac{\partial u}{\partial x}}}+G_4'\cdot \overset{X_2}{\boxed{\frac{\partial v}{\partial x}}}=0. \end{cases}$$

$$\begin{cases} F_3'\cdot X_1+F_4'\cdot X_2=-F_1' \\ G_3'\cdot X_1+G_4'\cdot X_2=-G_1' \end{cases}$$

由克拉默法则.

$$X_1=\frac{D_1}{D}=\frac{\begin{vmatrix} -F_1' & F_4' \\ -G_1' & G_4' \end{vmatrix}}{\begin{vmatrix} F_3' & F_4' \\ G_3' & G_4' \end{vmatrix}}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}.$$

微分法：
$$\begin{cases} dF(x,y,u,v)=0 \\ dG(x,y,u,v)=0 \end{cases}$$

$$\begin{cases} F_1'\cdot dx+F_2' dy+F_3' \overset{X_1}{\boxed{du}}+F_4'\overset{X_2}{\boxed{dv}}=0 \\ G_1' dx+G_2' dy+G_3' \boxed{du}+G_4'\boxed{dv}=0 \end{cases} \Rightarrow \begin{aligned} du&=(\quad)dx+(\quad)dy \\ dv&=(\quad)dx+(\quad)dy \end{aligned}$$

例.
$$\begin{cases} F(x,u,v)=0 \\ G(x,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x) \\ v=v(x) \end{cases} \text{ 则 } \frac{du}{dx},\ \frac{dv}{dx}$$

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$\frac{du}{dx}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} \qquad \frac{dv}{dx}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

例二
$$\begin{cases} F(x,y,u,v,w)=0 \\ G(x,y,u,v,w)=0 \\ H(x,y,u,v,w)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \\ w=w(x,y) \end{cases} \text{ 求 } \frac{\partial w}{\partial y}.$$

$$J=\frac{\partial(F,G,H)}{\partial(u,v,w)}=\begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

$$\frac{\partial w}{\partial y}=-\frac{1}{J}\frac{\partial(F,G,H)}{\partial(u,v,y)}=-\frac{1}{J}\begin{vmatrix} \vdots & \vdots & F_y \\ \vdots & \vdots & G_y \\ \vdots & \vdots & H_y \end{vmatrix}$$

例
$$\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=1. \end{cases} \longrightarrow \begin{cases} x=x(z) \\ y=y(z). \end{cases} \text{ 则 } \frac{dx}{dz},\ \frac{dy}{dz}.$$

解：①法：公式法
$F(x,y,z)=x+y+z$

解：①

$$F(x,y,z)=x+y+z$$

$$G(x,y,z)=x^2+y^2+z^2-1$$

$$J=\frac{\partial(F,G)}{\partial(x,y)}=\begin{vmatrix}F_x & F_y\\ G_x & G_y\end{vmatrix}=\begin{vmatrix}1 & 1\\ 2x & 2y\end{vmatrix}=2(y-x)$$

$$\frac{dx}{dz}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(z,y)}=-\frac{1}{2(y-x)}\begin{vmatrix}F_z & F_y\\ G_z & G_y\end{vmatrix}$$

$$=-\frac{1}{2(y-x)}\cdot\begin{vmatrix}1 & 1\\ 2z & 2y\end{vmatrix}=-\frac{y-z}{y-x}$$

$$\frac{dy}{dz}=-\frac{1}{2(y-x)}\frac{\partial(F,G)}{\partial(x,z)}=-\frac{1}{2(y-x)}\begin{vmatrix}1 & 1\\ 2x & 2z\end{vmatrix}$$

$$=-\frac{z-x}{y-x}$$ 含有 x,y,z

②法：由 $x+y+z=0$ 可得 $x=-y-z$ 代入

$(-y-z)^2+y^2+z^2-1=0$ $F(y,z)=0$ 确定 $y=y(z)$

则 $\dfrac{dy}{dz}=-\dfrac{F_z(y,z)}{F_y(y,z)}$ (含有 y,z)

例. $u=f(x,y,xyz)$ $z=z(x,y)$ 由

$$\int_{xy}^{z}g(xy+z-t)\,dt=e^{xyz}$$ 确定的.

特值法：$\int_{xy}^{z}(xy+z-t)\,dt=e^{xyz}$

则 $x\dfrac{\partial u}{\partial x}-y\dfrac{\partial u}{\partial y}=$ __ $xf_1'-yf_2'$ __.

解

解: $\frac{\partial u}{\partial x} = f_1'\cdot 1 + f_2'\cdot 0 + f_3'\cdot y(1\cdot z + x\cdot\frac{\partial z}{\partial x})$

$\frac{\partial u}{\partial y} = f_1'\cdot 0 + f_2'\cdot 1 + f_3'\cdot x(1\cdot z + y\frac{\partial z}{\partial y})$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xf_1' - yf_2' + xyf_3'\big(\underbrace{x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}}_{=0}\big)$$

$F(x,y,z) = \int_{xy}^{z} g(xy+z-t)\,dt - e^{xyz} = 0$ 则 $\frac{\partial z}{\partial x} = \cdots$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y}$ $\qquad \frac{\partial \int_{xy}^{z} g(xy+z-t)\,dt}{\partial z}$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f[b(x)]\cdot b'(x) - f[a(x)]\,a'(x)$$

$c(x)f(t)$

变量 变量

$\int_{xy}^{z} g(xy+z-t)\,dt \xrightarrow{s = xy+z-t} \int_{z}^{xy} g(s)\,d(xy+z-s)$

变量

$= -\int_{z}^{xy} g(s)\,ds = \int_{xy}^{z} g(s)\,ds$

$F(x,y,z) = \int_{xy}^{z} g(s)\,ds - e^{xyz} = 0$

$x\ F_x = -g(xy)\cdot y x - e^{xyz}\cdot yz\,x$

$y\ F_y = -g(xy)\cdot x y - e^{xyz}\cdot xz\,y$

$\therefore$

F_z

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=-\frac{xF_x-yF_y}{F_z}$$