2024年05月16日 星期四10-11

2024年5月16日 9:03

1. $\iiint\limits_{\Omega} f(x,y,z)\,dv \xlongequal{\text{投影法}} \iint\limits_{D_{xy}} dxdy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz$

$\xlongequal{\text{截面法}} \int_{z_1}^{z_2} dz \iint\limits_{D_z} f(x,y,z)\,dxdy$

$\iiint\limits_{\Omega} g(z)\,dv \xlongequal{\text{用}z\text{=常数.}} \int_{z_1}^{z_2} g(z)\cdot \sigma(z)\,dz$

常用截面法的区域：

(1) $z=\sqrt{x^2+y^2} \longrightarrow x^2+y^2=z^2$

截面面积 $\sigma(z)=\pi z^2$. 截面区域: D_z: $x^2+y^2\le z^2$.

$z=2-\sqrt{x^2+y^2}$.

$\sigma(z)=\pi(2-z)^2$ D_z: $x^2+y^2\le(2-z)^2$

(2) $z=x^2+y^2$.

$\sigma(z)=\pi z$ D_z: $x^2+y^2\le z$

$z=1-x^2-y^2$

$\sigma(z)=\pi(1-z)$. D_z: $x^2+y^2\le 1-z$.

(3) $z=\sqrt{1-x^2-y^2}$.

$\sigma(z)=\pi(1-z^2)$ D_z: $x^2+y^2\le 1-z^2$.

例. $x^2+y^2=az$ 与 $z=2a-\sqrt{x^2+y^2}$ 所围成的体积.

$(0,0,2a)$.

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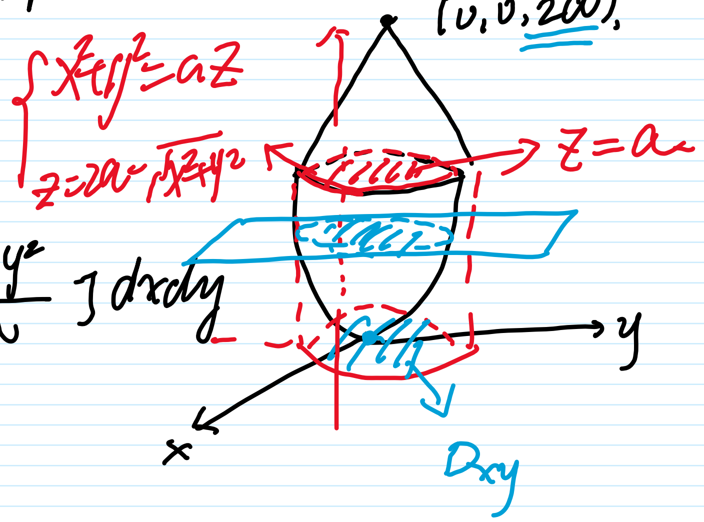


解: ①法:

$$V=V_{上曲}-V_{下曲}$$

$$=\iint_D\left[(2a-\sqrt{x^2+y^2})-\frac{x^2+y^2}{a}\right]dxdy$$

$D:\ x^2+y^2\le a^2$.

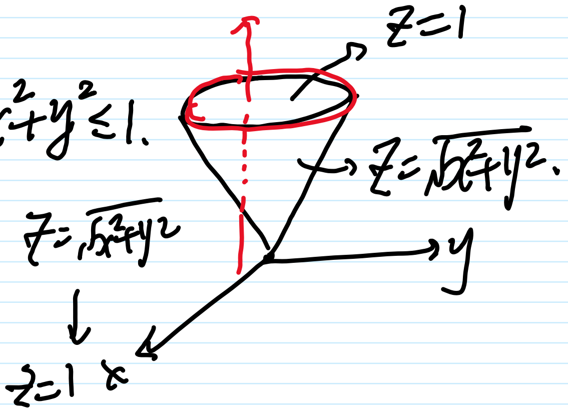
②法: $V=\iiint_\Omega 1\,dv$.

$$=\iint_{D_{xy}}dxdy\int_{\frac{x^2+y^2}{a}}^{2a-\sqrt{x^2+y^2}}1\,dz$$

③法: $V=\iiint_\Omega 1\,dv=\int_0^{2a}1\cdot\sigma(z)\,dz$

$$=\int_0^a \pi az\,dz+\int_a^{2a}\pi(2a-z)^2dz=\cdots$$

例: $I=\iiint_\Omega(x^2+y^2)\,dv$. Ω: $z=\sqrt{x^2+y^2}$ 与 $z=1$ 所围成的



解: ①法: 投影法

① Ω 在 xoy 面上投影 D_{xy}: $x^2+y^2\le1$.

② 平行于z轴射线穿过Ω. $z=\sqrt{x^2+y^2}$ $\downarrow$ $z=1$

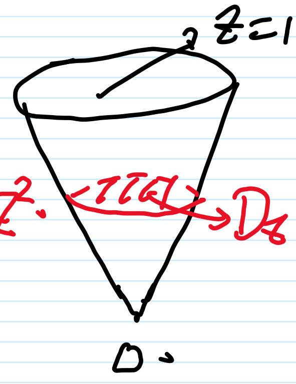
$$I=\iint_{D_{xy}}dxdy\int_{\sqrt{x^2+y^2}}^{1}(x^2+y^2)\,dz$$

$\iint \cdots\ (\ \sqrt{x^2+y^2}\ )\,dxdy$ 极坐标

$$= \iint_{D_{xy}} (x^2+y^2)\cdot(1-\sqrt{x^2+y^2})\,dxdy$$ 投影法 ……

②法：截面法.

用平行于xoy面去截. D_z: $x^2+y^2\le z^2$.



$$I=\int_0^1 dz\iint_{D_z}(x^2+y^2)\,dxdy$$

$$=\int_0^1\left[\int_0^{2\pi}d\theta\int_0^z \rho^2\,\rho\,d\rho\right]dz = \cdots$$

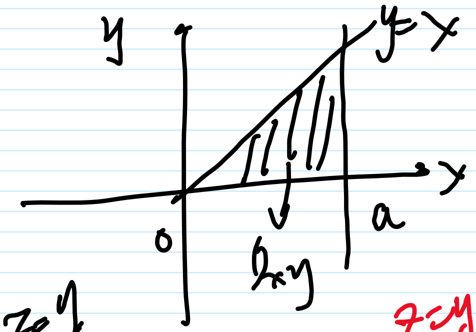
例：$I=\int_0^a dx\int_0^x dy\int_0^y f(z)\,dz$ 化为定积分

解：①法：

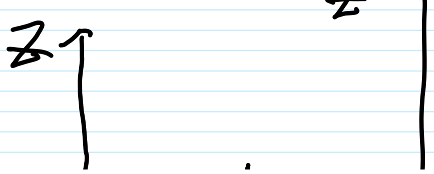
Ω: $\underbrace{0\le x\le a,\ 0\le y\le x}_{D_{xy}}$, $0\le z\le y$

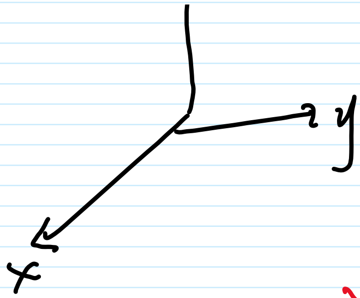
$$I=\iiint_\Omega f(z)\,dv \xlongequal{截面法} \int_0^a f(z)\,\sigma(z)\,dz,\quad \sigma(z)=\frac{1}{2}(a-z)^2.$$

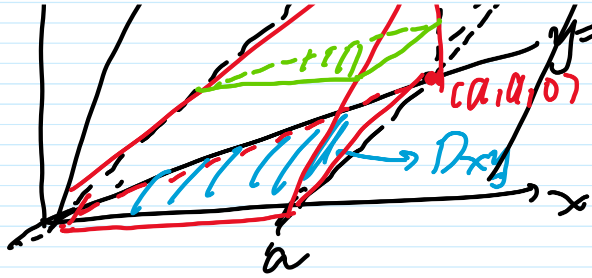
投影法 $\iint_{D_{xy}}dxdy\int_0^y f(z)\,dz$.



$$=\int_0^a dx\int_0^y dy\int_0^y f(z)\,dz$$





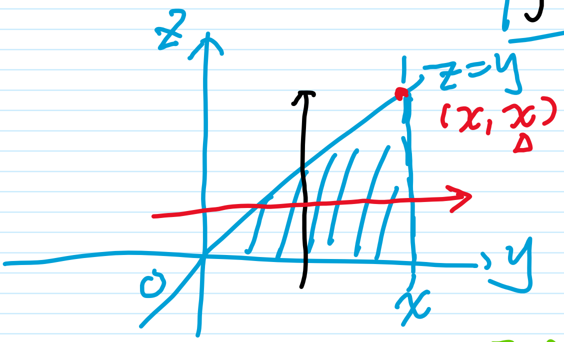


③法：交换积分次序

$$I=\int_0^a dx\boxed{\int_0^x dy\int_0^y f(z)\,dz}=\int dz\int dx\int f(z)\,dy$$

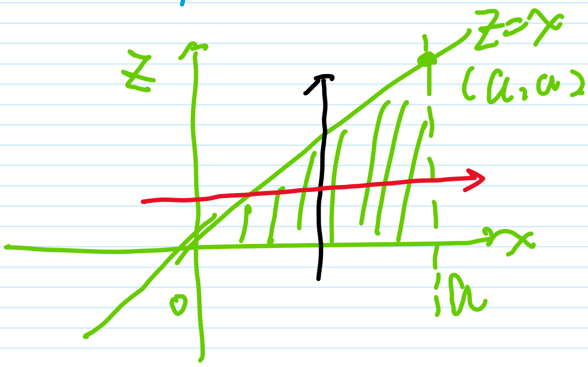
$(\boxed{z\to y}\to x)$, $\qquad (y\to \boxed{x\to z})$

$$(\boxed{y\to z}\to x)$$



$$\boxed{\int_0^a dx\int_0^x dz}\int_z^x f(z)\,dy$$

$$=\int_0^a dz\int_z^a dx\int_z^x f(z)\,dy$$



$$=\int_0^a dz\int_z^a f(z)(x-z)\,dx$$

$$=\int_0^a f(z)\,\frac{(a-z)^2-(z-z)^2}{2}\,dz$$

$$=\int_0^a f(z)\,\frac{(a-z)^2}{2}\,dz$$

2. 柱面坐标系下的计算 ⟶ 投影法 + 极坐标

$$\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta.\end{cases}\qquad dv=\rho\,d\rho\,d\theta\,dz.$$

$$\begin{cases} y = \rho\sin\theta \\ z = z \end{cases}$$

$dv = \rho\, d\rho\, d\theta\, dz$

$\rho =$ 常数 圆柱面

$\theta =$ 常数 半平面

$z =$ 常数 平面

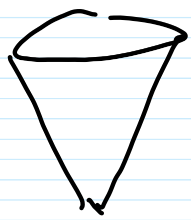
$$\iiint_\Omega f(x,y,z)dv = \iiint_\Omega f(\rho\cos\theta, \rho\sin\theta, z)\,\rho\, d\rho\, d\theta\, dz$$

$$= \int_\alpha^\beta d\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \rho\, d\rho \int_{z_1(\rho,\theta)}^{z_2(\rho,\theta)} f(\rho\cos\theta, \rho\sin\theta, z)\, dz$$

例. $I = \iiint_\Omega (x^2+y^2)dv$ $\Omega: z=\sqrt{x^2+y^2}$ 与 $z=1$ 所围成区域

解. 此法

投影法: $D_{xy}: x^2+y^2 \le 1$



射线: $z=\sqrt{x^2+y^2}$

$\downarrow$

$z=1$

$$I = \iint_{D_{xy}} dxdy \int_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)\,dz$$

$$= \iint_{D_{xy}} (x^2+y^2)(1-\sqrt{x^2+y^2})\,dxdy$$

极坐标 $= \int_0^{2\pi} d\theta \int_0^1 \rho^2 (1-\rho)\ \rho d\rho$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^2 (1-\rho)$$

③法:

D_{xy}: $x^2+y^2 \le 1$ $\quad \begin{cases} \theta: 0 \to 2\pi \\ \rho: 0 \to 1 \end{cases}$

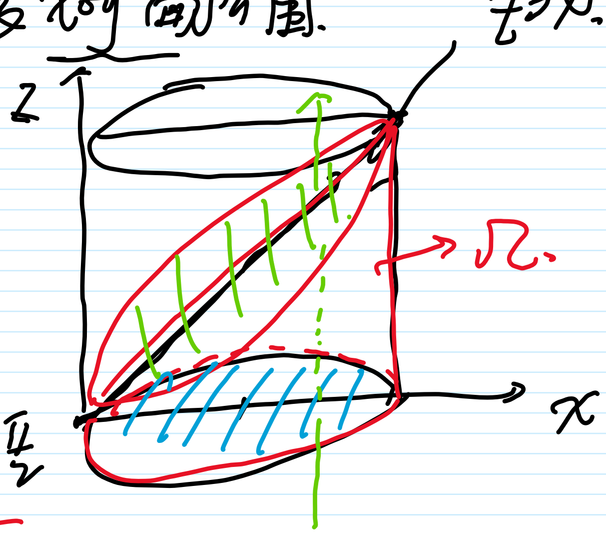
射线: $z=\sqrt{x^2+y^2} \downarrow z=1 \xrightarrow{\text{柱.}} z=\rho \downarrow z=1$

$$I = \int_0^{2\pi} d\theta \int_0^1 \rho\, d\rho \int_\rho^1 \rho^2\, dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho \; \rho^2 (1-\rho)\, d\rho$$

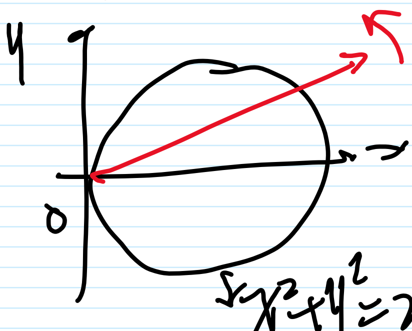
例. $I = \iiint\limits_\Omega z\sqrt{x^2+y^2}\, dv$.

Ω: $x^2+y^2=2x$. $z=x$ 及 xoy 面所围.



解: Ω 投影到 xoy 面上.

D_{xy}: $x^2+y^2 \le 2x$



$\theta: -\frac{\pi}{2} \to \frac{\pi}{2}$

$\rho: 0 \to 2\cos\theta$.

$x^2+y^2=2x \xrightarrow{\text{极}} \rho^2 = 2\rho\cos\theta$

射线: $z=0 \downarrow z=x$ $\xrightarrow{\text{柱}}$ $z=0 \downarrow z=\rho\cos\theta.$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho\, d\rho \int_0^{\rho\cos\theta} z\cdot\rho\, dz$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho\cdot\frac{\rho^2\cos^2\theta-0^2}{2}\,\rho\, d\rho$$

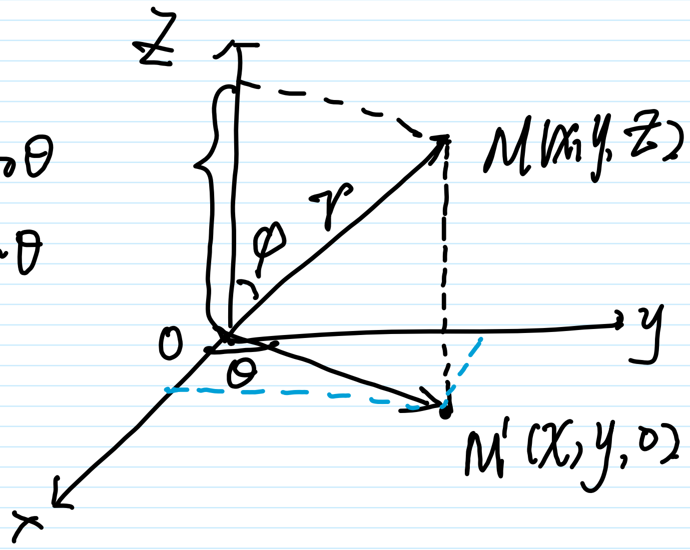
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{2}\cdot\frac{2^5\cdot\cos^5\theta}{5}\, d\theta$$

$$=\frac{32}{5}\int_0^{\frac{\pi}{2}}\cos^7\theta\, d\theta=\frac{32}{5}\cdot 1\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}$$

$$\int_0^{\frac{\pi}{2}}\cos^8\theta\, d\theta=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}$$

3. 球面坐标系下的计算.

$$\begin{cases} x=\boxed{r\sin\varphi}\cdot\cos\theta \\ y=\boxed{r\sin\varphi}\cdot\sin\theta \\ z=r\cos\varphi. \end{cases}$$



$$dv=r^2\sin\varphi\, d\theta\, d\varphi\, dz$$

$r=$常数 球面

$\varphi=$常数 圆锥面

$\varphi=$常数 圆锥面

$\theta=$常数 半平面

例：$I_1=\iiint\limits_{\Omega_1} f(x,y,z)\,dv$，$\Omega_1$：$x^2+y^2+z^2\leq R^2$.

解：$I_1=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\, d\varphi\int_0^{R}f(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)\,r^2dr$

Ω_2：$x^2+y^2+z^2\leq R^2$ 且 $z\geq 0$.

$I_2=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\, d\varphi\int_0^{R}f(\cdots\cdots)\,r^2dr$.

Ω_3：$x^2+y^2+z^2\leq R^2$ 且

$x\geq 0$、$y\geq 0$、$z\geq 0$

$I_3=\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\, d\varphi\int_0^{R}f(\cdots\cdots)\,r^2dr$.