

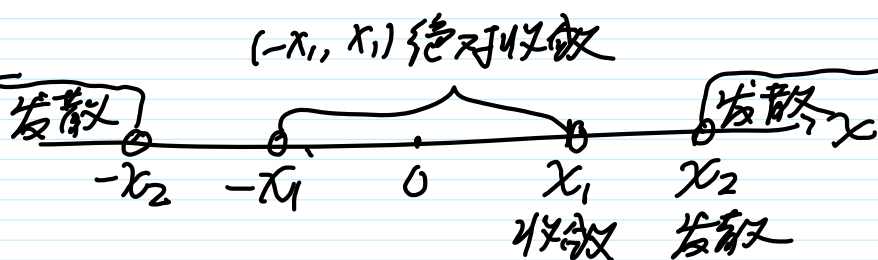
## 1. 幂级数收敛域.

$$\textcircled{1}. \rho(x) = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right|$$

由  $\rho(x) < 1$ . 得  $x \in (-R, R)$ .  $x = \pm R$  的敛散性另论

$$\textcircled{2} \sum_{n=0}^{\infty} a_n x^n \quad \rho = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} \quad \text{得 } R = \frac{1}{\rho}. \dots$$

## 2. Abel定理



$$x_1 \leq R \leq x_2.$$

条件收敛点  $\Rightarrow$  分界点.

例:  $\sum_{n=1}^{\infty} a_n x^{2n}$  与  $\sum_{n=1}^{\infty} a_n x^{2n+1}$  有相同收敛域.

↓  
设  $x_1 \in$  收敛域.

$\sum_{n=1}^{\infty} a_n x_1^{2n}$  与  $\sum_{n=1}^{\infty} a_n x_1^{2n+1} = x_1 \sum_{n=1}^{\infty} a_n x_1^{2n}$  收敛.

$$\rho_1(x) = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{2n+2}}{a_n x^{2n}} \right| \quad \rho_2(x) = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{2n+3}}{a_n x^{2n+1}} \right|$$

例:  $\sum_{n=1}^{\infty} (1 + \frac{1}{2} + \dots + \frac{1}{n}) (x-1)^n$  的收敛域  $(0, 2)$ .

解: 令  $t = x-1$ .  $\rho_t = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

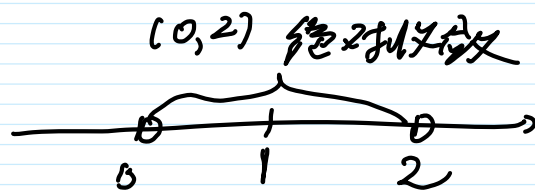
$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1}}{1 + \frac{1}{2} + \dots + \frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{1 + \frac{1}{2} + \dots + \frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \left( 1 + \frac{\frac{1}{n+1}}{1 + \frac{1}{2} + \dots + \frac{1}{n}} \right) = 1$$

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \dots + \frac{1}{n}) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

$$R_- = R_+ = \frac{1}{\rho} = 1$$



当  $x=0$  或  $x=2$  时

$$\sum_{n=1}^{\infty} \frac{(1 + \frac{1}{2} + \dots + \frac{1}{n})(\pm 1)^n}{u_n} \quad \text{发散}$$

$$\lim_{n \rightarrow \infty} |u_n| \neq 0 \Rightarrow \text{发散}$$

### 三. 幂级数运算及和函数

1. 运算 (四则运算).  $\sum_{n=0}^{\infty} a_n x^n$   $\sum_{n=0}^{\infty} b_n x^n$

① 加 (减).  $\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

(收敛域公共部分)

② 乘.  $(\sum_{n=0}^{\infty} a_n x^n) (\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n x^n$

$$c_0 = a_0 b_0, \quad c_1 = a_0 b_1 + a_1 b_0$$

$$\dots \quad c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

(收敛区间的公共部分)

③ 除.  $\frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \sum_{n=0}^{\infty} c_n x^n$

$$\frac{1+2x+x^2}{1+x} = 1+x. \quad \frac{1}{1+x} = \dots$$

$$a_0 = b_0 \cdot c_0. \quad a_1 = b_0 c_1 + b_1 c_0. \dots$$

例.  $\sum_{n=0}^{\infty} a_n x^n = 1 + 0 \cdot x + 0 \cdot x^2 + \dots + 0 \cdot x^n + \dots$   
 $[a_0=1, a_n=0 (n=1, \dots)]$  收敛域  $(-\infty, +\infty)$

$\sum_{n=0}^{\infty} b_n x^n = 1 - x + 0 \cdot x^2 + \dots + 0 \cdot x^n + \dots$   
 $[b_0=1, b_1=-1, b_n=0 (n \geq 2)]$  收敛域  $(-1, +\infty)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} c_n x^n = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n.$$

收敛域  $(-1, 1)$ .

$$1 = 1 \cdot c_0 \Rightarrow c_0 = 1. \quad 0 = 1 \cdot c_1 + (-1) \cdot c_0 \Rightarrow c_1 = 1. \dots$$

$$\frac{1}{1-x} \xrightarrow{\text{展开}} \sum_{n=0}^{\infty} x^n \quad x \in (-1, 1)$$

求和

2. 求和函数.  $S(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in \text{收敛域.}$

性质:

(1)  $S(x)$  在收敛域上连续 (单侧连续).

$$\lim_{x \rightarrow x_0} \sum_{n=0}^{\infty} a_n x^n = \lim_{x \rightarrow x_0} S(x) = S(x_0) = \sum_{n=0}^{\infty} a_n x_0^n = \sum_{n=0}^{\infty} \lim_{x \rightarrow x_0} a_n x^n.$$

极限与求和可交换.

(2)  $S(x)$  在收敛域上可积. 可逐项积分.

$$\int_0^1 (x+x^2) dx = \int_0^1 x dx + \int_0^1 x^2 dx.$$

“先积后导”  $\int_0^x s(t) dt = \int_0^x \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt.$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

积分与求和可交换

||  $a_n = n+1$  (例).

逐项求导  $\sum_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x}.$

$$\int_0^x s(t) dt = \frac{x}{1-x}$$

$$s(x) = \left( \frac{x}{1-x} \right)', \quad x \in \text{收敛域}.$$

③.  $s(x)$  在收敛区间上可导. 可逐项求导  $(x+x^2)'$   
 $= x' + (x^2)'$

“先导后积”  $s'(x) = \left( \sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} (a_n x^n)'$

$$= \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

||  $a_n = \frac{1}{n}$  (例)

求导与求和可交换

逐项求导  $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}, \quad x \in (-1, 1)$

$$s'(x) = \frac{1}{1-x} \quad \text{两边求导}.$$

$$\int_0^x s'(t) dt = \int_0^x \frac{1}{1-t} dt.$$

$$s(x) - s(0) = -\ln|1-x| = -\ln(1-x).$$

$$s(x) = s(0) - \ln(1-x) = a_0 - \ln(1-x)$$

例.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \ln(1+x), \quad x \in (-1, 1].$

解  $\therefore n=1 \rightarrow 1$  收敛域  $(-1, 1]$

解:  $\rho = 1, R = \frac{1}{\rho} = 1$ . 收敛域  $(-1, 1]$

"先导" ( $a_n$  中分母含  $n$ )

$$\text{设 } S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

在收敛区间  $(-1, 1)$  上两边求导.

$$S'(x) = \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \right)' = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}$$

$$= \frac{1}{1 - (-x)}$$

即  $\underline{S'(x) = \frac{1}{1+x}}$

"后积"

$$\int_0^x S'(t) dt = \int_0^x \frac{1}{1+t} dt.$$

$$S(x) - S(0) = \ln(1+x).$$

$$S(0) = 0$$

$$\underline{S(x) - S(\frac{1}{2}) = \ln(1+x) - \ln \frac{3}{2}}.$$

故  $S(x) = \ln(1+x), x \in (-1, 1).$

由性质 1,

$$S(1) = \lim_{x \rightarrow 1^-} S(x) = \lim_{x \rightarrow 1^-} \ln(1+x) = \ln 2.$$

综上,  $S(x) = \ln(1+x), x \in (-1, 1].$

$$\int S'(x) dx = \int \frac{1}{1+x} dx \Rightarrow S(x) = \ln(1+x) + C$$

代入  $x=0$ .  $S(0) = \ln(1+0) + C \Rightarrow C=0$

$$\Rightarrow S(x) = \ln(1+x).$$

证

由

...

...

$$\int \frac{\ln(1+x)}{x} dx, x \in (-1, 0) \cup (0, 1]$$

例.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n =$   $\left\{ \frac{\ln(1+x)}{x}, x \in (-1, 0) \cup (0, 1] \right.$   
 $\left. \begin{matrix} 1 \\ x=0 \end{matrix} \right\}$

解:  $S(x) = \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n \right)' = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n+1} x^{n-1}$   
 不是等比级数

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cdot x^{n+1} \cdot \left( \frac{1}{x} \right) \quad (x \neq 0)$$

$$S(x) = \frac{1}{x} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \frac{1}{x} S_1(x)$$

对  $S_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$  在  $(-1, 1)$  上求导.

"求导"

$$S_1'(x) = \sum_{n=0}^{\infty} (-1)^n \cdot x^n = \frac{1}{1+x}$$

"积分"

$$S_1(x) = \ln(1+x) + S_1(0) = \ln(1+x)$$

得  $S(x) = \frac{1}{x} S_1(x) = \frac{\ln(1+x)}{x}, x \in (-1, 0) \cup (0, 1]$

由连续性.

$$S(0) = \lim_{x \rightarrow 0} S(x) = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$S(1) = \lim_{x \rightarrow 1} \frac{\ln(1+x)}{x} = \ln 2.$$

综上:  $S(x) = \begin{cases} \frac{\ln(1+x)}{x}, & x \in (-1, 0) \cup (0, 1] \\ 1, & x=0 \end{cases}$

例.  $\sum_{n=1}^{\infty} n(n+1) x^n =$   $x \in (-1, 1)$

解: ① 求导.

"求导"  $1, n, n+1, n+2, \dots$

解:

①法.

"先积" (Ans中分子含n)

$$\int_0^x S(t) dt = \int_0^x \sum_{n=1}^{\infty} n(n+1)t^n dt.$$

$$= \sum_{n=1}^{\infty} \frac{n}{n+1} x^{n+1}$$

$$S(x) = x \frac{d}{dx} \sum_{n=1}^{\infty} (n+1)n x^{n-1} = x S_1(x)$$

$$S_2(x) = \int_0^x S_1(t) dt = \int_0^x \sum_{n=1}^{\infty} (n+1)n t^{n-1} dt$$

$$S_2(x) = \sum_{n=1}^{\infty} (n+1)x^n.$$

$$\int_0^x S_2(t) dt = \int_0^x \sum_{n=1}^{\infty} (n+1)t^n dt.$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = \frac{x^2}{1-x}$$

"导导".

$$S_2(x) = \left( \frac{x^2}{1-x} \right)'$$

$$S_1(x) = \left( \frac{x^2}{1-x} \right)''$$

$$\text{综上 } S(x) = x \left( \frac{x^2}{1-x} \right)''' = x \left( \frac{x^2-1}{1-x} + \frac{1}{1-x} \right)''' = \dots$$

②法: (写法).

$$S(x) = x \sum_{n=1}^{\infty} (n+1) \boxed{n x^{n-1}}$$

$$= x \sum_{n=1}^{\infty} (n+1) (x^n)'$$

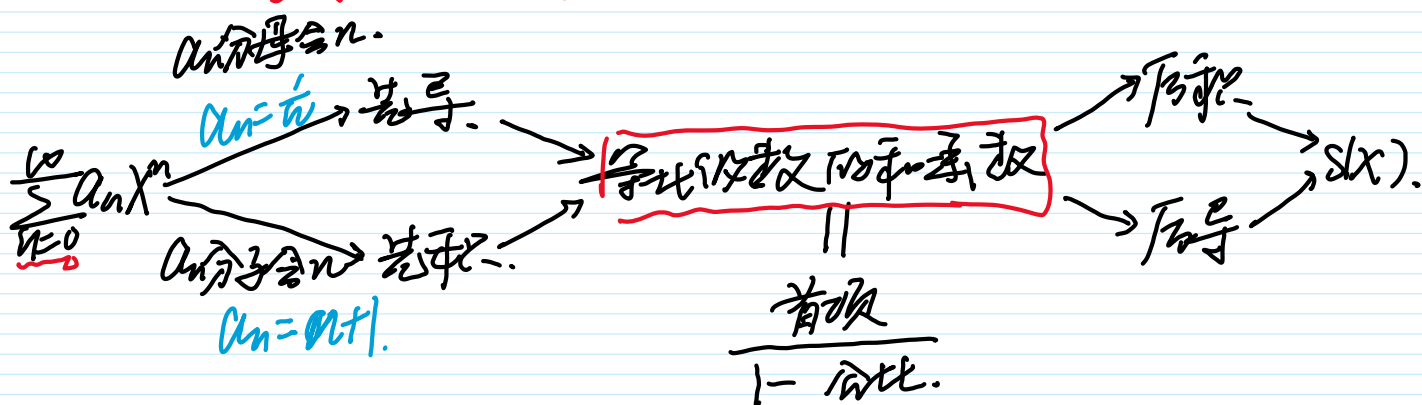
$$= x \left( \sum_{n=1}^{\infty} \boxed{(n+1)x^n} \right)'$$

$$= x \left( \sum_{n=1}^{\infty} (x^{n+1})' \right)'$$

$$= x \left( \sum_{n=1}^{\infty} (x^{n+1})' \right)' = x \left( x^2 \right)'''$$

$$= x \left( \sum_{n=1}^{\infty} x^{n+1} \right)' = x \left( \frac{x^2}{1-x} \right)'$$

如何求  $S(x) = \sum_{n=0}^{\infty} a_n x^n$ . (先求收敛域).



例.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)} = \underline{\frac{\pi}{2} - \ln 2}$

解: ① 设  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)} x^n = S(x)$  收敛域  $[-1, 1]$

$S = S(1) = \lim_{x \rightarrow 1^-} S(x)$  “洛必达法则”

$$S'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^{n-1}}{2n-1}$$

②  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)} x^{2n-1} = S(x)$  收敛域  $[-1, 1]$

$$S'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^{2n-2} = \frac{1}{x^2} S_1(x)$$

$$S_1'(x) = \sum_{n=1}^{\infty} 2 \cdot (-1)^{n-1} x^{2n-1} = 2 \cdot \frac{x}{1-x^2} \dots$$

设  $S(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)} x^{2n}$   $S'(x) = \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n-1}}{2n-1} x^{2n-1}$

在  $(-1, 1)$  上求导两次.

$$S''(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot 2 \cdot x^{2n-2} = 2 \cdot \frac{1}{1-x^2}$$



$$S''(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot 2 \cdot x^{2n-2} = 2 \cdot \frac{1}{1+x^2}.$$

一次积分  $\int_0^x S''(t) dt = \int_0^x \frac{2}{1+t^2} dt.$

$$S'(x) - S'(0) = 2 \arctan x.$$

$$S'(x) = 2 \arctan x.$$

再一次积分

$$\int_0^x S'(t) dt = 2 \int_0^x \arctan t \cdot dt.$$

$$S(x) = S(x) - S(0)$$

$$= 2 \left( \arctan t \cdot t \Big|_0^x - \int_0^x \frac{t}{1+t^2} dt \right).$$

$$= 2x \arctan x - \int_0^x \frac{1}{1+t^2} d(1+t^2)$$

$$S(x) = 2x \arctan x - \ln(1+x^2). \quad x \in (-1, 1).$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot (2n-1)} = S(1) = \lim_{x \rightarrow 1^-} S(x).$$

$$= 2 \arctan 1 - \ln 2 = \frac{\pi}{2} - \ln 2$$

②法:  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$  (裂项)

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \quad (\text{分子有理化})$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)} = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n} - \frac{2}{2n-1} \right)$$

$$= \sum_{n=1}^{\infty} \left[ \underbrace{(-1)^n \cdot \frac{1}{n}}_{\text{交错级数}} - 2 \cdot \underbrace{\frac{(-1)^n}{2n-1}}_{\text{交错级数}} \right]$$

$$- \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{收敛} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \quad \text{收敛}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

$$= -\ln 2 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

$$\text{例} \quad S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} x^{2n-1}$$

$$\text{例} \quad \sum_{n=1}^{\infty} \frac{1}{n \cdot (2n-1)} = \underline{\hspace{2cm}}$$

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n \cdot (2n-1)} x^{2n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ 均发散}$$

$$\text{例} \quad \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x, \quad x \in (-\infty, +\infty)$$

$$\text{解:} \quad \rho = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$$R = +\infty$$

$a_n$  分母含  $n$ . “ $\frac{0}{0}$  型”

$$S'(x) = \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right)' = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1}$$

$$\text{即} \quad \begin{cases} S'(x) = S(x) \\ S(0) = 1 \end{cases} \quad \begin{cases} S(x) = C e^x \\ \Rightarrow C = 1 \end{cases}$$

$$\Rightarrow f(x) = e^x.$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

↖ 收敛  
↘ 收敛

$$x \in (-\infty, +\infty)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 + x + x^2 + \dots, \quad x \in (-1, 1).$$

$x \neq 1$  ↘ 收敛

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \boxed{1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n} + \dots$$

泰勒多项式. 泰勒系数  $a_n = \frac{f^{(n)}(0)}{n!}$

$$(e^x)^{(n)} \Big|_{x=0} = e^x \Big|_{x=0} = 1.$$