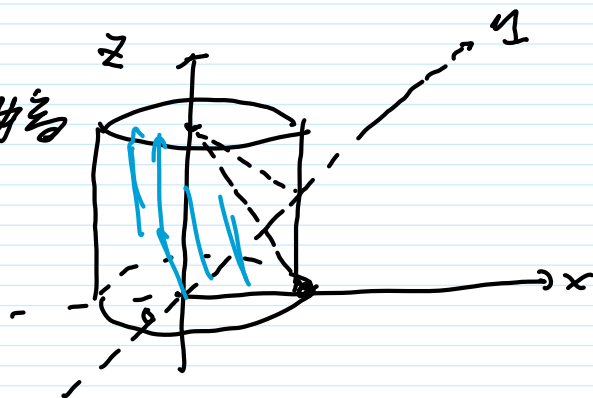


$\int_L f(x, y) ds$, 以 L 为底, 以 $f(x, y)$ 为高的柱面

例. $x^2 + y^2 = a^2$ 被 $z=0$ 和 $z=1-x$ 截面的面积.

解: 以 $x^2 + y^2 = a^2$ 为底, 以 $z=1-x$ 为高的柱面



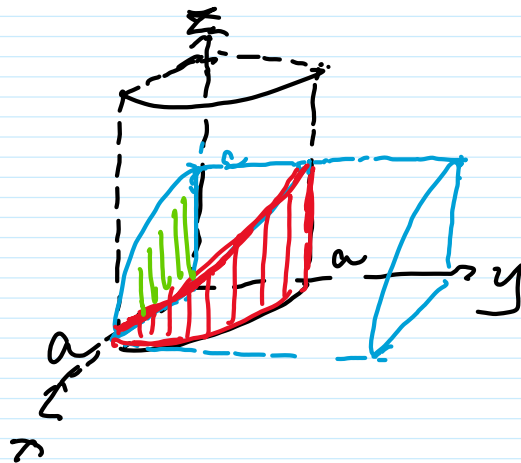
$$I = \int_L f(x) ds = \int_L 1 \cdot ds$$

$$= 2\pi a$$

$$L: x^2 + y^2 = a^2$$

例: $x^2 + y^2 = a^2$ 被 $z^2 = a^2$ 所截部分的面积 (I 卦限).

解: ① 以 z 为底, 曲线 $x^2 + y^2 = a^2$ 为底, 以 $z = \sqrt{a^2 - x^2}$ 为高的柱面



$$I = \int_2 \sqrt{a^2 - x^2} ds$$

$$= \int_0^{\frac{\pi}{2}} a \cdot \sin t \cdot \sqrt{(a \cos t)^2 + (a \sin t)^2} dt$$

② 以 x 为底, 曲线 $y = \sqrt{a^2 - x^2}$ 为底, 以 $z = \sqrt{a^2 - x^2}$ 为高的柱面

$$L: \begin{cases} x = a \cos t \\ y = a \sin t \end{cases} \quad 0 \leq t \leq \frac{\pi}{2}$$

③ 以 y 为底, 曲线 $x = \sqrt{a^2 - y^2}$ 为底, 以 $z = \sqrt{a^2 - y^2}$ 为高的柱面

$$D: x^2 + z^2 \leq a^2, \text{ 且 } x \geq a, z \geq a$$

$$y_z = -\frac{x}{\sqrt{a^2 - x^2}} \quad y_z = 0$$

$$J = \iint_D \sqrt{1 + \frac{x^2}{a^2 - x^2} + 0^2} \, dx dy.$$

例: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. 在 $x^2 + y^2 + z^2 = 1$ 内的面积

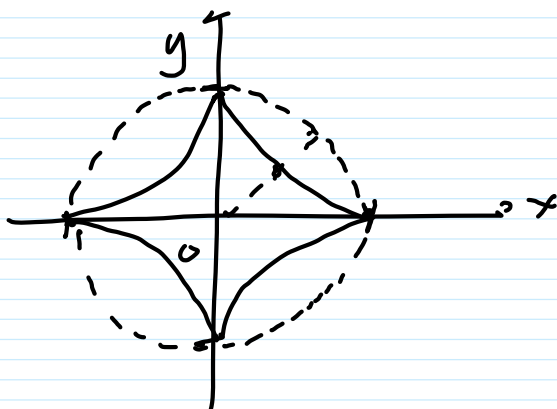
解: 由 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ 为底. 由 $z = \sqrt{1 - x^2 - y^2}$ 为高的
柱面面积 $2\pi \int_0^1$

$$I = 2 \int_L \sqrt{1 - x^2 - y^2} \, ds.$$

$$L: x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$$

$$(x^{\frac{1}{3}})^2 + (y^{\frac{1}{3}})^2 = 1.$$

二...

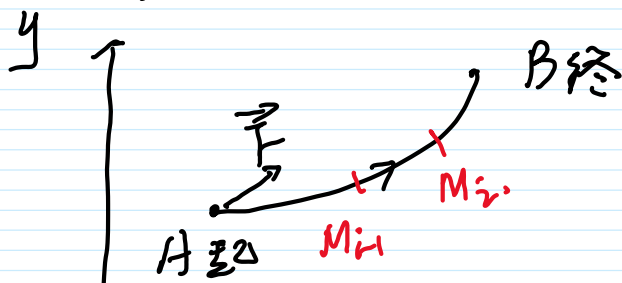


$$\begin{cases} x = \cos^3 t & 0 \leq t \leq \frac{\pi}{2} \\ y = \sin^3 t \end{cases}$$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

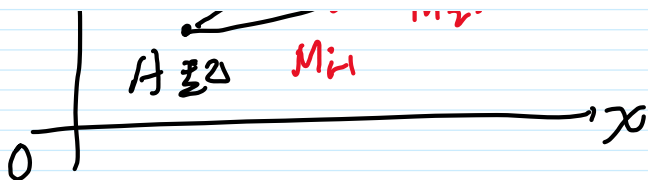
第 2 节 II 型曲线积分

引例: 变力沿有向曲线所做的功

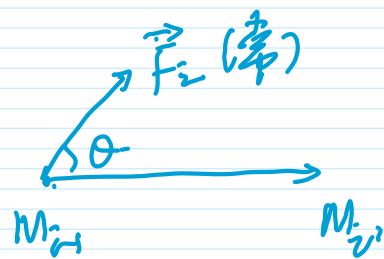


$$\vec{F} = (P(x, y), Q(x, y))$$

→ 1.2.1



$$\overrightarrow{M_{i-1}M_i} \approx \overrightarrow{M_{i-1}M_i} = (\Delta x_i, \Delta y_i)$$



$$\vec{F}_i = (P(x_i, y_i), Q(x_i, y_i))$$

$$W = |\vec{F}_i| \cdot \cos \theta \cdot |\overrightarrow{M_{i-1}M_i}|$$

$$W = \vec{F}_i \cdot \overrightarrow{M_{i-1}M_i}$$

$$W_i \approx \vec{F}_i \cdot \overrightarrow{M_{i-1}M_i}$$

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \vec{F}_i \cdot \overrightarrow{M_{i-1}M_i} = \lim_{n \rightarrow \infty} \sum_{i=1}^n (P_i, Q_i) \cdot (\Delta x_i, \Delta y_i) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P_i \Delta x_i + Q_i \Delta y_i \end{aligned}$$

定义: $P(x, y), Q(x, y)$ 在有向曲线 L 上有界
① 任意分割 ② 任意取点

若 $\lim_{n \rightarrow \infty} \sum_{i=1}^n P_i \Delta x_i$ 和 $\lim_{n \rightarrow \infty} \sum_{i=1}^n Q_i \Delta y_i$ 均存在 则分别称为

$P(x, y)$ 在 L 上对 x 的曲线积分

$Q(x, y)$ 在 L 上对 y 的曲线积分

$\vec{F} = (P(x, y), Q(x, y))$ 在 L 上对 \vec{F} 的曲线积分

记为 $\int_L P(x, y) dx \quad \int_L Q(x, y) dy$

$$\int P dx + \int Q dy = \int P(x, y) dx + \int Q(x, y) dy = \int \vec{F} \cdot \vec{ds}$$

$$\int_L p dx + \int_L Q dy = \int_L p(x,y) dx + Q(x,y) dy = \int_L \vec{F} \cdot d\vec{s}$$

其中 $\vec{F} = (P, Q)$, $d\vec{s} = (dx, dy)$

$$\int_V p(x,y,z) dx + Q(x,y,z) dy + R(x,y,z) dz$$

推广: ① $\int_L p dx + Q dy = - \int_{L^-} p dx + Q dy$

其中 L^- : 与 L 同路径, 起与终对换

② 不需要用对称性

I 型曲线积分的计算 \longrightarrow 定积分

(1) $L: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t: \underset{\text{起}}{\alpha} \rightarrow \underset{\text{终}}{\beta}$

$$I = \int_L p(x,y) dx + Q(x,y) dy$$

$$= \int_{\underset{\text{起}}{\alpha}}^{\underset{\text{终}}{\beta}} [p(x(t), y(t)) \cdot \underline{x'(t)} + Q(x(t), y(t)) \cdot \underline{y'(t)}] dt$$

(2) $L: y = y(x), \quad x: a \rightarrow b$
或 $x = x(y), \quad y: c \rightarrow d$

$$I = \int_a^b [p(x, y(x)) \cdot 1 + Q(x, y(x)) \cdot y'(x)] dx$$

$$I = \int_a^b [p(x, y(x)) \cdot 1 + Q(x, y(x)) \cdot y'(x)] dx.$$

$$\text{或 } I = \int_c^d [p(x(y), y) \cdot x'(y) + Q(x(y), y) \cdot 1] dy$$

(3). $L: r = r(\theta) \quad \theta: \alpha \rightarrow \beta$

$$I = \int_{\alpha}^{\beta} [p(r(\theta) \cos \theta, r(\theta) \sin \theta) [r(\theta) \cos \theta]' + Q(\dots) [r(\theta) \sin \theta]'] d\theta.$$

x	y	dx	dy	ds
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
$x(t)$	$y(t)$	$x'(t)dt$	$y'(t)dt$	$\sqrt{x'^2 + y'^2} dt$

I型: 起点 → 终点

I型: 下限 ≤ 上限

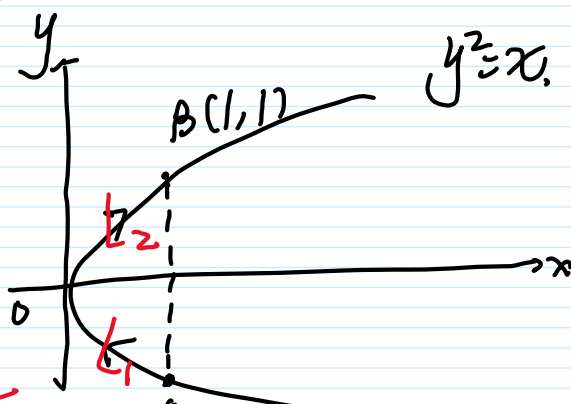
例 $I = \int_L xy dx$ $I_1 = \int_L xy ds$ 对称性 I型: 下限 ≤ 上限

$L: y^2 = x \quad A(1, -1) \rightarrow B(1, 1)$

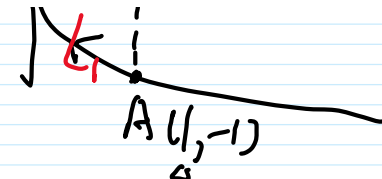
解: ① 法: 可加性

$L_1: y = -\sqrt{x}, \quad x: 1 \rightarrow 0$

$L_2: y = \sqrt{x}, \quad x: 0 \rightarrow 1$



$L_2: y = \sqrt{x}, x: 0 \rightarrow 1$



$$I = \int_1^0 x \cdot (-\sqrt{x}) \cdot 1 dx + \int_0^1 x \cdot \sqrt{x} \cdot 1 dx$$

$$= 2 \int_0^1 x^{\frac{3}{2}} dx = \dots$$

②法: $L: x = y^2, y: -1 \rightarrow 1$

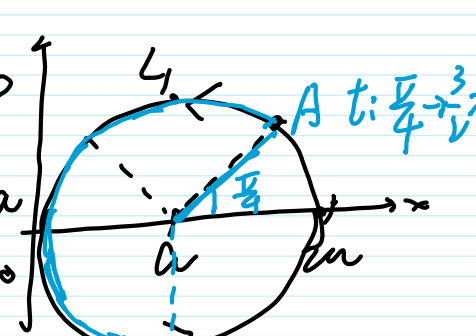
$$\int_L xy dx = \int_{-1}^1 y^2 \cdot y \cdot 2y dy$$

$$= 4 \int_0^1 y^4 dy = \frac{4}{5}$$

例: $I = \oint_L xy dx$ $L: x^2 + y^2 = 2ax$ 逆时针

解: ①法: $L_1: y = \sqrt{2ax - x^2}, x: 0 \rightarrow 2a$

$L_2: y = -\sqrt{2ax - x^2}, x: 0 \rightarrow 2a$



$$I = \int_0^{2a} x(\sqrt{2ax - x^2}) \cdot 1 dx + \int_0^{2a} x(-\sqrt{2ax - x^2}) \cdot 1 dx$$

$$= -2 \int_0^{2a} x \sqrt{2ax - x^2} dx = \dots$$

②法: $L: \begin{cases} x = a + a \cos t \\ y = a \sin t \end{cases} t: 0 \rightarrow 2\pi$

例 1: $L: \begin{cases} x = a + a \cos t \\ y = a \sin t \end{cases} \quad t: 0 \rightarrow 2\pi$

$$I = \int_0^{2\pi} (a + a \cos t) \cdot a \sin t \cdot (-a \sin t) dt$$

$$= -a^3 \int_0^{2\pi} (1 + \cos t) \sin^2 t dt$$

$$= -a^3 \left(\int_0^{2\pi} \sin^2 t dt + \int_0^{2\pi} \sin^2 t d(\sin t) \right) = \dots$$

例 2: $f(\theta) = 2a \cos \theta \quad \theta: -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2a \cos \theta)^2 \cos \theta \cdot \sin \theta \cdot (2a \cos^2 \theta)' d\theta$$

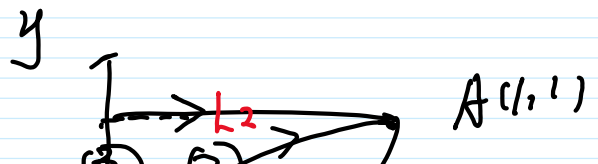
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2a \cos \theta)^2 \cos \theta \cdot \sin \theta \cdot (2a \cdot 2 \cos \theta \cdot (-\sin \theta)) d\theta$$

$$= -32a^3 \int_0^{\frac{\pi}{2}} \cos^4 \theta \sin^2 \theta d\theta$$

$$= -32a^3 \left(\int_0^{\frac{\pi}{2}} \cos^4 \theta \cdot 1 d\theta - \int_0^{\frac{\pi}{2}} \cos^6 \theta d\theta \right)$$

例 3: $I = \int_L 2xy dx - (3x+y) dy$

$L: O(0,0) \rightarrow A(1,1)$

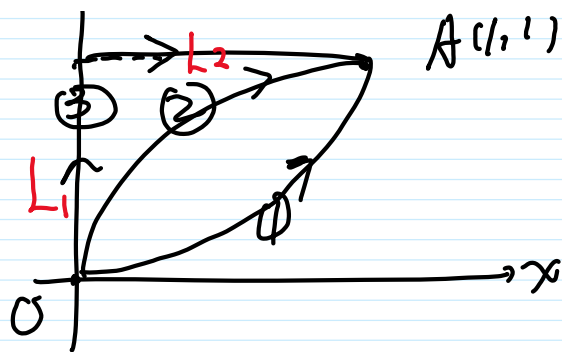


$$L: O(0,0) \rightarrow A(1,1)$$

$$(1): y = x^2$$

$$(2): x = y^2$$

$$(3): (0,0) \rightarrow (0,1) \rightarrow (1,1)$$



解: (1) $L: y = x^2, x: 0 \rightarrow 1$

$$I = \int_0^1 [2x \cdot x^2 \cdot 1 + (-3x - x^2) \cdot 2x] dx$$

$$= -2$$

(2) $L: x = y^2, y: 0 \rightarrow 1$

$$I = \int_0^1 [2 \cdot y^2 \cdot y \cdot 2y + (-3y^2 - y) \cdot 1] dy \Big|_0^1$$

(3) $L = L_1 + L_2$

$$L_1: x=0, \quad y: 0 \rightarrow 1$$

$$L_2: y=1, \quad x: 0 \rightarrow 1$$

$$I = \int_0^1 [2 \cdot 0 \cdot y \cdot 0 + (-3 \cdot 0 - y) \cdot 1] dy + \int_0^1 [2x \cdot 1 \cdot 1 + (-3x - 1) \cdot 0] dx$$

$$= \int_0^1 -y dy + \int_0^1 2x dx = \frac{1}{2}$$

推论: (1) $L \perp x$ 轴 $(x = x_0 \text{ 常数}, y: c \rightarrow d)$
 $dx = 0$ $\int_L p dx = 0$

$$\int_L p(x, y) dx + Q(x, y) dy = \int_c^d Q(x_0, y) dy$$

$$= \int_c^d [p(x_0, y) \cdot 0 + Q(x_0, y) \cdot 1] dy.$$

(2) $L \perp y$ 轴 $(y = y_0 \text{ 常数}, x: a \rightarrow b)$
 $dy = 0$

$$\int_L Q dy = 0$$

$$\int_L p dx + Q dy = \int_a^b p(x, y_0) dx$$

(3) Γ : 在 $z = z_0$ 平面上 $\int_\Gamma R dz = 0$
 $dz = 0$

$$\int_\Gamma p dx + Q dy + R dz = \int_\Gamma p dx + Q dy.$$

Γ : 在 x 轴上 $x: a \rightarrow b$

$$\int_\Gamma p dx + Q dy + R dz = \int_a^b p(x, 0, 0) dx$$

例

$$I = \int_\Gamma xy dx + yz dy + zx dz$$

Γ : 从 $A(3, 2, 1)$ 到 $B(0, 0, 0)$ 的有向线段

解: $\vec{S} = \vec{BA} = (3, 2, 1)$

过点 A, B 的直线: $\frac{x-0}{3} = \frac{y-0}{2} = \frac{z-0}{1} = t.$

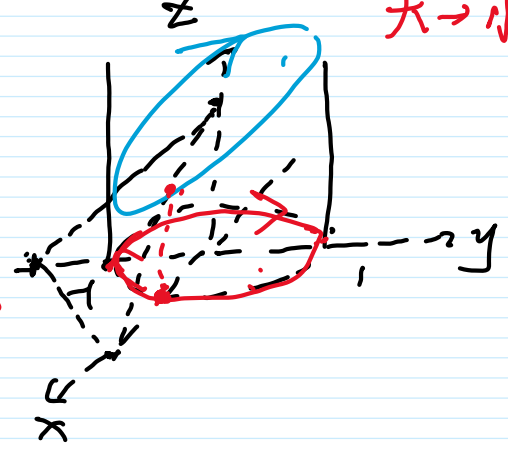
参数方程形式:
$$\begin{cases} x = 3t \\ y = 2t \\ z = t \end{cases} \quad t: 1 \rightarrow 0$$

$$I = \int_1^0 (3t \cdot 2t \cdot 3 + 2t \cdot t \cdot 2 + 3t \cdot t \cdot 1) dt$$

例:
$$I = \int_{\Gamma} (z-y)dx + (x-z)dy + (x-y)dz$$

$$\Gamma: \begin{cases} x^2 + y^2 = 1 \\ x-y+z=2 \end{cases}$$
 从 z 轴正方向看, 顺时针. 大 \rightarrow 小.

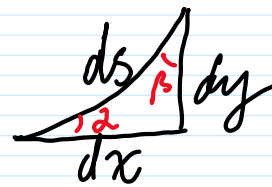
解: 参数方程:
$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 2 - \cos t + \sin t \end{cases} \quad t: 2\pi \rightarrow 0$$



$$I = \int_{2\pi}^0 (\dots) dt$$

两类曲线积分的联系. II 型 \rightarrow I 型.

$$(ds)^2 = (dx)^2 + (dy)^2$$



$$\begin{cases} dx = ds \cdot \cos \alpha \\ dy = ds \cdot \cos \beta \end{cases}$$

$$\int p dx + q dy = \int_C (p \cos \alpha + q \cos \beta) ds$$

② 有问 利用

$$\int_C p dx + q dy + r dz = \int_C (p \cdot \cos \alpha + q \cdot \cos \beta + r \cdot \cos \gamma) ds$$

$$\cos \alpha = \frac{dx}{ds} = \pm \frac{x'(t)}{\sqrt{x'^2 + y'^2}}$$

$$\cos \beta = \frac{dy}{ds} = \pm \frac{y'(t)}{\sqrt{x'^2 + y'^2}}$$