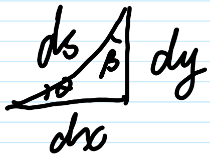
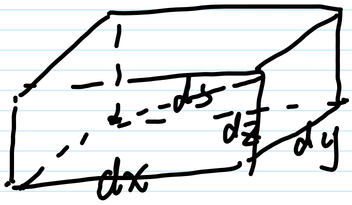




$dx = \cos\alpha \cdot ds$. $dy = \cos\beta \cdot ds$

$$\int_L P\,dx + Q\,dy = \int_L (P\cdot\cos\alpha + Q\cdot\cos\beta)\,ds$$

有向 无向

$$\int_\Gamma P\,dx + Q\,dy + R\,dz = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds$$

$$\cos\alpha = \frac{dx}{ds} = \pm \frac{x'(t)\cancel{dt}}{\sqrt{x'^2+y'^2}\,\cancel{dt}}$$

$$\cos\beta = \frac{dy}{ds} = \pm \frac{y'(t)\cancel{dt}}{\sqrt{x'^2+y'^2}\,\cancel{dt}}$$

有向L: $t: \alpha \to \beta$，若 $\alpha \le \beta$，则取"+"

若 $\alpha \ge \beta$，则取"−"

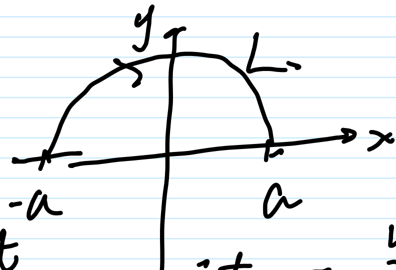
$$\int x\,dy - y\,dx$$

例. 将 $I = \int_L (-y)\,dx + x\,dy$ $L: y=\sqrt{a^2-x^2}$ $(a>0)$

化为了第1类线积分 $(-a,0) \to (a,0)$

解. 由法L: $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $t: \pi \to 0$

起→终



$$\cos\alpha = -\frac{x'(t)}{\sqrt{x'^2+y'^2}} = -\frac{-a\sin t}{a} = \sin t = \frac{y}{a}$$

$$\cos\beta = -\frac{y'(t)}{a} = -\frac{a\cos t}{a} = -\cos t = -\frac{x}{a}$$

$$I=\int_L[-y\cdot\sin t+x\cdot(-\cos t)]\,ds$$

$$=\int_L\left(-y\cdot\frac{y}{a}-x\cdot\frac{x}{a}\right)ds=\int_L\frac{x^2+y^2}{a}\,ds$$

$$=\int_L\frac{-a^2}{a}\,ds=-a\int_L 1\,ds=-a\cdot\pi a=-\pi a^2.$$

例2: $L:\ y=\sqrt{a^2-x^2}$. $\quad x:\ -a\longrightarrow a$

起 $\le$ 终

$$\cos\alpha=+\frac{1}{\sqrt{1+y'^2}}=\cdots$$

$$\cos\beta=+\frac{y'}{\sqrt{1+y'^2}}=\cdots$$

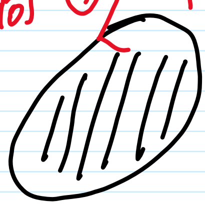
第三节 格林公式

1. 单连通区域: "无洞"  $x^2+y^2\le1$. 单.

复连通区域: "有洞"  $1\le x^2+y^2\le4$ 复. $0<x^2+y^2\le1$ 复.

2. L. 正向.

正向 逆时针



正向 逆.

3. 定理. (格林公式).

(1). D 单连通.

(2) L 正向.

(3) P, Q 偏导连续.

$$\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ P & Q\end{vmatrix}$$

③ P, Q 偏导连续. $\left|\begin{array}{cc}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q\end{array}\right|$

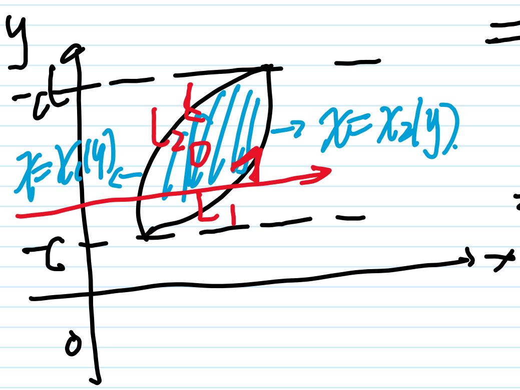
复杂

$$\Rightarrow \oint_L P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

简单

证明: $\oint_L Q\,dy = \iint_D \frac{\partial Q}{\partial x} dx\,dy.$

$$\iint_D \frac{\partial Q}{\partial x} dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial Q}{\partial x}\, dy$$



$$= \int_c^d dy \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\, dx$$

$$= \int_c^d \left[\int_{x_1(y)}^{x_2(y)} d\,Q(x,y)\right] dy$$

$$= \int_c^d \left[Q(x_2(y),y) - Q(x_1(y),y)\right] dy$$

$L = L_1 + L_2$. 其中 $L_1:\ x = x_2(y).\ y: c \to d$

$L_2:\ x = x_1(y).\ y: d \to c$

$$\oint_L Q\,dy = \int_{L_1} Q(x,y)\,dy + \int_{L_2} Q(x,y)\,dy$$

$$= \int_c^d Q(x_2(y),y)\cdot 1\ dy + \int_d^c Q(x_1(y),y)\cdot 1\ dy$$

$$= \int_c^d \left[Q(x_2(y),y) - Q(x_1(y),y)\right] dy = \text{右}.$$

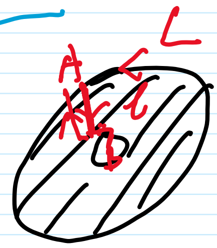
推论1. ① 单连通.

推论1. ① 单连通.

② P, Q 偏导连续.

$$\Rightarrow \oint_L P\,dx + Q\,dy = \pm \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

L 正向，取"+"；L 负向，取"−".



推论2. ① D 复连通.

② L, l 正向

③ P, Q 偏导连续

$$\Rightarrow \oint_{L+l} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_{L + \widehat{AB} + l + \widehat{BA}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\| $$

$$\oint_{L+l} P\,dx + Q\,dy + \underbrace{\oint_{\widehat{AB}+\widehat{BA}} P\,dx + Q\,dy}_{=0}$$

应用: $\iint_D 1\,dx\,dy = \sigma$.

当 $Q = x$, $P = 0$, 时 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

或 $Q = \frac{x}{2}$, $P = -\frac{y}{2}$ 时

我 $Q=\frac{x}{2}$、$P=-\frac{y}{2}$ 则

$$\boxed{\oint_C \frac{x}{2}dy-\frac{y}{2}dx=\iint_D 1\,dxdy=\sigma.}$$

例. $L:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\quad(a>0,b>0)$ 所围成面积

解: $S=\frac{1}{2}\oint_L x\,dy-y\,dx.$

其中 L: 逆时针.

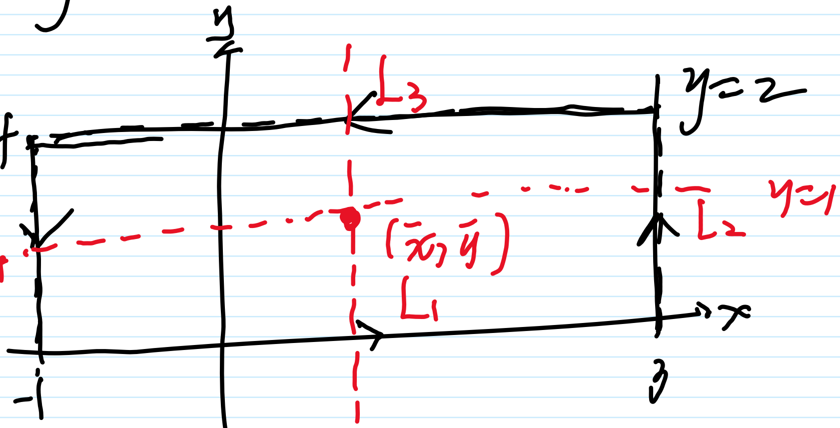
$L:\begin{cases}x=a\cos t\\ y=b\sin t,\end{cases}\quad t:0\to2\pi$

$$S=\frac{1}{2}\int_0^{2\pi}[a\cos t\cdot b\cos t-b\sin t\cdot(a(-\sin t))]\,dt$$

$$=\frac{1}{2}\int_0^{2\pi}ab\,dt=\pi ab$$

例: $I=\oint_L 4xy\,dx+3x^2dy$

其中 L: 逆时针



①方法:

解: $L=L_1+L_2+L_3+L_4.$

$L_1:\ y=0,\quad x:-1\to3\quad(dy=0)$

$I=\int 4xy\,dx+3x^2dy=\int 4xy\,dx$

$$I_{L_1}=\int_{L_1} 4xy\,dx+\underset{\sim}{3x^2}\,dy=\int_{L_1}4xy\,dx$$

$$=\int_{-1}^{3}4x\cdot 0\,dx=0$$

$$I_{L_2}\overset{dx=0}{=\!=}\int_{L_2}Q\,dy=\int_0^2 Q(3,y)\cdot 1\,dy$$

$$=\int_0^2 3\cdot 3^2\,dy=\cdots$$

$$I_{L_3}\overset{dy=0}{=\!=}\int_{L_3}P\,dx=\int_3^{-1}P(x,2)\cdot 1\,dx$$

$$=\int_{-3}^{1}4x\,2\,dx=\cdots$$

$$I_{L_4}=\int_{L_4}Q\,dy=\int_2^0 3\cdot(-1)^2\,dy=\cdots$$

②法：格林公式

$$I=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=\iint_D(6x-4x)\,dxdy$$

形心公式

$$=2\underbrace{\iint_D x\,dxdy}=2\cdot\bar{x}\cdot 6=2\times1\times4\times2=16.$$

$$\bar{x}=\frac{\iint_D x\,d\sigma}{\sigma}$$

$$\bar{x}=\frac{\iiint_\Omega x\,dv}{V}$$

$$=2\int_{-1}^{3}dx\int_0^2 x\,dy$$

$$=2\int^{3}2x\cdot dx=2\cdot(9-1)=16.$$

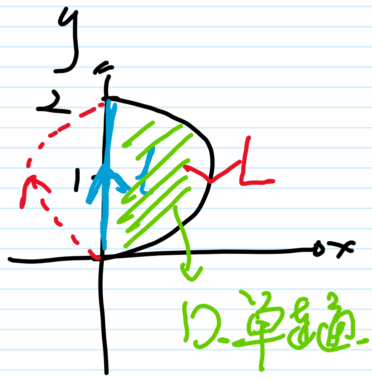
$\frac{\quad}{V}$ $\quad = 2\int_{-1}^{3} 2x \cdot dx = 2\cdot(\ \ )= 0.$

$$\bar{x} = \frac{\int_L x\,ds}{S}$$

例. $I = \int_L (x^3 - e^x\cos y)dx + (e^x \sin y + 4x)dy$

其中 L: $x^2+(y-1)^2=1$. 右半圆. 从 $(0,2)\to(0,0)$

解: $L:\begin{cases} x = \cos t \\ y = 1+\sin t \end{cases}$ $t: \frac{\pi}{2} \to -\frac{\pi}{2}$



用定积分计算非常繁琐. 格林公式

添加辅助线: 平行于坐标轴直线

构造封闭曲线 $L' = L + l$. 负向

$$\oint_{L'} P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = -\iint_D 4\,dxdy$$

$$= -4\cdot\frac{1}{2}\pi 1^2$$

$$= -2\pi.$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^3 - e^x\cos y) = e^x \sin y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(e^x\sin y + 4x) = e^x \sin y + 4.$$

$$\oint_{L'} P\,dx + Q\,dy = I + \int_l P\,dx + Q\,dy = -2\pi.$$

其中 l: $x=0$ $y: 0\to 2.$

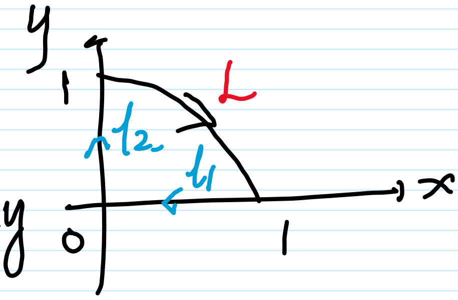
其中 l: $x=0$ $y: 0\to 2$.

$$\int_{l} P\,dx+Q\,dy = \int_{l} Q\,dy = \int_0^2 Q(0,y)\,dy$$

$$= \int_0^2 (e^0 \sin y + 4\times 0)\,dy$$

例. $I=\int_L x\,dy$. $L: y=\sqrt{1-x^2}$.

从 $(0,1)\to(1,0)$



解、构造 $L'=L+l_1+l_2$. 负向

$$\oint_{L'} x\,dy = -\iint_D \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & x \end{vmatrix} dx\,dy$$

$$= -\iint_D 1\,dx\,dy = -\frac{\pi}{4}.$$

$$\oint_{L'} x\,dy = I + \int_{l_1} x\,dy + \int_{l_2} x\,dy$$

l_1: $y=0$ $x: 1\to 0$ $dy=0$. $\int_{l_1} x\,dy=0$

l_2: $x=0$, $y: 0\to 1$. $\int_{l_2} x\,dy = \int_0^1 0\,dy = 0$

$I=-\frac{\pi}{4}$.