

$$1. f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{d^2}{dx^2} (\dots)$$

$$f_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$\frac{d^2 y}{dx^2}$$

2. 高阶偏导.

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \lim_{\Delta x \rightarrow 0} \frac{f_x(x+\Delta x, y) - f_x(x, y)}{\Delta x} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \lim_{\Delta y \rightarrow 0} \frac{f_x(x, y+\Delta y) - f_x(x, y)}{\Delta y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

= 二阶  
混合  
偏导

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\text{类似: } \frac{\partial^3 f}{\partial x^3} = f_{xxx} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \right)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \right)$$

$$\text{例: } z = e^{x+2y} \text{ 求所有二阶偏导及 } \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial y \partial x^2}$$

$$\text{解: } \frac{\partial z}{\partial x} = e^{x+2y} \cdot 1, \quad \frac{\partial z}{\partial y} = e^{x+2y} \cdot 2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (e^{x+2y}) = e^{x+2y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (e^{x+2y}) = 2e^{x+2y}$$

$$\left\{ \frac{\partial z}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = 2y \right.$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (2e^{x+2y}) = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (2e^{x+2y}) = 4e^{x+2y}$$

$$\left\{ \begin{aligned} \frac{\partial^3 z}{\partial x^2 \partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial}{\partial y} (e^{x+2y}) = 2e^{x+2y} \\ \frac{\partial^3 z}{\partial y \partial x^2} &= \frac{\partial^2}{\partial x^2} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2}{\partial x^2} (2e^{x+2y}) = 2e^{x+2y} \end{aligned} \right.$$

$$\frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \right) = \frac{\partial}{\partial x} \left( \frac{\partial^2 z}{\partial y \partial x} \right) = \frac{\partial}{\partial x} (2e^{x+2y})$$

$$\frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial^3 z}{\partial y \partial x \partial y}$$

例:  $f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

求  $f_{xy}(0, 0)$ ,  $f_{yx}(0, 0)$

$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$   
 $(x_0, y_0) \neq (0, 0)$

解:

$$f_{xy}(0, 0) = \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial x} \right] \Big|_{(0, 0)}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{f_x(0, 0 + \Delta y) - f_x(0, 0)}{\Delta y}$$

求  $f_x(x, y)$

当  $(x, y) \neq (0, 0)$ .

$$f_x(x, y) = \frac{\partial}{\partial x} \left( xy \cdot \frac{x^2 - y^2}{x^2 + y^2} \right)$$

y 常数

$$= y \cdot \frac{x^2 - y^2}{x^2 + y^2} + (xy) \cdot \frac{2x(x^2 + y^2) - (x^2 - y^2) \cdot 2x}{(x^2 + y^2)^2}$$

$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

例.  $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

当  $(x, y) = (0, 0)$

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = 0$$

综上.  $f_x(x, y) = \begin{cases} \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$$f_{xy}(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{\frac{\Delta y (0^4 + 4 \cdot 0^2 (\Delta y)^2 - (\Delta y)^4)}{(0^2 + (\Delta y)^2)^2} - 0}{\Delta y}$$

$$= -1.$$

同理  $f_{yx}(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f_y(0 + \Delta x, 0) - f_y(0, 0)}{\Delta x} = 1.$

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

定理. 二阶偏导连续  $\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$

推广: 三阶偏导连续  $\Rightarrow \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial^3 f}{\partial x \partial y \partial x}$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y \partial x \partial y}$$

$$\frac{\partial^2 z}{\partial x^2 \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

例  $\frac{\partial^2 z}{\partial x \partial y} = x+y$ .  $z(x,0)=x$ ,  $z(0,y)=y^2$ .

$$\frac{d f(y)}{d y} = \tan y$$

例 = 求函数  $z =$

解:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = x+y = xy + \frac{y^2}{2} + \varphi(x)$$

$$\frac{\partial z}{\partial x} = \int (x+y) dy$$

$$\frac{\partial z}{\partial x} = x \cdot y + \frac{1}{2} y^2 + \varphi(x)$$

$$z(x,y) = \int [xy + \frac{1}{2} y^2 + \varphi(x)] dx$$

$$= \frac{1}{2} x^2 y + \frac{1}{2} x y^2 + \frac{\Phi(x) + \psi(y)}{1}$$

$$\Phi'(x) = \varphi(x)$$

$$z(x,0) = \Phi(x) + \psi(0) = x$$

$$z(0,y) = \Phi(0) + \psi(y) = y^2$$

$$\Phi(x) + \psi(y) + \frac{\psi(0) + \Phi(0)}{1} = x + y^2$$

$$\text{综上 } z(x,y) = \frac{x^2 y + x y^2}{2} + x + y^2$$

### 第三节 全微分

一元函数微分: 不连续  $\Rightarrow$  不可导/不可微

$$y = f(x) \xrightarrow{\text{可微}} \Delta y = f(x+\Delta x) - f(x) = f'(x) \Delta x + o(\Delta x)$$

$$\text{可微} \leftarrow \text{可导} \quad \text{微分} \quad dy = f'(x) \cdot dx$$

函数 ← 可导 12.2.20 - 0.07 - 1.1.1.1  
二元函数微分

★ 定义:  $z = f(x, y)$ .  $\Delta x, \Delta y$

全增量  $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$

若  $\Delta z = A(x, y) \Delta x + B(x, y) \Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$

其中  $A(x, y), B(x, y)$  与  $\Delta x, \Delta y$  无关  
则  $z = f(x, y)$  在  $(x, y)$  可微. 记为

全微分  $dz = A(x, y) \Delta x + B(x, y) \Delta y$

$dz = A(x, y) dx + B(x, y) dy$

定理 (可微的必要条件).

偏导存在, 但不可微

可微  $\Rightarrow$  偏导存在

且  $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

例  $z = x + 2y$ .  $dz|_{(x_0, y_0)} = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$   
则  $dz = 1 \cdot dx + 2 dy$

证明: 证:  $f(x + \Delta x, y + \Delta y) - f(x, y) = A(x, y) \Delta x + B(x, y) \Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$

证:  $f_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$  证

$= \lim_{\Delta x \rightarrow 0} \frac{A(x, y) \Delta x + B(x, y) \cdot 0 + o(\sqrt{(\Delta x)^2 + 0^2})}{\Delta x}$

$= A(x, y) + \lim_{\Delta x \rightarrow 0} \frac{o(|\Delta x|)}{\Delta x} = A(x, y)$

$$\lim_{\Delta x \rightarrow 0} \underbrace{\frac{\Delta |x|}{|\Delta x|}}_{\text{无意义}} \cdot \underbrace{\frac{|\Delta x|}{\Delta x}}_{\text{有解}} = 0 \quad \text{例: } f_y(x, y) \neq B(x, y)$$

例:  $z = \sqrt{x^2 + y^2}$ , 在  $(0, 0)$  处是否可微.

解: 在  $(0, 0)$  连续,  $f_x(0, 0)$ ,  $f_y(0, 0)$  不存在  
偏导不存在  $\Rightarrow$  不可微

$$\text{例: } f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

在  $(0, 0)$  处不连续,  $f_x(0, 0) = f_y(0, 0) = 0$ .

讨论在  $(0, 0)$  是否可微.  
不可微.

解: 若可微.

$$\begin{aligned} \sin x \quad (x \rightarrow 0) \quad x^2 &\approx o(\sin x) \\ x^2 \quad (x \rightarrow 0) \\ \lim_{x \rightarrow 0} \frac{x^2}{\sin x} &= 0 \end{aligned}$$

$$f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = f_x(0, 0) \Delta x + f_y(0, 0) \Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

$$f(\Delta x, \Delta y) - 0 = 0 + 0 + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

$$\text{去证明: } \frac{\Delta x \cdot \Delta y}{(\Delta x^2 + \Delta y^2)^2} = o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

$$\text{去证 } \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{\Delta x \cdot \Delta y}{[(\Delta x)^2 + (\Delta y)^2]^{\frac{3}{2}}} = 0$$

但: 不存在 ( $\Delta y = \Delta x$ )

但. 不存在. ( $\Delta y = \Delta x$ )

$$\text{可微} \Leftrightarrow f(x+\Delta x, y+\Delta y) - f(x, y) = f_x(x, y)\Delta x + f_y(x, y)\Delta y + o(\rho)$$

$$\text{其中 } \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Leftrightarrow o(\rho) = [f(x+\Delta x, y+\Delta y) - f(x, y)] - [f_x \Delta x + f_y \Delta y]$$

★ 可微  $\Leftrightarrow \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{[f(x+\Delta x, y+\Delta y) - f(x, y)] - [f_x \Delta x + f_y \Delta y]}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$

推论: 可微  $\Rightarrow$  连续; 不连续  $\Rightarrow$  不可微

证明: 去证  $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$

$$\Leftrightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} [f(x, y) - f(x_0, y_0)] = 0$$

$u = f(p)$

$$\begin{aligned} x &= x_0 + \Delta x \\ y &= y_0 + \Delta y \end{aligned} \downarrow$$

$$\Leftrightarrow \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta z = 0$$

$$\lim_{\Delta p \rightarrow 0} \Delta u = 0$$

另证: 可微

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} [f_x \Delta x + f_y \Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})] = 0$$

例:  $f(x, y) = \sqrt{|xy|}$  在  $(0, 0)$  是否可微.

解: 连续.  $f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{\sqrt{|0 \cdot (0+\Delta y)|} - \sqrt{|0 \cdot 0|}}{\Delta y} = 0$

同理  $f_x(0, 0) = 0$

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \sqrt{|\Delta x \Delta y|} - [f(0, 0) \Delta x + f(0, 0) \Delta y]$$

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] - [f'_x(x_0) \Delta x + f'_y(x_0) \Delta y]}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \quad \checkmark$$

$$= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\sqrt{(\Delta x)^2 + (\Delta y)^2} - 0) - (0 \cdot \Delta x + 0 \cdot \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \sqrt{\frac{|\Delta x \cdot \Delta y|}{(\Delta x)^2 + (\Delta y)^2}}$$

不趋于 0  
( $\Delta y = k \Delta x$ )

故在 (0,0) 不可微

推广:  $u = f(x_1, x_2, \dots, x_n)$  可微.

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \dots + \frac{\partial u}{\partial x_n} dx_n$$

(全微分)

例:  $z = x + 2y$

$$g(x) = x + 2a$$

$$h(y) = b + 2y$$

$$dz = \underline{1 \cdot dx} + \underline{2 \cdot dy}$$

$$dg = g'(x) dx = \underline{1 \cdot dx}$$

$$dh = h'(y) dy = \underline{2 \cdot dy}$$

定理 (可微的充分条件):

偏导连续  $\Rightarrow$  可微

证明: 类似.

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{[f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] - [f'_x(x_0) \Delta x + f'_y(x_0) \Delta y]}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$



1. 中值定理

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

拉格朗日中值定理

$$\begin{aligned} f(x+\Delta x) - f(x) &= f'(\xi) \cdot \Delta x \\ &= f'(x + \theta \Delta x) \cdot \Delta x \end{aligned}$$

$$\begin{aligned} &f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y) \\ &= f_x(x + \theta_1 \Delta x, y + \Delta y) \cdot \Delta x + f_y(x, y + \theta_2 \Delta y) \cdot \Delta y. \end{aligned}$$

$$\begin{aligned} \Delta z &= [f_x(x + \theta_1 \Delta x, y + \Delta y) - f_x(x, y)] \cdot \Delta x \\ &\quad + [f_y(x, y + \theta_2 \Delta y) - f_y(x, y)] \cdot \Delta y. \end{aligned}$$

$$\lim_{x \rightarrow x_0} f(x) = A \Leftrightarrow \underline{f(x) = A + \alpha}$$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \Leftrightarrow \underline{f(x) = f(x_0) + \alpha}$$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = f(x_0,y_0) \Leftrightarrow f(x,y) = f(x_0,y_0) + \alpha \quad \checkmark$$

$$\text{其中 } \lim_{(x,y) \rightarrow (x_0,y_0)} \alpha = 0.$$

$$\begin{aligned} f_x(x + \underbrace{\theta_1 \Delta x}_0, y + \underbrace{\Delta y}_0) &= f_x(x, y) + \underbrace{\alpha_1}_0 \quad (\Delta x, \Delta y) \rightarrow 0? \end{aligned}$$

$$f_y(x, y + \theta_2 \Delta y) = f_y(x, y) + \alpha_2.$$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\alpha_1 \Delta x + \alpha_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \Rightarrow \text{可微}$$

例  $f(x, y) = \begin{cases} (x^2+y^2)^2 \sin \frac{1}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

讨论在  $(0, 0)$  是否连续、偏导数存在、可微及偏导数连续。

解:  $f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{[(\Delta x)^2 + 0^2] \sin \frac{1}{(\Delta x)^2 + 0^2} - 0}{\Delta x} = 0$

同理  $f_y(0, 0) = 0$ .

$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{[(\Delta x)^2 + (\Delta y)^2] \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2} - 0}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

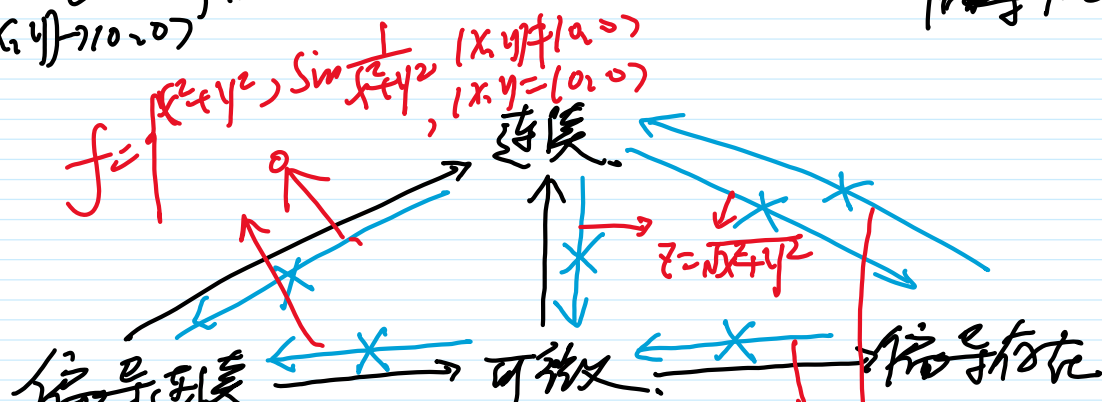
$= \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} \cdot \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2} = 0$

当  $(x, y) \neq (0, 0)$  时

$f_x = \frac{\partial}{\partial x} \left( (x^2+y^2)^2 \sin \frac{1}{x^2+y^2} \right)$

$= 2x \sin \frac{1}{x^2+y^2} + (x^2+y^2)^2 \cdot \cos \frac{1}{x^2+y^2} \cdot (-1) \cdot \frac{2x}{(x^2+y^2)^2}$

$\lim_{(x, y) \rightarrow (0, 0)} f_x(x, y) \neq 0 = f_x(0, 0) \Rightarrow$  偏导数不连续



偏导连续  $\xrightarrow{\text{可微}}$  偏导存在

$$f = \int \frac{xy}{x^2+y^2}, \quad (x,y) \neq (0,0)$$

0,  $(x,y) = (0,0)$

例.  $(axy^3 - y^2 \cos x) dx + (1 + by \sin x + 3x^2 y^2) dy$

是  $z = f(x,y)$  的全微分, 则  $a = \underline{2}$ ,  $b = \underline{-2}$ .

解:  $\therefore dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ .

方法: 偏积分法

$$\int (axy^3 - y^2 \cos x) dx = \int (1 + by \sin x + 3x^2 y^2) dy$$

$+ \phi(y)$   $+ \psi(x)$

方法: 混合偏导相等

$$\text{ii) } \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)$$

$$3axy^2 - 2y \cos x = 0 + by \cdot \cos x + 6x y^2$$

$$3a = 6, \quad -2 = b$$