二重极限的计算：（一元函数极限的计算方法）

四则、复合、重要极限、等价无穷小替换、

泰勒公式、夹逼定理、无穷小x有界量等

例 $\lim\limits_{(x,y)\to(0,0)}\left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)=0+0=0$

例 $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)^{(x^2+y^2)}=$ 1

换元 $t=x^2+y^2$

$\lim\limits_{x\to0^+}x^x=1$

$x^x=e^{x\ln x}\to e^0\ (x\to0^+)$

$\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)=0$

连续 $\Leftrightarrow\lim\limits_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0)$

特值法：$\lim\limits_{(x,y)\to(0,0)}(x^2+0^2)^0=\lim\limits_{x\to0}1=1$

例. $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)^{xy}=$ 1 ($y=0$)

解：$(x^2+y^2)^{xy}=e^{xy\ln(x^2+y^2)}$ $\quad\lim\limits_{(x,y)\to(0,0)}xy\ln(x^2+y^2)=0$

夹逼定理

$0\le|xy\ln(x^2+y^2)|\le\frac{(x^2+y^2)}{2}\ln(x^2+y^2)$

$\downarrow\qquad\qquad\downarrow(x,y)\to(0,0)$

$0\longrightarrow0\longleftarrow0$

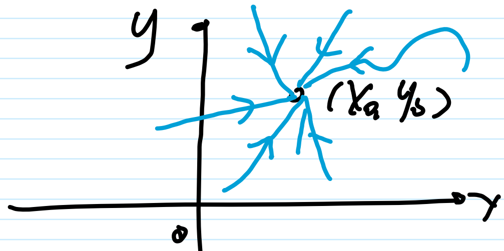
例. $\lim\limits_{(x,y)\to(0,0)}\frac{\sin(xy)}{x}$

解，原式 $=\lim\frac{\sin(\boxed{xy})}{\boxed{xy}}\cdot y=1\cdot0=0.$

例1 原式 $= \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{\boxed{xy}} \cdot y = 1\cdot 0 = 0.$

原式 $= \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x} = 0.$

例 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ 不存在



解: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{x\cdot 0}{x^2+0^2} = \lim\limits_{x\to 0} 0 = 0$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} \dfrac{0\cdot y}{0^2+y^2} = 0.$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} \dfrac{x\cdot x}{x^2+x^2} = \lim\limits_{x\to 0} \dfrac{x^2}{2x^2} = \dfrac{1}{2}.$

$$\boxed{\lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ \varphi_1(x,y)=0}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(x_0,y_0)\\ \varphi_2(x,y)=0}} f(x,y) \Rightarrow \text{极限不存在}}$$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=kx}} \dfrac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{kx^2}{(1+k^2)x^2} = \dfrac{k}{1+k^2}$

与k有关
路径

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{kx^3}{x^2+k^2x^4}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx^2}} \frac{x\cdot 0}{x^2+y^2} = \lim_{x\to 0}\frac{1}{x^2+k^2x^4}$$

$$= \lim_{x\to 0}\frac{kx}{1+k^2x^2} = \frac{0}{1+0} = 0$$

例. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ 不存在 取 $y=kx^2$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$$

解: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^3\, 0}{x^4+0^2} = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^4}{x^4+x^2} = \lim_{x\to 0}\frac{x^2}{x^2+1} = 0.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} \frac{x^5}{x^4+x^4} = \lim_{x\to 0}\frac{x}{2} = 0$$

$$0 \le \left|\frac{x^3y}{x^4+y^2}\right| \le \frac{|x|^3|y|}{2x^2|y|} = \frac{|x|}{2}$$

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$0 \qquad \to 0 \leftarrow \qquad 0$

例. $\lim\limits_{(x,y)\to(0,0)} \frac{(|x|+|y|)^3}{x^2+y^2} = 0$

例. $x=1$

解：极坐标换元 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$

$\downarrow$

$\rho\cos\theta=1$

$\Downarrow$

$\rho=\frac{1}{\cos\theta}$.

$$原式=\lim_{\substack{\rho\to 0^+\\ \theta任意}}\frac{\rho^3(|\cos\theta|^3+|\sin\theta|^3)}{\rho^2}=\lim_{\rho\to 0^+}\underset{无穷小}{\rho}\,(\underbrace{|\cos\theta|^3+|\sin\theta|^3}_{有界})=0$$

例. $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$

在 $(0,0)$ 处 <u>不连续</u>. 在除 $(0,0)$ 外 R^2 上都连续

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}\neq 0=f(0,0)$$

$$\lim_{(x,y)\to(x_0,y_0)}\frac{xy}{x^2+y^2}=\frac{x_0y_0}{x_0^2+y_0^2}\quad[(x_0,y_0)\neq(0,0)]$$

例. $f(x,y)=\begin{cases}\frac{\sin(xy)}{x(y^2+1)}, & x\neq 0 \\ \boxed{0, \quad x=0} \end{cases}$

在 $(0,0)$ 不连续

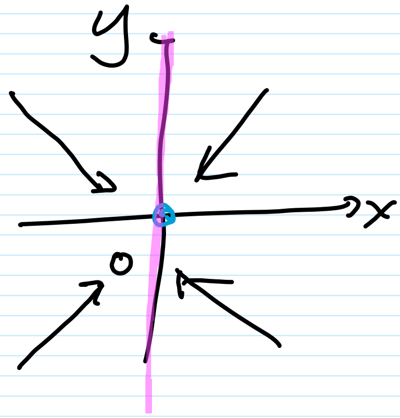
$\begin{cases} 0, & (x,y)=(0,0) \\ 1, & x=0 且 y\neq 0\end{cases}$

在 $(0,0)$ 处 <u>连续</u>. $\lim_{(x,y)\to(x_0,y_0)}f(x,y)=f(x_0,y_0)$.

解：i) $\lim_{(x,y)\to(0,0)}f(x,y)$

$$\overset{x\neq 0}{=}\lim_{(x,y)\to(0,0)}\frac{\sin(xy)}{x(y^2+1)}=\lim_{(x,y)\to(0,0)}\frac{xy}{x(y^2+1)}\cdots$$

$= 0 = f(0,0)$



ii) $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y)$

$= \lim\limits_{(x,y)\to(0,0)} 0 = 0$

综上. $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ 故连续

有界闭区域的连续函数,

① 有界、最值

② 介值 $m \le \mu \le M$ $\exists P_0 \in D$ 使 $f(P_0) = \mu$

③ 一致连续: $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$, 当 $0 < |P_1 P_2| < \delta$.

有 $|f(P_1) - f(P_2)| < \varepsilon$ 成立

第二节 偏导数

一. 偏导数概念 (二元函数)

定义: $z = f(x,y)$ 在 $U(P_0)$ 有定义、当固定 y (x) 於 y_0 (x_0) 时

x (y) 在 x_0 (y_0) 处有增量 Δx (Δy) 时、函数也有相应的增量

$\Delta_y z = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$

关于 x (y) 的偏增量 $\Delta_x z = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$ $\lim\limits_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y}$

若 $\lim \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{}$ 存在.

若 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ 存在.

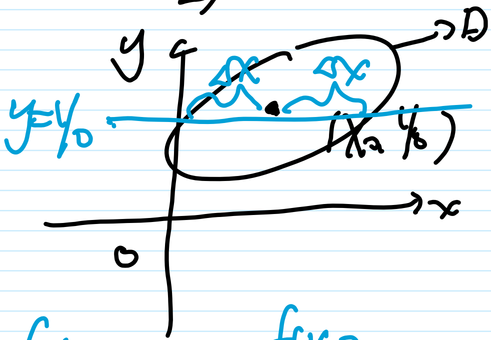
则称为 $z=f(x,y)$ 在 (x_0,y_0) 关于 x（y）的偏导数

记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$ $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$. $z_x(x_0,y_0)$. $f_x(x_0,y_0)$

$\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ $f_y(x_0,y_0)$

否则，偏导不存在

$$\boxed{\begin{gathered}\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}\\ \| \\ \left.\dfrac{d f(x,y_0)}{dx}\right|_{x=x_0}\end{gathered}} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$



一元函数导数：$f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

例：$y=2x$. $y'=2$. $y=ax$, $y'=a$

$f(x,y)=x\cdot y$.

$\dfrac{\partial f}{\partial x}$ 把y看成常数 y

$\dfrac{\partial f}{\partial y}$ 把x看成常数 x

$\frac{\partial}{\partial y}$ ——— x

$f(x,y)=x^y \quad (x>0 且 x\neq 1)$

$\frac{\partial f}{\partial x} \overset{幂函数}{=\!=\!=} y x^{y-1}$

$\frac{\partial f}{\partial y} \overset{指数函数}{=\!=\!=} x^y \cdot \ln x.$

定义：偏导(函)数

$$\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,\boxed{y})-f(x,\boxed{y})}{\Delta x}.$$

例 $f(x,y)=x+(y-1)\arctan\sqrt{\frac{x}{y}}$.

求 $\frac{\partial f}{\partial x}$，$\frac{\partial f}{\partial y}$，$\left.\frac{\partial f}{\partial x}\right|_{(5,1)}$，$\left.\frac{\partial f}{\partial y}\right|_{(5,1)}$ "$'$" $\to \frac{\partial}{\partial x}$

解：$\frac{\partial f}{\partial x} \overset{y是常数}{=\!=\!=} (x)'+\left[(y-1)\arctan\sqrt{\frac{x}{y}}\right]'$

"先导再代"

$=1+(y-1)\frac{1}{1+\frac{x}{y}}\left(\sqrt{\frac{x}{y}}\right)'$

(连续)

$=1+(y-1)\frac{1}{1+\frac{x}{y}}\cdot\frac{1}{\sqrt{y}}\cdot\frac{1}{2\sqrt{x}}$

"$'$" $\to \frac{\partial}{\partial y}$

$\frac{\partial f}{\partial y}=0+(y-1)'\arctan\sqrt{\frac{x}{y}}+(y-1)\left(\arctan\sqrt{\frac{x}{y}}\right)'$

$=\arctan\sqrt{\frac{x}{y}}+(y-1)\frac{1}{1+\frac{x}{y}}\sqrt{x}\cdot\left(-\frac{1}{2}\right)y^{-\frac{3}{2}}.$

$$=\arctan\sqrt{\frac{x}{y}}+(y-1)\frac{1}{1+\frac{x}{y}}\sqrt{x}\cdot(-\tfrac{1}{2})y$$

代入 $(5,1)$ 分别得到 $f_x(5,1)$，$f_y(5,1)$.

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(5,1)}=\left.\frac{d f(x,1)}{dx}\right|_{x=5}$$ "先代后导"

$$=\left.\frac{dx}{dx}\right|_{x=5}=1$$

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{(5,1)}=\left.\frac{d f(5,y)}{dy}\right|_{y=1}$$ ✓

例 $$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

则 $$f_x=\begin{cases}\dfrac{y(y^2-x^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}\quad f_y=\begin{cases}\dfrac{x(x^2-y^2)}{(y^2+x^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

解：当 $(x,y)\neq(0,0)$

$$f_x=\frac{y(x^2+y^2)-(xy)\cdot 2x}{(x^2+y^2)^2}=\frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

当 $(x,y)=(0,0)$

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\frac{\Delta x\cdot 0}{(\Delta x)^2+0^2}-0}{\Delta x}=0$$

$$\lim\frac{y(y^2-x^2)}{\ \ }=\lim_{(x,y)\to(0,0)}\frac{y^3}{y^4}$$ 不存在

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2} = \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{y^3}{y^4} \text{ 不存在}$$

$$\lim_{(x,y)\to(0,0)} \frac{y(y^2-x^2)}{(x^2+y^2)^2} \neq 0 = f_x(0,0)$$

$\Rightarrow f_x(x,y)$ 在(0,0)不连续.

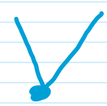
若 $f(x,y)=f(y,x)$, $f_x(x,y)=\varphi(x,y)$

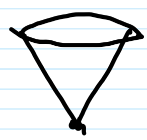
则 $f_y(x,y)=\varphi(y,x)$

在(0,0)处偏导存在,但不连续

一元函数极限　　　　重极限

一元函数; 可导 $\Rightarrow$ 连续.

连续 $\not\Rightarrow$ 可导. 例 $y=|x|$ 

例. $z=\sqrt{x^2+y^2}$. 在(0,0)连续但偏导不存在. 

解: $$f_x(0,0)=\lim_{\Delta x\to 0}\frac{\sqrt{(0+\Delta x)^2+0^2}-\sqrt{0^2+0^2}}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x} \text{ 不存在}$$

例. $$f(x,y)=\begin{cases}\dfrac{\sin(xy)-y}{xy^2}, & xy\neq 0\\ 0, & xy=0\end{cases}$$

则 $f_y(1,0)=$ $\underline{\quad -\frac{1}{6} \quad}$.

解: $$\lim \frac{\sin(1\cdot\Delta y)-\Delta y}{1\cdot\Delta y^2} - 0$$

解：$f_y(1,0) = \lim\limits_{\Delta y \to 0} \dfrac{\dfrac{\sin(1\cdot\Delta y) - \Delta y}{1\cdot \Delta y^2} - 0}{\Delta y}$

$= \lim\limits_{\Delta y \to 0} \dfrac{\sin\Delta y - \Delta y}{\Delta y^3} = -\dfrac{1}{6}.$

例 $u(x,y,z) = \int_{xz}^{yz} e^{t^2}\,dt$. 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

$u = \int_{xz}^{yz} xyz e^{t^2} dt.$

解：$\frac{\partial u}{\partial x} \xlongequal{y,z常数} -e^{(xz)^2}\cdot z.$

$\frac{\partial u}{\partial y} = e^{(yz)^2}\cdot z$

$\frac{\partial u}{\partial z} = e^{(yz)^2}\cdot y - e^{(xz)^2}\cdot x$

$\left[\int_{a(x)}^{b(x)} f(t)\,dt\right]' = f[b(x)]\cdot b'(x) - f[a(x)]\cdot a'(x).$

$\left[\int_{a(x)}^{b(x)} \underbrace{c(x)}_{常数}\cdot \underbrace{f(t)\,dt}_{积分变量}\right]' = \left[c(x)\cdot\int_{a(x)}^{b(x)} f(t)\,dt\right]'$

$= c'(x)\int_{a(x)}^{b(x)} f(t)\,dt + c(x)\cdot[\ \cdots\ ]$

例：$\frac{PV}{T} = C.$ (C常数)

$y'(x) = \frac{dy}{dx}$ 微商

则 $\frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p}=$ ___-1___.

解：$\frac{\partial p}{\partial V}=\frac{\partial}{\partial V}\left(\frac{CT}{V}\right)=-\frac{CT}{V^{\cancel{2}}}$

$\times$

$\frac{\partial V}{\partial T}=\frac{\partial}{\partial T}\left(\frac{CT}{p}\right)=\frac{\cancel{C}}{p}$

$\times$

$\frac{\partial T}{\partial p}=\frac{\partial}{\partial p}\left(\frac{pV}{C}\right)=\frac{V}{\cancel{C}}$

$=$

-1

例 $\frac{d(\sin x^2)}{d(x^2)}=\frac{\cos x^2\, d(x^2)}{d(x^2)}=\cos x^2$

二、偏导数（二元函数）的几何意义：$\begin{cases} z=f(x,y) \\ y=y_0 \end{cases}$ 在 $(x_0,y_0,f(x_0,y_0))$ 切线斜率

↓定义

导数：切线斜率

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_0,y_0)}=\left.\frac{d f(x,y_0)}{dx}\right|_{x=x_0}$$

空间曲线 $\begin{cases} z=f(x,y) \\ y=y_0. \end{cases}$