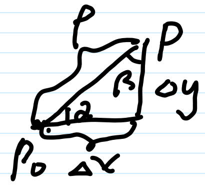
1. $\left.\dfrac{\partial f(x,y)}{\partial \vec{l}}\right|_{(x_0,y_0)} = \lim\limits_{\rho\to 0^+} \dfrac{f(x_0+\rho\cos\alpha,\ y_0+\rho\cos\beta) - f(x_0,y_0)}{\rho}$



其中$(\cos\alpha, \cos\beta) = \dfrac{1}{|\vec{l}|}\vec{l}$.

2. f 可微 $\Rightarrow$ 方向导数存在

$$\boxed{\left.\frac{\partial f}{\partial \vec{l}}\right|_{(x_0,y_0)} = f_x(x_0,y_0)\cdot\cos\alpha + f_y(x_0,y_0)\cdot\cos\beta.}$$ ✓

例. ① $u = x^2y + yz - z$. 在 $P_0(1,-1,0)$ 沿以 P_0 为起点以 $P_1(2,0,-1)$ 为终点向量的方向导数.

② u 在 P_0 沿哪个方向的方向导数最大. 且求值.

解: $u_x(1,-1,0) = 2xy\big|_{(1,-1,0)} = -2$ $\quad \nabla u(1,-1,0)$

$u_y(1,-1,0) = x^2+z\big|_{(1,-1,0)} = 1$ $\quad = (-2,1,-2)$

$u_z(1,-1,0) = y-1\big|_{(1,-1,0)} = -2$

② $\left.\dfrac{\partial u}{\partial \vec{l}}\right|_{max} = \sqrt{(-2)^2+1^2+(-2)^2} = 3.$

$\vec{l} = \overrightarrow{P_0P_1} = (1, 1, -1)$ $\quad \dfrac{1}{|\vec{l}|}\vec{l} = \dfrac{1}{\sqrt{3}}(1,1,-1).$

$\cos\alpha = \dfrac{\sqrt{3}}{3} \quad \cos\beta = \dfrac{\sqrt{3}}{3} \quad \cos\gamma = -\dfrac{\sqrt{3}}{3}$

$\left.\dfrac{\partial u}{\partial \vec{l}}\right|_{(1,-1,0)} = (-2)\times\dfrac{\sqrt{3}}{3} + 1\times\dfrac{\sqrt{3}}{3} + (-2)\times\left(-\dfrac{\sqrt{3}}{3}\right) = \dfrac{\sqrt{3}}{3}.$

$$\frac{\partial f}{\partial \vec{l}} = f_x\cdot\cos\alpha + f_y\cdot\cos\beta$$

定义: $\mathrm{grad} f$ $= (f_x, f_y)\cdot(\cos\alpha, \cos\beta)$

定义: grad $f = (f_x, f_y)\cdot(\cos\alpha, \cos\beta)$

$= \nabla f = \nabla f \cdot \vec{e_l}$

$= (f_x, f_y)$

$\vec{\alpha}\cdot\vec{\beta} = |\vec{\alpha}|\cdot|\vec{\beta}|\cdot\cos\theta.$

$$\boxed{\frac{\partial f}{\partial l} = |\nabla f|\cdot 1\cdot\cos\theta}$$

$\theta = \langle \nabla f, \vec{l} \rangle$

$\theta \in [0, \pi]$

(1) $\theta = 0$, $\cos\theta = 1$. 方向导数最大 $|\nabla f| = \sqrt{f_x^2 + f_y^2}$.

$\vec{l}$与梯度同向、方向导数沿梯度方向最大

函数沿梯度方向增加最快.

函数沿梯度其反方向变化最快:

(2) $\theta = \pi$. $\cos\theta = -1$. 方向导数最小 $-|\nabla f| = -\sqrt{f_x^2 + f_y^2}$

$\vec{l}$与梯度反向. 方向导数沿负梯度方向最小.

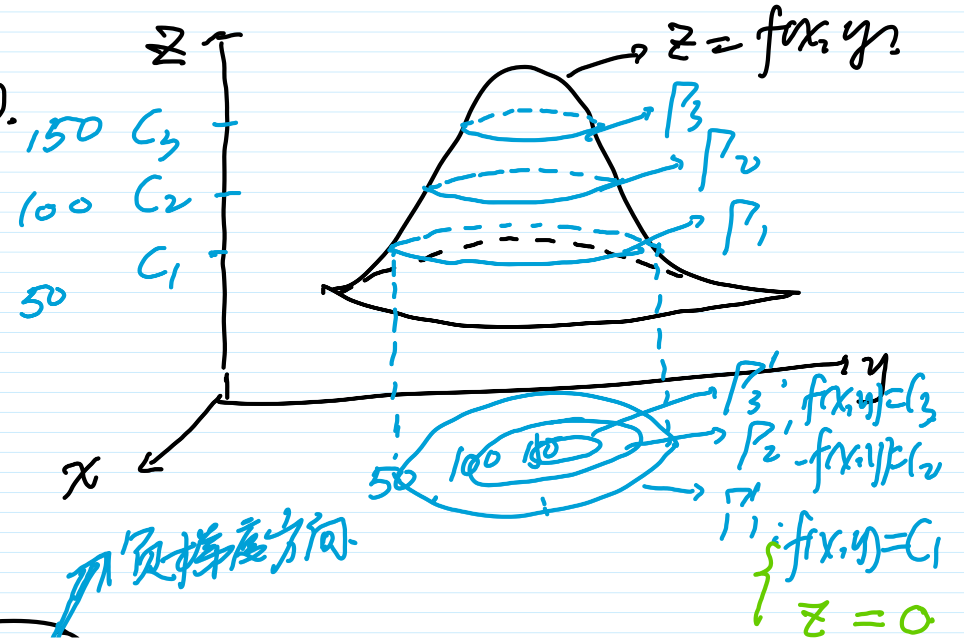
函数沿负梯度方向减少最快.

(3) $\theta = \frac{\pi}{2}$. $\cos\theta = 0$. 方向导数为0.

$\vec{l} \perp \nabla f$. 方向导数沿与梯度垂直为0

函数沿与梯度垂直的方向不变.

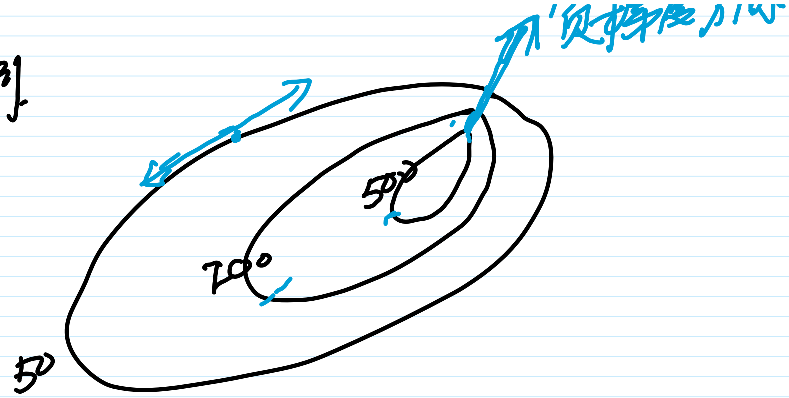
等值线: $z = f(x, y)$.



例

$$\begin{cases} f(x,y)=C \\ z=0 \end{cases}$$

例.



$\vec{l}=(1,2)$

例. $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 沿 x轴正半轴的方向导数最大，且为25，则 ________

解. ∇f 与 x轴正半轴同方向. $\vec{i}=(1,0)$.

设 $\nabla f = a\vec{i} = a(1,0)$ 且 $a>0$.

$\nabla f = a(1,2) = (a,2a)$, $a>0$.

$\nabla f = (25,0) = (f_x(x_0,y_0), f_y(x_0,y_0))$

$|\nabla f| = \nabla f = \sqrt{a^2+(2a)^2} = a\cdot\sqrt{5}$.

例. $\vec{l}=(0,2,2)$ $\quad \nabla f\cdot\vec{l}=1$.

则 $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0,y_0)} = \dfrac{1}{\sqrt{0^2+2^2+2^2}}\cdot 1 = \dfrac{1}{2\sqrt{2}}$.

解: $\left.\dfrac{\partial f}{\partial l}\right|_{(x_0,y_0)} = \nabla f\cdot\dfrac{1}{|\vec{l}|}\vec{l} = \dfrac{1}{|\vec{l}|}\nabla f\cdot\vec{l}$

第八节 极值.

- 极值
- 最值
- 条件极值 → 最值

、条件极值 —— [illegible]

一元函数 $y=f(x)$. 极值点 $\begin{cases} 驻点\ f'(x)=0. & 例: y=x^3,\ x=0 \\ 不可导点 & 例: y=|x|.\ x=0 \end{cases}$

$f'(x_0)=0$ $f''(x_0)\neq 0$. $\begin{cases} f''(x_0)>0 \Rightarrow 极小值点 & 例\ y=x^2 \\ f''(x_0)<0 \Rightarrow 极大值点 & 例\ y=-x^2. \end{cases}$

定理. $z=f(x,y)$. 在 $U(P_0)$ 各阶偏导连续.

泰勒公式. 令 $x-x_0=h$. $y-y_0=k$.

$$f(x,y) = f(x_0,y_0)+\frac{1}{1!}\left[f_x(x_0,y_0)\cdot(x-x_0)+f_y(x_0,y_0)(y-y_0)\right]$$
$$+\frac{1}{2!}\left[f_{xx}(x_0,y_0)\underbrace{(x-x_0)^2}_{h^2}\ 2f_{xy}(x_0,y_0)\underbrace{(x-x_0)(y-y_0)}_{hk}+f_{yy}(x_0,y_0)\underbrace{(y-y_0)^2}_{k^2}\right]+R.$$

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$
$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R.$$

定义: $z=f(x,y)$. $U(P_0)$ 有定义. $P_0(x_0,y_0)$. $P(x,y)\in U(P_0)$

$\forall P\in U(P_0)$. 都有 $f(P)\geq f(P_0)$ (或 $f(P)\leq f(P_0)$).

称 $P_0(x_0,y_0)$ 为 极小(大)值点.

$f(P_0)$ 为 极小(大)值

例. $z=f(x,y)=x^2+y^2$. $(0,0)$ 是极小值点.

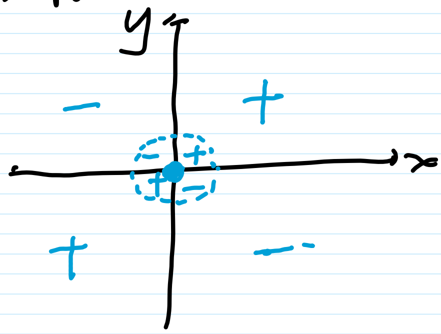
例. $z=f(x,y)=x^2+y^2$. $(0,0)$ 是极小值点.

例. $z=f(x,y)=xy$. $(0,0)$ 是否极值点.

$(0,0)$ 驻点

解: $f(0,0)=0$

但 $(0,0)$ 不是极值点.

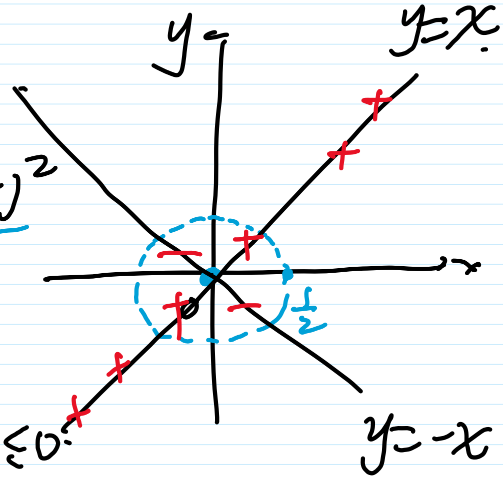


例. $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-xy}{(x^2+y^2)^2}=1$. 且 $f(x,y)$ 连续.

判断 $(0,0)$ <u>不是</u> 极值点.

解: $f(0,0)=0$

①法: 特值法: 设 $f(x,y)=\boxed{xy}+(x^2+y^2)^2$



$y=x$ $f(x,x)=x^2+4x^4\geq 0$

$y=-x$ $f(x,-x)=-x^2+4x^4=x^2(4x^2-1)\leq 0$

②法: $f(x,y)=xy+\boxed{\underset{\triangle}{(x^2+y^2)^2}+o\left[(x^2+y^2)^2\right]}$

$\lim\limits_{x\to x_0}f(x)=A$

$f(x)=A+\alpha$.

定理1 极值必要条件. 可微的极值点必为驻点.

$z=f(x,y)$ 可微且取得极值

$\Rightarrow \nabla f=(f_x,f_y)=\vec{0}$

定义：驻点. $\nabla f = \vec{0}$.

证明：$\frac{\partial f(x_0, y_0)}{\partial x} = \frac{d f(x, y_0)}{dx}\Big|_{x=x_0} = 0$

记 $g(x) = f(x, y_0)$　　$\forall (x,y) \in U(P_0)$

不妨设 (x_0, y_0) 是极小值点　　$f(x,y) \geq f(x_0, y_0)$

$f(x, y_0) \geq f(x_0, y_0)$

即　$g(x) \geq g(x_0)$

$x = x_0$ 是 $g(x)$ 可导的极值点 $\Rightarrow g'(x_0) = 0$

定理. 极值的充分条件.

$\nabla f(x_0, y_0) = \vec{0}$.　　二阶偏导连续

$A = f_{xx}(x_0, y_0)$　　$B = f_{xy}(x_0, y_0)$.　　$C = f_{yy}(x_0, y_0)$

则有

(i) $AC - B^2 > 0$. 取得极值 $\begin{cases} A > 0 \Rightarrow \text{极小值点} & z = x^2 + y^2. \\ A < 0 \Rightarrow \text{极大值点} & z = -x^2 - y^2. \end{cases}$

(ii) $AC - B^2 < 0$　不是极值点　例: $f(x,y) = xy$　$(0,0)$

(iii) $AC - B^2 = 0$　无法判断.

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \frac{1}{2!}(A \cdot h^2 + 2B h \cdot k + C \cdot k^2) + R$$

$$A h^2 + 2B h \cdot k + C \cdot k^2 = (h, k) \underbrace{\begin{pmatrix} A & B \\ B & C \end{pmatrix}}_{\text{黑塞矩阵}} \begin{pmatrix} h \\ k \end{pmatrix}$$

$A > 0$. $AC - B^2 > 0$　　$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ 正定矩阵.

$A>0$、$AC-B^2>0$ $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ 正定矩阵.

$$\Rightarrow Ah^2+2Bh\cdot k+C\cdot k^2\geq 0$$

$$f(x_0+h,y_0+k)-f(x_0,y_0)=\frac{1}{2!}(\qquad\qquad)\geq 0.$$

$\Rightarrow (x_0,y_0)$ 极小值点

$A<0$、$AC-B^2>0$ $-\begin{pmatrix} A & B \\ B & C \end{pmatrix}=\begin{pmatrix} -A & -B \\ -B & -C \end{pmatrix}$ 正定矩阵.

$$\Rightarrow -(Ah^2+2Bh\cdot k+Ck^2)\geq 0$$

即. $Ah^2+2Bh\cdot k+C\cdot k^2\leq 0$

$f(x_0+h,y_0+k)-f(x_0,y_0)\leq 0 \Rightarrow (x_0,y_0)$ 极大值点