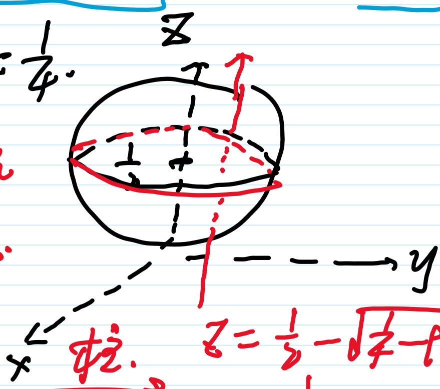
1. 柱面坐标. $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$ $dv = \rho\, d\rho\, d\theta\, dz.$

2. 球面坐标 $\begin{cases} x = r\sin\varphi\cdot\cos\theta \\ y = r\sin\varphi\cdot\sin\theta \\ z = r\cos\varphi \end{cases}$ $dv = r^2\sin\varphi\, d\varphi\, d\theta\, dr.$

例 $I = \iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\, dv.$ Ω: $x^2+y^2+z^2 \le z$ $x^2+y^2 \le z - z^2$

$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}.$



解: ①法: 投影法.

D_{xy}: $x^2+y^2 \le \frac{1}{4}.$ $\begin{cases} \theta: 0 \to 2\pi \\ \rho: 0 \to \frac{1}{2} \end{cases}$

射线: $z = \frac{1}{2} - \sqrt{\frac{1}{4}-x^2-y^2}$ $\downarrow$ $z = \frac{1}{2} + \sqrt{\frac{1}{4}-x^2-y^2}$

$z = \frac{1}{2} - \sqrt{\frac{1}{4}-\rho^2}$ $\downarrow$ $z = \frac{1}{2} + \sqrt{\frac{1}{4}-\rho^2}.$

$$I = \iint\limits_{D_{xy}} dxdy \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2-y^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz.$$

②法: 截面法 $I = \int_0^1 dz \iint\limits_{D_z} \sqrt{x^2+y^2+z^2}\, dxdy$

其中: D_z: $x^2+y^2 \le z - z^2$

③法: 柱面坐标 $\begin{cases} \theta: \\ \rho: \\ z: \end{cases}$

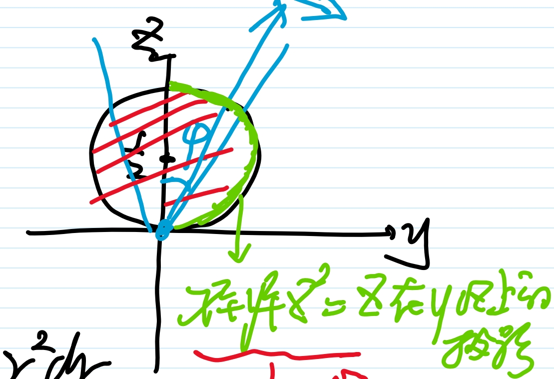
$$I = \int^{2\pi} d\theta \int^{\frac{1}{2}} \rho\, d\rho \int^{\frac{1}{2}+\sqrt{\frac{1}{4}-\rho^2}} \sqrt{\rho^2+z^2}\, dz$$

$$I=\int_0^{2\pi}d\theta\int_0^{\frac{1}{2}}\rho\, d\rho\int_{\frac{1}{2}-\sqrt{\frac{1}{4}-\rho^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-\rho^2}}\sqrt{\rho^2+z^2}\, dz$$

④法：球面坐标法.

Ω投影到xoy面：$D_{xy}: x^2+y^2\le\frac{1}{4}$.　　$\theta: 0\to 2\pi$

Ω投影到yoz面：$\begin{cases}\varphi: 0\to\frac{\pi}{2}\\ r: 0\to\cos\varphi\end{cases}$



$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\, d\varphi\int_0^{\cos\varphi} r\cdot r^2 dr$$

$=\cdots$

※、换元公式：

$$\begin{cases}x=x(u,v,w)\\ y=y(u,v,w)\\ z=z(u,v,w).\end{cases}$$

$$\iiint_{\Omega} f(x,y,z)\,dxdydz=\iiint_{\Omega'} f(x(u,v,w),\ y(u,v,w),\ z(u,v,w))\underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{\text{绝对值}}dudvdw$$

dv

例　$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ 围成的体积 $(a>0,\ b>0,\ c>0)$.　广义球面　$r=1$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\boxed{1-\frac{z^2}{c^2}}$$

解：①法：投影法.

$$V=\iiint_{\Omega}1\,dv=\iint_{D_{xy}}dxdy\int_{-\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,c}^{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,c}dz.=\cdots$$

广义极坐标

其中 $D_{xy}: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1$

$$\begin{cases}x=a\rho\cos\theta\\ y=b\rho\sin\theta\end{cases}$$

$d\sigma=ab\rho\, d\rho d\theta$

$$d\sigma = ab\rho\, d\rho\, d\theta$$

⑤法：截面法.

$$V=\int_{-c}^{c}dz\iint_{D_z}1\,dxdy=\int_{-c}^{c}1\cdot\sigma(z)\,dz.$$

$$=\int_{-c}^{c}1\cdot\pi ab\left(1-\frac{z^2}{c^2}\right)dz$$

⑥法：广义柱面坐标

$$\begin{cases}x=a\rho\cos\theta\\ y=b\rho\sin\theta\\ z=z\end{cases}\qquad dv=ab\rho\,d\rho\,d\theta\,dz$$

$$I=\int_0^{2\pi}d\theta\int_0^1 ab\rho\,d\rho\int_{-c\sqrt{1-\rho^2}}^{c\sqrt{1-\rho^2}}1\cdot dz$$

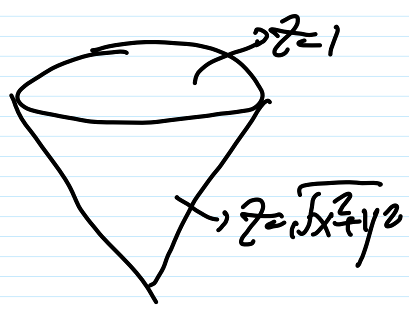
⑦法：广义球面坐标.

$$\begin{cases}x=a\,r\sin\varphi\cos\theta\\ y=b\,r\sin\varphi\sin\theta\\ z=c\,r\cos\varphi\end{cases}$$

$$dv=abc\cdot r^2\sin\varphi\,d\varphi\,d\theta\,dr.$$

$$I=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^1 1\cdot abc\,r^2\,dr$$

例. $I=\iiint_{\Omega}\left|\sqrt{x^2+y^2+z^2}-1\right|dv.$ Ω: $z=\sqrt{x^2+y^2}$ 与 $z=1$ 所围成



解1

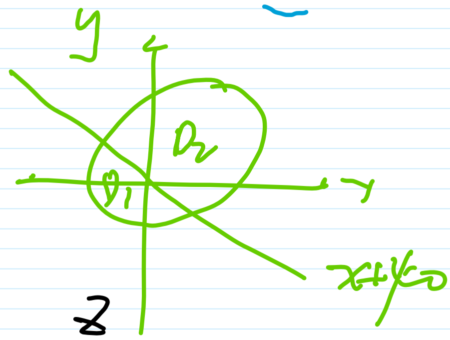
$$I=\iint_{D_{xy}}dxdy\int_{\sqrt{x^2+y^2}}^{1}\left|\sqrt{x^2+y^2+z^2}-1\right|dz$$

其中 D_{xy}: $x^2+y^2\le 1$

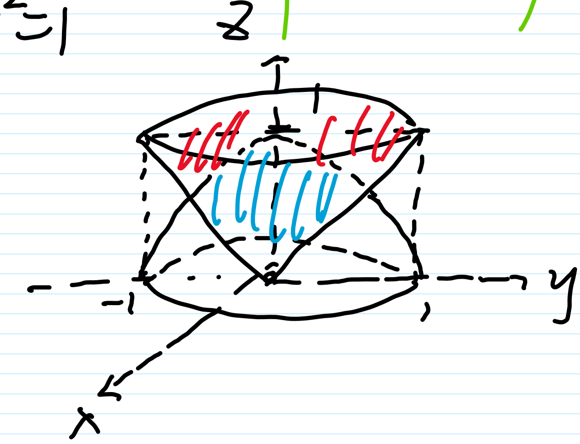
其中 $D_{xy}: x^2+y^2\le 1$.

$$I=\int_0^{2\pi}d\theta\int_0^1\rho\, d\rho\int_\rho^1 |\sqrt{\rho^2+z^2}-1|\,dz$$

$\iint_{D_1}|x+y|\,dxdy$ $\qquad x+y=0$.



$\sqrt{x^2+y^2+z^2}-1=0$ 即 $x^2+y^2+z^2=1$



$\Omega=\Omega_{蓝}+\Omega_{红}$

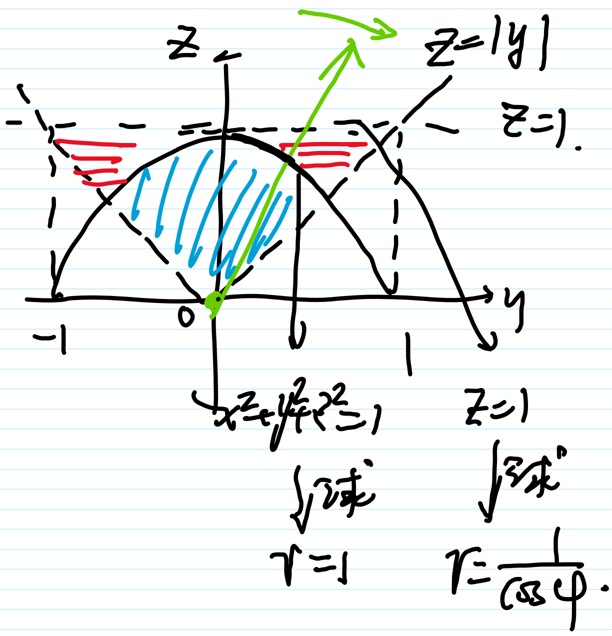
$\Omega_{蓝}: x^2+y^2+z^2\le 1$.

$\Omega_{红}: x^2+y^2+z^2\ge 1$

$$I=\iiint_{\Omega_{蓝}}(1-\sqrt{x^2+y^2+z^2})\,dv+\iiint_{\Omega_{红}}(\sqrt{x^2+y^2+z^2}-1)\,dv.$$

$\Omega_{蓝}: \begin{cases}\theta: 0\to 2\pi\\ \varphi: 0\to\frac{\pi}{4}\\ r: 0\to 1\end{cases}$

$\Omega_{红}: \begin{cases}\theta: 0\to 2\pi\\ \varphi: 0\to\frac{\pi}{4}\\ r: 1\to\frac{1}{\cos\varphi}\end{cases}$



$$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\, d\varphi\int_0^1(1-r)\,r^2dr$$

$$+\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\, d\varphi\int_1^{\frac{1}{\cos\varphi}}(r-1)\,r^2dr.$$

$$+\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi \int_1^{\frac{1}{\cos\varphi}} (r-1)\, r^2 dr.$$

第四节 几何应用

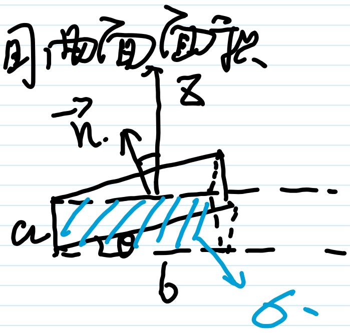
定积分：曲边梯形面积．面积．旋转体体积．体积．

二重积分：曲顶柱体体积．平面面积．曲面面积．

三重积分：空间几何体体积．

（质心形式） 形心公式．

一、空间曲面面积



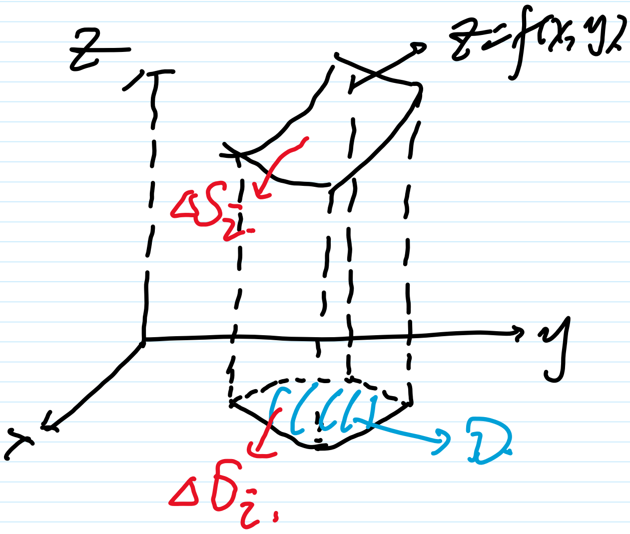
投影区域 $\sigma = a \times b$

$$S = a \times \frac{b}{\cos\theta} = \frac{\sigma}{\cos\theta}$$

$\vec{n} = (A, B, C)$. 方向余弦 $\left(\frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}}\right)$.

$$S = \frac{\sigma}{\left|\frac{C}{\sqrt{A^2+B^2+C^2}}\right|}$$

(i) 曲面 $z = f(x, y)$, $(x, y) \in D$



$$\Delta S_i \approx \frac{\Delta\sigma_i}{|\cos\theta|}$$

$$= \sqrt{f_x^2 + f_y^2 + 1^2}\, \Delta\sigma_i.$$

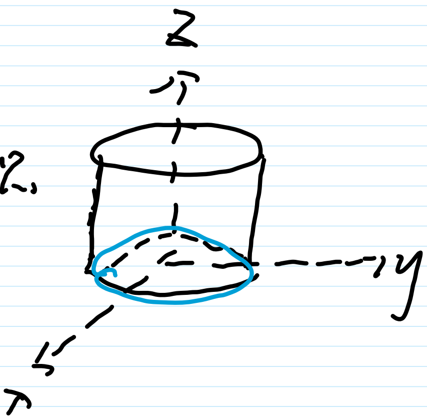
$\vec{n} = (\underset{A}{f_x}, \underset{B}{f_y}, \underset{C}{-1})$

$= \sqrt{f_x^2+f_y^2+1^2}\,\Delta\sigma_i.$ $\quad \vec{n}=(\underset{A}{f_x},\ \underset{B}{f_y},\ \underset{C}{-1})$

$$\boxed{dS=\sqrt{1+f_x^2+f_y^2}\,d\sigma}$$

$$S=\iint_D\sqrt{1+f_x^2+f_y^2}\,d\sigma.$$

$x^2+y^2=1$ 被 $z=0$ 和 $z=1$ 所截下的面积.



(ii). 曲面 $x=x(y,z)$ $\quad (y,z)\in D$

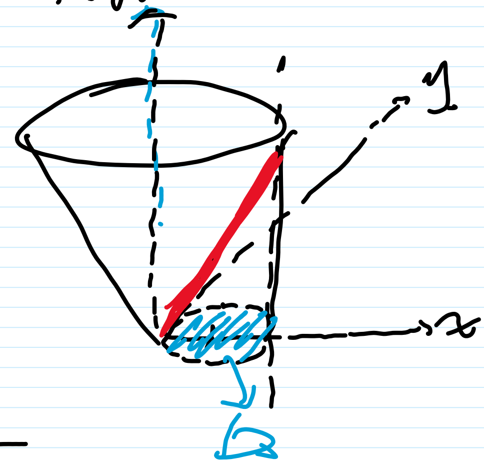
或 $y=y(z,x)$ $\quad (z,x)\in D$

$$S=\iint_D\sqrt{1+x_y^2+x_z^2}\,dy\,dz$$

或 $$S=\iint_D\sqrt{1+y_z^2+y_x^2}\,dz\,dx$$

例. $z=\sqrt{x^2+y^2}$、被 $x^2+y^2=x$ 截下的面积.

圆柱面



解:

$$z_x=\frac{x}{\sqrt{x^2+y^2}}$$

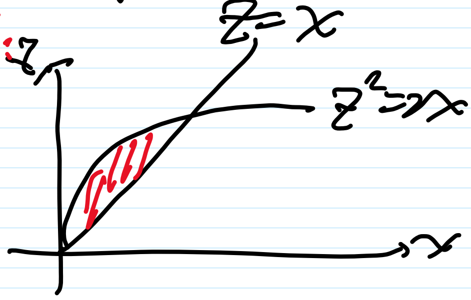
$$z_y=\frac{y}{\sqrt{x^2+y^2}}$$

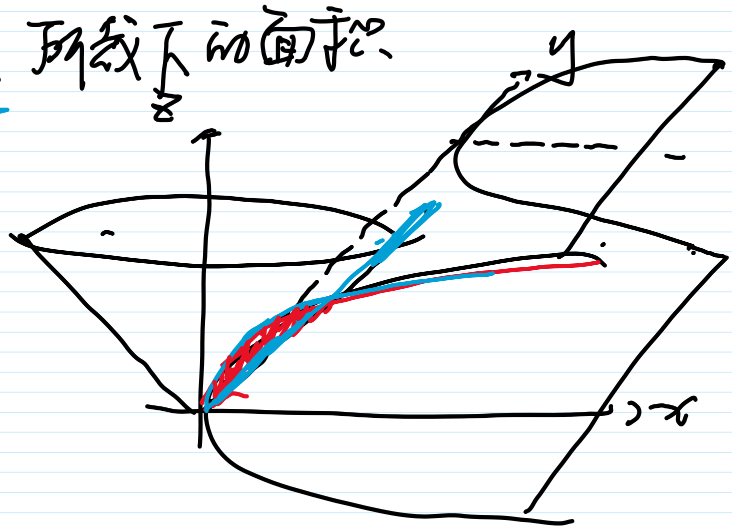
$$\sqrt{1+z_x^2+z_y^2}=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}=\sqrt{2}.$$

$$S=\iint_D\sqrt{2}\,dx\,dy=\sqrt{2}\iint_D 1\,dx\,dy=\sqrt{2}\cdot\pi\cdot\left(\frac12\right)^2=\frac{\sqrt{2}}{4}\pi$$

例 曲面 $z=\sqrt{x^2+y^2}$ 被 $z^2=2x$ 所截下的面积。

解：(1)法：





$$S=2S_1$$

$$y=\sqrt{z^2-x^2} \qquad \therefore S_1=\iint_D \sqrt{1+y_z^2+y_x^2}\,dz\,dx.$$

(2)法：

$$\sqrt{1+z_x^2+z_y^2}=\sqrt{2}.$$

交线 $\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases} \xrightarrow[\text{消去}z]{\text{在}xoy\text{面上}} \begin{cases} x^2+y^2=2x \\ z=0 \end{cases}$

D: $x^2+y^2\le 2x$

$$S=\iint_D \sqrt{2}\,dx\,dy=\sqrt{2}\cdot\pi\cdot 1^2=\sqrt{2}\pi.$$

二、质心公式（形心公式）

平面物体 $\rho=\mu(x,y)$ 的质心 $(\bar{x},\bar{y})$

$$\bar{x}=\frac{\iint_D x\cdot\mu(x,y)\,d\sigma}{m} \qquad \bar{y}=\frac{\iint_D y\,\mu(x,y)\,d\sigma}{m}$$

$$m=\iint_D \mu(x,y)\,d\sigma.$$

空间物体 $\rho=\mu(x,y,z)$ 的质心 $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{z}=\frac{\iiint\limits_{\Omega} z\cdot\mu(x,y,z)\,dv}{m}.$$

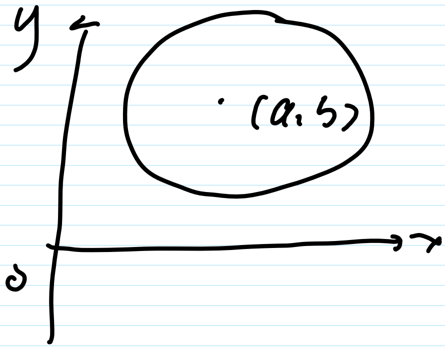
密度均匀的物体的质心，即为形心 即 $\rho=$常数.

形心公式:

平面图形 $$\bar{x}=\frac{\iint\limits_{D} x\,d\sigma}{\sigma}.$$

空间几何体 $$\bar{z}=\frac{\iiint\limits_{\Omega} z\,dv}{V}.$$

例. 密度均匀的圆片的质心 (圆形的形心).



解: $D:\ (x-a)^2+(y-b)^2\le R^2.$

质心 $(\bar{x}, \bar{y})$

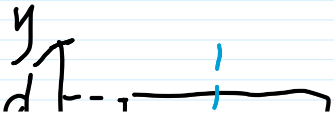
$$\bar{x}=\frac{\iint\limits_{D} x\cdot\mu\,d\sigma}{\mu\cdot\sigma}=\frac{\iint\limits_{D} x\,d\sigma}{\sigma}$$

$$\begin{cases} x=a+\rho\cos\theta \\ y=b+\rho\sin\theta \end{cases}$$

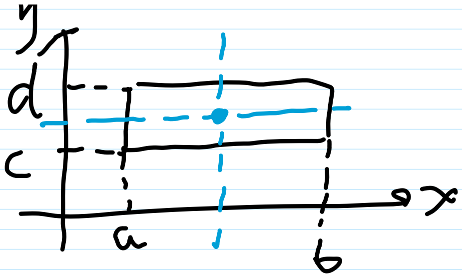
$$=\frac{1}{\pi R^2}\int_0^{2\pi}d\theta\int_0^R[a+\rho\cos\theta]\,\rho\,d\rho.$$

$d\sigma=\rho\,d\rho\,d\theta$

$$=\frac{1}{\pi R^2}\int_0^{2\pi}d\theta\int_0^R a\rho\,d\rho=\frac{1}{\pi R^2}\cdot 2\pi\cdot a\cdot\frac{R^2}{2}=a.$$

例. 

矩形的形心 $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$

例. 

矩形的形心 $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$

平面图形的形心必在对称轴(若有)
空间几何体的形心必在对称面(若有)

例. $I=\iint\limits_D (5x+3y)\,dxdy$. $D: x^2+y^2+2x-4y\le 4$.

$(x+1)^2+(y-2)^2\le 3^2$.

形心 $(-1, 2)$

解: $I = 5\iint\limits_D x\,dxdy + 3\iint\limits_D y\,dxdy$

$=5\cdot\bar{x}\cdot\sigma + 3\cdot\bar{y}\cdot\sigma$.

$=[5\cdot(-1)+3\cdot 2]\cdot\pi\cdot 3^2 = 9\pi$.

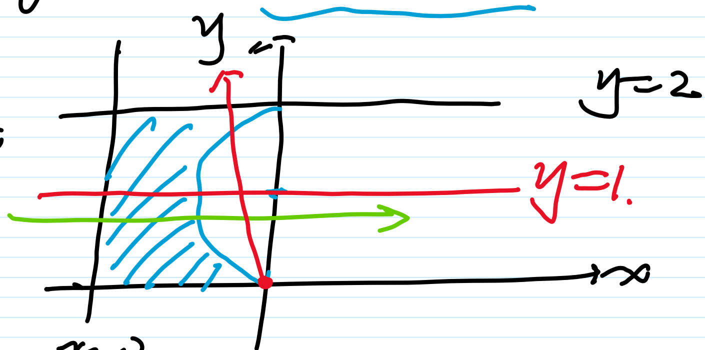
$I=\iint\limits_D x\,dxdy = \bar{x}\cdot\sigma$ 不能使用形心公式($\bar{x}$未知)

例. $I=\iint\limits_D y\,dxdy = \bar{y}\cdot\sigma$.

$x^2=2y-y^2 \xrightarrow{\text{极}} \rho = 2\sin\theta$

D: $x=-2$. $y=0$. $y=2$ 与 $x=-\sqrt{2y-y^2}$ 所围成

解: ①法 $I=\int_0^2 dy\int_{-2}^{-\sqrt{2y-y^2}} y\,dx$



②法: 形心公式

形心 $(\bar{x}, \bar{y})$ 在 $y=1$ 上. $x=-2$.

即 $\bar{y}=1$.

$2 \quad \frac{1}{2}\pi\cdot 1^2 \quad \pi$.

即 $y=1$,

$$I = \bar{y}\cdot 6 = 1\cdot 6 = 2^2 - \frac{1}{2}\pi\cdot 1^2 = 4 - \frac{\pi}{2}.$$

③法: 可加性. $D = D_{方} - D_{半圆}$

$$I = \iint\limits_{D_{方}} y\,dxdy - \iint\limits_{D_{半圆}} y\,dxdy$$

$$= \int_{-2}^{0} dx \int_{0}^{2} y\,dy - \int_{\frac{\pi}{2}}^{\pi} d\theta \int_{0}^{2\sin\theta} \rho\sin\theta\cdot\rho\,d\rho$$

第十章 曲线积分与曲面积分

曲线积分 $\left\{\begin{array}{l}\text{对弧长的} \cdots\cdots \ \text{I型} \\ \text{对坐标的} \cdots\cdots \ \text{II型}\end{array}\right.$

↓ 定积分

曲面积分 $\left\{\begin{array}{l}\text{对面积的} \cdots\cdots \ \text{I型} \\ \text{对坐标的} \cdots\cdots \ \text{II型}\end{array}\right.$

↓ 二重积分

I型: 无向

II型: 有向

复杂 简单

格林公式. II型曲线积分 ⟶ 二重积分

高斯公式. II型曲面积分 ⟶ 三重积分

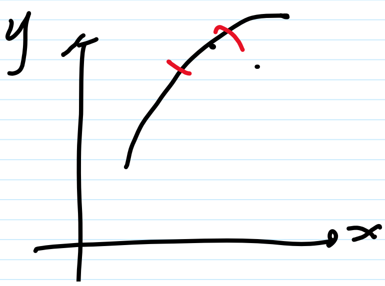
* 斯托克斯公式: II型曲线积分 ⟷ II型曲面积分

第一节 I型曲线积分

空间 $\rho = f(x,y,z)$

引例: 平面曲线形状的物体. $\rho = f(x,y)$

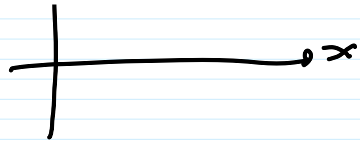
$f(\xi_i,\eta_i,\zeta_i)$



质量 $M = \lim \sum f(\xi_i,\eta_i)\cdot\Delta s_i$

$f(\xi_i,\eta_i,\zeta_i)$

质量 $m = \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i,\eta_i)\cdot\Delta s_i$



$\Delta m_i \approx f(\xi_i,\eta_i)\cdot\Delta s_i$ 弧长