

1.  $z = f(x, y)$

$$f_{xx}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right), f''_{11}$$

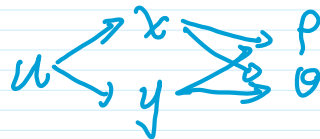
例. 证明:  $\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \left( \frac{\partial u}{\partial \rho} \right)^2 + \frac{1}{\rho^2} \left( \frac{\partial u}{\partial \theta} \right)^2$

$$u = u(x, y) \checkmark$$

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$$

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$$

$$u = u(\rho \cos \theta, \rho \sin \theta) \checkmark$$



$$\frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \theta}$$

$$u''_{11} = \frac{\partial^2 u(s, t)}{\partial s^2}$$

$$u''_{22} = \frac{\partial^2 u(s, t)}{\partial t^2}$$

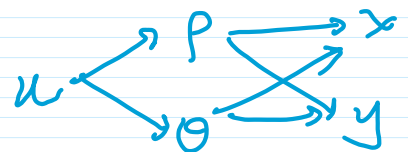
证明:  $\frac{\partial u}{\partial \rho} = u'_1 \cdot \cos \theta + u'_2 \cdot \sin \theta$

$$\frac{\partial u}{\partial \theta} = u'_1 \cdot \rho \cdot (-\sin \theta) + u'_2 \cdot \rho \cdot \cos \theta$$

$$\text{右} = \left( \frac{\partial u}{\partial \rho} \right)^2 + \left( \frac{\partial u}{\partial \theta} \right)^2 \frac{1}{\rho^2} = u''_{11} + u''_{22} = \text{左}$$

今知  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \theta^2}$

解:  $u = u(\rho \cos \theta, \rho \sin \theta)$



$$u = u(\rho(x, y), \theta(x, y))$$

$$\frac{\partial u}{\partial \rho} = u'_1 \cos \theta + u'_2 \sin \theta$$

$$\frac{\partial u}{\partial \rho} = u_1' \cos \theta + u_2' \sin \theta$$

$$\frac{\partial u}{\partial \theta} = -u_1' \rho \sin \theta + u_2' \rho \cos \theta$$

$$u_i' = u_i'(\rho \cos \theta, \rho \sin \theta)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \rho^2} &= \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right) = \frac{\partial}{\partial \rho} (u_1' \cos \theta + u_2' \sin \theta) = \cos \theta \frac{\partial u_1'}{\partial \rho} + \sin \theta \frac{\partial u_2'}{\partial \rho} \\ &= \cos \theta (u_{11}'' \cos \theta + u_{12}'' \sin \theta) + \sin \theta (u_{21}'' \cos \theta + u_{22}'' \sin \theta) \end{aligned}$$

$$u_{12}'' = u_{21}''$$

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial}{\partial \theta} (-u_1' \rho \sin \theta + u_2' \rho \cos \theta) \\ &= \rho \left[ \frac{\partial u_2'}{\partial \theta} \cos \theta + u_2' (-\sin \theta) - \frac{\partial u_1'}{\partial \theta} \sin \theta - u_1' \cos \theta \right] \end{aligned}$$

$$\begin{aligned} &= \rho \cos \theta (-u_{21}'' \rho \sin \theta + u_{22}'' \rho \cos \theta) - \rho \sin \theta (-u_{11}'' \rho \sin \theta + u_{12}'' \rho \cos \theta) \\ &\quad - \rho (u_1' \cos \theta + u_2' \sin \theta) \end{aligned}$$

$$g \rightarrow W \begin{cases} x \\ y \end{cases}$$

$$\text{例. } z = f(xy, x^2 + y^2) + g(xy^2), \quad \text{求 } \frac{\partial^2 z}{\partial x^2}$$

$$\text{解: } \frac{\partial z}{\partial x} = f_1' \cdot y + f_2' \cdot 2x + g' \cdot y^2$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (f_1' y + f_2' \cdot 2x + g' \cdot y^2)$$

$$= y \frac{\partial f_1'}{\partial x} + 2 \left( \frac{\partial f_2'}{\partial x} \cdot x + f_2' \cdot 1 \right) + y^2 \frac{\partial g'}{\partial x}$$

$$= y (f_{11}'' \cdot y + f_{12}'' \cdot 2x) + 2x (f_{21}'' y + f_{22}'' \cdot 2x) + y^2 (g_{11}'' y + g_{12}'' \cdot 2x)$$

$$= y(f_{11}'' y + f_{12}'' 2x) + 2x(f_{21}'' y + f_{22}'' 2x)$$

$$+ 2f_2' + y^2 (g'' y^2)$$

特值法:  $f(x, y) = ax + by - a - b + 1$

例  $z = f(x, y)$   $f(1, 1) = 1$   $f_x(1, 1) = a$   $f_y(1, 1) = b$

$$\varphi(x) = f\left[x, f\left\{x, f\left[x, f(x, x)\right]\right\}\right] \quad \text{求 } \varphi'(1) = \underline{\hspace{2cm}}$$

解:  $\varphi'(x) = f_1' \cdot 1 + f_2' \cdot \frac{d}{dx} f[x, f(x, x)]$

$$= f_1' + f_2' \cdot \left( f_1' \cdot 1 + f_2' \cdot \frac{d f(x, x)}{dx} \right)$$

$$= f_1' + f_2' \cdot \left( f_1' + f_2' (f_1' + f_2') \right)$$

故  $\varphi'(1) = a + b [a + b(a + b)]$

$$f_1'[x, f(x, f(x, x))]$$

$$\downarrow x=1$$

$$f_1'[1, f(1, f(1, 1))] = f_1'(1, 1) = f_1'(1, 1)$$

$$f_1'(x, x)$$

$$\downarrow x=1$$

例.  $u = u(x, y)$  = 二阶偏导数  $u_{xy} = u_{yx}$

$$u_{xx} = u_{yy}, \quad u(x, 2x) = x, \quad u_x(x, 2x) = x^2$$

求  $u_{xx}(x, 2x), \quad u_{xy}(x, 2x)$

例.  $u(x, y) = -x + y$

解.

$u(x, 2x) = x$

解:

$$u(x, y) |_{y=2x} = x$$

$$u(x, 2x) = x$$

$$u_x(x, y) |_{y=2x} = x^2 = \frac{\partial u(x, y)}{\partial x} |_{(x, 2x)}$$

$$\text{设 } u(x, y) = \frac{1}{3}x^3 + y$$

对  $u(x, 2x) = x$  两边关于  $x$  求导

$$u'_1 \cdot 1 + u'_2 \cdot 2 = 1 \quad \checkmark$$

$$\underbrace{u'_1(x, 2x)}_{u_x(x, 2x)} + 2 \underbrace{u'_2(x, 2x)}_{u_y(x, 2x)} = 1$$

对  $u'_2 = \frac{-x^2}{2}$  两边关于  $x$  求导

$$\underbrace{u''_{21} \cdot 1}_{u_{yx}(x, 2x)} + \underbrace{u''_{22} \cdot 2}_{u_{yy}(x, 2x)} = -x \quad u''_{12} + 2u''_{11} = -x \quad \textcircled{1}$$

$$u_{yx}(x, 2x) = u_{xy}(x, 2x) = u_{xx}(x, 2x)$$

对  $u_x(x, 2x) = x^2$  两边求导

$$\underbrace{u''_{11} \cdot 1 + u''_{12} \cdot 2}_{u''_{11} + 2u''_{12}} = 2x \quad \textcircled{2}$$

故得  $u''_{11} = u_{xx}(x, 2x) \quad u''_{12} = u_{xy}(x, 2x)$

2. 一阶全微分形式不变性 基于复合函数求导法则

一元函数:

一元函数:

$$y = f(u), \quad u \text{ 自} \quad dy = f'(u) du$$

$$y = f(u), \quad u = \varphi(x), \quad u \text{ 中} \quad dy = [f(\varphi(x))]' dx \\ = f'(u) \varphi'(x) dx = f'(u) du$$

二元函数

$$z = f(u, v), \quad u, v \text{ 自} \quad dz = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv.$$

$$z = f(u, v), \quad u = \varphi(x, y), \quad v = \psi(x, y), \quad u, v \text{ 中} \\ dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \quad z = f[\varphi(x, y), \psi(x, y)]$$

$$= (f'_1 \cdot \frac{\partial \varphi}{\partial x} + f'_2 \cdot \frac{\partial \psi}{\partial x}) dx + (f'_1 \cdot \frac{\partial \varphi}{\partial y} + f'_2 \cdot \frac{\partial \psi}{\partial y}) dy$$

$$= f'_1 (\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy) + f'_2 (\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy) \\ = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv.$$

例.

$$z = \frac{\cos(xy)}{e^{2x+y}} \quad \text{求} \quad dz = \underline{\hspace{2cm}}$$

解.

$$dz = d \frac{\cos(xy)}{e^{2x+y}}$$

$$= \frac{e^{2x+y} d \cos(xy) - \cos(xy) d e^{2x+y}}{e^{2x+y} \cdot 2}$$

$$\frac{d \frac{u}{v}}{v^2} \\ = \frac{v du - u dv}{v^2} \\ (-\frac{u}{v})' = \frac{u'v - u \cdot v'}{v^2}$$

$$= \frac{e^{-\sin(x^2y)}}{(e^{2x+y})^2}$$

$$= \frac{e^{2x+y} \cdot (-\sin(x^2y)) d(x^2y) - \cos(x^2y) \cdot e^{2x+y} d(2x+y)}{(e^{2x+y})^2}$$

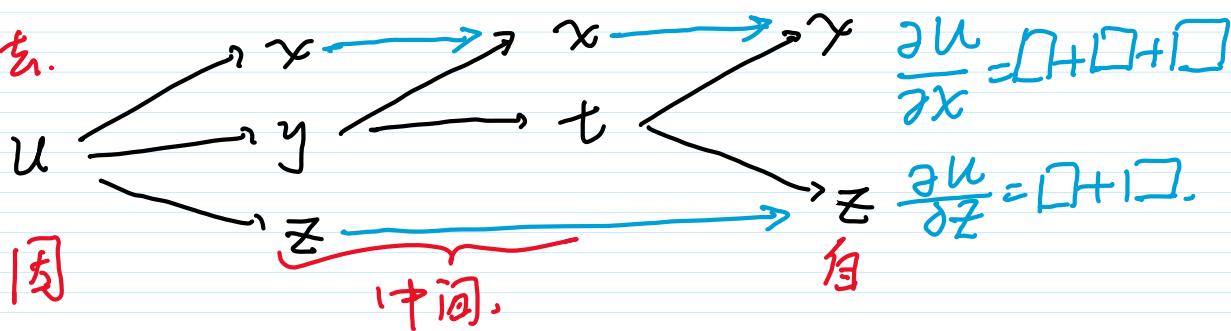
$$= \frac{-\sin(x^2y) [y \cdot 2x dx + x^2 dy] - \cos(x^2y) \cdot (2dx + dy)}{e^{2x+y}}$$

$$= \left[ \frac{\partial z}{\partial x} \right] dx + \left[ \frac{\partial z}{\partial y} \right] dy$$

例:  $u = f(x, y, z)$ ,  $y = \varphi(x, t)$ ,  $t = \psi(x, z)$ .

求:  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial z}$ ,  $du$ .

解: ①法.



$$u = f[x, \varphi[x, \psi[x, z]], z]$$

$$\frac{\partial u}{\partial x} = f'_1 \cdot 1 + f'_2 [\varphi'_1 \cdot 1 + \varphi'_2 (\psi'_1 + \psi'_2 \cdot 0)] + f'_3 \cdot 0$$

$$\frac{\partial u}{\partial z} = f'_1 \cdot 0 + f'_2 [\varphi'_1 \cdot 0 + \varphi'_2 (\psi'_1 \cdot 0 + \psi'_2 \cdot 1)] + f'_3 \cdot 1$$

②法: 微分法.

$$du = df(x, y, z) = f'_1 dx + f'_2 dy + f'_3 dz$$

$$du = df(x, y, z) = f'_1 dx + f'_2 dy + f'_3 dz$$

$$dy = d\varphi(x, t) = \varphi'_1 dx + \varphi'_2 dt$$

$$dt = d\psi(x, z) = \psi'_1 dx + \psi'_2 dz$$

$$du = \left( \frac{\partial u}{\partial x} \right) dx + \left( \frac{\partial u}{\partial z} \right) dz$$

## 第五节 隐函数求导

例:  $\sin y + y - e^x = 1$ . 确定  $y = y(x)$ .

则  $y' =$  \_\_\_\_\_

解: 两边关于  $x$  求导  $\cos y \cdot y' + y' - e^x = 0$

$$y' = \frac{e^x}{1 + \cos y}$$

定理: ①  $F(x, y) = 0$

② 在  $U(p_0)$  偏导连续.  $F(x_0, y_0) = 0$

③  $F_x(x_0, y_0) \neq 0$   $\Rightarrow$   $F_y(x, y) \neq 0$  (偏导连续)

则可确定一隐函数  $y = f(x)$ . 唯一连续可导.

$y_0 = f(x_0)$ . 且.

$$\frac{dy}{dx} = - \frac{F_x(x, y)}{F_y(x, y)}$$

公式法

$dx \rightarrow F_y$

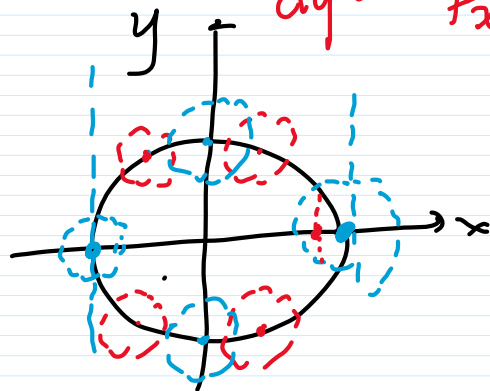
例.  $x^2 + y^2 - 1 = 0.$

$\Rightarrow y = \pm \sqrt{1-x^2}.$

$$\frac{dy}{dx} = \frac{d\sqrt{1-x^2}}{dx} = \frac{-x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{d(-\sqrt{1-x^2})}{dx} = \frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

$F(x, y) = x^2 + y^2 - 1. \quad F_x = 2x \quad F_y = 2y \quad \frac{dy}{dx} = -\frac{x}{y}$



直接法

对  $F(x, y) = 0$  隐函数不求导

$$F'_1 + F'_2 \cdot f'(x) = 0$$

$$f'(x) = \frac{dy}{dx} = -\frac{F'_1}{F'_2} = -\frac{F_x}{F_y}$$

定理:

①  $F(x, y, z) = 0$

②  $U(p_0)$  上偏导连续  $F(x_0, y_0, z_0) = 0$

③  $F_z(x_0, y_0, z_0) \neq 0$

则确定一邻域内  $z = f(x, y)$  且.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

公式法

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$



$$\overline{\partial x} \rightarrow F_z$$

对  $F(x, y, z(x, y)) = 0$  两边关于  $x, (y)$  求偏导

关于  $x$ :  $F'_1 \cdot 1 + F'_2 \cdot 0 + F'_3 \frac{\partial z}{\partial x} = 0 \Rightarrow \dots$

关于  $y$ :  $F'_1 \cdot 0 + F'_2 \cdot 1 + F'_3 \frac{\partial z}{\partial y} = 0$

例.  $x^3 y - e^z = z$  求  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$  用  $dz = \dots$   
 解: ① 公式法: 设  $F(x, y, z) = x^3 y - e^z - z = 0$  求偏导

$F_x = 3x^2 y, F_y = x^3, F_z = -e^z - 1.$

$$\frac{\partial z}{\partial x} = - \frac{3x^2 y}{-e^z - 1} = \frac{3x^2 y}{1 + e^z} \cdot \frac{\partial z}{\partial y} = \frac{x^3}{1 + e^z}$$

X  $\frac{\partial z}{\partial x} = - \frac{F_x}{F_z} = - \frac{3x^2 y - e^z \frac{\partial z}{\partial x} - \frac{\partial z}{\partial x}}{-e^z - 1} \Rightarrow \frac{\partial z}{\partial x}$

② 直接法: 对  $x^3 y - e^z = z$  关于  $x$  求偏导  
 是  $x, y$  函数

$$3x^2 y - e^z \cdot \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \dots$$

③ 微分法:

对  $x^3 y - e^z = z$  两边微分

$$1 \cdot 3x^2 y \cdot dx - e^z dz = dz$$

$$d(x^3y - e^z) = dz$$

$$d(x^3y) - de^z = dz$$

$$3x^2y dx + x^3 dy - e^z dz = dz$$

$$\text{得 } dz = \underbrace{\frac{3x^2y}{1+e^z}}_{\frac{\partial z}{\partial x}} dx + \underbrace{\frac{x^3}{1+e^z}}_{\frac{\partial z}{\partial y}} dy$$

$$\begin{aligned} dz|_{(1,1)} &= \frac{3 \times 1^2 \times 1}{1+e^0} dx + \frac{1^3}{1+e^0} dy \\ &= \frac{3}{2} dx + \frac{1}{2} dy \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{3x^2y}{1+e^z} \right) \quad \text{--- } x, y \text{ 不变} \\ &= 3x^2 \frac{1 \cdot (1+e^z) - y \cdot (0 + e^z \frac{\partial z}{\partial y})}{(1+e^z)^2} \end{aligned}$$

例  $z = z(x, y)$  由  $\varphi(x - az, y - bz) = 0$  确定隐函数

$$\text{则 } a \cdot \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \underline{\quad 1 \quad}$$

解: 公式法:  $f(x, y, z) = \varphi(x - az, y - bz)$

$$f_x = \varphi'_1 \cdot (1 - 0) + \varphi'_2 \cdot (0 - a) = \varphi'_1$$

$$f_y = \varphi'_1 \cdot (0 - b) + \varphi'_2 \cdot (1 - 0) = \varphi'_2$$

$$F_x = 1 - 0 = 1$$

$$F_y = \varphi_1' (0-0) + \varphi_2' (1-0) = \varphi_2'$$

$$F_z = \varphi_1' (0-a) + \varphi_2' (a-b) = -a\varphi_1' + b\varphi_2'$$

$$\text{则 } \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{\varphi_1'}{a\varphi_1' + b\varphi_2'} \quad \Rightarrow \quad a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{\varphi_2'}{a\varphi_1' + b\varphi_2'}$$

③ 微分法:

$$d\varphi(x-az, y-bz) = 0$$

$$\varphi_1' d(x-az) + \varphi_2' d(y-bz) = 0$$

$$\varphi_1' dx - a\varphi_1' dz + \varphi_2' dy - b\varphi_2' dz = 0$$

$$dz = \frac{\varphi_1'}{a\varphi_1' + b\varphi_2'} dx + \frac{\varphi_2'}{a\varphi_1' + b\varphi_2'} dy$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\varphi_1'}{a\varphi_1' + b\varphi_2'} \right) \quad \begin{matrix} \varphi_1' = \varphi_1'(x-az, y-bz) \\ \varphi_2' = \varphi_2'(x-az, y-bz) \end{matrix}$$

$$= \frac{\frac{\partial \varphi_1'}{\partial x} (a\varphi_1' + b\varphi_2') - \varphi_1' \frac{\partial}{\partial x} (a\varphi_1' + b\varphi_2')}{(a\varphi_1' + b\varphi_2')^2}$$

$$\frac{\partial \varphi_1'}{\partial x} = \varphi_{11}'' (1 - a \frac{\partial z}{\partial x}) + \varphi_{12}'' (0 - b \frac{\partial z}{\partial x})$$

$$\frac{\partial \varphi_2'}{\partial x} = \varphi_{21}'' (1 - a \frac{\partial z}{\partial x}) + \varphi_{22}'' (0 - b \frac{\partial z}{\partial x})$$

$$\frac{\partial \phi_2'}{\partial x} = \phi_{21}'' (1 - u \frac{\partial x}{\partial x}) + \phi_{22}'' (v - \frac{\partial x}{\partial x})$$

方程组

隐函数组

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \rightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

定理: ①  $F(x, y, u, v) = 0$   $G(x, y, u, v) = 0$

②  $U(p_0)$  邻域存在  $F(x_0, y_0, u_0, v_0) = 0$   
 $G(x_0, y_0, u_0, v_0) = 0$

③  $J = \frac{\partial(F, G)}{\partial(u, v)}|_{p_0} \neq 0.$

其中  $\frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$

则存在一隐函数组  $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$