

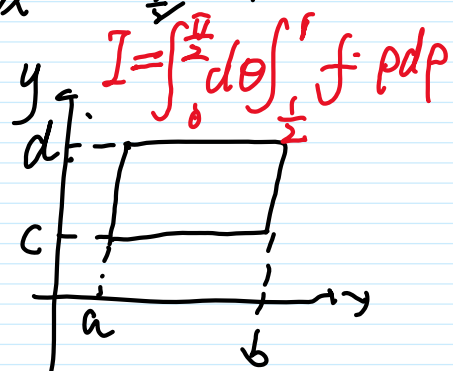
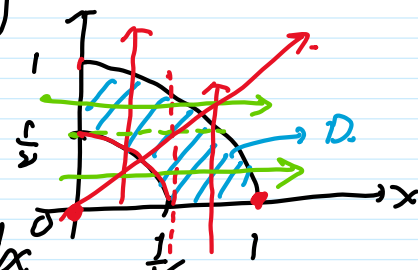
## 1. 直角坐标系下的计算

$$\iint_D f(x,y) dx dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) dx.$$

$$I = \iint_D f(x,y) dx dy \quad D: \frac{1}{4} \leq x^2 + y^2 \leq 1, \quad x \geq 0 \text{ 且 } y \geq 0.$$

$$I = \int_{\frac{1}{2}}^1 dx \int_{\sqrt{\frac{1}{4}-x^2}}^{\sqrt{1-x^2}} f(x,y) dy + \int_{\frac{1}{2}}^1 dx \int_0^{\sqrt{1-x^2}} f(x,y) dy$$

$$I = \int_0^{\frac{1}{2}} dy \int_{\sqrt{\frac{1}{4}-y^2}}^{\sqrt{1-y^2}} f(x,y) dx + \int_{\frac{1}{2}}^1 dy \int_0^{\sqrt{1-y^2}} f(x,y) dx$$



推广1:  $D: a \leq x \leq b, c \leq y \leq d.$

$$\iint_D f(x,y) dx dy = \int_a^b dx \int_c^d f(x,y) dy$$

$$= \int_c^d dy \int_a^b f(x,y) dx$$

推广2:  $D: a \leq x \leq b, c \leq y \leq d.$  且  $f(x,y) = g(x) \cdot h(y).$

$$\iint_D f(x,y) dx dy = \int_a^b dx \int_c^d f(x,y) dy$$

$$= \int_a^b \left[ \int_c^d \underline{g(x) \cdot h(y)} dy \right] dx$$

$$= \int_a^b \left[ g(x) \int_c^d \underline{h(y)} dy \right] dx.$$

$$= \left[ \int_a^b g(x) dx \right] \left[ \int_c^d h(y) dy \right] \quad \text{累次}$$

$$= \left[ \int_a^b g(x) dx \right] \left[ \int_a^b f(x) dx \right]$$

例:

$$\left[ \int_a^b f(x) \cdot g(x) dx \right]^2 \leq \left[ \int_a^b f^2(x) dx \right] \left[ \int_a^b g^2(x) dx \right]$$

$$(\vec{\alpha}, \vec{\beta})^2 \leq |\vec{\alpha}|^2 |\vec{\beta}|^2 \quad \checkmark$$

证明:

①法:

$$h(x) = f(x) + t \cdot g(x)$$

$$\Delta \leq 0$$

$$\int_a^b h(x) dx \geq 0$$

②法:

$$I_2 = \left[ \int_a^b f(x) g(x) dx \right] \left[ \int_a^b f(y) g(y) dy \right]$$

$$= \iint_D [f(x) g(x) \cdot f(y) g(y)] dx dy \quad \text{其中 } D: a \leq x \leq b, a \leq y \leq b$$

$$I_2 = \left[ \int_a^b f^2(x) dx \right] \left[ \int_a^b g^2(y) dy \right]$$

$$= \iint_D [f^2(x) - g^2(y)] dx dy$$

$$= \left[ \int_a^b f^2(y) dy \right] \left[ \int_a^b g^2(x) dx \right]$$

$$= \iint_D [f^2(y) \cdot g^2(x)] dx dy$$

$$= \iint_D \frac{f^2(x) g^2(y) + f^2(y) g^2(x)}{2} dx dy$$

D. 关于  $y=x$  对称

$\Downarrow$

$$\iint_D f(x) g(x) d\sigma$$

$$= \iint_D f(y) g(y) d\sigma$$

$$= \iint_D \frac{f(x, y) + f(y, x)}{2} d\sigma$$

$$\frac{f(x) g^2(y) + f^2(y) g(x)}{2} \geq f(x) g(x) f(y) g(y) \quad \therefore \text{右} \geq I_2$$

2. 极坐标系的计算

## 2. 极坐标系的计算

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$$

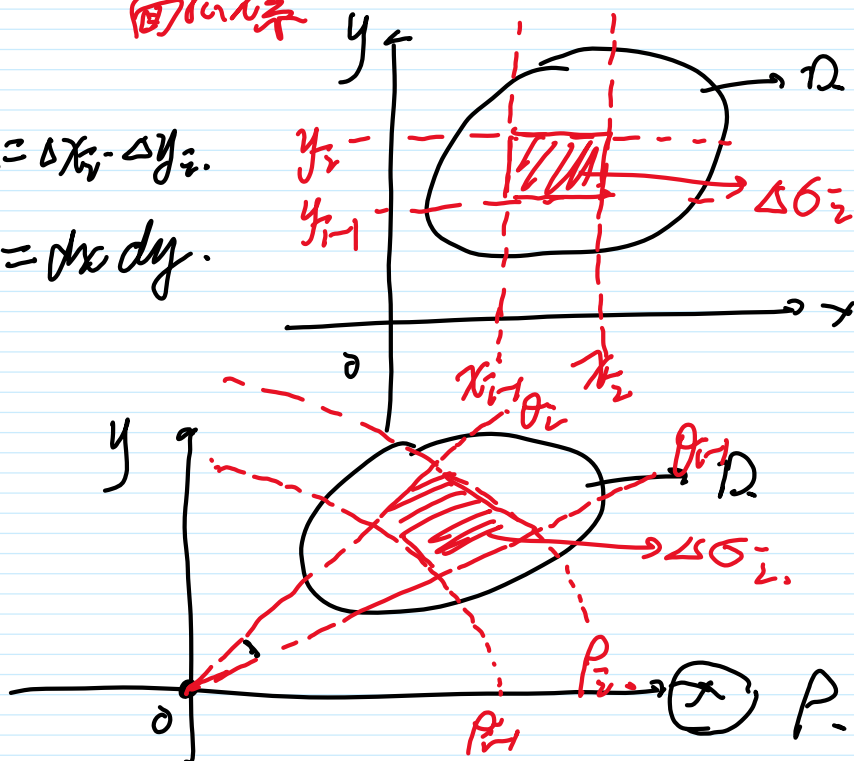
$$\iint_D f(x, y) d\sigma = \iint_D f(x, y) \underbrace{dx dy}_{\text{面积元素}} = \iint_D f(\rho \cos \theta, \rho \sin \theta) \underline{d\sigma}$$

$x = \text{常数}$   
 $y = \text{常数}$

$$\Delta \sigma_i = \Delta x_i \cdot \Delta y_i$$

$$d\sigma = dx dy$$

$\theta = \text{常数}$  射线  
 $\rho = \text{常数}$  圆



$$\begin{aligned} \Delta \sigma_i &= (\pi \rho_i^2 - \pi \rho_{i-1}^2) \cdot \frac{\theta_i - \theta_{i-1}}{2\pi} \\ &= \frac{(\rho_i + \rho_{i-1}) \cdot \Delta \rho_i}{2} \cdot \Delta \theta_i \\ &= \frac{(2\rho_{i-1} + \Delta \rho_i)}{2} \cdot \Delta \rho_i \cdot \Delta \theta_i \\ &= \rho_i \Delta \rho_i \cdot \Delta \theta_i + \frac{1}{2} (\Delta \rho_i)^2 \Delta \theta_i \\ \Delta \sigma_i &\approx \rho_i \Delta \rho_i \Delta \theta_i \end{aligned}$$

$$d\sigma = \boxed{p} dp d\theta.$$

$$\iint_D f(x, y) dx dy = \iint_D f(\rho \cos \theta, \rho \sin \theta) \cdot \boxed{p} dp d\theta$$

$$= \int d\theta \int f(\rho, \theta) p dp$$

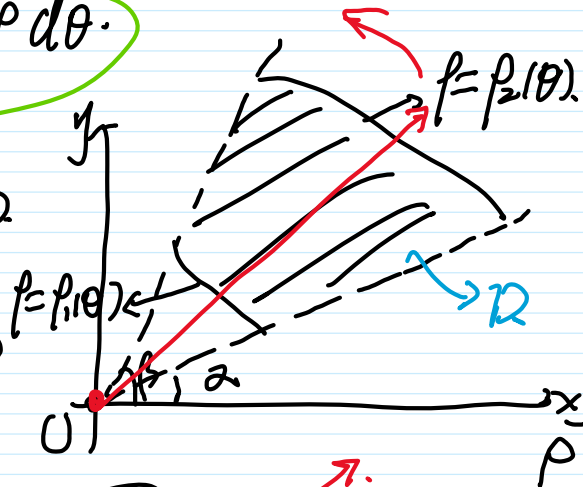
$$= \int dp \int f(\rho, \theta) p d\theta.$$

射线法:

(i)

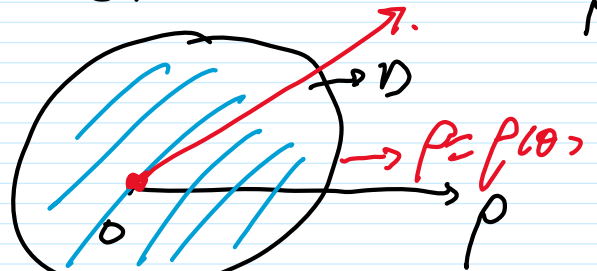
$$D: \alpha \leq \theta \leq \beta \quad \rho_1(\theta) \leq \rho \leq \rho_2(\theta)$$

$$I = \int_{\alpha}^{\beta} d\theta \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(\rho \cos \theta, \rho \sin \theta) p dp$$



(ii)

$$I = \int_0^{2\pi} d\theta \int_0^{\rho(\theta)} f(\rho \cos \theta, \rho \sin \theta) p dp$$



例.  $I = \iint_D 1 d\sigma:$

$$D': x^2 + y^2 \leq 2Ry.$$

$$D: x^2 + y^2 \leq 2Rx.$$

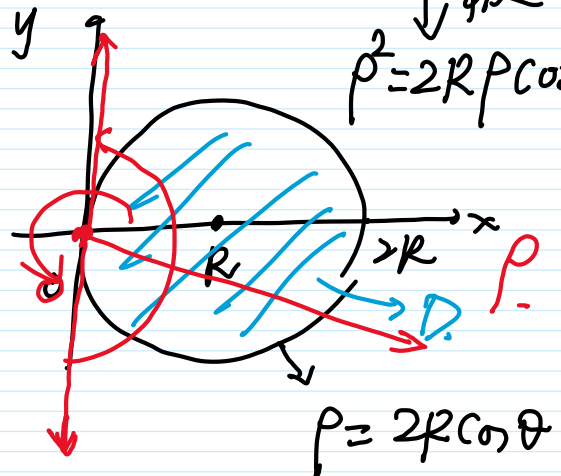
$$x^2 + y^2 = 2Rx$$

↓ 极

$$\rho^2 = 2R\rho \cos \theta$$

解法 1: 直线

$$I = \int_0^{2R} dx \int_{-\sqrt{2Rx-x^2}}^{\sqrt{2Rx-x^2}} dy$$



② 法:

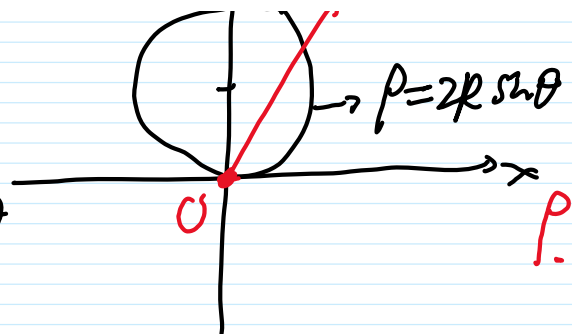
$$I = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2R \cos \theta} 1 p dp$$

$$I' = \iint_{D'} 1 d\sigma = \int_0^{\pi} d\theta \int_0^{2R \sin \theta} 1 p dp$$



$$I = \int_{b'}^{b''} \int_{a'}^{a''} f(x, y) dx dy = \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho d\theta$$

$$x^2 + y^2 = 2Ry \quad \text{极坐标} \quad \rho^2 = 2R\rho \sin \theta$$

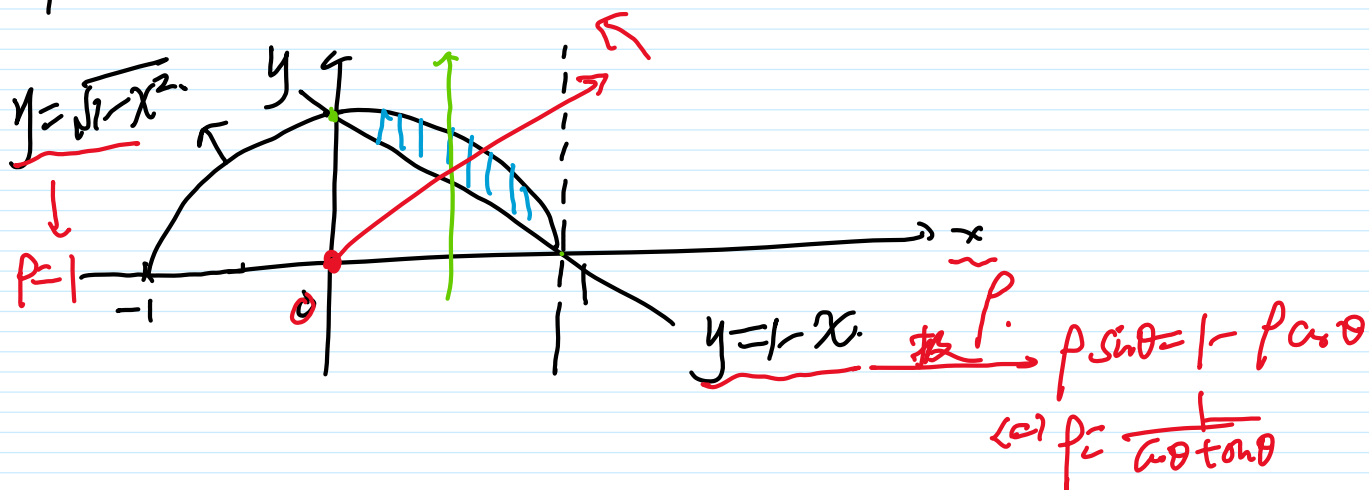


极坐标计算:

- ① D 与圆有关.
- ②  $f(x, y)$  含有  $x^2 + y^2$  或  $\frac{y}{x}$  形式  
 $\rho^2$   $\tan \theta$

$$\text{例: } \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f(x, y) dy = \int_0^{\frac{\pi}{2}} d\theta \int_{\cos \theta}^1 f(\rho \cos \theta, \rho \sin \theta) \rho d\rho \quad (\text{极坐标})$$

解: 确定 D:  $x=0, x=1, y=1-x, y=\sqrt{1-x^2}$



$$\text{例: } \int_0^1 dy \int_{-y}^{\sqrt{1-y^2}} f(x, y) dx$$

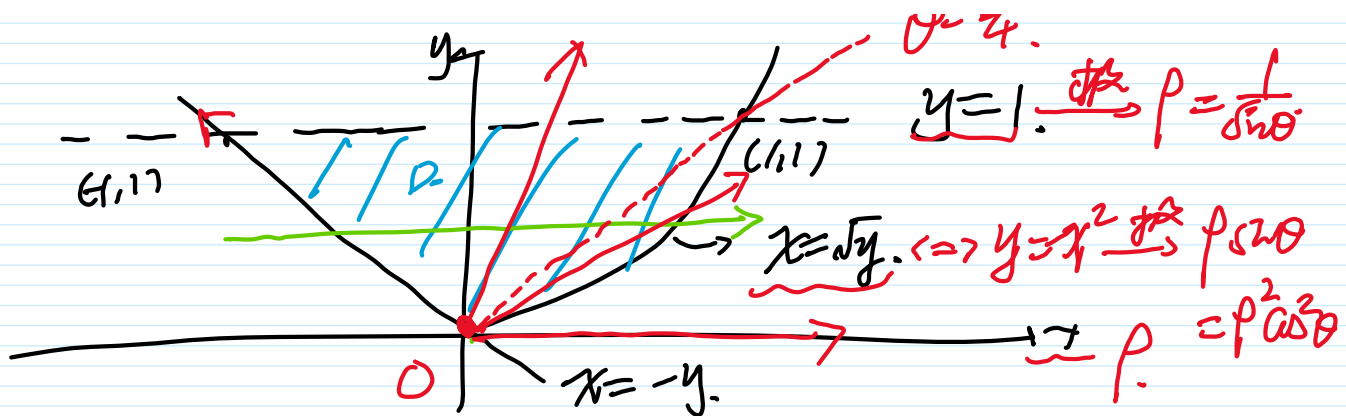
$$= \int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{\sin \theta}}^{\frac{1}{\cos \theta}} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\frac{1}{\sin \theta}} f(\rho \cos \theta, \rho \sin \theta) \rho d\rho \quad (\text{极坐标})$$

解:

$y \uparrow$

$\theta = \frac{\pi}{4}$   
 $y=1$  极坐标  $\rho = \frac{1}{\sin \theta}$

解:



例:  $I = \iint_D x^2 dx dy$ .  $D: x^2 + y^2 \leq 1$ .

解:  $D: 0 \leq \theta \leq 2\pi, 0 \leq \rho \leq 1$

$I = \int_0^{2\pi} d\theta \int_0^1 \rho^2 \cos^2 \theta \rho d\rho$

$= \left( \int_0^{2\pi} \cos^2 \theta d\theta \right) \left( \int_0^1 \rho^3 d\rho \right) = \dots$

$D$  关于  $y=x$  对称

$\frac{1}{2} \iint_D 1 dx dy = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2}$

则  $I = \iint_D y^2 dx dy = \frac{1}{2} \iint_D (x^2 + y^2) dx dy$

$= \frac{1}{2} \left( \int_0^{2\pi} d\theta \int_0^1 \rho^2 \cdot \rho d\rho \right)$

$= \frac{1}{2} \times 2\pi \times \frac{1}{4} = \frac{\pi}{4}$

例:  $I = \iint_D e^{-x^2-y^2} d\sigma$ .  $D: x^2 + y^2 \leq a^2$  ( $a > 0$ )

$D: x^2 + y^2 \leq a^2, x \geq 0$  且  $y \geq 0$

解:  $I = \left( \int_0^{\frac{\pi}{2}} d\theta \right) \int_0^a e^{-\rho^2} \rho d\rho$

$= \frac{\pi}{2} \cdot \left( -\frac{1}{2} \right) \Big|_0^a e^{-\rho^2} d(-\rho^2)$

$$= 2\pi i \left(-\frac{1}{2}\right) \int_0^a e^{-p^2} \underbrace{d(-p^2)}$$

$$= -\pi \cdot e^{-p^2} \Big|_0^a = \pi (1 - e^{-a^2}).$$

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$I_1 = \iint_{D_1} e^{-x^2-y^2} d\sigma = \frac{\pi}{4} (1 - e^{-a^2})$$

$$D_1 \subseteq D_2 \subseteq D_3$$

$$e^{-x^2-y^2} \geq 0.$$

$$\int_{D_1} e^{-x^2-y^2} d\sigma \leq \int_{D_2} e^{-x^2-y^2} d\sigma \leq \int_{D_3} e^{-x^2-y^2} d\sigma$$

$$\frac{\pi}{4}(1-e^{-a^2}) \leq I_2 \leq \frac{\pi}{4}(1-e^{-2a^2})$$

$\downarrow a \rightarrow +\infty$

$\frac{1}{4}$

$$\leq I_2 \leq$$

$\downarrow a \rightarrow +\infty$

$$I_4$$

$$\lim_{q \rightarrow +\infty} I_2 = \frac{I}{\phi}.$$

$$I_2 = \iint_{D_2} (e^{-x^2} \cdot e^{-y^2}) \, dx \, dy$$

$$= \underbrace{\left( \int_0^a e^{-x^2} dx \right)}_{\frac{\sqrt{\pi}}{2}} \underbrace{\left( \int_0^a e^{-y^2} dy \right)}_{\frac{\sqrt{\pi}}{2}}.$$

12: 05X5a  
05Y5a

$$= \left( \int_0^{\infty} e^{-x} dx \right) \left( \int_0^{\infty} e^{-y} dy \right).$$

$$I_2 = \left( \int_0^a e^{-x^2} dx \right)^2$$

$$\lim_{a \rightarrow +\infty} \int_0^a e^{-x^2} dx = \lim_{a \rightarrow +\infty} \sqrt{I_2} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

例. 求  $x^2 + y^2 = az$  与  $z = 2a - \sqrt{x^2 + y^2}$  所围成体积 ( $a > 0$ )

解,  $V = V_{\text{上曲}} - V_{\text{下曲}}.$

