

BayesLenseSPTpol

Abstract: At this point there is nothing in here but a few derivations.

Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

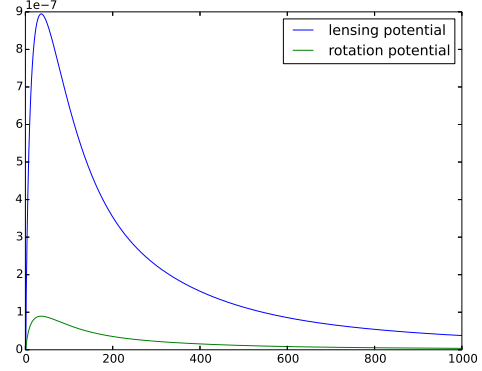
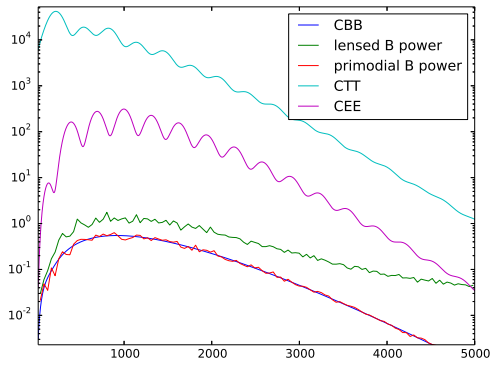
$$\tilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^\perp\psi(x))$$

$$\tilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^\perp\psi(x))$$

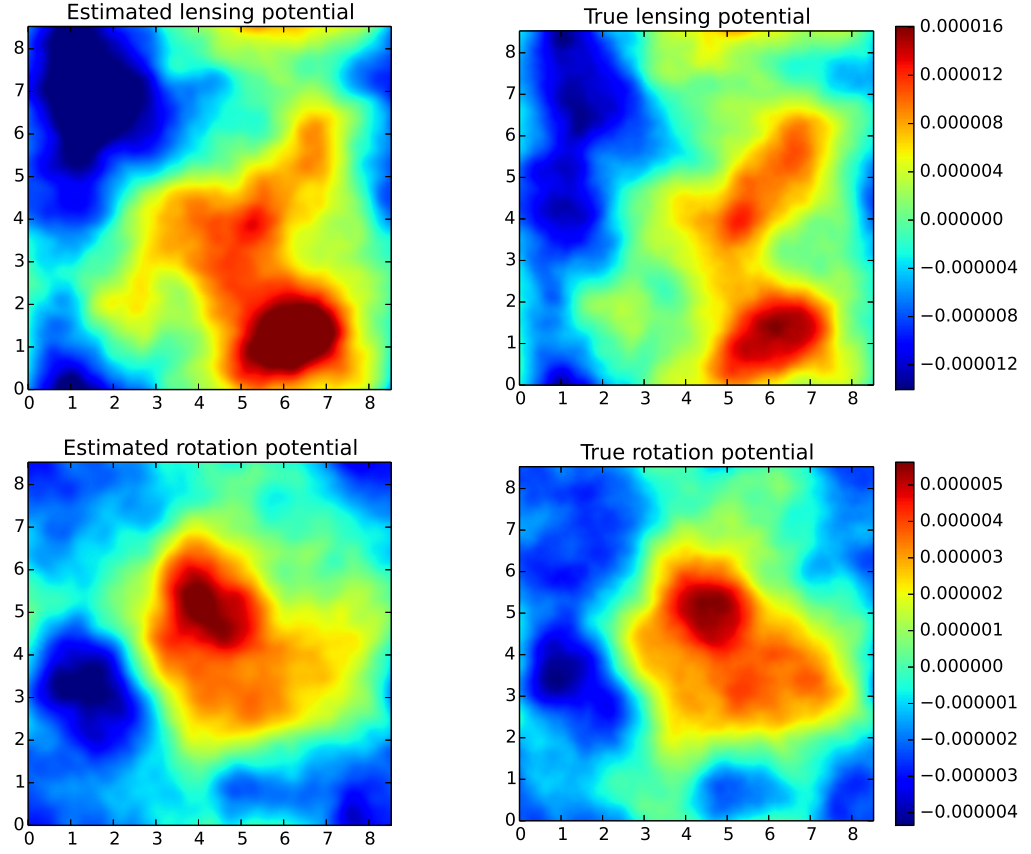
where Q, U denote the unlensed CMB polarization fields, ϕ denotes the lensing potential, ψ denotes a field rotation potential and $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. We observe \tilde{Q}, \tilde{U} and try to estimate ϕ, ψ, Q and U .

1. Old stuff:

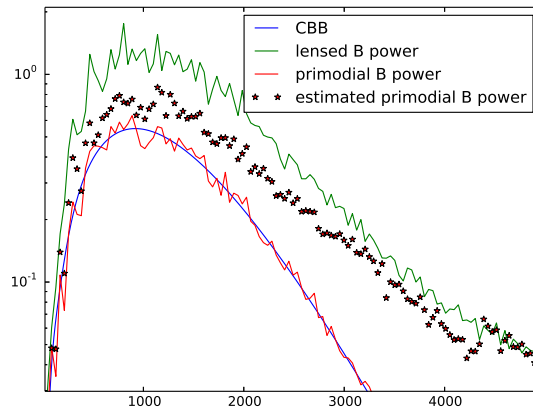
Here is a plot of the power spectrums I was using for the simulations



Here is a plot of the maximum likelihood estimate of ϕ and ψ from \tilde{Q}, \tilde{U}



Finally, here is a plot of the power of the estimated unlensed primordial B mode. In particular I use the estimates $\hat{\phi}$ and $\hat{\psi}$ and unlense the observed \tilde{Q}, \tilde{U} .



Appendix A

If we exclude the zero frequency $l = 0$ we get

$$\begin{aligned}
-\log P(\tilde{Q}, \tilde{U}|\phi, \psi) - c_1 &= \frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{|\tilde{E}_k^{\phi\psi}|^2}{C_k^{EE}} + \frac{|\tilde{B}_k^{\phi\psi}|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{|-\cos(2\varphi_k)\tilde{Q}_k^{\phi\psi} - \sin(2\varphi_k)\tilde{U}_k^{\phi\psi}|^2}{C_k^{EE}} + \frac{|\sin(2\varphi_k)\tilde{Q}_k^{\phi\psi} - \cos(2\varphi_k)\tilde{U}_k^{\phi\psi}|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{\cos^2(2\varphi_k)}{C_k^{EE}} + \frac{\sin^2(2\varphi_k)}{C_k^{BB}} \right] |\tilde{Q}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{\cos^2(2\varphi_k)}{C_k^{BB}} + \frac{\sin^2(2\varphi_k)}{C_k^{EE}} \right] |\tilde{U}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \left[\frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C_k^{BB} - C_k^{EE})}{C_k^{EE}C_k^{BB}} \right] \tilde{Q}_k^{\phi\psi}(\tilde{U}_k^{\phi\psi})^* dk
\end{aligned}$$

where c_1 is a constant, $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that in the last term in the above display, I've implicitly used the fact that

$$\int_{\mathbb{R}^2} M_k \tilde{Q}_k^{\phi\psi} (\tilde{U}_k^{\phi\psi})^* dk = \int_{\mathbb{R}^2} M_k (\tilde{Q}_k^{\phi\psi})^* \tilde{U}_k^{\phi\psi} dk$$

whenever $M_k = M_{-k}$.

Claim 1. *Let M_k real and symmetric in k . Let $X(x)$ and $Y(x)$ be real fields. Define $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and similarly for $Y^{\phi\psi}$. Then*

$$\begin{aligned}
\frac{\partial}{\partial \phi_l} \int_{\mathbb{R}^2} X_k^{\phi\psi} (Y_k^{\phi\psi})^* M_k dk &= il_q 2dk \int_{\mathbb{R}^2} \left([(\nabla^q X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi*} + [(\nabla^q Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi*} \right) M_k \frac{dk}{2\pi} \\
\frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} X_k^{\phi\psi} (Y_k^{\phi\psi})^* M_k dk &= -il_q 2dk \int_{\mathbb{R}^2} \left([(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi*} + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi*} \right) M_k \frac{dk}{2\pi}
\end{aligned}$$

Lemma 1.

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} = il_q [(\nabla^q X)^{\phi\psi}]_{k+l} \frac{dk}{\pi} \quad \frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = -il_q [(\nabla^q X)^{\phi\psi}]_{k-l} \frac{dk}{\pi} \quad (1)$$

$$\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} = -il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} \frac{dk}{\pi} \quad \frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} = il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l} \frac{dk}{\pi}. \quad (2)$$

where $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$.

Proof. First notice

$$\frac{\partial}{\partial \text{re}\phi_l} \nabla^q \phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix \cdot k} \frac{\partial \phi_k}{\partial \text{re}\phi_l} \frac{dk}{2\pi} = [il_q e^{ix \cdot l} - il_q e^{-ix \cdot l}] \frac{dk}{2\pi} \quad (3)$$

$$\frac{\partial}{\partial \text{im}\phi_l} \nabla^q \phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix \cdot k} \frac{\partial \phi_k}{\partial \text{im}\phi_l} \frac{dk}{2\pi} = [-l_q e^{ix \cdot l} - l_q e^{-ix \cdot l}] \frac{dk}{2\pi}. \quad (4)$$

Therefore

$$\frac{\partial}{\partial \phi_l} \nabla^q \phi(x) = -il_q e^{-ix \cdot l} \frac{dk}{\pi} \quad (5)$$

$$\frac{\partial}{\partial \phi_l^*} \nabla^q \phi(x) = il_q e^{ix \cdot l} \frac{dk}{\pi}. \quad (6)$$

This implies

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^q X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-\frac{\partial}{\partial \phi_l} \nabla^q \phi(x) \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^q X)^{\phi\psi}(x)] \left[il_q e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

Similarly

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^q X)^{\phi\psi}(x)] \left[-il_q e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}. \quad (7)$$

Conversely

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-\frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[\frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[il_2 e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-il_1 e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[-il_q e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned} \quad (8)$$

$$\begin{aligned}
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-il_1 e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[-il_q e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned} \quad (9)$$

since $\nabla^{\perp,1} = \nabla^2$ and $\nabla^{\perp,2} = -\nabla^1$. Similary

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l^*} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-\frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[\frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[-il_2 e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[il_1 e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[il_q e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned} \quad (10)$$

$$\begin{aligned}
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[il_1 e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[il_q e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned} \quad (11)$$

□

Lemma 2. If $A(x)$ and $B(x)$ are real scalar fields then the two integrals, $\int_{\mathbb{R}^2} i\{A_{k-l} - A_{k+l}\} B_k^* dk$ and $\int_{\mathbb{R}^2} \{A_{k-l} + A_{k+l}\} B_k^* dk$, are both real numbers.

Proof. By a simple change of variables it is clear that $\int_{\mathbb{R}^2} (i\{A_{k-l} - A_{k+l}\}B_k^*)^* dk = \int_{\mathbb{R}^2} i\{A_{k'-l} - A_{k'+l}\}B_{k'}^* dk'$ and $\int_{\mathbb{R}^2} (\{A_{k-l} + A_{k+l}\}B_k^*)^* dk = \int_{\mathbb{R}^2} \{A_{k'-l} + A_{k'+l}\}B_{k'}^* dk'$. \square

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

Lemma 3. *If $A(x)$ and $B(x)$ are real scalar fields then $\int_{\mathbb{R}^2} A_{k+l}B_k^* \frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix \cdot l} A(x)B(x) \frac{dx}{2\pi}$.*

Appendix B

$$\tilde{X}_\ell = \int \frac{d^2 m}{(2\pi)^2} D_{\ell m} X_m \quad (12)$$

$$\frac{\partial \tilde{X}_\ell}{\partial X_{\ell'}} = \int \frac{d^2 m}{(2\pi)^2} D_{\ell m} \delta(m - \ell') \quad (13)$$

$$= \int d^2 m \delta^2(\ell - m - \mathcal{P}) \delta(\ell' - m) \exp \left\{ - \int \frac{d^2 k}{(2\pi)^2} (k \cdot m) \phi_k \right\} \quad (14)$$

$$= \int d^2 m \left[\delta(\ell - m) \delta(\ell' - m) - \int \frac{d^2 k}{(2\pi)^2} (k \cdot m) \phi_k \delta(\ell - m - k) \delta(\ell' - m) \right. \quad (15)$$

$$\left. + \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} (k \cdot m) (k \cdot m) \phi_{k_1} \phi_{k_2} \delta(\ell - m - k_1 - k_2) \delta(\ell' - m) \right] \quad (16)$$

$$= \int d^2 m \left[\delta(\ell - m) \delta(\ell' - m) - \frac{(\ell - m) \cdot m}{(2\pi)^2} \phi_{\ell-m} \delta(\ell' - m) \right. \quad (17)$$

$$\left. + \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} (k \cdot m) (k \cdot m) \phi_{k_1} \phi_{k_2} \delta(\ell - m - k_1 - k_2) \delta(\ell' - m) \right] \quad (18)$$

$$= \delta(\ell - \ell') - \frac{(\ell - \ell') \cdot \ell'}{(2\pi)^2} \phi_{\ell-\ell'} \quad (19)$$

$$+ \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} (k_1 \cdot \ell') (k_2 \cdot \ell') \phi_{k_1} \phi_{k_2} \delta(\ell - \ell' - k_1 - k_2) \quad (20)$$

$$? = \delta(\ell - \ell') - \frac{(\ell - \ell') \cdot \ell'}{(2\pi)^2} \phi_{\ell-\ell'} \quad (21)$$

$$+ \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} (k_1 \cdot \ell') (k_2 \cdot \ell') (2\pi)^2 \delta(k_1 + k_2) C_k^{\phi\phi} \delta(\ell - \ell' - k_1 - k_2) \quad (22)$$

$$= \delta(\ell - \ell') - \frac{(\ell - \ell') \cdot \ell'}{(2\pi)^2} \phi_{\ell-\ell'} - \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} (k_1 \cdot \ell')^2 C_k^{\phi\phi} \delta(\ell - \ell') \quad (23)$$

$$= \delta(\ell - \ell') \left(1 - \frac{\ell'^2}{8\pi} \int k^3 C_k^{\phi\phi} dk \right) - \frac{(\ell - \ell') \cdot \ell'}{(2\pi)^2} \phi_{\ell-\ell'} \quad (24)$$