BayesLenseSPTpol

Abstract: At this point there is nothing in here but a few derivations.

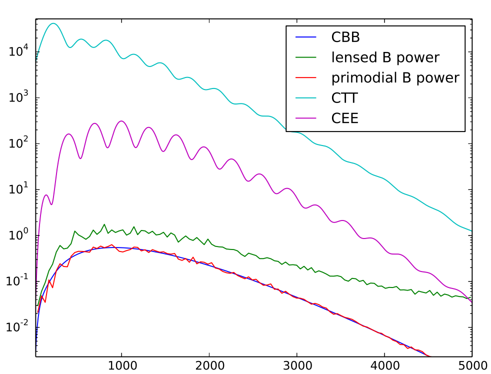
Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

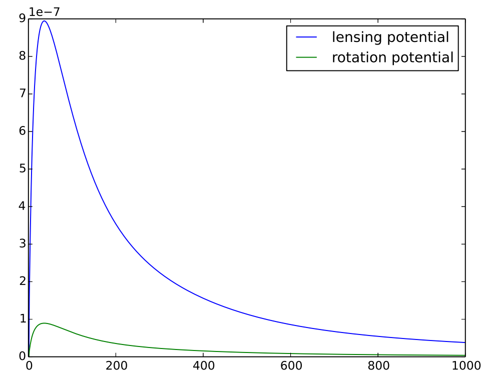
$$\widetilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$
$$\widetilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$

where Q, U denote the unlensed CMB polarization fields, ϕ denotes the lensing potential, ψ denotes a field rotation potential and $\nabla^{\perp} \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. We observe $\widetilde{Q}, \widetilde{U}$ and try to estimate ϕ, ψ, Q and U.

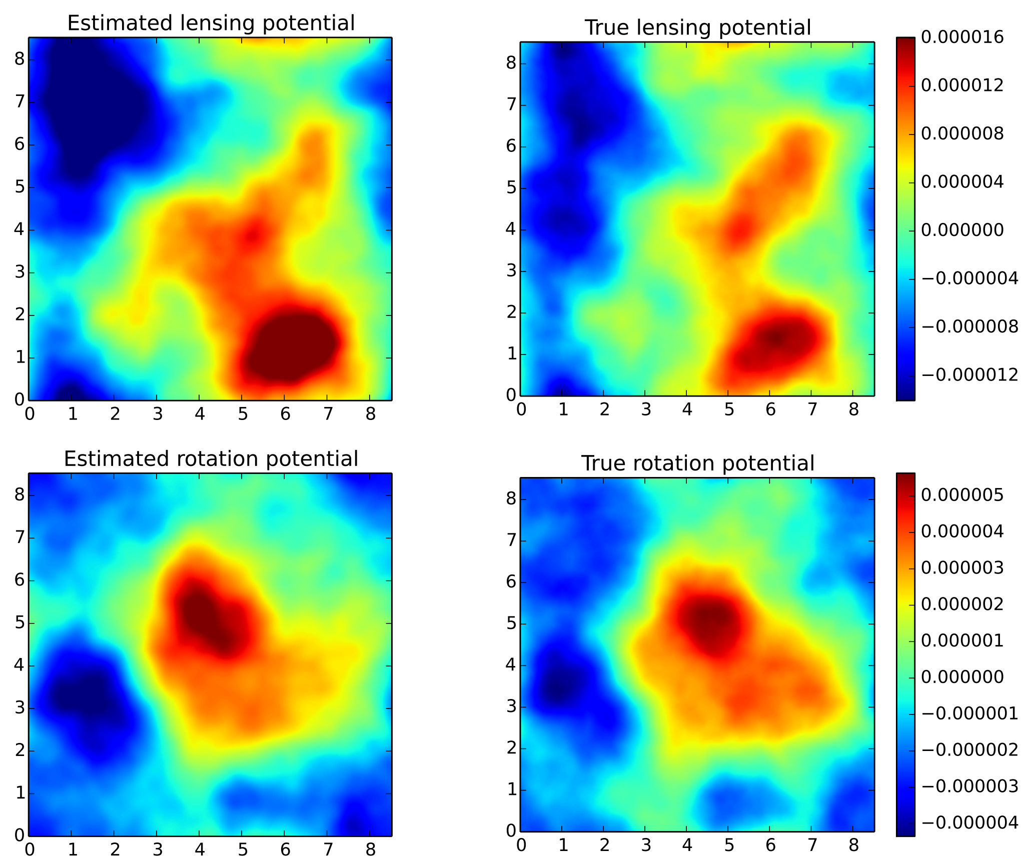
1. Old stuff:

Here is a plot of the power spectrums I was using for the simulations

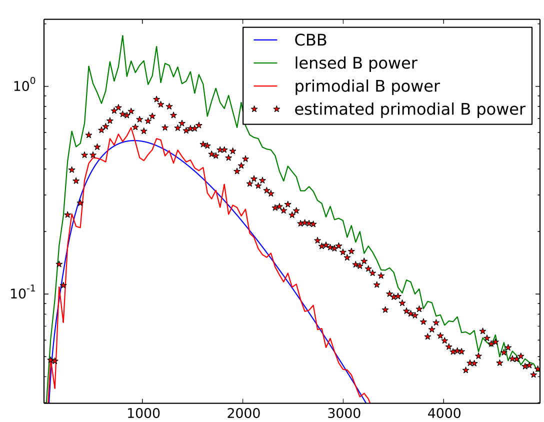




Here is a plot of the maximum likelihood estimate of ϕ and ψ from $\tilde{Q}, \tilde{U}$



Finally, here is a plot of the power of the estimated unlensed primodial B mode. In particular I use the estimates $\hat{\phi}$ and $\hat{\psi}$ and unlense the observed $\tilde{Q}, \tilde{U}$.



Appendix A

If we exclude the zero frequency $l = 0$ we get

$$
\begin{aligned}
-\log P(\widetilde{Q},\widetilde{U}|\phi,\psi) - c_1 &= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|\widetilde{E}^{\phi\psi}_k\big|^2}{C^{EE}_k} + \frac{\big|\widetilde{B}^{\phi\psi}_k\big|^2}{C^{BB}_k}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|-\cos(2\varphi_k)\widetilde{Q}^{\phi\psi}_k - \sin(2\varphi_k)\widetilde{U}^{\phi\psi}_k\big|^2}{C^{EE}_k} + \frac{\big|\sin(2\varphi_k)\widetilde{Q}^{\phi\psi}_k - \cos(2\varphi_k)\widetilde{U}^{\phi\psi}_k\big|^2}{C^{BB}_k}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C^{EE}_k} + \frac{\sin^2(2\varphi_k)}{C^{BB}_k}\right]|\widetilde{Q}^{\phi\psi}_k|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C^{BB}_k} + \frac{\sin^2(2\varphi_k)}{C^{EE}_k}\right]|\widetilde{U}^{\phi\psi}_k|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C^{BB}_k - C^{EE}_k)}{C^{EE}_k C^{BB}_k}\right]\widetilde{Q}^{\phi\psi}_k(\widetilde{U}^{\phi\psi}_k)^* dk
\end{aligned}
$$

where c_1 is a constant, $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^{\perp}\psi(x))$ and $\nabla^{\perp} \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that in the last term in the above display, I've implicity used the fact that

$$
\int_{\mathbb{R}^2} M_k\widetilde{Q}^{\phi\psi}_k(\widetilde{U}^{\phi\psi}_k)^* dk = \int_{\mathbb{R}^2} M_k(\widetilde{Q}^{\phi\psi}_k)^*\widetilde{U}^{\phi\psi}_k dk
$$

whenever $M_k = M_{-k}$.

Claim 1. *Let M_k real and symmetric in k. Let $X(x)$ and $Y(x)$ be real fields. Define $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x))$ and similarly for $Y^{\phi\psi}$. Then*

$$
\begin{aligned}
\frac{\partial}{\partial\phi_l}\int_{\mathbb{R}^2} X^{\phi\psi}_k(Y^{\phi\psi}_k)^* M_k dk &= il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^q X)^{\phi\psi}]_{k+l}Y^{\phi\psi *}_k + [(\nabla^q Y)^{\phi\psi}]_{k+l}X^{\phi\psi *}_k\Big)M_k\frac{dk}{2\pi} \\
\frac{\partial}{\partial\psi_l}\int_{\mathbb{R}^2} X^{\phi\psi}_k(Y^{\phi\psi}_k)^* M_k dk &= -il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}Y^{\phi\psi *}_k + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l}X^{\phi\psi *}_k\Big)M_k\frac{dk}{2\pi}
\end{aligned}
$$

Lemma 1.

$$
\frac{\partial X^{\phi\psi}_k}{\partial\phi_l} = il_q[(\nabla^q X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X^{\phi\psi}_k}{\partial\phi^*_l} = -il_q[(\nabla^q X)^{\phi\psi}]_{k-l}\frac{dk}{\pi} \tag{1}
$$

$$
\frac{\partial X^{\phi\psi}_k}{\partial\psi_l} = -il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X^{\phi\psi}_k}{\partial\psi^*_l} = il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l}\frac{dk}{\pi}. \tag{2}
$$

where $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$.

Proof. First notice

$$
\frac{\partial}{\partial\,\mathrm{re}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{re}\phi_l}\frac{dk}{2\pi} = \left[il_q e^{ix\cdot l} - il_q e^{-ix\cdot l}\right]\frac{dk}{2\pi} \tag{3}
$$

$$
\frac{\partial}{\partial\,\mathrm{im}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{im}\phi_l}\frac{dk}{2\pi} = \left[-l_q e^{ix\cdot l} - l_q e^{-ix\cdot l}\right]\frac{dk}{2\pi}. \tag{4}
$$

Therefore

$$
\frac{\partial}{\partial\phi_l}\nabla^q\phi(x) = -il_q e^{-ix\cdot l}\frac{dk}{\pi} \tag{5}
$$

$$
\frac{\partial}{\partial\phi^*_l}\nabla^q\phi(x) = il_q e^{ix\cdot l}\frac{dk}{\pi}. \tag{6}
$$

This implies

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^q X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\phi_l}\nabla^q\phi(x)\right]\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

Similarly

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[-il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}. \tag{7}$$

Conversely

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad+\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (8)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_2 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad+\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_1 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (9)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[-il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

since $\nabla^{\perp,1}=\nabla^2$ and $\nabla^{\perp,2}=-\nabla^1$. Similary

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad+\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (10)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_2 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad+\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_1 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (11)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

□

Lemma 2. *If $A(x)$ and $B(x)$ are real scalar fields then the two integrals, $\int_{\mathbb{R}^2} i\{A_{k-l}-A_{k+l}\}B_k^* dk$ and $\int_{\mathbb{R}^2}\{A_{k-l}+A_{k+l}\}B_k^* dk$, are both real numbers.*

Proof. By a simple change of variables it is clear that $\int_{\mathbb{R}^2}\left(i\{A_{k-l}-A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2} i\{A_{k'-l}-A_{k'+l}\}B_{k'}^* dk'$ and $\int_{\mathbb{R}^2}\left(\{A_{k-l}+A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2}\{A_{k'-l}+A_{k'+l}\}B_{k'}^* dk'$.

□

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

Lemma 3. *If $A(x)$ and $B(x)$ are real scalar fields then $\int_{\mathbb{R}^2} A_{k+l}B_k^* \frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix\cdot l}A(x)B(x)\frac{dx}{2\pi}$.*

Appendix B

Eq.(4.9) of Lewis 2006

$$\begin{aligned}\tilde{T}_\ell = T_\ell &- \int \frac{d^2\ell'}{2\pi}\ell'\cdot(\ell-\ell')\phi_{\ell-\ell'}T_{\ell'} \\ &- \frac{1}{2}\int \frac{d^2\ell_1}{2\pi}\frac{d^2\ell_2}{2\pi}\ell_1\cdot(\ell_1+\ell_2-\ell)\ (\ell_1\cdot\ell_2)T_{\ell_1}\phi_{\ell_1}\phi_{\ell-\ell_1-\ell_2}\end{aligned} \tag{12}$$

$$\langle\phi_\ell\phi_{\ell'}^*\rangle = \delta(\ell-\ell')C_\ell = \frac{\delta_{\ell,\ell'}}{(\Delta\ell)^2}C_\ell \equiv \delta_{\ell,\ell'}\delta_0 C_\ell \tag{13}$$

$$\begin{aligned}A_{\ell k} \equiv \frac{\partial\tilde{T}_\ell}{\partial T_k} =& \delta_{\ell,k} + \sum_{\ell'}\frac{(\Delta\ell')^2}{2\pi}\ell'\cdot(\ell-\ell')\phi_{\ell-\ell'}\delta_{\ell',k} \\ &- \frac{1}{2}\sum_{\ell_1,\ell_2}\frac{(\Delta\ell_1)^2}{2\pi}\frac{(\Delta\ell_2)^2}{2\pi}\ell_1\cdot(\ell_1+\ell_2-\ell)\ (\ell_1\cdot\ell_2)\delta_{\ell_1,k}\phi_{\ell_1}\phi_{\ell-\ell_1-\ell_2}\end{aligned} \tag{14}$$

$$\begin{aligned}=&\delta_{\ell,k} + \frac{1}{2\pi}k\cdot(\ell-k)\ (\Delta\ell')^2\phi_{\ell-k} \\ &- \frac{1}{2}\frac{(\Delta\ell_1)^2}{2\pi}\frac{(\Delta\ell_2)^2}{2\pi}\sum_{\ell_2}k\cdot(k+\ell_2-\ell)\ (k\cdot\ell_2)\phi_k\phi_{\ell-k-\ell_2}\end{aligned} \tag{15}$$

$$\frac{\partial\log\det(A)}{\partial\phi} = \frac{1}{\det(A)}\frac{\partial}{\partial\phi}\det(A) \tag{16}$$

Keep $O(\phi)$ for off-diagnal terms and $O(\phi^2)$ for diagnal terms, and accurate to $O(\phi^2)$ for $\det(A)$

$$A_{\ell k} = \delta_{\ell,k}\left(1-\frac{(\Delta\ell)^4}{8\pi^2}\sum_{\ell_2}(k\cdot\ell_2)^2\phi_k\phi_{-\ell_2}\right) + \frac{(\Delta\ell)^2}{2\pi}k\cdot(\ell-k)\phi_{\ell-k} \tag{17}$$

$$\det A = 1 - \frac{(\Delta\ell)^4}{8\pi^2}\sum_k\sum_{\ell_2}(k\cdot\ell_2)^2\phi_k\phi_{-\ell_2} + \frac{(\Delta\ell)^4}{4\pi^2}\sum_\ell\sum_{k\neq\ell}k\cdot(\ell-k)\ \ell\cdot(k-\ell)\phi_{\ell-k}\phi_{k-\ell}$$

Note that $\frac{\partial}{\partial\phi_l} \equiv \frac{\partial}{\partial\mathrm{re}\phi_l} + i\frac{\partial}{\partial\mathrm{Im}\phi_l}$, then $\frac{\partial}{\partial\phi_l}\phi_k = 2\delta_{k,-l}$

$$\begin{aligned}\frac{\partial}{2\partial\phi_i}\det(A) &= -\frac{(\Delta\ell)^4}{8\pi^2}\left(\sum_{\ell_2}(i\cdot\ell_2)^2\phi_{-\ell_2} + \sum_k(i\cdot k)^2\phi_k\right) + \frac{(\Delta\ell)^4}{4\pi^2}\sum_{\ell-k=-i} + \sum_{k-\ell=-i}k\cdot(\ell-k)\ \ell\cdot(k-\ell)\phi_i \\ &= -\frac{(\Delta\ell)^4}{8\pi^2}\sum_k(i\cdot k)^2\left(\phi_{-k}+\phi_k\right) - \frac{(\Delta\ell)^4}{2\pi^2}\phi_i\sum_\ell \ell\cdot i\ i^2 + (\ell\cdot i)^2 \\ &= -\frac{(\Delta\ell)^4}{4\pi^2}\sum_k(i\cdot k)^2\phi_k - \frac{(\Delta\ell)^4}{2\pi^2}\phi_i\sum_\ell(\ell\cdot i)^2 \leftarrow \text{diverging?}\end{aligned} \tag{18}$$

$$\langle \det A \rangle_\phi = 1 - \frac{(\Delta\ell)^2}{8\pi^2} \sum_k k^4 C_k^{\phi\phi} + 0 = 1 - \frac{1}{4\pi} \int dk \; k^5 C_k^{\phi\phi} \tag{19}$$

$\det A$ is not directly used in the Hamiltonian Markov chain.

$$\frac{\partial \log \det(A)}{\partial \phi} = \mathrm{Tr}(A^{-1} \frac{\partial A}{\partial \phi}) \text{ seems useless} \tag{20}$$