BayesLenseSPTpol

Abstract: At this point there is nothing in here but a few derivations.

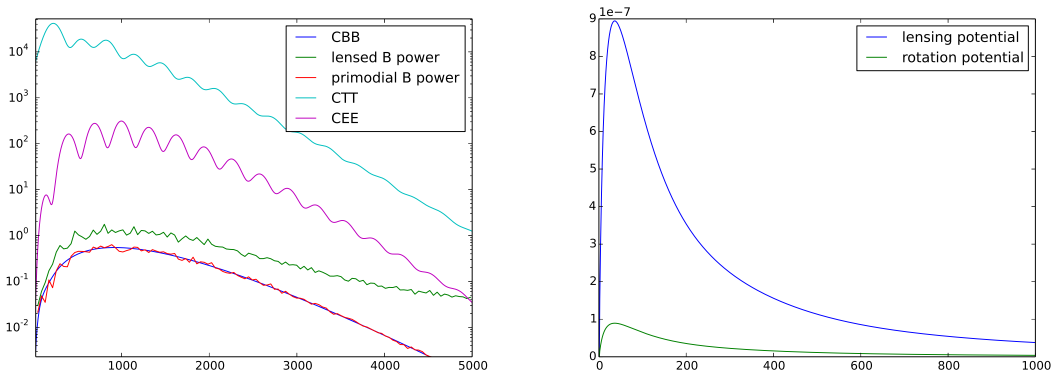
Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

$$\widetilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$
$$\widetilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$

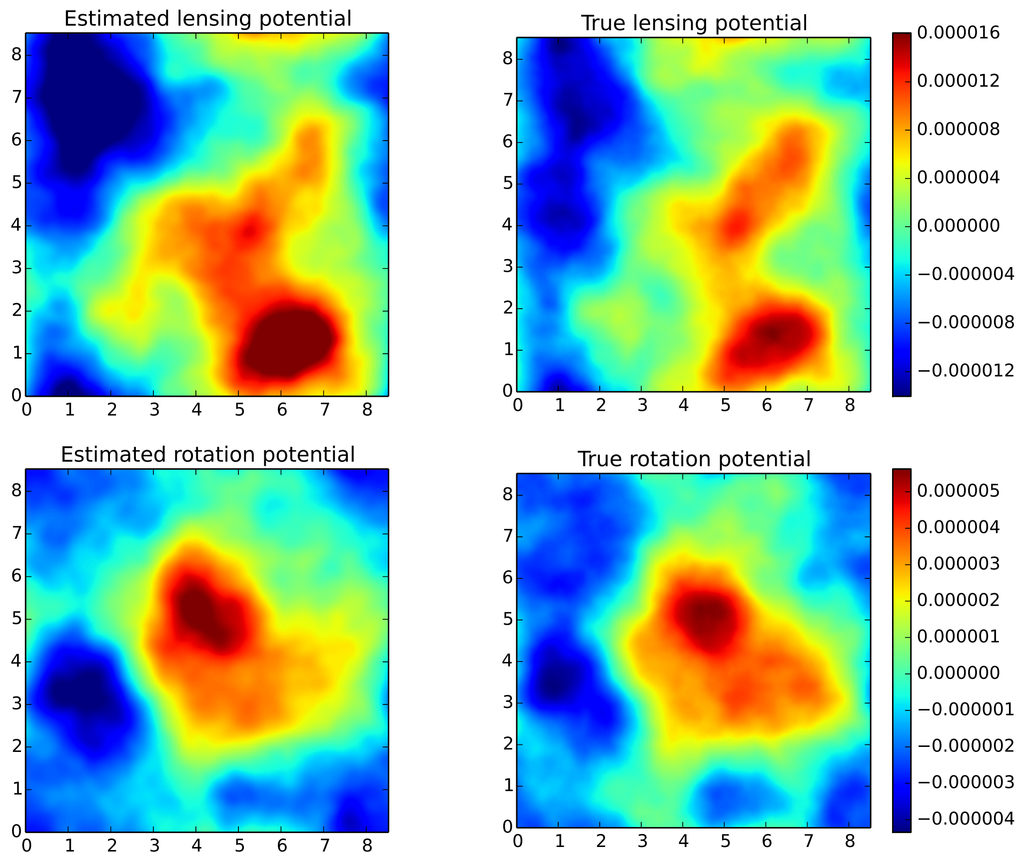
where Q, U denote the unlensed CMB polarization fields, ϕ denotes the lensing potential, ψ denotes a field rotation potential and $\nabla^{\perp} \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. We observe $\widetilde{Q}, \widetilde{U}$ and try to estimate ϕ, ψ, Q and U.

1. Old stuff:

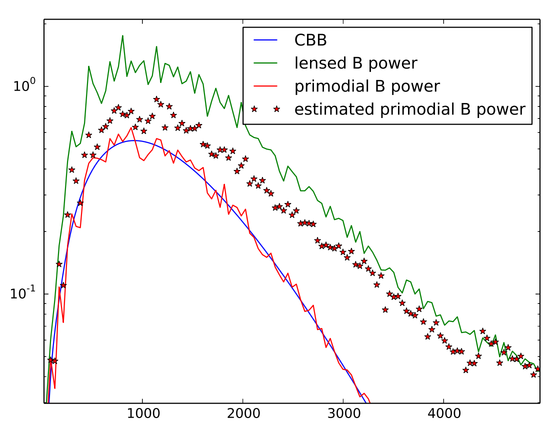
Here is a plot of the power spectrums I was using for the simulations



Here is a plot of the maximum likelihood estimate of ϕ and ψ from $\tilde{Q}, \tilde{U}$



Finally, here is a plot of the power of the estimated unlensed primodial B mode. In particular I use the estimates $\hat{\phi}$ and $\hat{\psi}$ and unlense the observed $\tilde{Q}, \tilde{U}$.



Appendix A

If we exclude the zero frequency $l = 0$ we get

$$\begin{aligned}
-\log P(\widetilde{Q},\widetilde{U}|\phi,\psi) - c_1 &= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|\widetilde{E}_k^{\phi\psi}\big|^2}{C_k^{EE}} + \frac{\big|\widetilde{B}_k^{\phi\psi}\big|^2}{C_k^{BB}}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|-\cos(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \sin(2\varphi_k)\widetilde{U}_k^{\phi\psi}\big|^2}{C_k^{EE}} + \frac{\big|\sin(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \cos(2\varphi_k)\widetilde{U}_k^{\phi\psi}\big|^2}{C_k^{BB}}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C_k^{EE}} + \frac{\sin^2(2\varphi_k)}{C_k^{BB}}\right]|\widetilde{Q}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C_k^{BB}} + \frac{\sin^2(2\varphi_k)}{C_k^{EE}}\right]|\widetilde{U}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C_k^{BB} - C_k^{EE})}{C_k^{EE}C_k^{BB}}\right]\widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk
\end{aligned}$$

where c_1 is a constant, $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that in the last term in the above display, I've implicity used the fact that

$$\int_{\mathbb{R}^2} M_k \widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk = \int_{\mathbb{R}^2} M_k (\widetilde{Q}_k^{\phi\psi})^* \widetilde{U}_k^{\phi\psi} dk$$

whenever $M_k = M_{-k}$.

Claim 1. *Let M_k real and symmetric in k. Let $X(x)$ and $Y(x)$ be real fields. Define $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and similarly for $Y^{\phi\psi}$. Then*

$$\begin{aligned}
\frac{\partial}{\partial\phi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= \quad il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^q X)^{\phi\psi}]_{k+l}Y_k^{\phi\psi *} + [(\nabla^q Y)^{\phi\psi}]_{k+l}X_k^{\phi\psi *}\Big)M_k\frac{dk}{2\pi} \\
\frac{\partial}{\partial\psi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= -il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}Y_k^{\phi\psi *} + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l}X_k^{\phi\psi *}\Big)M_k\frac{dk}{2\pi}
\end{aligned}$$

Lemma 1.

$$\frac{\partial X_k^{\phi\psi}}{\partial\phi_l} = \quad il_q[(\nabla^q X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial\phi_l^*} = -il_q[(\nabla^q X)^{\phi\psi}]_{k-l}\frac{dk}{\pi} \tag{1}$$

$$\frac{\partial X_k^{\phi\psi}}{\partial\psi_l} = -il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial\psi_l^*} = \quad il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l}\frac{dk}{\pi}. \tag{2}$$

where $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$.

Proof. First notice

$$\frac{\partial}{\partial\,\mathrm{re}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{re}\phi_l}\frac{dk}{2\pi} = \left[il_q e^{ix\cdot l} - il_q e^{-ix\cdot l}\right]\frac{dk}{2\pi} \tag{3}$$

$$\frac{\partial}{\partial\,\mathrm{im}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{im}\phi_l}\frac{dk}{2\pi} = \left[-l_q e^{ix\cdot l} - l_q e^{-ix\cdot l}\right]\frac{dk}{2\pi}. \tag{4}$$

Therefore

$$\frac{\partial}{\partial\phi_l}\nabla^q\phi(x) = -il_q e^{-ix\cdot l}\frac{dk}{\pi} \tag{5}$$

$$\frac{\partial}{\partial\phi_l^*}\nabla^q\phi(x) = \quad il_q e^{ix\cdot l}\frac{dk}{\pi}. \tag{6}$$

This implies

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^q X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\phi_l}\nabla^q\phi(x)\right]\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

Similarly

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[-il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}. \tag{7}$$

Conversely

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (8)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_2 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_1 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (9)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[-il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

since $\nabla^{\perp,1}=\nabla^2$ and $\nabla^{\perp,2}=-\nabla^1$. Similary

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (10)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_2 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_1 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (11)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}$$

□

Lemma 2. *If $A(x)$ and $B(x)$ are real scalar fields then the two integrals, $\int_{\mathbb{R}^2} i\big\{A_{k-l}-A_{k+l}\big\}B_k^* dk$ and $\int_{\mathbb{R}^2}\big\{A_{k-l}+A_{k+l}\big\}B_k^* dk$, are both real numbers.*

Proof. By a simple change of variables it is clear that $\int_{\mathbb{R}^2} \left(i\left\{A_{k-l} - A_{k+l}\right\}B_k^*\right)^* dk = \int_{\mathbb{R}^2} i\left\{A_{k'-l} - A_{k'+l}\right\}B_{k'}^* dk'$ and $\int_{\mathbb{R}^2} \left(\left\{A_{k-l} + A_{k+l}\right\}B_k^*\right)^* dk = \int_{\mathbb{R}^2} \left\{A_{k'-l} + A_{k'+l}\right\}B_{k'}^* dk'$.

□

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

Lemma 3. *If $A(x)$ and $B(x)$ are real scalar fields then* $\int_{\mathbb{R}^2} A_{k+l}B_k^* \frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix\cdot l} A(x)B(x)\frac{dx}{2\pi}$.