

# BayesLenseSPTpol

**Abstract:** At this point there is nothing in here but a few derivations.

Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

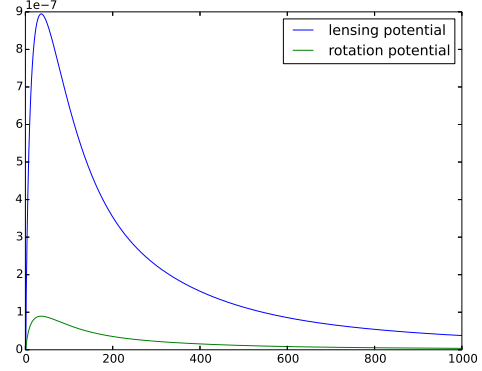
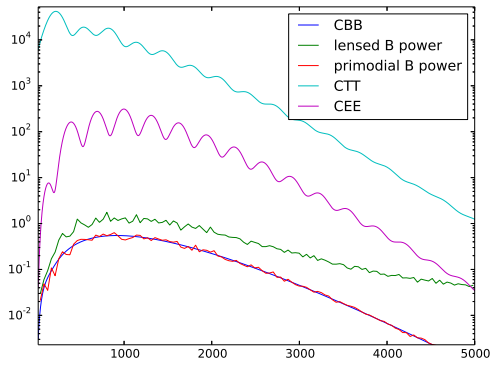
$$\tilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^\perp\psi(x))$$

$$\tilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^\perp\psi(x))$$

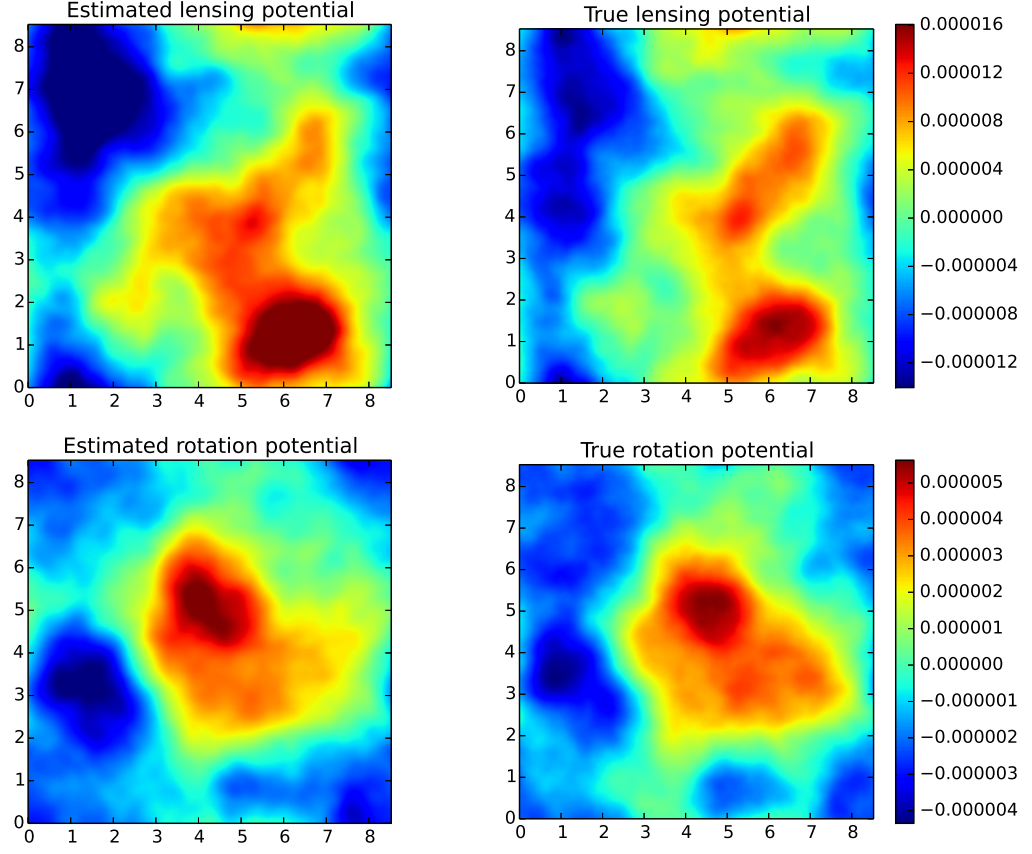
where  $Q, U$  denote the unlensed CMB polarization fields,  $\phi$  denotes the lensing potential,  $\psi$  denotes a field rotation potential and  $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$ . We observe  $\tilde{Q}, \tilde{U}$  and try to estimate  $\phi, \psi, Q$  and  $U$ .

## 1. Old stuff:

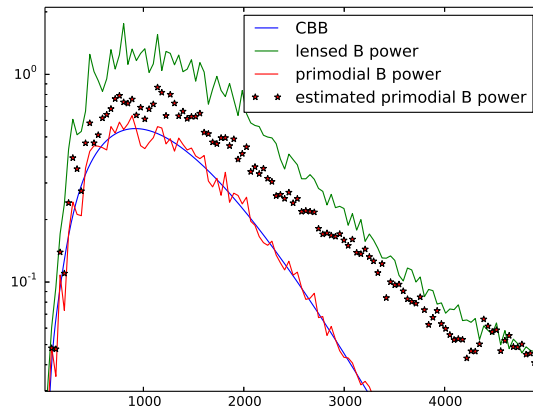
Here is a plot of the power spectrums I was using for the simulations



Here is a plot of the maximum likelihood estimate of  $\phi$  and  $\psi$  from  $\tilde{Q}, \tilde{U}$



Finally, here is a plot of the power of the estimated unlensed primordial  $B$  mode. In particular I use the estimates  $\hat{\phi}$  and  $\hat{\psi}$  and unlense the observed  $\tilde{Q}, \tilde{U}$ .



## Appendix A

If we exclude the zero frequency  $l = 0$  we get

$$\begin{aligned}
-\log P(\tilde{Q}, \tilde{U}|\phi, \psi) - c_1 &= \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{|\tilde{E}_k^{\phi\psi}|^2}{C_k^{EE}} + \frac{|\tilde{B}_k^{\phi\psi}|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{|-\cos(2\varphi_k)\tilde{Q}_k^{\phi\psi} - \sin(2\varphi_k)\tilde{U}_k^{\phi\psi}|^2}{C_k^{EE}} + \frac{|\sin(2\varphi_k)\tilde{Q}_k^{\phi\psi} - \cos(2\varphi_k)\tilde{U}_k^{\phi\psi}|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{\cos^2(2\varphi_k)}{C_k^{EE}} + \frac{\sin^2(2\varphi_k)}{C_k^{BB}} \right] |\tilde{Q}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{\cos^2(2\varphi_k)}{C_k^{BB}} + \frac{\sin^2(2\varphi_k)}{C_k^{EE}} \right] |\tilde{U}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \left[ \frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C_k^{BB} - C_k^{EE})}{C_k^{EE}C_k^{BB}} \right] \tilde{Q}_k^{\phi\psi}(\tilde{U}_k^{\phi\psi})^* dk
\end{aligned}$$

where  $c_1$  is a constant,  $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^\perp\psi(x))$  and  $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$ . Notice that in the last term in the above display, I've implicitly used the fact that

$$\int_{\mathbb{R}^2} M_k \tilde{Q}_k^{\phi\psi} (\tilde{U}_k^{\phi\psi})^* dk = \int_{\mathbb{R}^2} M_k (\tilde{Q}_k^{\phi\psi})^* \tilde{U}_k^{\phi\psi} dk$$

whenever  $M_k = M_{-k}$ .

**Claim 1.** *Let  $M_k$  real and symmetric in  $k$ . Let  $X(x)$  and  $Y(x)$  be real fields. Define  $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^\perp\psi(x))$  and similarly for  $Y^{\phi\psi}$ . Then*

$$\begin{aligned}
\frac{\partial}{\partial \phi_l} \int_{\mathbb{R}^2} X_k^{\phi\psi} (Y_k^{\phi\psi})^* M_k dk &= il_q 2dk \int_{\mathbb{R}^2} \left( [(\nabla^q X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi*} + [(\nabla^q Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi*} \right) M_k \frac{dk}{2\pi} \\
\frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} X_k^{\phi\psi} (Y_k^{\phi\psi})^* M_k dk &= -il_q 2dk \int_{\mathbb{R}^2} \left( [(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi*} + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi*} \right) M_k \frac{dk}{2\pi}
\end{aligned}$$

**Lemma 1.**

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} = il_q [(\nabla^q X)^{\phi\psi}]_{k+l} \frac{dk}{\pi} \quad \frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = -il_q [(\nabla^q X)^{\phi\psi}]_{k-l} \frac{dk}{\pi} \quad (1)$$

$$\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} = -il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} \frac{dk}{\pi} \quad \frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} = il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l} \frac{dk}{\pi}. \quad (2)$$

where  $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$ .

*Proof.* First notice

$$\frac{\partial}{\partial \text{re}\phi_l} \nabla^q \phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix \cdot k} \frac{\partial \phi_k}{\partial \text{re}\phi_l} \frac{dk}{2\pi} = [il_q e^{ix \cdot l} - il_q e^{-ix \cdot l}] \frac{dk}{2\pi} \quad (3)$$

$$\frac{\partial}{\partial \text{im}\phi_l} \nabla^q \phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix \cdot k} \frac{\partial \phi_k}{\partial \text{im}\phi_l} \frac{dk}{2\pi} = [-l_q e^{ix \cdot l} - l_q e^{-ix \cdot l}] \frac{dk}{2\pi}. \quad (4)$$

Therefore

$$\frac{\partial}{\partial \phi_l} \nabla^q \phi(x) = -il_q e^{-ix \cdot l} \frac{dk}{\pi} \quad (5)$$

$$\frac{\partial}{\partial \phi_l^*} \nabla^q \phi(x) = il_q e^{ix \cdot l} \frac{dk}{\pi}. \quad (6)$$

This implies

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^q X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ -\frac{\partial}{\partial \phi_l} \nabla^q \phi(x) \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^q X)^{\phi\psi}(x)] \left[ il_q e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

Similarly

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^q X)^{\phi\psi}(x)] \left[ -il_q e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}. \quad (7)$$

Conversely

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ -\frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ \frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \quad (8)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ il_2 e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ -il_1 e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \quad (9) \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[ -il_q e^{-ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

since  $\nabla^{\perp,1} = \nabla^2$  and  $\nabla^{\perp,2} = -\nabla^1$ . Similary

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l^*} \int_{\mathbb{R}^2} e^{-ix \cdot k} X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ -\frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ \frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \quad (10)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^1 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ -il_2 e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix \cdot k} \nabla^2 X(x - \nabla \phi(x) - \nabla^\perp \psi(x)) \left[ il_1 e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \quad (11) \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix \cdot k} [(\nabla^{\perp,q} X)^{\phi\psi}(x)] \left[ il_q e^{ix \cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

□

**Lemma 2.** If  $A(x)$  and  $B(x)$  are real scalar fields then the two integrals,  $\int_{\mathbb{R}^2} i\{A_{k-l} - A_{k+l}\} B_k^* dk$  and  $\int_{\mathbb{R}^2} \{A_{k-l} + A_{k+l}\} B_k^* dk$ , are both real numbers.

*Proof.* By a simple change of variables it is clear that  $\int_{\mathbb{R}^2} (i\{A_{k-l} - A_{k+l}\}B_k^*)^* dk = \int_{\mathbb{R}^2} i\{A_{k'-l} - A_{k'+l}\}B_{k'}^* dk'$  and  $\int_{\mathbb{R}^2} (\{A_{k-l} + A_{k+l}\}B_k^*)^* dk = \int_{\mathbb{R}^2} \{A_{k'-l} + A_{k'+l}\}B_{k'}^* dk'$ .  $\square$

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

**Lemma 3.** *If  $A(x)$  and  $B(x)$  are real scalar fields then  $\int_{\mathbb{R}^2} A_{k+l}B_k^* \frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix \cdot l} A(x)B(x) \frac{dx}{2\pi}$ .*

## Appendix B: Determinant

Eq.(4.9) of Lewis 2006

$$\begin{aligned} \tilde{T}_\ell &= T_\ell - \int \frac{d^2 \ell'}{2\pi} \ell' \cdot (\ell - \ell') \phi_{\ell - \ell'} T_{\ell'} \\ &\quad - \frac{1}{2} \int \frac{d^2 \ell_1}{2\pi} \frac{d^2 \ell_2}{2\pi} \ell_1 \cdot (\ell_1 + \ell_2 - \ell) (\ell_1 \cdot \ell_2) T_{\ell_1} \phi_{\ell_2} \phi_{\ell - \ell_1 - \ell_2} \end{aligned} \quad (12)$$

$$\langle \phi_\ell \phi_{-\ell'} \rangle = \langle \phi_\ell \phi_{\ell'}^* \rangle = \delta(\ell - \ell') C_\ell = \frac{\delta_{\ell, \ell'}}{dl} C_\ell \quad (13)$$

$$\begin{aligned} A_{\ell k} &\equiv \frac{\partial \tilde{T}_\ell}{\partial T_k} = \delta_{\ell, k} + \sum_{\ell'} \frac{dl}{2\pi} \ell' \cdot (\ell - \ell') \phi_{\ell - \ell'} \delta_{\ell', k} \\ &\quad - \frac{1}{2} \sum_{\ell_1, \ell_2} \frac{dl}{2\pi} \frac{dl}{2\pi} \ell_1 \cdot (\ell_1 + \ell_2 - \ell) (\ell_1 \cdot \ell_2) \delta_{\ell_1, k} \phi_{\ell_2} \phi_{\ell - \ell_1 - \ell_2} \end{aligned} \quad (14)$$

$$= \delta_{\ell, k} + \frac{dl}{2\pi} k \cdot (\ell - k) \phi_{\ell - k} - \frac{1}{2} \frac{dl}{2\pi} \frac{dl}{2\pi} \sum_{\ell_2} k \cdot (k + \ell_2 - \ell) (k \cdot \ell_2) \phi_{\ell_2} \phi_{\ell - k - \ell_2} \quad (15)$$

Keep  $O(\phi)$  for off-diagonal terms and  $O(\phi^2)$  for diagonal terms, and accurate to  $O(\phi^2)$  for  $\det(A)$

$$A_{\ell k} = \delta_{\ell, k} \left( 1 - \frac{dl^2}{8\pi^2} \sum_{\ell_2} (k \cdot \ell_2)^2 \phi_{\ell_2} \phi_{-\ell_2} \right) + \frac{dl}{2\pi} k \cdot (\ell - k) \phi_{\ell - k} \quad (16)$$

$$\begin{aligned} \det A &= 1 - \frac{dl^2}{8\pi^2} \sum_k \sum_{\ell_2} (k \cdot \ell_2)^2 \phi_{\ell_2} \phi_{-\ell_2} - \frac{1}{2} \frac{dl^2}{4\pi^2} \sum_\ell \sum_{k \neq \ell} k \cdot (\ell - k) \ell \cdot (k - \ell) \phi_{\ell - k} \phi_{k - \ell} \\ &= 1 - \frac{dl^2}{8\pi^2} \sum_k \sum_{\ell_2} (k \cdot \ell_2)^2 \phi_{\ell_2} \phi_{-\ell_2} + \frac{dl^2}{8\pi^2} \sum_k \sum_{\ell'} (k \cdot \ell') (k + \ell') \cdot \ell' \phi_{\ell'} \phi_{-\ell'} \\ &= 1 + \frac{dl^2}{8\pi^2} \sum_k \sum_{\ell'} (k \cdot \ell') \ell'^2 \phi_{\ell'} \phi_{-\ell'} \\ &= 1 \end{aligned} \quad (17)$$

Note that  $\frac{\partial}{\partial \phi_l} \equiv \frac{\partial}{\partial \text{Re} \phi_l} + i \frac{\partial}{\partial \text{Im} \phi_l}$ , then  $\frac{\partial}{\partial \phi_i} \phi_l = 2\delta_{i, -l}$

## Appendix C: $C_l^{\hat{\phi}\hat{\phi}}, C_l^{\hat{\phi}\hat{\psi}}, C_l^{\hat{\psi}\hat{\psi}}$ and $C_l^{\phi\phi}$

Using  $f(x) = x + \nabla\phi(x)$ ,  $\nabla\hat{\phi} + \nabla^\perp\hat{\psi} = \nabla\phi(f^{-1}(x)) = \nabla\phi(x - \nabla\hat{\phi} - \nabla^\perp\hat{\psi})$  and  $f^{-1}(x) \simeq x - \nabla\phi(x) + (\nabla\phi \cdot \nabla)\nabla\phi$ ,

$$\nabla^2\hat{\phi} = \nabla \cdot (\nabla\phi|_{x-\nabla\phi(x)+(\nabla\phi \cdot \nabla)\nabla\phi}) \quad (18)$$

$$\begin{aligned} \nabla^2\hat{\phi} &= \nabla \cdot (d|_{x-d+(d \cdot \nabla)d}) \\ &\simeq \nabla \cdot \left[ d - (d \cdot \nabla)d + [(d \cdot \nabla)d \cdot \nabla]d + \frac{1}{2}(d \cdot \nabla)^2 d \right] \\ &= \nabla^2\phi - \partial_1(\vec{d} \cdot \nabla)d_1 - \partial_2(\vec{d} \cdot \nabla)d_2 + \nabla \cdot [(d \cdot \nabla)d \cdot \nabla]d + \frac{1}{2}\partial_1(\vec{d} \cdot \nabla)^2 d_1 + \frac{1}{2}\partial_2(\vec{d} \cdot \nabla)^2 d_2 \quad (19) \end{aligned}$$

$$\begin{aligned} -l^2\hat{\phi}_l &= -l^2\phi_l - \int (k \cdot l)(k \cdot (k+l))\phi_{k+l}\phi_k^* \frac{d^2k}{2\pi} \\ &\quad + \frac{1}{2} \int (m+k+l) \cdot k \ m \cdot k \ (k+2m) \cdot k \ \phi_{m+k+l}\phi_m^*\phi_k^* \frac{d^2k}{2\pi} \frac{d^2m}{2\pi} \quad (20) \end{aligned}$$

$$\begin{aligned} l^4 C_l^{\hat{\phi}\hat{\phi}} &= l^4 C_l^{\phi\phi} - l^2 \int \frac{d^2k}{(2\pi)^2} (k \cdot l)(k^2 + k \cdot l)^2 C_k C_{k+l} \\ &\quad - l^2 C_l^{\phi\phi} \int \frac{d^2k}{(2\pi)^2} (k \cdot l)^2 (l^2 - k \cdot l - 2k^2) C_k \quad (21) \end{aligned}$$

$$\begin{aligned} (\nabla^\perp)^2\hat{\psi} &= \nabla^\perp \cdot (\nabla\phi|_{x-\nabla\hat{\phi}-\nabla^\perp\hat{\psi}}) \\ &\simeq \nabla^\perp \cdot [\nabla\phi - (\nabla\phi \cdot \nabla)\nabla\phi] \\ &= \partial_2(\vec{d} \cdot \nabla)d_1 - \partial_1(\vec{d} \cdot \nabla)d_2 \\ &= d_1\partial_{12}d_1 + d_2\partial_{22}d_1 - d_1\partial_{11}d_2 - d_2\partial_{12}d_2 \\ &= 0 \leftarrow (\vec{d} = \nabla\phi) \quad (22) \end{aligned}$$