BayesLenseSPTpol

Abstract: At this point there is nothing in here but a few derivations.

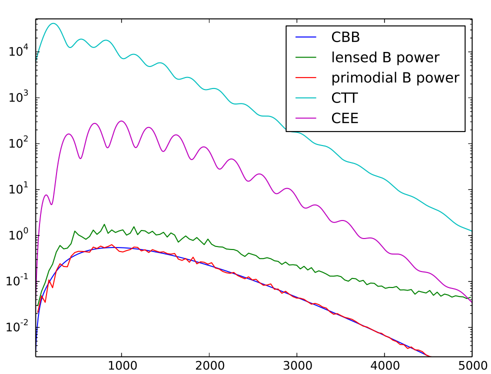
Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

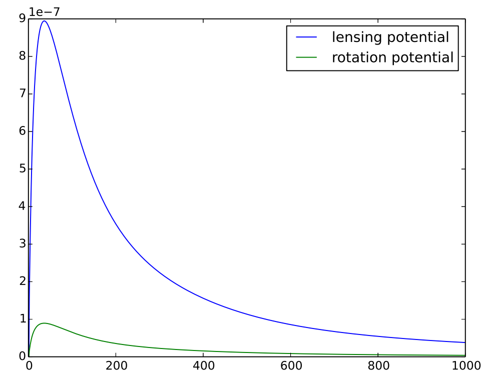
$$\widetilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$
$$\widetilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$

where Q, U denote the unlensed CMB polarization fields, ϕ denotes the lensing potential, ψ denotes a field rotation potential and $\nabla^{\perp} \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. We observe $\widetilde{Q}, \widetilde{U}$ and try to estimate ϕ, ψ, Q and U.

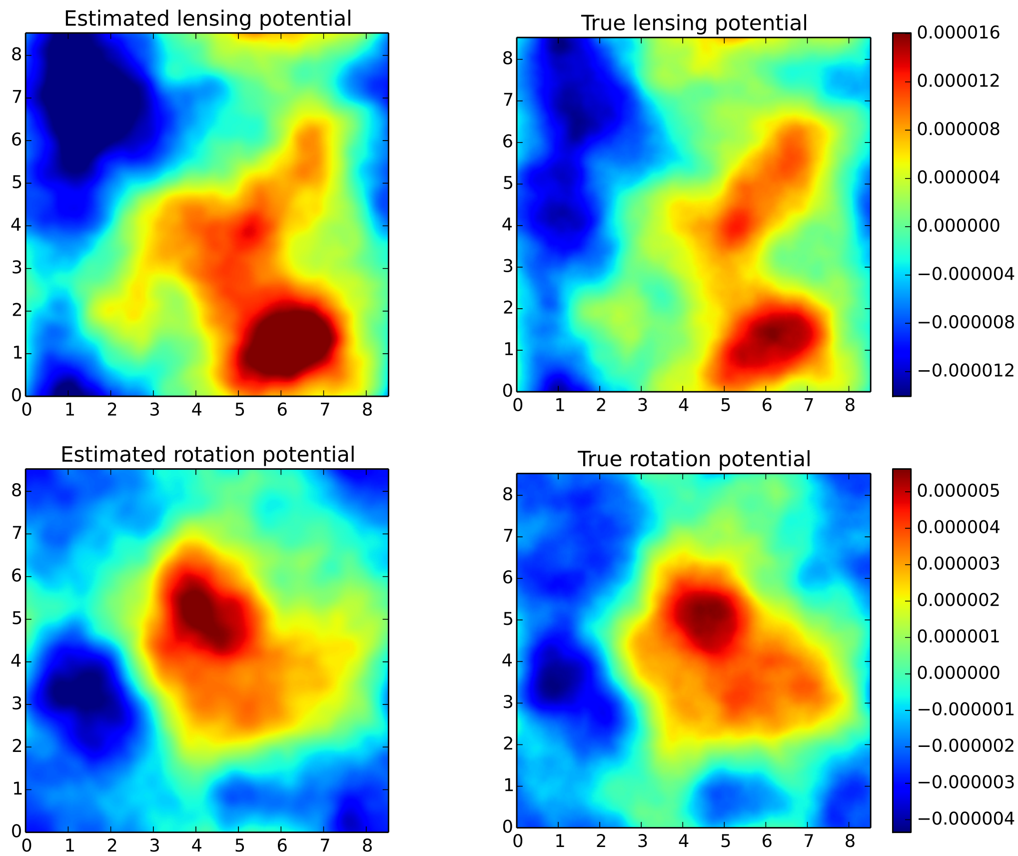
1. Old stuff:

Here is a plot of the power spectrums I was using for the simulations

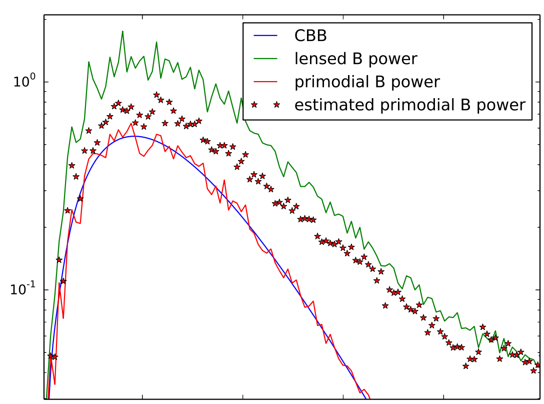




Here is a plot of the maximum likelihood estimate of ϕ and ψ from $\tilde{Q}, \tilde{U}$



Finally, here is a plot of the power of the estimated unlensed primodial B mode. In particular I use the estimates $\hat{\phi}$ and $\hat{\psi}$ and unlense the observed $\tilde{Q}, \tilde{U}$.



Appendix A

If we exclude the zero frequency $l = 0$ we get

$$
\begin{aligned}
-\log P(\widetilde{Q}, \widetilde{U}|\phi, \psi) - c_1 &= \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\left|\widetilde{E}_k^{\phi\psi}\right|^2}{C_k^{EE}} + \frac{\left|\widetilde{B}_k^{\phi\psi}\right|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\left|-\cos(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \sin(2\varphi_k)\widetilde{U}_k^{\phi\psi}\right|^2}{C_k^{EE}} + \frac{\left|\sin(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \cos(2\varphi_k)\widetilde{U}_k^{\phi\psi}\right|^2}{C_k^{BB}} \right] dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\cos^2(2\varphi_k)}{C_k^{EE}} + \frac{\sin^2(2\varphi_k)}{C_k^{BB}} \right] |\widetilde{Q}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\cos^2(2\varphi_k)}{C_k^{BB}} + \frac{\sin^2(2\varphi_k)}{C_k^{EE}} \right] |\widetilde{U}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C_k^{BB} - C_k^{EE})}{C_k^{EE}C_k^{BB}} \right] \widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk
\end{aligned}
$$

where c_1 is a constant, $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that in the last term in the above display, I've implicity used the fact that

$$
\int_{\mathbb{R}^2} M_k \widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk = \int_{\mathbb{R}^2} M_k (\widetilde{Q}_k^{\phi\psi})^* \widetilde{U}_k^{\phi\psi} dk
$$

whenever $M_k = M_{-k}$.

Claim 1. *Let M_k real and symmetric in k. Let $X(x)$ and $Y(x)$ be real fields. Define $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and similarly for $Y^{\phi\psi}$. Then*

$$
\begin{aligned}
\frac{\partial}{\partial \phi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= il_q 2dk \int_{\mathbb{R}^2} \Big([(\nabla^q X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi^*} + [(\nabla^q Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi^*} \Big) M_k \frac{dk}{2\pi} \\
\frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= -il_q 2dk \int_{\mathbb{R}^2} \Big([(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi^*} + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi^*} \Big) M_k \frac{dk}{2\pi}
\end{aligned}
$$

Lemma 1.

$$
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} = il_q [(\nabla^q X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = -il_q [(\nabla^q X)^{\phi\psi}]_{k-l}\frac{dk}{\pi} \tag{1}
$$

$$
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} = -il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} = il_q [(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l}\frac{dk}{\pi}. \tag{2}
$$

where $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$.

Proof. First notice

$$
\frac{\partial}{\partial\, \mathrm{re}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial \phi_k}{\partial\, \mathrm{re}\phi_l}\frac{dk}{2\pi} = \left[il_q e^{ix\cdot l} - il_q e^{-ix\cdot l}\right]\frac{dk}{2\pi} \tag{3}
$$

$$
\frac{\partial}{\partial\, \mathrm{im}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial \phi_k}{\partial\, \mathrm{im}\phi_l}\frac{dk}{2\pi} = \left[-l_q e^{ix\cdot l} - l_q e^{-ix\cdot l}\right]\frac{dk}{2\pi}. \tag{4}
$$

Therefore

$$
\frac{\partial}{\partial \phi_l}\nabla^q\phi(x) = -il_q e^{-ix\cdot l}\frac{dk}{\pi} \tag{5}
$$

$$
\frac{\partial}{\partial \phi_l^*}\nabla^q\phi(x) = il_q e^{ix\cdot l}\frac{dk}{\pi}. \tag{6}
$$

This implies

$$
\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^q X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\phi_l}\nabla^q\phi(x)\right]\frac{dx}{2\pi} \\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}
$$

Similarly

$$
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^q X)^{\phi\psi}(x)\right]\left[-il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}. \tag{7}
$$

Conversely

$$
\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (8)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_2 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_1 e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (9)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[-il_q e^{-ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}
$$

since $\nabla^{\perp,1}=\nabla^2$ and $\nabla^{\perp,2}=-\nabla^1$. Similary

$$
\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l}\int_{\mathbb{R}^2} e^{-ix\cdot k} X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_2}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[\frac{\partial}{\partial\psi_l^*}\frac{\partial\psi(x)}{\partial x_1}\right]\frac{dx}{2\pi} \qquad (10)\\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^1 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[-il_2 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k}\nabla^2 X(x-\nabla\phi(x)-\nabla^\perp\psi(x))\left[il_1 e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi} \qquad (11)\\
&= \sum_{q=1,2}\int_{\mathbb{R}^2} e^{-ix\cdot k}\left[(\nabla^{\perp,q} X)^{\phi\psi}(x)\right]\left[il_q e^{ix\cdot l}\frac{dk}{\pi}\right]\frac{dx}{2\pi}
\end{aligned}
$$

□

Lemma 2. *If $A(x)$ and $B(x)$ are real scalar fields then the two integrals, $\int_{\mathbb{R}^2} i\{A_{k-l}-A_{k+l}\}B_k^* dk$ and $\int_{\mathbb{R}^2}\{A_{k-l}+A_{k+l}\}B_k^* dk$, are both real numbers.*

Proof. By a simple change of variables it is clear that $\int_{\mathbb{R}^2}\left(i\{A_{k-l}-A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2} i\{A_{k'-l}-A_{k'+l}\}B_{k'}^* dk'$ and $\int_{\mathbb{R}^2}\left(\{A_{k-l}+A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2}\{A_{k'-l}+A_{k'+l}\}B_{k'}^* dk'$. □

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

Lemma 3. *If $A(x)$ and $B(x)$ are real scalar fields then* $\int_{\mathbb{R}^2} A_{k+l}B_k^*\frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix\cdot l}A(x)B(x)\frac{dx}{2\pi}$.

Appendix B: Determinant

Eq.(4.9) of Lewis 2006

$$\begin{aligned}\tilde{T}_\ell = T_\ell &- \int \frac{d^2\ell'}{2\pi}\ell'\cdot(\ell-\ell')\phi_{\ell-\ell'}T_{\ell'} \\ &- \frac{1}{2}\int\frac{d^2\ell_1}{2\pi}\frac{d^2\ell_2}{2\pi}\ell_1\cdot(\ell_1+\ell_2-\ell)\ (\ell_1\cdot\ell_2)T_{\ell_1}\ \phi_{\ell_2}\phi_{\ell-\ell_1-\ell_2}\end{aligned} \tag{12}$$

$$\langle\phi_\ell\phi_{-\ell'}\rangle = \langle\phi_\ell\phi_{\ell'}^*\rangle = \delta(\ell-\ell')C_\ell = \frac{\delta_{\ell,\ell'}}{dl}C_\ell \tag{13}$$

$$\begin{aligned}A_{\ell k}\equiv\frac{\partial\tilde{T}_\ell}{\partial T_k} =&\delta_{\ell,k} + \sum_{\ell'}\frac{dl}{2\pi}\ell'\cdot(\ell-\ell')\phi_{\ell-\ell'}\delta_{\ell',k} && (14)\\ &-\frac{1}{2}\sum_{\ell_1,\ell_2}\frac{dl}{2\pi}\frac{dl}{2\pi}\ell_1\cdot(\ell_1+\ell_2-\ell)\ (\ell_1\cdot\ell_2)\delta_{\ell_1,k}\phi_{\ell_2}\phi_{\ell-\ell_1-\ell_2} \\ =&\delta_{\ell,k}+\frac{dl}{2\pi}k\cdot(\ell-k)\phi_{\ell-k} - \frac{1}{2}\frac{dl}{2\pi}\frac{dl}{2\pi}\sum_{\ell_2}k\cdot(k+\ell_2-\ell)\ (k\cdot\ell_2)\phi_{\ell_2}\phi_{\ell-k-\ell_2} && (15)\end{aligned}$$

Keep $O(\phi)$ for off-diagnal terms and $O(\phi^2)$ for diagnal terms, and accurate to $O(\phi^2)$ for $\det(A)$

$$\begin{aligned}A_{\ell k} &= \delta_{\ell,k}\left(1-\frac{dl^2}{8\pi^2}\sum_{\ell_2}(k\cdot\ell_2)^2\phi_{\ell_2}\phi_{-\ell_2}\right)+\frac{dl}{2\pi}k\cdot(\ell-k)\phi_{\ell-k} && (16)\\ \det A &= 1-\frac{dl^2}{8\pi^2}\sum_k\sum_{\ell_2}(k\cdot\ell_2)^2\phi_{\ell_2}\phi_{-\ell_2} - \frac{1}{2}\frac{dl^2}{4\pi^2}\sum_\ell\sum_{k\neq\ell}k\cdot(\ell-k)\ \ell\cdot(k-\ell)\phi_{\ell-k}\phi_{k-\ell} \\ &= 1-\frac{dl^2}{8\pi^2}\sum_k\sum_{\ell_2}(k\cdot\ell_2)^2\phi_{\ell_2}\phi_{-\ell_2}+\frac{dl^2}{8\pi^2}\sum_k\sum_{\ell'}(k\cdot\ell')\ (k+\ell')\cdot\ell'\phi_{\ell'}\phi_{-\ell'} \\ &= 1+\frac{dl^2}{8\pi^2}\sum_k\sum_{\ell'}(k\cdot\ell'){\ell'}^2\phi_{\ell'}\phi_{-\ell'} \\ &= 1 && (17)\end{aligned}$$

Note that $\frac{\partial}{\partial\phi_l}\equiv\frac{\partial}{\partial\mathrm{re}\phi_l}+i\frac{\partial}{\partial\mathrm{Im}\phi_l}$, then $\frac{\partial}{\partial\phi_i}\phi_l = 2\delta_{i,-l}$