BayesLenseSPTpol

Abstract: At this point there is nothing in here but a few derivations.

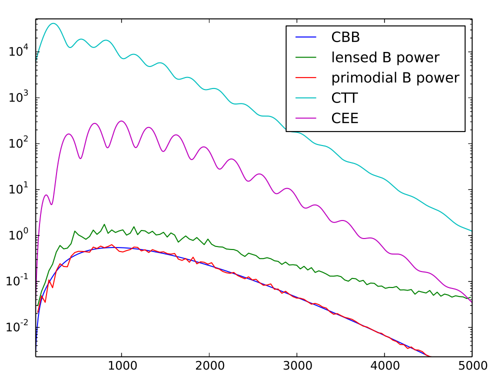
Here is the general polarization lensing setup which includes both a gravitational potential and a rotational potential.

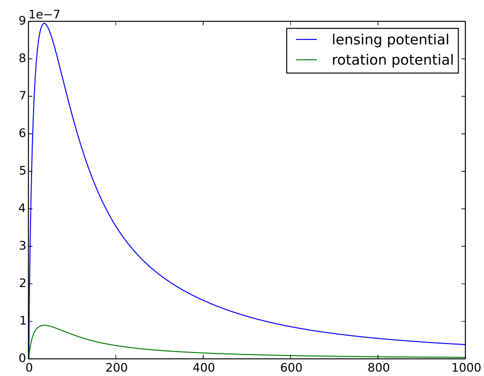
$$\widetilde{Q}(x) \equiv Q(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$
$$\widetilde{U}(x) \equiv U(x + \nabla\phi(x) + \nabla^{\perp}\psi(x))$$

where Q, U denote the unlensed CMB polarization fields, ϕ denotes the lensing potential, ψ denotes a field rotation potential and $\nabla^{\perp} \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. We observe $\widetilde{Q}, \widetilde{U}$ and try to estimate ϕ, ψ, Q and U.

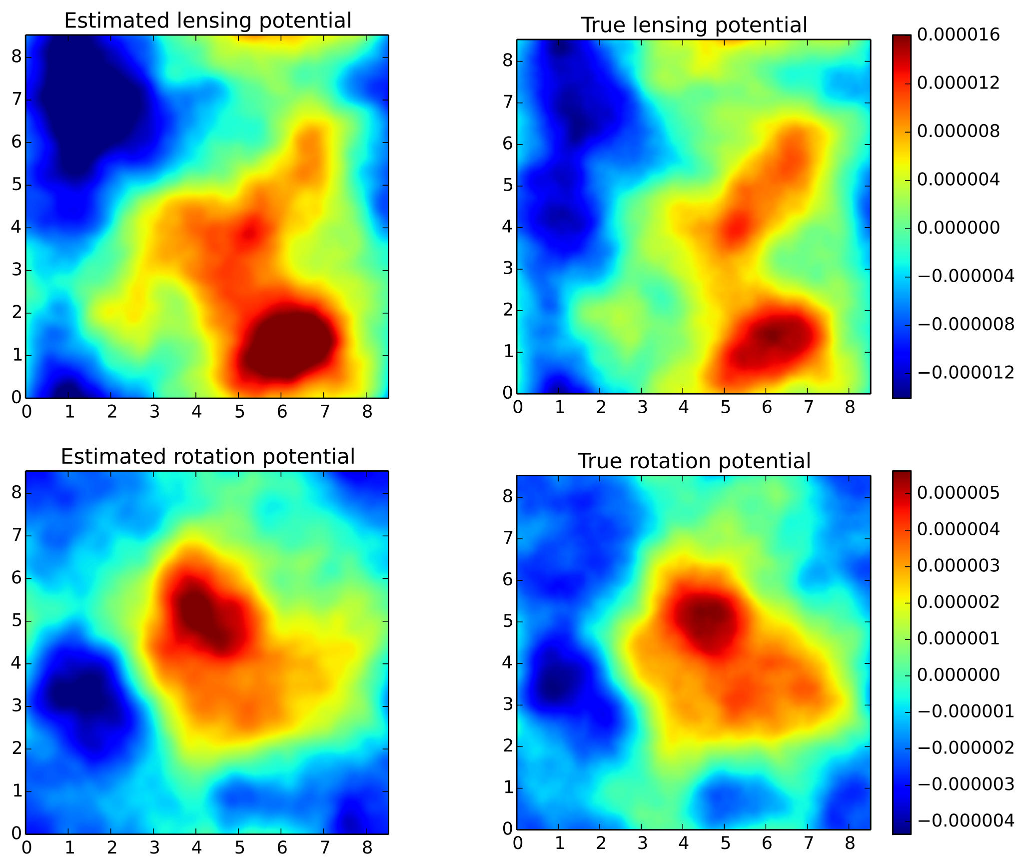
1. Old stuff:

Here is a plot of the power spectrums I was using for the simulations

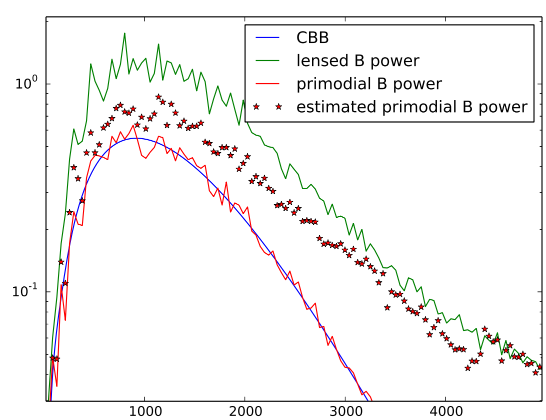




Here is a plot of the maximum likelihood estimate of ϕ and ψ from $\tilde{Q}, \tilde{U}$



Finally, here is a plot of the power of the estimated unlensed primodial B mode. In particular I use the estimates $\hat{\phi}$ and $\hat{\psi}$ and unlense the observed $\tilde{Q}, \tilde{U}$.



Appendix A

If we exclude the zero frequency $l = 0$ we get

$$
\begin{aligned}
-\log P(\widetilde{Q}, \widetilde{U}|\phi,\psi) - c_1 &= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|\widetilde{E}_k^{\phi\psi}\big|^2}{C_k^{EE}} + \frac{\big|\widetilde{B}_k^{\phi\psi}\big|^2}{C_k^{BB}}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\big|-\cos(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \sin(2\varphi_k)\widetilde{U}_k^{\phi\psi}\big|^2}{C_k^{EE}} + \frac{\big|\sin(2\varphi_k)\widetilde{Q}_k^{\phi\psi} - \cos(2\varphi_k)\widetilde{U}_k^{\phi\psi}\big|^2}{C_k^{BB}}\right]dk \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C_k^{EE}} + \frac{\sin^2(2\varphi_k)}{C_k^{BB}}\right]|\widetilde{Q}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{\cos^2(2\varphi_k)}{C_k^{BB}} + \frac{\sin^2(2\varphi_k)}{C_k^{EE}}\right]|\widetilde{U}_k^{\phi\psi}|^2 dk \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left[\frac{2\sin(2\varphi_k)\cos(2\varphi_k)(C_k^{BB} - C_k^{EE})}{C_k^{EE}C_k^{BB}}\right]\widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk
\end{aligned}
$$

where c_1 is a constant, $Y^{\phi\psi}(x) \equiv Y(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and $\nabla^\perp \equiv (\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1})$. Notice that in the last term in the above display, I've implicity used the fact that

$$\int_{\mathbb{R}^2} M_k \widetilde{Q}_k^{\phi\psi}(\widetilde{U}_k^{\phi\psi})^* dk = \int_{\mathbb{R}^2} M_k (\widetilde{Q}_k^{\phi\psi})^* \widetilde{U}_k^{\phi\psi} dk$$

whenever $M_k = M_{-k}$.

Claim 1. *Let M_k real and symmetric in k. Let $X(x)$ and $Y(x)$ be real fields. Define $X^{\phi\psi}(x) \equiv X(x - \nabla\phi(x) - \nabla^\perp\psi(x))$ and similarly for $Y^{\phi\psi}$. Then*

$$
\begin{aligned}
\frac{\partial}{\partial\phi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^q X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi^*} + [(\nabla^q Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi^*}\Big) M_k \frac{dk}{2\pi} \\
\frac{\partial}{\partial\psi_l}\int_{\mathbb{R}^2} X_k^{\phi\psi}(Y_k^{\phi\psi})^* M_k dk &= -il_q 2dk\int_{\mathbb{R}^2}\Big([(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l} Y_k^{\phi\psi^*} + [(\nabla^{\perp,q} Y)^{\phi\psi}]_{k+l} X_k^{\phi\psi^*}\Big) M_k \frac{dk}{2\pi}
\end{aligned}
$$

Lemma 1.

$$
\frac{\partial X_k^{\phi\psi}}{\partial\phi_l} = il_q[(\nabla^q X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial\phi_l^*} = -il_q[(\nabla^q X)^{\phi\psi}]_{k-l}\frac{dk}{\pi} \tag{1}
$$

$$
\frac{\partial X_k^{\phi\psi}}{\partial\psi_l} = -il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k+l}\frac{dk}{\pi} \qquad \frac{\partial X_k^{\phi\psi}}{\partial\psi_l^*} = il_q[(\nabla^{\perp,q} X)^{\phi\psi}]_{k-l}\frac{dk}{\pi}. \tag{2}
$$

where $\nabla^q X \equiv \frac{\partial X}{\partial x_q}$.

Proof. First notice

$$
\frac{\partial}{\partial\,\mathrm{re}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{re}\phi_l}\frac{dk}{2\pi} = \left[il_q e^{ix\cdot l} - il_q e^{-ix\cdot l}\right]\frac{dk}{2\pi} \tag{3}
$$

$$
\frac{\partial}{\partial\,\mathrm{im}\phi_l}\nabla^q\phi(x) = \int_{\mathbb{R}^2} ik_q e^{ix\cdot k}\frac{\partial\phi_k}{\partial\,\mathrm{im}\phi_l}\frac{dk}{2\pi} = \left[-l_q e^{ix\cdot l} - l_q e^{-ix\cdot l}\right]\frac{dk}{2\pi}. \tag{4}
$$

Therefore

$$
\frac{\partial}{\partial\phi_l}\nabla^q\phi(x) = -il_q e^{-ix\cdot l}\frac{dk}{\pi} \tag{5}
$$

$$
\frac{\partial}{\partial\phi_l^*}\nabla^q\phi(x) = il_q e^{ix\cdot l}\frac{dk}{\pi}. \tag{6}
$$

This implies

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \phi_l} &= \frac{\partial}{\partial \phi_l} \int_{\mathbb{R}^2} e^{-ix\cdot k} X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^q X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[-\frac{\partial}{\partial \phi_l} \nabla^q \phi(x) \right] \frac{dx}{2\pi} \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix\cdot k} \left[(\nabla^q X)^{\phi\psi}(x) \right] \left[i l_q e^{-ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

Similarly

$$\frac{\partial X_k^{\phi\psi}}{\partial \phi_l^*} = \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix\cdot k} \left[(\nabla^q X)^{\phi\psi}(x) \right] \left[-i l_q e^{ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}. \tag{7}$$

Conversely

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l} &= \frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} e^{-ix\cdot k} X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^1 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[-\frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^2 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[\frac{\partial}{\partial \psi_l} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \qquad (8) \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^1 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[i l_2 e^{-ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^2 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[-i l_1 e^{-ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \qquad (9) \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix\cdot k} \left[(\nabla^{\perp,q} X)^{\phi\psi}(x) \right] \left[-i l_q e^{-ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

since $\nabla^{\perp,1} = \nabla^2$ and $\nabla^{\perp,2} = -\nabla^1$. Similary

$$\begin{aligned}
\frac{\partial X_k^{\phi\psi}}{\partial \psi_l^*} &= \frac{\partial}{\partial \psi_l} \int_{\mathbb{R}^2} e^{-ix\cdot k} X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \frac{dx}{2\pi} \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^1 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[-\frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_2} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^2 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[\frac{\partial}{\partial \psi_l^*} \frac{\partial \psi(x)}{\partial x_1} \right] \frac{dx}{2\pi} \qquad (10) \\
&= \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^1 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[-i l_2 e^{ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \\
&\quad + \int_{\mathbb{R}^2} e^{-ix\cdot k} \nabla^2 X(x - \nabla\phi(x) - \nabla^{\perp}\psi(x)) \left[i l_1 e^{ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi} \qquad (11) \\
&= \sum_{q=1,2} \int_{\mathbb{R}^2} e^{-ix\cdot k} \left[(\nabla^{\perp,q} X)^{\phi\psi}(x) \right] \left[i l_q e^{ix\cdot l} \frac{dk}{\pi} \right] \frac{dx}{2\pi}
\end{aligned}$$

□

Lemma 2. *If $A(x)$ and $B(x)$ are real scalar fields then the two integrals, $\int_{\mathbb{R}^2} i\{A_{k-l} - A_{k+l}\} B_k^* dk$ and $\int_{\mathbb{R}^2} \{A_{k-l} + A_{k+l}\} B_k^* dk$, are both real numbers.*

Proof. By a simple change of variables it is clear that $\int_{\mathbb{R}^2}\left(i\{A_{k-l}-A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2} i\{A_{k'-l}-A_{k'+l}\}B_{k'}^* dk'$ and $\int_{\mathbb{R}^2}\left(\{A_{k-l}+A_{k+l}\}B_k^*\right)^* dk = \int_{\mathbb{R}^2}\{A_{k'-l}+A_{k'+l}\}B_{k'}^* dk'$.

□

The following lemma is equivalent to the so-called Convolution Theorem. We state it here for reference.

Lemma 3. *If $A(x)$ and $B(x)$ are real scalar fields then* $\int_{\mathbb{R}^2} A_{k+l}B_k^*\frac{dk}{2\pi} = \int_{\mathbb{R}^2} e^{-ix\cdot l}A(x)B(x)\frac{dx}{2\pi}$.

Appendix B

$$\tilde{X}_\ell = \int \frac{d^2m}{(2\pi)^2} D_{\ell m} X_m \tag{12}$$

$$\frac{\partial \tilde{X}_\ell}{\partial X_{\ell'}} = \int \frac{d^2m}{(2\pi)^2} D_{\ell m}\delta(m-\ell') \tag{13}$$

$$= \int d^2m\; \delta^2(\ell - m - \mathcal{P})\delta(\ell'-m)\exp\left\{-\int \frac{d^2k}{(2\pi)^2}(k\cdot m)\phi_k\right\} \tag{14}$$

$$= \int d^2m\Bigg[\delta(\ell-m)\delta(\ell'-m) - \int \frac{d^2k}{(2\pi)^2}(k\cdot m)\phi_k\delta(\ell-m-k)\delta(\ell'-m) \tag{15}$$

$$+\frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}(k\cdot m)(k\cdot m)\phi_{k_1}\phi_{k_2}\delta(\ell-m-k_1-k_2)\delta(\ell'-m)\Bigg] \tag{16}$$

$$= \int d^2m\Bigg[\delta(\ell-m)\delta(\ell'-m) - \frac{(\ell-m)\cdot m}{(2\pi)^2}\phi_{\ell-m}\delta(\ell'-m) \tag{17}$$

$$+\frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}(k\cdot m)(k\cdot m)\phi_{k_1}\phi_{k_2}\delta(\ell-m-k_1-k_2)\delta(\ell'-m)\Bigg] \tag{18}$$

$$= \delta(\ell-\ell') - \frac{(\ell-\ell')\cdot\ell'}{(2\pi)^2}\phi_{\ell-\ell'} \tag{19}$$

$$+\frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}(k_1\cdot\ell')(k_2\cdot\ell')\phi_{k_1}\phi_{k_2}\delta(\ell-\ell'-k_1-k_2) \tag{20}$$

$$? = \delta(\ell-\ell') - \frac{(\ell-\ell')\cdot\ell'}{(2\pi)^2}\phi_{\ell-\ell'} \tag{21}$$

$$+\frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}(k_1\cdot\ell')(k_2\cdot\ell')(2\pi)^2\delta(k_1+k_2)C_k^{\phi\phi}\delta(\ell-\ell'-k_1-k_2) \tag{22}$$

$$= \delta(\ell-\ell') - \frac{(\ell-\ell')\cdot\ell'}{(2\pi)^2}\phi_{\ell-\ell'} - \frac{1}{2}\int\frac{d^2k_1}{(2\pi)^2}(k_1\cdot\ell')^2C_k^{\phi\phi}\delta(\ell-\ell') \tag{23}$$

$$= \delta(\ell-\ell')\left(1-\frac{\ell^2}{8\pi}\int k^3C_k^{\phi\phi}dk\right) - \frac{(\ell-\ell')\cdot\ell'}{(2\pi)^2}\phi_{\ell-\ell'} \tag{24}$$