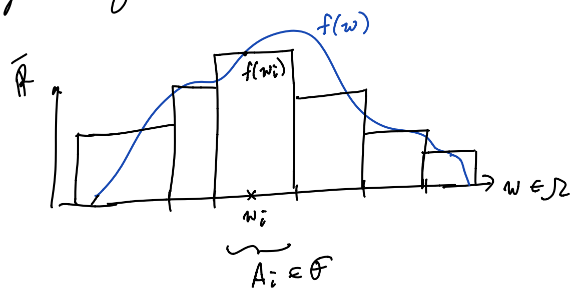
Lecture 10: Integration and expected value

In this lecture we will define

$$\int_\Omega f(w)\,d\mu(w)$$

where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f: \Omega \to \overline{\mathbb{R}}$ s.t. f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

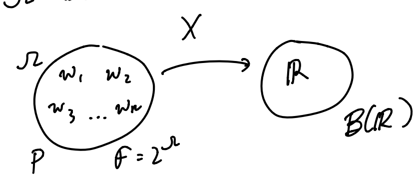
The notation $\int_\Omega f(w)\,d\mu(w)$ is extremely suggestive of Riemann integration



where one might guess

$$\int_\Omega f(w)\,d\mu(w) \approx \sum_i \underbrace{f(w_i)\mu(A_i)}_{\substack{\text{area of block } i \text{ when} \\ \text{width of } A_i \text{ is measured} \\ \text{with } \mu}}$$

To see the connection with expected value suppose Ω has n members:



In this case we would want the definition of "expected value of X", denoted $E(X)$, to, at the very least, satisfy:

$$E(X) = \left\{ \begin{array}{l} \text{the weighted average of the} \\ \text{numbers } \{X(w_1), X(w_2), \ldots, X(w_n)\} \\ \text{with weights } P(\{w_i\}). \end{array} \right.$$

$$= \sum_{i=1}^n X(w_i) \underbrace{P(\underbrace{\{w_i\}}_{\text{partition}})}_{\text{measure}}$$

$$= \int_\Omega X(w)\,dP(w).$$

Assumption: For the rest of this lecture suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space

Game plan:

$\int_\Omega f\,d\mu$ as shorthand for $\int_\Omega f(w)\,d\mu(w)$

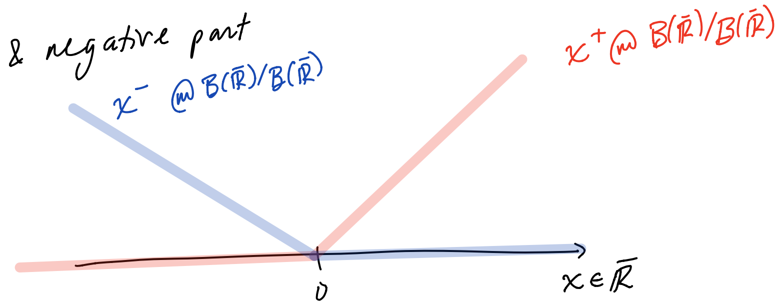
Step 1: Define $\int_\Omega f\,d\mu$ for $f \in \mathcal{M}_s(\Omega, \mathcal{F})$ ← Non-negative simple functions.

Step 2: extend to $f \in \mathcal{M}(\Omega, \mathcal{F})$ ← Non-negative measurable functions.

Step 3: extend to some, but not all, f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by

$$\int_\Omega f\,d\mu := \int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu$$

where $(\cdot)^+, (\cdot)^-$ denotes the positive part & negative part



so that $|x| = x^+ + x^-$ & $x = x^+ - x^-$

Remark: Although this construction seems tedious & annoying, the method of construction is general & broadly applicable. For example, is use for defining the same game plan

$$\int_0^t \underbrace{f(s)}_{f:\mathbb{R} \xrightarrow{ⓜ} \mathbb{R}}\, \underbrace{dB(s)}_{\text{Brownian Motion}}$$

Step 1

Def: If $f \in \mathcal{M}_s(\Omega, \mathcal{F})$ has the form $f = \sum_{i=1}^n c_i I_{A_i}$ then

$$\int_\Omega f(w)\,d\mu(w) := \sum_{i=1}^n c_i \mu(A_i)$$

$c_i \in [0, \infty]$; A_i forms a measurable partition

Here are the four basic properties of $\int_\Omega f\,d\mu$ we will show at each step in the game plan:

Thm (simple 3)

(0) $\int_\Omega f\,d\mu$ is well defined over $\mathcal{M}_s(\Omega,\mathcal{F})$

(1) Monotonicity:

If $f,g\in\mathcal{M}_s(\Omega,\mathcal{F})$ & $f(\omega)\le g(\omega)$ $\forall\omega\in\Omega$ then
$$\int_\Omega f\,d\mu \le \int_\Omega g\,d\mu.$$

(2) linearity:

If $f,g\in\mathcal{M}_s(\Omega,\mathcal{F})$ & $\alpha,\beta\in[0,\infty]$ then
$$\int_\Omega \alpha f+\beta g\,d\mu = \alpha\int_\Omega f\,d\mu + \beta\int_\Omega g\,d\mu$$

(3) Continuity from below (CFB):

If $f_n(\omega)\uparrow f(\omega)$ as $n\to\infty$ for all $\omega\in\Omega$ where $f_n,f\in\mathcal{M}_s(\Omega,\mathcal{F})$ then
$$\int_\Omega f_n\,d\mu \uparrow \int_\Omega f\,d\mu.$$

Proof:

Suppose $f=\sum_{i=1}^n c_i I_{A_i}$, $g=\sum_{k=1}^m d_k I_{B_k}$ both in $\mathcal{M}_s(\Omega,\mathcal{F})$

$\mathcal{F}$-sets which partition Ω

$$\therefore\quad f=\sum_{ik} c_{ik} I_{A_i\cap B_k} \quad\text{where } c_{ik}:=c_i$$

$$g=\sum_{ik} d_{ik} I_{A_i\cap B_k} \quad\text{where } d_{ik}:=d_k$$

a finer partition of Ω.

To show (0) & (1) it is sufficient to show
$$f\le g \Rightarrow \sum_{i=1}^n c_i\mu(A_i) \le \sum_{k=1}^m d_k\mu(B_k)$$

$$f\le g \Rightarrow \sum_{ik} c_{ik} I_{A_i\cap B_k} \le \sum_{ik} d_{ik} I_{A_i\cap B_k}$$

exactly one term is non-zero (assuming $A_i\neq\emptyset$ & $B_k\neq\emptyset$).

$$\Rightarrow\quad c_{ik}\le d_{ik}\quad\forall i,k$$

$$\Rightarrow\quad \underbrace{\sum_{ik} c_{ik}\mu(A_i\cap B_k)}_{=\sum_i c_i\mu(A_i)} \le \underbrace{\sum_{ik} d_{ik}\mu(A_i\cap B_k)}_{=\sum_k d_k\mu(B_k)}$$

by additivity of μ.

For (2)

$$\int_\Omega \alpha f+\beta g\,d\mu = \int_\Omega \sum_{ik}(\alpha c_{ik}+\beta d_{ik}) I_{A_i\cap B_k}\,d\mu$$
$$= \sum_{ik}(\alpha c_{ik}+\beta d_{ik})\mu(A_i\cap B_k)$$

$\vdots$ use additivity of μ & linearity of $\sum_{ik}$

$$= \alpha\int_\Omega f\,d\mu + \beta\int_\Omega g\,d\mu$$

For (3)

Suppose $\underbrace{f_n\uparrow f}_{\text{all in } \mathcal{M}_s(\Omega,\mathcal{F})}$.

Notice $\int_\Omega f_n\,d\mu \uparrow$ by (1) so just show
$$\lim_n \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu.$$

Case 1: $f=cI_A$ for $c\in(0,\infty]$. The case $c=0$ is trivial.

Let $0<b<c$ so that
$$bI_{\{f_n\ge b\}} \le f_n \le f = cI_A.$$

Now integrate each term & use (1) to get
$$b\mu(f_n\ge b) \le \int_\Omega f_n\,d\mu \le \int_\Omega f\,d\mu = c\mu(A)$$

Left column:

$\therefore$ $\underbrace{b \lim_n \mu(f_n \geq b)}_{=\mu(A)} \leq \lim_n \int_\Omega f_n d\mu \leq \int_\Omega f\, d\mu = c\mu(A)$

$= \mu(A)$ by CFB since $0<b<c$ & $f_n \uparrow f$ implies $\{f_n \geq b\} \uparrow \{f = c\} = A$ as $n \to \infty$

Now take the limit as $b \uparrow c$ to get

$$c\mu(A) \leq \underbrace{\lim_n \int_\Omega f_n d\mu \leq \int_\Omega f\, d\mu}_{} = c\mu(A)$$

$\therefore$ these are all equal.

<u>Case</u> 2: $f = \sum_{i=1}^{n} c_i I_{A_i}$, where A_i's partition Ω.

Fix $k \in \{1, 2, \ldots, m\}$

$\therefore$ Now $f_n I_{A_k} \uparrow f I_{A_k}$ so that Case 1 applies (since $f I_{A_k} = c_k I_{A_k}$) to give

$$\int_\Omega f_n I_{A_k} d\mu \uparrow \int_\Omega f I_{A_k} d\mu$$

Now sum over $k = 1, \ldots, m$ to get.

$$\int_\Omega f_n \underbrace{\sum_{k=1}^{m} I_{A_k}}_{=1} d\mu \uparrow \int_\Omega f \underbrace{\sum_{k=1}^{m} I_{A_k}}_{=1} d\mu$$

<u>QED</u>

Step 2

Recall the structure theorem:

If $f \in \mathcal{M}(\Omega, \mathcal{F})$ then $\exists f_n \in \mathcal{M}_s(\Omega, \mathcal{F})$ s.t.

$$f_n \uparrow f$$

<u>Def</u>: if $f \in \mathcal{M}(\Omega, \mathcal{F})$ define

$$\int_\Omega f d\mu := \lim_n \uparrow \int_\Omega f_n d\mu$$

$f_n \in \mathcal{M}_s$ s.t. $f_n \uparrow f$.

Thm (little 3)

Statements (0)–(3) in "simple 3" hold when $\mathcal{M}_s(\Omega, \mathcal{F})$ is replaced with $\mathcal{M}(\Omega, \mathcal{F})$.

<u>Proof</u>:

To show (0) & (1), i.e. $\int_\Omega f d\mu$ is well defined & monotonic, start by assuming

$$\underbrace{f \leq g}_{\text{both in } \mathcal{M}(\Omega, \mathcal{F})}$$

$\therefore$ $\exists f_n, g_n \in \mathcal{M}_s(\Omega, \mathcal{F})$ s.t.

$$\lim_n \uparrow f_n = f \leq g = \lim_n \uparrow g_n$$

Notice the following "trick"

$$\lim_m \uparrow f_n \wedge g_m = f_n \wedge (\lim_m \uparrow g_m)$$
$$= f_n \wedge g$$
$$= f_n \quad \text{since } f_n \leq f \leq g$$

$$\therefore \int_\Omega f_n d\mu = \int_\Omega \lim_m \uparrow f_n \wedge g_m \, d\mu$$
$$= \lim_m \uparrow \int_\Omega f_n \wedge g_m \, d\mu \quad \text{by "simple 3"}$$
$$\leq \lim_m \uparrow \int_\Omega g_m \, d\mu$$

Now take a limit as $n \to \infty$ to get

$$\lim_n \uparrow \int_\Omega f_n d\mu \leq \lim_m \uparrow \int_\Omega g_m d\mu.$$

This shows (0) & (1).

The proof of (2), i.e. that

$$\int_\Omega \alpha f + \beta g \, d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g \, d\mu$$

when $\alpha, \beta \in [0, \infty]$ is easy (using the fact that $\alpha f + \beta g = \lim_n \uparrow (\alpha f_n + \beta g_n)$ which implies $\int_\Omega \alpha f + \beta g \, d\mu = \lim_n \uparrow \int \alpha f_n + \beta g_n d\mu$ by def).

For (3):

Show $\underbrace{f_n \uparrow f}_{\text{all in } \mathcal{M}(\Omega,\mathcal{F})} \implies \int_\Omega f_n\,d\mu \uparrow \int_\Omega f\,d\mu$

Suppose $f_n \uparrow f$ & let $f_n = \lim_m \uparrow \underbrace{\phi_{nm}}_{\in \mathcal{M}_s(\Omega,\mathcal{F})}$ so that

$$
\begin{array}{ccccccccc}
\phi_{11} & \le & \phi_{12} & \le & \cdots & \le \phi_{1n} & \le & \longrightarrow & f_1 \\
\vdots & & & & & \vdots & & & \vdots\ \text{\rotatebox{90}{$\le$}} \\
\phi_{k1} & \le & \phi_{k2} & \le & \cdots & \le \phi_{kn} & \le & \longrightarrow & f_k \\
\vdots & & & & & \vdots & & & \vdots\ \text{\rotatebox{90}{$\le$}} \\
\phi_{n1} & \le & \phi_{n2} & \le & \cdots & \le \phi_{nn} & \le & \longrightarrow & f_n
\end{array}
$$

define $\phi_n := \max_{1 \le i,j \le n} \phi_{ij} \in \mathcal{M}_s(\Omega,\mathcal{F})$

Now $\phi_{kn} \le \phi_n \le f_n \le f, \quad \forall k \le n \quad (*)$

Taking limits as $n \to \infty$ in $(*)$ gives

$$f_k = \lim_n \uparrow \phi_{kn} \le \lim_n \uparrow \phi_n \le \lim_n \uparrow f_n \le f$$

Taking limits as $k \to \infty$

$$f = \lim_k \uparrow f_k = \lim_n \uparrow \phi_n = \lim_n \uparrow f_n = f$$

$\in \mathcal{M}_s(\Omega,\mathcal{F})$

$\therefore\ f = \lim_n \uparrow \phi_n$ where $\phi_n \in \mathcal{M}_s(\Omega,\mathcal{F})$ so

$$\int f\,d\mu := \lim_n \uparrow \int_\Omega \phi_n\,d\mu \quad \text{by def.}$$

Now just show $\lim_n \int_\Omega \phi_n\,d\mu = \lim_n \int_\Omega f_n\,d\mu$

Instead of taking limits in $(*)$ first, integrate to get

$$\int_\Omega \phi_{kn}\,d\mu \le \int_\Omega \phi_n\,d\mu \le \int_\Omega f_n\,d\mu, \quad \forall k \le n$$

Now let $n \to \infty$ for

$$\int_\Omega f_k\,d\mu \le \lim_n \int_\Omega \phi_n\,d\mu \le \lim_n \int f_n\,d\mu$$

where $\int_\Omega f_k\,d\mu = \lim_n \uparrow \int_\Omega \phi_{kn}\,d\mu$ by def.

Finally let $k \to \infty$ to give

$$\lim_k \int_\Omega f_k\,d\mu = \lim_n \int_\Omega \phi_n\,d\mu.$$

QED

Before we move to step 3 we need some useful facts.

Def: $f = g$ μ-a.e. means $\mu(f \neq g) = 0$

$f \le g$ μ-a.e. means $\mu(f \not\le g) = 0$

Thm (a.e. useful facts)

(i) $f \in \mathcal{M}(\Omega,\mathcal{F})$ & $\int_\Omega f\,d\mu < \infty \implies f < \infty$ μ-a.e.

(ii) If $f \in \mathcal{M}(\Omega,\mathcal{F})$ then

$$\int_\Omega f\,d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}$$

(iii) If $f, g \in \mathcal{M}(\Omega,\mathcal{F})$ and $f = g$ μ-a.e then $\int_\Omega f\,d\mu = \int_\Omega g\,d\mu$.

which implies I can change f on μ-null sets without changing $\int_\Omega f\,d\mu$.

Proof:

For (i) Notice that $f \in \mathcal{M}(\Omega,\mathcal{F})$ implies

$$\infty I_{\{f=\infty\}} \le f$$

using our convention that $\infty \cdot 0 = 0$

$$\therefore \int_\Omega f\,d\mu < \infty \overset{\text{little 3}}{\Longrightarrow} \infty\,\mu(f=\infty) \le \int_\Omega f\,d\mu < \infty$$

$$\Longrightarrow \underbrace{\mu(f=\infty)=0}_{\text{i.e. } f<\infty\ \mu\text{-a.e.}}$$

For (ii) suppose $f \in \mathcal{M}(\Omega,\mathcal{F})$.

$$\int f\,d\mu = 0 \iff \int_\Omega f I_{\{f \ge \frac{1}{n}\}}\,d\mu = 0,\ \forall n$$

The direction $\Leftarrow$ follows since
$\{f \ge \frac{1}{n}\} \uparrow \{f>0\}$
$\therefore f I_{\{f\ge\frac{1}{n}\}} \uparrow f I_{\{f>0\}} = f$
$\therefore \int_\Omega f I_{\{f \ge \frac{1}{n}\}}\,d\mu \uparrow \int_\Omega f\,d\mu$

$$\iff \mu(f \ge \tfrac{1}{n}) = 0 \quad \forall n$$

since $\frac{1}{n} I_{\{f\ge\frac{1}{n}\}} \le f I_{\{f\ge\frac{1}{n}\}} \le \infty I_{\{f\ge\frac{1}{n}\}}$
$\therefore \frac{1}{n}\mu(f\ge\frac{1}{n}) \le \int_\Omega f I_{\{f\ge\frac{1}{n}\}}\,d\mu \le \infty\,\mu(f\ge\frac{1}{n})$

$$\iff \mu(f>0)=0$$

since $\mu(f\ge\frac{1}{n}) \uparrow \mu(f>0)$ by CFB

$$\iff f=0 \ \mu\text{-a.e.}$$

For (iii) suppose $f,g \in \mathcal{M}(\Omega,\mathcal{F})$ & $f=g$ μ-a.e.

$$\therefore \int_\Omega f\,d\mu \overset{\ell 3}{=} \int_\Omega f I_{\{f=g\}}\,d\mu + \underbrace{\int_\Omega f I_{\{f\ne g\}}\,d\mu}_{=0 \text{ by (ii)}}$$

$$= \int_\Omega g I_{\{f=g\}}\,d\mu + \underbrace{\int_\Omega g I_{\{f\ne g\}}\,d\mu}_{=0 \text{ by (ii)}}$$

$$= \int_\Omega g\,d\mu.$$

QED.

Step 3

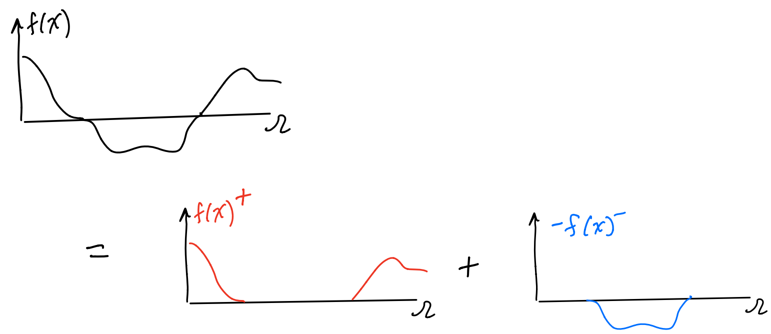
Recall

$$x^+ := \begin{cases} x & \text{if } x \ge 0 \\ 0 & \text{o.w.} \end{cases} \quad \& \quad x^- = \begin{cases} 0 & \text{if } x \ge 0 \\ |x| & \text{if } x<0. \end{cases}$$

For any $f:\Omega \to \bar{\mathbb{R}}$ s.t. f ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$ we have

- $f^+, f^- \in \mathcal{M}(\Omega,\mathcal{F})$ by composition of ⓜ is ⓜ
- $f = f^+ - f^-$
- $|f| = f^+ + f^-$.

Picture:



Def: If $f:\Omega\to\bar{\mathbb{R}}$ s.t. f ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$ and either $\int_\Omega f^+\,d\mu < \infty$ or $\int_\Omega f^-\,d\mu<\infty$ then define

$$\int_\Omega f\,d\mu := \int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu.$$

Notation:

$$\mathcal{Q}^+(\Omega,\mathcal{F},\mu) := \left\{ f:\Omega\to\bar{\mathbb{R}} \;\middle|\; f \text{ ⓜ } \mathcal{F}/\mathcal{B}(\bar{\mathbb{R}}) \ \& \ \int_\Omega f^+\,d\mu < \infty \right\}$$

$$\mathcal{Q}^-(\Omega,\mathcal{F},\mu) := \left\{ f:\Omega\to\bar{\mathbb{R}} \;\middle|\; f \text{ ⓜ } \mathcal{F}/\mathcal{B}(\bar{\mathbb{R}}) \ \& \ \int_\Omega f^-\,d\mu < \infty \right\}$$

$$\mathcal{Q}(\Omega,\mathcal{F},\mu) := \mathcal{Q}^+(\Omega,\mathcal{F},\mu) \cup \mathcal{Q}^-(\Omega,\mathcal{F},\mu)$$

$$L_1(\Omega,\mathcal{F},\mu) := \mathcal{Q}^+(\Omega,\mathcal{F},\mu) \cap \mathcal{Q}^-(\Omega,\mathcal{F},\mu)$$

(11)

$Q^+ \equiv$ quasi-integrable from above

$Q^- \equiv$ quasi-integrable from below

$Q \equiv$ quasi-integrable

$L_1 \equiv$ integrable.

Thm (Big 3):

(1) If $f, g \in Q(\Omega, \mathcal{F}, \mu)$ then

$$f \le g \ \mu\text{-a.e.} \Rightarrow \int f d\mu \le \int g d\mu$$

(2)

[a] $f \in \mathcal{M}(\Omega, \mathcal{F}, \mu)$ & $\alpha \in [0, \infty]$

$$\Rightarrow \int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$$

[b] $f \in Q(\Omega, \mathcal{F}, \mu)$ & $\alpha \in \mathbb{R}$

$$\Rightarrow \alpha f \in Q(\Omega, \mathcal{F}, \mu) \text{ and } \int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$$

[c] $f, g \in Q^+(\Omega, \mathcal{F}, \mu)$ or $f, g \in Q^-(\Omega, \mathcal{F}, \mu)$

$$\Rightarrow f + g \in Q(\Omega, \mathcal{F}, \mu) \text{ and } \int_\Omega f + g \, d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$$

(3) If $f_1, f_2, \ldots \in \mathcal{M}(\Omega, \mathcal{F})$ then

$$\lim_n \uparrow f_n = f \ \mu\text{-a.e.} \Rightarrow \lim_n \uparrow \int_\Omega f_n d\mu = \int_\Omega f d\mu$$

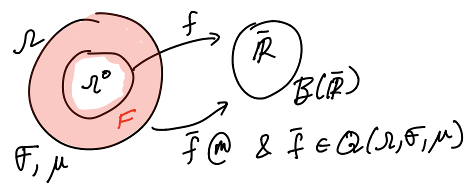
The only differerence from little 3 is this μ-a.e.

(12)

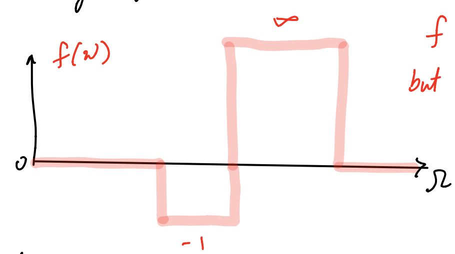
Remark:

- In (2)[c] it could happen that
$$f(\omega) + g(\omega) = \infty - \infty$$
but since $\mu(\{f = \infty\} \cap \{g = -\infty\}) = 0$ we can modify f & g to be defined everywhere.
In fact this allows us to define $\int_\Omega f d\mu$ for all functions $f: \Omega^\circ \to \mathbb{R}$ s.t.
  - $*$ $\exists F \in \mathcal{F}$ s.t. $(\Omega^\circ)^c \subset F$ & $\mu(F) = 0$
  - $*$ $\bar{f}(\omega) = \begin{cases} f(\omega) & \omega \in F^c \\ 0 & \omega \in F \end{cases} \in Q(\Omega, \mathcal{F}, \mu)$.

Picture:



- In (2)[b] the restriction $\alpha \in \mathbb{R}$ is necessary. e.g.



$f \in Q(\Omega, \mathcal{F}, \mu)$
but $\infty f \notin Q(\Omega, \mathcal{F}, \mu)$

Proof:

For (1)

Suppose $f, g \in Q(\Omega, \mathcal{F}, \mu)$ & $f \le g$ μ-a.e.

Modify f & g on a μ-null set so that $f \le g$ everywhere.

$\therefore$ $f^+ - f^- \le g^+ - g^-$

$\therefore$ $\underbrace{f^+ \le g^+ \text{ and } f^- \ge g^-}$

To see why notice that if not there is a contradiction.

$f^-(\omega) < g^-(\omega)$ or $f^+(\omega) > g^+(\omega)$

$f^-(\omega) < g^-(\omega)$:
- $g^-(\omega) > 0$ $\Rightarrow$ $f(\omega) \le g(\omega) < 0$ $\Rightarrow$ $f^-(\omega) \ge g^-(\omega)$ $\Rightarrow$ Contradiction
- $g^-(\omega) = 0$ $\Rightarrow$ $f^-(\omega) < 0$ $\Rightarrow$ Contradiction

$f^+(\omega) > g^+(\omega)$:
- $f^+(\omega) > 0$ $\Rightarrow$ $0 < f(\omega) \le g(\omega)$ $\Rightarrow$ $f^+(\omega) \le g^+(\omega)$ $\Rightarrow$ Contradiction
- $f^+(\omega) = 0$ $\Rightarrow$ $g^+(\omega) < 0$ $\Rightarrow$ Contradiction

Now $f^+ \le g^+$ and $f^- \geq g^-$ implies

$$\int_\Omega f^+ d\mu \le \int_\Omega g^+ d\mu$$ by little 3.

$$\int_\Omega f^- d\mu \geq \int_\Omega g^- d\mu.$$

$\therefore$ side fact:

$g \in Q^+$ & $f \le g \Rightarrow \int_\Omega f^+ d\mu < \infty \Rightarrow f \in Q^+$

$f \in Q^-$ & $f \le g \Rightarrow \int_\Omega g^- d\mu < \infty \Rightarrow g \in Q^-$

$$\therefore \underbrace{\int_\Omega f^+ d\mu - \int_\Omega f^- d\mu}_{=: \int_\Omega f d\mu} \le \underbrace{\int_\Omega g^+ d\mu - \int_\Omega g^- d\mu}_{=: \int_\Omega g\, d\mu}$$

<u>For (2)[a]</u>. This is just little (3)

<u>For (2)[b]</u>. Suppose $f \in Q(\Omega, \mathcal{F}, \mu)$ & $\alpha \in \mathbb{R}$.

Case 1: $\alpha \in (-\infty, 0)$

$$\therefore \int_\Omega (\alpha f)^+ d\mu = \int_\Omega |\alpha| f^- d\mu \overset{[a]}{=} |\alpha| \int_\Omega f^- d\mu < \infty$$

or

$$\int_\Omega (\alpha f)^- d\mu = \int_\Omega |\alpha| f^+ d\mu \overset{[a]}{=} |\alpha| \int_\Omega f^+ d\mu < \infty$$

$\therefore \alpha f \in Q(\Omega, \mathcal{F}, \mu)$ and

$$\int_\Omega \alpha f d\mu = \int_\Omega (\alpha f)^+ d\mu - \int_\Omega (\alpha f)^- d\mu$$
$$= |\alpha| \left[\int_\Omega f^- d\mu - \int_\Omega f^+ d\mu \right] = \alpha \int_\Omega f d\mu$$

Case 2 is similar.

Side fact:

$f \in Q^\pm$ & $\alpha \in (-\infty, 0) \Rightarrow \alpha f \in Q^\mp$

$f \in Q^\pm$ & $\alpha \in (0, \infty) \Rightarrow \alpha f \in Q^\pm$

<u>For (2)[c]</u>:

Suppose $f, g \in Q^+(\Omega, \mathcal{F}, \mu)$ or $f, g \in Q^-(\Omega, \mathcal{F}, \mu)$.

Show $f + g \in Q(\Omega, \mathcal{F}, \mu)$ and

$$\int_\Omega f + g\, d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$$

Notice that if $a, b \in \overline{\mathbb{R}}$ s.t. $a+b$ is defined then $(a+b)^+ - (a+b)^- = a + b = a^+ - a^- + b^+ - b^-$ so that $(a+b)^+ + a^- + b^- = (a+b)^- + a^+ + b^+$. (*)

Warning! Be careful moving terms to the other side. Here it is ok since:

$(a+b) = \infty \Rightarrow a^+$ or b^+ is $\infty \Rightarrow RHS = LHS = \infty$

$(a+b) = -\infty \Rightarrow a^-$ or b^- is $\infty \Rightarrow RHS = LHS = \infty$

Now (*) implies that

$$\underbrace{(f+g)^+ + f^- + g^-}_{\in \mathcal{M}(\Omega, \mathcal{F})} = \underbrace{(f+g)^- + f^+ + g^+}_{\in \mathcal{M}(\Omega, \mathcal{F})}.$$

$\therefore$ little 3 implies

$$\int_\Omega (f+g)^+ d\mu + \int_\Omega f^- d\mu + \int_\Omega g^- d\mu$$
$$\overset{(*)}{=} \int_\Omega (f+g)^- d\mu + \int_\Omega f^+ d\mu + \int_\Omega g^+ d\mu$$

Case 1: $f, g \in Q^-(\Omega, \mathcal{F}, \mu)$.

The idea is to show $\int_\Omega (f+g)^- d\mu < \infty$ so one can move it, along with $\underbrace{\int_\Omega f^- d\mu \text{ \& } \int_\Omega g^- d\mu}_{\text{both finite since } f, g \in Q^-}$, to the opposite side in (*).

Indeed $(f+g)^- \le f^- + g^-$ by convexity.

$$\therefore \int_\Omega (f+g)^- d\mu \le \int_\Omega f^- d\mu + \int_\Omega g^- d\mu < \infty \text{ by little 3.}$$

Now move the three finite terms in (*) to get

$$\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g\, d\mu.$$

Case 2: $f, g \in Q^+(\Omega, \mathcal{F}, \mu)$ is similar.

For (3)

Suppose $f_1, f_2, \ldots \in \mathcal{M}(\Omega, \mathcal{F})$ and

$$\lim_n \uparrow f_n = f \quad \mu\text{-a.e.}$$

all the f_n's and f on a μ-null set (note: countable unions of μ-nulls is μ-null) so that

$$0 \le f_n \uparrow f \quad \text{everywhere.}$$

Now (3) follows directly by little (3).

QED

Corollary to Big 3:

If $f \in Q(\Omega, \mathcal{F}, \mu)$ then

$$\left| \int_\Omega f\, d\mu \right| \le \int_\Omega |f|\, d\mu.$$

If $f \,\text{ⓜ}\, \mathcal{F}/B(\mathbb{R})$ and $\int_\Omega |f|\, d\mu < \infty$ then $f \in L_1(\Omega, \mathcal{F}, \mu)$ and if $g \in L_1(\Omega, \mathcal{F}, \mu)$ also then

$$\int_\Omega \alpha f + \beta g\, d\mu = \alpha \int_\Omega f\, d\mu + \beta \int_\Omega g\, d\mu$$

$\forall \alpha, \beta \in \mathbb{R}$.

Proof:

Suppose $f \in Q(\Omega, \mathcal{F}, \mu)$.

$$\therefore \quad \underbrace{-|f|}_{\in Q^+} \le f \le \underbrace{|f|}_{\in Q^-}$$

$$\therefore \quad -\int_\Omega |f|\, d\mu \underset{\text{Big 3 (1) \& (2)}}{\le} \int_\Omega f\, d\mu \underset{\text{Big 3 (1)}}{\le} \int |f|\, d\mu$$

$$\therefore \quad \left| \int_\Omega f\, d\mu \right| \le \int |f|\, d\mu.$$

Also $\underbrace{\int_\Omega |f|\, d\mu}_{<\infty} = \underbrace{\int_\Omega f^+ d\mu}_{<\infty} + \underbrace{\int_\Omega f^-}_{<\infty} \implies f \in Q^+ \cap Q^- = L_1$

Finally $f, g \in L_1(\Omega, \mathcal{F}, \mu)$ & $\alpha, \beta \in \mathbb{R}$

$$\implies \alpha f, \alpha g \in L_1(\Omega, \mathcal{F}, \mu)$$

$$\implies \int_\Omega \alpha f + \beta g\, d\mu \overset{B3}{=} \int_\Omega \alpha f\, d\mu + \int_\Omega \beta g\, d\mu$$

$$\overset{B3}{=} \alpha \int_\Omega f\, d\mu + \beta \int_\Omega g\, d\mu$$

QED

Using the linear part of Big (3)

An application typically looks like this:

$$\cdots = \int_\Omega \alpha f + \beta g\, d\mu$$

You've got to a point where this is well defined i.e. $\alpha f + \beta g \in Q(\Omega, \mathcal{F}, \mu)$

$$= \int_\Omega \alpha f\, d\mu + \int_\Omega \beta g\, d\mu$$

You can make this "move" if the terms on the right are defined & their sum isn't $\infty - \infty$ or $-\infty + \infty$.

e.g. $-\infty < \int_\Omega \alpha f\, d\mu$ & $\int_\Omega \beta g\, d\mu = \infty$

$\implies \int_\Omega (\alpha f)^- d\mu < \infty$ & $\int_\Omega (\beta g)^- d\mu < \infty$

$\implies \alpha f, \beta g \in Q^-(\Omega, \mathcal{F}, \mu)$

$\implies$ Big 3(2) applies

$$= \alpha \int_\Omega f\, d\mu + \int_\Omega \beta g\, d\mu$$

Now you can make this "move" if $\alpha \in \mathbb{R}$ or $f \ge 0$ or $f \le 0$.

e.g. Suppose $f \le 0$ so that $-f \in \mathcal{M}(\Omega, \mathcal{F})$

$\therefore \int_\Omega -\infty f\, d\mu = \infty \int_\Omega -f\, d\mu$ by Big 3(2)a

$= \infty(-1) \int_\Omega f\, d\mu$ by Big 3(2)b

Notation

In the liturature the following are all synonymous:

$$\int f\, d\mu \equiv \int_\Omega f\, d\mu \equiv \int_\Omega f(\omega)\, d\mu(\omega) \equiv \int_\Omega f(\omega)\, \mu(d\omega)$$

This one annoys me for some reason.

If $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \overline{B(\mathbb{R})}, \overline{\mathcal{L}}')$ then

$$\int f\, d\mu \equiv \int_\mathbb{R} f(x)\, dx \equiv \text{"Lebesgue integral"}$$

Counting measure and infinite series

Notice that $\int f\, d\mu$ is flexible enough to unify integration theory with part of (but not all) infinite series theory.

Let $\Omega = \mathbb{N} := \{1, 2, 3, \ldots\}$

$\mathcal{F} = 2^\Omega$

$\lambda =$ counting measure

Any $f(k)$ mapping $k \in \mathbb{N}$ to $\bar{\mathbb{R}}$ is ⓜ

Claim: $\displaystyle\int_{\mathbb{N}} f(k)\, d\lambda(k) = \sum_{k=1}^{\infty} f(k)$

whenever $\displaystyle\sum_{k=1}^{\infty} f^+(k) < \infty$ or $\displaystyle\sum_{k=1}^{\infty} f^-(k) < \infty$.

Proof: For any fixed $N \in \mathbb{N}$

$$f_N(k) := \begin{cases} f(k) & \text{for } 1 \le k \le N \\ 0 & \text{for } k > N \end{cases}$$

$$= \underbrace{f(1) I_{\{1\}}(k) + \cdots + f(N) I_{\{N\}}(k)}_{\text{has the form } \sum_{i=1}^{N} c_i I_{A_i}(k)}$$

with similar def for $f_N^+(k)$, $f_N^-(k)$.

Notice $f_N^+, f_N^- \in \mathcal{M}_s(\Omega, \mathcal{F})$.

$$\therefore \int_{\mathbb{N}} f_N^{\pm}(k)\, d\lambda(k) \overset{\text{def}}{=} \sum_{i=1}^{N} \overset{c_i}{f^{\pm}(i)}\, \underbrace{\lambda(\overset{A_i}{\{i\}})}_{=1 \text{ by counting measure}}$$

$$= \sum_{k=1}^{N} f^{\pm}(k)$$

$$\therefore \lim_{N\to\infty}\uparrow \int_{\mathbb{N}} f_N^{\pm}(k)\, d\lambda(k) = \sum_{k=1}^{\infty} f^{\pm}(k)$$

$\quad \| \ B3(3)$ since $f_N^{\pm}\uparrow$

$$\int_{\mathbb{N}} \lim_{N\to\infty}\uparrow f_N^{\pm}(k)\, d\lambda(k)$$

$$\|$$

$$\int_{\mathbb{N}} f^{\pm}(k)\, d\lambda(k)$$

$\therefore$ if $\displaystyle\sum_{k=1}^{\infty} f^+(k) < \infty$ or $\displaystyle\sum_{k=1}^{\infty} f^-(k) < \infty$

then $f \in \mathcal{Q}(\Omega, \mathcal{F}, \mu)$ &

$$\int_{\mathbb{N}} f(k)\, d\lambda(k) := \int_{\mathbb{N}} f^+(k)\, d\lambda(k) - \int_{\mathbb{N}} f^-(k)\, d\lambda(k)$$

$$= \sum_{k=1}^{\infty} \underbrace{[f^+(k) - f^-(k)]}_{f(k)}$$

QED

Warning! $\int_{\mathbb{N}} f(k)\, d\lambda(k)$ isn't defined for some convergent series $\sum_{k=1}^{\infty} f(k) < \infty$.

e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} < \infty$ but

$$\int_{\mathbb{N}} \left(\frac{(-1)^k}{k}\right)^{\pm} d\lambda(k) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

$\therefore \frac{(-1)^k}{k} \notin \mathcal{Q}(\Omega, \mathcal{F}, \lambda)$ so $\int_{\mathbb{N}} \frac{(-1)^k}{k}\, d\lambda(k)$ is not defined. The reason is that our $\int_\Omega f\, d\mu$ allows limiting infinite cancellation.

Lebesgue integration vrs. Riemann Integration.

Suppose $[a,b]$ is a bdd interval of $\mathbb{R}$ and $f: [a,b] \to \mathbb{R}$

Let

$$\underline{R}(f) := \sup\left\{ \sum_{A\in\pi} \left(\inf_{x\in A} f(x)\right) \cdot \mathcal{L}'(A) : \pi \text{ is a partition of } [a,b] \right\}$$

$$\bar{R}(f) := \inf\left\{ \sum_{A\in\pi} \left(\sup_{x\in A} f(x)\right) \cdot \mathcal{L}'(A) : \pi \text{ is a partition of } [a,b] \right\}$$

Def: The Riemann integral of f exists if

$$\underline{R}(f) = \bar{R}(f) < \infty$$

In which case, the common value, denoted $R(f)$ is the Riemann integral of f.

Thm (Lebesgue) If $[a,b]$ is a bdd subinterval of $\mathbb{R}$ and $f: [a,b] \to \mathbb{R}$ then $R(f)$ exists iff

- f is bounded and
- $\mu(\{x \in [a,b] : f \text{ is discontinuous } @ x\}) = 0$.

Moreover, if $R(f)$ exists then

$$R(f) = \int_{[a,b]} f(x)\,dx. \leftarrow \text{Lebesgue integral.}$$

Proof: Note ... there is no measurability assumption on f. Proof left as an exercise.

Note:

We can use this fact to compute the Lebesgue integral via the fundamental Thm of calculus:

f' is continuous & $a,b<\infty$

$$\Rightarrow \underbrace{\int_a^b f'(x)\,dx}_{\text{Lebesgue integral}} = R(f') = f(b)-f(a)$$

Important advantage of $\int f\,d\mu$ vrs $R(f)$

The Riemann integral of f is not invariant to changing f on a set of measure 0 ... but Lebesgue integration is.

Here are some examples to illustrate this & that Lebesgue integration is non-trivially more general than Riemann integration (for bdd $[a,b]$).

example 1: $f(x)=0$ on $x\in[0,1]$

$\therefore$ trivially $R(f)=0=\int_0^1 f(x)\,dx$.

example 2: $f(x)=I_{\mathbb{Q}}(x)$ on $x\in[0,1]$

since f is discontinuous at all $x\in[0,1]$, which has non-zero Lebesgue measure,

$R(f)$ does not exist.

but

$$\int_0^1 f(x)\,dx = 0.$$

example 3: $f(x)=I_C(x)$ on $x\in[0,1]$ where C is the Cantor set.

Now

uncountable set.

$$C=\{x\in[0,1]: f \text{ is discontinuous @ } x\}$$

$\mathcal{L}'(C)=0$ and f is bdd

$\therefore R(f)$ exists and equals $\int_0^1 f(x)\,dx=0$

At this point one might conjecture that for any Lebesgue integrable $f:[a,b]\to\mathbb{R}$ one can modify f on a $\mathcal{L}'$-null set to get a Riemann integrable f. The next example shows this is not true (implying that $\int f\,d\mathcal{L}'$ is non-trivially more general than $R(f)$).

example 4: $f(x)=I_V(x)$ on $x\in[0,1]$ where V is the fat cantor set

(this set is constructed by removing proportion $\frac{1}{3^n}$ at step n instead of proportion $1/3$ used to construct the cantor set).

In this case

$$V=\{x\in[0,1]: f \text{ is discontinuous @ } x\}$$

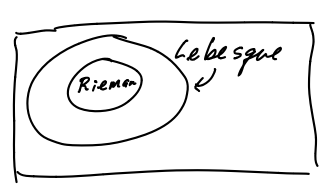
and

$$\mathcal{L}'(V)>0.$$

$\therefore R(f)$ does not exist ... yet $\int_0^1 f(x)\,dx$ does (and is non-zero).

Also notice that f can't be modified on a $\mathcal{L}'$-null set to get a Riemann integrable function.

Therefore we have the following picture

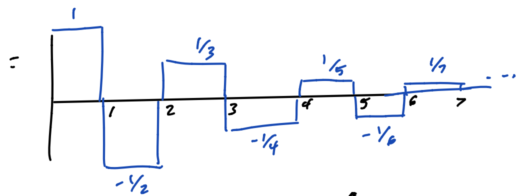


functions on bdd intervals of $\mathbb{R}$

The story is different for functions defined on non-bdd subintervals of $\mathbb{R}$.

In particular, there do exist improper Riemann integrals which are not Lebesgue integrable.

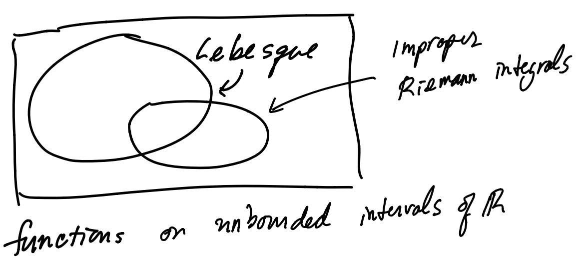
e.g. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} I_{[n-1,n)}(x)$

$=$ 

Has a finite improper Riemann integral

... but this isn't Lebesgue integrable.

so that ...



Integration to the limit

For this section let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f_1, f_2, \ldots$ & f be ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ mapping Ω into $\overline{\mathbb{R}}$.

Fatou's lemma

If $f_n \geq 0$ μ-a.e. then

$$\int_\Omega \underbrace{\liminf_{n\to\infty} f_n}_{\text{ⓜ by closure}} \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

Note: the fact that we don't have the "sup version" is related to the fact that general measures are not guarenteed to be continuous from above (... & if we did then $\lim_n \int f_n d\mu$ would always equal $\int \lim_n f_n d\mu$).

Proof:

$$LHS = \int_\Omega \lim_{q\to\infty}\uparrow \inf_{n\geq q} f_n \, d\mu$$

$$= \lim_{q\to\infty}\uparrow \underbrace{\int_\Omega \inf_{n\geq q} f_n \, d\mu}_{\text{smaller than } \int f_n d\mu \ \forall n\geq q \text{ by Big 3}} \quad \text{By Big 3.}$$

$$\leq \lim_{q\to\infty}\uparrow \inf_{n\geq q} \int_\Omega f_n \, d\mu$$

$$= \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$

QED.

The following theorems give conditions when

$$\lim_n \int = \int \lim_n$$

and can all be thought of as conditions for extending Fatou's lemma to include the limsup upper bounds.

Thm (DCT)

If (a) $f_n \to f$ μ-a.e. as $n\to\infty$

(b) $\sup_n |f_n| \underset{\mu\text{-a.e.}}{\leq} g \in L_1(\Omega, \mathcal{F}, \mu)$

then (A) $f_n, f \in L_1(\Omega, \mathcal{F}, \mu)$

(B) $\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$

Proof:

(b) implies that $f_n, f, \liminf_n f_n$ & $\limsup_n f_n$ are all in $L_1(\Omega, \mathcal{F}, \mu)$

since $|\liminf_n f_n| \le \liminf_n |f_n| \le g$

$\therefore$ (B) is true

To show (A) notice

$$\int_\Omega g\,d\mu + \int_\Omega f\,d\mu \overset{\text{Big 3}}{=} \int_\Omega g+f\,d\mu$$

$$= \int_\Omega \liminf_n \underbrace{(g+f_n)}_{\ge 0\ \mu\text{-a.e since } -g \le f_n \le g\ \mu\text{-a.e.}} d\mu$$

$$\overset{\text{Fatou}}{\le} \liminf_n \int_\Omega g+f_n\,d\mu$$

$$\overset{\text{Big 3}}{=} \int_\Omega g\,d\mu + \liminf_n \int_\Omega f_n\,d\mu$$

These cancel since $\int_\Omega g\,d\mu < \infty$

$$\therefore \int_\Omega f\,d\mu \le \liminf_n \int_\Omega f_n\,d\mu$$

Side Note: this gives us an extension to Fatou:

$-g \le f_n$ & $g \in L_1$

$\Rightarrow \int_\Omega \liminf_n f_n\,d\mu \le \liminf_n \int_\Omega f_n\,d\mu$

Now all we need is $\limsup_n \int_\Omega f_n\,d\mu \le \int_\Omega f\,d\mu$.

$$\cancel{\int_\Omega g\,d\mu} - \int_\Omega f\,d\mu \overset{\text{Big 3}}{=} \int_\Omega \liminf_n \underbrace{(g-f_n)}_{\ge 0\ \mu\text{-a.e.}} d\mu$$

$$\overset{\text{Fatou}}{\le} \liminf_n \int_\Omega g - f_n\,d\mu$$

$$\overset{\text{Big 3}}{=} \cancel{\int_\Omega g\,d\mu} - \limsup_n \int_\Omega f_n\,d\mu$$

$$\therefore \limsup_n \int_\Omega f_n\,d\mu \le \int_\Omega f\,d\mu.$$

QED

Corollary (BCT)

If $f_n \to f$ μ-a.e. & $\exists B \in \mathbb{R}$ s.t. $|f_n| \le B$ μ-a.e. $\forall n$ and $\mu(\Omega) < \infty$ then $f \in L_1(\Omega, \mathcal{F}, \mu)$ and

$$\lim_n \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu.$$

Def: $f_1, f_2, \ldots$ are *uniformly integrable* (UI) if

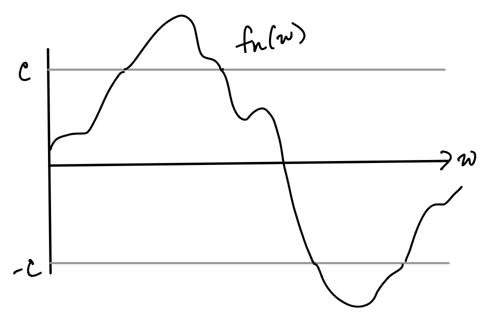
$$\lim_{c \to \infty} \sup_n \int_\Omega |f_n| I_{\{|f_n| \ge c\}}\,d\mu = 0.$$

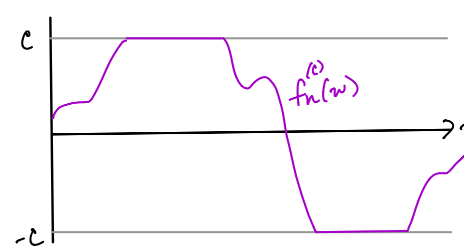
This looks strange untill one notices that it allows one to control

$$\int_\Omega |f_n - f_n^{(c)}|\,d\mu$$

where $f_n^{(c)}$ is the "clamped f at c":

$$f_n^{(c)}(\omega) := f_n I_{\{|f_n| < c\}}$$





$$\therefore \int_\Omega |f_n - f_n^{(c)}|\,d\mu = \int_\Omega |f_n| I_{\{|f_n| \ge c\}}\,d\mu$$

The original definition of UI is a bit clumsy to work with. The following theorem gives a more managable criterion.

Thm (Dilatation criterion for UI)

If $\exists \varepsilon > 0$ s.t.

$$\sup_n \int_\Omega |f_n|^{1+\varepsilon} d\mu < \infty$$

then the f_n's are UI.

Proof:

$$\int_\Omega |f_n| I_{\{|f_n|\geq c\}} d\mu \overset{\text{Big 3}}{\leq} \int_\Omega |f_n| \underbrace{\left(\frac{|f_n|}{c}\right)^\varepsilon}_{\geq 1 \text{ on } \{|f_n|\geq c\}} I_{\{|f_n|\geq c\}} d\mu$$

$$\leq \frac{1}{c^\varepsilon} \int_\Omega |f_n|^{1+\varepsilon} d\mu$$

Take $\sup_n$ then $\lim_{c\to\infty}$ to get UI.

QED.

Thm (UI condition for $\lim \int = \int \lim$)

If (a) $f_n \to f$ μ-a.e.
(b) the f_n's are UI
(c) $\mu(\Omega) < \infty$

then (A) $f_n, f \in L_1(\Omega, \mathcal{F}, \mu)$
(B) $\lim_n \int_\Omega f_n d\mu = \int_\Omega f d\mu$

Proof:

To prove (A) notice

$$\int_\Omega |f_n| d\mu \overset{\text{Big 3}}{=} \underbrace{\int_\Omega |f_n| I_{\{|f_n|\geq c\}} d\mu}_{\substack{\leq \sup_n \int_\Omega |f_n| I_{\{|f_n|\geq c\}} d\mu \\ \to 0 \text{ as } c\to\infty}} + \underbrace{\int_\Omega |f_n| I_{\{|f_n|< c\}} d\mu}_{\leq c\mu(\Omega) < \infty}$$

$\therefore \int_\Omega |f_n| d\mu < \infty$ so $f_n \in L_1(\Omega, \mathcal{F}, \mu)$.

$f \in L_1(\Omega, \mathcal{F}, \mu)$ since

$$\int_\Omega |f| d\mu = \int_\Omega \liminf_n |f_n| d\mu$$

$$\leq \liminf_n \int |f_n| d\mu \quad \text{by Fatou}$$

$$\leq \sup_n \int_\Omega |f_n| I_{\{|f_n|\geq c\}} d\mu + c\mu(\Omega)$$

$< \infty$ since $\int$ is finite for large enough c

Now to show (B) notice

Fact: $\exists c_1, c_2, \ldots$ s.t. $\mu(|f| = c_k) = 0$ and $\lim_{k\to\infty} c_k = \infty$.

This follows by a Thm in Lecture 5 which states that μ can not assign non-zero mass to uncountably many disjoint sets in $\mathcal{F}$. Since $\{|f| = c\}$ forms disjoint sets for different $c \in \mathbb{R}$, $\mu(|f| = c) > 0$ for countably many $c \in \mathbb{R}$.

$$\therefore \limsup_n \left| \int_\Omega f_n d\mu - \int_\Omega f d\mu \right| \quad \pm \int_\Omega f_n^{(c_k)} d\mu \pm \int_\Omega f^{(c_k)} d\mu$$

$$\leq \underbrace{\limsup_n \left| \int_\Omega (f_n - f_n^{(c_k)}) d\mu \right|}_{\mathrm{I}}$$

$$+ \underbrace{\limsup_n \left| \int_\Omega f_n^{(c_k)} d\mu - \int_\Omega f^{(c_k)} d\mu \right|}_{\mathrm{II}} + \underbrace{\left| \int_\Omega (f - f^{(c_k)}) d\mu \right|}_{\mathrm{III}}$$

where

term $\mathrm{I} \leq \sup_n \int_\Omega |f_n| I_{\{|f_n|\geq c_k\}} d\mu \xrightarrow{k\to\infty} 0$, since $c_k \xrightarrow{k\to\infty} \infty$.

term $\mathrm{II} = 0$ To see this notice that

$I_{\{|f_n| < c\}} \xrightarrow{n\to\infty} I_{\{|f| < c\}}$ μ-a.e. whenever $\mu(|f| = c) = 0$.

$\therefore f_n^{(c_k)} \xrightarrow{n\to\infty} f^{(c_k)}$ μ-a.e. $\forall k$

$\therefore$ BCT applies $\Rightarrow \int_\Omega f_n^{(c_k)} d\mu \xrightarrow{n\to\infty} \int_\Omega f^{(c_k)} d\mu$

$$\text{term III} \le \int_\Omega |f| I_{\{|f|\ge c_k\}}\, d\mu$$

$$\xrightarrow{k\to\infty} \infty\, \mu(|f|=\infty)$$

since $I_{\{|f|\ge c_k\}} \uparrow I_{\{|f|=\infty\}}$

Since $f\in L_1(\Omega,\mathcal{F},\mu)$ we have $\mu(|f|=\infty)=0$ by (i) of "useful facts".

$\therefore$ term III $\to 0$ as $k\to\infty$.

Now we have

$$\limsup_n \left| \int_\Omega f_n\, d\mu - \int_\Omega f\, d\mu \right| = 0.$$

QED

We will need to differentiate under the integral when working with moment generating functions, etc. Here are sufficient conditions for reference:

Thm $\left(\frac{d}{dt}\int f_t\, d\mu = \int \frac{d}{dt} f\, d\mu \right)$

Suppose $a<b$ are real numbers & $\forall t\in(a,b)$ $f_t \in L_1(\Omega,\mathcal{F},\mu)$. Let $t_0\in(a,b)$. If $\exists\, \Omega_0\in\mathcal{F}$ s.t.

(a) $\mu(\Omega_0^c)=0$

(b) $\frac{d}{dt} f_t(\omega)\Big|_{t=t_0}$ exists $\forall \omega\in\Omega_0$

(c) $\sup_{\substack{t\in N\\ t\neq t_0}} \left| \frac{f_t(\omega)-f_{t_0}(\omega)}{t-t_0} \right| \le g(\omega)$ $\forall\omega\in\Omega_0$

for some $g\in L_1(\Omega,\mathcal{F},\mu)$ & open $N\subset(a,b)$ containing t_0

Then

(A) $\frac{df_t}{dt}\Big|_{t=t_0} \in L_1(\Omega,\mathcal{F},\mu)$

(B) $\frac{d}{dt}\int_\Omega f_t\, d\mu\Big|_{t=t_0}$ exists

(C) $\frac{d}{dt}\int_\Omega f_t\, d\mu\Big|_{t=t_0} = \int_\Omega \frac{d}{dt} f_t\Big|_{t=t_0} d\mu$

The proof is left as an exercise. However it should be noted that the mean value thm implies that (b) & (c) can be replaced with the stronger conditions:

(b′) $\frac{d}{dt} f_t(\omega)$ exists $\forall\omega\in\Omega_0$ neighborhood N of t_0

(c′) $\sup_{t\in N} \left| \frac{d f_t(\omega)}{dt} \right| \le g(\omega)$ for some $g\in L_1(\Omega,\mathcal{F},\mu)$.

Change of Variables

The last theorem of this lecture covers an extremely useful theorem for the following setup:

$$\begin{tikzcd}
\underset{\mathcal{F}_1,\mu}{\Omega_1} \arrow[r, "T\ (m)"] \arrow[rr, bend left, "f\circ T\ (m)"] & \underset{\mathcal{F}_2,\mu T^{-1}}{\Omega_2} \arrow[r, "f\ (m)"] & \underset{\mathcal{B}(\bar{\mathbb{R}})}{\bar{\mathbb{R}}}
\end{tikzcd}$$

The following thm says you can integrate f w.r.t. μT^{-1} or $f\circ T$ w.r.t. μ ... both give the same answer

Thm (Change of Variarables)

In the set up above

(a) $f\in \mathcal{Q}^+(\Omega_2,\mathcal{F}_2,\mu T^{-1}) \iff f\circ T\in \mathcal{Q}^+(\Omega_1,\mathcal{F}_1,\mu)$

(b) $f\in \mathcal{Q}^-(\Omega_2,\mathcal{F}_2,\mu T^{-1}) \iff f\circ T\in \mathcal{Q}^-(\Omega_1,\mathcal{F}_1,\mu)$

(c) If any of the four statements in (a), (b) hold then

$$\int_{\Omega_1} f\circ T\, d\mu = \int_{\Omega_2} f\, d\mu T^{-1}.$$

Proof:

This is proved using the "1-2-3 argument", i.e.

Let $\mathcal{G}$ denote the f's which satisfy (a), (b) & (c).

Step 1: show $\mathcal{M}_s(\Omega_2, \mathcal{F}_2) \subset \mathcal{G}$

Step 2: show $\mathcal{M}(\Omega_2, \mathcal{F}_2) \subset \mathcal{G}$

Step 3: show $f \in \mathcal{G}$ whenever $f: \Omega_2 \to \overline{\mathbb{R}}$.

For step 1, suppose $f \in \mathcal{M}_s(\Omega_2, \mathcal{F}_2)$.

Clearly $f \in \mathcal{Q}^-(\Omega_2, \mathcal{F}_2, \mu)$ & $f \circ T \in \mathcal{Q}^-(\Omega_1, \mathcal{F}_1, \mu)$ so (b) holds.

Now the following two integrals are defined:

$$\int_{\Omega_2} f\, d\mu T^{-1} = \int_{\Omega_2} \sum_{i=1}^n c_i I_{A_i}\, d\mu T^{-1} := \sum_{i=1}^n c_i \mu T^{-1}(A_i)$$

$$\parallel$$

$$\int_{\Omega_1} f \circ T\, d\mu = \int_{\Omega_1} \sum_{i=1}^n c_i I_{A_i} \circ T\, d\mu = \sum_{i=1}^n c_i \mu(T \in A_i)$$

$\therefore$ (c) holds.

To show (a) notice that

$$f \in \mathcal{Q}^+(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$
$$\iff \int_{\Omega_2} f\, d\mu T^{-1} < \infty \quad \text{since } f^+ = f$$
$$\parallel \text{ by (b)}$$
$$\iff \int_{\Omega_1} f \circ T\, d\mu < \infty$$
$$\iff f \circ T \in \mathcal{Q}^+(\Omega_1, \mathcal{F}_1, \mu)$$

$\therefore \mathcal{M}_s(\Omega_2, \mathcal{F}_2) \subset \mathcal{G}$.

For step 2:

Again, any $f \in \mathcal{M}(\Omega_2, \mathcal{F}_2)$ satisfies (b). Also

$$\int_{\Omega_2} f\, d\mu T^{-1} = \int_{\Omega_2} \lim_n \uparrow \underbrace{f_n}_{\in \mathcal{M}_s}\, d\mu T^{-1}$$
$$= \lim_n \uparrow \int_{\Omega_2} f_n\, d\mu T^{-1} \quad \text{by Big 3}$$
$$= \lim_n \uparrow \int_{\Omega_1} f_n \circ T\, d\mu \quad \text{by step 1}$$
$$= \int_{\Omega_1} f \circ T\, d\mu \quad \text{by Big 3.}$$

$\therefore$ (c) holds

Now (a) holds by similar reasoning as in step 2.

$\therefore \mathcal{M}(\Omega_2, \mathcal{F}_2) \subset \mathcal{G}$.

For step 3. Let $f: \Omega_2 \to \overline{\mathbb{R}}$.

To show (a) & (b) notice that

$$f \in \mathcal{Q}^\pm(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$
$$\iff \int_{\Omega_2} f^+ d\mu T^{-1} < \infty \text{ or } \int_{\Omega_2} f^- d\mu T^{-1} < \infty$$
$$\parallel \text{ step 2} \qquad\qquad \parallel \text{ step 2}$$
$$\iff \int_{\Omega_1} f^+ \circ T\, d\mu < \infty \text{ or } \int_{\Omega_1} f^- \circ T\, d\mu < \infty$$
$$\iff f \circ T \in \mathcal{Q}^\pm(\Omega_1, \mathcal{F}_1, \mu).$$

Also note that (c) follows since $f^+, f^- \in \mathcal{M}(\Omega_2, \mathcal{F}_2) \subset \mathcal{G}$.

QED

Side fact: For any $A \in \mathcal{F}_2$

$$f \in \mathcal{Q}^\pm(\Omega_2, \mathcal{F}_2, \mu T^{-1}) \Rightarrow f I_A \in \mathcal{Q}^\pm(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$

since $f^+ I_A \le f^+$ and $f^- I_A \le f^-$.

$\therefore$ If (a) or (b) hold & $A \in \mathcal{F}_2$ then

$$\underbrace{\int_A f\, d\mu T^{-1}}_{= \int_{\Omega_2} f I_A\, d\mu} = \underbrace{\int_{T^{-1}(A)} f \circ T\, d\mu}_{= \int_{\Omega_1} (f \circ T)(I_A \circ T)\, d\mu}$$

e.g. Let X be a random variable defined on $(\Omega, \mathcal{F}, P)$

$$\begin{tikzcd} \underset{\mathcal{F}, P}{\Omega} \arrow[r, bend left, "X(\omega)"] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

We have mentioned that

$$E(X) := \int_\Omega X \, dP \qquad (*)$$

Unfortunately this looks nothing like the undergrad definition

$$E(X) = \int_{\mathbb{R}} x f(x) \, dx \qquad (**)$$

where f is "the probability density of X".

We will cover densities in the next lecture but we can halfway to $(**)$ from $(*)$ with the change of variables thm.

In particular let $T \equiv X$ and $f = id$ in the change of variables Thm to get the following picture:

$$\begin{tikzcd} \underset{\mathcal{F}, P}{\Omega} \arrow[r, bend left, "X(\omega)"] \arrow[rr, bend left=50, "id \circ X(\omega)"] & \underset{\mathcal{B}(\mathbb{R}),\, PX^{-1}}{\mathbb{R}} \arrow[r, bend left, "id(\omega)"] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

$$\therefore \quad E(X) = \int_\Omega X(\omega) \, dP(\omega)$$

$$= \int_\Omega id \circ X(\omega) \, dP(\omega)$$

by change of variables.

$$= \int_{\mathbb{R}} id(x) \, dPX^{-1}(x)$$

$$= \int_{\mathbb{R}} x \, \underbrace{dPX^{-1}(x)}$$

" this step will be covered in the next lecture

$$= \int_{\mathbb{R}} x \, \overbrace{f(x) dx}$$