Lecture 2: Classes of sets & the "Good sets" technique.

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Part of the goal of this class is to get exposure to the measure theory that underpins probability theory.

This means that we will have to endure a fair amount of technical definitions etc before we can get to the good stuff.

Fix some set Ω, the sample space.

Let $2^\Omega :=$ power set of Ω.
$= \underline{\text{set}}$ of all subsets of Ω.

Def: A $\mathcal{F} \subset 2^\Omega$ is non-empty and
$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

Then $\mathcal{F}$ is a...

field if $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

σ-field if $A_1, A_2, \ldots \in \mathcal{F} \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

λ-system if $\underbrace{A_1, A_2, \ldots}_{\text{disjoint}} \in \mathcal{F} \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

Note: Many textbooks replace the non-empty requirement by $\Omega \in \mathcal{F}$ or $\emptyset \in \mathcal{F}$ but closure under complementation makes them all equivalent definitions

Def: $\mathcal{M} \subset 2^\Omega$ is a monotone class

$A_1, A_2 \ldots \in \mathcal{M}$ & $A_n \uparrow A \Longrightarrow A \in \mathcal{M}$ &

$A_1, A_2, \ldots \in \mathcal{M}$ & $A_n \downarrow A \Longrightarrow A \in \mathcal{M}$

where $A_n \uparrow A$ means $\begin{cases} A_1 \subset A_2 \subset \cdots \\ A = \bigcup_{k=1}^{\infty} A_k \end{cases}$

and $A_n \downarrow A$ means $\begin{cases} A_1 \supset A_2 \supset \cdots \\ A = \bigcap_{k=1}^{\infty} A_k \end{cases}$.

Note: we will sometimes write $\lim_n \uparrow A_n$ & $\lim_n \downarrow A_n$ for the monotonic limits above.

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Def: $\mathcal{P} \subset 2^\Omega$ is a π-system if
$$A, B \in \mathcal{P} \Longrightarrow A \cap B \in \mathcal{P}.$$

Overview of important Results.

a class that is a field & a monotonic

1) $\sigma = \lambda + \pi = f + \mathcal{M}$

2) $\sigma\langle\pi\rangle = \lambda\langle\pi\rangle$ (Dynkin's π-λ Thm)

3) $\sigma\langle f\rangle = \mathcal{M}\langle f\rangle$ (Halmo's monotone class thm)

the monotone class generated by a field. useful for Carathéodory Thm

4) if probs $P = Q$ on π then $P = Q$ on $\sigma\langle\pi\rangle$.

Lets look at some examples first.

Examples and Observations

e.g. $(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}] \uparrow (\frac{1}{2}, \frac{1}{4})$ since $(\frac{1}{2}, \frac{1}{4}) = \bigcup_{n=1}^{\infty} \underbrace{(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}]}_{\text{increase as sets.}}$

$(\frac{1}{4} - \frac{1}{n}, \frac{1}{4} + \frac{1}{n}] \downarrow \{\frac{1}{4}\}$ ...

e.g. $\Omega = (0, 1]$

$\mathcal{F}$ = Finite disjoint unions of intervals $(a, b] \subset \Omega$

$\mathcal{F}$ is a field but not a σ-field since

$(\frac{1}{2}, \frac{1}{4}) \notin \mathcal{F}$ but $(\frac{1}{2}, \frac{1}{4}) = \bigcup_{n=1}^{\infty} \underbrace{(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}]}_{\in \mathcal{F}}$

e.g. 2^Ω satisfies the conditions of all 5 defs.

... so they are not vacuous.

Note: fields & σ-fields are closed under

$A \cap B = (A^c \cup B^c)^c$

$A - B = A \cap B^c$

$A \Delta B = (A - B) \cup (B - A)$

Note: λ-systems are not necessarily closed under intersection.

e.g. $\Omega = \{1, 2, 3, 4\}$

$\mathcal{F} = \{\{1,2\}, \{2,3\}, \{1,4\}, \{3,4\}, \emptyset, \Omega\}$

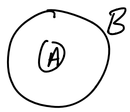
is a λ-sys but $\{2\} = \{1,2\} \cap \{2,3\} \notin \mathcal{F}$.

Note: λ-systems are closed under nested set subtraction.

i.e. if $\mathcal{F}$ is a λ-system then

$A, B \in \mathcal{F}$ & $A \subset B \implies B - A = B \cap A^c$

$= (\underbrace{B^c \cup A}_{\text{disjoint}})^c$



$\in \mathcal{F}$.

Note: For any $A_1, A_2, \ldots \in 2^\Omega$

$$\bigcup_{k=1}^{n} A_k \uparrow \bigcup_{k=1}^{\infty} A_k$$

$$\bigcap_{k=1}^{n} A_k \downarrow \bigcap_{k=1}^{\infty} A_k.$$

so monotone classes are closed under countable intersection & union.

$\sigma = \lambda + \pi = f + m$

Thm: If $\mathcal{F} \subset 2^\Omega$ then

$\mathcal{F}$ is a σ-field $\iff$ $\mathcal{F}$ is a λ-system & a π-system

$\iff$ $\mathcal{F}$ is a field & a monotone class.

Proof:

$(\sigma \Rightarrow \lambda + \pi)$: Trivial

$(\sigma \Leftarrow \lambda + \pi)$: Suppose $\mathcal{F}$ is a λ-sys & a π-sys.

All we need to show is that $\mathcal{F}$ is closed under countable non-disjoint unions.

We use a trick that we'll use later.

Let $A_1, A_2, \ldots \in \mathcal{F}$.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \underbrace{A_n - (A_1 \cup \cdots \cup A_{n-1})}_{\text{only add in unique points} \implies \text{these are disjoint.}}$$

$$= \bigcup_{n=1}^{\infty} A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c$$

in $\mathcal{F}$ by λ-sys.

in $\mathcal{F}$ by π-sys.

$\in \mathcal{F}$.

$(\sigma \Rightarrow f + m)$: Suppose $\mathcal{F}$ is a σ-field

Clearly $\mathcal{F}$ is a field.

Let $A_1, A_2, \ldots \in \mathcal{F}$

$A_n \uparrow A \implies A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ by σ-field props

$A_n \downarrow A \implies A = \bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$

$(\sigma \Leftarrow f + m)$: All we need to show is

$$A_1, A_2, \ldots \in \mathcal{F} \implies \underbrace{\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}}$$

follows by $\bigcup_{n=1}^{N} A_n \uparrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ by m props.

QED.

Generators

Let $\mathcal{C} \subset 2^\Omega$ & define

$\sigma\langle \mathcal{C} \rangle =$ σ-field generated by $\mathcal{C}$

$$:= \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-field} \\ \mathcal{C} \subset \mathcal{F}}} \mathcal{F}$$

Define $f\langle \mathcal{C} \rangle$, $\lambda\langle \mathcal{C} \rangle$, $m\langle \mathcal{C} \rangle$ similarly.

Thm: $\sigma\langle \mathcal{C} \rangle$ is a σ-field containing $\mathcal{C}$.

Proof:

- $\mathcal{C} \subset \sigma\langle \mathcal{C} \rangle$ since $\exists$ a σ-field $\mathcal{F}$ s.t. $\mathcal{C} \subset \mathcal{F}$ i.e. $\mathcal{F} = 2^\Omega$
- $\Omega \in \mathcal{F}$, $\forall$ σ-fields $\mathcal{F}$ so $\sigma\langle \mathcal{C} \rangle$ is not empty
- $A \in \sigma\langle \mathcal{C} \rangle \implies A \in \mathcal{F}$, $\forall$ such $\mathcal{F}$
  $\implies A^c \in \mathcal{F}$, $\forall$ such $\mathcal{F}$
  $\implies A^c \in \sigma\langle \mathcal{C} \rangle$.
- countable additivity is similar QED.

Same goes for $\lambda\langle \mathcal{C}\rangle$, $f\langle \mathcal{C}\rangle$ & $\mathcal{M}\langle \mathcal{C}\rangle$.

e.g. $B_0^{(0,1]}$ = finite disjoint unions of sets $(a,b]\subset(0,1]$.

$\mathcal{B}^{(0,1]} = \sigma\langle B_0^{(0,1]}\rangle$ is called the Borel σ-field of $(0,1]$

$\mathcal{B}^{(0,1]}$ is very rich. It contains all closed, open, one point, countable sets.

Also Note, $N \in \mathcal{B}^{(0,1]}$
set of Normal numbers.

To see why:

$$w\in N \iff \lim_n \frac{S_n(w)}{n} = 0$$

$$\iff \forall k\, \exists m \text{ s.t. } \forall n\geq m\ \left|\frac{S_n(w)}{n}\right| < \frac{1}{k}$$

$$\iff w \in \bigcap_k \bigcup_m \bigcap_{n\geq m} \underbrace{\left\{\left|\frac{S_n}{n}\right| < \frac{1}{k}\right\}}_{\in B_0^{(0,1]}}$$

<u>Important Fact</u>: There is no simple recipe for general $A\in\sigma\langle\mathcal{C}\rangle$, i.e. $\exists A\in\sigma\langle\mathcal{C}\rangle$ s.t.

$$A \neq \bigcup_{n=1}^{\infty} C_n$$

for any choice of $C_n \in \mathcal{C}$.

This makes it hard to prove things about $\sigma\langle\mathcal{C}\rangle$. Here is the main tool you can use.

Suppose you want to show each $A\in\sigma\langle\mathcal{C}\rangle$ satisfies some property.
Let $\mathcal{G}\subset 2^{\Omega}$ be all sets that have this property ("the good sets").

<u>Thm</u> (good sets):

$$\mathcal{C}\subset \underbrace{\mathcal{G}}_{\text{a } \sigma\text{-field}} \implies \sigma\langle\mathcal{C}\rangle\subset\mathcal{G}$$

i.e. if the generators $\mathcal{C}$ are "good" & the "good sets" have enough closure properties then everything in $\sigma\langle\mathcal{C}\rangle$ is "good."

<u>Proof</u>: $$\sigma\langle\mathcal{C}\rangle = \bigcap_{\left\{\substack{\mathcal{F} \text{ is a } \sigma\text{-field} \\ \mathcal{C}\subset\mathcal{F}}\right.} \mathcal{F}$$

$\mathcal{G}$ is one of these so $\sigma\langle\mathcal{C}\rangle$ is a further restriction

QED.

Lets finish with an easy example of this technique.
Next time we will use it to full effect to prove Dynkin's π-λ Thm.

Thm: Let Ω be a sample space

$\Omega_0 \subset \Omega$

$\mathcal{E} \subset 2^{\Omega}$. a σ-field of Ω_0 sets

Then $\sigma\langle \mathcal{E} \cap \Omega_0 \rangle = \sigma\langle \mathcal{E} \rangle \cap \Omega_0$.

Note: $\mathcal{F} \cap \Omega_0 := \{ F \cap \Omega_0 : F \in \mathcal{F} \}$

Proof:

$\sigma\langle \mathcal{E} \cap \Omega_0 \rangle \subset \sigma\langle \mathcal{E} \rangle \cap \Omega_0$:

Follows by good sets since

$$\underbrace{\mathcal{E} \cap \Omega_0}_{\text{generators}} \subset \underbrace{\sigma\langle \mathcal{E} \rangle \cap \Omega_0}_{\text{good sets.}}$$

Form a σ-field by an exercise

$\therefore \sigma\langle \mathcal{E} \cap \Omega_0 \rangle \subset \sigma\langle \mathcal{E} \rangle \cap \Omega_0$ by good sets.

$\sigma\langle \mathcal{E} \rangle \cap \Omega_0 \subset \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$:

Let $\mathcal{G} \subset 2^{\Omega}$ include all sets s.t.

$\mathcal{G} \cap \Omega_0 \subset \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$, i.e. $G \in \mathcal{G}$ iff $G \cap \Omega_0 \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$

Clearly $\mathcal{E} \subset \mathcal{G}$. Now just show $\mathcal{G}$ is a σ-field.

✓ $\Omega \in \mathcal{G}$ since $\sigma\langle \mathcal{E} \cap \Omega_0 \rangle$ is a σ-field on Ω_0 & $\Omega \cap \Omega_0 = \Omega_0 \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$

✓ $A \in \mathcal{G} \Rightarrow A \cap \Omega_0 \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$

$$\Rightarrow \underbrace{A^c \cap \Omega_0}_{\text{Complement of } A \text{ in } \Omega_0} \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$$

$\Rightarrow A^c \in \mathcal{G}$

✓ $A_1, A_2, \ldots \in \mathcal{G} \Rightarrow A_k \cap \Omega_0 \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle, \forall k$

$\Rightarrow \left[\bigcup_{k=1}^{\infty} A_k \right] \cap \Omega_0 \in \sigma\langle \mathcal{E} \cap \Omega_0 \rangle$ can add these

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{G}$

QED.