Lemmas for the Carathéodory Extension Theorem.

Definitions

(i) P_0 is a probability measure on the field $\mathcal{F}_0$ of subsets in Ω.

(ii) $\mathcal{F}^{\uparrow} := \{\lim_k\!\uparrow A_k : A_k \in \mathcal{F}_0\}$ and $P^{\uparrow}(\lim_k\!\uparrow A_k) := \lim_k P_0(A_k)$

(iii) $\mathcal{F}^{\downarrow} := \{\lim_k\!\downarrow A_k : A_k \in \mathcal{F}_0\}$ and $P^{\downarrow}(\lim_k\!\downarrow A_k) := \lim_k P_0(A_k)$

(iv) $P^*(A) = \inf\{P^{\uparrow}(B) : A \subset B \in \mathcal{F}^{\uparrow}\}$

(v) $P_*(A) = \sup\{P^{\downarrow}(B) : A \supset B \in \mathcal{F}^{\downarrow}\}$

(vi) $\overline{\mathcal{F}} := \{A \in 2^{\Omega} : P^*(A) = P_*(A)\}$ and $\overline{P}(A) := P^*(A) = P_*(A)$ for $A \in \overline{\mathcal{F}}$

(vii) $\mathcal{F} := \sigma\langle \mathcal{F}_0 \rangle$. Note: we will show $\mathcal{F} \subset \overline{\mathcal{F}}$.

(viii) $P(A) := \overline{P}(A)$ for all $A \in \mathcal{F}$.

Lemma 1.

(i) $\lim_k\!\uparrow (A_k \cap B) = (\lim_k\!\uparrow A_k) \cap B$

(ii) $\lim_k\!\uparrow (A_k \cup B) = (\lim_k\!\uparrow A_k) \cup B$

Lemma 2. Properties of $\mathcal{F}^{\uparrow}$ and $\mathcal{F}^{\downarrow}$.

(i) $A \in \mathcal{F}^{\uparrow} \Leftrightarrow A^c \in \mathcal{F}^{\downarrow}$

(ii) $\mathcal{F}^{\uparrow} = \{\bigcup_{k=1}^{\infty} A_k \colon A_k \in \mathcal{F}_0\}$

(iii) $\mathcal{F}^{\downarrow} = \{\bigcap_{k=1}^{\infty} A_k \colon A_k \in \mathcal{F}_0\}$

(iv) If $A, B \in \mathcal{F}^{\uparrow}$ then $A \cap B \in \mathcal{F}^{\uparrow}$ and $A \cup B \in \mathcal{F}^{\uparrow}$.

(v) $\mathcal{F}^{\uparrow}$ is closed under countable unions and increasing limits of $\mathcal{F}^{\uparrow}$ sets.

Lemma 3. Properties of $P^{\uparrow}$ and $P^{\downarrow}$.

(i) If $A \in \mathcal{F}^{\uparrow}$ then $P^{\uparrow}(A) + P^{\downarrow}(A^c) = 1$.

(ii) If $A, B \in \mathcal{F}^{\uparrow}$ then $P^{\uparrow}(A \cup B) = P^{\uparrow}(A) + P^{\uparrow}(B) - P^{\uparrow}(A \cap B)$.

(iii) If $A \subset B$ and $A, B \in \mathcal{F}^{\uparrow}$ then $P^{\downarrow}(A) \le P^{\uparrow}(A) \le P^{\uparrow}(B)$.

(iv) If $\lim_n\!\uparrow A_n = A$ and $A_n \in \mathcal{F}^{\uparrow}$ then $P^{\uparrow}(A_n) \nearrow P^{\uparrow}(A)$.

Lemma 4. Properties of P^* and P_*.

Fact 1: If $A \in 2^{\Omega}$ then $P^*(A) + P_*(A^c) = 1$. Almost the complement rule.

Fact 2: If $A \subset B \subset C$ and $A, B, C \in 2^{\Omega}$ then $P_*(A) \le P_*(B) \le P^*(B) \le P^*(C)$.

Fact 3: If $A, B \in 2^{\Omega}$ then $P^*(A \cup B) \le P^*(A) + P^*(B) - P^*(A \cap B)$. Almost inclusion-exclusion.

Fact 4: If $A, B \in 2^{\Omega}$ then $P_*(A \cup B) \ge P_*(A) + P_*(B) - P_*(A \cap B)$. Almost inclusion-exclusion.

(i) If $A_k \subset B_k$ then $P^*(\cup_{k=1}^n B_k) - P^*(\cup_{k=1}^n A_k) \le \sum_{k=1}^n \big[P^*(B_k) - P^*(A_k)\big]$. Approximating unions term-by-term.

Fact 5: If $\lim_n\!\uparrow A_n = A$ then $P^*(A_n) \nearrow P^*(A)$. Continuous from below.

Proof of Lemma 1.

(i) Notice that $A_k \subset A_{k+1}$ implies $A_k \cap B \subset A_{k+1} \cap B$ for all $k = 1, 2, \ldots$. Therefore

$$\lim_k\uparrow (A_k \cap B) = \bigcup_{k=1}^{\infty} (A_k \cap B) = B \cap \bigcup_{k=1}^{\infty} A_k = B \cap \lim_k\uparrow A_k.$$

(ii) Similar to (i).

Proof of Lemma 2.

(i)

$$\begin{aligned} A \in \mathcal{F}^{\uparrow} &\iff \exists A_k \text{ s.t. } A_k \subset A_{k+1} \text{ and } \bigcup_{k=1}^{\infty} A_k = A \\ &\iff \exists A_k^c \text{ s.t. } A_k^c \supset A_{k+1}^c \text{ and } \bigcap_{k=1}^{\infty} A_k^c = A^c \\ &\iff A^c \in \mathcal{F}^{\downarrow}. \end{aligned}$$

(ii) The fact that $\mathcal{F}^{\uparrow} \subset \{\bigcup_{k=1}^{\infty} A_k \colon A_k \in \mathcal{F}_0\}$ is trival since $\lim_k\uparrow A_k := \bigcup_{k=1}^{\infty} A_k$. For the other inclusion let $\bigcup_{k=1}^{\infty} A_k \in \{\bigcup_{k=1}^{\infty} A_k \colon A_k \in \mathcal{F}_0\}$. Then

$$\underbrace{\bigcup_{k=1}^{n} A_k}_{\text{in } \mathcal{F}_0} \uparrow \underbrace{\bigcup_{k=1}^{\infty} A_k}_{\therefore \text{ this is in } \mathcal{F}^{\uparrow}}$$

Therefore $\{\bigcup_{k=1}^{\infty} A_k \colon A_k \in \mathcal{F}_0\} \subset \mathcal{F}^{\uparrow}$.

(iii) This proof is similar to (ii).

(iv) Let $A, B \in \mathcal{F}^{\uparrow}$. Then there increasing exists $\mathcal{F}_0$ sets A_k and B_k such that $A = \bigcup_{k=1}^{\infty} A_k$ and $B = \bigcup_{k=1}^{\infty} B_k$. Notice that $A_k \cap B_k$ and $A_k \cup B_k$ are both increasing sets in k. To see why $\lim_k\uparrow (A_k \cap B_k) = A \cap B$ notice

$$\begin{aligned} w \in \lim_k\uparrow (A_k \cap B_k) &\iff w \in \bigcup_{k=1}^{\infty} (A_k \cap B_k) \\ &\iff \text{there exists a } k_0 \text{ such that } w \in A_{k_0} \text{ and } w \in B_{k_0} \\ &\implies w \in \bigcup_{k=1}^{\infty} A_k = A \text{ and } w \in \bigcup_{k=1}^{\infty} B_k = B \qquad (1) \\ &\iff w \in A \cap B \end{aligned}$$

Therefore $\lim_k\uparrow (A_k \cap B_k) \subset A \cap B$. To see the other inclusion notice that the '$\implies$' in conditional (1) can be turned into '$\iff$'. This follows since $w \in \bigcup_{k=1}^{\infty} A_k$ implies there exists a k_1 such that $w \in A_{k_1} \subset A_{k_1+1} \subset \cdots$ by monotonicity and similarly $w \in \bigcup_{k=1}^{\infty} B_k$ implies there exists a k_2 such that $w \in B_{k_2} \subset B_{k_2+1} \subset \cdots$. Now taking $k_0 := max(k_1, k_2)$ shows that $w \in A_{k_0}$ and $w \in B_{k_0}$. Therefore

$$A, B \in \mathcal{F}^{\uparrow} \implies A \cap B = \lim_k\uparrow \underbrace{(A_k \cap B_k)}_{\in \mathcal{F}_0} \in \mathcal{F}^{\uparrow}$$

The proof that $A \cup B \in \mathcal{F}^{\uparrow}$ is similar.

(v) Let $A_k \subset A_{k+1}$ be $\mathcal{F}^{\uparrow}$ sets for $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$. We show $\lim_k\uparrow A_k := \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}^{\uparrow}$. First write

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} A_{k,m}$$

where for each $k \in \mathbb{N}$, $A_{k,m} \in \mathcal{F}_0$ and $\lim_m\uparrow A_{k,m} = A_k$. We show $\lim_N\uparrow \bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m} = A$ which would show that $A \in \mathcal{F}^{\uparrow}$ since $\bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m} \in \mathcal{F}_0$. Clearly $\bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m}$ increases in N. Notice also that for any $M \leq N$

$$A_{M,N} \subset \bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m} \subset \bigcup_{k=1}^{N} \bigcup_{m=1}^{\infty} A_{k,m} = \bigcup_{k=1}^{N} A_k = A_N. \tag{2}$$

Taking limits as $N \to \infty$ gives

$$A_M \subset \lim_N\uparrow \bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m} \subset \lim_N\uparrow A_N = A.$$

Taking limits as $M \to \infty$ gives

$$A = \lim_N\uparrow \underbrace{\bigcup_{k=1}^{N} \bigcup_{m=1}^{N} A_{k,m}}_{\in \mathcal{F}_0} \in \mathcal{F}^{\uparrow}. \tag{3}$$

Now it's easy to also show that $\mathcal{F}^{\uparrow}$ is also closed under countable unions of $\mathcal{F}^{\uparrow}$ sets (since partial unions increase up to infinite unions and since partial unions of $\mathcal{F}^{\uparrow}$ sets are also $\mathcal{F}^{\uparrow}$ sets by Lemma 2(iv)).

Proof of Lemma 3.

(i) Suppose $A \in \mathcal{F}^{\uparrow}$. Let $A_k \in \mathcal{F}_0$ such that $A_k \uparrow A$. Then $A_k^c \downarrow A^c$ and

$$P^{\uparrow}(A) + P^{\downarrow}(A^c) = \lim_k \underbrace{[P_0(A_k) + P_0(A_k^c)]}_{=1} = 1. \tag{4}$$

(ii) Suppose $A, B \in \mathcal{F}^{\uparrow}$. Let $A_k, B_k \in \mathcal{F}_0$ such that $A_k \uparrow A$ and $B_k \uparrow B$. Notice that the proof of Lemma 2(iv) shows that $A_k \cup B_k \uparrow A \cup B \in \mathcal{F}^{\uparrow}$ and $A_k \cap B_k \uparrow A \cap B \in \mathcal{F}^{\uparrow}$. Therefore

$$\begin{aligned} P^{\uparrow}(A \cup B) &= \lim_k\uparrow P_0(A_k \cup B_k) \\ &= \lim_k\uparrow [P_0(A_k) + P_0(B_k) - P_0(A_k \cap B_k)] \\ &= P^{\uparrow}(A) + P^{\uparrow}(B) - P^{\uparrow}(A \cap B). \end{aligned}$$

(iii) Suppose $A, B \in \mathcal{F}^{\uparrow}$ and $A \subset B$. To see why $P^{\downarrow}(A) \leq P^{\uparrow}(A)$ notice that $A \cup A^c = \Omega$ implies

$$\begin{aligned} 1 = P^{\uparrow}(\Omega) &= P^{\uparrow}(A \cup A^c) \\ &\leq P^{\uparrow}(A) + P^{\uparrow}(A^c) \quad \text{by (ii)} \\ &\leq P^{\uparrow}(A) + 1 - P^{\downarrow}(A) \quad \text{by (i)} \end{aligned}$$

Next, to see why $P^{\uparrow}(A) \leq P^{\uparrow}(B)$ notice that $A \subset B$ impiles that $B = A \cup (B - A)$ is a disjoint decomposition of B (i.e. $B =$ hole + ring). Therefore

$$\begin{aligned} P^{\uparrow}(B) &= P^{\uparrow}(A) + P^{\uparrow}(B - A) - 0, \quad \text{by (ii)} \\ &\geq P^{\uparrow}(A). \end{aligned}$$

(iv) Suppose $\lim_n\uparrow A_n = A$ and $A_n \in \mathcal{F}^\uparrow$. Then from (iii) $P^\uparrow(A_n)$ is monotonically increasing and bounded about by $P^\uparrow(A)$. We just need to show the limit is $P^\uparrow(A)$. By equation (2) and (iii)

$$P^\uparrow\Big(\underbrace{\bigcup_{k=1}^{N}\bigcup_{m=1}^{N} A_{k,m}}_{\in\mathcal{F}_0}\Big) \le P^\uparrow(A_N) \le P^\uparrow(A). \tag{5}$$

Notice that

$$\begin{aligned}\lim_N P^\uparrow\Big(\bigcup_{k=1}^{N}\bigcup_{m=1}^{N} A_{k,m}\Big) &= \lim_N P_0\Big(\bigcup_{k=1}^{N}\bigcup_{m=1}^{N} A_{k,m}\Big), \quad \text{since } \bigcup_{k=1}^{N}\bigcup_{m=1}^{N} A_{k,m} \in \mathcal{F}_0 \\ &= P^\uparrow\Big(\underbrace{\lim_N\uparrow \bigcup_{k=1}^{N}\bigcup_{m=1}^{N} A_{k,m}}_{=A,\text{ by (3)}}\Big), \quad \text{by definition of } P^\uparrow.\end{aligned}$$

Therefore taking limits in N in (5) one obtains $P^\uparrow(A) \le \lim_N P^\uparrow(A_N) \le P^\uparrow(A)$.

Lemma 4.

(i) We show this in two stages.

First we show that for any $A, B, C \in 2^\Omega$ such that $A \subset B$ then

$$P^*(B \cup C) - P^*(A \cup C) \le P^*(B) - P^*(A). \tag{6}$$

To see why notice first that (6) is equivalent to $P^*(B) + P^*(A \cup C) \ge P^*(B \cup C) + P^*(A)$ which is true because

$$\begin{aligned}P^*(B) + P^*(A \cup C) &\ge P^*[B \cap (A \cup C)] + P^*[B \cup (A \cup C)], \quad \text{by fact 3} \\ &= P^*[(B \cap A) \cup (B \cap C)] + P^*[A \cup B \cup C] \\ &\ge P^*[A \cup (B \cap C)] + P^*[B \cup C], \quad \text{since } A \subset B \\ &\ge P^*[A] + P^*[B \cup C].\end{aligned}$$

Secondly we use (6) to show that for any $A_1 \subset B_1$ and $A_2 \subset B_2$

$$P^*(B_1 \cup B_2) - P^*(A_1 \cup A_1) \le \big[P^*(B_1) - P^*(A_1)\big] + \big[P^*(B_2) - P^*(A_2)\big]. \tag{7}$$

Notice the two following inequalities which follow directly from (6)

$$\begin{aligned}P^*(B_1 \cup A_2) - P^*(A_1 \cup A_2) &\le \big[P^*(B_1) - P^*(A_1)\big] \\ P^*(B_1 \cup B_2) - P^*(B_1 \cup A_2) &\le \big[P^*(B_2) - P^*(A_2)\big].\end{aligned}$$

Adding the above two inequalities gives (7). Now induction proves the claim.