

Exercise 21. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be π -systems of \mathcal{F} -sets such that

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k) \quad (28)$$

for each choice of $A_k \in \mathcal{A}_k$ for $k = 1, \dots, n$. (a) Show by simple example that the \mathcal{A}_k 's need not be independent. (b) Show that the \mathcal{A}_k 's will be independent if for each k , Ω is the countable union of \mathcal{A}_k -sets. Hint: for fixed A_2, \dots, A_n use the following general inclusion-exclusion formula to show that (28) with A_1 replaced by any finite union of \mathcal{A}_1 -sets. Here is the general inclusion-exclusion formula:

$$P\left(\bigcup_{i=1}^n B_i\right) = \sum_{m=1}^n (-1)^{m-1} \sum_{i_1 < i_2 < \dots < i_m} P(B_{i_1} \cap B_{i_2} \cap \dots \cap B_{i_m})$$

for any \mathcal{F} -sets B_1, B_2, \dots, B_n .

Exercise 22. (a) For each $k = 1, \dots, n$ let \mathcal{P}_k be a partition of Ω into countably many \mathcal{F} -sets. Show that the σ -fields $\sigma(\mathcal{P}_1), \dots, \sigma(\mathcal{P}_n)$ are independent if and only if (28) holds for each choice of A_k from \mathcal{P}_k for $k = 1, \dots, n$. (b) Use part (a) to show that \mathcal{F} -sets A_1, \dots, A_n are independent if and only if

$$P\left(\bigcap_{k=1}^n B_k\right) = \prod_{k=1}^n P(B_k)$$

for each choice of B_k as A_k or A_k^c for $k = 1, \dots, n$. (c) Use part (b) to show that the events H_1, \dots, H_n in Theorem 44 are independent.

Exercise 23. Let A_1, A_2, \dots be \mathcal{F} -sets. Show that $P(A_n \text{ i.o.}) = 1$ if and only if $\sum_{n=1}^{\infty} P(A_n|A)$ diverges for every \mathcal{F} -set A of nonzero probability. Hint: show $P(A_n \text{ i.o.}) < 1 \iff \sum_{n=1}^{\infty} P(A_n|A) < \infty$ for some \mathcal{F} -set A with $P(A) > 0$.

Exercise 24. Let P and Q be probability measures on a σ -field \mathcal{F} of subsets of a sample space Ω .

- P and Q are said to be **singular**, denoted $P \perp Q$, if and only if there exists a set $F \in \mathcal{F}$ such that

$$P(F^c) = 0 = Q(F).$$

- P is said to be **absolutely continuous with respect to** Q , denoted $P \ll Q$, if and only if

$$P(F) = 0 \text{ for every } \mathcal{F}\text{-set } F \text{ for which } Q(F) = 0.$$

Show that

$$P \perp Q \iff \left[\begin{array}{l} \text{there exists } \mathcal{F}\text{-sets } F_1, F_2, \dots \text{ such that} \\ P(F_n^c) \rightarrow 0 \text{ and } Q(F_n) \rightarrow 0 \text{ as } n \rightarrow \infty \end{array} \right]$$

and

$$P \ll Q \iff \lim_{\delta \downarrow 0} \left(\sup \{P(F) : F \in \mathcal{F} \text{ with } Q(F) \leq \delta\} \right) = 0.$$

4.2 Application: law of the iterated logarithm for coin flips

Section Assumption. Throughout this section let $P : \mathcal{B}^{(0,1]} \rightarrow [0, 1]$ be the probability model developed in Section 1 (and extended from the Carathéodory) for a uniform random number in $w \in (0, 1]$. Also let s_n be defined as in Section 1.

To motivate the law of the iterated logarithm lets start by discussing the difference between the weak law and strong law of large numbers.

$$\text{Weak law: } \lim_{n \rightarrow \infty} P\left(\left|\frac{s_n}{n}\right| < \epsilon\right) = 1, \text{ for all } \epsilon > 0;$$

$$\text{Strong law: } P\left(\lim_{n \rightarrow \infty} \frac{s_n}{n} = 0\right) = 1.$$

In some sense the difference can be explained as follows. The weak law fixes each n then analyzes the ensemble of $s_n(\omega)/n$ over $\omega \in (0, 1]$. In particular, the weak law says that for large n it becomes increasingly rare to find ω 's which satisfy $|s_n(\omega)/n| \geq \epsilon$. Conversely the strong law fixes each $\omega \in (0, 1]$ and analyzes the ensemble of $s_n(\omega)/n$ over n . In particular for almost all ω , $s_n(\omega)/n \rightarrow 0$ as $n \rightarrow \infty$.

Now lets make a similar analogy with the central limit theorem and the law of the iterated logarithm. From the strong law we know that $s_n(\omega)/n$ converges to 0 for nearly every ω . We can then ask: at what rate? In particular, can we find a smaller denominator than n , call it ℓ_n , so that s_n/ℓ_n doesn't converge to zero. An answer is given by the central limit theorem

$$\text{CLT: } \lim_{n \rightarrow \infty} P\left(\frac{s_n}{\sqrt{n}} < x\right) = \Phi(x)$$

where $\Phi(x) = P(Z \leq x)$ and Z is a standard normal random variable. Notice two things. First, this suggests that the $s_n(\omega)/\sqrt{n}$ reaches up to ∞ and down to $-\infty$ for different values of ω and n . In particular for every cut-off M , there exists n large enough so that $P(s_n/\sqrt{n} \geq M) \approx 1 - \Phi(M) > 0$ to arbitrary precision. Also, the central limit theorem is similar to the weak law in that it fixes each n then analyzes the ensemble of $s_n(\omega)/\sqrt{n}$ over $\omega \in (0, 1]$. The question then becomes, can one find an analogous form of the strong law such that for each fixed ω one analyzes the ensemble rate of $s_n(\omega)$ as $n \rightarrow \infty$. The law of the iterated logarithm gives the right rate

$$\text{LIL: } P\left(\limsup_{n \rightarrow \infty} \frac{s_n}{\sqrt{2n \log \log n}} = 1\right) = 1.$$

Another way to think about the rate $\sqrt{2n \log \log n}$ is the effect due to the correlation of between s_n across different values of n . The expected maximum of n independent standard Gaussian random variables behaves as $\sqrt{2 \log n}$. That maximum occurs uniformly on $\{1, 2, \dots, n\}$ and is therefore is expected to occur at index $n/2$. If s_n/\sqrt{n} was not correlated across n , the central limit theorem might suggest that the maximum of s_k/\sqrt{k} over $k \in \{1, 2, \dots, n\}$ behaves on the order of $\sqrt{2 \log n}$. This loosely suggests the maximum of s_k over $k \in \{1, 2, \dots, n\}$ behaves on

the order $\sqrt{n \log n}$. Now, in some sense, the LIL says that the correlation across n will dampen the maximum excursions to be at most $\sqrt{n \log n}$.

Lemma 1 (Half of large deviation result). *For all $n \in \mathbb{N}$ and $x > 0$ one has*

$$P(s_n/\sqrt{n} \geq x) \leq \exp\left(-\frac{x^2}{2}\right)$$

Proof. This was established in exercise 2. \square

The other half of the large deviation result we need is Lemma 2, below. Combined these two lemmas give us good approximations to $P(s_n/\sqrt{n} \geq x_n)$ for large-ish values of x_n : large compared to 0 but still small compared to \sqrt{n} (if x_n was larger then \sqrt{n} then $P(s_n/\sqrt{n} > x_n) = 0$). This is the key for deriving the Law of the Iterated Logarithm. Also note that Lemma 1 was proved as an exercise but it is typically established using Markov's inequality, the moment generating function and the fact that $\frac{e^x + e^{-x}}{2} \leq \exp(x^2/2)$.

Lemma 2 (Other half of large deviation result). *If the sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfies $0 \leq x_n \rightarrow \infty$ and $x_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$ then*

$$P(s_n/\sqrt{n} \geq x_n) \geq \exp\left(-\frac{x_n^2}{2}(1 + o(1))\right).$$

Proof. The general idea is to use the fact that $s_n = 2(\sum_{k=1}^n d_k) - n$ where $\sum_{k=1}^n d_k \sim \text{Bin}(n, 1/2)$. Therefore we can write $P(s_n \geq \sqrt{n}x_n)$ as a sum $\sum_{i \in \mathcal{I}_n} P(s_n = i)$ where i is the set of integers greater than $\sqrt{n}x_n$ and less than or equal to n . In fact, since we are trying to construct a lower bound we are free to discard terms in \mathcal{I}_n which will give

$$P(s_n \geq \sqrt{n}x_n) \geq \sum_{i \in \mathcal{I}_n} P(s_n = i).$$

The main problem is how to find \mathcal{I}_n so that the right hand side is $\exp(-\frac{x_n^2}{2}(1 + o(1)))$.

Lets start by getting some idea of how many integers we should include in \mathcal{I}_n by analysing how $P(s_n = i)$ behaves when $i \approx \sqrt{n}x_n$. To make things a bit more precise let i_n be a sequence of integers depending on n such that $i_n \rightarrow \infty$ but $i_n/n \rightarrow 0$.

$$\begin{aligned} P(s_n = i_n) &= P\left(\sum_{k=1}^n d_k = (i_n + n)/2\right) \\ &= \binom{n}{(i_n + n)/2} \frac{1}{2^n}, \text{ if } (i_n + n)/2 \text{ is an integer} \\ &= \frac{n!}{\frac{i_n + n}{2}! \frac{n - i_n}{2}!} \frac{1}{2^n} \\ &= \frac{2}{\sqrt{2\pi n}} [1 + o(1)] \exp\left(-\frac{(1 + o(1))i_n^2}{2n}\right) \end{aligned} \quad (29)$$

To see why (29) is true notice that by Stirling's formula we have $n! = (1 + o(1))\sqrt{2\pi n}n^n e^{-n}$. Therefore

$$\begin{aligned} \frac{n!}{\frac{i_n + n}{2}! \frac{n - i_n}{2}!} \frac{1}{2^n} &= \frac{(1 + o(1))}{(1 + o(1))^2} \\ &\quad \times \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n + i_n)/2} \sqrt{2\pi(n - i_n)/2}} \\ &\quad \times \frac{n^n}{(n + i_n)^{(n + i_n)/2} (n - i_n)^{(n - i_n)/2}} \\ &\quad \times \frac{e^{-n}}{e^{-(n + i_n)/2} e^{-(n - i_n)/2}} \\ &= \frac{(1 + o(1))}{(1 + o(1))^2} \\ &\quad \times \frac{1}{\underbrace{\sqrt{2\pi(n + i_n)(n - i_n)/(4n)}}_{=:I}} \\ &\quad \times \underbrace{\left(\frac{n}{n + i_n}\right)^{(n + i_n)/2} \left(\frac{n}{n - i_n}\right)^{(n - i_n)/2}}_{=:II} \end{aligned}$$

Notice

$$\begin{aligned} I &= \frac{1}{\sqrt{2\pi(n + i_n)(n - i_n)/(4n)}} \\ &= \frac{2}{\sqrt{2\pi n}} \times \frac{1}{\sqrt{(n + i_n)(n - i_n)/(n^2)}} \\ &= \frac{2}{\sqrt{2\pi n}} \times \frac{1}{\sqrt{1 - (i_n/n)^2}} = \frac{2}{\sqrt{2\pi n}} (1 + o(1)). \end{aligned}$$

Secondly notice that $(1 + x) \log(1 + x) = x + \frac{1}{2}x^2 + O(x^3)$ as $x \rightarrow 0$. Therefore

$$\begin{aligned} \log II &= -\frac{1}{2} \left[(n + i_n) \log\left(1 + \frac{i_n}{n}\right) + (n - i_n) \log\left(1 - \frac{i_n}{n}\right) \right] \\ &= -\frac{n}{2} \left[\frac{i_n}{n} + \frac{1}{2} \frac{i_n^2}{n^2} - \frac{i_n}{n} + \frac{1}{2} \frac{i_n^2}{n^2} + O(i_n^3/n^3) \right] \\ &= -\frac{n}{2} \left[\frac{i_n^2}{n^2} + O(i_n^3/n^3) \right] = -\frac{i_n^2}{2n} \left[1 + \underbrace{O(i_n/n)}_{o(1)} \right]. \end{aligned}$$

To finish notice that $\frac{(1 + o(1))^2}{(1 + o(1))^2} = [1 + o(1)]$ which implies (29).

Now, looking at (29) it is clear that we want \mathcal{I}_n to contain about $\sqrt{2\pi n}$ terms that are near $\sqrt{n}x_n$ (so that we can apply (29)). In particular \mathcal{I}_n denote the set of indices between $\sqrt{n}x_n$ and $\sqrt{n}x_n + 2\sqrt{\pi n}$ which have the same parity at n (i.e. that $(i_n + n)/2$ is an integer). Also let i_n be the maximum integer in \mathcal{I}_n , which implies $i_n = \sqrt{n}x_n + 2\sqrt{\pi n} + O(1)$. Now

$$\begin{aligned} P(s_n \geq \sqrt{n}x_n) &\geq \sum_{i \in \mathcal{I}_n} P(s_n = i) \\ &\geq [\#\mathcal{I}_n] P(s_n = i_n), \text{ since } \binom{n}{(i + n)/2} \geq \binom{n}{(i_n + n)/2} \end{aligned}$$

$$\begin{aligned}
&= [\sqrt{\pi n} + O(1)]P(s_n = i_n) \\
&\geq \sqrt{2} \exp\left(-\frac{(1+o(1))i_n^2}{2n}\right), \quad \text{by (29)} \\
&\geq \exp\left(-\frac{(1+o(1))i_n^2}{2n}\right) \\
&= \exp\left(-\frac{(1+o(1))[\sqrt{n}x_n + 2\sqrt{\pi n} + O(1)]^2}{2n}\right) \\
&= \exp\left(-\frac{x_n^2}{2}(1+o(1))\right), \quad \text{since } x_n \rightarrow \infty.
\end{aligned}$$

□

Lemma 3 (Maximal inequality). *For all $n \in \mathbb{N}$ and every nonnegative integer c*

$$P\left(\max_{1 \leq j \leq n} s_j \geq c\right) \leq 2P(s_n \geq c).$$

Proof. First write

$$\begin{aligned}
P\left(\max_{1 \leq j \leq n} s_j \geq c\right) &= P\left(\max_{1 \leq j \leq n} s_j \geq c, s_n \geq c\right) \\
&\quad + P\left(\max_{1 \leq j \leq n} s_j \geq c, s_n < c\right) \\
&= P(s_n \geq c) + P\left(\max_{1 \leq j \leq n} s_j \geq c, s_n < c\right).
\end{aligned}$$

Therefore all that remains is to show $P(\max_{1 \leq j \leq n} s_j \geq c, s_n < c) \leq P(s_n \geq c)$. Start by segmenting the event $\{\max_{1 \leq j \leq n} s_j \geq c\}$ corresponding to the first indice j for which $s_j = c$ (this must occur when $\max_{1 \leq j \leq n} s_j \geq c$ since s_n goes up or down with jumps of size 1 and c is a nonnegative integer). In particular, write

$$\left\{\max_{1 \leq j \leq n} s_j \geq c\right\} = \bigcup_{j=1}^n \underbrace{\{s_1 < c, \dots, s_{j-1} < c, s_j = c\}}_{=: F_j}$$

Now

$$\begin{aligned}
&\left\{\max_{1 \leq j \leq n} s_j \geq c\right\} \cap \{s_n < c\} \\
&= \bigcup_{j=1}^n F_j \cap \{s_n < c\} \\
&= \bigcup_{j=1}^n \underbrace{F_j \cap \{s_n - s_j < 0\}}_{\text{disjoint since the } F_j \text{'s are}}, \quad \text{since } \omega \in F_j \text{ implies } s_j(\omega) = c.
\end{aligned}$$

Now notice two things. First $P(s_n - s_j < 0) = P(s_n - s_j > 0)$ by symmetry. Secondly, since $\{s_n - s_j < 0\} \in \sigma\langle z_{j+1}, \dots, z_n \rangle$ and $F_j \in \sigma\langle z_1, \dots, z_j \rangle$, the event $\{s_n - s_j < 0\}$ is independent of F_j (by ANOVA). Therefore

$$\begin{aligned}
&P\left(\max_{1 \leq j \leq n} s_j \geq c, s_n < c\right) \\
&= \sum_{j=1}^n P(F_j \cap \{s_n - s_j < 0\})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n P(F_j)P(s_n - s_j < 0) \text{ by independence} \\
&= \sum_{j=1}^n P(F_j)P(s_n - s_j > 0) \text{ by symmetry} \\
&= \sum_{j=1}^n P(F_j \cap \{s_n - s_j > 0\}) \text{ by independence again} \\
&= \sum_{j=1}^n P(F_j \cap \{s_n > c\}) \text{ since } \omega \in F_j \text{ implies } s_j(\omega) = c \\
&= P\left(\max_{1 \leq j \leq n} s_j \geq c, s_n > c\right) \\
&\leq P(s_n > c) \\
&\leq P(s_n \geq c).
\end{aligned}$$

Therefore $P(\max_{1 \leq j \leq n} s_j \geq c) \leq 2P(s_n \geq c)$. □

Theorem 49 (Law of the iterated logarithm for coin flips).

1. $P\left[\limsup_{n \rightarrow \infty} \frac{s_n}{\sqrt{2n \log \log n}} = 1\right] = 1;$
2. $P\left[\liminf_{n \rightarrow \infty} \frac{s_n}{\sqrt{2n \log \log n}} = -1\right] = 1.$

Proof. To make the formulas more readable let $\ell_n := \sqrt{2n \log \log n}$. Notice first that

$$\begin{aligned}
&\left\{\limsup_n \frac{s_n}{\ell_n} = 1\right\} \\
&= \bigcap_{\epsilon \in (0,1) \cap \mathbb{Q}} \{s_n/\ell_n > (1-\epsilon) \text{ i.o.}_n\} \cap \{s_n/\ell_n < (1+\epsilon) \text{ a.a.}_n\}.
\end{aligned}$$

This implies that $\{\limsup_n (s_n/\ell_n) = 1\}$ and $\{\liminf_n (s_n/\ell_n) = -1\}$ are Borel measurable. Secondly notice that by symmetry we have

$$\begin{aligned}
&P\left[\limsup_{n \rightarrow \infty} (s_n/\ell_n) = 1\right] \\
&= P\left[\liminf_{n \rightarrow \infty} (-s_n/\ell_n) = -1\right] \\
&= P\left[\liminf_{n \rightarrow \infty} (s_n/\ell_n) = -1\right].
\end{aligned}$$

We can also simplify our proof by using countable sub-additivity

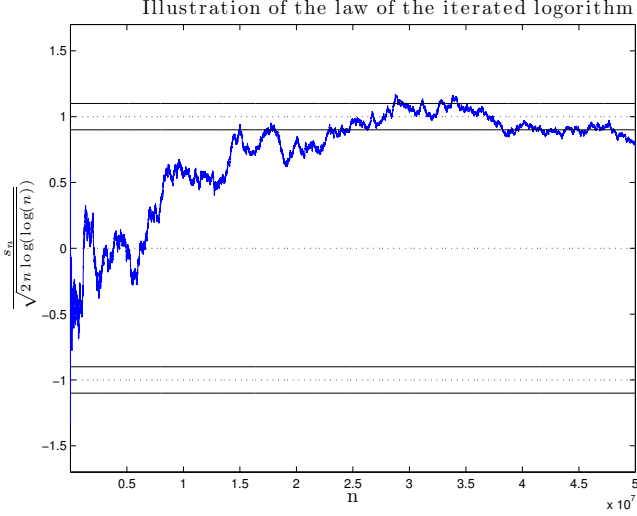
$$\begin{aligned}
&P\left[\left\{\limsup_{n \rightarrow \infty} (s_n/\ell_n) = 1\right\}^c\right] \\
&\leq \sum_{\epsilon \in (0,1) \cap \mathbb{Q}} P[\{s_n/\ell_n \leq (1-\epsilon) \text{ a.a.}_n\}] \\
&\quad + \sum_{\epsilon \in (0,1) \cap \mathbb{Q}} P[\{s_n/\ell_n \geq (1+\epsilon) \text{ i.o.}_n\}].
\end{aligned}$$

Therefore the proof will follow by establishing Lemmas 4 and 5 which state that for all $\epsilon > 0$

$$\begin{aligned}
&P[s_n/\ell_n \geq (1+\epsilon) \text{ i.o.}_n] = 0 \\
&P[s_n/\ell_n > (1-\epsilon) \text{ i.o.}_n] = 1
\end{aligned}$$

□

Theorem 49 is interesting for a number of reasons. First, it gives a very detailed analysis of the Central Limit Theorem. Second, it shows the power of the Borel-Cantelli lemmas. Third, it is one of those theorems in probability which is extremely hard to see in simulations: $\sqrt{2n \log \log n}$ grows too slow for modern computers to probe. The following simulation is an attempt at illustrating Theorem 186 but we still don't see the required fluctuation from $+1$ to -1 .



Lemma 4. For all $\epsilon > 0$

$$P[s_n/\ell_n \geq (1+\epsilon) \text{ i.o.}_n] = 0$$

Proof. The obvious strategy is to use the first Borel-Cantelli lemma. In particular, it would be nice if we could show

$$\sum_{n=1}^{\infty} P[s_n/\ell_n \geq (1+\epsilon)] < \infty$$

which would give us the lemma. By the first half of the large deviation result we know $P[s_n/\ell_n \geq (1+\epsilon)]$ converge to zero as $n \rightarrow \infty$. Unfortunately, they do not converge fast enough for the first Borel-Cantelli. Taking a sub-sequence will get something summable but we would need to show that the events are well behaved between the sub-sequence. Fortunately we can do this since the sets $\{s_n/\ell_n \geq (1+\epsilon)\}$ overlap a lot. The strategy is to group the events $\{s_n/\ell_n \geq (1+\epsilon)\}$ by unioning them into blocks, then apply Borel-Cantelli on the blocks. This will be sufficient since the blocks occur infinity often if and only if the events $s_n/\ell_n \geq (1+\epsilon)$ occur infinity often. Controlling the probability of the blocks is done with the maximal inequality.

Let the k^{th} block be defined

$$B_k := \bigcup_{j=n_{k-1}}^{n_k} \{s_j/\ell_j \geq (1+\epsilon)\}$$

where n_k is a (yet to be determined) subsequence. We use maximal inequality to bound $P[B_k]$ as follows

$$P[B_k] \leq P\left[\max_{n_{k-1} \leq j \leq n_k} s_j \geq (1+\epsilon) \min_{n_{k-1} \leq j \leq n_k} \ell_j\right]$$

$$\begin{aligned} &\leq P\left[\max_{n_{k-1} \leq j \leq n_k} s_j \geq (1+\epsilon)\ell_{n_{k-1}}\right] \\ &\leq P\left[\max_{j \leq n_k} s_j \geq (1+\epsilon)\ell_{n_{k-1}}\right] \\ &\leq P\left[\max_{j \leq n_k} s_j \geq \lceil (1+\epsilon)\ell_{n_{k-1}} \rceil\right], \quad \text{since } s_j \in \mathbb{Z} \\ &\leq 2P[s_{n_k} \geq \lceil (1+\epsilon)\ell_{n_{k-1}} \rceil], \quad \text{maximal ineq} \\ &\leq 2P[s_{n_k} \geq (1+\epsilon)\ell_{n_{k-1}}] \\ &\leq \exp\left(-\frac{1}{2}(1+\epsilon)^2 \ell_{n_{k-1}}^2 / n_k\right), \quad \text{half of large deviation} \\ &\leq \exp\left(-(1+\epsilon)^2 n_{k-1} \log \log n_{k-1} / n_k\right) \\ &= \left(\frac{1}{\log n_{k-1}}\right)^{(1+\epsilon)^2 \frac{n_{k-1}}{n_k}}. \end{aligned}$$

Now we just find n_k which makes the last term summable over k . If $n_k \approx \theta^k$ one gets

$$\left(\frac{1}{\log n_{k-1}}\right)^{(1+\epsilon)^2 \frac{n_{k-1}}{n_k}} \approx \left(\frac{1}{(k-1) \log \theta}\right)^{(1+\epsilon)^2 \frac{1}{\theta}}$$

which is summable if $(1+\epsilon)^2 \frac{1}{\theta} > 1$. We also need that $\theta > 1$ since we need $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Luckily there does exist such a θ for which $(1+\epsilon)^2 > \theta > 1$. \square

Lemma 5. For all $\epsilon > 0$

$$P[s_n/\ell_n > (1-\epsilon) \text{ i.o.}_n] = 1$$

Proof. In the previous lemma we presented a technique to adjust the first Borel-Cantelli lemma in the case the summability condition doesn't hold. For this lemma we want to use the second Borel-Cantelli lemma but, again, it doesn't directly apply since the condition that the events $s_n/\ell_n > (1-\epsilon)$ are independent does not hold. Here is a generic technique to get around this obstacle. Find subsequence n_k and subsets

$$I_k \subset \{s_{n_k}/\ell_{n_k} > (1-\epsilon)\}$$

such that I_k 's are independent and $\sum_k P[I_k] = \infty$ so that $P[I_k \text{ i.o.}_k] = 1$ (which would then give the lemma). Unfortunately, even this doesn't work. What ends up working is to find two sets A_k, I_k such that

$$A_k \cap I_k \subset \{s_{n_k}/\ell_{n_k} > (1-\epsilon)\} \quad (30)$$

$$I_k \text{ are independent and } \sum_k P[I_k] = \infty \quad (31)$$

$$A_k \text{ are not independent but } P[A_k \text{ a.a.}_k] = 1. \quad (32)$$

To see why this is sufficient notice that (31) implies $P[I_k \text{ i.o.}_k] = 1$ by the second Borel-Cantelli lemma. Then

$$\begin{aligned} &P[A_k \text{ a.a.}_k] = 1 \text{ and } P[I_k \text{ i.o.}_k] = 1 \\ &\implies P[A_k \cap I_k \text{ i.o.}_k] = 1 \\ &\stackrel{(30)}{\implies} P[s_{n_k}/\ell_{n_k} > (1-\epsilon) \text{ i.o.}_k] = 1 \end{aligned}$$

Therefore (30), (31) and (32) are sufficient to establish the lemma.

Figuring out how to define I_k and A_k are the tricky parts. The intuition is that if your going to get independent events you need to look at increments of s_n . Define

$$I_k := \{s_{n_k} - s_{n_{k-1}} \geq (1 - \epsilon/2)\ell_{n_k}\}$$

$$A_k := \{s_{n_{k-1}} > -(\epsilon/2)\ell_{n_k}\}.$$

Clearly $I_k \cap A_k \subset \{s_{n_k}/\ell_{n_k} > (1 - \epsilon)\}$ so that (30) holds. Moreover, the I_k 's are independent. To show (32)

$$\begin{aligned} P[A_k \text{ a.a. } k] &= P[s_{n_{k-1}} > -(\epsilon/2)\ell_{n_k} \text{ a.a. } k] \\ &= P[s_{n_{k-1}} < (\epsilon/2)\ell_{n_k} \text{ a.a. } k], \quad \text{by symmetry} \\ &= P\left[\frac{s_{n_{k-1}}}{\ell_{n_{k-1}}} < (\epsilon/2)\frac{\ell_{n_k}}{\ell_{n_{k-1}}} \text{ a.a. } k\right] \\ &= 1 - P\left[\frac{s_{n_{k-1}}}{\ell_{n_{k-1}}} \geq (\epsilon/2)\frac{\ell_{n_k}}{\ell_{n_{k-1}}} \text{ i.o. } k\right] \\ &= 1, \text{ by Lemma 4 if } \frac{\ell_{n_k}}{\ell_{n_{k-1}}} \rightarrow \infty. \end{aligned}$$

To show (31) notice

$$\begin{aligned} P[I_k] &= P[s_{n_k} - s_{n_{k-1}} \geq (1 - \epsilon/2)\ell_{n_k}] \\ &= P[s_{n_k - n_{k-1}} \geq (1 - \epsilon/2)\ell_{n_k}] \\ &\geq P\left[\frac{s_{n_k - n_{k-1}}}{\sqrt{n_k - n_{k-1}}} \geq \frac{(1 - \epsilon/2)\ell_{n_k}}{\sqrt{n_k - n_{k-1}}}\right] \\ &\geq \exp\left(-\frac{1}{2}\left[\frac{(1 - \epsilon/2)\ell_{n_k}}{\sqrt{n_k - n_{k-1}}}\right]^2 (1 + o(1))\right), \quad (33) \\ &\quad \text{by Lemma 2 if:} \\ &\quad \frac{\ell_{n_k}}{\sqrt{n_k - n_{k-1}}} \rightarrow \infty \text{ and} \\ &\quad \frac{1}{\sqrt{n_k - n_{k-1}}} \frac{\ell_{n_k}}{\sqrt{n_k - n_{k-1}}} \rightarrow 0 \text{ and} \\ &\quad n_k - n_{k-1} \rightarrow \infty. \end{aligned}$$

To finish the proof of (32) and (31) we need to find $n_k \rightarrow \infty$ such that ℓ_{n_k} satisfies the above conditions and the [sum of \(33\) diverges](#).

A subsequence of the form $n_k := \lfloor \exp(k^\theta) \rfloor$ will work. To check the conditions notice

$$\begin{aligned} \frac{\ell_{n_k}}{\ell_{n_{k-1}}} &\sim \frac{\sqrt{2 \exp(k^\theta) \log k^\theta}}{\sqrt{2 \exp((k-1)^\theta) \log (k-1)^\theta}} \\ &= \exp\left(\frac{k^\theta - (k-1)^\theta}{2}\right) \frac{\log k}{\log(k-1)} \\ &= \exp\left(\frac{\theta(k^*)^{\theta-1}}{2}\right) (1 + o(1)), \quad \text{where } k-1 \leq k^* \leq k \\ &\rightarrow \infty, \quad \text{if } \theta > 1. \end{aligned}$$

Also

$$n_k - n_{k-1} \sim \exp(k^\theta) - \exp((k-1)^\theta)$$

$$= \theta(k^*)^{(\theta-1)} \exp((k^*)^\theta), \quad \text{where } k-1 \leq k^* \leq k$$

$$\rightarrow \infty, \quad \text{if } \theta > 0.$$

And therefore

$$\begin{aligned} \frac{\ell_{n_k}}{\sqrt{n_k - n_{k-1}}} &\sim \frac{\sqrt{2 \exp(k^\theta) \log k^\theta}}{\sqrt{\exp(k^\theta) - \exp((k-1)^\theta)}} \\ &= \frac{\sqrt{2\theta \log k}}{\sqrt{1 - \exp((k-1)^\theta - k^\theta)}} \\ &= \frac{\sqrt{2\theta \log k}}{\sqrt{1 - o(1)}} \\ &\rightarrow \infty. \end{aligned}$$

Clearly we now have that $\frac{\ell_{n_k}}{n_k - n_{k-1}} \rightarrow 0$. Finally we need to show that the sum of (33) over k diverges. The individual terms are

$$\begin{aligned} &\exp\left(-\frac{1}{2}\left[\frac{(1 - \epsilon/2)\ell_{n_k}}{\sqrt{n_k - n_{k-1}}}\right]^2 (1 + o(1))\right) \\ &\sim \exp\left(-\frac{(1 - \epsilon/2)^2}{2} 2\theta \log k\right) \\ &= \exp\left(-(1 - \epsilon/2)^2 \theta \log k\right) \\ &= k^{-(1 - \epsilon/2)^2 \theta} \end{aligned}$$

the sum of the above terms over k diverges if $(1 - \epsilon/2)^\theta < 1$, i.e. [if \$\theta < \frac{1}{\(1 - \epsilon\)^2}\$](#) . Now putting all the conditions on θ together says that we simply need to choose θ such that

$$1 < \theta < \frac{1}{(1 - \epsilon/2)^2}.$$

□