Lecture 9: σ-fields generated by functions. The structure thm. Applications to R.V.s

1

σ-fields generated by functions

or r.v.s are extremely useful for cleaning up & generalizing some of the stuff we did for the coin flip model & also allow us to define conditional expected value etc.

e.g. in previous lectures we said things like $\underbrace{\{S_n - S_k > c\}}_{\in\sigma\langle H_{k+1},\ldots,H_n\rangle}$ is indep of $\underbrace{\{S_k > c\}}_{\in\sigma\langle H_1,\ldots,H_k\rangle}$

... while true it is a bit annoying & implicitly due to facts like:

$$\{S_k > c\} = \bigcup_{\substack{r_1,\ldots,r_k\in\{-1,1\} \\ \text{s.t. } r_1+\cdots+r_k > c}} \underbrace{\{R_1 = r_1\}\cap\cdots\cap\{R_k = r_k\}}_{\in\sigma\langle H_1,\ldots,H_k\rangle}$$

countable

which are not very generalizable.

e.g. Recall "just check the coords": $\vec f = (f_1,\ldots,f_d)$

$$\underset{\mathcal F}{\Omega} \xrightarrow{\vec f\ ⓜ} \underset{\mathcal B(\bar{\mathbb R}^d)}{\bar{\mathbb R}^d} \quad \text{iff} \quad \underset{\mathcal F}{\Omega} \xrightarrow{f_i\ ⓜ} \underset{\mathcal B(\bar{\mathbb R})}{\bar{\mathbb R}} \quad \forall i$$

appears to use $\mathcal B(\bar{\mathbb R}^d)$ as the natural σ-field on $\bar{\mathbb R}^d = \bar{\mathbb R}\times\cdots\times\bar{\mathbb R}$. What about when f_i maps into $\underset{\mathcal F_i}{\Omega_i}$ ... what is the σ-field on $\Omega_1\times\cdots\times\Omega_n$?

2

e.g. it would be nice if a r.v. Y

$$\underset{\mathcal B([0,1])}{[0,1]} \xrightarrow{Y} \underset{\mathcal B(\mathbb R)}{\mathbb R}$$

which satisfied $\{Y \le c\} \in \sigma\langle H_1,\ldots,H_k\rangle$ $\forall c$ could be shown to be a ⓜ function of $R_1,\ldots,R_k$ i.e. $\exists g$ ⓜ $\mathcal B(\mathbb R^k)/\mathcal B(\mathbb R)$ s.t. $Y = g(R_1,\ldots,R_k)$

e.g. we want to extend the notion of independence to non-discrete R.V.s, i.e. if B_t is a Brownian motion conclude that B_t, $t<t_0$ is indep of $B_{t>t_0}$ given B_{t_0}.

Basic definition: $\sigma\langle f_i, \mathcal F_i : i\in\mathcal I\rangle$

Let $\mathcal I$ be a general index set (any cardinality allowed).

Let $(\Omega_i, \mathcal F_i)$ be a measurable space, $\forall i \in\mathcal I$.

Let $f_i : \Omega \to \Omega_i$, $\forall i\in\mathcal I$

$$\underset{?}{\Omega} \xrightarrow{f_i} \underset{\mathcal F_i}{\Omega_i} \quad \vdots \quad \underset{?}{\Omega} \xrightarrow{f_k} \underset{\mathcal F_k}{\Omega_k}$$

Def: $\sigma\langle f_i, \mathcal F_i : i\in\mathcal I\rangle$

$\equiv \sigma\langle f_i : i\in\mathcal I\rangle$ when $\mathcal F_i$ is implicit

$$:= \bigcap_{\substack{\sigma\text{-fields } \mathcal G \\ \text{on } \Omega \text{ s.t.} \\ f_i\ ⓜ\ \mathcal G/\mathcal F_i,\ \forall i\in\mathcal I}} \mathcal G$$

= smallest σ-field on Ω making all the f_i's measurable.

Thm: $\sigma\langle f_1, \mathcal{F}_1\rangle = \underbrace{f_1^{-1}(\mathcal{F}_1)}_{\substack{\text{the pull backs}\\ \text{of each } F\in\mathcal{F}_1}}$

Warning: this only works for the σ-field generated by a single function.

Proof:
This follows easily by "good sets" & the fact that $f_1^{-1}(\mathcal{F}_1)$ is a σ-field.

QED.

Thm (Generators are enough).

If
$$\begin{tikzcd} & \Omega_i \;\; \sigma\langle\mathcal{A}_i\rangle \\ \Omega \arrow[ur, "f_i"] \arrow[dr, "f_k"'] & \vdots \\ & \Omega_k \;\; \sigma\langle\mathcal{A}_k\rangle \end{tikzcd}$$

then
$$\sigma\langle f_i, \sigma\langle\mathcal{A}_i\rangle : i\in\mathcal{I}\rangle = \sigma\langle f_i^{-1}(\mathcal{A}_i) : i\in\mathcal{I}\rangle$$

Proof:
To show $\subset$ Notice that clearly
$$f_k \;(m)\; \sigma\langle f_i^{-1}(\mathcal{A}_i) : i\in\mathcal{I}\rangle/\mathcal{A}_k,\ \forall k\in\mathcal{I}$$
∴ "check (m) on generators" implies
$$f_k \;(m)\; \sigma\langle f_i^{-1}(\mathcal{A}_i) : i\in\mathcal{I}\rangle/\sigma\langle\mathcal{A}_k\rangle,\ \forall k\in\mathcal{I}$$
∴ $\sigma\langle f_i^{-1}(\mathcal{A}_i) : i\in\mathcal{I}\rangle$ is a $\mathcal{G}$ in the def of $\sigma\langle f_i, \sigma\langle\mathcal{A}_i\rangle : i\in\mathcal{I}\rangle$.

To show $\supset$ Notice that clearly
$$f_k^{-1}(\mathcal{A}_k) \subset \sigma\langle f_i, \sigma\langle\mathcal{A}_i\rangle : i\in\mathcal{I}\rangle,\ \forall k\in\mathcal{I}$$
$$\therefore\ \underbrace{\sigma\langle f_k^{-1}(\mathcal{A}_k) : k\in\mathcal{I}\rangle}_{\substack{\text{since this is the}\\ \text{"smallest" σ-field}\\ \text{containing } f_k^{-1}(\mathcal{A}_k),\ \forall k\in\mathcal{I}}} \subset \sigma\langle f_i, \sigma\langle\mathcal{A}_i\rangle : i\in\mathcal{I}\rangle$$

QED.

Note: This trivially implies
$$\boxed{\sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle = \sigma\langle f_i^{-1}(\mathcal{F}_i) : i\in\mathcal{I}\rangle}$$
since $\mathcal{F}_i$ generates itself.

Product σ-field

Now we can define the natural "Product σ-field" on $\Omega_1\times\cdots\times\Omega_n\times\cdots$ using the "coordinate projections"
$$\pi_k(\omega) = \omega_k$$

Def: Let $(\Omega_i, \mathcal{F}_i)$ be a measurable space $\forall i\in\mathcal{I}$. Define
$$\bigotimes_{i\in\mathcal{I}} \mathcal{F}_i := \sigma\langle \pi_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$$
a σ-field on $\Omega = \prod_{i\in\mathcal{I}} \Omega_i$.

Picture:
$$\begin{tikzcd} & \Omega_1 \;\; \mathcal{F}_1 \\ \underset{\mathcal{F}_1\otimes\mathcal{F}_2\otimes\cdots}{\Omega_1\times\Omega_2\times\cdots} \arrow[ur, "\pi_1\,(m)"] \arrow[r, "\pi_2\,(m)"] & \Omega_2 \;\; \mathcal{F}_2 \\ & \vdots \end{tikzcd}$$

Thm (just check the coordinates)

Suppose $f_i : \Omega \to \Omega_i$ where $(\Omega, \mathcal{F})$ and $(\Omega_i, \mathcal{F}_i)$ are measurable spaces $\forall i \in \mathcal{I}$.

Define the vector map $\vec{f}(\omega) = (f_i(\omega))_{i \in \mathcal{I}}$

$$\begin{tikzcd} \underset{\mathcal{F}}{\Omega} \arrow[r, "\vec{f}"] & \underset{i\in\mathcal{I}}{\prod}\Omega_i \quad \underset{i\in\mathcal{I}}{\bigotimes}\mathcal{F}_i \end{tikzcd}$$

Then $\vec{f} \textcircled{m} \mathcal{F} / \bigotimes_{i \in \mathcal{I}} \mathcal{F}_i \iff f_i \textcircled{m} \mathcal{F}/\mathcal{F}_i \quad \forall i \in \mathcal{I}$.

Proof:

Notice that

$$\bigotimes_{i\in\mathcal{I}} \mathcal{F}_i := \sigma\langle \pi_i, \mathcal{F}_i : i \in \mathcal{I}\rangle$$
$$= \sigma\langle \pi_i^{-1}(\mathcal{F}_i) : i \in \mathcal{I}\rangle$$

$\therefore$

$$\vec{f} \textcircled{m} \mathcal{F}/\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i \iff \vec{f} \textcircled{m} \mathcal{F} / \{\pi_i^{-1}(F_i) : F_i \in \mathcal{F}_i, i \in \mathcal{I}\}$$

by "generators are enough"

$$\iff \underbrace{\vec{f}^{-1}(\pi_i^{-1}(F_i))}_{} \in \mathcal{F}, \ \forall F_i \in \mathcal{F}_i \ \forall i \in \mathcal{I}.$$
$$= (\pi_i \circ \vec{f})^{-1}(F_i)$$
$$= f_i^{-1}(F_i)$$
$$\iff f_i \textcircled{m} \mathcal{F}/\mathcal{F}_i \quad \forall i \in \mathcal{I}.$$

QED

Remark: $\bigotimes_{k=1}^{d} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^d)$

and $\bigotimes_{k=1}^{\infty} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^\infty)$ where $\mathcal{B}(\mathbb{R}^\infty)$ is defined with metric

$$d\left((x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty}\right) := \sum_{k=1}^{\infty} 2^{-k}\left(|x_k - y_k| \wedge 1\right)$$

Remark: The previous Thm implies

$$\vec{f}^{-1}\Big(\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i\Big) = \sigma\langle \vec{f}, \bigotimes_{i\in\mathcal{I}}\mathcal{F}_i\rangle = \sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I}\rangle$$

single map pullback

by "good sets" since $\sigma\langle \vec{f}, \bigotimes_i \mathcal{F}_i\rangle$ makes each $f_i \textcircled{m}$ and $\sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I}\rangle$ makes $\vec{f} \textcircled{m}$

What it means for Y to be $\textcircled{m}$ $\sigma\langle X_1, \dots, X_n\rangle / \mathcal{B}(\mathbb{R})$ & the structure Thm

To modivate this result consider

$$\Omega = (0,1]$$
$$\mathcal{F} = \mathcal{B}((0,1])$$
$$X_1(\omega) = I_{(0,\frac{1}{2}]}(\omega)$$
$$X_2(\omega) = I_{(\frac{1}{2},1]}(\omega)$$

Suppose $Y : \Omega \to \mathbb{R}$ is another r.v. on Ω

$$\begin{tikzcd} \underset{\mathcal{F}}{\Omega} \arrow[r, bend left, "X_1 \textcircled{m}"] \arrow[r, "X_2 \textcircled{m}"] \arrow[r, bend right, "Y \textcircled{m}"'] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

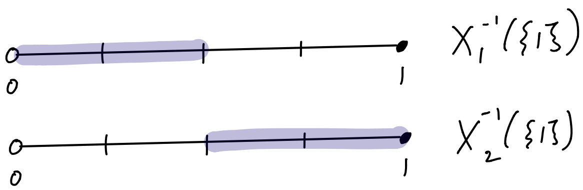
which additionally satisfies

$$Y \textcircled{m} \sigma\langle X_1, X_2\rangle / \mathcal{B}(\mathbb{R})$$

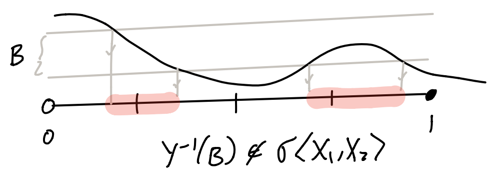
So that

$$\begin{tikzcd} \underset{\sigma\langle X_1, X_2\rangle}{\Omega} \arrow[r, bend left, "X_1 \textcircled{m}"] \arrow[r, "X_2 \textcircled{m}"] \arrow[r, bend right, "Y \textcircled{m}"'] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

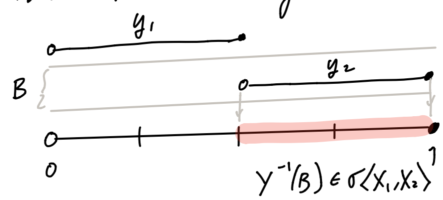
Notice that $\sigma\langle X_1, X_2\rangle$ contains $\emptyset$, Ω & 7



$\therefore$ Y can't look like



In fact Y must only look like



i.e. $Y(\omega) = y_1 I_{\{X_1=1\}}(\omega) + y_2 I_{\{X_2=1\}}(\omega)$

$$= y_1 I_{\{1\}}(X_1(\omega)) + y_2 I_{\{1\}}(X_2(\omega))$$

$$= g(X_1, X_2)$$

g is Ⓜ $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$

This holds in complete generality.

e.g. $Y, X_1, X_2, \ldots$ are r.v.s on $(\Omega, \mathcal{F}, P)$. Then

Y Ⓜ $\sigma\langle X_1, X_2, \ldots\rangle \iff Y = g(X_1, X_2, \ldots)$ where g Ⓜ $\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$

also extends to uncountable collections X_i, $i \in \mathcal{I}$.

To prove this we need an important thm 8 which is also used for defining $\int_\Omega f(\omega)\,d\mu(\omega)$ when f Ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$.

<u>Def</u>: $f: \Omega \to \bar{\mathbb{R}}$ is a <u>simple function</u> if range(f) is a finite set & f Ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$.

<u>Thm</u>:
Suppose $f: \Omega \to \bar{\mathbb{R}}$ is Ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$ where $(\Omega, \mathcal{F})$ is a measurable space. Then f is a simple function iff $f = \sum_{k=1}^{n} c_k I_{A_k}$ where $n < \infty$, $c_k \in \bar{\mathbb{R}}$, $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are disjoint & $\Omega = \bigcup_{k=1}^{n} A_k$.

<u>Proof</u>:

$\Leftarrow$: