# Lecture 3: Dynkin's $\pi$-$\lambda$ Theorem and Borel $\sigma$-fields



Thm (Dynkins' $\pi$-$\lambda$)

$\mathcal{P}$ is a $\pi$-system $\Longrightarrow$ $\lambda\langle\mathcal{P}\rangle = \sigma\langle\mathcal{P}\rangle$

Remark:
The most important use of Dynkin's thm is in the proof that probability measures are characterized by their values on a $\pi$-system of generators.

For example, in undergrad probability we tell students that the CDF characterizes probability distributions ... so if $P$ & $Q$ are probabilities on $((0,1], \mathcal{B}^{(0,1]})$ then $P = Q$ if

$$P((0,x]) = Q((0,x]) \quad \forall x \in (0,1].$$

This follows since (by a HWK)

$$\mathcal{B}^{(0,1]} = \sigma\langle\mathcal{P}\rangle$$

where $\mathcal{P} = \{(0,x] : 0 < x \le 1\}$ is a $\pi$-system.

Remark: Dynkin's $\pi$-$\lambda$ thm also allows us to extend the "good sets" technique

i.e. $\underset{\text{a }\pi\text{-system}}{\mathcal{P}} \subset \underset{\text{a }\lambda\text{-system}}{\mathcal{G}} \Longrightarrow \lambda\langle\mathcal{P}\rangle \subset \mathcal{G} \Longrightarrow \sigma\langle\mathcal{P}\rangle \subset \mathcal{G}$

since these are equal.

This allows you to prove a little less for $\mathcal{G}$ but a little more for $\mathcal{P}$.

Remark: The proof of Dynkin's $\pi$-$\lambda$ thm is an excellent example of using the "good sets" technique.

## Proof of Dynkin's $\pi$-$\lambda$ Thm:



**show $\lambda\langle\mathcal{P}\rangle \subset \sigma\langle\mathcal{P}\rangle$:** Follows immediatly by good sets.

$\therefore$ just show $\sigma\langle\mathcal{P}\rangle \subset \lambda\langle\mathcal{P}\rangle$

$\therefore$ just show $\lambda\langle\mathcal{P}\rangle$ is a $\sigma$-field (by good sets)

$\therefore$ just show $\lambda\langle\mathcal{P}\rangle$ is closed under "$\cap$" (by $\sigma = \lambda + \pi$)

$\therefore$ just show $A, B \in \lambda\langle\mathcal{P}\rangle \Longrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle$

For $A \in \lambda\langle\mathcal{P}\rangle$ let

$$\boxed{\mathcal{G}_A := \{B \subset \Omega : A \cap B \in \lambda\langle\mathcal{P}\rangle\}}. \quad (*)$$

$\therefore$ just show $\forall A \in \lambda\langle\mathcal{P}\rangle$, $\lambda\langle\mathcal{P}\rangle \subset \mathcal{G}_A$

$\therefore$ just show $\forall A \in \lambda\langle\mathcal{P}\rangle$ $\boxed{\mathcal{P} \subset \mathcal{G}_A \text{ \& } \mathcal{G}_A \text{ is a } \lambda\text{-sys}}$ $(**)$

which is sufficient by "good sets."

We will show $(**)$ first under the case $A \in \mathcal{P}$.

However first Notice

$$\boxed{B \in \mathcal{G}_A \Leftrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle \Leftrightarrow A \in \mathcal{G}_B} \quad (***)$$

**Show $(**)$ when $A \in \mathcal{P}$:**

- $\mathcal{P} \subset \mathcal{G}_A$ since

  $B \in \mathcal{P} \Longrightarrow A \cap B \in \mathcal{P}$, by $\pi$-sys.

  $\Longrightarrow B \in \mathcal{G}_A$, by $(*)$

- $\mathcal{G}_A$ is not $\emptyset$ since $A \in \mathcal{G}_A$.

- $\mathcal{G}_A$ is closed under complementation since $B \in \mathcal{G}_A \Longrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle$

  $\Longrightarrow \underbrace{A - A \cap B}_{= A \cap (A \cap B)^c = A \cap B^c} \in \lambda\langle\mathcal{P}\rangle$, nested set subtract

  $\Longrightarrow B^c \in \mathcal{G}_A$

- $\mathcal{G}_A$ is closed under countable disjoint union since

$$\underbrace{B_1, B_2, \ldots}_{\text{disjoint}} \in \mathcal{G}_A \Rightarrow \underbrace{A \cap \bigcup_{k=1}^{\infty} B_k}_{= \bigcup_{k=1}^{\infty} (B_k \cap A)} \in \mathcal{G}_A$$

where $B_k \cap A$ are disjoint members of $\mathcal{G}_A$

Show (**) for general $A \in \lambda\langle\mathcal{P}\rangle$

- $\mathcal{P} \subset \mathcal{G}_A$ since

$$B \in \mathcal{P} \Rightarrow A \in \mathcal{G}_B, \text{ since (**) holds over } \mathcal{P}$$
$$\Leftrightarrow B \in \mathcal{G}_A$$

- The proof that $\mathcal{G}_A$ is a $\lambda$-sys is exactly similar as previous case.

QED

The following thm is similar to Dynkin's $\pi$-$\lambda$ but for fields & monotonic classes.

Thm (Halmos's monotone class thm)

$$\mathcal{F} \text{ is a field} \Rightarrow \mathcal{M}\langle\mathcal{F}\rangle = \sigma\langle\mathcal{F}\rangle$$

Proof: exercise

Remark: This thm is used when extending a prob $P$ on a field $\mathcal{F}$ to $\sigma\langle\mathcal{F}\rangle$ by adding monotonic limits to $\mathcal{F}$ & defining the extension to $P$ with limits.



# Borel $\sigma$-fields

Def:

If $\Omega$ is a metric space with distance $d: \Omega \times \Omega \to [0,\infty]$ then

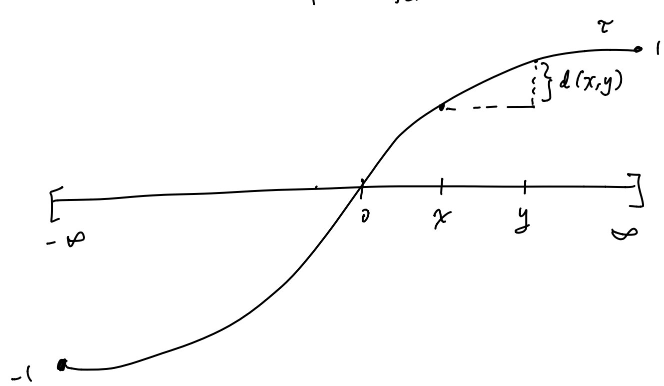
$$\mathcal{B}^{\Omega} := \underbrace{\text{Borel } \sigma\text{-field}}_{\text{w.r.t } d} := \sigma\langle \underbrace{\text{open subsets of } \Omega}_{\text{w.r.t } d} \rangle.$$

This defines $\mathcal{B}^{\mathbb{R}}, \mathcal{B}^{\mathbb{R}^d}, \mathcal{B}^{\bar{\mathbb{R}}}, \mathcal{B}^{\bar{\mathbb{R}}^d}$ etc ...

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ uses metric

$$d(x,y) = |\tau(x) - \tau(y)|$$

$$\tau(x) := \begin{cases} \frac{x}{1+|x|} & \text{for } |x| < \infty \\ -1 & \text{for } x = -\infty \\ 1 & \text{for } x = \infty \end{cases}$$




Remark: Even though $\mathcal{B}^{\Omega} = \sigma\langle\text{open sets}\rangle$ there exists other generators exist & are useful for different purposes.

e.g. The FAP $([0,1], \mathcal{B}_0^{(0,1]}, P)$ from first lecture will be extended to $((0,1], \mathcal{B}^{(0,1]}, P)$ using $\mathcal{B}^{(0,1]} = \sigma\langle \mathcal{B}_0^{(0,1]} \rangle$ ... which will give Lebesgue measure on $(0,1]$.

e.g. we discussed $\mathcal{B}^{(0,1]} = \sigma\langle (0,x] : 0 \le x \le 1 \rangle$ is useful for proving two prob measures on $(0,1]$ are equal

Remark: It is good practice to prove a few equivalent generators for Borel σ-fields. This is typically done with "good sets" i.e.

$\sigma\langle \mathcal{E}_1 \rangle \subset \sigma\langle \mathcal{E}_2 \rangle$ follows by $\mathcal{E}_1 \subset \sigma\langle \mathcal{E}_2 \rangle$

Most are easy ... but a few can be slightly subtle:

$$\left.\begin{aligned} \mathcal{B}^{\bar{\mathbb{R}}} &= \sigma\langle [-\infty, a] : a \in \mathbb{R} \rangle \\ &= \sigma\langle [-\infty, a) : a \in \mathbb{R} \rangle \\ &\neq \sigma\langle (-\infty, a) : a \in \mathbb{R} \rangle \end{aligned}\right\} \begin{array}{l} \sigma\text{-fields} \\ \text{on } \Omega = \bar{\mathbb{R}} \end{array}$$

Remark: The Lebesgue σ-field of $\mathbb{R}$ extends $\mathcal{B}^{\mathbb{R}}$ using the Lebesgue measure by adding sets with "outer Lebesgue measure 0".

Thm: Suppose $\Omega$ is a metric space.

(i) $\Omega_0 \subset \Omega \Longrightarrow \underbrace{\mathcal{B}^{\Omega_0}}_{\substack{\text{w.r.t the} \\ \text{induced metric} \\ \text{on } \Omega_0}} = \mathcal{B}^{\Omega} \cap \Omega_0$

(ii) $\Omega_0 \subset \Omega$ & $\Omega_0 \in \mathcal{B}^{\Omega}$

$\Longrightarrow \mathcal{B}^{\Omega_0} = \{ B : B \in \mathcal{B}^{\Omega} \text{ \& } B \subset \Omega_0 \}$

Proof: see notes.

Thm: If $\Omega$ is a separable metric space then $\mathcal{B}^{\Omega} = \sigma\langle \text{open balls in } \Omega \rangle$.

Proof: exercise

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