

Lecture 16 : Brownian motion and Brownian bridge

(1)

In lecture 13 we discussed

The scaled and linearly interpolated random walk for $t \in [0, 1]$

$$W_t^n(w) := \frac{1}{\sqrt{n}} \left((1 - \alpha_t^n) S_{\lfloor nt \rfloor} + \alpha_t^n S_{\lceil nt \rceil} \right)$$

Scale the y -axis by $\frac{1}{\sqrt{n}}$ and scale the x -axis by $\frac{1}{n}$ and interpolate

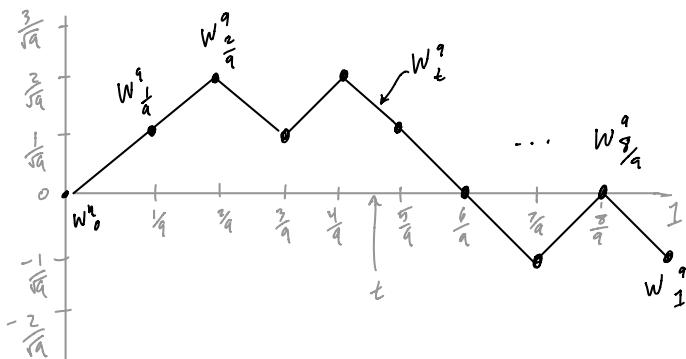
$$\text{where } S_k := R_1 + \dots + R_k \quad \&$$

$$R_1, R_2, \dots \stackrel{\text{iid}}{\sim} \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$\alpha_t^n = \text{distance to nearest left int } \lfloor nt \rfloor$$

$$= nt - \lfloor nt \rfloor$$

$$\therefore W_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} + \frac{nt - \lfloor nt \rfloor}{\sqrt{n}} R_{\lceil nt \rceil}$$



Theorem : (fdd converge)

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For any $0 \leq t_1 < t_2 < \dots < t_m \leq 1$

$$(W_{t_1}^n, \dots, W_{t_m}^n) \xrightarrow{\mathcal{D}} N(0, \Sigma) \text{ as } n \rightarrow \infty$$

$$\text{where } \Sigma_{ij} = t_i \wedge t_j$$

Proof:

First notice that the linear interpolation term $\frac{nt - \lfloor nt \rfloor}{\sqrt{n}} R_{\lceil nt \rceil} \xrightarrow{P} 0$ for any term.

\therefore By Slutsky it will be sufficient to show

$$\frac{1}{\sqrt{n}} (S_{\lfloor nt_1 \rfloor}, \dots, S_{\lfloor nt_m \rfloor}) \xrightarrow{\mathcal{D}} N(0, \Sigma).$$

Notice that

$$\begin{aligned} S_{\lfloor nt_1 \rfloor} &= \sum_{k=0}^{\lfloor nt_1 \rfloor} R_k \\ S_{\lfloor nt_2 \rfloor} - S_{\lfloor nt_1 \rfloor} &= \sum_{k=\lfloor nt_1 \rfloor + 1}^{\lfloor nt_2 \rfloor} R_k \\ &\vdots \\ S_{\lfloor nt_m \rfloor} - S_{\lfloor nt_{m-1} \rfloor} &= \sum_{k=\lfloor nt_{m-1} \rfloor + 1}^{\lfloor nt_m \rfloor} R_k \end{aligned} \quad \left. \begin{array}{l} \text{indep} \\ \text{sums} \end{array} \right\}$$

$$\text{Since } \sqrt{\frac{\lfloor nt_k \rfloor - \lfloor nt_{k-1} \rfloor}{n}} \xrightarrow{n \rightarrow \infty} \sqrt{t_k - t_{k-1}}$$

Slutsky & the CLT implies

$$(W_{t_1}^n, W_{t_2}^n - W_{t_1}^n, \dots, W_{t_m}^n - W_{t_{m-1}}^n)$$

$$\xrightarrow{\mathcal{D}} N\left(0, \begin{pmatrix} t_1 & & & \\ & (t_2 - t_1) & & \\ & & \ddots & \\ & & & (t_m - t_{m-1}) \end{pmatrix}\right)$$

The continuous mapping Theorem then gives

$$(W_{t_1}^n, W_{t_2}^n, \dots, W_{t_m}^n) \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

$$\text{where } \Sigma_{ij} = t_i \wedge t_j$$

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The last thing to show is that

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The sequence $\{W^n\}_{n=1}^{\infty}$ is tight

for the existence of a limit measure.

We use Billingsley's sufficient condition given on page 88 of his weak convergence book (the proof isn't difficult, just tedious)

Lemma:

$$\lim_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \lambda^2 P \left(\max_{p \leq n} |S_p| \geq \lambda \sqrt{n} \right) = 0$$

$$\Rightarrow \{W^n\}_{n=1}^{\infty} \text{ is tight}$$

Thm: (Brownian Motion)

Let W_t^n be defined as above. Then there exists a stochastic process W on $(C[0,1], \mathcal{B}(C[0,1]))$, with unique induced distribution, satisfying

$$(i) \quad W^n \xrightarrow{\mathcal{D}} W$$

$$(ii) \quad (W_{t_1}, \dots, W_{t_m}) \sim N(0, (t_i \wedge t_j)_{ij=1}^n)$$

$$\forall t_1, \dots, t_m \in [0,1] \quad \forall m \in \mathbb{N}$$

Proof:

By Etemadi's maximal inequality from Lecture 11 we have

$$\begin{aligned} P \left(\max_{1 \leq p \leq n} |S_p| \geq 3\lambda \sqrt{n} \right) &\leq 3 \max_{1 \leq p \leq n} P(|S_p| \geq \lambda \sqrt{n}) \\ &\leq 3 \max_{1 \leq p \leq n} P \left(\left| \frac{S_p}{\sqrt{n}} \right| \geq \frac{\lambda \sqrt{n}}{\sqrt{n}} \right) \end{aligned}$$

Now we use the fact that the R_i 's are bdd so Hoeffding's neg:

$$P \left(\left| \frac{S_k}{\sqrt{n}} \right| \geq \varepsilon \right) \leq 2 \exp \left(- \frac{2k\varepsilon^2}{(b-a)^2} \right)$$

applies to give

$$P \left(\left| \frac{S_k}{\sqrt{n}} \right| \geq \frac{\lambda \sqrt{n}}{\sqrt{n}} \right) \leq 2 \exp \left(- \frac{2}{4} k \left(\frac{\lambda \sqrt{n}}{\sqrt{n}} \right)^2 \right)$$

$$= 2 \exp \left(- \frac{1}{2} \lambda^2 \frac{n}{k} \right)$$

$$\leq 2 \exp(-\lambda^2 / 2) \quad \text{when } 1 \leq k \leq n$$

$$\begin{aligned} \therefore \lim_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \lambda^2 P \left(\max_{p \leq n} |S_p| \geq 3\lambda \sqrt{n} \right) &= 0 \\ &\underbrace{\quad}_{\leq 6 \lambda^2 \exp(-\lambda^2 / 2)} \end{aligned}$$

$\therefore \{W^n\}_{n=1}^{\infty}$ is tight by prev lemma.

Prohorov's Thm implies there exists a subseq n_k and a measure P on $(C[0,1], \mathcal{B}(C[0,1]))$

s.t.

$$\mathcal{L}(W^{n_k}) \xrightarrow{\mathcal{D}} P \quad \text{as } k \rightarrow \infty$$

Let W_t denote a stochastic process with induced distribution P .

$$\therefore W^{n_k} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} W$$

$$\therefore (W_{t_1}^{n_k}, \dots, W_{t_m}^{n_k}) \xrightarrow{\mathcal{D}} (W_{t_1}, \dots, W_{t_m}) \quad \text{since finite projections are continuous}$$

by the CLT //
 by uniqueness

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$\therefore W$ satisfies (ii)

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To show (i) we use the sub-sub-seg check: $\forall n_p \exists n_p$ s.t. $W^{n_p} \xrightarrow{j \rightarrow \infty} W$.

Let n_p be arbitrary. By Prohorov's theorem $\exists W'$ (possibly different from W at this point)

$$\text{s.t. } W^{n_p} \xrightarrow{j \rightarrow \infty} W'$$

By the CLT we already know that for any finite dimensional projection $\pi = \pi_{t_1, \dots, t_m}$

$$\begin{aligned} \pi(W^{n_p}) &\xrightarrow{\mathcal{D}} \pi(W'), \text{ by continuous} \\ &\text{mapping} \\ &\text{by CLT} \quad \downarrow \mathcal{D} \end{aligned}$$

$$N(0, (\langle t_i \wedge t_j \rangle)_{i,j=1}^m)$$

$$\text{by (ii) } \pi \stackrel{\mathcal{D}}{\longrightarrow} \pi(W)$$

$\therefore \pi(W) \stackrel{\mathcal{D}}{\longrightarrow} \pi(W')$ for all $\pi = \pi_{t_1, \dots, t_m}$
by uniqueness of limits

$\therefore W \stackrel{\mathcal{D}}{\longrightarrow} W'$ since

$$\underbrace{\langle \pi_{t_1, \dots, t_m} : t_i \in [0,1] \rangle}_{\pi\text{-system}} = B(C[0,1])$$

$\therefore W^{n_p} \xrightarrow{j \rightarrow \infty} W$ and since n_p was arb $W \xrightarrow{n \rightarrow \infty} W$.

QED

Remark: Notice the proof works for any iid R_1, R_2, \dots s.t.

$$E(R_i) = 0$$

$$E(R_i^2) = 1$$

R_i is bounded

The last condition isn't required but makes the maximal inequality easy with Hoeffding

The above result, for general R_i , is sometimes referred to as the "Functional CLT".

Example:

Let's see an example of something the functional CLT can do that the finite dimensional CLT can't.

The goal is to derive the asymptotic distribution of

$$\begin{aligned} \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} S_k &= \max_{1 \leq k \leq n} W_{\frac{k}{n}}^n \\ &= \sup_{t \in [0,1]} W_t^n, \quad \text{since } W^n \text{ is linear b/w} \\ &\quad \frac{k-1}{n} \text{ & } \frac{k}{n} \end{aligned}$$

Notice that the map

$$f \in C[0,1] \mapsto \sup_{t \in [0,1]} f(t) \in \mathbb{R}$$

is obviously continuous w.r.t. the sup-norm metric on $C[0,1]$.

∴ The continuous mapping theorem implies

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} S_k = \sup_{t \in [0,1]} W_t^n \xrightarrow[n \rightarrow \infty]{D} \sup_{t \in [0,1]} W_t$$

which you can then use stopping time theory to show

$$\sup_{t \in [0,1]} W_t \stackrel{D}{=} |Z| \text{ where } Z \sim N(0,1)$$

so that

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} S_k \xrightarrow[n \rightarrow \infty]{D} |Z|.$$

Extending W_t to $[0, \infty)$

In the previous section we constructed a stochastic process W_t using the independent r.v.s R_1, R_2, \dots

By re-arranging the R_k 's in a triangular sequence

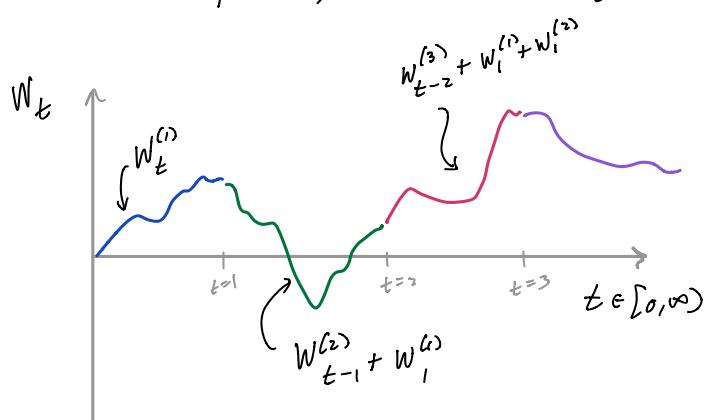
$$\left. \begin{array}{ccccccc} R_{11} & R_{12} & R_{13} & \cdots \\ R_{21} & R_{22} & R_{23} & \cdots \\ R_{31} & R_{32} & R_{33} & \cdots \\ \vdots & & & & & & \end{array} \right\} \text{all indep}$$

we can establish the existence of an infinite sequence of independent copies of W_t :

$$W_t^{(1)}, W_t^{(2)}, W_t^{(3)} \dots$$

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Stitching these together gives us a new stochastic process, also denoted W_t :



This new process satisfies

(i) W_t takes values in $C[0, \infty)$

(ii) $W_0 = 0$

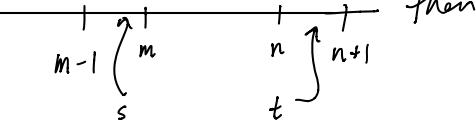
(iii) For any $0 \leq t_1 < t_2 < \dots < t_m < \infty$, $m \in \mathbb{N}$

$$(W_{t_1}, \dots, W_{t_m}) \sim N\left(0, (t_i \wedge t_j)_{i,j=1}^m\right)$$

The fact that $\text{Cov}(W_t, W_s) = t \wedge s$ is easy to verify by direct computation.

For example if $n \leq t \leq n+1$ then

$$\begin{aligned} \text{Var}(W_t) &= \text{Var}\left(W_{t-n}^{(n+1)} + W_1^{(n)} + \dots + W_1^{(1)}\right) \\ &= \text{Var}\left(W_{t-n}^{(n+1)}\right) + \text{Var}\left(W_1^{(n)}\right) + \dots + \text{Var}\left(W_1^{(1)}\right) \\ &= t - n + \underbrace{1 + \dots + 1}_{n-\text{terms}} \\ &= t \end{aligned}$$

and if  then

$$\text{var}(W_t - W_s)$$

$$= \text{var}[(W_t - W_n) + (W_n - W_{n-1}) + \dots + (W_{m+1} - W_m) \\ + (W_m - W_s)]$$

$$= \text{var}(W_t - W_n) \leftarrow (t-n) \\ + \text{var}(W_n - W_{n-1}) \leftarrow 1 \\ \vdots \\ + \text{var}(W_{m+1} - W_m) \leftarrow 1 \quad \left\{ \begin{array}{l} n-m \text{ terms} \\ \end{array} \right. \\ + \text{var}(W_m - W_s) \leftarrow m-s \\ \\ = (t-s)$$

Now

$$\text{var}(W_t - W_s) = \underbrace{\text{var}(W_t)}_{t-s} + \underbrace{\text{var}(W_s)}_s - 2 \text{cov}(W_t, W_s)$$

$$\therefore \text{cov}(W_t, W_s) = \frac{1}{2}(t+s - (t-s)) \\ = s \wedge t.$$

This stochastic process $(W_t : t \in [0, \infty))$
is called Brownian Motion.

Proposition 1 (Increments)

W_t has stationary & independent increments, i.e.

(a) If $t > s$ then

$$\text{var}(W_t - W_s) = t - s$$

$$(b) W_{t+s} - W_t \stackrel{D}{=} W_s - W_0 \stackrel{D}{=} N(0, s)$$

(c) $W_t - W_s$ is independent of $W_u - W_v$
whenever $t > s \geq u > v$

Proof: This follows directly from

$$\text{cov}(W_t, W_s) = t \wedge s. \quad \underline{\text{QED.}}$$

Proposition 2: (non-differentiability)

For $\omega \in \Omega$, the function
 $t \mapsto W_t(\omega)$ is nowhere differentiable.

Proof:

It is sufficient to show that for each $s \in [0, \infty)$

$$\mathcal{D}_s := \{ \omega : W_s(\omega) \text{ is differentiable at some } t < s \}$$

is contained in $A \in \mathcal{F}$ with $P(A) = 0$.

Consider \mathcal{D}_1 . Suppose $W_1(\omega)$ is differentiable at some $t < 1$ (i.e. $\omega \in \mathcal{D}_1$). Then

$\exists c \geq 0 \text{ & } \delta > 0 \text{ s.t.}$

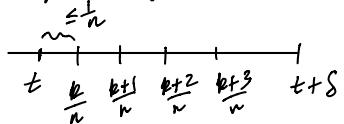
$$|W_u - W_t| \leq c(u-t)$$

$\forall u \in (t, t+\delta)$

Indeed whenever $u, v \in (t, t+s)$

$$|W_u - W_v| \leq |W_u - W_t| + |W_v - W_t| \\ \stackrel{\text{def}}{=} \leq c(u-t) + c(v-t)$$

Now for large enough $n \exists k \in \mathbb{N}$ s.t.



and hence

$$|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}| \leq c\left(\underbrace{\frac{k+1}{n} - t}_{\leq \frac{2}{n}}\right) + c\left(\underbrace{\frac{k}{n} - t}_{\leq \frac{1}{n}}\right) \leq \frac{8c}{n}$$

$$\left|W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}\right| \leq \frac{8c}{n}$$

$$\left|W_{\frac{k+3}{n}} - W_{\frac{k+2}{n}}\right| \leq \frac{8c}{n}$$

Let $\Delta_{k,n} := W_{\frac{k+1}{n}} - W_{\frac{k}{n}}$ and notice

that even though f & c depend on $w \in \mathcal{R}$
there will always exist a $C \in \mathbb{Q}^+$ &
 $N = N(w)$ s.t. $\forall n \geq N(w) \exists k \leq n-1$

$$\max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n}$$

In particular

$$w \in D_1 \Rightarrow$$

$$w \in \bigcup_{C \in \mathbb{Q}^+} \left\{ \bigcup_{k=0}^{n-1} \left\{ \max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n} \right\} \text{ a.a.n } \right\}$$

measurable set

Now for any $c \in \mathbb{Q}^+$, Fatou gives

$$P \left(\bigcup_{k=0}^{n-1} \left\{ \max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n} \right\} \text{ a.a.n} \right) \\ \leq \liminf_{n \rightarrow \infty} P \left(\bigcup_{k=0}^{n-1} \left\{ \max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n} \right\} \right) \\ \leq \liminf_{n \rightarrow \infty} \sum_{k=0}^{n-1} P \left[\max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n} \right] \\ \leq \liminf_{n \rightarrow \infty} \sum_{k=0}^{n-1} P \left(|\Delta_{k,n}| \leq \frac{1}{n}, |\Delta_{k+1,n}| \leq \frac{1}{n}, |\Delta_{k+2,n}| \leq \frac{C}{n} \right)$$

but these are indep increments with variance $\frac{1}{n}$

Since $\sqrt{n} \Delta_{k,n}, \sqrt{n} \Delta_{k+1,n}, \sqrt{n} \Delta_{k+2,n} \stackrel{iid}{\sim} N(0,1)$

we have that $\forall c \in \mathbb{Q}^+$

$$\liminf_{n \rightarrow \infty} \sum_{k=0}^{n-1} \underbrace{\left[P(|z| \leq \frac{c}{\sqrt{n}}) \right]^3}_{\begin{array}{l} o(\frac{1}{\sqrt{n}}) \\ \curvearrowright \\ o(n \frac{1}{n^{3/2}}) \end{array}} = 0$$

$\therefore D_1$ is covered by

$$\bigcup_{C \in \mathbb{Q}^+} \left\{ \bigcup_{k=0}^{n-1} \left\{ \max(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|) \leq \frac{C}{n} \right\} \text{ a.a.n } \right\}$$

each of these is P -null.

$\therefore D_1$ is P -neg.

Similar arguments show D_n is P -null
 $\forall n \in \mathbb{N}$.

QED.

Up to this point we know $W_t(w)$ is continuous in t for all $w \in \Omega$ and have characterized the f.d.d. of W_t .

However to classify W_t as inducing a measure on $C[0, \infty)$ we need a σ -field on $C[0, \infty)$, preferably given by a Polish metric.

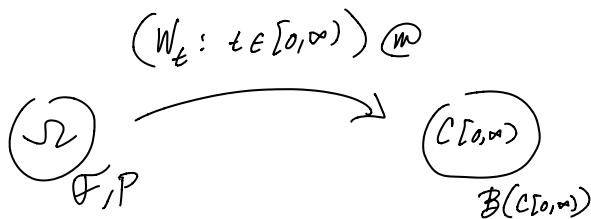
For $f, g \in C[0, \infty)$ define

$$d_n(f, g) := \sup_{t \in [0, n]} |f(t) - g(t)|$$

and set

$$d(f, g) := \sum_{n=1}^{\infty} \frac{d_n(f, g) \wedge 1}{2^n}$$

Now with this metric $C[0, \infty)$ is Polish and our process W_t satisfies



Moreover all our old results carry over. In particular if W_t & W'_t are stochastic processes taking values in $(C[0, \infty), B(C[0, \infty)))$ s.t.

$$(W_{t_1}, \dots, W_{t_n}) \stackrel{d}{=} (W'_{t_1}, \dots, W'_{t_n}) \quad t_{t_1}, \dots, t_m, m$$

then $(W_t : t \in [0, \infty)) \stackrel{d}{=} (W'_t : t \in [0, \infty)).$

See Klenke for details.

Proposition 3 (Scaling & time reversal)

- $(W_t : t \in [0, \infty)) \stackrel{d}{=} (\sqrt{c} W_{ct} : t \in [0, 1])$ for any $c > 0$.

- If $W_t^* := \begin{cases} t W_{\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$

then $(W_t : t \in [0, \infty)) \stackrel{d}{=} (W_t^* : t \in [0, \infty))$

Proof: All that needs to be checked is that

$$\text{cov}(\sqrt{c} W_{ct}, \sqrt{c} W_{cs}) = t \cdot s \quad \text{and}$$

$$\text{cov}(t W_{\frac{1}{t}}, s W_{\frac{1}{s}}) = t \cdot s$$

which is a simple exercise.

QED.

When solving a problem about W_t it often helps to see how scaling and time reversal might effect the answer.

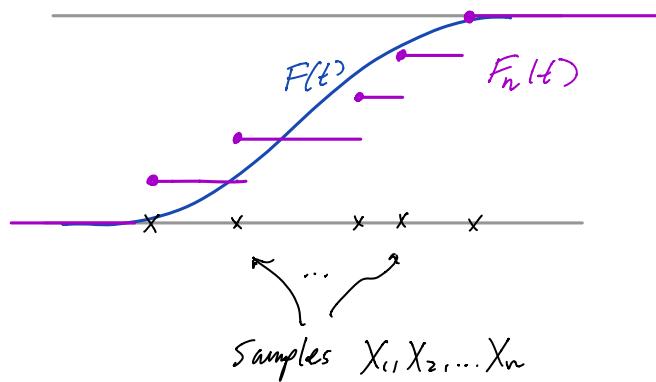
Brownian Bridge

Let X be a r.v. with c.d.f. $F(t)$.

Given iid copies of X : X_1, X_2, \dots, X_n

Construct an estimate of $F(t)$ as follows

$$F_n(t) = \frac{1}{n} \sum_{k=1}^n I_{\{X_k \leq t\}}$$



In Lecture 11 we proved the Glivenko-Cantelli Thm which says:

$$\sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \xrightarrow{\text{P-a.e.}} 0 \quad \text{as } n \rightarrow \infty$$

Notice that if we re-scale by \sqrt{n} we get

$$\sqrt{n}(F_n(t) - F(t)) = \sqrt{n} \left(\underbrace{\frac{1}{n} \sum_{k=1}^n (I_{\{X_k \leq t\}} - F(t))}_{:= Y_{1,t}} \right)$$

where $Y_{1,t}, \dots$ are iid r.v.s with

$$E(Y_{k,t}) = E(I_{\{X_k \leq t\}}) - F(t)$$

$$= P(X_k \leq t) - F(t) = 0$$

