

Lecture 21: Martingales

(1)

Martingales are models for a system of wagers coming from a sequence of fair games.

submartingales model advantageous games

supermartingales model disadvantageous games

Another view point is that sub & super martingales model the stochastic equivalent to monotonic sequences of numbers. Indeed we will prove that under "bdd" type conditions sub & super martingales are guaranteed to converge.

Indeed, a basic theme in the next lecture is that "Martingales like to converge" so "looking for a Martingale" is a technique for establishing convergence.

Assumption: For the rest of this lecture fix some probability space (Ω, \mathcal{F}, P) .

Definition: A filtration is an increasing sequence of sub σ -fields

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$$

Definition: A sequence of r.v.s X_1, X_2, \dots defined on (Ω, \mathcal{F}, P) is adapted to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ if X_n is \mathcal{F}_n -measurable for each $n \in \mathbb{N}$.

Definition:

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Suppose $(X_n)_{n \in \mathbb{N}}$ is adapted to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ & $X_n \in L_1(\Omega, \mathcal{F}, P)$ for all $n \in \mathbb{N}$. Then

- $(X_n)_{n \in \mathbb{N}}$ is a martingale if

$$E(X_{n+1} | \mathcal{F}_n) = X_n \quad \text{f } n \in \mathbb{N}$$

- $(X_n)_{n \in \mathbb{N}}$ is a submartingale if

$$E(X_{n+1} | \mathcal{F}_n) \geq X_n \quad \text{f } n \in \mathbb{N}$$

- $(X_n)_{n \in \mathbb{N}}$ is a supermartingale if

$$E(X_{n+1} | \mathcal{F}_n) \leq X_n \quad \text{f } n \in \mathbb{N}$$

when \mathcal{F}_n represents the current state of information & X_n represents a gambler fortune, this says future fortune decreases in expected value given \mathcal{F}_n .

Definition: If $(X_n)_{n \in \mathbb{N}}$ are r.v.s on (Ω, \mathcal{F}, P) then the natural filtration is

$$\mathcal{F}_n := \sigma(X_1, \dots, X_n)$$

Notation:

As a shorthand we let

"subM" = submartingale

"supM" = super martingale

"M" = martingale

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Notice that if $(X_n)_{n \geq 1}$ is a subM
then clearly $(-X_n)_{n \geq 1}$ is a supM by (e1).
Also if $(X_n)_{n \geq 1}$ is both a subM & supM
it is then a M.

Here is another immediate consequence of the definitions

Proposition

$(X_n)_{n \geq 1}$ is a subM w.r.t. filtration $(\mathcal{F}_n)_{n \geq 1}$



$$E(X_m | \mathcal{F}_n) \stackrel{\text{a.e.}}{=} X_n \quad \forall m > n$$

Proof:

(\Leftarrow): Trivial

(\Rightarrow): If $m > n$ then $\mathcal{F}_n \subset \mathcal{F}_m$ $\xrightarrow{\text{a more dramatic smoother}}$

$$\begin{aligned} E(X_{m+1} | \mathcal{F}_n) &\stackrel{\text{a.e.}}{=} E(\underbrace{E(X_{m+1} | \mathcal{F}_m)}_{\text{an intermediate smoothing}} | \mathcal{F}_n) \\ &\stackrel{\text{a.e.}}{\geq} E(X_m | \mathcal{F}_n). \end{aligned}$$

The result now follows by induction.

QED

Corollary:

$$(X_n)_{n \geq 1} \text{ is a subM} \implies E(X_m) \geq E(X_n) \quad \forall m > n$$

$$(X_n)_{n \geq 1} \text{ is a supM} \implies E(X_m) \leq E(X_n) \quad \forall m > n$$

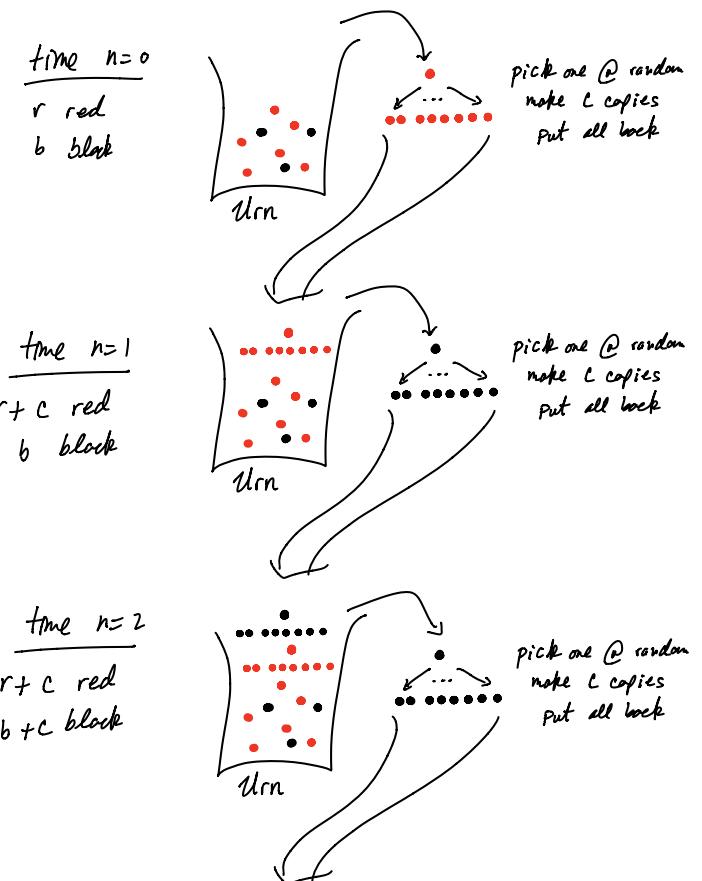
$$(X_n)_{n \geq 1} \text{ is a M} \implies E(X_m) = E(X_n) \quad \forall m > n$$

Proof: Simple application of (S1)

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Pólya's Urn

This is a classic example of a martingale (and exchangeable sequences of r.v.s) and a good canonical example of a martingale to have in your mind.



Repeat

:

Let $Y_n := \begin{cases} 1 & \text{if the } n^{\text{th}} \text{ draw is red.} \\ 0 & \text{if } \end{cases}$

$R_n := \text{proportion of reds}$
 $\text{in the urn at time } n$

$$\mathcal{F}_n := \sigma(R_1, R_2, \dots, R_n)$$

Question: Does R_n have a limiting distribution?

First thing to notice is that

$$P(Y_n = 1 | R_1, \dots, R_n) = R_n$$

so that

$$E(Y_n | \mathcal{F}_n) = E(Y_n | R_1, \dots, R_n)$$

$$= \sum_{y \in \Omega_n} y P(Y_n = y | R_1, \dots, R_n)$$

$$= R_n$$

Moreover $R_n = \frac{r + c(Y_0 + \dots + Y_{n-1})}{r+b+nC}$
which implies

$$E(R_{n+1} | \mathcal{F}_n)$$

$$\begin{aligned} &= E\left(\frac{r+c(Y_0+\dots+Y_n)}{r+b+(n+1)c} \middle| \mathcal{F}_n\right) \\ &= E\left(\frac{r+b+nC}{r+b+(n+1)c} \underbrace{\left(\frac{r+c(Y_0+\dots+Y_{n-1})}{r+b+nC}\right)}_{\text{you add } C \text{ more balls}} \middle| \mathcal{F}_n\right) \\ &\quad + E\left(\frac{c Y_n}{r+b+(n+1)c} \middle| \mathcal{F}_n\right) \\ &= \frac{r+b+nC}{r+b+(n+1)c} R_n + \frac{c}{r+b+(n+1)c} R_n \\ &= R_n \end{aligned}$$

Since R_n is trivially \mathbb{M} w.r.t. \mathcal{F}_n we therefore have that R_n is a martingale. We will see that under mild conditions martingales converge.

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Transformation of Martingales

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Theorem: (Transformations)

- (i) If $(X_n)_{n \geq 1}$ & $(Y_n)_{n \geq 1}$ are subMs w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, then so are $(X_n + Y_n)_{n \geq 1}$ and $(X_n \vee Y_n)_{n \geq 1}$.
 - (ii) If $(X_n)_{n \geq 1}$ is a M w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ is convex and $f(X_n) \in L_1(\Omega, \mathcal{F}, P)$ then $(f(X_n))_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$.
 - (iii) If $(X_n)_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ is convex, f is non-decreasing and $f(X_n) \in L_1(\Omega, \mathcal{F}, P)$ then $(f(X_n))_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$.
 - (iv) If $(X_{1n})_{n \geq 1}, (X_{2n})_{n \geq 1}, \dots, (X_{mn})_{n \geq 1}$ are m subM wrt $(\mathcal{F}_n)_{n \geq 1}$ and $w_1, w_2, \dots, w_m \geq 0$ then $Y_n := \sum_{i=1}^m w_i X_{in}$ is a subM wrt $(\mathcal{F}_n)_{n \geq 1}$.
- Proof:
- For (i):
- Remember that the def of subM, supM & M require $X_n, Y_n \in L_1(\Omega, \mathcal{F}, P)$ so the fact that $(X_n + Y_n)_{n \geq 1}$ is a subM is a trivial consequence of (ex) from lecture 20.

To show $(X_n \vee Y_n)_{n \geq 1}$ is a subM
notice that

$$\begin{aligned} X_{n+1} &\leq X_{n+1} \vee Y_{n+1} \\ Y_{n+1} &\leq X_{n+1} \vee Y_{n+1} \\ \therefore X_n &\stackrel{\text{P.a.e.}}{\leq} E(X_{n+1} | \mathcal{F}_n) \stackrel{\text{P.a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n) \quad (\star) \\ Y_n &\stackrel{\text{P.a.e.}}{\leq} E(Y_{n+1} | \mathcal{F}_n) \stackrel{\text{P.a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n) \\ &\quad \text{by monotonicity (e3).} \\ \therefore X_n \vee Y_n &\stackrel{\text{P.a.e.}}{\leq} E(X_{n+1} | \mathcal{F}_n) \vee E(Y_{n+1} | \mathcal{F}_n) \\ &\quad \text{by subM prop.} \\ &\stackrel{\text{P.a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n) \end{aligned}$$

Also $|X_n \vee Y_n| \leq |X_n| + |Y_n| \in L_1(\Omega, \mathcal{F}, P)$
so that $(X_n \vee Y_n)_{n \geq 1}$ satisfies the subM properties.

For (ii) & (iii)

$$E(f(X_{n+1}) | \mathcal{F}_n) \stackrel{\text{P.a.e.}}{\geq} f(E(X_{n+1} | \mathcal{F}_n)) = f(X_n)$$

by Jensen for conditional expected value (proved in the exact same way as was done in Lecture 11)

The proves (ii) & (iii) is similar.

For (iv)

Follows easily by linear properties of $E(\cdot | \mathcal{F}_n)$.

QED

⑦

Notice a few simple consequences of the previous thm:

$(X_n)_{n \geq 1}$ is a M $\Rightarrow (|X_n|)_{n \geq 1}$ is a subM

$(X_n)_{n \geq 1}$ is a M } $\Rightarrow (X_n^2)_{n \geq 1}$ is a subM
& $X_n \in L_2$

$(X_n)_{n \geq 1}$ is a subM $\Rightarrow (X_n^+)_{n \geq 1}$ is a subM

A collection of examples

Example: (strategy against a bad game)

X_0 = a gambler's initial fortune

X_n = fortune after n-plays of a game when always betting \$1 on each play,
i.e. $X_n - X_{n-1}$ = winnings obtained on the n^{th} play

Suppose $(X_n)_{n \geq 1}$ is a supM wrt $(\mathcal{F}_n)_{n \geq 1}$.

Since X_n is a supM your expected winnings are monotonically decreasing with n

Question:

Can I wager differently (bet more on hot streaks) to convert X_n into a subM?

Let W_n be the wager on n^{th} play s.t.

$$W_n \geq 0$$

W_n is \mathcal{F}_{n-1} -measurable \hookrightarrow I can change bet based on the future

W_n is bounded \hookrightarrow i.e. there is a house max bet

⑧

Winnings on the n^{th} play is

$$W_n (X_n - X_{n-1}) \\ = \Delta_n = \text{winnings on } n^{\text{th}} \text{ play wagering \$1}$$

The total fortune after n plays with wagers W_n is given by

$$X_n^* = X_0 + W_1 \Delta_1 + W_2 \Delta_2 + \dots + W_n \Delta_n \\ \quad \uparrow \quad \uparrow \\ \quad \text{at } \mathcal{F}_{n-1} \quad \text{at } \mathcal{F}_n$$

claim: $(X_n^*)_{n \geq 1}$ is a supM w.r.t \mathcal{F}_n

Proof: so no matter the strategy still a losing game
Clearly X_n^* is adapted to \mathcal{F}_n and $L_1(\mathcal{F}, \mathcal{G}, P)$ by definition.

Also

$$\begin{aligned} E(X_{n+1}^* | \mathcal{F}_n) &\stackrel{\text{P-a.e.}}{=} E(X_n^* + W_{n+1} \Delta_{n+1} | \mathcal{F}_n) \\ &\stackrel{\text{P-a.e.}}{=} X_n^* + W_{n+1} E(\Delta_{n+1} | \mathcal{F}_n) \\ &= X_n^* + W_{n+1} \left[E(X_{n+1} | \mathcal{F}_n) - E(X_n | \mathcal{F}_n) \right] \\ &\quad \uparrow \quad \underbrace{\leq X_n}_{\text{positive}} \quad \underbrace{\geq X_n}_{\text{negative}} \\ &\leq X_n^* \end{aligned}$$

\therefore indeed $(X_n^*)_{n \geq 1}$ is a supM.

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Now suppose $(X_n)_{n \geq 1}$ is a subM (10)

By similar reasoning $(X_n^*)_{n \geq 1}$ is also a subM. timid play

Suppose $W_n \in [0, 1]$

Now

$$\begin{aligned} E(X_{n+1}^* - X_n^* | \mathcal{F}_n) &\stackrel{\text{a.e.}}{=} E(W_{n+1}(X_{n+1} - X_n) | \mathcal{F}_n) \\ &\quad \text{at } \mathcal{F}_{n+1} \\ &\stackrel{\text{a.e.}}{=} W_{n+1} E(X_{n+1} - X_n | \mathcal{F}_n) \\ &\in [0, 1] \quad \geq 0 \text{ by subM} \\ &\stackrel{\text{a.e.}}{\leq} E(X_{n+1} - X_n | \mathcal{F}_n) \end{aligned}$$

Take E of both sides gives

$$E(X_{n+1}^* - X_n^*) \leq E(X_{n+1} - X_n)$$

$$\therefore \sum_{n=0}^{N-1} E(X_{n+1}^* - X_n^*) \leq \sum_{n=0}^{N-1} E(X_{n+1} - X_n)$$

||

$$E(X_N^*) - E(X_0^*) \leq E(X_N) - E(X_0)$$

same initial fortune ↑

$$\therefore E(X_N^*) \leq E(X_N)$$

i.e. timid play reduces expected fortune.

Example: Smoothing Martingale

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$(\mathcal{F}_n)_{n \geq 1}$ is a filtration.

Let $X \in L_1(\Omega, \mathcal{F}, P)$ &

$$X_n := E(X | \mathcal{F}_n)$$

Clearly X_n is \mathcal{F}_n -measurable and in $L_1(\Omega, \mathcal{F}, P)$ by smoothing property (53) from Lecture 20. Also

$$E(X_{n+1} | \mathcal{F}_n) \stackrel{\text{a.e.}}{=} E(E(X | \mathcal{F}_{n+1}) | \mathcal{F}_n)$$

$$\stackrel{\text{a.e.}}{(52)} = E(X | \mathcal{F}_n)$$

$$\stackrel{\text{a.e.}}{=} X_n$$

$\therefore (X_n)_{n \geq 1}$ is a M w.r.t. $(\mathcal{F}_n)_{n \geq 1}$.

Next lecture we will investigate if

$$X_n \xrightarrow{?} X$$

Example: Martingale difference sequences

If $(X_n)_{n \geq 1}$ is a M w.r.t. $(\mathcal{F}_n)_{n \geq 1}$ we can always write

$$X_n = X_1 + \underbrace{(X_2 - X_1)}_{\in \mathcal{F}_1} + \cdots + \underbrace{(X_n - X_{n-1})}_{\in \mathcal{F}_n}$$

where the terms in the sum satisfy

$$\begin{aligned} E((X_{i+1} - X_i) | \mathcal{F}_i) &\stackrel{\text{a.e.}}{=} E(X_{i+1} | \mathcal{F}_i) - X_i \\ &\stackrel{\text{a.e.}}{=} 0 \end{aligned}$$

Definition: If $(\Delta_n)_{n \geq 1}$ are real

valued r.v.s on (Ω, \mathcal{F}, P) then $(\Delta_n)_{n \geq 1}$

is a subM, M or supM difference sequence w.r.t. $(\mathcal{F}_n)_{n \geq 1}$ iff

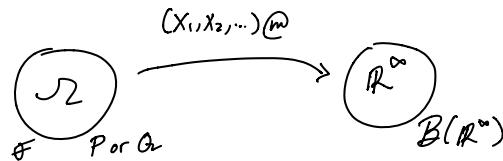
(i) $\Delta_n \in L_1(\Omega, \mathcal{F}_n, P|_{\mathcal{F}_n})$ and

$$\text{(ii)} \quad E(\Delta_{n+1} | \mathcal{F}_n) \begin{cases} \leq 0 & \text{for supM} \\ = 0 & \text{for M} \\ \geq 0 & \text{for subM} \end{cases}$$

Example: Lebesgue decomposition supM

Suppose you have two models P & Q for an infinite sequence of r.v.s

$$X = (X_1, X_2, \dots)$$



Question 1: How can one check if

$$PX^{-1} \perp \underbrace{QX^{-1}}_{\text{if this holds}}$$

if this holds then one sample from X allows you to determine which method generated it

$$n \sim P \Rightarrow X^{(n)}$$

$$\text{or}$$

$$n \sim Q \Rightarrow X^{(n)}$$

$$\underbrace{PX^{-1} = QX^{-1}}_{\text{if this holds it will be impossible to distinguish P or Q}}$$

from one sample of $X = (X_1, \dots)$

Question 2: If $PX^{-1} \perp QX^{-1}$ can one distinguish P from Q as $n \rightarrow \infty$ when observing X at the first n-coordinates:

$$X_{1:n} := (X_1, X_2, \dots, X_n) \in \mathbb{R}^n$$

Let

$$\mathcal{F}_n := \sigma(X_1, \dots, X_n)$$

$$\mathcal{F}_\infty := \sigma(X_i : i \geq 1)$$

$$Q_n := Q|_{\mathcal{F}_n} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ & similarly for } P$$

$$Q_\infty := Q|_{\mathcal{F}_\infty}$$

$$Q_n = \underbrace{Q_n^a}_{\text{The Lebesgue decomposition w.r.t } P_n} + \underbrace{Q_n^s}_{\text{where } Q_n^a \ll P_n \text{ & } Q_n^s \perp P_n}$$

$$Q_\infty = \underbrace{Q_\infty^a}_{\text{The Lebesgue decomposition w.r.t } P_\infty} + \underbrace{Q_\infty^s}_{\text{where } H(Q_\infty, P_\infty) \rightarrow 0}$$

$$\text{Note: } Q_\infty \perp P_\infty \iff Q_\infty^a \equiv 0$$

Suppose g_n & P_n are densities of $Q_n(X_1, \dots, X_n)^{-1}$ and $P_n(X_1, \dots, X_n)^{-1}$ w.r.t. some dominating measure μ_n (could be \mathbb{Z}^n or $Q_n(X_1, \dots, X_n)^{-1} + P_n(X_1, \dots, X_n)^{-1}$ e.g.)

Define g_∞ , P_∞ & Q_∞ similarly.

Now

$$\frac{dQ_n^a}{dP_n} = \frac{P_n^{-a-e}}{f_n(X_1, \dots, X_n)} I_{\{P_n(X_1, \dots, X_n) > 0\}}$$

and

$$\frac{dQ_\infty^a}{dP_\infty} = \frac{P_\infty^{-a-e}}{f_\infty(X_1, X_2, \dots)} I_{\{P_\infty(X_1, X_2, \dots) > 0\}}$$

which follow by the proof of the Lebesgue Decomposition Thm & change of variables.

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Now

$$P X^{-1} \perp Q X^{-1}$$

↙ since events in \mathcal{G}_n are of the form $X^{-1}(B)$ for $B \in \mathcal{B}(\mathbb{R}^n)$

$$\iff Q_\infty \perp P_\infty$$

$$\iff Q_\infty^a \equiv 0$$

$$\iff \frac{dQ_\infty^a}{dP_\infty} = 0 \quad P_\infty\text{-a.e.}$$

$$\overset{?}{\iff} \frac{dQ_n^a}{dP_n} \xrightarrow{?} 0$$

see next lecture ..

$$\overset{?}{\iff} H(Q_n, P_n) \rightarrow 0$$

where $H(\cdot, \cdot)$ is the Hellinger distance on probability measures:

$$H(Q_n, P_n) := \int_{\mathbb{R}^n} (f_n - f_n) d\mu_n = 2(1 - \int_{\mathbb{R}^n} f_n d\mu_n)$$

$$\overset{?}{\iff} KL(Q_n || P_n) \rightarrow 0$$

where $KL(\cdot || \cdot)$ is the symmetrized Kullback-Liebler dist b/w P_n & Q_n :

$$KL(Q_n || P_n) := \int_{\Omega} \log \frac{dQ_n^a}{dP_n} dP_n$$

$$+ \int_{\Omega} \log \frac{dP_n^s}{dQ_n} dQ_n$$

We will resolve the question marks later but notice that we can get partway to resolving some by noticing

$\frac{dQ_n^a}{dP_n}$ is a supM w.r.t. \mathcal{F}_n on (Ω, \mathcal{F}, P)

Indeed $\frac{dQ_n^a}{dP_n} \in L_1(\Omega, \mathcal{F}_n, P_n)$ and $\forall A \in \mathcal{F}_n$

$$\int_A E\left(\frac{dQ_{n+1}^a}{dP_{n+1}} \mid \mathcal{F}_n\right) dP = \int_A \frac{dQ_{n+1}^a}{dP_{n+1}} dP$$

can replace with $d\mu_{n+1}$ by C.V.

$$= Q_{n+1}^a(A)$$

$$\leq Q_{n+1}(A), \quad Q_n = Q_n^a + Q_n^s$$

$$= Q_n(A) \text{ since } A \in \mathcal{F}_n$$

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$$\therefore \int_A E\left(\frac{dQ_n^a}{dP_{n+1}} \mid \mathcal{F}_n\right) dP_n \leq Q_n(A), \quad \forall A \in \mathcal{F}_n \quad (15)$$

but we also have that

$$\int_A \frac{dQ_n^a}{dP_n} dP_n \leq Q_n(A), \quad \forall A \in \mathcal{F}_n$$

where this is the P_n -largest such, by Lebesgue density theorem

$$\therefore E\left(\frac{dQ_n^a}{dP_{n+1}} \mid \mathcal{F}_n\right) \stackrel{P_n-a.e.}{\leq} \frac{dQ_n^a}{dP_n}$$

i.e. $\left(\frac{dQ_n^a}{dP_n}\right)_{n \geq 1}$ is a seqM wrt- $(\mathcal{F}_n)_{n \geq 1}$ on (Ω, \mathcal{F}, P) .

Also notice that $\frac{dQ_n^a}{dP_n}$ is a likelihood ratio density

Since $\forall A \in \mathcal{F}_n$

$$\begin{aligned} \int_A E\left(\frac{dQ_{n+1}}{dP_{n+1}} \mid \mathcal{F}_n\right) dP &= \int_A \frac{dQ_{n+1}}{dP_{n+1}} dP \\ &= Q_{n+1}(A) \\ &= Q_n(A) \\ &= \int_A \frac{dQ_n}{dP_n} dP \end{aligned}$$

so that $E\left(\frac{dQ_{n+1}}{dP_{n+1}} \mid \mathcal{F}_n\right) = \frac{dQ_n}{dP_n}$ by

uniqueness of densities.

Stopping times

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Stopping times are a dynamic rule for when to stop gambling ... but needs to be computable before the next game is about to occur.

$\therefore \text{defn}$

Definition: A function $\tau : \Omega \rightarrow \bar{\mathbb{N}}$ is a **stopping time** for a filtration $(\mathcal{F}_n)_{n \geq 1}$, if $\{\tau = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$.

Determine to stop or not after n^{th} game is played but before $n+1^{\text{st}}$ is played

When X_n represents gamblers total fortune think of X_τ as "final fortune".

Note: τ is \mathcal{F} -measurable since

$B(\bar{\mathbb{N}}) = \sigma\{\xi_n : n \in \bar{\mathbb{N}}\}$ and by generators are enough we just check that

$\{\tau = n\} \in \mathcal{F}_n \in \mathcal{F}$

$$\{\tau = \infty\} = \{\tau < \infty\}^c = \bigcap_{n=1}^{\infty} \{\tau = n\}^c \in \mathcal{F}$$

For continuous time martingales it will be convenient to use the condition

$\{\tau \leq n\} \in \mathcal{F}_n$ rather than $\{\tau = n\} \in \mathcal{F}_n$.

In our case they are equivalent:

$$\{\tau = n\} \in \mathcal{F}_n, \quad \forall n \in \mathbb{N} \Rightarrow \{\tau \leq n\} = \bigcup_{k=1}^n \{\tau = k\} \in \mathcal{F}_n$$

$\in \mathcal{F}_n \subset \mathcal{F}_n$

$$\{\tau \leq n\} \in \mathcal{F}_n, \quad \forall n \in \mathbb{N} \Rightarrow \{\tau = n\} = \{\tau \leq n\} - \{\tau \leq n-1\}$$

$\in \mathcal{F}_n$

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Proposition: (The X_n values determine when to stop) for the natural filtration

If $(\mathcal{F}_n)_{n \geq 1}$ is the natural filtration for $(X_n)_{n \geq 1}$ and τ is a stopping time w.r.t. $(\mathcal{F}_n)_{n \geq 1}$ then $\forall w_1, w_2 \in \Omega$

$$\left. \begin{array}{l} \tau(w_1) = n < \infty \quad \& \\ X_k(w_1) = X_k(w_2), \quad \& k \leq n \end{array} \right\} \Rightarrow \tau(w_2) = n$$

Proof:

Since $\{\tau = n\} \in \mathcal{F}_n = \sigma(X_1, \dots, X_n)$ we have

$$\{\tau = n\} = \{w \in \Omega : (X_1(w), \dots, X_n(w)) \in B\}$$

for some $B \in \mathcal{B}(\mathbb{R}^n)$.

$$\begin{aligned} \tau(w_1) = n < \infty &\Leftrightarrow (X_1(w_1), \dots, X_n(w_1)) \in B \\ &\Leftrightarrow (X_1(w_2), \dots, X_n(w_2)) \in B \\ &\Leftrightarrow \tau(w_2) = n < \infty \end{aligned}$$

QED

Example: Defining $\tau = 5$ gives a stopping time since $\mathcal{F}_n \in \mathcal{N}$

$$\{\tau = n\} = \begin{cases} \Omega & \text{if } n=5 \\ \emptyset & \text{o.w.} \end{cases} \in \mathcal{F}_n$$

Example:

Let $c \in \mathbb{R}$ & $(X_n)_{n \geq 1}$ be adapted to $(\mathcal{F}_n)_{n \geq 1}$

& $\tau :=$ The first time X_n enters $[c, \infty)$

$$= \inf \left(\{n : X_n \geq c\} \cup \{\infty\} \right)$$

↑
so $\tau = \infty$ if
no $n \in \mathbb{N}$ s.t. $X_n \geq c$

τ is a stopping time since

$$\{\tau \leq n\} = \underbrace{\{X_1 \geq c\}}_{\in \mathcal{F}_1} \cup \dots \cup \underbrace{\{X_n \geq c\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n$$

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Proposition: ($\sigma \wedge \tau$ is a ST)

If $\sigma \wedge \tau$ are stopping times for a filtration $(\mathcal{F}_n)_{n \geq 1}$, then so is

$$\sigma \wedge \tau := \min(\sigma, \tau).$$

Proof: $\{\sigma \wedge \tau \leq n\} = \underbrace{\{\sigma \leq n\}}_{\in \mathcal{F}_n} \cup \underbrace{\{\tau \leq n\}}_{\in \mathcal{F}_n}.$

QED

The above proposition is particularly useful for truncating a ST $\tau \geq k$:

$\tau \wedge k$ is a ST for any fixed $k < \infty$
b.d. stopping.

Recall that \mathcal{F}_n represents information available to you at time n .

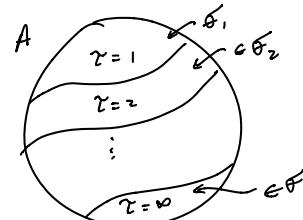
What represents information at time τ ?

Definition: If τ is a stopping time w.r.t filtration $(\mathcal{F}_n)_{n \geq 1}$, define \mathcal{F}_τ to be the collection of all A.C.R.s.t.

$A \cap \{\tau = n\} \in \mathcal{F}_n, \quad \forall n \in \mathbb{N}$ &

$A \cap \{\tau = \infty\} \in \mathcal{F} \rightarrow$ so that $\mathcal{F}_\tau \subset \mathcal{F}$

So $A \in \mathcal{F}_\tau$ has the form



Proposition: (equiv def of \mathcal{F}_∞)

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If τ is a stopping time w.r.t filtration $(\mathcal{F}_n)_{n \geq 1}$, then

$$A \in \mathcal{F}_\infty \stackrel{(a)}{\iff} A \in \mathcal{F} \text{ & } A \cap \{\tau = n\} \in \mathcal{F}_n, \forall n \in \mathbb{N}$$

$$\stackrel{(b)}{\iff} A \in \mathcal{F} \text{ & } A \cap \{\tau \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{N}$$

Proof:

$\stackrel{(a)}{\Rightarrow}$: Easy since

$$A = \underbrace{(A \cap \{\tau = \infty\})}_{\in \mathcal{F}} \cup \underbrace{(A \cap \{\tau = 1\})}_{\in \mathcal{G}_1} \cup \underbrace{(A \cap \{\tau = 2\})}_{\in \mathcal{G}_2} \cup \dots \in \mathcal{F}$$

$\stackrel{(a)}{\Leftarrow}$: Easy since

$$A \cap \{\tau = \infty\} = A \cap \underbrace{\{\tau < \infty\}}_{\in \mathcal{F}}^c \in \mathcal{F}.$$

$$\stackrel{(b)}{\Rightarrow}: A \cap \{\tau \leq n\} = A \cap \bigcup_{k=1}^n \{\tau = k\} \in \mathcal{F}_n$$

$$\stackrel{(b)}{\Leftarrow}: A \cap \{\tau = n\} = \underbrace{A \cap \{\tau \leq n\}}_{\in \mathcal{F}_n} - \underbrace{A \cap \{\tau \leq n-1\}}_{\in \mathcal{F}_{n-1} \subset \mathcal{F}_n} \in \mathcal{F}_n$$

QED

Proposition: (X_τ is \mathcal{F}_∞ -measurable & \mathcal{F}_∞ is a σ -field)

If τ is a stopping time w.r.t filtration

$(\mathcal{F}_n)_{n \geq 1}$ and $(X_n)_{n \geq 1}$ is adapted to

$(\mathcal{F}_n)_{n \geq 1}$ then \mathcal{F}_∞ is a sub σ -field of \mathcal{F} .

If, in addition, $\tau < \infty$ then

$X_\tau = X_{\tau(\omega)}(\omega)$ is \mathcal{F}_∞ -measurable.

Proof:

Clearly $\mathcal{F}_\infty \subset \mathcal{F}$ by the previous proposition. To show \mathcal{F}_∞ is a σ -field notice ...

$\bullet \mathcal{N} \in \mathcal{F}_\infty$ since $\mathcal{N} \in \mathcal{F}$ &

$$\mathcal{N} \cap \{\tau = n\} = \{\tau = n\} \in \mathcal{F}_n$$

$\bullet A \in \mathcal{F}_\infty \implies A^c \in \mathcal{F}_\infty$

since $A^c \in \mathcal{F}$ &

$$A^c \cap \{\tau = n\} = \{\tau = n\} - A \cap \{\tau = n\} \in \mathcal{F}_n$$

$\bullet A_1, \dots \in \mathcal{F}_\infty \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_\infty$

since $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ &

$$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap \{\tau = n\} = \bigcup_{k=1}^{\infty} A_k \cap \{\tau = n\} \in \mathcal{F}_n$$

$\therefore \mathcal{F}_\infty$ is a sub σ -field of \mathcal{F} .

Now suppose $\tau(\omega) < \infty$, $\forall \omega \in \Omega$.

Let $B \in \mathcal{B}(\mathbb{R})$ and show $\{X_\tau \in B\} \in \mathcal{F}_\infty$

i.e. show

$$(i) \{X_\tau \in B\} \cap \{\tau = n\} \in \mathcal{F}_n, \forall n \in \mathbb{N} \&$$

$$(ii) \{X_\tau \in B\} \cap \{\tau = \infty\} \in \mathcal{F}$$

Condition (i) clearly holds since the LHS is

$$\underbrace{\{X_n \in B\}}_{\in \mathcal{F}_n} \cap \underbrace{\{\tau = n\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n$$

Condition (ii) holds since $\{\tau = \infty\} = \emptyset$

by assumption.

QED

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Proposition: $(\sigma \leq \tau \Rightarrow \mathcal{F}_\sigma \subset \mathcal{F}_\tau)$

If σ & τ are ST wrt filtration $(\mathcal{F}_n)_{n \geq 1}$

then (i) $\{\sigma \leq \tau\} \in \mathcal{F}_\tau$

(ii) $\sigma(\omega) \leq \tau(\omega) \forall \omega \in \Omega \Rightarrow \mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

Proof:

For (i)

$$\underbrace{\{\sigma \leq \tau\}}_{\in \mathcal{F} \text{ since both are ST}} \cap \{\tau = n\} = \{\sigma \leq n\} \in \mathcal{F}_n$$

For (ii)

$$A \in \mathcal{F}_\sigma \Rightarrow A \in \mathcal{F} \text{ & } A \cap \{\sigma = n\} \in \mathcal{F}_n$$

$$\Rightarrow A \in \mathcal{F} \text{ & }$$

$$\begin{aligned} A \cap \{\tau = n\} &= A \cap \{\tau = n\} \cap \underbrace{\{\sigma \leq \tau\}}_{\substack{\in \mathcal{F}_n \\ \text{the intersection of these } \in \mathcal{F}_n}} \\ &= A \cap \{\tau = n\} \cap \{\sigma \leq n\} \end{aligned}$$

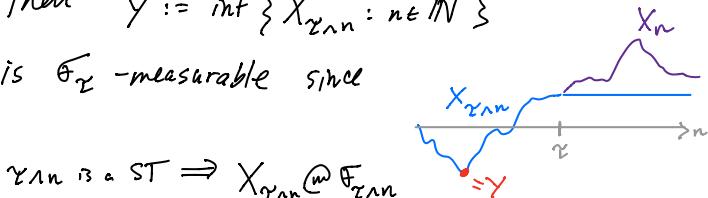
$$\Rightarrow A \in \mathcal{F}_\tau$$

GEP

Example: Let $(X_n)_{n \geq 1}$ be adapted to the filtration $(\mathcal{F}_n)_{n \geq 1}$ & τ be a ST.

Then $Y := \inf \{X_{\tau \wedge n} : n \in \mathbb{N}\}$

is \mathcal{F}_τ -measurable since



$\tau \wedge n$ is a ST $\Rightarrow X_{\tau \wedge n} \in \mathcal{F}_{\tau \wedge n}$

$\Rightarrow X_{\tau \wedge n} \in \mathcal{F}_\tau$ since $\mathcal{F}_{\tau \wedge n} \subset \mathcal{F}_\tau$

$\Rightarrow \inf_n X_{\tau \wedge n} \in \mathcal{F}_\tau$ by closure.

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Example: Strong Markov property of Brownian Motion

Theorem: (Finite optional sampling)

Let (X_1, X_2, \dots, X_n) be a subM w.r.t. filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$ and σ, τ be stopping times s.t. $\sigma \leq \tau \leq n$. Then

(X_σ, X_τ) is a subM wrt $(\mathcal{F}_\sigma, \mathcal{F}_\tau)$

Proof:

Previous propositions establish that $(\mathcal{F}_\sigma, \mathcal{F}_\tau)$ is a filtration and (X_σ, X_τ) is adapted to $(\mathcal{F}_\sigma, \mathcal{F}_\tau)$.

Integrability follows since

$$|X_\sigma|, |X_\tau| \leq |X_1| + \dots + |X_n| \in L_1(\Omega, \mathcal{F}, P).$$

Now

$$E(X_\tau | \mathcal{F}_\sigma) \stackrel{\text{a.e.}}{\geq} X_\sigma.$$

$$\Leftrightarrow \int_A E(X_\tau | \mathcal{F}_\sigma) dP \geq \int_A X_\sigma dP \quad \forall A \in \mathcal{F}_\sigma$$

$$\Leftrightarrow \int_A X_\tau dP \geq \int_A X_\sigma dP, \quad \forall A \in \mathcal{F}_\sigma$$

$$\Leftrightarrow E((X_\tau - X_\sigma) I_A) \geq 0 \quad \forall A \in \mathcal{F}_\sigma$$

Note that

$$(X_\tau - X_\sigma) I_A = \sum_{k=\sigma}^{\tau-1} (X_{k+1} - X_k) I_A$$

$$= \sum_{k=1}^{n-1} (X_{k+1} - X_k) I_{A \cap \mathcal{F}_k}$$

where $A_\sigma = A \cap \{\sigma \leq k \leq \tau-1\}$

$$= \underbrace{A \cap \{\sigma \leq k\}}_{\in \mathcal{F}_\sigma} \cap \underbrace{\{\tau \leq k\}}_{\in \mathcal{F}_\tau}^c \in \mathcal{F}_\tau$$

since $A \in \mathcal{F}_\sigma$

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$$\therefore E((X_{k+1} - X_k) I_{A_\sigma} | \mathcal{F}_\sigma)$$

$$= I_{A_\sigma} E(X_{k+1} - X_k | \mathcal{F}_\sigma) \stackrel{\text{a.e.}}{\geq} 0$$

$\underbrace{\phantom{I_{A_\sigma} E(X_{k+1} - X_k | \mathcal{F}_\sigma)}}$
 ≥ 0
a.e.

$$\therefore E((X_{k+1} - X_k) I_{A_\sigma}) \geq 0$$

$$\therefore E((X_\tau - X_\sigma) I_A) \geq 0 \quad \text{as was to be shown}$$

QED

The Finite optional sampling Thm is a tool for generating some useful inequalities. Here is an example which generalizes Kolmogorov's Maximal inequality.

Theorem: (Kolmogorov's max for subM)

Let (X_1, X_2, \dots, X_n) be a subM w.r.t a filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$.

If $M_n := \max(X_1, X_2, \dots, X_n)$ then

$$P(M_n \geq c) \leq \frac{E(X_n I_{\{M_n \geq c\}})}{c} \leq \frac{E(X_n^+)}{c}$$

for all $c > 0$.

Proof:

Let $\tau := \text{first time } X_k \text{ enters } [c, \infty)$
(or n if it never does)

$$= \begin{cases} \min \{k : X_k \geq c\}, & \text{if } M_n \geq c \\ n & \text{o.w.} \end{cases}$$

A previous example showed ω is a ST.

$\therefore (X_\omega, X_n)$ is a subM wrt $(\mathcal{F}_\omega, \mathcal{F}_n)$

$\therefore E(X_n | \mathcal{F}_\omega) \stackrel{\text{P-a.e.}}{\geq} X_\omega$

$\therefore \int_A X_n dP \geq \int_A X_\omega dP \quad \forall A \in \mathcal{F}_\omega$

Since $\{M_n \geq c\} = \{X_\omega \geq c\} \in \mathcal{F}_\omega$ we have

$$\int_{M_n \geq c} X_n dP \geq \int_{X_\omega \geq c} X_\omega dP \geq c P(\underbrace{X_\omega \geq c}_{= M_n \geq c})$$

$$\therefore P(M_n \geq c) \leq \frac{1}{c} \int_{M_n \geq c} X_n dP \leq \frac{1}{c} \int_{X_\omega \geq c} X_\omega dP$$

QED.

\therefore Kolmogorov's Max-Mg for subM gives

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq c\right) = P\left(\max_{1 \leq k \leq n} S_k^2 \geq c^2\right) \\ \leq \frac{E(S_n^2)^+}{c^2} = \frac{\text{var}(S_n)}{c^2}$$

A similar method generalizes Hoeffding's inequality.

Theorem: (Azuma)

Example: This example shows that Kolmogorov's Max-Mg for subM is a generalization of the earlier version.

Let X_1, X_2, \dots be indep r.v.s s.t.

$$E(X_i) = 0$$

$$\text{var}(X_i) < \infty$$

Set $S_n = X_1 + \dots + X_n$.

Since the X_i 's form a M diff seq w.r.t. the filtration $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$ we have

$$(S_k)_{k=1}^n \text{ is a M w.r.t. } (\mathcal{F}_k)_{k=1}^n$$

$$\therefore (S_k^2)_{k=1}^n \text{ is a subM w.r.t. } (\mathcal{F}_k)_{k=1}^n$$

↖ used $\text{var}(X_i) < \infty$ for integrability condition.