Lecture 6: Independence

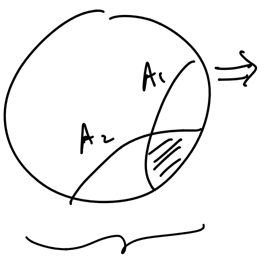
Let $(\Omega, \mathcal{F}, P)$ be a probability space

sample space, σ-field, prob measure

Suppose $A_1, A_2 \in \mathcal{F}$ with $P(A_1) > 0$ & $P(A_2) > 0$.

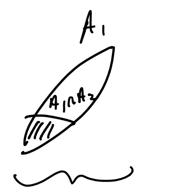
Recall from undergrad probability

$$P(A_2 | A_1) \equiv \text{prob of } A_2 \text{ given } A_1 := \frac{P(A_1 \cap A_2)}{P(A_1)}$$



Sample space Ω

$\Rightarrow$ Sample a random $\omega \in \Omega$ according to P. Your told $\omega \in A_1$ but nothing else $\Rightarrow$



New sample space.

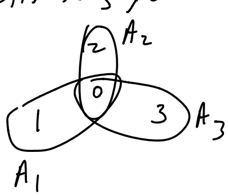
New Prob $\dfrac{P(\cdot \cap A_1)}{P(A_1)}$

A_1 is <u>independent</u> of A_2 if $P(A_2|A_1) = P(A_2)$

i.e. if $P(A_1 \cap A_2) = P(A_1)P(A_2)$.

<u>Question</u>: How to make sense of independence among a collection of events (possibly uncountably many)? Is pairwise independence enough?

e.g. $\Omega = \{0,1,2,3\}$, $\mathcal{F} = 2^\Omega$, $P = $ unif on Ω.



$$i \neq j \Rightarrow P(\underbrace{\underbrace{A_i \cap A_j}_{=\{0\}}}_{=1/4}) = \underbrace{P(A_i)}_{1/2} \underbrace{P(A_j)}_{1/2}$$

so A_1, A_2, A_3 are pairwise independent.

But A_1, A_2, A_3 are not jointly indep:

$$P(A_1 \cap A_2 \cap A_3) = 1/4$$

$$P(A_1)P(A_2)P(A_3) = \frac{1}{8}$$

(Note: $P(A_1 | A_2 \cap A_3) = 1 \neq P(A_1)$)

e.g. Let $A_1, \dots, A_n$ be events (i.e. $A_i \in \mathcal{F}$) s.t. $A_1 = \emptyset$. Then

$$P(A_1 \cap \cdots \cap A_n) = 0 = P(A_1) \cdots P(A_n)$$

So the full factorization criterian will not work as a def of independence either

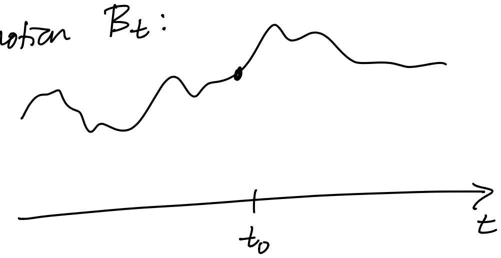
Here is the "right" def of indep for a collection of events.

<u>Def</u>: A collection of events $\{A_k\}_{k \in \mathcal{K}}$ are <u>independent events</u> iff $\forall$ finite $\mathcal{H} \subset \mathcal{K}$

$$P\Big(\bigcap_{h \in \mathcal{H}} A_h\Big) = \prod_{h \in \mathcal{H}} P(A_h).$$

<u>Note</u>: $\mathcal{K}$ is allowed to be <u>any</u> index set.

we will also need the notion of independent σ-fields to make sense of things like the strong markov property of Brownian motion B_t:



$\sigma\langle B_t : t < t_0 \rangle$ is indep of $\sigma\langle B_t : t > t_0 \rangle$ given $\sigma\langle B_{t_0} \rangle$.

<u>Def</u>: Let $\mathcal{K}$ be an arbitrary index set. $\forall k \in \mathcal{K}$, let $\mathcal{A}_k$ be a collection of events.

The $\mathcal{A}_k$'s are <u>independent collections</u> if $\{A_k\}_{k \in \mathcal{K}}$ are indedendent events for each choice $A_k \in \mathcal{A}_k$.

Thm: Let $\mathcal{A}_k, \mathcal{B}_k$ be collections of events for each $k \in \mathcal{K}$ (arb index set). Then ③

(i) (subclasses):

If $\mathcal{A}_k \subset \mathcal{B}_k$ $\forall k \in \mathcal{K}$ & the $\mathcal{B}_k$'s are indep then the $\mathcal{A}_k$'s are indep.

(ii) (augmentation):

$\mathcal{A}_k$'s are indep iff $\mathcal{A}_k \cup \{\Omega\}$'s are indep.

(iii) (simplified product):

If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ all contain Ω then the $\mathcal{A}_k$'s are indep iff

$$P\Big(\bigcap_{k=1}^{n} A_k\Big) = \prod_{k=1}^{n} P(A_k)$$

$\forall A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$.

Proof:

(i): trivial

(ii): $\Leftarrow$ follows by "subclasses".

For $\Rightarrow$ choose $A_k \in \mathcal{A}_k \cup \{\Omega\}$ & finite $\mathcal{H} \subset \mathcal{K}$. Let $\mathcal{K}_0 = \{k : A_k \in \mathcal{A}_k\}$.

$$\therefore\ P\Big(\bigcap_{h \in \mathcal{H}} A_h\Big)$$

$$= P\Big(\bigcap_{h \in \mathcal{H} \cap \mathcal{K}_0} A_h\Big), \quad A_k = \Omega \text{ when } k \in \mathcal{K} - \mathcal{K}_0$$

$$= \prod_{h \in \mathcal{H} \cap \mathcal{K}_0} P(A_h), \quad \mathcal{A}_k\text{'s indep}$$

$$= \prod_{h \in \mathcal{H}} P(A_h), \quad P(A_k) = P(\Omega) = 1 \text{ when } k \in \mathcal{K} - \mathcal{K}_0$$

$\therefore$ $\mathcal{A}_k \cup \{\Omega\}$'s are indep.

④

(iii) $P\Big(\bigcap_{k=1}^{n} A_k\Big) = \prod_{k=1}^{n} P(A_k)$

$$\Rightarrow P\Big(\bigcap_{h \in \mathcal{H}} A_h\Big) = \prod_{h \in \mathcal{H}} P(A_h)$$

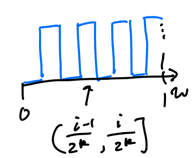
for $\mathcal{H} \subset \{1, 2, \dots, n\}$ by replacing A_k with $\Omega \in \mathcal{A}_k$, $k \notin \mathcal{H}$.

QED

e.g. Coin flip model from lecture 1:

$\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}((0,1])$, P = uniform measure.

$X_k(\omega) :=$ k^{th} binary digit of ω =

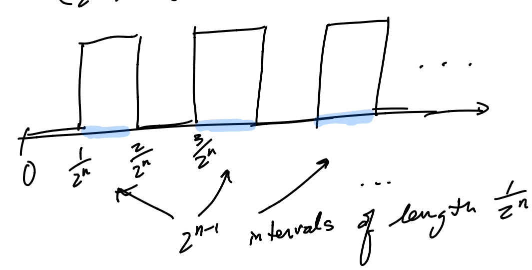


$H_k := \{\omega \in \Omega : X_k(\omega) = 1\}$

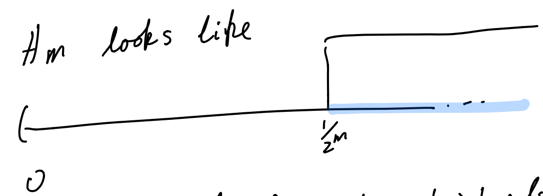
↖ event of flipping a heads on the k^{th} toss if we want X_k to model fair coin flips.

Claim: $H_1, H_2, H_3, \dots$ are indep events.

Proof: $H_n = (\frac{1}{2^n}, \frac{2}{2^n}] \cup (\frac{3}{2^n}, \frac{4}{2^n}] \cup (\frac{5}{2^n}, \frac{6}{2^n}] \cdots$



If $m < n$ then H_m looks like



$\therefore$ $H_n \cap H_m$ = union of half of the disjoint intervals that make up H_n

Let $1 \le i_1 < i_2 < \dots < i_n$ & show

$$P(H_{i_1} \cap \dots \cap H_{i_n}) = \underbrace{P(H_{i_1}) \cdots P(H_{i_n})}_{= \frac{1}{2^n}}$$

Now $H_{i_n} \cap H_{i_{n-1}} \cap \cdots \cap H_{i_1} = \frac{2^{i_n -1}}{2^{n-1}}$ disjoint intervals of length $\frac{1}{2^{i_n}}$

$2^{i_n -1}$ intervals of length $\frac{1}{2^{i_n}}$

reduce the # of intervals by $\frac{1}{2}$ for each further intersection

$\therefore\ P(H_{i_n} \cap H_{i_{n-1}} \cap \cdots \cap H_{i_1}) = \frac{2^{i_n-1}}{2^{n-1}} \cdot \frac{1}{2^{i_n}} = \frac{1}{2^n}$

as was to be shown QED.

π-generators are enough & ANOVA

At this point checking two σ-fields are indep would be a daunting task since we have no representation for general events in a σ-field.

The following thm helps this.

Thm (π-generators are enough):

Let $\mathcal{A}_k \subset \mathcal{F}$, $\forall k \in \mathcal{K}$. Then

$\mathcal{A}_k$'s are indep π-systems $\Longrightarrow \sigma\langle \mathcal{A}_k \rangle$'s are independent.

Proof: Let $\mathcal{B}_k := \mathcal{A}_k \cup \{\Omega\}$

Suppose the $\mathcal{A}_k$'s are indep π-sys

$\therefore$ the $\mathcal{B}_k$'s are indep π-sys, by augmentation

$\therefore\ \forall$ distinct $k_1, k_2, \dots, k_n \in \mathcal{K}$

the $\mathcal{B}_{k_1}, \mathcal{B}_{k_2}, \dots, \mathcal{B}_{k_n}$ are indep π-sys

show $\sigma\langle \mathcal{B}_{k_1} \rangle, \mathcal{B}_{k_2}, \dots, \mathcal{B}_{k_n}$ are indep π-sys and we will be done (by induction)

By the simplified product criterion this is equivalent to showing

$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n) \quad (*)$$

$\forall B_1 \in \sigma\langle \mathcal{B}_{k_1} \rangle, B_2 \in \mathcal{B}_{k_2}, \dots, B_n \in \mathcal{B}_{k_n}$

Fixing $B_2, \dots, B_n$ let

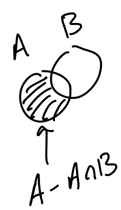
$$\mathcal{G} := \{B_1 \in \mathcal{F} : (*) \text{ holds}\}$$

& show $\sigma\langle \mathcal{B}_{k_1} \rangle \subset \mathcal{G}$.

- $\mathcal{B}_{k_1} \subset \mathcal{G}$: yes, since $\mathcal{B}_k$'s are indep.
- $\Omega \in \mathcal{G}$: yes, since $\Omega \in \mathcal{B}_k\ \forall k \in \mathcal{K}$.
- $B \in \mathcal{G} \Rightarrow$

$$P(B^c \cap \underbrace{B_2 \cap \cdots}_{A}) = P(B_2 \cap \cdots) - P(B \cap B_2 \cap \cdots)$$

since $P(B^c \cap A) = P(A - B \cap A)$



$$= P(\Omega \cap B_2 \cap \cdots) - P(B \cap B_2 \cap \cdots)$$

$$= P(\Omega) \cdot P(B_2) \cdots - P(B)P(B_2) \cdots$$

since $\Omega, B \in \mathcal{G}$

$$= \underbrace{[P(\Omega) - P(B)]}_{P(B^c)} P(B_2) \cdots P(B_n)$$

$\Longrightarrow B^c \in \mathcal{G}$

- $\underbrace{A_1, A_2, \dots}_{\text{disjoint}} \in \mathcal{G}$

$$\Longrightarrow P\Big(\big(\bigcup_k A_k\big) \cap B_2 \cap \cdots \cap B_n\Big)$$

$$= \sum_k P(A_k \cap B_2 \cap \cdots \cap B_n)$$

$$= \sum_k P(A_k) P(B_2) \cdots P(B_n)$$

$$= P(B_2) \cdots P(B_n) \underbrace{\Big[\sum_k P(A_k)\Big]}_{P(\bigcup_k A_k)}$$

$\Longrightarrow \bigcup_k A_k \in \mathcal{G}$

$\therefore \mathcal{L}$ is a λ-sys & B_{k_1} is a π-sys. 7

$\therefore \sigma\langle B_{k_1}\rangle \subset \mathcal{L}$.

QED.

e.g. coin flip example showed $H_1, H_2, \ldots$ are indep

since $\{H_k\}$ is a π-sys for each k, $\sigma\langle H_1\rangle, \sigma\langle H_2\rangle, \ldots$ are indep σ-fields (where $\sigma\langle H_k\rangle = \{\emptyset, \Omega, H_k, H_k^c\}$)

$\therefore$ Any sequence $H_1, H_2^c, H_3, H_4^c, H_5^c, \ldots$ are indep.

tails on the 5^{th} toss

To modivate the next thm let

A = the event $\sum_{k=1}^{n}(1-2X_{2k})=0$ for infinitely many n

B = the event $\sum_{k=1}^{n}(1-2X_{2k+1})=0$ for infinitely many n

is A indep of B?

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Thm (ANOVA): Matrix of π-systems

$$\begin{matrix} \mathcal{O}_{11} & \mathcal{O}_{12} & \mathcal{O}_{13} & \cdots \\ \mathcal{O}_{21} & \mathcal{O}_{22} & \mathcal{O}_{23} & \cdots \\ \mathcal{O}_{31} & \mathcal{O}_{32} & \mathcal{O}_{33} & \cdots \\ \vdots & & & \ddots \end{matrix}$$

Let $R_i = \sigma\langle \underbrace{\mathcal{O}_{i1}, \mathcal{O}_{i2}, \ldots}_{i^{th}\text{ Row}}\rangle$

Then

all the $\mathcal{O}_{ik}$'s are indep $\iff$ (i) R_k's are indep (ii) the $\mathcal{O}_{ik}$'s within each Row are independent

Proof:

($\Longrightarrow$) Suppose all the $\mathcal{O}_{ik}$'s are indep.

$\therefore$ (ii) clearly holds

To show (i) Note

$R_k = \sigma\langle \underbrace{\mathcal{O}_{k1} \cup \mathcal{O}_{k2} \cup \cdots}_{\substack{\text{would like to}\\ \text{use } \pi\text{-generators}\\ \text{but this isn't a}\\ \pi\text{-sys}}}\rangle = \sigma\langle P_k\rangle$

where P_k = the closure of $\mathcal{O}_{k_1} \cup \mathcal{O}_{k_2} \cup \cdots$ under finite intersection

Clearly P_k's are π-systems.

Lets show the P_k's are indep.

Select one P_k from $\mathcal{P}_k$ and note:

$$\underbrace{P_{i_1} \cap \cdots \cap P_{i_n}}$$

write this as $\underbrace{(A_1 \cap \cdots)}_{P_{i_1}}^{\text{Row } i_1} \cap \overbrace{(B_1 \cap \cdots)}^{\text{Row } i_2} \cap \overbrace{(C_1 \cap \cdots)}^{\text{Row } i_3} \cap \cdots$

each event in here is from a unique $\mathcal{A}_{i_2,j}$.

merging (via "$\cap$") multiple sets from the same $\mathcal{A}_{k_j}$ if necessary ... still a $\mathcal{A}_{ki}$ set by π-sys assumption

Now,

$$P(P_{i_1} \cap \cdots \cap P_{i_n}) = P(A_1) \cdots P(B_1) \cdots P(C_1) \cdots$$
$$= P(P_{i_1}) \cdots P(P_{i_n})$$

e.g.

$$P(\overbrace{\underbrace{A_1 \cap A_2}_{\text{both in } \mathcal{A}_{i_1,1}}}^{P_{i_1}} \cap \overbrace{\underbrace{B_1}_{\in \mathcal{A}_{i_2,1}} \cap \underbrace{B_2}_{\in \mathcal{A}_{i_2,2}}}^{P_{i_2}})$$

$= P(\underbrace{A_1 \cap A_2}_{\in \mathcal{A}_{i_1,1} \text{ by } \pi\text{-sys}}) P(B_1) P(B_2)$ since $\mathcal{A}_{i_1,1}, \mathcal{A}_{i_2,1}$ & $\mathcal{A}_{i_2,2}$ are indep

$= P(A_1 \cap A_2) P(B_1 \cap B_2)$ since $\mathcal{A}_{i_2,1}$ & $\mathcal{A}_{i_2,2}$ are indep

$= P(P_{i_1}) P(P_{i_2})$

$\therefore$ $\mathcal{P}_k$'s are indep π-sys.

$\therefore$ The σ-fields $R_k := \sigma\langle \mathcal{P}_k \rangle$ are independent by π-generators.

($\Leftarrow$)

Suppose the row σ-fields R_k are indep & the $\mathcal{A}_{k_i}$'s within each Row are indep.

Let $\mathcal{H}$ be a finite set of (Row, col) index tuples

For each $(i,k) \in \mathcal{H}$ choose one $A_{ik} \in \mathcal{A}_{ik}$.

$$\therefore P\left(\bigcap_{(i,k) \in \mathcal{H}} A_{ik}\right) = P\left(\bigcap_{\text{Rows } i \text{ in } \mathcal{H}} \overbrace{\bigcap_{\text{Cols } k \text{ s.t. } (i,k) \in \mathcal{H}} A_{ik}}^{\in R_i}\right)$$

$$\overset{R_i\text{'s indep}}{=} \prod_{\text{Rows } i \text{ in } \mathcal{H}} P\left(\bigcap_{\text{Cols } k \text{ s.t. } (i,k) \in \mathcal{H}} A_{ik}\right)$$

$$\overset{\text{within Row indep}}{=} \prod_{\text{Rows } i \text{ in } \mathcal{H}} \prod_{\text{Cols } k \text{ s.t. } (i,k) \in \mathcal{H}} P(A_{ik})$$

$$= \prod_{(i,k) \in \mathcal{H}} P(A_{ik})$$

QED

Kolmogorov's 0-1 law

Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a sequence of collections of $\mathcal{F}$-sets (i.e. $\mathcal{A}_k \subset \mathcal{F}$)

Def: The tail σ-field of the $\mathcal{A}_k$'s is defined as

$$\mathcal{T} := \bigcap_{m=1}^{\infty} \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \ldots \rangle$$
$$= \{ A \in \mathcal{F} : A \in \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \ldots \rangle \ \forall m \}$$

($\mathcal{T}$ is a σ-field for the same reason $\sigma\langle \mathcal{E} \rangle$ is)

e.g. Let A = the abnormal numbers in $(0,1]$ from Lecture 1.

Let $\varepsilon_n \searrow 0$ as $n \to \infty$

$$\therefore A = \{ w : \tfrac{S_n(w)}{n} \not\to 0 \text{ as } n \to \infty \}$$
$$\subset \{ w : \left|\tfrac{S_n(w)}{n}\right| > \varepsilon_n \text{ for infinitely many } n \}$$
$$= \bigcap_k \bigcup_{n \geq k} \underbrace{\{ w : \left|\tfrac{S_n(w)}{n}\right| > \varepsilon_n \}}_{A_n}$$
$$= \Big(\bigcup_{n\geq 1} A_n\Big) \cap \Big(\bigcup_{n\geq 2} A_n\Big) \cap \Big(\bigcup_{n\geq 3} A_n\Big) \cap \cdots$$

These get smaller

Since any restriction provided by the first $k-1$ terms is already found in the k^{th} term

$$= \bigcap_{k=m}^{\infty} \bigcup_{n \geq k} A_n \quad \text{for any } m$$
$$\in \sigma\langle \{A_m\}, \{A_{m+1}\}, \ldots \rangle$$

for any m

$\therefore A \in \mathcal{T}$ The tail σ-field generated by $\{A_1\}, \{A_2\}, \ldots$

Thm (Kolmogorov's 0-1 law)

If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are indep π-systems then $\forall A \in \mathcal{T}$, $P(A) = 0$ or $P(A) = 1$.

Proof:

$\mathcal{A}_1, \ldots, \mathcal{A}_{m-1}, \mathcal{A}_m, \mathcal{A}_{m+1} \ldots$ are indep π-sys.

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \ldots \sigma\langle \mathcal{A}_{m-1} \rangle, \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \ldots \rangle$ are indep π-sys by anova.

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \ldots, \sigma\langle \mathcal{A}_{m-1} \rangle, \mathcal{T}$ are indep π-sys by subclasses.

holds for all m

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \sigma\langle \mathcal{A}_2 \rangle, \ldots, \mathcal{T}$ are indep π-sys by the finite selection requirement of the def of indep.

$\therefore \sigma\langle \mathcal{A}_1, \mathcal{A}_2, \ldots \rangle, \mathcal{T}$ are indep π-sys by Anova.

$\therefore \mathcal{T}, \mathcal{T}$ are indep π-sys by subclasses

$\therefore \forall A \in \mathcal{T}$, $P(A \cap A) = P(A)P(A)$

$\therefore P(A) = 0$ or 1.

QED

Borel-Cantelli and Fatou

Let $A_1, A_2, \ldots \in \mathcal{F}$.

Def:

$$\{A_n \text{ i.o.}_n\} := \{\omega \in \Omega : \omega \in A_n \text{ infinitely often in } n\}$$

$$:= \underbrace{\bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n}_{\forall m \, \exists n \geq m \text{ s.t. } \omega \in A_n .}$$

$$\{A_n \text{ a.a.}_n\} := \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many } n\}$$

$$:= \underbrace{\bigcup_{m=1}^{\infty} \bigcap_{n \geq m} A_n}_{\exists m \text{ s.t. } \forall n \geq m ,\ \omega \in A_n}$$

Note: $\{A_n \text{ i.o.}_n\} \in \mathcal{F}$ & $\{A_n \text{ a.a.}_n\} \in \mathcal{F}$

Sometimes people write

$\limsup_{n \to \infty} A_n$ for $\{A_n \text{ i.o.}_n\}$

$\liminf_{n \to \infty} A_n$ for $\{A_n \text{ a.a.}_n\}$

since

indicator of A_n

$$\limsup_n I_{A_n}(\omega) = I_{\{A_n \text{ i.o.}_n\}}(\omega)$$

$$\liminf_n I_{A_n}(\omega) = I_{\{A_n \text{ a.a.}_n\}}(\omega)$$

Some Facts:

$\{A_n \text{ a.a.}_n\} \subset \{A_n \text{ i.o.}_n\}$

$\{A_n \text{ a.a.}_n\}^c = \{A_n^c \text{ i.o.}_n\}$ & vice-versa

$$\{A_n \text{ a.a.}_n\} = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} A_n$$

$$= \left(\bigcap_{n \geq 1} A_n\right) \cup \left(\bigcap_{n \geq 2} A_n\right) \cup \cdots$$

these grow since your removing restrictions

$$= \bigcup_{m=k}^{\infty} \bigcap_{n \geq m} A_n , \text{ for any } k$$

since anything in the first $k-1$ terms are included in the latter.

$\in$ tail σ-field generated by $\{A_1\}, \{A_2\}, \ldots$

$$A_n \uparrow A \Rightarrow A = \bigcup_{m=1}^{\infty} A_m \ \& \ A_1 \subset A_2 \subset \cdots$$

$$\Rightarrow A = \bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} \underbrace{\bigcap_{n \geq m} A_n}_{= A_m} = \{A_n \text{ a.a.}_n\}$$

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$$\{A_n \text{ i.o.}_n\} = \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n$$

$$= \Big(\bigcup_{n\geq 1} A_n\Big) \cap \Big(\bigcup_{n \geq 2} A_n\Big) \cap \cdots$$

these decrease as sets

$$= \bigcap_{m=k}^{\infty} \bigcup_{n \geq m} A_n, \quad \forall k$$

since the restrictions found in the first $k-1$ terms is already in the k^{th} term.

$\in$ tail σ-field generated by $\{A_1\}, \{A_2\}, \ldots$

$$A_n \downarrow A \Leftrightarrow A_n^c \uparrow A^c$$

$$\Rightarrow A^c = \{A_n^c \text{ a.a.}_n\}$$

$$\Rightarrow A = \{A_n \text{ i.o.}_n\}$$

Note: The 0-1 law already implies

$A_1, A_2 .. \in \mathcal{F}$ are indep

$$\Rightarrow P(A_n \text{ i.o.}_n) = 0 \text{ or } 1$$

$$P(A_n \text{ a.a.}_n) = 0 \text{ or } 1.$$

Thm (First Borel Cantelli lemma)

$$\underbrace{\sum_{n=1}^{\infty} P(A_n) < \infty}_{\text{if the } A_n\text{'s become sufficiently rare}} \Rightarrow \underbrace{P(A_n \text{ i.o.}_n) = 0}_{\text{it is impossible for } A_n\text{'s to happen i.o.}}$$

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Proof:

$$P(A_n \text{ i.o.}_n) = P\Big(\bigcap_m \bigcup_{n\geq m} A_n\Big)$$

$$\leq P\Big(\bigcup_{n \geq m} A_n\Big), \quad \forall m$$

$$\leq \sum_{n=m}^{\infty} P(A_n), \quad \forall m$$

$$\longrightarrow 0 \quad \text{as } m \to \infty \text{ if } \sum_{n=1}^{\infty} P(A_n) < \infty$$

QED.

Warning: $P(A_n \text{ i.o.}_n) = 0 \not\Rightarrow \sum_{n=1}^{\infty} P(A_n) < \infty$

e.g. $\Omega = (0,1]$

$A_n = (0, 1/n]$

P = uniform measure

$P(A_n \text{ i.o.}_n) = 0$ but

$\sum P(A_n) = \infty$

If however the A_n's are independent then $P(A_n \text{ i.o.}) = 1$ or 0.

The contrapositive of the first Borel-Cantelli says

$$P(A_n \text{ i.o.}_n) \neq 0 \Rightarrow \sum P(A_n) = \infty$$

$$\Updownarrow \text{ indep}$$

$$P(A_n \text{ i.o.}_n) = 1$$

The reverse implication is given by the next result.

Thm (Second Borel Cantelli lemma) 17

If $A_1, A_2, \dots$ are independent then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \iff P(A_n \text{ i.o.n}) = 1$$

Proof: We just need to show $\Rightarrow$ by previous comments.

Suppose $\sum P(A_n) = \infty$.

Show $P(A_n^c \text{ a.a.}) = 0$.

$$P(A_n^c \text{ a.a.n}) = P\Big(\bigcup_m \bigcap_{n\geq m} A_n^c\Big)$$

$$= P\Big(\Big(\bigcap_{n\geq 1} A_n^c\Big) \cup \Big(\bigcap_{n\geq 2} A_n^c\Big) \cup \cdots\Big)$$

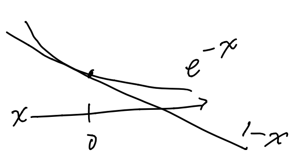
these grow

$$= P\Big(\lim_m{\uparrow} \bigcap_{n\geq m} A_n^c\Big)$$

$$= \lim_m P\Big(\bigcap_{n\geq m} A_n^c\Big)$$

$$= \lim_m \lim_p P\Big(\bigcap_{n\geq m}^{p} A_n^c\Big)$$

$$= \lim_m \lim_p \prod_{n\geq m}^{p} \underbrace{P(A_n^c)}_{=1-P(A_n)\,\leq\, e^{-P(A_n)}}$$



$$\leq \lim_m \lim_p \exp\Big(-\sum_{n\geq m}^{p} P(A_n)\Big)$$

$$= \lim_m \exp\Big(\underbrace{-\sum_{n\geq m}^{\infty} P(A_n)}_{-\infty}\Big)$$

$$= 0$$

QED

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Restatement:

$$\sum P(A_n) < \infty \overset{FBCL}{\Longrightarrow} P(A_n \text{ i.o.n}) = 0$$

If A_n's are indep then

$$\sum P(A_n) < \infty \overset{SBCL}{\iff} P(A_n \text{ i.o.n}) = 0$$

Thm Fatou's lemma

$$P(A_n \text{ a.a.n}) \leq \liminf_n P(A_n)$$

$$\leq \limsup_n P(A_n)$$

$$\leq P(A_n \text{ i.o.n}).$$

Note: for measures you don't have this inequality always

Proof:

$$P(A_n \text{ a.a.n}) = P\Big(\lim_m{\uparrow} \bigcap_{n\geq m} A_n\Big)$$

$$= \lim_m \underbrace{P\Big(\bigcap_{n\geq m} A_n\Big)}_{\leq P(A_n),\ \forall n\geq m}$$

$$\leq \lim_m \inf_{n\geq m} P(A_n)$$

$$= \liminf_n P(A_n)$$

Now take complements for the other inequality.

$$\limsup_n P(A_n) \leq P(A_n \text{ i.o.n})$$

$$\Updownarrow$$

$$\limsup_n \big(1 - P(A_n^c)\big) \leq 1 - P(A_n^c \text{ a.a.n})$$

$$\Updownarrow$$

$$1 - \liminf_n P(A_n^c) \leq 1 - P(A_n^c \text{ a.a.n})$$

↑ holds by first inequality

QED

Using the first Borel-Cantelli lemma for showing strong laws

The FBCL (first borell cantelli lemma) is useful for showing things like

$$P\left(\lim_n X_n = c\right) = 1$$

when you have bounds of the form

$$P\left(|X_n - c| \geq \varepsilon\right) \leq b(\varepsilon, n)$$

where $b(\varepsilon, n)$ has fast decay in n.

e.g. suppose $\exists\, \varepsilon_n \downarrow 0$ s.t. $\sum_{n=1}^{\infty} b(\varepsilon_n, n) < \infty$

$$\therefore \sum_{n=1}^{\infty} P(|X_n - c| \geq \varepsilon_n) < \infty$$

$$\therefore P\left(|X_n - c| \geq \varepsilon_n \text{ i.o.}_n\right) = 0 \quad \text{by FBCL}$$

$$\therefore P\left(\underbrace{|X_n - c| < \varepsilon_n \text{ a.a.}_n}_{\substack{\text{imply that eventually} \\ |X_n - c| \to 0 \text{ at} \\ \text{rate } \leq \varepsilon_n}}\right) = 1$$

$$\therefore 1 = P\left(|X_n - c| < \varepsilon_n \text{ a.a.}_n\right)$$

$$\leq \underbrace{P\left(\lim_n X_n = c\right)}_{\text{so this is } 1.} \leq 1$$

Here is another way...

Suppose $\sum_{n=1}^{\infty} b(\varepsilon, n) < \infty \quad \forall \varepsilon > 0$

$$\therefore \sum_{k=1}^{\infty} P(|X_n - c| \geq \varepsilon) < \infty$$

$$\therefore P\left(|X_n - c| \geq \varepsilon \text{ i.o.}_n\right) = 0 \quad \forall \varepsilon$$

$$\therefore P\left(\bigcup_{\varepsilon \in \mathbb{Q}^+} \{|X_n - c| \geq \varepsilon \text{ i.o.}_n\}\right) = 0$$

by sub additivity

$$\therefore P\left(\underbrace{\bigcap_{\varepsilon \in \mathbb{Q}^+} \{|X_n - c| < \varepsilon \text{ a.a.}_n\}}_{\text{equals the event } \{X_n \to c\}}\right) = 1$$

Note: the above two arguments do not require independence of the X_n's.

SLLN $\Rightarrow$ WLLN via Fatou

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Hewitt-Savage 0-1 law for coin flips

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Random Coin flip walks

Series with random signs

$\sum_n \frac{1}{n} = \infty$ } No cancelation at all

$\sum_n R_n \frac{1}{n} = ?$

$\sum_n R_n \frac{1}{\sqrt{n}} = ?$

} cancelation is given by random coin flips

$\sum_n (-1)^n \frac{1}{\sqrt{n}} < \infty$ } cancelation is finely tuned using ANOVA for generating an infinite sequence of uniform random variables

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Porous medium

Maximal inequality for S_n

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$$P\left(\frac{S_n}{\ell_n} > 1-\varepsilon \ \ i.o.\, n\right)$$

$$P\left(\frac{S_n}{\ell_n} < 1+\varepsilon \ \ a.a.\, n\right)$$