Lecture 9: σ-fields generated by functions. The structure thm. Applications to R.V.s

①

σ-fields generated by functions or r.v.s are extremely useful for cleaning up & generalizing some of the stuff we did for the coin flip model & also allow us to define conditional expected value etc.

e.g. in previous lectures we said things like $\underbrace{\{S_n - S_k > c\}}_{\in\sigma\langle H_{k+1},\dots,H_n\rangle}$ is indep of $\underbrace{\{S_k > c\}}_{\in\sigma\langle H_1,\dots,H_k\rangle}$

... while true it is a bit annoying & implicitly due to facts like:

$$\{S_k > c\} = \bigcup_{\substack{r_1,\dots,r_k\in\{-1,1\}\\ \text{s.t. } r_1+\dots+r_k > c}} \underbrace{\{R_1 = r_1\}\cap\dots\cap\{R_k = r_k\}}_{\in\sigma\langle H_1,\dots,H_k\rangle}$$

↑ countable

which are not very generalizable.

e.g. Recall "just check the coords": $\vec{f} = (f_1,\dots,f_d)$

$$(\Omega,\mathcal{F}) \xrightarrow{\vec{f}\ \text{(m)}} (\overline{\mathbb{R}}^d, \mathcal{B}(\overline{\mathbb{R}}^d)) \quad \text{iff} \quad (\Omega,\mathcal{F}) \xrightarrow{f_i\ \text{(m)}} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))\ \ \forall i$$

appears to use $\mathcal{B}(\overline{\mathbb{R}}^d)$ as the natural σ-field on $\overline{\mathbb{R}}^d = \overline{\mathbb{R}}\times\dots\times\overline{\mathbb{R}}$. What about when f_i maps into $(\Omega_i, \mathcal{F}_i)$ ... what is the σ-field on $\Omega_1\times\dots\times\Omega_n$?

②

e.g. it would be nice if a r.v. Y

$$([0,1], \mathcal{B}([0,1])) \xrightarrow{Y} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

which satisfied $\{Y \le c\} \in \sigma\langle H_1,\dots,H_k\rangle$ $\forall c$ could be shown to be a (m) function of $R_1,\dots,R_k$ i.e. $\exists g$ (m) $\mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$ s.t. $Y = g(R_1,\dots,R_k)$

e.g. we want to extend the notion of independence to non-discrete R.V.s, i.e. conclude that if B_t is a Brownian motion $B_t,\ t<t_0$ is indep of $B_{t>t_0}$ given B_{t_0}.

Basic definition: $\sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$

Let $\mathcal{I}$ be a general index set (any cardinality allowed).

Let $(\Omega_i, \mathcal{F}_i)$ be a measurable space, $\forall i\in\mathcal{I}$.

Let $f_i : \Omega \to \Omega_i$, $\forall i\in\mathcal{I}$

$$\begin{array}{ccc} & \xrightarrow{f_i} & (\Omega_i, \mathcal{F}_i) \\ (\Omega, ?) & \vdots & \\ & \xrightarrow{f_k} & (\Omega_k, \mathcal{F}_k) \end{array}$$

Def: $\sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$

$\equiv \sigma\langle f_i : i\in\mathcal{I}\rangle$ when $\mathcal{F}_i$ is implicit

$$:= \bigcap_{\substack{\sigma\text{-fields } \mathcal{G} \\ \text{on } \Omega \text{ s.t.} \\ f_i\ \text{(m)}\ \mathcal{G}/\mathcal{F}_i,\ \forall i\in\mathcal{I}}} \mathcal{G}$$

= smallest σ-field on Ω making all the f_i's measurable.

Thm: $\sigma\langle f_1, \mathcal{F}_1\rangle = \underbrace{f_1^{-1}(\mathcal{F}_1)}_{\text{the pull backs of each } F\in\mathcal{F}_1}$ (3)

Warning: This only works for the σ-field generated by a single function.

Proof:
This follows easily by "good sets" & the fact that $f_1^{-1}(\mathcal{F}_1)$ is a σ-field.

QED.

Thm (Generators are enough).

If

$$\begin{array}{ccc} & \xrightarrow{f_i} & \Omega_i \quad \sigma\langle \mathcal{Q}_i\rangle \\ \Omega & \vdots & \\ & \xrightarrow{f_k} & \Omega_k \quad \sigma\langle \mathcal{Q}_k\rangle \end{array}$$

then

$$\sigma\langle f_i, \sigma\langle \mathcal{Q}_i\rangle : i\in\mathcal{I}\rangle = \sigma\langle f_i^{-1}(\mathcal{Q}_i) : i\in\mathcal{I}\rangle$$

Proof:
To show $\subset$ Notice that clearly

$$f_k \ (m)\ \sigma\langle f_i^{-1}(\mathcal{Q}_i): i\in\mathcal{I}\rangle / \mathcal{Q}_k, \ \forall k\in\mathcal{I}$$

$\therefore$ "check (m) on generators" implies

$$f_k \ (m)\ \sigma\langle f_i^{-1}(\mathcal{Q}_i): i\in\mathcal{I}\rangle / \sigma\langle \mathcal{Q}_k\rangle, \ \forall k\in\mathcal{I}$$

$\therefore$ $\sigma\langle f_i^{-1}(\mathcal{Q}_i): i\in\mathcal{I}\rangle$ is a $\mathcal{Y}$ in the def of $\sigma\langle f_i, \sigma\langle\mathcal{Q}_i\rangle : i\in\mathcal{I}\rangle$.

To show $\supset$ Notice that clearly (4)

$$f_k^{-1}(\mathcal{Q}_k) \subset \sigma\langle f_i, \sigma\langle\mathcal{Q}_i\rangle : i\in\mathcal{I}\rangle, \ \forall k\in\mathcal{I}$$

$$\therefore \underbrace{\sigma\langle f_k^{-1}(\mathcal{Q}_k) : k\in\mathcal{I}\rangle}_{\substack{\text{since this is the} \\ \text{"smallest" } \sigma\text{-field} \\ \text{containing } f_k^{-1}(\mathcal{Q}_k),\ \forall k\in\mathcal{I}}} \subset \sigma\langle f_i, \sigma\langle\mathcal{Q}_i\rangle : i\in\mathcal{I}\rangle$$

QED.

Note: This trivially implies

$$\sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle = \sigma\langle f_i^{-1}(\mathcal{F}_i) : i\in\mathcal{I}\rangle$$
$$= \sigma\langle \sigma\langle f_i\rangle : i\in\mathcal{I}\rangle$$

since $\mathcal{F}_i$ generates itself & $f_i^{-1}(\mathcal{F}_i) = \sigma\langle f_i\rangle$

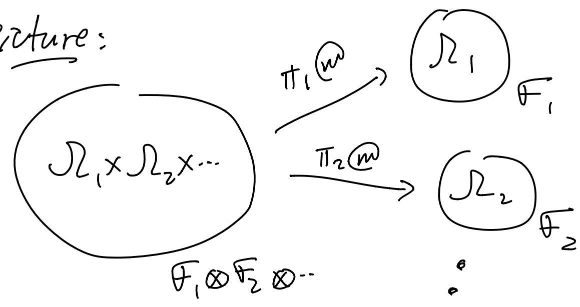
Product σ-field

Now we can define the natural "Product σ-field" on $\Omega_1\times\cdots\times\Omega_n\times\cdots$ using the "coordinate projections" $\pi_k(\omega) = \omega_k$

Def: Let $(\Omega_i, \mathcal{F}_i)$ be a measurable space $\forall i\in\mathcal{I}$. Define

$$\underbrace{\bigotimes_{i\in\mathcal{I}} \mathcal{F}_i}_{\text{a } \sigma\text{-field on } \Omega = \prod_{i\in\mathcal{I}}\Omega_i} := \sigma\langle \pi_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$$

Picture:



Thm (just check the coordinates)

Suppose $f_i : \Omega \to \Omega_i$ where $(\Omega, \mathcal{F})$ and $(\Omega_i, \mathcal{F}_i)$ are measurable spaces $\forall i \in \mathcal{I}$.

Define the vector map $\vec{f}(\omega) = (f_i(\omega))_{i\in\mathcal{I}}$

$$\begin{tikzcd} \underset{\mathcal{F}}{\Omega} \arrow[r, "\vec{f}"] & \underset{\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i}{\prod_{i\in\mathcal{I}} \Omega_i} \end{tikzcd}$$

Then $\vec{f}\ \textcircled{m}\ \mathcal{F}/\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i \iff f_i\ \textcircled{m}\ \mathcal{F}/\mathcal{F}_i \quad \forall i \in \mathcal{I}$.

Proof:

Notice that

$$\bigotimes_{i\in\mathcal{I}} \mathcal{F}_i := \sigma\langle \pi_i, \mathcal{F}_i : i \in \mathcal{I}\rangle$$
$$= \sigma\langle \pi_i^{-1}(\mathcal{F}_i) : i \in \mathcal{I}\rangle$$

$\therefore$

$$\vec{f}\ \textcircled{m}\ \mathcal{F}/\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i \iff \vec{f}\ \textcircled{m}\ \mathcal{F}/\{\pi_i^{-1}(F_i) : F_i \in \mathcal{F}_i,\ i\in\mathcal{I}\}$$

by "generators are enough"

$$\iff \underbrace{\vec{f}^{-1}(\pi_i^{-1}(F_i))} \in \mathcal{F}, \ \forall F_i \in \mathcal{F}_i\ \forall i \in \mathcal{I}.$$
$$= (\pi_i \circ \vec{f})^{-1}(F_i)$$
$$= f_i^{-1}(F_i)$$
$$\iff f_i\ \textcircled{m}\ \mathcal{F}/\mathcal{F}_i \quad \forall i \in \mathcal{I}.$$

QED

Remark: $\bigotimes_{k=1}^{d} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^d)$

and $\bigotimes_{k=1}^{\infty} \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^\infty)$ where $\mathcal{B}(\mathbb{R}^\infty)$ is defined with metric

$$d\left((x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty\right) := \sum_{k=1}^{\infty} 2^{-k}\left(|x_k - y_k| \wedge 1\right)$$

Remark: The previous Thm implies

$$\vec{f}^{-1}\Big(\bigotimes_{i\in\mathcal{I}}\mathcal{F}_i\Big) = \sigma\langle \vec{f}, \bigotimes_{i\in\mathcal{I}}\mathcal{F}_i\rangle = \sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$$

single map pullback

by "good sets" since $\sigma\langle \vec{f}, \bigotimes_i \mathcal{F}_i\rangle$ makes each $f_i\ \textcircled{m}$ and $\sigma\langle f_i, \mathcal{F}_i : i\in\mathcal{I}\rangle$ makes $\vec{f}\ \textcircled{m}$

What it means for Y to be $\textcircled{m}\ \sigma\langle X_1,\dots,X_n\rangle/\mathcal{B}(\mathbb{R})$ & the structure Thm

To motivate this result consider

$$\Omega = (0,1]$$
$$\mathcal{F} = \mathcal{B}((0,1])$$
$$X_1(\omega) = I_{(0,\frac{1}{2}]}(\omega)$$
$$X_2(\omega) = I_{(\frac{1}{2},1]}(\omega)$$

Suppose $Y : \Omega \to \mathbb{R}$ is another r.v. on Ω

$$\begin{tikzcd} \underset{\mathcal{F}}{\Omega} \arrow[r, bend left, "X_1\ \textcircled{m}"] \arrow[r, "X_2\ \textcircled{m}"] \arrow[r, bend right, "Y\ \textcircled{m}"'] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

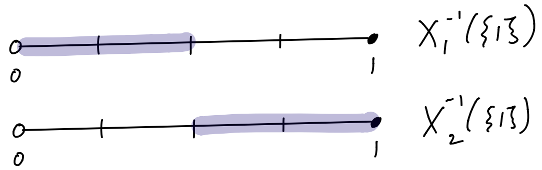
which additionally satisfies

$$Y\ \textcircled{m}\ \sigma\langle X_1, X_2\rangle/\mathcal{B}(\mathbb{R})$$

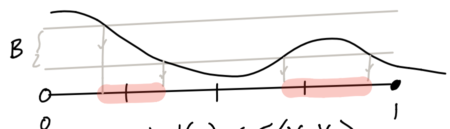
So that

$$\begin{tikzcd} \underset{\sigma\langle X_1, X_2\rangle}{\Omega} \arrow[r, bend left, "X_1\ \textcircled{m}"] \arrow[r, "X_2\ \textcircled{m}"] \arrow[r, bend right, "Y\ \textcircled{m}"'] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

Notice that $\sigma\langle X_1, X_2\rangle$ contains $\emptyset, \Omega$ & 7

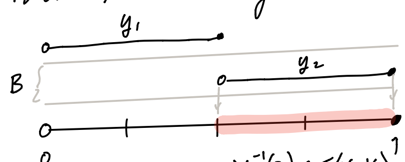


$\therefore$ Y can't look like



$Y^{-1}(B) \notin \sigma\langle X_1, X_2\rangle$

In fact Y must only look like



$Y^{-1}(B) \in \sigma\langle X_1, X_2\rangle$

i.e. $Y(\omega) = y_1 I_{\{X_1=1\}}(\omega) + y_2 I_{\{X_2=1\}}(\omega)$

$$= y_1 I_{\{1\}}(X_1(\omega)) + y_2 I_{\{1\}}(X_2(\omega))$$

$$= g(X_1, X_2)$$

g is ⓜ $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$

This holds in complete generality.

e.g. $Y, X_1, X_2, \ldots$ are r.v.s on $(\Omega, \mathcal{F}, P)$. Then

Y ⓜ $\sigma\langle X_1, X_2, \ldots\rangle \iff Y = g(X_1, X_2, \ldots)$ where g ⓜ $\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$

also extends to uncountable collections X_i, $i \in \mathcal{I}$.

To prove this we need an important thm 8 which is also used for defining $\int_\Omega f(\omega)\,d\mu(\omega)$ when f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Def: $f: \Omega \to \overline{\mathbb{R}}$ is a simple function if range(f) is a finite set & f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Thm: Suppose $f: \Omega \to \overline{\mathbb{R}}$ is ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ where $(\Omega, \mathcal{F})$ is a measurable space. Then f is a simple function iff $f = \sum_{k=1}^{n} c_k I_{A_k}$ where $n < \infty$, $c_k \in \overline{\mathbb{R}}$, $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are disjoint & $\Omega = \bigcup_{k=1}^{n} A_k$.

Proof:

$\Leftarrow$: Clearly $f: \Omega \to \overline{\mathbb{R}}$ & the range of f is finite. To see why f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ let $B \in \mathcal{B}(\overline{\mathbb{R}})$ and note:

$$f^{-1}(B) = \bigcup_{\substack{k \text{ s.t.} \\ c_k \in B}} A_k \in \mathcal{F}$$

since $A_k \in \mathcal{F}$

$\therefore f$ is simple

$\Rightarrow$: Suppose f is simple. Let $\underbrace{\{c_1, c_2, \ldots, c_n\}}_{\text{unique}} = \text{range}(f)$

Define $A_k := \{\omega : f(\omega) = c_k\}$.

$\therefore$ A_k's are disjoint since c_k's are unique. $A_k \in \mathcal{F}$ since f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ & $\{c_k\} \in \mathcal{B}(\overline{\mathbb{R}})$

$$\Omega = f^{-1}(\{c_1, \ldots, c_n\}) = \bigcup_{k=1}^{n} A_k$$

QED.

Def: Let $(\Omega, \mathcal{F})$ be a measurable space.

Define

$\mathcal{M}_s(\Omega,\mathcal{F}) :=$ all non-negative simple functions on Ω

$\mathcal{M}(\Omega,\mathcal{F}) :=$ all non-negative $\textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ functions on Ω.

Thm (Structure theorem)

Let $(\Omega,\mathcal{F})$ be a measurable space & $f:\Omega \to \overline{\mathbb{R}}$. For each $n=1,2,\dots$ define

$$f_n(w) := \begin{cases} \lfloor 2^n f(w) \rfloor 2^{-n} & \text{if } -n \le f(w) < n \\ n & \text{if } f(w) \ge n \\ -n & \text{if } f(w) < -n \end{cases}$$

Then

(i) If $f\ \textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ then

$$\underbrace{f_n(w)}_{\text{bdd \& simple}} \to f(w) \text{ as } n\to\infty,\ \forall w \in \Omega$$

(ii) If $f\ \textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ and bdd then

$$\sup_{w\in\Omega} |\underbrace{f_n(w)}_{\text{bdd \& simple}} - f(w)| \to 0 \text{ as } n\to\infty,\ \forall w\in\Omega$$

(iii) If $f \in \mathcal{M}(\Omega,\mathcal{F})$ then

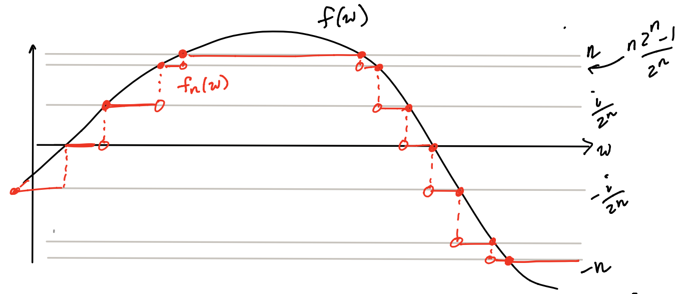
$$\underbrace{f_n(w)}_{\text{bdd \& in } \mathcal{M}_s(\Omega,\mathcal{F})} \uparrow f(w) \text{ as } n\to\infty,\ \forall w\in\Omega$$

Proof:

f_n is clearly bdd.

If $f\ \textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ then f_n is a simple function since f_n has finite range and $f_n\ \textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by cut & paste & composition of $\textcircled{m}$ is $\textcircled{m}$.

Here is the picture:



where $i\in\mathbb{Z}$ s.t. $-n \le \frac{i}{2^n} < n$

Notice:

- if $f(w) < \infty$ then $\forall$ large n
$$|f(w) - f_n(w)| \le \frac{1}{2^n}$$
- If $f(w) = \infty$ then
$$n = f_n(w) \to f(w) \quad \text{as } n\to\infty$$
- If $f(w) = -\infty$ then
$$-n = f_n(w) \to f(w) \quad \text{as } n\to\infty$$

$\therefore$ (i) & (ii) holds.

Finally if $f\in\mathcal{M}(\Omega,\mathcal{F})$ then the fact that $\left\{\frac{i}{2^n} : i\in\mathbb{Z}\right\} \subset \left\{\frac{i}{2^{n+1}} : i \in\mathbb{Z}\right\}$ implies

$$f_n(w) \le f_{n+1}(w)$$

which proves (iii). QED

Now we have the tools to prove implications of $Y\ \textcircled{m}\ \sigma\langle X_1, X_2, \dots\rangle$.

Thm (Charaterizing $\textcircled{m}$ functions of $\textcircled{m}$ functions)

Let $Y, X_1, X_2, \dots$ be r.v.s defined on a measurable space $(\Omega,\mathcal{F})$. Then the following statements are equivalent:

(i) $Y\ \textcircled{m}\ \sigma\langle X_1, X_2,\dots\rangle/\mathcal{B}(\mathbb{R})$

(ii) There exists a $g:\mathbb{R}^\infty \to \mathbb{R}$ s.t. $g\ \textcircled{m}\ \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$ and

$$Y = g(X_1, X_2, \dots)$$

Note the fully general measure theoretic thm also holds & reads:

Theorem: Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{I}$ be an arbitrary index set. For each $i \in \mathcal{I}$ suppose $f_i : \Omega \to \Omega_i$ where $(\Omega_i, \mathcal{F}_i)$ is a measure space & f ⓜ $\mathcal{F}/\mathcal{F}_i$. Let $h : \Omega \to \bar{\mathbb{R}}$ be ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$. Then the following are equiv:

(i) h ⓜ $\sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I} \rangle$

(ii) there exists a function $g : \prod_{i \in \mathcal{I}} \Omega_i \to \bar{\mathbb{R}}$ s.t. g ⓜ $\bigotimes_{i \in \mathcal{I}} \mathcal{F}_i / \mathcal{B}(\bar{\mathbb{R}})$ s.t.

$$h(\omega) = g\big((f_i(\omega))_{i \in \mathcal{I}}\big).$$

Here is the picture

$$\begin{tikzcd}
 & \Omega_i \;\; \mathcal{F}_i \\
\underset{\sigma\langle f_i, i \in \mathcal{I}\rangle}{\Omega} \arrow[ur, "f_i \text{ ⓜ}"] \arrow[dr, "f_k \text{ ⓜ}"'] \arrow[dd, "h \text{ ⓜ}"] & \vdots \\
 & \Omega_k \;\; \mathcal{F}_k \\
\bar{\mathbb{R}} \;\; \mathcal{B}(\bar{\mathbb{R}}) &
\end{tikzcd}$$

iff $\exists g$ s.t. the following commutes

$$\begin{tikzcd}
\underset{\sigma\langle f_i : i \in \mathcal{I}\rangle}{\Omega} \arrow[r, "(f_i)_{i \in \mathcal{I}} \text{ ⓜ}"] \arrow[d, "h \text{ ⓜ}"'] & \prod_{i \in \mathcal{I}} \Omega_i \;\; \bigotimes_{i \in \mathcal{I}} \mathcal{F}_i \arrow[dl, "g \text{ ⓜ}"] \\
\bar{\mathbb{R}} \;\; \mathcal{B}(\bar{\mathbb{R}}) &
\end{tikzcd}$$

The proofs are exactly the same so lets do the less general one.

Proof:

(ii) $\Rightarrow$ (i): follows simply by composition of ⓜ fns is ⓜ.

(i) $\Rightarrow$ (ii): Suppose Y ⓜ $\sigma\langle X_1, X_2, \dots \rangle / \mathcal{B}(\mathbb{R})$.

Case 1: Y is a simple function

$\therefore$ $Y(\omega) = \sum_{k=1}^{n} c_k I_{A_k}(\omega)$ where $A_k \in \sigma\langle X_1, X_2, \dots\rangle$

Let $\vec{X}(\omega) := (X_1(\omega), X_2(\omega), \dots)$ & recall that $\vec{X}$ ⓜ $\sigma\langle X_1 \dots\rangle / \mathcal{B}(\mathbb{R}^\infty)$ by "just check the coords" &

$$A_k \in \sigma\langle X_1, X_2, \dots \rangle = \vec{X}^{-1}(\mathcal{B}(\mathbb{R}^\infty))$$

$\therefore$ each $A_k = \vec{X}^{-1}(B_k)$ for $B_k \in \mathcal{B}(\mathbb{R}^\infty)$.

Now
$$Y(\omega) = \sum_{k=1}^{n} c_k I_{\vec{X}^{-1}(B_k)}(\omega)$$
$$= \sum_{k=1}^{n} c_k I_{B_k}(\vec{X}(\omega))$$
$$= \underbrace{\sum_{k=1}^{n} c_k I_{B_k}}_{=g}(X_1, X_2, \dots)$$
which is clearly ⓜ $\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$.

Case 2: Y is not simple (... but ⓜ $\sigma\langle X_1, \dots \rangle / \mathcal{B}(\mathbb{R})$)

By the structure thm $\exists$ simple Y_n ⓜ $\sigma\langle X_1, X_2, \dots\rangle$ s.t. $Y_n(\omega) \to Y(\omega)$ as $n \to \infty$ $\forall \omega \in \Omega$.

For each n, case 1 applies to Y_n

$\therefore$ $\exists$ g_n ⓜ $\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$ s.t.

$$Y_n(\omega) = g_n(X_1(\omega), X_2(\omega), \dots)$$

Now it is tempting to try

$$Y(\omega) = \lim_n Y_n(\omega) = \underbrace{\lim_n}_{\text{and set this to } g.} g_n(X_1(\omega), X_2(\omega) \dots)$$

However $g_n(x_1, x_2, \dots)$ is only guarenteed to have a limit (to Y) when $(x_1, x_2, \dots)$ is in the range of $(X_1, X_2, \dots)$.

This is solved by setting

$$g(\vec{x}) := \begin{cases} \lim_n g_n(\vec{x}) & \text{if } \vec{x} \in A \\ 0 & \text{if } \vec{x} \in A^c. \end{cases}$$

where $A := \{\vec{x} \in \mathbb{R}^\infty : \limsup_n g_n(\vec{x}) = \liminf_n g_n(\vec{x})\}$

$\in \mathcal{B}(\mathbb{R}^\infty)$ by closure thm & that $g_n \in \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$.

Now $Y(\omega) = g(X_1(\omega), X_2(\omega), \dots)$

QED.

Independent R.V.s

Now, with the exception of expected value, we have the full theory of random variables at our disposal. Here are the extensions of independence of events to independence of random variables.

$$\underbrace{\Omega, \mathcal{F}, P}_{\text{Prob. space}} \xrightarrow[X_k(\omega)]{X_i(\omega)\ \vdots} \mathbb{R}, \mathcal{B}(\mathbb{R})$$

where $i, k \in \mathcal{I}$ a general index set.

Def: The r.v.s X_i for $i \in \mathcal{I}$ are independent if $\sigma\langle X_i \rangle$ for $i \in \mathcal{I}$ are independent σ-fields.

Thm (ANOVA):

Matrix of r.v.s (all defined on $(\Omega, \mathcal{F}, P)$)

$$\begin{matrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \vdots & & & \ddots \end{matrix}$$

Then the r.v.s $\{X_{ik}\}_{i,k}$ are indep if and only if

(i) the r.v.s within each row are indep. &

(ii) The rows $R_i = \sigma\langle X_{i1}, X_{i2}, \dots \rangle$ are indep.

Proof: Just like old ANOVA ... noting that $\sigma\langle X_i \rangle$ are π-systems and

$$\sigma\langle X_{i1}, X_{i2}, \dots \rangle = \sigma\langle X_{i1}^{-1}(\mathcal{B}(\mathbb{R})), X_{i2}^{-1}(\mathcal{B}(\mathbb{R})), \dots \rangle$$

$$= \sigma\langle \sigma\langle X_{i1} \rangle, \sigma\langle X_{i2} \rangle, \dots \rangle$$

QED.

Thm (existence of indep $X_1, X_2, \dots$)

Let $P_1, P_2, \dots$ be prob measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then there exists a single prob space $(\Omega, \mathcal{F}, P)$ and r.v.s Y_i all defined on Ω s.t.

(i) $PY_i^{-1}(B) = P_i(B)$, $\forall B \in \mathcal{B}(\mathbb{R}), \forall i = 1, 2, \dots$

(ii) $Y_1, Y_2, \dots$ are independent.

Proof:

Let $(\Omega, \mathcal{F}, P)$ be our old friend: "Borel's coinflip model on $\Omega = (0,1]$".

Let $X_k(\omega) = k^{th}$ binary digit of ω & re-arrange them in an infinite matrix:

$$\left.\begin{matrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \vdots & \vdots & & \ddots \end{matrix}\right\} \text{all indep since } \sigma\langle X_{ik}\rangle = \sigma\langle H_{ik}\rangle$$

For each row i define

$$U_i(\omega) := \sum_{k=1}^{\infty} \frac{1}{2^k} X_{ik}(\omega).$$

Notice $U_i = g(X_{i1}, X_{i2}, X_{i3}, \dots)$

where $g(\vec{x}) = \limsup_n \sum_{k=1}^{n} \frac{1}{2^k} \pi_k(\vec{x})$

is m $\mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\bar{\mathbb{R}})$ by closure & the fact that $\mathcal{B}(\mathbb{R}^\infty)$ makes each π_k m.

$\therefore$ U_i m $\sigma\langle X_{i1}, X_{i2}, \dots\rangle / \mathcal{B}(\bar{\mathbb{R}})$

$\therefore$ $\sigma\langle U_i\rangle \subset \underbrace{\sigma\langle X_{i1}, X_{i2}, \dots\rangle}_{\text{These are indep by AnovA}}$

$\therefore$ $U_1, U_2, \dots$ are independent by sub-class thm & each is uniform on $[0,1]$ by HWK 4.

Now set $Y_i := F_i^{-1}(U_i)$ ← where we are allowed to modify U_i so it is uniform on $(0,1)$.

where $F_i(x) := P_i((-\infty, x])$.

Switching lemma shows Y_i is a r.v. s.t.

$$PY_i^{-1} = P_i \text{ on } \mathcal{B}(\mathbb{R}).$$

To show the Y_i's are independent notice that Y_i is a m function of U_i which implies so that Y_i m $\sigma\langle U_i\rangle$

$$\sigma\langle Y_i\rangle \subset \underbrace{\sigma\langle U_i\rangle}_{\text{indep.}}$$

QED

Thm (Kolmogorov's 0-1 law for R.V.s)

If $X_1, X_2, \dots$ are indep r.v.s on $(\Omega, \mathcal{F}, P)$ then all events in $\mathcal{T} := \bigcap_{n=1}^{\infty} \sigma\langle X_n, X_{n+1}, \dots\rangle$ have prob 0 or 1. Moreover, if Y is another r.v. on $(\Omega, \mathcal{F}, P)$ which is m $\mathcal{T}/\mathcal{B}(\mathbb{R})$ then $\exists c \in \mathbb{R}$ s.t. $P(Y = c) = 1$. ← i.e. Y is constant with prob 1.

Proof: Notice that

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma\langle \underbrace{\sigma\langle X_n\rangle, \sigma\langle X_{n+1}\rangle, \dots}_{\text{independent } \pi\text{-systems.}} \rangle$$

$\therefore$ $\forall A \in \mathcal{T}$, $P(A) = 1$ or 0 by "old 0-1 law".

Suppose Y m $\mathcal{T}$.

$\therefore$ $\{Y \le c\} \in \mathcal{T}$, $\forall c \in \mathbb{R}$

$\therefore$ $P(Y \le c) = 0$ or 1

But since $P(Y\le c)$ is monotonic in c right continuous, $\lim_{c\to\infty} P(Y\le c)=1$ and $\lim_{c\to-\infty} P(Y\le c)=0$ there must exist a $c_0\in\mathbb{R}$ s.t. (17)

$$P(Y\le c_0)=1 \quad \& \quad P(Y<c_0)=0$$

$$\therefore P(Y=c_0)=P(Y\le c_0)-P(Y<c_0)=1.$$

QED

e.g. Let $X_1, X_2, \ldots$ be indep r.v.s on $(\Omega, \mathcal{F}, P)$. Let $S_n = X_1+\cdots+X_n$.

Suppose a_n is any sequence of real numbers s.t. $\lim_n a_n=\infty$.

Now $\limsup_n \frac{S_n}{a_n} = \limsup_n \frac{X_m+X_{m+1}+\cdots+X_n}{a_n}$ for any m

$$\in \underbrace{\bigcap_{m=1}^{\infty} \sigma\langle X_m, X_{m+1},\ldots\rangle}_{\text{tail } \sigma\text{-field of indep r.v.s}}$$

$\therefore \exists c$ s.t. $P\left(\limsup_n \frac{S_n}{a_n}=c\right)=1$.

In the special case $X_i = \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$

we have:

$$a_n^{-1} := \frac{1}{n} \overset{SLLN}{\Longrightarrow} c=0$$

$$a_n^{-1} := \frac{1}{\sqrt{n}} \overset{CLT}{\Longrightarrow} c=\infty$$

$$a_n^{-1} := \frac{1}{\sqrt{2n\log\log n}} \overset{LIL}{\Longrightarrow} c=1$$

In general, events of the form $\left\{\sum_{k=1}^{\infty} X_k = c\right\}$ are not tail events since the value of $\sum_k X_k$ depends on X_1 for example.

However events of the form (18)

$$\left\{\sum_{k=1}^{\infty} X_k = \pm\infty\right\} \text{ or}$$

$$\left\{\sum_{k=1}^{\infty} X_k \text{ converges}\right\}$$

are tail events, and therefore have probability 0 or 1 when the X_k's are indep.

Kolmogorov's 3-series thm gives necessary & sufficient conditions when $P\left(\sum_{k=1}^{\infty} X_k \text{ converges}\right)=1$ or 0 for independent X_k's.

but we need integration before we can state it.

We can still get something out of this observation.

We used Kolmogorov's maximal inequality to show

$$P\left(\sum_{n=1}^{\infty} \frac{R_n}{n} \text{ converges}\right)=1$$

where $R_n = \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$ are indep.

$\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}}$ was left unsolved but at least we can now conclude

$$P\left(\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}} \text{ converges}\right) = 0 \text{ or } 1.$$

There is also a nice extension to the Hewitt-Savage 0-1 law.

Def: $f : \mathbb{R}^\infty \to \mathbb{R}$ is a **symmetric function** if $f \in \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$ and $f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots)$ whenever π is a permutation of $\mathbb{N}$ that permutes a finite number of coordinates (i.e. $\exists N$ s.t. $\pi(n) = n$ for all $n \geq N$).

Thm (Hewitt-Savage)

indep & $PX_i^{-1} = PX_j^{-1}$

If $X_1, X_2, \dots$ are iid r.v.s on $(\Omega, \mathcal{F}, P)$ then $f(X_1, X_2, \dots)$ is constant with probability 1 whenever f is a symmetric function.

Moreover, any event $A \in \mathcal{F}$, which has the form

$$(*) \qquad I_A(\omega) = f(X_1(\omega), X_2(\omega), \dots)$$

for a symmetric function f, satisfies $P(A) = 0$ or 1.

Sketch of Proof:

If A satisfies $(*)$ then

$$A \in \sigma\langle X_1, X_2, \dots \rangle = \sigma\Big\langle \underbrace{\bigcup_{m=1}^{\infty} \sigma\langle X_1, \dots, X_m\rangle}_{\text{This is a field.}} \Big\rangle$$

Approximate A with $A_n \in \sigma\langle X_1, \dots, X_{m_n}\rangle$ s.t. $P(A \,\Delta\, A_n) \to 0$ as $n \to \infty$

Now notice 3 key facts

1) $A = A^{\pi_n}$
2) $P(A_n) = P(A_n^{\pi_n})$
3) A_n is indep of $A_n^{\pi_n}$

with an appropriately chosen π_n

where $I_{A^{\pi_n}} = f(X_{\pi_n(1)}, X_{\pi_n(2)}, \dots)$

and $I_{A_n^{\pi_n}} = f_n(X_{\pi_n(1)}, \dots, X_{\pi_n(m_n)})$

where $I_{A_n} = f_n(X_1, \dots, X_{m_n})$

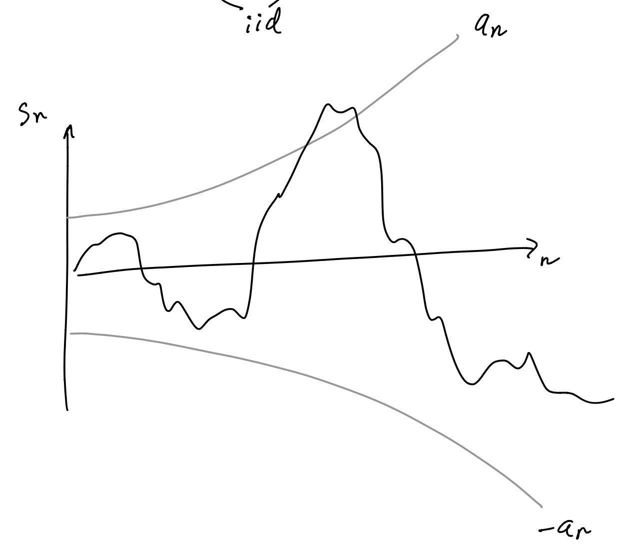
existance since $A_n \in \sigma\langle X_1, \dots, X_{m_n}\rangle$

QED

Recall that Hewitt-Savage 0-1 law is usefull for showing things like

$$P(|S_n| \geq a_n \text{ i.o.}) = 0 \text{ or } 1$$

when $S_n = X_1 + \cdots + X_n$ is a random walk
iid



Notice that Hewitt-Savage applies to more events than Kolmogorov's 0-1 law since any tail event is automatically a "symmetric event". However Hewitt-Savage requires more assumptions (that the X_i are iid).

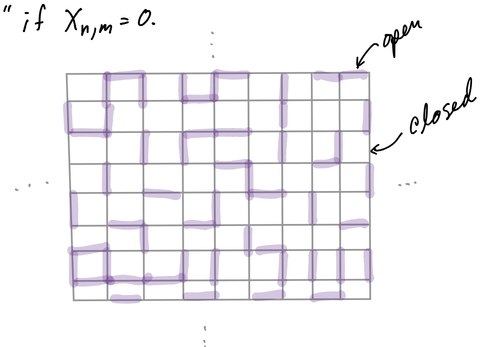
Bond percolation

Consider a lattice on $\mathbb{Z}^2$ where each edge is assigned a lable $(n,m) \in \mathbb{Z}^2$.

Let $X_{n,m}$ be a sequence of iid random variables, all defined on a probability space $(\Omega, \mathcal{F}, P)$ s.t.

$$P(X_{n,m}=1)=\theta \ \& \ P(X_{n,m}=0)=1-\theta.$$

Let edge (n,m) be "open" if $X_{n,m}=1$ & "closed" if $X_{n,m}=0$.



Let $\vec{X} = (X_{n,m})_{n,m\in\mathbb{Z}}$ & note that

$$\underset{\mathcal{F},P}{\Omega} \xrightarrow{\vec{X}(\omega)} \prod_{n,m\in\mathbb{Z}} \{0,1\} \quad \bigotimes_{n,m\in\mathbb{Z}} \sigma\langle\{1\}\rangle$$

and $B \in \bigotimes_{n,m\in\mathbb{Z}} \sigma\langle\{1\}\rangle$ where

$$B := \left\{ \vec{x} : \begin{array}{l} \text{there exists an infinite connected} \\ \text{open path in } \vec{x} \in \prod_{n,m\in\mathbb{Z}^2} \{0,1\} \end{array} \right\}$$

$$= \bigcup_{n,m} \{ |C_{n,m}| = \infty \}$$

where $C_{n,m}$ is the collection of edges connected to (n,m) via an open path, i.e the open connected cluster containing (n,m))

Now what is

$$P(\vec{X} \in B) =: P(\theta) = ?$$

Notice that for any finite permutation $\pi(n,m)$ of $(n,m) \in \mathbb{Z}^2$

$$\{\omega\in\Omega : \vec{X}(\omega) \in B\} = \{\omega\in\Omega : \vec{X}^{\pi}(\omega) \in B\}$$

where $\vec{X}^{\pi} = (X_{\pi(n,m)}(\omega))_{n,m\in\mathbb{Z}}$.

This follows since the existance of an infinite connected open path is invariant to any finite "scramble" of the edges encoded by $\vec{X}$.

$\therefore \{\omega\in\Omega : \vec{X}(\omega)\in B\}$ is a symmetric event.

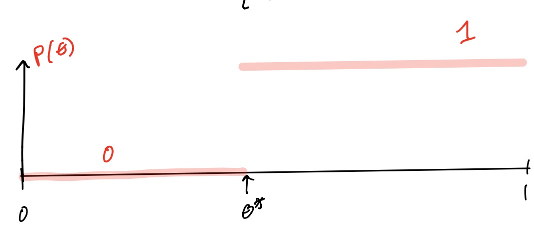
Since $X_{n,m}$ are iid, Hewitt-Savage applies.

$\therefore P(\theta) \in \{0,1\}$.

Also notice $P(\theta)$ is monotonically increasing in θ... this follows since one can simulate $\vec{X}'$ for $\theta' < \theta$ by "thinning" a simulation of $\vec{X}$ based on θ (use an inverse c.d.f. construction).

This implies there exists a critical θ^* s.t.

$$P(\theta) = \begin{cases} 1 & \text{if } \theta > \theta^* \\ 0 & \text{if } \theta < \theta^* \end{cases}$$



The hard part is finding the value of θ^*!

Ask Dan Romik for more details.

Here is an example of $\underbrace{\text{"site percolation"}}$ simulations.

instead of opening or closing edges
you open or close the whole pixel area.

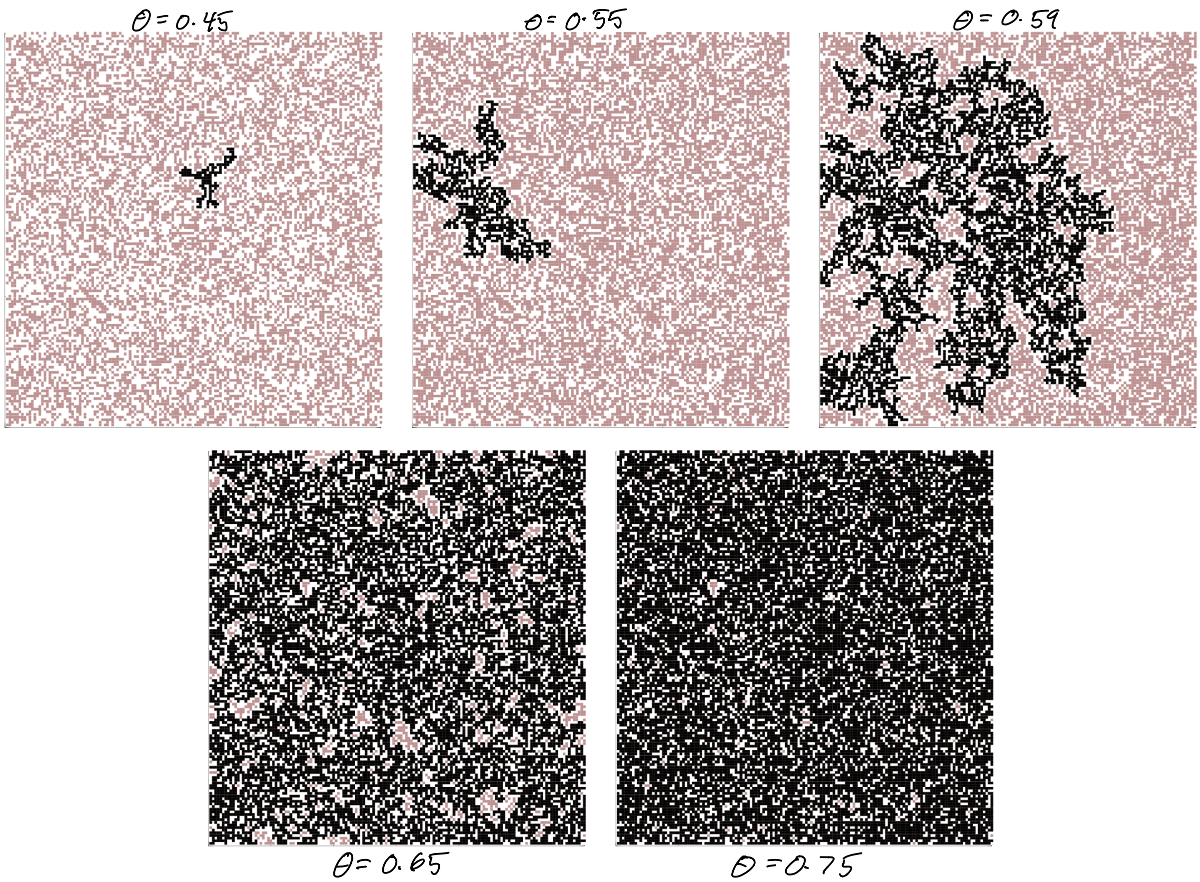


Figure 1.2: Percolation in $2d$ square lattices with system size $L \times L = 150 \times 150$. Occupation probability $p = 0.45, 0.55, 0.59, 0.65,$ and 0.75, respectively. Notice, that the largest cluster *percolates* through the lattice from top to bottom in this example when $p \geq 0.59$.

The black regions show the largest cluster.
The phase transition θ^* for site percolation is different than bond percolation (site percolation: $\theta^* \simeq 0.59$, bond percolation: $\theta^* = 1/2$).

Lets have a little more fun before we move on to integration.

Law of Pure types

In HWK 4 you studied the random series $W = (1-\theta)\sum_{n=1}^{\infty} \theta^{n-1} X_n$

where $X_n = \begin{cases} 1 & \text{w.p. } 1/2 \\ 0 & \text{w.p. } 1/2 \end{cases}$ are iid.

Notice there are 3 cases

$\theta = 0 \Rightarrow$ W is concentrated on 0.

$0 < \theta < 1/2 \Rightarrow$ W is concentrated on $B \in \mathcal{B}([0,1])$ with $\mathcal{L}^1(B) = 0$

$\theta = 1/2 \Rightarrow$ W is uniform on $[0,1]$

These are "pure types"

Def: Let X be a r.v.. Then

- X is *purely atomic* if $\exists$ a countable $B \subset \mathbb{R}$ s.t. $PX^{-1}(B) = 1$

  $P(X \in B)$

- X is *purely singular* if $PX^{-1}(\{x\}) = 0 \quad \forall x \in \mathbb{R}$ & $\exists B \in \mathcal{B}(\mathbb{R})$ s.t. $PX^{-1}(B) = 1$ & $\mathcal{L}^1(B) = 0$
- X is *purely absolutely continuous* if $\forall B \in \mathcal{B}(\mathbb{R})$, $\mathcal{L}^1(B) = 0 \Rightarrow PX^{-1}(B) = 0$.
- X is of *pure type* if X is purely atomic, purely singular or purely absolutely continuous.

e.g. Let U_1 & U_2 be two indep uniform r.v.s on $(\Omega, \mathcal{F}, P)$. Then

$$X = \frac{1}{5} I_{\{U_1 \le 1/2\}} + U_2 I_{\{U_1 > 1/2\}}$$

is not of pure type.

Thm (Jessen-Wintner law of pure types)

Let $X_1, X_2, \ldots$ be independent r.v.s defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose each X_n takes its values in a countable set $\subset \mathbb{R}$ and $\sum_{n=1}^{\infty} X_n$ converges (with prob 1) to a finite limit X. Then X is of pure type.

Proof:

By changing the X_n's to 0 on a set of P-prob 0 we may assume

$$X(w) = \sum_{n=1}^{\infty} X_n(w), \quad \forall w \in \Omega.$$

By assumption $\exists$ a countable $\mathcal{C} \subset \mathbb{R}$ s.t.

$$X_n(w) \in \mathcal{C}, \quad \forall w \in \Omega \text{ and } \forall n \in \mathbb{N}.$$

Let $G := \{ n_1 x_1 + \cdots + n_p x_p : p \ge 1, x_i \in \mathcal{C}, n_i \in \mathbb{Z} \}$

Notice that $\mathcal{C} \subset G$, G is countable and G is closed under addition, subtraction and $G = -G$.

We will show that $\forall B \in \mathcal{B}(\mathbb{R})$,

$$P(X \in B + G) = 0 \text{ or } 1.$$

where $B + G := \{ b + g : b \in B, g \in G \}$

$$= \bigcup_{g \in G} (B + g) \in \mathcal{B}(\mathbb{R}).$$

Notice the following fact:

Fact: If $x, y \in \mathbb{R}$ satisfy $x-y \in G$ then

$$x \in B+G \Leftrightarrow x = b+g, \text{ for } b\in B, g \in G$$

$$\Leftrightarrow y = b + \underbrace{g-x+y}_{\in G}, \text{ for } b \in B, g\in G$$

$$\Leftrightarrow y \in B+G$$

Now since

$$X(w) - \sum_{n=m}^{\infty} X_n(w) = \sum_{n=1}^{m-1} X_n(w) \in G$$

the above fact implies

$$\{X \in B+G\} = \left\{\sum_{n=m}^{\infty} X_n \in B+G\right\}$$

$$\in \sigma\langle X_m, X_{m+1}, \dots \rangle$$

holds $\forall m$

$\therefore$ Kolmogorov's 0-1 law implies

$$P(X \in B+G) = 1 \text{ or } 0.$$

which holds for any $B \in \mathcal{B}(\mathbb{R})$.

study the following exaustive cases:

Case 1: $P(X \in B+G) = 1$ for some countable set $B \in \mathcal{B}(\mathbb{R})$.

$\therefore PX^{-1}$ is purely atomic.

Case 2: $P(X \in B+G) = 0$, $\forall$ countable $B \in \mathcal{B}(\mathbb{R})$ but $\exists B' \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{L}'(B') = 0$ and $P(X \in B'+G) = 1$.

Notice that

$$\mathcal{L}'(B'+G) = \mathcal{L}'\Big(\bigcup_{g\in G}(B'+g)\Big) \le \sum_{g \in G} \underbrace{\mathcal{L}'(B'+g)}_{=\mathcal{L}'(B')=0} = 0.$$

$\therefore PX^{-1}$ is purely singular.

case 3: $P(X \in B+G) = 0$, $\forall$ countable $B \in \mathcal{B}(\mathbb{R})$ and $P(X \in B'+G) = 0$, $\forall B' \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{L}'(B') = 0$.

Now if $\mathcal{L}'(B) = 0$ then $\mathcal{L}'(B-G) = 0$ and therefore

$$P(X \in B) \le P(X \in \underbrace{(B-G)}_{B' \text{ with } \mathcal{L}'(B')=0} + G) = 0$$

$\therefore PX^{-1}$ is purely abs continuous.

Q.E.D.

e.g. we know $P\left(\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}} \text{ converges}\right) = 1 \text{ or } 0.$ iid Radamacher R.V.S

if you can show it is 1 then the limit can not be of "mixed type".