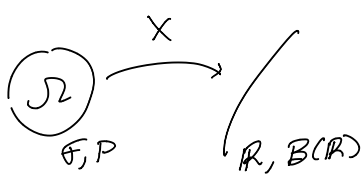
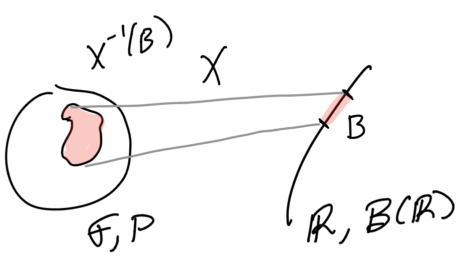
Lecture 8: Measurable functions, Random variables and distribution functions

In this lecture we will start by developing the measure theoretic notion of measurable functions & define random variables X as measurable functions $X: \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{F}, P)$ is a prob. space.



Measurability of X is required since we want $P(X \in B)$ to be defined where $B \in \mathcal{B}(\mathbb{R})$ and

$$\{X \in B\} \equiv \{\omega \in \Omega : X(\omega) \in B\} =: \underbrace{X^{-1}(B)}_{\text{pre-image of } B \text{ under } X.}$$



Measurable functions

Let $(\Omega_1, \mathcal{F}_1)$ & $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces & $f: \Omega_1 \to \Omega_2$.

Def: f is measurable between $\mathcal{F}_1$ & $\mathcal{F}_2$ (written $f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{F}_2$ for short) iff

$$f^{-1}(A) \in \mathcal{F}_1, \quad \forall A \in \mathcal{F}_2. \qquad (*)$$

Note: It will sometimes be convinent to write $f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{F}_2$ when f satisfies $(*)$ even when $\mathcal{F}_1$ or $\mathcal{F}_2$ are Not σ-fields ... just collections of sets.

A few basic facts about $f^{-1}(A)$

(1) $f^{-1}(\Omega_2) = \Omega_1$ since f maps into Ω_2

(2) $f^{-1}(\emptyset) = \emptyset$

(3) $f^{-1}(A^c) = (f^{-1}(A))^c$

since $\omega \in f^{-1}(A^c) \iff f(\omega) \in A^c$
$\iff f(\omega) \notin A$
$\iff \omega \notin f^{-1}(A)$

(4) $f^{-1}(\bigcup_k A_k) = \bigcup_k f^{-1}(A_k)$ even A_k's are not disjoint

since $\omega \in f^{-1}(\bigcup_k A_k) \iff f(\omega) \in A_k$ some k
$\iff \omega \in f^{-1}(A_k)$ some k
$\iff \omega \in \bigcup_k f^{-1}(A_k)$

Thm (Generators are enough)

If $\underset{\mathcal{F}_1}{\Omega_1} \xrightarrow{f} \underset{\sigma\langle \mathcal{A} \rangle}{\Omega_2}$ & $\mathcal{F}_1$ is a σ-field

then $f \ \textcircled{m}\ \mathcal{F}_1/\sigma\langle \mathcal{A} \rangle \iff f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{A}$.

Proof:

$\Rightarrow$: trivial

$\Leftarrow$: Good sets on

$$\mathcal{G} = \{A \subset \Omega_2 : f^{-1}(A) \in \mathcal{F}_1\}.$$

$\mathcal{A} \subset \mathcal{G}$ by assumption & $\mathcal{G}$ is a σ-field by facts (1), (3), (4).

QED.

e.g.

$$\begin{tikzcd} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \arrow[r, "f"] & \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \end{tikzcd}$$

f is monotone $\Longrightarrow f^{-1}((-\infty, x])$ is an interval $\forall x$

show that $a, b \in f^{-1}((-\infty, x])$ & $a \le y \le b \Rightarrow y \in \underbrace{f^{-1}((-\infty, x])}_{\text{could be open or closed}}$

$\Longrightarrow f^{-1}((-\infty, x]) \in \mathcal{B}(\mathbb{R}), \forall x$

$\Longrightarrow f$ m $\mathcal{B}(\mathbb{R}) / \{(-\infty, x] : x \in \mathbb{R}\}$

$\Longleftrightarrow f$ m $\mathcal{B}(\mathbb{R}) / \underbrace{\sigma\langle (-\infty, x] : x \in \mathbb{R}\rangle}_{= \mathcal{B}(\mathbb{R})}$

$\therefore$ All monotone funs are measurable.

e.g.

$$\begin{tikzcd} \underset{\mathcal{B}(\mathbb{R}^d)}{\mathbb{R}^d} \arrow[r, "f"] & \underset{\mathcal{B}(\mathbb{R}^k)}{\mathbb{R}^k} \end{tikzcd}$$

f is continuous

$\Longleftrightarrow f^{-1}(G)$ is open $\forall$ open $G \subset \mathbb{R}^k$

$\Longrightarrow f$ m $\mathcal{B}(\mathbb{R}^d) /$ opens in $\mathbb{R}^k$

$\Longleftrightarrow f$ m $\mathcal{B}(\mathbb{R}^d) / \underbrace{\sigma\langle \text{opens in } \mathbb{R}^k \rangle}_{\mathcal{B}(\mathbb{R}^k)}$

This extends to metric spaces Ω_1 & Ω_2 so that f m $\mathcal{B}(\Omega_1) / \mathcal{B}(\Omega_2)$ for all metric continuous $f : \Omega_1 \to \Omega_2$

Thm (composition of m is m).

If

$$\begin{tikzcd} \underset{\mathcal{F}_1}{\Omega_1} \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & \underset{\mathcal{F}_2}{\Omega_2} \arrow[r, "g"] & \underset{\mathcal{F}_3}{\Omega_3} \end{tikzcd}$$

where f m $\mathcal{F}_1 / \mathcal{F}_2$ & g m $\mathcal{F}_2 / \mathcal{F}_3$

then $g \circ f$ m $\mathcal{F}_1 / \mathcal{F}_3$

Proof:

If $B \in \mathcal{F}_3$ then

$$(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$$

$$= \underbrace{f^{-1}(\underbrace{g^{-1}(B)}_{\in \mathcal{F}_2})}_{\in \mathcal{F}_1}$$

QED.

e.g. if f m $\mathcal{F} / \mathcal{B}(\mathbb{R})$ then $|f|, f^2, \sin(f) \ldots$ are all m $\mathcal{F} / \mathcal{B}(\mathbb{R})$ by continuity.

Corollary (Just check the coordinates)

If

$$\begin{tikzcd} \underset{\mathcal{F}}{\Omega} \arrow[r, "f"] & \underset{\mathcal{B}(\mathbb{R}^n)}{\mathbb{R}^n} \end{tikzcd}$$

works for $\bar{\mathbb{R}}^n$, $\mathcal{B}(\bar{\mathbb{R}}^n)$ too

where $\mathcal{F}$ is a σ-field and $f(\omega) = (f_1(\omega), \ldots, f_n(\omega))$

then

$$f \text{ m } \mathcal{F} / \mathcal{B}(\mathbb{R}^n) \Longleftrightarrow \text{each } f_k \text{ is m } \mathcal{F} / \mathcal{B}(\mathbb{R})$$

Proof:

$\Longrightarrow$: follows since the coordinate mappings $\pi_k(x_1,\dots,x_n) := x_k$ are continuous & $f_k = \pi_k \circ f$

$\Longleftarrow$:

$$f^{-1}\big(\underbrace{(a_1,b_1]\times\cdots\times(a_n,b_n]}_{\text{rectangle}}\big) = \underbrace{\bigcap_{k=1}^{n} \underbrace{f_k^{-1}((a_k,b_k])}_{\in\mathcal{F} \text{ since } f_k \text{ ⓜ } \mathcal{F}/\mathcal{B}(\mathbb{R})}}_{\in\mathcal{F}}$$

$\therefore$ f ⓜ $\mathcal{F}/$rectangles

$\therefore$ f ⓜ $\mathcal{F}/\underbrace{\sigma\langle \text{rectangles}\rangle}_{\mathcal{B}(\mathbb{R}^n)}$.

QED

Thm (cut & paste over countable & ⓜ pieces)

If $\underset{\mathcal{F}_1}{\Omega_1} \xrightarrow{f} \underset{\mathcal{F}_2}{\Omega_2}$ where $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-fields and $\Omega_1 = \bigcup_{k=1}^{\infty} A_k$ s.t. $\forall k$, $A_k \in \mathcal{F}_1$

then

f ⓜ $\mathcal{F}_1/\mathcal{F}_2 \iff f|_{A_k}$ ⓜ $(\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$ for all $k = 1, 2, \dots$

Proof:

$\Longrightarrow$: suppose f ⓜ $\mathcal{F}_1/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$w \in f|_{A_k}^{-1}(B) \iff f|_{A_k}(w) \in B \;\&\; w \in A_k$$
$$\iff f(w) \in B \;\&\; w \in A_k$$

$$\therefore\; f|_{A_k}^{-1}(B) = \underbrace{A_k}_{\in\mathcal{F}_1} \cap \underbrace{f^{-1}(B)}_{\in\mathcal{F}_1} \in \mathcal{F}_1 \cap A_k$$

$\Longleftarrow$: suppose $f|_{A_k}$ ⓜ $(\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B) \cap \Omega_1 \\ &= f^{-1}(B) \cap \bigcup_k A_k \\ &= \bigcup_k \big(f^{-1}(B) \cap A_k\big) \\ &= \underbrace{\bigcup_k \underbrace{f|_{A_k}^{-1}(B)}_{\in \mathcal{F}_1 \cap A_k \text{ by assumption}}}_{\in\mathcal{F}_1 \text{ since } \mathcal{F}_1 \cap A_k \subset \mathcal{F}_1 \;\&\; \mathcal{F}_1 \text{ is a } \sigma\text{-field}} \end{aligned}$$

QED.

Corollary: Piecewise metric continuous functions are ⓜ if the "pieces" are countable & Borel measurable.

Thm: (just check ⓜ on the range)

If $f : \Omega_1 \to \Omega_2^{\circ} \subset \Omega_2$ ← metric space

and $\mathcal{F}_1$ is a σ-field on Ω_1 then

f ⓜ $\mathcal{F}_1/\mathcal{B}(\Omega_2) \iff f$ ⓜ $\mathcal{F}_1/\mathcal{B}(\Omega_2^{\circ})$.

Proof: The borel restriction thm says (7)

$$\mathcal{B}(\Omega_2^\circ) = \mathcal{B}(\Omega_2)\cap\Omega_2^\circ .$$

$\therefore$

$$f \text{ ⓜ } \mathcal{F}_1/\mathcal{B}(\Omega_2^\circ) \iff f \text{ ⓜ } \mathcal{F}_1/\mathcal{B}(\Omega_2)\cap\Omega_2^\circ$$

$$\iff \underbrace{f^{-1}(B\cap\Omega_2^\circ)}_{=\ f^{-1}(B)} \in \mathcal{F}_1,\ \forall B\in\mathcal{B}(\Omega_2)$$

$$\iff f^{-1}(B)\in\mathcal{F}_1,\ \forall B\in\mathcal{B}(\Omega_2)$$

since f maps into Ω_2°

$$\iff f \text{ ⓜ } \mathcal{F}_1/\mathcal{B}(\Omega_2).$$

QED.

e.g. $\sin(x) \text{ ⓜ } \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$

$\iff \sin(x) \text{ ⓜ } \mathcal{B}(\mathbb{R})/\mathcal{B}([-1,1])$

Question:

is $f(x) = \begin{cases} \frac{1}{x}, & x>0 \\ -\infty, & x=0 \\ \sin(x), & x<0 \end{cases}$ ⓜ $\mathcal{B}(\mathbb{R})/\mathcal{B}(\bar{\mathbb{R}})$

Yes since f is metric continuous on countably many measurable pieces.

(8)

Notation:

- f is "$\mathcal{F}$-measurable" if

$$\Omega_{\mathcal{F}} \xrightarrow{f \text{ ⓜ}} \bar{\mathbb{R}}_{\mathcal{B}(\bar{\mathbb{R}})}$$

indicates $f \text{ ⓜ } \mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$

- f is "Borel measurable" if

$$\mathbb{R}^d_{\mathcal{B}(\mathbb{R}^d)} \xrightarrow{f \text{ ⓜ}} \bar{\mathbb{R}}_{\mathcal{B}(\bar{\mathbb{R}})}$$

sometimes just "measurable"

- f is "Lebesgue measurable" if

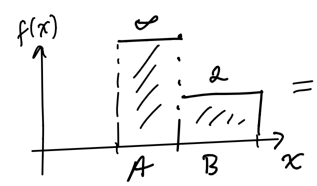
$$\mathbb{R}^d_{\overline{\mathcal{B}(\mathbb{R}^d)}} \xrightarrow{f \text{ ⓜ}} \bar{\mathbb{R}}_{\mathcal{B}(\bar{\mathbb{R}})}$$

when $\overline{\mathcal{B}(\mathbb{R}^d)}$ is the completion w.r.t. Lebesgue measure.

Convention for ∞

- $\infty + x = \infty$ when $x\in(-\infty,\infty]$
- $\infty\cdot 0 = 0$
- $\infty\cdot\infty = \infty$
- $\frac{x}{\infty} = 0$ when $x\in\mathbb{R}$
- $\frac{x}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty-\infty$ are not defined.

Note: we use the convention $\infty\cdot 0 = 0$ so



$= \infty I_A + 2 I_B$

Thm (closure thm)

Let $(\Omega, \mathcal{F})$ be a measurable space.

(i) if $\Omega \underset{g\,(m)}{\overset{f\,(m)}{\rightrightarrows}} \overline{\mathbb{R}}$ (with $\mathcal{F}$ on Ω and $\mathcal{B}(\overline{\mathbb{R}})$ on $\overline{\mathbb{R}}$) then

$f+g$, $f\cdot g$, f/g, $\max(f,g)$, $\min(f,g)$, f^+, f^-, $|f|$ are all (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

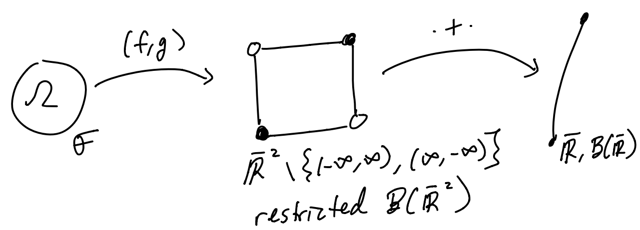
provided they are defined $\forall w \in \Omega$ ← i.e. No $\infty - \infty$, $\frac{\infty}{\infty}$ …

(ii) if $\Omega \xrightarrow{f_1\,(m)} \overline{\mathbb{R}}$, …, $\Omega \xrightarrow{f_n\,(m)} \overline{\mathbb{R}}$, … (with $\mathcal{F}$ on Ω and $\mathcal{B}(\overline{\mathbb{R}})$ on $\overline{\mathbb{R}}$)

then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Proof:

(i) Just show $f+g$ … the others are similar. Since $f(w)+g(w)$ is defined $\forall w \in \Omega$ we have



f & g are (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

$\Leftrightarrow$ (f,g) is (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}}^2)$ by "just check cords"

$\Leftrightarrow$ (f,g) is (m) on the restricted space, by "just check the range"

$+$ is (m) by metric continuity

$\therefore$ $f+g$ (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by "Composition of (m) is (m)".

To show (ii) just notice

$$\left(\sup_n f_n\right)^{-1}([-\infty, c]) = \{w : \sup_n f_n(w) \le c\}$$

$$= \bigcap_{n=1}^{\infty} \underbrace{\{w : f_n(w) \le c\}}_{\in \mathcal{F} \text{ since } f_n \text{ (m) } \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})}$$

$\in \mathcal{F}$ since $\mathcal{F}$ is a σ-field.

Note: this will not be true for $<$

$\therefore$ $\sup_n f_n$ (m) $\mathcal{F}/\{[-\infty, c] : c \in \overline{\mathbb{R}}\}$

$\therefore$ $\sup_n f_n$ (m) $\mathcal{F}/\underbrace{\sigma\langle [-\infty,c] : c \in \overline{\mathbb{R}}\rangle}_{=\mathcal{B}(\overline{\mathbb{R}})}$.

For the others

$\inf_n f_n = -\sup_n (-f_n)$

$$\limsup_{n\to\infty} f_n = \lim_{m\to\infty} \underbrace{\sup_{n \ge m} f_n}_{\text{decreases as } m\to\infty} = \underbrace{\inf_m \underbrace{\sup_{n\ge m} f_n}_{(m)}}_{(m)}$$

$\liminf_{n\to\infty} f_n$ … similar.

QED.

e.g. Coin flip model from lecture 1.

$S_n(w) = \sum_{k=1}^{n} R_k(w)$ maps $(0,1] \to \overline{\mathbb{R}}$

Since S_n is constant over intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$

we have $S_n^{-1}([-\infty, x]) =$ finite disjoint union of dyatics $\in \mathcal{B}((0,1])$

By "generators are enough"

$(0,1] \xrightarrow{S_n\,(m)} \overline{\mathbb{R}}$ (with $\mathcal{B}((0,1])$ and $\mathcal{B}(\overline{\mathbb{R}})$) $\forall n$

$\therefore$ $\limsup_n \frac{S_n}{\sqrt{2n \log\log n}}$ is (m) $\mathcal{B}((0,1])/\mathcal{B}(\overline{\mathbb{R}})$

$\therefore$ $\left\{\limsup_n \frac{S_n}{\sqrt{2n\log\log n}} = 1\right\} \in \mathcal{B}((0,1])$ since it is the pre-image of $\{1\} \in \mathcal{B}(\overline{\mathbb{R}})$.

Random variables, induced measures and c.d.f's

Def: X is a <u>random variable</u> if there exists a probability space $(\Omega, \mathcal{F}, P)$ where $X: \Omega \to \mathbb{R}$ s.t. $X \ (m)\ \mathcal{F}/\mathcal{B}(\mathbb{R})$.

X is an <u>extended random variable</u> if $X: \Omega \to \overline{\mathbb{R}}$ & $X \ (m)\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

<u>Picture</u>:

$$\underset{\mathcal{F}, P}{\Omega} \xrightarrow{X\,(m)} \underset{\mathcal{B}(\mathbb{R}).}{\mathbb{R}}$$

we write $X(\omega)$ instead of $f(\omega)$ to indicate $(\Omega, \mathcal{F})$ has a probability measure attached. Think of P as modeling a random draw $\omega \in \Omega$ & $X(\omega)$ as a "variable" or "label" associated with each $\omega \in \Omega$.

Since $X \ (m)\ \mathcal{F}/\mathcal{B}(\mathbb{R})$ it makes sense to talk about quantities like:

$$P(X=1) = P(\underbrace{X^{-1}(\underbrace{\{1\}}_{\in \mathcal{B}(\mathbb{R})})}_{\in \mathcal{F}})$$

$$P(X \le x) = P(\underbrace{X^{-1}((-\infty, x])}_{\in \mathcal{F}})$$

$$P(X \in \mathbb{Q}) = P(X^{-1}(\mathbb{Q})).$$

Def: If X is a random variable defined on $(\Omega, \mathcal{F}, P)$, the <u>distribution</u> of X (also called the <u>induced probability measure</u>) is a set function $PX^{-1}: \mathcal{B}(\mathbb{R}) \to [0,1]$ given by

$$PX^{-1}(B) := P(X^{-1}(B)) = P(X \in B).$$

More generally if

$$\underset{\mathcal{F}_1, \mu}{\Omega_1} \xrightarrow{f\,(m)} \underset{\mathcal{F}_2}{\Omega_2}$$

where $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-fields & $(\Omega_1, \mathcal{F}_1, \mu)$ is a measure then

$$\mu f^{-1}(F) := \mu(f^{-1}(F)), \ \forall F \in \mathcal{F}_2$$

<u>induced measure</u> on $(\Omega_2, \mathcal{F}_2)$.

Thm: In the setup above μf^{-1} is a measure on $(\Omega_2, \mathcal{F}_2)$

<u>Proof</u>:
Follows immediately since pre-images & set operations commute. QED

(13)

Since $\mu f^{-1}(\Omega_2) = \mu(\Omega_1)$ we have the following facts:

- $\mu(\Omega_1) = 1 \Rightarrow \mu f^{-1}(\Omega_2) = 1$

  i.e. PX^{-1} is a probability measure

- $\mu(\Omega_1) < \infty \Rightarrow \mu f^{-1}(\Omega_2) < \infty$

- Warning:

  μ is a σ-finite measure

  $\not\Rightarrow \mu f^{-1}$ is a σ-finite measure

e.g.

$$\underset{\mathcal{B}(\mathbb{R}),\, \mathcal{L}^1}{\mathbb{R}} \xrightarrow{f \equiv 0} \Omega_2 = \{0\}, \quad \mathcal{F}_2 = \{\phi, \Omega_2\}$$

f is $\mathcal{B}(\mathbb{R})/\mathcal{F}_2$ measurable and $\mathcal{L}^1$ is σ-finite but $\mathcal{L}^1 f^{-1}$ is not σ-finite since

$$\mathcal{L}^1 f^{-1}(\phi) = 0 \text{ and}$$

$$\mathcal{L}^1 f^{-1}(\Omega_2) = \mathcal{L}^1(\mathbb{R}) = \infty.$$

Notice that even if two r.v.s X & Y are defined on different probability spaces the induced distributions are both on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Def: If X & Y are two r.v.s s.t.

$$\underset{\mathcal{F}_1, P}{\Omega_1} \xrightarrow{X} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \quad \& \quad \underset{\mathcal{F}_2, Q}{\Omega_2} \xrightarrow{Y} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}}$$

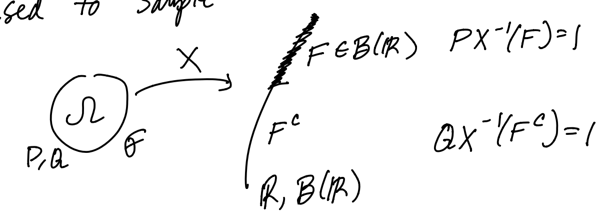
we write $X \sim Y$ or $X \overset{\mathcal{D}}{=} Y$ if $PX^{-1} = QY^{-1}$ over $\mathcal{B}(\mathbb{R})$.

(14)

In the simple statistics setup there are two possible measures P & Q, both defined on $(\Omega, \mathcal{F})$. $\omega \in \Omega$ is picked at random either from P or Q & the "observer" only gets to see the value of $X(\omega)$ (i.e. the data).

The observer then trys to figure out which P or Q ω was sampled from.

Note: using the notation from HWK3 if $PX^{-1} \perp QX^{-1}$ then the observer will know exactly which P or Q was used to sample ω



- If $X(\omega) \in F$ then ω was drawn from P.
- If $X(\omega) \in F^c$ then ω was drawn from Q.

Def: The distribution function (sometimes called cumulative distribution function .. c.d.f) for a random variable X defined on $(\Omega, \mathcal{F}, P)$ is the function $F: \mathbb{R} \to [0,1]$ defined as

$$F(t) = P(X \leq t), \quad \forall t \in \mathbb{R}.$$
$$= PX^{-1}((-\infty, t])$$

(15)

Thm: If X & Y are two r.v.s with c.d.f.s F_X & F_Y, respectively, then

$$X \overset{\mathcal{D}}{=} Y \iff F_X(t) = F_Y(t) \ \forall t \in \mathbb{R}.$$

Proof:

$\Rightarrow$: trivial when taking $B = (-\infty, t]$ since $X^{-1}((-\infty,t]) = \{X \le t\}$

$\Leftarrow$:

$F_X(t) = F_Y(t) \ \forall t \in \mathbb{R}$

$\Rightarrow PX^{-1}((-\infty,t]) = QY^{-1}((-\infty,t]) \ \forall t \in \mathbb{R}.$

$\Rightarrow PX^{-1} = QY^{-1}$ on $\mathcal{P} = \{(-\infty,t] : t \in \mathbb{R}\}$

$\overset{\pi\text{-uniq}}{\Longrightarrow} PX^{-1} = QY^{-1}$ on $\sigma\langle \underbrace{\mathcal{P}}_{\pi\text{-sys}} \rangle = \mathcal{B}(\mathbb{R})$.

QED

Thm (Properties of c.d.f.s)
Let $F(t) = P(X \le t)$ for a r.v. X defined on $(\Omega, \mathcal{F}, P)$. Then

(I) $F(x) \le F(y)$, $\forall x \le y$

(II) $\lim_{x \downarrow y} F(x) = F(y)$

(III) $\lim_{x \to \infty} F(x) = 1$ & $\lim_{x \to -\infty} F(x) = 0$

(16)

Proof:

(I): clear since $A \subset B \Rightarrow PX^{-1}(A) \le PX^{-1}(B)$

(II):

$\{X \le x\} \downarrow \{X \le y\}$ as $x \downarrow y$

which follows since if $x_n \downarrow y$ as $n \to \infty$ then clearly $\{X \le y\} \subset \bigcap_{n=1}^{\infty} \{X \le x_n\}$ & $w \in \bigcap_{n=1}^{\infty} \{X \le x_n\} \Rightarrow X(w) \le x_n \ \forall n$
$\Rightarrow X(w) \le \lim_n x_n = y$

$\therefore P(X \le x) \downarrow P(X \le y)$

warning: It is not true that $\{X < x\} \downarrow \{X < y\}$ as $x \downarrow y$
since $X(w) < x_n \Rightarrow X(w) \le \lim x_n = y$, $X(w) < x_n \not\Rightarrow X(w) < \lim x_n = y$
e.g. take $X(w) = 0$, $x_n = \frac{1}{n}$, $y = 0$

(III): Follows since

$\{X \le x\} \uparrow \mathbb{R}$ & $\{X \le x\} \downarrow \emptyset$.

QED.

It turns out properties (I), (II), (III) are are characterizing properties of c.d.f.s i.e. if $F: \mathbb{R} \to [0,1]$ satisfies (I), (II), (III) then $\exists$ a r.v. X s.t. $F(t) = P(X \le t)$.
In fact, this will be the main tool we use to show the existance of an infinite sequence of indep. r.v.s all with a specified distribution.

Def: if $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) of the above thm define

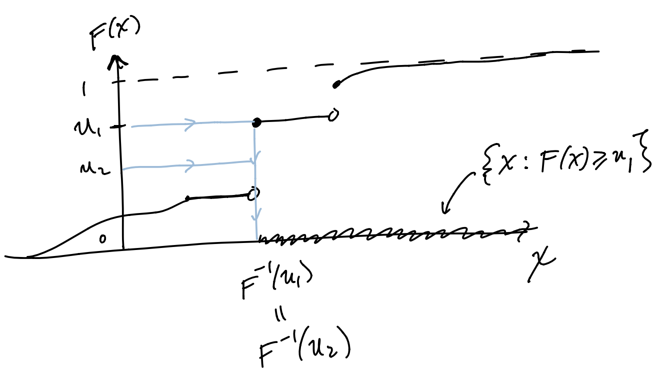
$$F^{-1}(u) := \inf\{x\in\mathbb{R} : F(x)\geq u\}$$

often called the inverse c.d.f or the quantile function.

for $u\in(0,1)$.

Note: $F^{-1}:(0,1)\to\mathbb{R}$ is well defined since $\{x: F(x)\geq u\}\neq\emptyset$ when $0<u<1$ by (III) & (I).

Picture:



There are two important facts about F & F^{-1} which are useful to remember.

Suppose $\mathcal{U}$ is a r.v. which is uniform on $(0,1)$, i.e. $\exists\,(\Omega,\mathcal{F},P)$ & $\underset{\mathcal{F},P}{\Omega} \xrightarrow{\mathcal{U}\,(m)} \underset{\mathcal{B}((0,1))}{(0,1)}$

s.t. $P(\mathcal{U}\in B) = \mathcal{L}'(B)$.

For example, digit coin flip model:

$\Omega=(0,1]$, $\mathcal{F}=\mathcal{B}((0,1])$, $P=\mathcal{L}'\big|_{(0,1]}$

where

$$\mathcal{U}(w)=\begin{cases} w & \text{if } w<1\\ \tfrac12 & \text{if } w=1\end{cases}$$

is (m) by cut & paste &

$P(\mathcal{U}\in B) = P(B) = \mathcal{L}'(B)$

since $P(\mathcal{U}=\tfrac12)=0$

Now the two important facts are ... if F is the c.d.f of a r.v. X then

(a) $F^{-1}(\mathcal{U}) \overset{\mathcal{D}}{=} X$

(b) $F(X) \overset{\mathcal{D}}{=} \mathcal{U}$ when F is continuous.

Lemma (switching formula):

if $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) above then

$(*)\quad F(x)\geq u \iff x\geq F^{-1}(u)$

$\forall u\in(0,1)$ & $x\in\mathbb{R}$.

Proof:

A useful restatement of $(*)$ is simply that

$(**)\quad \{x: F(x)\geq u\} = [F^{-1}(u),\infty)$.

To show $(**)$ notice that $\{x: F(x)\geq u\}$ must be an interval of the form $(F^{-1}(u),\infty)$ or $[F^{-1}(u),\infty)$ since

$$F^{-1}(u) \overset{def}{=} \inf\{x: F(x)\geq u\}$$

and

$$x'\in\{x: F(x)\geq u\} \overset{by (I)}{\Longrightarrow} \begin{cases} x''\in\{x: F(x)\geq u\}\\ \forall x''>x'.\end{cases}$$

∴ to show $\{x: F(x)\ge u\} = [F^{-1}(u),\infty)$ just prove the inf of the LHS is attained.

Let $x_p \in \{x: F(x)\ge u\}$ s.t. $x_p \downarrow \underbrace{F^{-1}(u)}_{=\text{inf of LHS}}$

$$\therefore\quad F\left(\lim_p x_p\right) \overset{(II)}{=} \lim_p F(x_p) \ge u$$

since x_p is in $\{x: F(x)\ge u\}$

$$\therefore\quad F^{-1}(u) = \lim_p x_p \in \{x: F(x)\ge u\}$$

so the inf is attained.

(Note we used III implicity to show $F^{-1}(u)$ is well defined and a r.v.)

QED

Lemma (c.d.f sandwich)

If $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) then $\forall u\in(0,1)$

$$F(F^{-1}(u)-) \le u \le F(F^{-1}(u)).$$

Proof: $u\le F(F^{-1}(u))$ holds since $F^{-1}(u)\in\{x: F(x)\ge u\}$.

The contrapositive of the switching formula is

$$F(x)<u \iff x<F^{-1}(u)$$

$$\therefore\quad F(F^{-1}(u)-) = \lim_{\underbrace{x\uparrow F^{-1}(u)}_{\substack{\text{these satisfy } x<F^{-1}(u) \\ \text{so } F(x)<u \text{ by switch}}}} \underbrace{F(x)}_{<u} \le u$$

QED

Maybe the best way to remember these

$$\{x: F(x)\ge u\} = [F^{-1}(u),\infty)$$

$$F(F^{-1}(u)-) \le u \le \underbrace{F(F^{-1}(u)+)}_{=F(F^{-1}(u)) \text{ by (II)}}$$

Thm (c.d.f representation)

If $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) then $\exists$ a r.v. X on a prob space $(\Omega,\mathcal{F},P)$ s.t.

(1) $P(X\le x) = F(x)\quad \forall x\in\mathbb{R}$

and, moreover, any r.v. $\mathcal{U}$ which is uniformly distributed on $(0,1)$ satisfies

(2) $F^{-1}(\mathcal{U}) \overset{\mathcal{D}}{=} X$

(3) $P(F(X)\le u)\le u$ for all $u\in(0,1)$.

If, in addition, F is continuous then

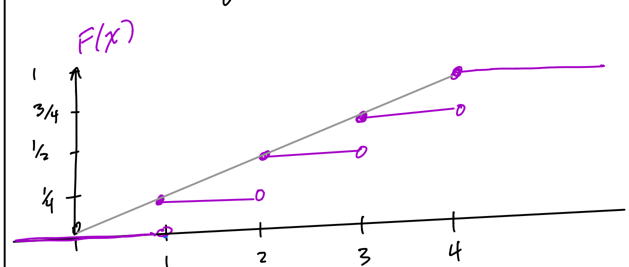
(4) $F(X) \overset{\mathcal{D}}{=} \mathcal{U}$.

Here is the picture of why we don't always get (4).

Suppose $P(X=i)=\frac{1}{4}$ for $i=1,2,3,4$

So X is uniformly distributed on $\{1,2,\dots,4\}$

The c.d.f of X is



∴ $F(X)$ assigns $\frac{1}{4}$ prob to $\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$.

i.e. $P(F(X)\le 0.25) = 0.25$

but $P(F(X)\le 0.3) = 0.25 \le 0.3$

Proof:

First note that F^{-1} is $\mathcal{C}^m$ by monotonicity.

Let $\mathcal{U}$ be a r.v. uniform on $(0,1)$.

$$\therefore \; P(F^{-1}(\mathcal{U}) \le x) = P(\mathcal{U} \le F(x))$$

by switching lemma

$$= \mathcal{J}'((0, F(x))) \quad \mathcal{U} \text{ is uniform}$$

$$= F(x)$$

Now set $X := F^{-1}(\mathcal{U})$ to get (1) & (2) using π-uniqness.

For (3),

$$P(F(X) \le u) = P(F(F^{-1}(\mathcal{U})) \le u)$$

since $X \overset{d}{=} F^{-1}(\mathcal{U})$

$$\le P(\mathcal{U} \le u)$$

since $\mathcal{U} \le F(F^{-1}(\mathcal{U})) \le u$

by c.d.f sandwich

$$= u, \quad \forall u \in (0,1)$$

If F is continuous c.d.f sandwich gives $F(F^{-1}(\mathcal{U})-) = \mathcal{U} = F(F^{-1}(\mathcal{U}))$ so

$$P(F(X) \le u) = P(\mathcal{U} \le u)$$

which shows (4) QED