Lecture 16: Brownian motion and Brownian brigde

In lecture 13 we discussed

The scaled an linearly interpolated random walk for $t \in [0,1]$

$$W_t^n(\omega) := \frac{1}{\sqrt{n}}\underbrace{\left((1-\alpha_t^n)\, S_{\lfloor nt\rfloor} + \alpha_t^n S_{\lceil nt\rceil}\right)}$$

Scale the y-axis by $\frac{1}{\sqrt{n}}$

Scale the x-axis by $\frac{t}{n}$ and interpolate

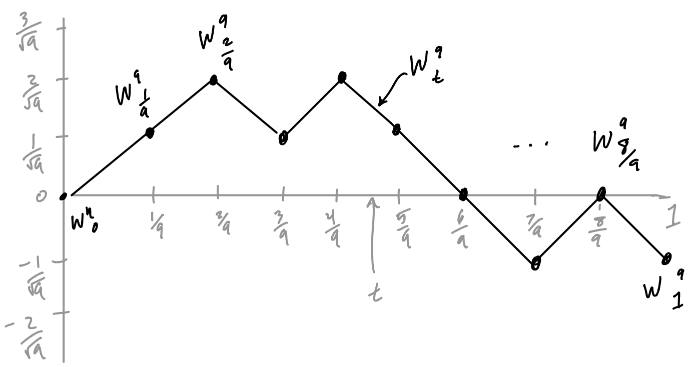
where $S_k := R_1 + \cdots + R_k$ &

$$R_1, R_2, \ldots \overset{iid}{\sim} \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$$

α_t^n = distance to nearest left int $\lfloor nt \rfloor$

$= nt - \lfloor nt \rfloor$

$$\therefore \quad W_t^n = \frac{1}{\sqrt{n}} S_{\lfloor nt\rfloor} + \frac{nt - \lfloor nt\rfloor}{\sqrt{n}} R_{\lceil nt\rceil}$$



Theorem: (f.d.d converge)

For any $0 \le t_1 < t_2 < \cdots < t_m \le 1$

$$(W_{t_1}^n, \ldots, W_{t_m}^n) \xrightarrow{\mathcal{D}} N(0, \Sigma) \quad \text{as } n \to \infty$$

where $\Sigma_{ij} = t_i \wedge t_j$

Proof:

First notice that the linear interpolation term $\frac{nt - \lfloor nt\rfloor}{\sqrt{n}} R_{\lceil nt\rceil} \xrightarrow{P} 0$ for any term.

$\therefore$ By slutsky it will be sufficient to show

$$\frac{1}{\sqrt{n}}\left(S_{\lfloor nt_1\rfloor}, \ldots, S_{\lfloor nt_m\rfloor}\right) \xrightarrow{\mathcal{D}} N(0, \Sigma).$$

Notice that

$$\left.\begin{aligned} S_{\lfloor nt_1\rfloor} &= \sum_{k=0}^{\lfloor nt_1\rfloor} R_k \\ S_{\lfloor nt_2\rfloor} - S_{\lfloor nt_1\rfloor} &= \sum_{k=\lfloor nt_1\rfloor+1}^{\lfloor nt_2\rfloor} R_k \\ &\vdots \\ S_{\lfloor nt_m\rfloor} - S_{\lfloor nt_{m-1}\rfloor} &= \sum_{k=\lfloor nt_{m-1}\rfloor+1}^{\lfloor nt_m\rfloor} R_k \end{aligned}\right\} \text{indep sums}$$

Since $\sqrt{\frac{\lfloor nt_k\rfloor - \lfloor nt_{k-1}\rfloor}{\sqrt{n}}} \xrightarrow{n\to\infty} \sqrt{t_k - t_{k-1}}$

Slutsky & the CLT implies

$$\left(W_{t_1}^n, W_{t_2}^n - W_{t_1}^n, \ldots, W_{t_m}^n - W_{t_{m-1}}^n\right) \xrightarrow{\mathcal{D}} N\left(0, \begin{pmatrix} t_1 & & 0 \\ & (t_2 - t_1) & \\ 0 & & \ddots \; (t_m - t_{m-1}) \end{pmatrix}\right)$$

The continuous mapping Theorem then gives

$$(W_{t_1}^n, W_{t_2}^n, \ldots, W_{t_m}^n) \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

where $\Sigma_{ij} = t_i \wedge t_j$

QED

The last thing to show is that the sequence $\{W^n\}_{n=1}^{\infty}$ is tight for the existance of a limit measure.

We use Billingsley's sufficient condition given on page 88 of his weak convergence ~~book~~ (the proof isn't difficult, just tedious)

Lemma:

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \lambda^2 P\left(\max_{k\le n} |S_k| \ge \lambda\sqrt{n}\right) = 0$$

$$\Rightarrow \{W^n\}_{n=1}^{\infty} \text{ is tight}$$

Thm: (Brownian Motion)

Let W_t^n be defined as above. Then there exists a stochastic process W on $(C[0,1], \mathcal{B}(C[0,1]))$, with unique induced distribution, satisfying

(i) $W^n \xrightarrow{\mathcal{D}} W$

(ii) $(W_{t_1}, \dots, W_{t_m}) \sim N\left(0, (t_i \wedge t_j)_{ij=1}^{n}\right)$

$$\forall t_1, \dots, t_m \in [0,1] \quad \forall m \in \mathbb{N}$$

Proof:

By Etemadi's maximal inequality from Lecture 11 we have

$$P\left(\max_{1\le k\le n} |S_k| \ge 3\lambda\sqrt{n}\right) \le 3 \max_{1\le k\le n} P\left(|S_k| \ge \lambda\sqrt{n}\right)$$

$$\le 3 \max_{1\le k\le n} P\left(\left|\frac{S_k}{k}\right| \ge \frac{\lambda\sqrt{n}}{k}\right)$$

Now we use the fact that the R_i's are bdd so Hoeffding's ineq:

$$P\left(\left|\frac{S_k}{k}\right| \ge \varepsilon\right) \le 2\exp\left(-\frac{2k\varepsilon^2}{(b-a)^2}\right)$$

applies to give

$$P\left(\left|\frac{S_k}{k}\right| \ge \frac{\lambda\sqrt{n}}{k}\right) \le 2\exp\left(-\frac{2}{4}k\left(\frac{\lambda\sqrt{n}}{k}\right)^2\right)$$

$$= 2\exp\left(-\frac{1}{2}\lambda^2\frac{n}{k}\right)$$

$$\le 2\exp\left(-\lambda^2/2\right) \quad \text{when } 1\le k\le n$$

$$\therefore \lim_{\lambda\to\infty} \limsup_{n\to\infty} \underbrace{\lambda^2 P\left(\max_{k\le n} |S_k| \ge 3\lambda\sqrt{n}\right)}_{\le 6\lambda^2 \exp(-\lambda^2/2)} = 0$$

$\therefore$ $\{W^n\}_{n\ge 1}^{\infty}$ is tight by prev lemma.

Prohorov's Thm implies there exists a sub seq n_k and a measure P on $(C[0,1], \mathcal{B}(C[0,1]))$ s.t.

$$\mathcal{L}(W^{n_k}) \xrightarrow{\mathcal{D}} P \quad \text{as } k\to\infty$$

Let W_t denote a stochastic process with induced distribution P.

$$\therefore W^{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} W$$

$$\therefore (W_{t_1}^{n_k}, \dots, W_{t_m}^{n_k}) \xrightarrow{\mathcal{D}} (W_{t_1}, \dots, W_{t_m})$$ since finite projections are continuous

by the CLT $\downarrow \mathcal{D}$

$$N(0, (t_i \wedge t_j)_{ij}) \quad \text{//}$$

by uniqness

$\therefore$ W satisfies (ii)

To show (i) we use the sub-sub-seq check: $\forall n_k \; \exists n_{k_j}$ s.t. $W^{n_{k_j}} \xrightarrow[j\to\infty]{\mathcal{D}} W$.

Let n_k be arbitrary. By Prohorov's thm $\exists W'$ (possibly different from W at this point)

s.t. $\quad W^{n_{k_j}} \xrightarrow[j\to\infty]{\mathcal{D}} W'$

By the CLT we already know that for any finite dimensional projection $\pi \equiv \pi_{t_1,\dots,t_m}$

$$\begin{array}{ccc} \pi(W^{n_{k_j}}) & \xrightarrow{\mathcal{D}} & \pi(W') \\ {\scriptstyle \text{by CLT}} \downarrow \mathcal{D} & & \\ N(0, (t_i \wedge t_j)_{i,j=1}^m) & & \\ {\scriptstyle \text{by (ii)}} \; \| \mathcal{D} & & \\ \pi(W) & & \end{array}$$

, by continuous mapping

$\therefore \quad \pi(W) \overset{\mathcal{D}}{=} \pi(W')$ for all $\pi = \pi_{t_1,\dots,t_m}$ by uniqueness of limits

$\therefore \quad W \overset{\mathcal{D}}{=} W'$ since

$$\underbrace{\sigma\langle \pi_{t_1,\dots,t_m} : t_i \in [0,1]\rangle}_{\pi\text{-system}} = \mathcal{B}(C[0,1])$$

$\therefore \quad W^{n_{k_j}} \xrightarrow[j\to\infty]{\mathcal{D}} W$ and since n_k was arb $W^n \xrightarrow[n\to\infty]{\mathcal{D}} W$.

QED

Remark: Notice the proof works for any iid $R_1, R_2, \dots$ s.t.

$$E(R_i) = 0$$
$$E(R_i^2) = 1$$
$$R_i \text{ is bounded}$$

The last condition isn't required but makes the maximal inequality easy with Hoeffding

The above result, for general R_i, is sometimes referred to as the "Functional CLT".

Example:

Lets see an example of something the functional CLT can do that the finite dimensional CLT can't.

The goal is to derive the asyptotic distribution of

$$\frac{1}{\sqrt{n}} \max_{1\le k\le n} S_k = \max_{1\le k\le n} W^n_{\frac{k}{n}}$$

$$= \sup_{t\in[0,1]} W^n_t, \quad \text{since } W^n \text{ is linear btwn } \tfrac{k-1}{n} \text{ \& } \tfrac{k}{n}$$

Notice that the map

$$f \in C[0,1] \longmapsto \sup_{t\in[0,1]} f(t) \in \mathbb{R}$$

is obviously continuous w.r.t. the sup-norm metric on $C[0,1]$.

$\therefore$ The continuous mapping theorem implies

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} S_k = \sup_{t \in [0,1]} W_t^n \xrightarrow[n \to \infty]{\mathcal{D}} \sup_{t \in [0,1]} W_t$$

which you can then use stopping time theory to show

$$\sup_{t \in [0,1]} W_t \overset{\mathcal{D}}{=} |Z| \quad \text{where } Z \sim N(0,1)$$

so that

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} S_k \xrightarrow[n \to \infty]{\mathcal{D}} |Z|.$$

Exending W_t to $[0,\infty)$

In the previous section we constructed a stochastic process W_t using the independent r.v.s $R_1, R_2, \dots$

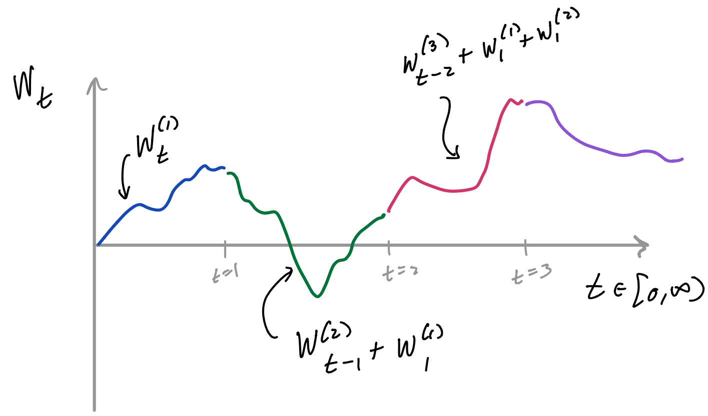
By re-arranging the R_k's in a triangular sequence

$$\text{all indep} \left\{ \begin{array}{llll} R_{11} & R_{12} & R_{13} & \cdots \\ R_{21} & R_{22} & R_{23} & \cdots \\ R_{31} & R_{32} & R_{33} & \cdots \\ \vdots & & & \end{array} \right.$$

we can establish the existence of an infinite sequence of independent copies of W_t:

$$W_t^{(1)}, \; W_t^{(2)}, \; W_t^{(3)} \; \dots$$

Stiching these together gives us a new stochastic process, also denoted W_t:



This new process satisfies

(i) W_t takes values in $C[0,\infty)$

(ii) $W_0 = 0$

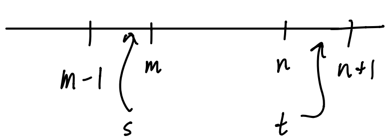
(iii) For any $0 \le t_1 < t_2 < \cdots < t_m < \infty$, $m \in \mathbb{N}$

$$(W_{t_1}, \dots, W_{t_m}) \sim N\left(0, \, (t_i \wedge t_j)_{i,j=1}^m\right)$$

The fact that $\operatorname{cov}(W_t, W_s) = t \wedge s$ is easy to verify by direct computation.

For example if $n \le t \le n+1$ then

$$\begin{aligned} \operatorname{var}(W_t) &= \operatorname{var}\left(W_{t-n}^{(n+1)} + W_1^{(n)} + \cdots + W_1^{(1)} \right) \\ &= \operatorname{var}\left(W_{t-n}^{(n+1)}\right) + \operatorname{var}\left(W_1^{(n)}\right) + \cdots + \operatorname{var}\left(W_1^{(1)}\right) \\ &= t - n + \underbrace{1 + \cdots + 1}_{n\text{-terms}} \\ &= t \end{aligned}$$

and if  then

$$\mathrm{var}(W_t - W_s)$$

$$= \mathrm{var}\Big[(W_t - W_n) + (W_n - W_{n-1}) + \cdots + (W_{m+1} - W_m) + (W_m - W_s)\Big]$$

$$= \mathrm{var}(W_t - W_n) \leftarrow (t-n)$$

$$+ \mathrm{var}(W_n - W_{n-1}) \leftarrow 1$$

$$\vdots \qquad n-m \text{ terms}$$

$$+ \mathrm{var}(W_{m+1} - W_m) \leftarrow 1$$

$$+ \mathrm{var}(W_m - W_s) \leftarrow m-s$$

$$= (t-s)$$

Now

$$\underbrace{\mathrm{var}(W_t - W_s)}_{t-s} = \underbrace{\mathrm{var}(W_t)}_{t} + \underbrace{\mathrm{var}(W_s)}_{s} - 2\,\mathrm{cov}(W_t, W_s)$$

$$\therefore \mathrm{cov}(W_t, W_s) = \frac{1}{2}\big(t+s-(t-s)\big)$$

$$= s \wedge t.$$

This stochastic process $(W_t : t \in [0,\infty))$ is called Brownian Motion.

Proposition 1 (Increments)

W_t has stationary & independent increments, i.e.

(a) If $t > s$ then

$$\mathrm{var}(W_t - W_s) = t - s$$

(b) $W_{t+s} - W_t \overset{\mathcal{D}}{=} W_s - W_0 \overset{\mathcal{D}}{=} N(0,s)$

(c) $W_t - W_s$ is independent of $W_u - W_v$ whenever $t > s \geq u > v$

Proof: This follows directly from $\mathrm{cov}(W_t, W_s) = t \wedge s$. QED.

Proposition 2: (non-differentiability)

For P-a.e. $w \in \Omega$, the function $t \mapsto W_t(w)$ is nowhere differentiable.

Proof:

It is sufficient to show that for each $s \in [0,\infty)$

$$D_s := \{w : W_\cdot(w) \text{ is differentiable at some } t < s\}$$

is contained an $A \in \mathcal{F}$ with $P(A) = 0$.

Consider D_1. Suppose $W_\cdot(w)$ is differentiable at some $t < 1$ (i.e. $w \in D_1$). Then $\exists\, c \geq 0$ & $\delta > 0$ s.t.

$$|W_u - W_t| \leq c(u-t)$$

$$\forall\ u \in (t, t+\delta)$$

Indeed whenever $u, v \in (t, t+\delta)$

$$\underbrace{|W_u - W_v|}_{\pm W_t} \le |W_u - W_t| + |W_v - W_t| \le c(u-t) + c(v-t)$$

Now for large enough n $\exists k \in \mathbb{N}$ s.t.

$$\underbrace{t \;\; \tfrac{k}{n}}_{\le \frac{1}{n}} \;\; \tfrac{k+1}{n} \;\; \tfrac{k+2}{n} \;\; \tfrac{k+3}{n} \;\; t+\delta$$

and hence

$$\left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right| \le c\underbrace{\left(\tfrac{k+1}{n} - t\right)}_{\le \frac{2}{n}} + c\underbrace{\left(\tfrac{k}{n} - t\right)}_{\le \frac{1}{n}} \le \frac{8c}{n}$$

$$\left|W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}\right| \le \frac{8c}{n}$$

$$\left|W_{\frac{k+3}{n}} - W_{\frac{k+2}{n}}\right| \le \frac{8c}{n}$$

Let $\Delta_{k,n} := W_{\frac{k+1}{n}} - W_{\frac{k}{n}}$ and notice that even though δ & c depend on $w \in \Omega$ there will always exist a $C \in \mathbb{Q}^+$ & $N \equiv N(w)$ s.t. $\forall n \ge N(w)$ $\exists k \le n-1$

$$\max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n}$$

In particular

$w \in D_1 \Rightarrow$

$$w \in \underbrace{\bigcup_{C \in \mathbb{Q}^+} \left\{ \bigcup_{k=0}^{n-1} \left\{ \max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n} \right\} \text{ a.a. } n \right\}}_{\text{measurable set}}$$

Now for any $c \in \mathbb{Q}^+$, Fatou gives

$$P\left(\bigcup_{k=0}^{n-1} \left\{ \max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n} \right\} \text{ a.a. } n \right)$$

$$\le \liminf_{n\to\infty} P\left(\bigcup_{k=0}^{n-1} \left\{ \max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n} \right\} \right)$$

$$\le \liminf_{n\to\infty} \sum_{k=0}^{n-1} P\left[\max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n} \right]$$

$$\le \liminf_{n\to\infty} \sum_{k=0}^{n-1} P\left(|\Delta_{k,n}| \le \frac{1}{n}, |\Delta_{k+1,n}| \le \frac{1}{n}, |\Delta_{k+2,n}| \le \frac{C}{n} \right)$$

but these are indep increments with variance $\frac{1}{n}$

since $\sqrt{n}\Delta_{k,n}, \sqrt{n}\Delta_{k+1,n}, \sqrt{n}\Delta_{k+2,n} \overset{iid}{\sim} N(0,1)$ we have that $\forall c \in \mathbb{Q}^+$

$$\liminf_{n\to\infty} \underbrace{\sum_{k=0}^{n-1} \Big[\underbrace{P\left(|Z| \le \tfrac{C}{\sqrt{n}}\right)}_{o\left(\frac{1}{\sqrt{n}}\right)} \Big]^3}_{o\left(n \frac{1}{n^{3/2}}\right)} = 0$$

$\therefore$ D_1 is covered by

$$\bigcup_{C \in \mathbb{Q}^+} \underbrace{\left\{ \bigcup_{k=0}^{n-1} \left\{ \max\left(|\Delta_{k,n}|, |\Delta_{k+1,n}|, |\Delta_{k+2,n}|\right) \le \frac{C}{n} \right\} \text{ a.a. } n \right\}}_{\text{each of these is } P\text{-null.}}$$

$\therefore$ D_1 is P-neg.

Similar arguments show D_n is P-null $\forall n \in \mathbb{N}$.

QED.

Up to this point we know $W_t(\omega)$ is continuous in t for all $\omega \in \Omega$ and have characterized the f.d.d of W_t.

However to classify W_t as inducing a measure on $C[0,\infty)$ we need a σ-field on $C[0,\infty)$, preferably given by a Polish metric.

For $f, g \in C[0,\infty)$ define

$$d_n(f,g) := \sup_{t\in[0,n]} |f(t)-g(t)|$$

and set

$$d(f,g) := \sum_{n=1}^{\infty} \frac{d_n(f,g)\wedge 1}{2^n}$$

Now with this metric $C[0,\infty)$ is Polish and our process W_t satisfies

$$\underset{\mathcal{F},P}{\Omega} \xrightarrow{(W_t\,:\,t\in[0,\infty))\ (m)} \underset{\mathcal{B}(C[0,\infty))}{C[0,\infty)}$$

Moreover all our old results carry over. In particular if W_t & W_t' are stochastic processes takeing values in $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ s.t.

$$(W_{t_1},\dots,W_{t_n}) \overset{\mathcal{D}}{=} (W'_{t_1},\dots,W'_{t_n}) \quad \forall t_1,\dots,t_m, m$$

then $(W_t : t\in[0,\infty)) \overset{\mathcal{D}}{=} (W'_t : t\in[0,\infty))$.

See Klenke for details.

Proposition 3 (Scaling & time reversal)

- $(W_t : t\in[0,\infty)) \overset{\mathcal{D}}{=} \left(\frac{1}{\sqrt{c}}W_{ct} : t\in[0,1)\right)$ for any $c>0$.
- If $W_t^* := \begin{cases} tW_{1/t} & \text{if } t>0 \\ 0 & \text{if } t=0 \end{cases}$

  then $(W_t : t\in[0,\infty)) \overset{\mathcal{D}}{=} (W_t^* : t\in[0,\infty))$

Proof: All that needs to be checked is that

$$\operatorname{cov}\left(\frac{1}{\sqrt{c}}W_{ct}, \frac{1}{\sqrt{c}}W_{ct}\right) = t\wedge s \quad \text{and}$$

$$\operatorname{cov}(tW_{1/t}, sW_{1/s}) = t\wedge s$$

which is a simple exercise.

QED.

When solving a problem about W_t it often helps to see how scaling and time reversal might effect the answer.

Example:

Let $M_t = \max_{0\le s\le t} W_s$

Notice that for any $c>0$

$$M_t \overset{\mathcal{D}}{=} \max_{0\le s\le t} \frac{1}{\sqrt{c}} W_{cs}$$

$$= \frac{1}{\sqrt{c}} \max_{0\le s\le ct} W_s$$

$$= \frac{1}{\sqrt{c}} M_{ct}$$

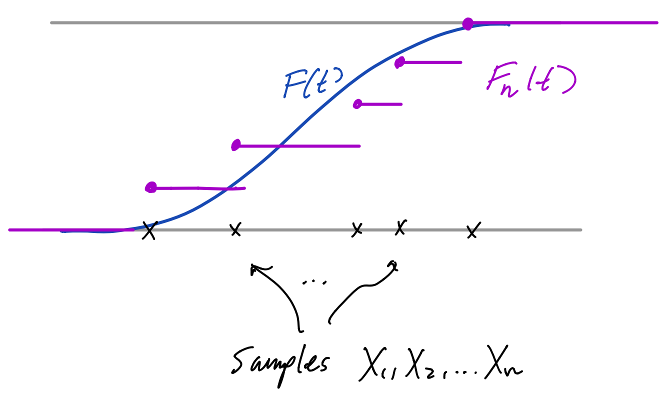
So M_t has the same scaling property as W_t.

Brownian Bridge

Let X be a r.v. with c.d.f $F(t)$.

Given iid copies of X: $X_1, X_2, \dots, X_n$ construct an estimate of $F(t)$ as follows

$$F_n(t) = \frac{1}{n}\sum_{k=1}^{n} I_{\{X_k \le t\}}$$



In Lecture 11 we proved the Glivenko-Cantelli Thm which says:

$$\sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \xrightarrow{P\text{-a.e.}} 0 \quad \text{as } n \to \infty$$

Notice that if we re-scale by $\sqrt{n}$ we get

$$\sqrt{n}\left(F_n(t) - F(t)\right) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n}\underbrace{\left(I_{\{X_k \le t\}} - F(t)\right)}_{:=Y_{kt}}\right)$$

where $Y_{1t}, Y_{2t}, \dots$ are iid r.v.s with

$$E(Y_{k,t}) = E\left(I_{\{X_k \le t\}} - F(t)\right)$$
$$= P(X_k \le t) - F(t) = 0$$

and $\operatorname{cov}(Y_{k,t}, Y_{k,s}) = E\left(I_{\{X_k \le t\}} - F(t)\right)\left(I_{\{X_k \le s\}} - F(s)\right)$

$$= F(t \wedge s) - F(t)F(s)$$

by monotonicity $= F(t) \wedge F(s) - F(t)F(s)$

Now by the CLT for any $t_1, \dots, t_m$

$$\begin{bmatrix} \sqrt{n}\left(F_n(t_1) - F(t_1)\right) \\ \vdots \\ \sqrt{n}\left(F_n(t_m) - F(t_m)\right) \end{bmatrix} = \sqrt{n}\sum_{k=1}^{n}\begin{bmatrix} Y_{kt_1} \\ \vdots \\ Y_{kt_m} \end{bmatrix} \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

where $\Sigma_{ij} = F(t_i) \wedge F(t_j) - F(t_i)F(t_j)$.

It is natural to ask if one has functional convergence here, i.e. if there exists a process $(B_t : t \in \mathbb{R})$ for which

$$F^n \xrightarrow{\mathcal{D}} B$$

where $F^n_t := \sqrt{n}\left(F_n(t) - F(t)\right)$.

The problem is that the sample paths of F^n are not continuous so we can't use our old theory. To get a functional limit theorem one must use the space of functions which are right-continuous & have left limits, the so called cadlag functions.

Working in this extended space the functional limit theorem holds, often called Donsker's theorem, and gives the existance of a process B_t s.t.

(i) $F^n \xrightarrow{\mathcal{D}} B$

(ii) $\forall t_1, \dots, t_m \in \mathbb{R}$

$$(B_{t_1}, \dots, B_{t_m}) \sim N\left(0, \left[F(t_i) \wedge F(t_j) - F(t_i)F(t_j)\right]_{i,j=1}^{m}\right)$$

For $t, s \in [0,1]$ notice that

$$\operatorname{cov}(W_t - tW_1, W_s - sW_1)$$
$$= t \wedge s - ts - ts + ts$$
$$= t \wedge s - ts$$

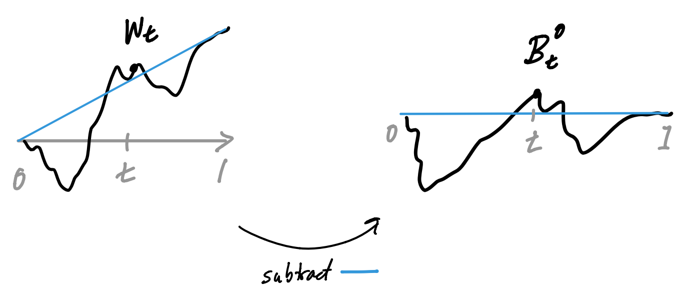
Let $B^0_t := W_t - tW_1$ and notice

$$\operatorname{cov}\left(B^0_{F(t)}, B^0_{F(s)}\right) = F(t) \wedge F(s) - F(t)F(s)$$

& $E B^0_{F(t)} = 0$. Therefore by Gaussianity of f.d.d we have

$$\left(B^0_{F(t)} : t \in \mathbb{R}\right) \overset{\mathcal{D}}{=} \underbrace{\left(B_t : t \in \mathbb{R}\right)}_{\substack{\text{the limiting dist} \\ \text{of } \sqrt{n}(F_n(t) - F(t)).}}$$

$\left(B^0_t : t \in [0,1]\right)$ is called a Brownian Bridge.



Also notice that

$$\sup_{t \in \mathbb{R}} B^0_{F(t)} = \sup_{t \in [0,1]} B^0_t \overset{\mathcal{D}}{=} \sup_{t \in [0,1]} \left(W_t - tW_1\right).$$

and by the functional CLT results

$$\sup_{t \in \mathbb{R}} \sqrt{n}\left(F_n(t) - F(t)\right) \xrightarrow{\mathcal{D}} \sup_{t \in [0,1]} B^0_t$$

This explains the Kolmogorov-Smirnov method of testing a given $F(t)$ whether or not

$$X_1, \ldots, X_n \overset{iid}{\sim} F(t) \quad (*)$$

If $\sup_{t \in [0,\infty]} \sqrt{n}\left(F_n(t) - F(t)\right)$ is sufficiently rare as a sample of $\sup_{t \in [0,1]} B^0_t$ then start questioning $(*)$.