# Lebesque measure on $\mathbb{R}^d$

## Contents



# 1 The Borel field $\mathcal{B}_0^{(0,1]^d}$

**Definition 1.** *The Borel field on $(0,1]^d$, denoted $\mathcal{B}_0^{(0,1]^d}$, is defined as the field generated by the rectangles in $(0,1]^d$ as follows*

$$\mathcal{B}_0^{(0,1]^d} := f\Big\langle (a_1,b_1]\times\cdots\times(a_d,b_d] : 0\le a_k < b_k \le 1\Big\rangle.$$

The most important fact about this field is the following characterization of a general set in $\mathcal{B}_0^{(0,1]^d}$.

**Claim 1.** *Any set in $\mathcal{B}_0^{(0,1]^d}$ is a finite (possibly empty) disjoint union of rectangles from $\{(a_1,b_1]\times\cdots\times(a_d,b_d] : 0\le a_k<b_k\le 1\}$.*

The reason this claim is so important is that it allows us to easily define the notion of $d$-dimensional volume on sets in $\mathcal{B}_0^{(0,1]}$. Then, by showing that this $d$-dimensional volume satisfies the axioms of a probability measure on $\mathcal{B}_0^{(0,1]}$ we can invoke the Carathéodory extension theorem to get a $d$-dimensional volume measurement over all subsets in the Borel $\sigma$-field $\mathcal{B}^{(0,1]^d} := \sigma\langle \mathcal{B}_0^{(0,1]^d}\rangle$ which will result in Lebesque measure on $(0,1]^d$.

# 2 The Borel $\sigma$-field $\mathcal{B}^{(0,1]^d}$

**Definition 2.** *The Borel $\sigma$-field on $(0,1]^d$, denoted $\mathcal{B}^{(0,1]^d}$, is defined as the field generated by the rectangles in $(0,1]^d$ as follows*

$$\sigma\Big\langle (a_1,b_1]\times\cdots\times(a_d,b_d] : 0\le a_k < b_k \le 1\Big\rangle.$$

This $\sigma$-field was the main object we studied in the first third of the class. In particular, we proved the SLLN and the law of the iterated logarithm using the uniform probability measure (i.e. 1-d volume) which we proved existed by defining it on $\mathcal{B}_0^{(0,1]}$ then extending with Carathéodory. The Borel $\sigma$-field is very rich and can be generated by many different sub-classes. In particular

$$\begin{aligned}
\mathcal{B}^{(0,1]} &= \sigma\langle \mathcal{B}_0^{(0,1]}\rangle \\
&= \sigma\Big\langle (a,b] : 0\le a\le b\le 1\Big\rangle \\
&= \sigma\Big\langle (a,b) : 0< a< b< 1\Big\rangle \\
&= \sigma\Big\langle [a,b] : 0< a< b< 1\Big\rangle \\
&= \sigma\Big\langle (0,a] : 0< a< 1\Big\rangle \\
&= \sigma\Big\langle \text{open subsets of } (0,1]\Big\rangle \\
&= \sigma\Big\langle \text{closed subsets of } (0,1]\Big\rangle.
\end{aligned}$$

All of the above equalities are shown using the good sets principle. In particular, to show that $\sigma\langle\mathcal{A}_1\rangle = \sigma\langle\mathcal{A}_2\rangle$ one simply needs to establish that $\mathcal{A}_1\subset\sigma\langle\mathcal{A}_2\rangle$ (which implies that $\sigma\langle\mathcal{A}_1\rangle\subset\sigma\langle\mathcal{A}_2\rangle$ by "good sets") and $\mathcal{A}_2\subset\sigma\langle\mathcal{A}_1\rangle$ (which implies that $\sigma\langle\mathcal{A}_2\rangle\subset\sigma\langle\mathcal{A}_1\rangle$ by "good sets").

**Example 1.** *To see an example lets show that*

$$\sigma\Big\langle (a,b] : 0<a<b<1\Big\rangle = \sigma\Big\langle (a,b) : 0<a<b<1\Big\rangle.$$

*It will be sufficient to show the following two statements (i) and (ii) for any arbitrary $0<a_0<b_0<1$.*
*(i) $(a_0,b_0]\in\sigma\langle(a,b):0<a<b<1\rangle$: This is follows from the identity*

$$(a_0,b_0] = \bigcap_{n=1}^\infty (a_0, b_0+n^{-1}).$$

*(ii) $(a_0,b_0)\in\sigma\langle(a,b]:0<a<b<1\rangle$: This is follows from the identity*

$$(a_0,b_0) = \bigcup_{n=1}^\infty (a_0, b_0-n^{-1}].$$

## 2.1 Some interesting sets in $\mathcal{B}^{(0,1]}$

**Example 2.** *The set of normal and abnormal numbers are in $\mathcal{B}^{(0,1]}$.*

**Example 3.** *All countable subsets of $(0,1]$ and all co-countable subsets of $(0,1]$ (i.e. the complements of countable sets) are in $\mathcal{B}^{(0,1]}$. In particular, the collection of irrational numbers in $(0,1]$ is a Borel set.*

**Example 4.** *Under the spinner model we have*

$$\{\limsup_n \frac{s_n}{\ell_n} = 1\}$$
$$= \bigcap_{\epsilon\in(0,1)\cap\mathbb{Q}} \{s_n/\ell_n > (1-\epsilon)\ i.o.\} \cap \{s_n/\ell_n < (1+\epsilon)\ a.a.\}$$

*where* $\ell_n := \sqrt{2n\log\log n}$. *Therefore the law of the iterated logarithm event* $\{w : \limsup_n \frac{s_n(w)}{\sqrt{2n\log\log n}} = 1\}$ *is in* $\mathcal{B}^{(0,1]}$.

**Example 5.** *The Cantor set is in* $\mathcal{B}^{(0,1]}$. *It is an instructive exercise to show that: the Cantor set is in* $\mathcal{B}^{(0,1]}$*; that it has Lebesque measure* $0$*; and that it is uncountable (a nice way to see that it is uncountable is to work with a base-3 digit characterization of the Cantor set).*

# 3 The Borel $\sigma$-field $\mathcal{B}^{\mathbb{R}^d}$

Now we construct the Borel $\sigma$-field on the whole Euclidean space $\mathbb{R}^d$ which will be used to define Lebesque measure which generalizes the $d$-dimensional volume measure on $\mathcal{B}^{(0,1]^d}$.

**Definition 3.** *The Borel* $\sigma$*-field of* $\mathbb{R}^d$*, denoted* $\mathcal{B}^{\mathbb{R}^d}$*, is defined as the* $\sigma$*-field generated by the class of all finite rectangles in* $\mathbb{R}^d$ *as follows*

$$\sigma\big\langle (a_1,b_1]\times\cdots\times(a_d,b_d] : -\infty < a_k < b_k < \infty\big\rangle.$$

## 3.1 How $\mathcal{B}^{\mathbb{R}^d}$ is related to $\mathcal{B}^{(0,1]^d}$

Since both classes of sets $\mathcal{B}^{\mathbb{R}^d}$ and $\mathcal{B}^{(0,1]^d}$ contain subsets of $(0,1]^d$ it is natural to ask about the nature of the overlap. In particular, if a set $A\in\mathcal{B}^{\mathbb{R}^d}$ and $A$ is also a subset of $(0,1]^d$ is it necessary that $A\in\mathcal{B}^{(0,1]^d}$ (and vise versa)? Another equality interesting question asks the following: if I take a set $A\in\mathcal{B}^{\mathbb{R}^d}$ and intersect it with $(0,1]^d$ is the result in $\mathcal{B}^{(0,1]^d}$ (and, if so, are all $\mathcal{B}^{(0,1]^d}$ sets structured in this manner)? It turns out the both answers are yes. To prove this we establish two more general statements which yield our previous questions as corollaries.

**Claim 2.** *Suppose* $\mathcal{F}$ *is a* $\sigma$*-field in* $\Omega$ *and let* $\Omega_0$ *be any subset of* $\Omega$ *(not necessarily in* $\mathcal{F}$*). Then*

$$\mathcal{F}\cap\Omega_0 := \{F\cap\Omega_0 : F\in\mathcal{F}\}$$

*is a* $\sigma$*-field of* $\Omega_0$.

*Proof.* It will be sufficient to show the following three facts (i), (ii), (iii):
(i) $\Omega_0\in\mathcal{F}\cap\Omega_0$: To see why notice that $\Omega\in\mathcal{F}$. Therefore $\Omega\cap\Omega_0\in\mathcal{F}\cap\Omega_0$. Now since $\Omega_0\subset\Omega$ implies $\Omega\cap\Omega_0=\Omega_0$ we have that $\Omega_0\in\mathcal{F}\cap\Omega_0$ as was to be shown.
(ii) $A\in\mathcal{F}\cap\Omega_0 \Rightarrow (\Omega_0 - A)\in\mathcal{F}\cap\Omega_0$: Let $A\in\mathcal{F}\cap\Omega_0$. Then $A = B\cap\Omega_0$ for some $B\in\mathcal{F}$. Therefore letting $A^c$ denote complementation within the larger space $\Omega$ we have

$$\begin{aligned}\Omega_0 - A &= \Omega_0\cap A^c\\ &= \Omega_0\cap(B^c\cup\Omega_0^c)\\ &= \underbrace{\Omega_0\cap B^c}_{\in\mathcal{F}\cap\Omega_0}.\end{aligned}$$

(iii) $A_1, A_2,\ldots\in\mathcal{F}\cap\Omega_0 \Rightarrow \cup_{k=1}^\infty A_k\in\mathcal{F}\cap\Omega_0$: Notice that each $A_k = B_k\cap\Omega_0$ for some $B_k\in\mathcal{F}$. Therefore

$$\begin{aligned}\bigcup_{k=1}^\infty A_k &= \bigcup_{k=1}^\infty (B_k\cap\Omega_0)\\ &= \Omega_0\cap\underbrace{\bigcup_{k=1}^\infty B_k}_{\in\mathcal{F}}\end{aligned}$$

Therefore $\bigcup_{k=1}^\infty A_k\in\mathcal{F}\cap\Omega_0$. ☐

**Claim 3.** *Let* $\Omega$ *be a sample space and* $\mathcal{A}$ *be a class of subsets of* $\Omega$*. If* $\Omega_0\subset\Omega$ *then*

$$\underbrace{\sigma\langle\mathcal{A}\cap\Omega_0\rangle}_{\substack{\sigma\text{-field}\\ \text{on }\Omega_0}} = \underbrace{\sigma\langle\mathcal{A}\rangle}_{\substack{\sigma\text{-field}\\ \text{on }\Omega}}\cap\Omega_0.$$

*Proof.* We first show that $\sigma\langle\mathcal{A}\cap\Omega_0\rangle\subset\sigma\langle\mathcal{A}\rangle\cap\Omega_0$. This easily follows by "good sets" since clearly $\mathcal{A}\cap\Omega_0\subset\sigma\langle\mathcal{A}\rangle\cap\Omega_0$ and Claim 2 shows that $\sigma\langle\mathcal{A}\rangle\cap\Omega_0$ is a $\sigma$-field.

Now we show $\sigma\langle\mathcal{A}\rangle\cap\Omega_0\subset\sigma\langle\mathcal{A}\cap\Omega_0\rangle$. Notice that this inclusion is equivalent to the statement that for every $A\in\sigma\langle\mathcal{A}\rangle$, $A\cap\Omega_0\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle$. To show this let

$$\mathcal{G} := \{A\subset\Omega : A\cap\Omega_0\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle\}.$$

It will then be sufficient to show (i)-(iv) below and then use good sets to conclude $\sigma\langle\mathcal{A}\rangle\subset\mathcal{G}$.
(i) $\mathcal{A}\subset\mathcal{G}$: This follows since for any $A\in\mathcal{A}$ one has that

$$A\cap\Omega_0\in\mathcal{A}\cap\Omega_0\subset\sigma\langle\mathcal{A}\cap\Omega_0\rangle.$$

(ii) $\Omega\in\mathcal{G}$ : This follows since $\Omega_0\subset\Omega$ implies that

$$\Omega\cap\Omega_0 = \Omega_0\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle$$

where the last inclusion follows since $\sigma\langle\mathcal{A}\cap\Omega_0\rangle$ is a $\sigma$-field of $\Omega_0$ so it must contain $\Omega_0$.

(iii) $A\in\mathcal{G}\Rightarrow A^c\in\mathcal{G}$ : Notice that $A^c$ here denotes complementation within $\Omega$. This axiom follows since

$$\begin{aligned}A\in\mathcal{G} &\Rightarrow A\cap\Omega_0\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle\\ &\Rightarrow \underbrace{\Omega_0 - A\cap\Omega_0}_{\text{complement in }\Omega_0}\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle\\ &\Rightarrow \underbrace{\Omega_0\cap(A^c\cup\Omega_0^c)}_{=A^c\cap\Omega_0}\in\sigma\langle\mathcal{A}\cap\Omega_0\rangle\\ &\Rightarrow A^c\in\mathcal{G}.\end{aligned}$$

(iv) $A_1, A_2, \ldots \in \mathcal{G} \Rightarrow \bigcup_k A_k \in \mathcal{G}$ :

$$\begin{aligned} A_1, A_2, \ldots \in \mathcal{G} &\Rightarrow A_k \cap \Omega_0 \in \sigma\langle \mathcal{A} \cap \Omega_0 \rangle, \forall k \\ &\Rightarrow \bigcup_k (A_k \cap \Omega_0) \in \sigma\langle \mathcal{A} \cap \Omega_0 \rangle \\ &\Rightarrow \Big(\bigcup_k A_k\Big) \cap \Omega_0 \in \sigma\langle \mathcal{A} \cap \Omega_0 \rangle \\ &\Rightarrow \bigcup_k A_k \in \mathcal{G}. \end{aligned}$$

□

*The above two claims immediately yield the desired corollary.*

**Corollary 1.** $\mathcal{B}^{\mathbb{R}^d} \cap (0,1]^d = \mathcal{B}^{(0,1]^d} = \{A \in \mathcal{B}^{\mathbb{R}^d} : A \subset (0,1]^d\}$

*Proof.* The first equality follows since

$$\begin{aligned} \mathcal{B}^{\mathbb{R}^d} \cap (0,1]^d &= \sigma\langle \textit{finite rectangles in } \mathbb{R}^d \rangle \cap (0,1]^d \\ &= \sigma\langle \textit{finite rectangles in } \mathbb{R}^d \cap (0,1]^d \rangle \\ &= \mathcal{B}^{(0,1]^d}. \end{aligned}$$

The second equality follows by noticing $\{A \in \mathcal{B}^{\mathbb{R}^d} : A \subset (0,1]^d\} = \mathcal{B}^{\mathbb{R}^d} \cap (0,1]^d$. The inclusion '$\subset$' is obvious. The other inclusion is also almost obvious since

$$\begin{aligned} A \in \mathcal{B}^{\mathbb{R}^d} \cap (0,1]^d &\Rightarrow A = B \cap (0,1]^d \text{ for some } B \in \mathcal{B}^{\mathbb{R}^d} \\ &\Rightarrow A \in \mathcal{B}^{\mathbb{R}^d} \text{ (since } (0,1]^d \text{ and } B \text{ are in } \mathcal{B}^{\mathbb{R}^d}) \\ &\quad\ \text{and } A \subset (0,1]^d \\ &\Rightarrow A \in \{A \in \mathcal{B}^{\mathbb{R}^d} : A \subset (0,1]^d\}. \end{aligned}$$

□

### 3.2 Equivalent generators of $\mathcal{B}^{\mathbb{R}^d}$

**Claim 4.**

$$\begin{aligned} \mathcal{B}^{\mathbb{R}^d} &= \sigma\big\langle (-\infty, c_1] \times \cdots \times (-\infty, c_d] : -\infty < c_k < \infty \big\rangle && (1) \\ &= \sigma\big\langle \textit{open subsets of } \mathbb{R}^d \big\rangle && (2) \\ &= \sigma\big\langle \textit{closed subsets of } \mathbb{R}^d \big\rangle && (3) \\ &= \sigma\big\langle \textit{compact subsets of } \mathbb{R}^d \big\rangle && (4) \\ &= \sigma\Big\langle f\langle \textit{finite rectangles of } \mathbb{R}^d \rangle \Big\rangle. && (5) \end{aligned}$$

*Proof.* We only show that $\mathcal{B}^{\mathbb{R}^d} = \sigma\langle \text{open subsets} \rangle$ and simply remark that the other proofs are similar. It will be sufficient to show the following two statements (i) and (ii):

(i) the open sets are in $\mathcal{B}^{\mathbb{R}^d}$: Let $G$ be an open subset of $\mathbb{R}^d$. Now for each element $y \in G$ there exists a finite rectangle within $G$, call it $R_y$, with rational edges which covers $y$ (i.e. $y \in R_y$ and $R_y \subset G$). Then clearly $G = \cup_{y \in G} R_y$. Notice that since there are only countably many rational rectangles the union $\cup_{y \in G} R_y$ must be a countable union of the generators of $\mathbb{R}^d$ and therefore $G \in \mathcal{B}^{\mathbb{R}^d}$.

(ii) the finite rectangles are in $\sigma\langle \text{open sets} \rangle$: This follows easily since

$$(a_1, b_1] \times \cdots \times (a_d, b_d] = \bigcap_{n=1}^{\infty} (a_1, b_1 + n^{-1}) \times \cdots \times (a_d, b_d + n^{-1}).$$

□

It is worth while mentioning that while some of these characterization of $\mathcal{B}^{\mathbb{R}^d}$ are useful (for example the left infinite generators in (1) form a $\pi$-system so any $\sigma$-finite measure is uniquely specified on these generators) I would suggest that the most important characterization is $\mathcal{B}^{\mathbb{R}^d} = \sigma\langle f\langle \textit{finite rectangles} \rangle \rangle$. The reason being is that one can use the extension theorem to uniquely extend a $\sigma$-finite measure on $f\langle \textit{finite rectangles} \rangle$ to one on $\sigma\langle f\langle \textit{finite rectangles} \rangle \rangle$. Moreover, we have an explicit understanding of the elements in $f\langle \textit{finite rectangles} \rangle$ which makes it relatively easy to specify a measure. Be careful, though, the structure of $f\langle \textit{finite rectangles in } \mathbb{R}^d \rangle$ is a bit different than $f\langle \textit{finite rectangles in } (0,1]^d \rangle$

## 4 Lebesque measure $\mathcal{L}^d$

For any $\boldsymbol{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ let $(\boldsymbol{i}, \boldsymbol{i}+1]$ be the unit cube in $\mathbb{R}^d$ translated up by $\boldsymbol{i}$ so that

$$(\boldsymbol{i}, \boldsymbol{i}+1] \equiv (i_1, i_1+1] \times \cdots \times (i_d, i_d+1].$$

Notice that these sets give a checker board decomposition, $\mathbb{R}^d = \bigcup_{\boldsymbol{i} \in \mathbb{Z}^d} (\boldsymbol{i}, \boldsymbol{i}+1]$, so that $\mathbb{R}^d$ is expressed as a countable disjoint union of the translated unit cubes. Let $\mathcal{B}_0^{(\boldsymbol{i},\boldsymbol{i}+1]}$ denote the field of finite disjoint unions of rectangles in $(\boldsymbol{i}, \boldsymbol{i}+1]$ and let $\mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]} \equiv \sigma\langle \mathcal{B}_0^{(\boldsymbol{i},\boldsymbol{i}+1]} \rangle$ denote the Borel $\sigma$-field of $(\boldsymbol{i}, \boldsymbol{i}+1]$. Finally let $P_{\boldsymbol{i}}$ denote the unique uniform probability measure on $\mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]}$ which assigns Euclidean volume to the rectangles in $(\boldsymbol{i}, \boldsymbol{i}+1]$, i.e.

$$P_{\boldsymbol{i}}\big((a_1, b_1] \times \cdots \times (a_d, b_d]\big) = \prod_{k=1}^{d} (b_k - a_k)$$

whenever $(a_1, b_1] \times \cdots \times (a_d, b_d] \subset (\boldsymbol{i}, \boldsymbol{i}+1]$. The construction of $P_{\boldsymbol{i}}$ is done in exactly the same way as the uniform probability measure was constructed on $(0,1]$ in the beginning of the class. Lets recall how this is done. One first shows that for any $A \in \mathcal{B}_0^{(\boldsymbol{i},\boldsymbol{i}+1]}$ one can define $P_{\boldsymbol{i}}(A)$ to be the sum of the disjoint rectangle volumes which make up $A$ (this is not trivial since there are different decompositions of $A$ into disjoint rectangles, but one can use a result similar to Theorem 1.3 of Billingsley to prove that $P_{\boldsymbol{i}}$ is well defined). Secondly, one shows that $P_{\boldsymbol{i}}$ is a probability measure on $((\boldsymbol{i}, \boldsymbol{i}+1], \mathcal{B}_0^{(\boldsymbol{i},\boldsymbol{i}+1]})$. The hard part of this step is to show the countable additivity. For $(0,1]$ we used the equivalent condition that $P_{\boldsymbol{i}}$ is continuous from above at $\varnothing$.

This argument carries over to $((\boldsymbol{i},\boldsymbol{i}{+}1], \mathcal{B}_0^{(\boldsymbol{i},\boldsymbol{i}+1]}, P_{\boldsymbol{i}})$. Finally one invokes the Carathéodory Extension theorem to get a uniform probability measure $((\boldsymbol{i},\boldsymbol{i}+1], \mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]}, P_{\boldsymbol{i}})$ (uniqueness follows by the fact that rectangles, including the empty ones, form a $\pi$-system).

Now, using the uniform probability measures $((\boldsymbol{i},\boldsymbol{i}+1], \mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]}, P_{\boldsymbol{i}})$ we can define Lebesque measure $\mathcal{L}^d$ on sets $A \in \mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]}$ by stitching these $P_{\boldsymbol{i}}$ together as follows

$$\mathcal{L}^d(A) := \sum_{\boldsymbol{i}\in\mathbb{Z}^d} P_{\boldsymbol{i}}\big((\boldsymbol{i},\boldsymbol{i}+1]\cap A\big). \tag{6}$$

Notice that each $(\boldsymbol{i},\boldsymbol{i}+1]\cap A$ is in the Borel $\sigma$-field $\mathcal{B}^{(\boldsymbol{i},\boldsymbol{i}+1]}$ by Claim 3 so that $P_{\boldsymbol{i}}\big((\boldsymbol{i},\boldsymbol{i}+1]\cap A\big)$ is defined. Lets see that $\mathcal{L}^d$ is indeed a measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$.

**Claim 5.** *$\mathcal{L}^d$ is a measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$.*

*Proof.* We show the following three axioms (i), (ii) and (iii):
(i) $\mathcal{L}^d(A) \in [0,\infty]$: Trivial.

(ii) $\mathcal{L}^d(\varnothing) = 0$: This is also easy since $P_{\boldsymbol{i}}\big((\boldsymbol{i},\boldsymbol{i}+1]\cap\varnothing\big) = 0$.

(iii) Countable additivity: Suppose $A_1, A_2, \ldots \in \mathcal{B}^{\mathbb{R}^d}$ are disjoint. Then

$$\begin{aligned}
\mathcal{L}^d\Big(\bigcup_{k=1}^\infty A_k\Big) &= \sum_{\boldsymbol{i}\in\mathbb{Z}^d} P_{\boldsymbol{i}}\Big((\boldsymbol{i},\boldsymbol{i}+1]\cap \bigcup_{k=1}^\infty A_k\Big)\\
&= \sum_{\boldsymbol{i}\in\mathbb{Z}^d} P_{\boldsymbol{i}}\Big(\bigcup_{k=1}^\infty (\boldsymbol{i},\boldsymbol{i}+1]\cap A_k\Big)\\
&= \sum_{\boldsymbol{i}\in\mathbb{Z}^d}\sum_{k=1}^\infty P_{\boldsymbol{i}}\Big((\boldsymbol{i},\boldsymbol{i}+1]\cap A_k\Big) \qquad (7)\\
&= \sum_{k=1}^\infty\sum_{\boldsymbol{i}\in\mathbb{Z}^d} P_{\boldsymbol{i}}\Big((\boldsymbol{i},\boldsymbol{i}+1]\cap A_k\Big) \qquad (8)\\
&= \sum_{k=1}^\infty \mathcal{L}^d(A_k)
\end{aligned}$$

where (7) follows since $P_{\boldsymbol{i}}$ is countably additive and the $(\boldsymbol{i},\boldsymbol{i}+1]\cap A_k$'s are disjoint; and (8) follows from general results about positive iterated sums. □

### 4.1 Uniqueness

Recall the uniqueness theorem for $\sigma$-finite measures

**Claim 6.** *If $\mu_1$ and $\mu_2$ are measures on $(\Omega, \sigma\langle\mathcal{P}\rangle)$ such that*

*(a) $\mu_1$ and $\mu_2$ agree on $\mathcal{P}$;*

*(b) $\mathcal{P}$ is a $\pi$-system;*

*(c) $\mu_1$ and $\mu_2$ are $\sigma$-finite on $\mathcal{P}$,*

*then $\mu_1$ and $\mu_2$ agree on all of $\sigma\langle\mathcal{P}\rangle$.*

*Proof.* See the class notes for a proof. □

To apply this to Lebesque measure define $\mathcal{P}$ to be the $\pi$-system composed of the finite rectangles $\{(a_1,b_1]\times\cdots\times(a_d,b_d] : -\infty < a_k < b_k < \infty\}$ and the empty set $\varnothing$. Notice that using methods discussion in Section 3.2 one can easily establish that $\mathcal{B}^{\mathbb{R}^d} = \sigma\langle\mathcal{P}\rangle$. Also notice that $\mathcal{L}^d$ is $\sigma$-finite on $\mathcal{P}$ since $\mathcal{L}^d\big((\boldsymbol{i},\boldsymbol{i}+1]\big) = 1$, $\mathbb{R}^d = \cup_{\boldsymbol{i}\in\mathbb{Z}^d}(\boldsymbol{i},\boldsymbol{i}+1]$ and each $(\boldsymbol{i},\boldsymbol{i}+1]\in\mathcal{P}$. Therefore Claim 6 establishes the following claim

**Corollary 2.** *$\mathcal{L}^d$ is the only measure on $(\mathbb{R}^d, \mathcal{B}^{(0,1]^d})$ which assigns standard Euclidean volume to the finite rectangles as follows*

$$\mathcal{L}^d\big((a_1,b_1]\times\cdots\times(a_d,b_d]\big) = \prod_{k=1}^d (b_k - a_k) \tag{9}$$

*for $-\infty < a_k < b_k < \infty$.*

### 4.2 Behavior under linear transformations

Lebesque measure $\mathcal{L}^d$ can be thought of as a uniform measure on $\mathbb{R}^d$. In this section we show some facts about $\mathcal{L}^d$ which one would expect from a uniform measure.

**Claim 7.** *For any $A \in \mathcal{B}^{\mathbb{R}^d}$ and $x \in \mathbb{R}^d$, the set $A + x := \{a + x : a \in A\}$ is in $\mathcal{B}^{\mathbb{R}^d}$ and*

$$\mathcal{L}^d(A + x) = \mathcal{L}^d(A) \tag{10}$$

*Proof.* To show $A + x \in \mathcal{B}^{\mathbb{R}^d}$ use the good sets principle. Fix $x \in \mathbb{R}^d$ and set $\mathcal{G}_x := \{A \subset \mathcal{B}^{\mathbb{R}^d} : A \in \mathcal{B}^{\mathbb{R}^d} \text{ and } A + x \in \mathcal{B}^{\mathbb{R}^d}\}$. It is easy to see that $\mathcal{G}_x$ is a $\sigma$-field since complementation and union is preserved under translation by $x$. For example,

$$\begin{aligned}
A \in \mathcal{G}_x &\Rightarrow A \in \mathcal{B}^{\mathbb{R}^d} \text{ and } A + x \in \mathcal{B}^{\mathbb{R}^d}\\
&\Rightarrow A^c \in \mathcal{B}^{\mathbb{R}^d} \text{ and } (A+x)^c \in \mathcal{B}^{\mathbb{R}^d}\\
&\Rightarrow A^c \in \mathcal{B}^{\mathbb{R}^d} \text{ and } A^c + x \in \mathcal{B}^{\mathbb{R}^d}\\
&\Rightarrow A^c \in \mathcal{G}_x.
\end{aligned}$$

The other axioms are established in a similar fashion. Moreover, clearly all the finite rectangles are in $\mathcal{G}_x$. Therefore good sets implies $\mathcal{B}^{\mathbb{R}^d} \subset \mathcal{G}_x$ which implies $A \in \mathcal{B}^{\mathbb{R}^d} \to A + x \in \mathcal{B}^{\mathbb{R}^d}$, as was to be shown.

Now to show (10) one can simply use the same arguments used in the Claim 2 on the uniqueness of $\mathcal{L}^d$. In particular, fix $x$ and define $\mu_x(A) := \mathcal{L}^d(A+x)$. It is easy to show that $\mu_x$ is a measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$. Moreover, since the volume of any rectangle in $\mathbb{R}^d$ is invariant under translation by $x$, the measures $\mu_x$ and $\mathcal{L}^d$ both agree on the $\pi$-system of finite, possibly empty, rectangles in $\mathbb{R}^d$. Since they are also both $\sigma$-finite on these rectangles one must have, by Claim 6, $\mathcal{L}^d(A) = \mu_x(A) := \mathcal{L}^d(A+x)$ for all $A \in \mathcal{B}^{\mathbb{R}^d}$, as was to be shown. □

The above claim illustrates how $\mathcal{L}^d$ behaves under translation of sets. The following theorem shows how $\mathcal{L}^d$ behaves under nonsingular linear transformations

**Claim 8.** *If $T : \mathbb{R}^d \to \mathbb{R}^d$ is linear and nonsingular, then $A \in \mathcal{B}^{\mathbb{R}^d}$ implies that $TA := \{T(a) : a \in A\} \in \mathcal{B}^{\mathbb{R}^d}$ and*

$$\mathcal{L}^d(TA) := |\det T|\mathcal{L}^d(A).$$

*Proof.* This proof is similar to the previous proof, albeit a bit more tedious. See Theorem 12.2 in Billingsley (page 173) the details. □

It is interesting to know that the assumption that $T : \mathbb{R}^d \to \mathbb{R}^d$ be nonsingular is necessary for the Borel measurability of $TA$. In particular, there exists a linear singular map $T : \mathbb{R}^2 \to \mathbb{R}^2$ and a Borel measurable set $A \in \mathcal{B}^{\mathbb{R}^2}$ such that $TA \notin \mathcal{B}^{\mathbb{R}}$!

## 4.3 Lower dimensional subsets

In this section we show that $\mathcal{L}^d$ assigns zero measure to low dimensional hyperplanes. In fact, this will be a consequence of the following more general theorem in combination with the translation invariance of $\mathcal{L}^d$.

**Claim 9.** *Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. Then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive $\mu$-measure*

*Proof.* Let $\{B_i : i \in \mathcal{I}\}$ a disjoint collection of sets of such that $\mu(B_i) > 0$ for each $i \in \mathcal{I}$. We show $\mathcal{I}$ must be countable.

Since $\mu$ is $\sigma$-finite there exists $A_1, A_2, \ldots \in \mathcal{F}$ such that $\mu(A_k) < \infty$ and $\Omega = \cup_k A_k$.

We show the following three facts.

<u>(i) $\{i \in \mathcal{I} : \mu(A_k \cap B_i) > \epsilon\}$ is finite for all $k$:</u> Let $\epsilon > 0$ and suppose by contradiction one can find a countably infinite set $\mathcal{I}_c \subset \mathcal{I}$ such that $\mu(A_k \cap B_i) > \epsilon$ for all $i \in \mathcal{I}_c$ and for this set of indices one has

$$\mu(A_k) \geq \mu(A_k \cap (\cup_{i \in \mathcal{I}_c} B_i)) = \sum_{i \in \mathcal{I}_c} \mu(A_k \cap B_i) > \sum_{i \in \mathcal{I}_c} \epsilon = \infty$$

which gives a contradiction.

<u>(ii) $\{i \in \mathcal{I} : \mu(A_k \cap B_i) > 0\}$ is countable for all $k$:</u> This follows from the identity

$$\{i \in \mathcal{I} : \mu(A_k \cap B_i) > 0\} = \bigcup_{\text{rational } \epsilon} \underbrace{\{i \in \mathcal{I} : \mu(A \cap B_i) > \epsilon\}}_{\text{finite by (i)}}.$$

<u>(iii) $\mathcal{I} = \bigcup_k \{i \in \mathcal{I} : \mu(A_k \cap B_i) > 0\}$:</u> To show $\mathcal{I} \cup \bigcup_k \{i \in \mathcal{I} : \mu(A_k \cap B_i) > 0\}$ notice that if $i \in \mathcal{I}$ then $\mu(B_i) > 0$. Now $\Omega = \cup_k A_k$ so there must exist a $k$ such that $\mu(A_k \cap B_i) > 0$. Therefore $i \in \bigcup_k \{i \in \mathcal{I} : \mu(A_k \cap B_i) > 0\}$. The other inclusion is obvious.

To finish the proof simply notice that (ii) and (iii) imply $\mathcal{I}$ is countable. □

**Corollary 3.** *If $k < d$ then $\mathcal{L}^d(A) = 0$ for any $k$-dimensional hyperplane $A \subset \mathbb{R}^d$ where $k < d$.*

*Proof.* Let $A$ be a $k$-dimensional hyperplane where $k < d$. Let $x$ be a point in $\mathbb{R}^d$ which is not contained in $A$. Then $\{A + xt : t \in \mathbb{R}\}$ is an uncountable class of disjoint subsets of $\mathcal{B}^{\mathbb{R}^d}$. Since $\mathcal{L}^d$ is translation invariance $\mathcal{L}^d(A) = \mathcal{L}^d(A + xt)$ for each $t \in \mathbb{R}$. Now by Claim 9, $\mathcal{L}^d(A) = 0$, for otherwise there would exists a uncountable, disjoint collection of sets of positive $\mathcal{L}^d$-measure. □

## 4.4 Regularity and Approximation

If $B \in \mathcal{B}^{\mathbb{R}^d}$ then

$$\begin{aligned}\mathcal{L}^d(B) &= \sup\{\mathcal{L}^d(C) : C \subset B,\, C \text{ closed}\} \\ &= \inf\{\mathcal{L}^d(O) : B \subset O,\, O \text{ open}\}\end{aligned}$$

To prove these claims we need some results that hold for general measure.

**Lemma 1.** *Suppose $\mathcal{F}_0$ is a field, $\mu$ is a measure on $\mathcal{F} := \sigma\langle\mathcal{F}_0\rangle$ and $\mu$ is $\sigma$-finite on $\mathcal{F}_0$. For all $B \in \mathcal{F}$ and $\epsilon > 0$ there exists a disjoint sequence of $\mathcal{F}_0$-sets $A_1, A_2, \ldots$ such that $B \subset \cup_{n=1}^\infty A_n$ and $\mu\big(\cup_{n=1}^\infty A_n - B\big) \leq \epsilon$.*

↓———begin long version———

*Proof.* Since $\mu$ is $\sigma$-finite on $\mathcal{F}_0$ and $\mathcal{F}_0$ is a field, one can find disjoint $\mathcal{F}_0$-sets $F_1, F_2, \ldots$ such that $\Omega = \cup_{n=1}^\infty F_n$ and $\mu(F_n) < \infty$. We start by supposing $\mu(F_n) > 0$ for all $n$ and show at the end of the proof to remove this assumption.

Fix $B \in \mathcal{F}$. Define $\mu_n(\cdot) := \frac{\mu(\cdot \cap F_n)}{\mu(F_n)}$ on $\mathcal{F}$. Notice that $\mu_n$ is a probability measure on $\mathcal{F}$ so that

$$\mu_n(B) = \inf\{\mu_n(D) : B \subset D \in \mathcal{F}^\uparrow\} \leq 1.$$

Therefore one can find a $D_n \in \mathcal{F}^\uparrow$, covering $B$, such that $B \subset D_n$ and

$$\underbrace{\mu_n(D_n) - \mu_n(B)}_{=\mu_n(D_n - B)} \leq \frac{\epsilon}{2^n \mu(F_n)}. \tag{11}$$

Now define $D := \bigcup_{n=1}^\infty D_n \cap F_n$ and notice that

$$\begin{aligned} B \subset D_n, \forall n &\Rightarrow (B \cap F_n) \subset (D_n \cap F_n), \forall n \\ &\Rightarrow \bigcup_{n=1}^\infty (B \cap F_n) \subset \bigcup_{n=1}^\infty (D_n \cap F_n) \\ &\Rightarrow B \subset D. \end{aligned}$$

Notice also that each $D_n \in \mathcal{F}^\uparrow$ can be written as a disjoint union of $\mathcal{F}_0$-sets (exercise). Let $D_n = \cup_{m=1}^\infty A_{n,m}$ be such a decomposition (i.e. the $A_{n,m}$'s are disjoint across $m$ and $A_{n,m} \in \mathcal{F}_0$). Then

$$D = \bigcup_{(n,m) \in \mathbb{N}^+ \times \mathbb{N}^+} A_{n,m} \cap F_n$$

where the sets $A_{n,m} \cap F_n$ are disjoint (the $F_n$'s are disjoint for different $n$'s and the $A_{n,m}$'s are disjoint for different $m$'s) and are $\mathcal{F}_0$-sets. Now

$$\begin{aligned} \mu(D - B) &= \mu\Big(\bigcup_{n=1}^{\infty} D_n \cap F_n \cap B^c\Big) \\ &= \sum_{n=1}^{\infty} \mu(D_n \cap F_n \cap B^c) \\ &= \sum_{n=1}^{\infty} \mu(F_n)\mu_n(D_n \cap B^c) \\ &= \sum_{n=1}^{\infty} \mu(F_n)\underbrace{\mu_n(D_n - B)}_{\le \epsilon/(2^n\mu(F_n))} \\ &\le \epsilon. \end{aligned} \tag{12}$$

Therefore the class $\{A_{n,m} \cap F_n\}_{(n,m)\in\mathbb{N}^+\times\mathbb{N}^+}$ gives a countable, disjoint $\mathcal{F}_0$-set covering of $B$ such that $\mu(\bigcup_{(n,m)\in\mathbb{N}^+\times\mathbb{N}^+} A_{n,m} \cap F_n - B) \le \epsilon$.

It's easy to extend to the case when some of the $\mu(F_n) = 0$ by defining $\mu_n(\cdot) := 0$ and $D_n := F_n$ for these $n$. Then (12) still follows. □

↑————end long version———

**Exercise 1.** *Let $\mu$ be any measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$ which assigns finite measure to bounded sets in $\mathcal{B}^{\mathbb{R}^d}$. Define the Borel field of $\mathbb{R}^d$ as follows.*

$$\mathcal{B}_0^{\mathbb{R}^d} := f\big\langle (a_1, b_1] \times \cdots \times (a_d, b_d] : -\infty < a_k < b_k < \infty \big\rangle.$$

*Show that for any $A \in \mathcal{B}_0^{\mathbb{R}^d}$ and any $\epsilon > 0$ there exists an open set $G$ such that $A \subset G$ and $\mu(G - A) \le \epsilon$ (Hint: Use a characterization of fields and simply make sure the boundaries of $A$ are finite).*

**Claim 10.** *Let $\mu$ be any measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$ which assigns finite measure to bounded sets in $\mathcal{B}^{\mathbb{R}^d}$. For any $B \in \mathcal{B}^{\mathbb{R}^d}$ and $\epsilon > 0$ there exists a closed set $C$ and an open set $O$ such that $C \subset B \subset O$ and*

$$\mu(O - C) < \epsilon.$$

↓———begin long version———

*Proof.* This is a relatively easy consequence of Lemma 1 and Exercise 1.

First notice the following fact: if $A, B, C$ are $\mathcal{F}$-sets then

$$A \subset B \subset C \Rightarrow \mu(C - A) = \mu(C - B) + \mu(B - A). \tag{13}$$

Once consequence of this fact is that it will be sufficient to approximate $B$ by an open covering $O$ such that $\mu(O - B) < \epsilon/2$ and by a closed subset $C$ such that $\mu(B - C) < \epsilon/2$.

Now notice that $\mu$ is $\sigma$-finite on $\mathcal{R}_0^d$ since $\mu$ assigns finite measure to bounded sets in $\mathcal{R}^d$ (just use the covers $(\boldsymbol{i}, \boldsymbol{i}+1]$). Therefore we can use Lemma 1 to get a disjoint sets $A_k \in \mathcal{R}_0^d$ such that

$$B \subset \bigcup_{k=1}^{\infty} A_k \text{ and } \mu\Big(\bigcup_{k=1}^{\infty} A_k - B\Big) \le \epsilon/4.$$

Now, use Lemma 1 to expand $A_k$ to an open cover $G_k$ in such a way that $\mu(G_k - A_k) \le \epsilon/(2^k 4)$. Now clearly $B \subset \bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} G_k$ and

$$\begin{aligned} \mu\Big(\bigcup_{k=1}^{\infty} G_k - B\Big) &= \mu\Big(\bigcup_{k=1}^{\infty} G_k - \bigcup_{k=1}^{\infty} A_k\Big) + \mu\Big(\bigcup_{k=1}^{\infty} A_k - B\Big) \\ &\le \sum_{k=1}^{\infty} \mu(G_k - A_k) + \epsilon/4 \\ &\le \epsilon/2. \end{aligned}$$

Since $\bigcup_{k=1}^{\infty} G_k$ is open have constructed the desired open set $O$.

To show the existence of the closed set we simply take complements. In particular, let $O$ be an open set such that

$$B^c \subset O \text{ and } \mu(O - B^c) \le \epsilon/2.$$

Notice that $O - B^c = O \cap B = B - O^c$ and that $O^c$ is a closed set. Therefore

$$O^c \subset B \text{ and } \mu(B - O^c) \le \epsilon/2$$

as was to be shown. □

↑————end long version———

The following claim now gives our desired results for Lebesque measure.

**Corollary 4.** *Let $\mu$ be any measure on $(\mathbb{R}^d, \mathcal{B}^{\mathbb{R}^d})$ which assigns finite measure to bounded sets in $\mathcal{B}^{\mathbb{R}^d}$. Then*

$$\begin{aligned} \mu(B) &= \sup\{\mu(C) : C \subset B,\ C \textit{ closed}\} \\ &= \inf\{\mu(O) : B \subset O,\ O \textit{ open}\} \end{aligned}$$

↓———begin long version———

*Proof.* This would be a good exercise too!! We first show that $\mu(B) = \inf\{\mu(O) : B \subset O,\ O \text{ open}\}$. Notice that this equality is immediately true when $\mu(B) = \infty$, since it implies $\mu(O) = \infty$ for any open cover of $B$. Now suppose $\mu(B) < \infty$. Clearly

$$\mu(B) \le \inf\{\mu(O) : B \subset O,\ O \text{ open}\}$$

since $\mu(B) \le \mu(O)$ for each open cover $O$ of $B$. Conversely, Claim 10 shows that there exists open sets $O_n$ such that $B \subset O_n$ and $\mu(O_n - B) \le n^{-1}$. Since $\mu(B) < \infty$ this implies that

$0 \leq \mu(O_n) - \mu(B) = \mu(O_n - B) \leq n^{-1}$ which implies that $\lim_{n\to\infty} \mu(O_n) = \mu(B)$ and therefore

$$\mu(B) \geq \inf\{\mu(O) : B \subset O,\, O \text{ open}\}.$$

Now we show that $\mu(B) = \sup\{\mu(C) : C \subset B,\, C \text{ closed}\}$. Again the inequality '$\geq$' is obvious. To show the other inequality let $C_n \subset B$ be closed sets such that $\mu(B - C_n) \leq n^{-1}$. Notice that if $\mu(B) = \infty$ then $\mu(C_n) = \infty$ and the inequality holds. On the other hand if $\mu(B) < \infty$, then $\mu(C_n) < \infty$ and we have that $0 \leq \mu(B) - \mu(C_n) \leq n^{-1}$. Therefore $\lim_{n\to\infty} \mu(C_n) = \mu(B)$ which implies

$$\mu(B) \leq \sup\{\mu(C) : C \subset B,\, C \text{ closed}\}$$

as was to be shown. □

↑————end long version———

*Remark:* In fact, a measure $\mu$ is called inner-regular if $\mu(A)$ Claim (4) holds.

*Remark:* If one uses the semiring theory given in Billingsly (Theorem 11.4) one can strengthen this a bit and show that if $B \in \mathcal{B}^{\mathbb{R}^d}$ and $\mathcal{L}^d(B) < \infty$ then

$$\mathcal{L}^d(B) = \sup\{\mathcal{L}^d(K) : K \subset B,\, K \text{ compact}\}.$$

**Exercise 2.** *Give an example of a $\sigma$-finite measure $\mu$ on $\mathcal{B}^{\mathbb{R}}$ and a Borel set $B$ such that*

$$\mu(B - C) = \infty = \mu(O - B)$$

*for every closed subset $C$ of $B$ and every open super set $O$ of $B$.*