Lecture 21: Martingales

Martingales are models for a system of wagers coming from a sequence of fair games.

submartingales model advantageous games

supermartingales model disadvantageous games

Another view point is that sub & super martingales model the stochastic equivalent to monotonic sequences of numbers. Indeed we will prove that under "bdd" type conditions sub & super martingales are guarenteed to converge.

Indeed, a basic theme in the next lecture is that "Martingales like to converge" so "looking for a Martingale" is a technique for establishing convergence.

Assumption: For the rest of this lecture fix some probability space $(\Omega, \mathcal{F}, P)$.

Definition: A **filtration** is an increasing sequence of sub σ-fields

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$$

Definition: A sequence of r.v.s $X_1, X_2, \ldots$ defined on $(\Omega, \mathcal{F}, P)$ is **adapted** to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if X_n is $\mathcal{F}_n$-measurable for each $n\in\mathbb{N}$.

Definition:

Suppose $(X_n)_{n\in\mathbb{N}}$ is adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ & $X_n \in L_1(\Omega, \mathcal{F}, P)$ for all $n\in\mathbb{N}$. Then

- $(X_n)_{n\in\mathbb{N}}$ is a **martingale** if
$$E(X_{n+1}|\mathcal{F}_n) \overset{P\text{-a.e.}}{=} X_n \quad \forall n\in\mathbb{N}$$
- $(X_n)_{n\in\mathbb{N}}$ is a **submartingale** if
$$E(X_{n+1}|\mathcal{F}_n) \overset{P\text{-a.e.}}{\geq} X_n \quad \forall n\in\mathbb{N}$$
- $(X_n)_{n\in\mathbb{N}}$ is a **supermartingale** if
$$\underbrace{E(X_{n+1}|\mathcal{F}_n) \overset{P\text{-a.e.}}{\leq} X_n}_{} \quad \forall n\in\mathbb{N}$$

when $\mathcal{F}_n$ represents the current state of information & X_n represents a gambler fortune, this says future fortune decreases in expected value given $\mathcal{F}_n$.

Definition: If $(X_n)_{n\in\mathbb{N}}$ are r.v.s on $(\Omega, \mathcal{F}, P)$ then the **natural filtration** is

$$\mathcal{F}_n := \sigma\langle X_1, \ldots, X_n\rangle$$

Notation: As a shorthand we let

"subM" ≡ submartingale

"supM" ≡ supermartingale

"M" ≡ martingale

Notice that if $(X_n)_{n\geq 1}$ is a subM then clearly $(-X_n)_{n\geq 1}$ is a supM by (e1).

Also if $(X_n)_{n\geq 1}$ is both a subM & supM it is then a M.

Here is another immediate consequence of the definitions

Proposition

$(X_n)_{n\geq 1}$ is a subM w.r.t. filtration $(\mathcal{F}_n)_{n\geq 1}$

$$\Updownarrow$$

$$E(X_m|\mathcal{F}_n) \overset{a.e.}{\geq} X_n \quad \forall m>n$$

Proof:

($\Leftarrow$): Trivial

a more dramatic smoother

($\Rightarrow$): If $m>n$ then $\mathcal{F}_n \subset \mathcal{F}_m$

$$E(X_{m+1}|\mathcal{F}_n) \underset{(s2)}{\overset{a.e.}{=}} E(\underbrace{E(X_{m+1}|\mathcal{F}_m)}_{\text{an intermediate smoothing}}|\mathcal{F}_n)$$

$$\underset{(e3)}{\overset{a.e.}{\geq}} E(X_m|\mathcal{F}_n).$$

The result now follows by induction.

QED

Corollary:

$(X_n)_{n\geq}$ is a subM $\Longrightarrow E(X_m) \geq E(X_n) \quad \forall m>n$

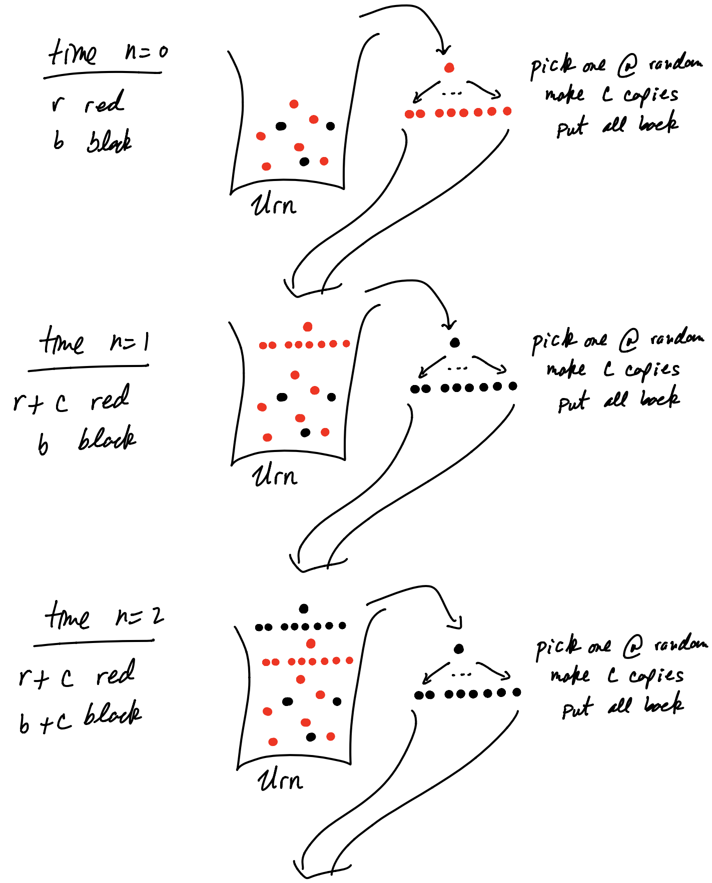
$(X_n)_{n\geq}$ is a supM $\Longrightarrow E(X_m) \leq E(X_n) \quad \forall m>n$

$(X_n)_{n\geq}$ is a M $\Longrightarrow E(X_m) = E(X_n) \quad \forall m>n$

Proof: Simple application of (s1)

Pólya's Urn

This is a classic example of a martingale (and exchangable sequences of r.v.s) and a good canonical example of a martingale to have in your mind.



Repeat

⋮

Let $Y_n := \begin{cases} 1 & \text{if the } n^{th} \text{ draw is red.} \\ 0 & \text{if} \end{cases}$

$R_n :=$ proportion of reds in the urn at time n

$\mathcal{F}_n := \sigma\langle R_1, R_2, \dots, R_n\rangle$

Question: Does R_n have a limiting distribution?

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First thing to notice is that

$$P(Y_n = 1 \mid R_1, \dots, R_n) = R_n$$

so that

$$E(Y_n \mid \mathcal{F}_n) = E(Y_n \mid R_1, \dots, R_n)$$

$$= \sum_{y \in 0,1} y\, P(Y_n = y \mid R_1, \dots, R_n)$$

$$= R_n$$

Moreover $R_n = \dfrac{r + c(Y_0 + \cdots + Y_{n-1})}{r + b + nc}$ ← each time you add c more balls

which implies

$E(R_{n+1} \mid \mathcal{F}_n)$

$$= E\left(\frac{r + c(Y_0 + \cdots + Y_n)}{r + b + (n+1)c} \,\middle|\, \mathcal{F}_n \right)$$

$$= E\left(\frac{r+b+nc}{r+b+(n+1)c} \underbrace{\frac{r + c(Y_0 + \cdots + Y_{n-1})}{r+b+nc}}_{= R_n \text{ (m) } \mathcal{F}_n} \,\middle|\, \mathcal{F}_n \right)$$

$$+ E\left(\frac{cY_n}{r+b+(n+1)c} \,\middle|\, \mathcal{F}_n \right)$$

$$= \frac{r+b+nc}{r+b+(n+1)c} R_n + \frac{c}{r+b+(n+1)c} R_n$$

$$= R_n$$

Since R_n is trivially (m) $\mathcal{F}_n$ we therefore have that R_n is a martingale.
We will see that under mild conditions martingales converge.

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Transformation of Martingales

Theorem: (Transformations)

(i) If $(X_n)_{n \geq 1}$ & $(Y_n)_{n \geq 1}$ are subMs w.r.t. $(\mathcal{F}_n)_{n \geq 1}$ then so are $(X_n + Y_n)_{n \geq 1}$ and $(X_n \vee Y_n)_{n \geq 1}$.

(ii) If $(X_n)_{n \geq 1}$ is a M w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, $f: \mathbb{R} \to \mathbb{R}$ is convex and $f(X_n) \in L_1(\Omega, \mathcal{F}, P)$ then $(f(X_n))_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$.

(iii) If $(X_n)_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$, $f: \mathbb{R} \to \mathbb{R}$ is convex, f is non-decreasing and $f(X_n) \in L_1(\Omega, \mathcal{F}, P)$ then $(f(X_n))_{n \geq 1}$ is a subM w.r.t. $(\mathcal{F}_n)_{n \geq 1}$.

(iv) If $(X_{1n})_{n \geq 1}, (X_{2n})_{n \geq 1}, \dots, (X_{mn})_{n \geq 1}$ are m subM wrt $(\mathcal{F}_n)_{n \geq 1}$ and $w_1, w_2, \dots, w_m \geq 0$ then

$$Y_n := \sum_{i=1}^{m} w_i X_{in}$$

is a subM wrt $(\mathcal{F}_n)_{n \geq 1}$.

Proof:

For (i):
Remember that the def of subM, supM & M require $X_n, Y_n \in L_1(\Omega, \mathcal{F}, P)$ so the fact that $(X_n + Y_n)_{n \geq 1}$ is a sub M is a trivial consequence of (e2) from lecture 20.

To show $(X_n \vee Y_n)_{n\geq 1}$ is a subM notice that

$$X_{n+1} \leq X_{n+1} \vee Y_{n+1}$$
$$Y_{n+1} \leq X_{n+1} \vee Y_{n+1}$$

$$\therefore \quad E(X_{n+1} | \mathcal{F}_n) \overset{p\text{-a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n)$$
$$E(Y_{n+1} | \mathcal{F}_n) \overset{p\text{-a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n) \quad (*)$$

by monotonicity (e3).

$$\therefore \quad X_n \vee Y_n \overset{p\text{-a.e.}}{\leq} E(X_{n+1} | \mathcal{F}_n) \vee E(Y_{n+1} | \mathcal{F}_n)$$

by subM prop.

$$\overset{p\text{-a.e.}}{\leq} E(X_{n+1} \vee Y_{n+1} | \mathcal{F}_n)$$

by $(*)$

Also $|X_n \vee Y_n| \leq |X_n| + |Y_n| \in L_1(\Omega, \mathcal{F}, P)$ so that $(X_n \vee Y_n)_{n\geq 1}$ satisfies the subM properties.

For (ii) & (iii)

$$E(f(X_{n+1}) | \mathcal{F}_n) \overset{p\text{-a.e.}}{\geq} f(E(X_{n+1} | \mathcal{F}_n)) \overset{p\text{-a.e.}}{=} f(X_n)$$

by Jensen for conditional expected value (proved in the exact same way as was done in Lecture 11.

The prores (ii) & (iii) is similar.

For (iv)

Follows easily by linear properties of $E(\cdot | \mathcal{F}_n)$.

QED

Notice a few simple consequences of the above thm:

$(X_n)_{n\geq 1}$ is a M $\Rightarrow$ $(|X_n|)_{n\geq 1}$ is a subM

$\left.\begin{array}{l}(X_n)_{n\geq 1} \text{ is a M} \\ \& \ X_n \in L_2\end{array}\right\} \Rightarrow (X_n^2)_{n\geq 1}$ is a subM

$(X_n)_{n\geq}$ is a subM $\Rightarrow$ $(X_n^+)_{n\geq 1}$ is a subM

A collection of examples

Example: (strategy against a bad game)

$X_0 =$ a gamblers initial fortune

$X_n =$ fortune after n-plays of a game when always betting \$1 on each play,
i.e. $X_n - X_{n-1} =$ winnings obtained on the n^{th} play

Suppose $(X_n)_{n\geq 1}$ is a supM wrt $(\mathcal{F}_n)_{n\geq 1}$.

Since X_n is a supM your expected winnings are monotonically decreasing with n

Question:

Can I wager differently (bet more on hot streaks) to convert X_n into a subM?

Let W_n be the wager on n-th play s.t.

$W_n \geq 0$

W_n is $\mathcal{F}_{n-1}$-measurable $\leftarrow$ I can change bet based on the future

W_n is bounded $\leftarrow$ i.e. there is a house max bet winnings on the n^{th} play is

$$W_n \underbrace{(X_n - X_{n-1})}_{=: \Delta_n \,=\, \text{winnings on } n^{th} \text{ play wagering } \$1}$$

The total fortune after n plays with wagers W_n is given by

$$X_n^* = X_0 + W_1\Delta_1 + W_2\Delta_2 + \cdots + W_n \Delta_n$$

W_n @ $\mathcal{F}_{n-1}$, Δ_n @ $\mathcal{F}_n$

claim: $(X_n^*)_{n\geq 1}$ is a supM w.r.t $\mathcal{F}_n$ — so no matter the strategy still a losing game

Proof:

clearly X_n^* is adapted to $\mathcal{F}_n$ an $L_1(\Omega,\mathcal{F},P)$ by bddness.

Also

$$E(X_{n+1}^* | \mathcal{F}_n) \overset{P\text{-a.e}}{=} E(\underbrace{X_n^*}_{@\mathcal{F}_n} + W_{n+1} \underbrace{\Delta_{n+1}}_{@\mathcal{F}_{n+1}} | \mathcal{F}_n)$$

$$\overset{P\text{-a.e.}}{=} X_n^* + W_{n+1} E(\Delta_{n+1}|\mathcal{F}_n)$$

$$\overset{P\text{-a.e.}}{=} X_n^* + \underbrace{W_{n+1}}_{\text{positive}} \underbrace{[\underbrace{E(X_{n+1}|\mathcal{F}_n)}_{\leq X_n} - \underbrace{E(X_n|\mathcal{F}_n)}_{= X_n}]}_{\text{negative}}$$

$$\leq X_n^*$$

$\therefore$ indeed $(X_n^*)_{n\geq 1}$ is a supM.

Now suppose $(X_n)_{n\geq 1}$ is a subM ← a favorable game

By similar reasoning $(X_n^*)_{n\geq 1}$ is also a subM.

Suppose $W_n \in [0,1]$ ← timid play

Now

$$E(\underbrace{X_{n+1}^* - X_n^*}_{n+1^{st} \text{ gain}} | \mathcal{F}_n) \overset{a.e.}{=} E(\underbrace{W_{n+1}}_{@\mathcal{F}_n}(X_{n+1} - X_n)|\mathcal{F}_n)$$

$$\overset{a.e.}{=} \underbrace{W_{n+1}}_{\leq 1} \underbrace{E(X_{n+1} - X_n | \mathcal{F}_n)}_{\geq 0 \text{ by subM}}$$

$$\overset{a.e.}{\leq} E(X_{n+1} - X_n | \mathcal{F}_n)$$

Take E of both sides gives

$$E(X_{n+1}^* - X_n^*) \leq E(X_{n+1} - X_n)$$

$$\therefore \sum_{n=0}^{N-1} E(X_{n+1}^* - X_n^*) \leq \sum_{n=0}^{N-1} E(X_{n+1} - X_n)$$

$$E(X_N^*) - \underbrace{\cancel{E(X_0^*)}}_{\text{same initial fortune}} \leq E(X_N) - \underbrace{\cancel{E(X_0)}}$$

$$\therefore E(X_N^*) \leq E(X_N)$$

i.e. timid play reduces expected fortune.

(11)

Example: Lebesgue decomposition sup M

(12)

Example: Strong Markov property of Brownian Motion