

Homework 6

Due March 22, 2017

Let X_1, X_2, \dots be random variables defined on a probability space (Ω, \mathcal{F}, P) and let Q be another probability measure on (Ω, \mathcal{F}) . Let $\mathcal{F}_n := \sigma\langle X_1, \dots, X_n \rangle$, $\mathcal{F}_\infty := \sigma\langle X_n : n \in \mathbb{N} \rangle$ and

$$P_n = P|_{\mathcal{F}_n} \text{ and } Q_n = Q|_{\mathcal{F}_n}$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Let $Q_n = Q_n^a + Q_n^s$ be the Lebesgue decomposition of Q_n with respect to P_n and

$$\ell_n := \frac{dQ_n^a}{dP_n} \text{ for all } n \in \mathbb{N} \cup \{\infty\}.$$

Exercise 1. Show that

$$\begin{aligned} Q_\infty \ll P_\infty &\iff E(\ell_\infty) = 1; \\ Q_\infty \perp P_\infty &\iff E(\ell_\infty) = 0 \end{aligned}$$

Exercise 2. Show that the process $(\sqrt{\ell_n})_{n \in \mathbb{N}}$ is a supermartingale, is UI and

$$E(\sqrt{\ell_n}) \downarrow E(\sqrt{\ell_\infty})$$

as $n \rightarrow \infty$.

Exercise 3. Show that

$$\begin{aligned} Q_\infty \perp P_\infty &\iff \lim_n E(\sqrt{\ell_n}) = 0 \\ &\iff \lim_n H(Q_n, P_n) = \sqrt{2} \end{aligned}$$

where $H(\cdot, \cdot)$ is the Hellinger distance on probability measures defined as

$$H(Q_n, P_n) = \left[\int_\Omega (\sqrt{q_n} - \sqrt{p_n})^2 d\mu \right]^{1/2}$$

where $q_n = \frac{dQ_n}{d\mu}$, $p_n = \frac{dP_n}{d\mu}$ and μ is any measure such that $Q_n, P_n \ll \mu$ (H is invariant to which μ you choose).

Hint: For the second iff show that $\int_\Omega \sqrt{q_n} \sqrt{p_n} d\mu = \int_\Omega \sqrt{\ell_n} dP_n$.

Exercise 4. Show that if $Q_n \ll P_n$ then

$$E(\ell_\infty | \mathcal{F}_n) \stackrel{a.e.}{=} \ell_n.$$

Exercise 5. Show that

$$Q_\infty \ll P_\infty$$

$$\iff Q_n \ll P_n \text{ for all } n \in \mathbb{N} \text{ and the } \ell_n \text{'s are UI}$$

$$\iff Q_n \ll P_n \text{ for all } n \in \mathbb{N} \text{ and the } \sqrt{\ell_n} \text{'s converge in } L_2$$

$$\iff Q_n \ll P_n \text{ for all } n \in \mathbb{N} \text{ and the } H(Q_n, P_n) \rightarrow 0.$$