Lecture 4: Measures

We are going to define measures and probability measures in tandem so we can compare their properties.

Def: Let $\mathcal{F} \subset 2^{\Omega}$ be a field over Ω

P is a <u>Probability measure</u> on $(\Omega, \mathcal{F})$ if

- $P: \mathcal{F} \to [0,1]$
- $P(\Omega) = 1$
- $P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$ when $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$ & A_k's are disjoint

μ is a <u>measure</u> on $(\Omega, \mathcal{F})$ if

- $\mu: \mathcal{F} \to [0, \infty]$
- $\mu(\emptyset) = 0$
- $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ when $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$ & A_k's are disjoint

<u>Note</u>: these definitions are over fields and allow one to state extension thms easily. (i.e. if P is a prob measure on $B_0^{(0,1]}$ then $\exists$ an extended prob measure on $\sigma\langle B_0^{(0,1]}\rangle$).

<u>Note</u>: Any Prob measure is also a measure since

$$\mathcal{F} \text{ is not empty} \Rightarrow \exists A \in \mathcal{F}$$
$$\Rightarrow \Omega = A \cup A^c \in \mathcal{F}$$
$$\emptyset = \Omega^c \in \mathcal{F}$$
$$\Rightarrow 1 = P(\Omega) = P(\emptyset) + \underbrace{P(\Omega)}_{=1}$$
$$\Rightarrow P(\emptyset) = 0$$

Def: $(\Omega, \mathcal{F})$ is a <u>measurable space</u> if $\mathcal{F} \subset 2^{\Omega}$ & $\mathcal{F}$ is a σ-field over Ω

Def: $(\Omega, \mathcal{F}, P)$ is called a <u>Probability space</u>.

σ-field ($\mathcal{F}$), Prob measure on $\mathcal{F}$ (P)

$(\Omega, \mathcal{F}, \mu)$ is called a <u>measure space</u>

measure on $\mathcal{F}$ (μ)

Def:

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

<u>μ is finite</u> if $\mu(\Omega) < \infty$.

<u>μ is infinite</u> if $\mu(\Omega) = \infty$.

<u>μ is σ-finite over $\mathcal{A} \subset \mathcal{F}$</u> if $\exists A_1, A_2, \ldots \in \mathcal{A}$ s.t.

$$\Omega = \bigcup_{k=1}^{\infty} A_k \quad \&$$

$$\mu(A_k) < \infty$$

<u>μ is σ-finite</u> if μ is σ-finite over $\mathcal{F}$.

<u>Remark</u>: Most of the measures people work with are σ-finite. In fact, we will show that all non-trivial σ-finite measures can be represented as integrals $\mu(A) = \int_A \delta \, dP$ where P is a prob (The probabilist's world view of measure theory).

Probability vrs Measure

Example

$\Omega = \{1, 2, \dots, n\}$

$\mathcal{F} = 2^{\Omega}$

$P(A) = \frac{1}{n} \sum_{w \in \Omega} \underbrace{I_A(w)}_{\text{indicator of } A} = \frac{\#\text{ of elements in } A}{n}$

Basic Properties

Suppose P is a probability measure on a field $\mathcal{F}_0$ of Ω-sets. Then...

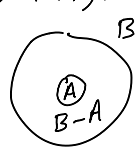
1. $P(A^c) = 1 - P(A)$

2. $P(\emptyset) = 0$

3. If $A \subset B$ then

$$P(B-A) = P(B) - P(A).$$

since $B = (B-A) \cup A$



4. $A \subset B \implies P(A) \leq P(B)$

by 3.

Example

Ω is any set

$\mathcal{F} = 2^{\Omega}$

$\mu(A) = \sum_{w \in \Omega} I_A(w)$

defined as the sup over finite sums. Can prove a version of Fubini to prove the measure axioms hold

μ is called counting measure.

Basic Properties

Suppose μ is a measure on a field $\mathcal{F}_0$ of Ω-sets. Then...

1. $\mu(A^c) = \mu(\Omega) - \mu(A)$ if $\mu(A) < \infty$

since $\mu(\Omega) = \mu(A) + \mu(A^c)$

but can only move $\mu(A)$ over if finite

2. SAME

3. If $A \subset B$ & $\mu(A) < \infty$ then

$$\mu(B-A) = \mu(B) - \mu(A).$$

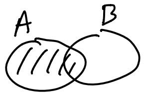
4. SAME, since we still have

$$\mu(B) = \underbrace{\mu(B-A)}_{\geq 0} + \mu(A)$$

Probability vrs Measure

Probability

5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

use  $= A \cup (B - A \cap B)$

and 3.

6. $P\left(\underbrace{\bigcup_{k=1}^{n} A_k}_{\text{can overlap}}\right) \le \sum_{k=1}^{n} P(A_k)$

can overlap

use 5 & induction.

7. If $A_n \uparrow A \in \mathcal{F}_0$ then $P(A_n) \uparrow P(A)$

Note: property 7 is called continuous from below

Proof:

$P(A_n) = P\left(\bigcup_{k=1}^{n} A_k\right)$ since $A_1 \subset A_2 \subset \ldots$

$= P\left(\bigcup_{k=1}^{n} \underbrace{[A_k - A_{k-1}]}\right)$

Same as $A_k - (A_1 \cup \cdots \cup A_k)$. So disjoint

$= \sum_{k=1}^{n} P(A_k - A_{k-1})$

$\uparrow \sum_{k=1}^{\infty} P(A_k - A_{k-1})$

since P is a Prob measure

$= P\left(\bigcup_{k=1}^{\infty} [A_k - A_{k-1}]\right)$

$= P(A)$

Measure

5. $\mu(A \cup B) = \mu(A) + \mu(B - A \cap B)$

$= \mu(A) + \mu(B) - \mu(A \cap B)$

$\uparrow$ if $\mu(A \cap B) < \infty$

6. SAME. For proof use the first part of 5 & induction

7. SAME, same proof too

Probability vrs Measure

Probability

8. If $A_n \downarrow A \in \mathcal{F}_0$ then $P(A_n) \downarrow P(A)$.

Called <u>Continuity from above</u>

Proof

First note that $A_n \downarrow A \iff A_n^c \uparrow A^c$

By 7. we have

$$\underbrace{P(A_n^c)}_{=1-P(A_n)} \uparrow \underbrace{P(A^c)}_{=1-P(A)}$$

$\therefore$ $P(A_n) \downarrow P(A)$.

9. $P\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} P(A_k)$.

Proof: $\bigcup_{k=1}^{n} A_k \uparrow \bigcup_{k=1}^{\infty} A_k$

$$\therefore \quad P\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_n P\left(\bigcup_{k=1}^{n} A_k\right)$$

$$= \lim_n \sum_{k=1}^{n} P(A_k - A_1 \cup \cdots \cup A_{k-1})$$

$$\le \lim_n \sum_{k=1}^{n} P(A_k)$$

Measure

8. If $A_n \downarrow A \in \mathcal{F}_0$ and $\mu(A_m) < \infty$ for some m

then $\mu(A_n) \downarrow \mu(A)$.

Note: the extra condition is necessary.

e.g. $\Omega = \mathbb{Z}$, $\mathcal{F}_0 = 2^{\Omega}$, μ = counting measure

$A_n = \{n, n+1, \dots\} \downarrow A = \emptyset$ but

$$\underbrace{\mu(A_n)}_{\infty} \not\downarrow \underbrace{\mu(A)}_{=0}$$

Proof:

Let $n \ge m$ & note $A_m \supset A_n \supset A$.

$\therefore$ $\mu(A) \le \mu(A_n) \le \mu(A_m) < \infty$

Define $P(B) := \dfrac{\mu(B \cap A_m)}{\mu(A_m)}$

P is a prob measure on $(\Omega, \mathcal{F}_0)$.

$\therefore$ $P(A_n) \downarrow P(A)$ by 8 for Probs

$\therefore$ $\dfrac{\mu(A_n \cap A_m)}{\cancel{\mu(A_m)}} \downarrow \dfrac{\mu(A \cap A_m)}{\cancel{\mu(A_m)}}$

$\therefore$ $\mu(A_n) \downarrow \mu(A)$ since $A_n \cap A_m = A_n$ & $A \cap A_m = A$.

9. SAME. SAME proof ~~too~~.

Probability vrs Measure

Probability

Thm (uniqueness)

Let P & Q be two prob measures on $(\Omega, \sigma\langle \mathcal{P} \rangle)$. If

(i) $P = Q$ on $\mathcal{P}$

(ii) $\mathcal{P}$ is a π-sys

Then $P = Q$ on $\sigma\langle \mathcal{P} \rangle$.

Proof: Use "good sets".

Let $\mathcal{G} = \{A \in \sigma\langle \mathcal{P} \rangle : P(A) = Q(A)\}$

we want to show $\sigma\langle \mathcal{P} \rangle \subset \mathcal{G}$.

By Dynkin's π-λ extension of goodsets we need $\underbrace{\mathcal{P}}_{\pi\text{-system}} \subset \underbrace{\mathcal{G}}_{\lambda\text{-sys.}}$

Clearly $\mathcal{P} \subset \mathcal{G}$ & $\mathcal{P}$ is a π-sys.

To show $\mathcal{G}$ is a λ-sys just note:

- $\Omega \in \mathcal{G}$, since $P(\Omega) = Q(\Omega) = 1$
- $A \in \mathcal{G} \Rightarrow P(A) = Q(A)$
  $\Rightarrow P(A^c) = Q(A^c)$
  $\Rightarrow A^c \in \mathcal{G}$
- $\underbrace{A_1, A_2 \ldots}_{\text{disjoint}} \in \mathcal{G} \Rightarrow P\left(\bigcup_k A_k\right) = \sum_{k=1}^{\infty} P(A_k)$
  $= \sum_{k=1}^{\infty} Q(A_k)$
  $= Q\left(\bigcup_k A_k\right)$
  $\Rightarrow \bigcup_k A_k \in \mathcal{G}$.

QED.

Measure

Thm (uniqueness)

Let μ & ν be two measures on $(\Omega, \sigma\langle \mathcal{P} \rangle)$. If

(i) $\mu = \nu$ on $\mathcal{P}$

(ii) $\mathcal{P}$ is a π-sys

(iii) μ, ν are σ-finite over $\mathcal{P}$

Then $\mu = \nu$ on $\sigma\langle \mathcal{P} \rangle$.

Note: The extra condition is necessary.

For example, $\Omega = (0,1]$, $\mathcal{P} = \{(0,x] : 0 < x \le 1\}$,

$\mu(A) = \sum_{w \in \Omega} I_A(w)$ i.e. counting measure

$\nu(A) = \sum_{w \in \Omega} I_{A \cap \text{rational}}(w)$

μ & ν both agree on $\mathcal{P}$ but not on $\sigma\langle \mathcal{P} \rangle$

since $\mu(\text{irrationals}) = \infty$
$\nu(\text{irrationals}) = 0$

Proof:

Let $C_1, C_2, \ldots \in \mathcal{P}$ which cover Ω and $\mu(C_k), \nu(C_k) < \infty$. Define

$$P_k(A) = \begin{cases} \mu(A \cap C_k)/\mu(C_k) & \text{if } \mu(C_k) > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$Q_k(A) = \begin{cases} \nu(A \cap C_k)/\nu(C_k) & \text{if } \nu(C_k) > 0 \\ 0 & \text{o.w.} \end{cases}$$

case 1: $\mu(C_k) = 0$

$\therefore \nu(C_k) = 0$

$\therefore P_k = Q_k$ on $\sigma\langle \mathcal{P} \rangle$

7

Case 2: $\mu(C_k) > 0$

$\therefore \nu(C_k) > 0$ & both P_k, Q_k are prob measures.

Since $A \in \mathcal{P} \Rightarrow A \cap C_k \in \mathcal{P}$ we have $P_k = Q_k$ on $\mathcal{P}$.

Both P_k & Q_k are prob measures.

$\therefore \quad P_k = Q_k$ on $\sigma\langle \mathcal{P} \rangle$

Now stitch P_k & Q_k together

$$\Omega = \bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} C_k - C_1 \cup \cdots \cup C_{k-1}$$

$$= \bigcup_{k=1}^{\infty} \underbrace{C_k \cap C_1^c \cap \cdots \cap C_{k-1}^c}_{\substack{\text{disjoint cover} \\ \text{... but not in} \\ \mathcal{P}.}}$$

$$\therefore \mu(A) = \mu\left(\bigcup_{k=1}^{\infty} A \cap C_k \cap C_1^c \cap \cdots \cap C_{k-1}^c \right)$$

$$= \sum_{k=1}^{\infty} \mu(A \cap C_k \cap C_1^c \cap \cdots \cap C_{k-1}^c)$$

$$= \sum_{k=1}^{\infty} \mu[C_k] P_k(A \cap C_1^c \cap \cdots \cap C_{k-1}^c)$$

$$= \sum_{k=1}^{\infty} \nu[C_k] Q_k(A \cap C_1^c \cap \cdots \cap C_{k-1}^c)$$

$$= \nu(A)$$

QED

8

Continuity is equivalent to countable additivity for P

The next thm is only needed for probability & is useful for showing a P is measurable

Thm:

If P is a finitely additive prob on a field $\mathcal{F}_0 \subset 2^{\Omega}$ then the following are equivalent

(i) P is a prob measure

(ii) P is continuous from below

(iii) P is continuous from below

(iv) $\forall A_1, A_2, \ldots \in \mathcal{F}_0$ s.t. $A_n \downarrow \emptyset$ one has $\underbrace{P(A_n) \downarrow 0}_{\text{continuous from above at } \emptyset}$

Proof:

we already have

$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

show $(iv) \overset{(a)}{\Rightarrow} (ii) \overset{(b)}{\Rightarrow} (i)$

For (a) suppose (iv) & let $\underbrace{A_n \uparrow A}_{\text{all in } \mathcal{F}_0}$

$\therefore \quad A_n^c \downarrow A^c$

$\therefore \quad A_n^c - A^c \downarrow \emptyset$

$\therefore \quad P(A_n^c - A^c) \downarrow 0$ by (iv)

$\therefore \quad P(A_n^c) - P(A^c) \downarrow 0$, since $A_n \subset A$ so that $A^c \subset A_n^c$

$\therefore \quad -P(A_n) + P(A) \downarrow 0$

$\therefore \quad P(A_n) \uparrow P(A)$

$\therefore$ (ii) holds

Note: these properties follow by FAP

For (b) suppose (ii) holds

Let $\underbrace{A_1, A_2, \ldots}_{\text{disjoint}} \in \mathcal{F}_0$ & $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_0$

$$\therefore P\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_n \uparrow P\Big(\bigcup_{k=1}^{n} A_k\Big) = \lim_n \sum_{k=1}^{n} P(A_k) = \sum_{k=1}^{\infty} P(A_k)$$

QED

Null & Negligible sets

Definitions: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

- A is μ-null iff $A \in \mathcal{F}$ & $\mu(A) = 0$.
- A is μ-negligible iff $\exists$ a μ-null cover of A.
- $(\Omega, \mathcal{F}, \mu)$ is complete iff all μ-neg subsets of Ω are in $\mathcal{F}$
- The completion of $\mathcal{F}$ w.r.t. μ is $\overline{\mathcal{F}} = \sigma\langle \mathcal{F}, \mathcal{N}_\mu \rangle$ where $\mathcal{N}_\mu := \{N \subset \Omega : N \text{ is } \mu\text{-neg}\}$.

Once we prove the existance of Lebesgue measure $\mathcal{L}$ on $\mathbb{R}$

$$\overline{\mathcal{B}^{\mathbb{R}}} := \sigma\langle \mathcal{B}^{\mathbb{R}}, \mathcal{N}_{\mathcal{L}} \rangle = \text{Lebesgue measurable sets of } \mathbb{R}$$

where $\mathcal{B}^{\mathbb{R}}$ = Borel measurable sets of $\mathbb{R}$.

The following thm shows how to extend $(\Omega, \mathcal{F}, \mu)$ to $(\Omega, \overline{\mathcal{F}})$.

Claim: if $(\Omega, \mathcal{F}, \mu)$ is a measure space then $\sigma\langle \mathcal{F}, \mathcal{N}_\mu \rangle = \underbrace{\{F \cup N : F \in \mathcal{F}, N \in \mathcal{N}_\mu\}}_{\text{"RHS"}}$

Proof:

The inclusion "$\supset$" is trivial.

For "$\subset$" use good sets.

- clearly $\mathcal{F}, \mathcal{N}_\mu \subset$ RHS since $\emptyset \in \mathcal{F}$ & $\emptyset \in \mathcal{N}_\mu$
- $\emptyset \in$ RHS.
- RHS is closed under countable unions since $\bigcup_k (F_k \cup N_k) = \underbrace{\Big(\bigcup_k F_k\Big)}_{\in \mathcal{F}} \cup \underbrace{\Big(\bigcup_k N_k\Big)}_{\in \mathcal{N}_\mu}$
- $A \in$ RHS $\Rightarrow$ $A = F \cup N$ for a μ-null $C \supset N$

$$\Rightarrow A^c = F^c \cap N^c = (F^c \cap N^c) \cap (C \cup C^c) = \underbrace{(F^c \cap \underbrace{N^c \cap C^c}_{= C^c})}_{\in \mathcal{F}} \cup \underbrace{(F^c \cap N^c \cap C)}_{\in \mathcal{N}_\mu \text{ since covered by } C}$$

since $C^c \subset N^c$ $\therefore$ $C^c \cap N^c = C^c$

$\Rightarrow A^c \in$ RHS

QED

Now we can define $\bar{\mu}$ on $\bar{\mathcal{F}}$ by

$$\bar{\mu}(\underbrace{F}_{\in \mathcal{F}} \cup \underbrace{N}_{\in \mathcal{N}_\mu}) = \mu(F)$$

Claim:

(i) $\bar{\mu}$ is well defined

(ii) $\bar{\mu}$ is a measure on $(\Omega, \bar{\mathcal{F}})$ extending μ on $(\Omega, \mathcal{F})$.

(iii) $\bar{\mu}$ is unique

(iv) $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ is complete.

Proof:

For (i): Suppose $F_1 \cup N_1 = F_2 \cup N_2$

$\therefore\ F_1 - F_2 \subset N_2$ & $F_2 - F_1 \subset N_1$

For suppose $\omega \in F_1 - F_2$

$\therefore\ \omega \in F_1$ & $\omega \notin F_2$

$\therefore\ \omega \in \underbrace{F_1 \cup N_1}_{= F_2 \cup N_2}$ & $\omega \notin F_2$

$\therefore\ \omega \in N_2$

$\therefore\ \mu(F_1 - F_2) = \mu(F_2 - F_1) = 0$

$$\begin{aligned}\therefore\ \mu(F_1) &\leq \mu(F_1 \cup F_2) \\ &= \mu(F_2 \cup (F_1 - F_2)) \\ &= \mu(F_2) + \underbrace{\mu(F_1 - F_2)}_{=0}\end{aligned}$$

& $\mu(F_2) \leq \mu(F_1)$ similarly

$\therefore\ \mu(F_1) = \mu(F_2)$

$\therefore\ \bar{\mu}(F_1 \cup N_1) = \bar{\mu}(F_2 \cup N_2)$.

For (ii): only need to show

$$\begin{aligned}\bar{\mu}\Big(\bigcup_k \underbrace{(F_k \cup N_k)}_{\text{disjoint over } k}\Big) &= \bar{\mu}\Big(\underbrace{\Big[\bigcup_k F_k\Big]}_{\in \mathcal{F}} \cup \underbrace{\Big[\bigcup_k N_k\Big]}_{\in \mathcal{N}_\mu}\Big) \\ &:= \mu\Big(\bigcup_k F_k\Big) \\ &= \sum_k \mu(F_k) \\ &= \sum_k \bar{\mu}(F_k \cup N_k)\end{aligned}$$

$F_k \cup N_k$ disjoint implies F_k's disjoint

For (iii):

Let ν be a measure on $(\Omega, \bar{\mathcal{F}})$ s.t. $\nu = \mu$ on $\mathcal{F}$.

$\therefore\ \nu = \mu$ on $\mathcal{N}_\mu$

$\therefore\ \nu(F \cup N) = \nu(F) = \mu(F) = \bar{\mu}(F \cup N)$

$\geq$ by "increasing"

$\leq$ by sub-additivity

$\therefore\ \nu = \mu$ on $\bar{\mathcal{F}}$.

For (iv) let $\bar{N} \subset \Omega$ be $\bar{\mu}$-neg

$\therefore\ \bar{N} \subset F \cup N \subset F \cup A$

for $F \in \mathcal{F}$ & $N \in \mathcal{N}_\mu$ s.t. $0 = \bar{\mu}(F \cup N) = \mu(F)$

for $A \in \mathcal{F}$ s.t. $\mu(A) = 0$

$\therefore\ \bar{N}$ is μ-neg so $\bar{\mathcal{F}}$ contains all $\bar{\mu}$-neg sets

QED

(11)

What are we going to use measures for

Random variables will be maps

$$\Omega \xrightarrow{X} \mathbb{R}$$

where $(\Omega, \mathcal{F}, P)$ is a prob space & X is $\mathcal{F}$-measurable.

$$EX := \int_\Omega X(\omega)\, dP(\omega)$$

integration w.r.t. measure P

For two measures P & Q on $(\Omega, \mathcal{F})$ the Radon-Nikodym derivative $\frac{dPX^{-1}}{dQX^{-1}}$ will be the likelihood ratio of P to Q for X.

Radon-Nikodym derivatives will also give use conditional expected values, etc.

(12)

Why does probability seem so rich a subject when its essentially just measure theory under the constraint $\mu(\Omega) = 1$

I think the answer is that once $\mu(\Omega) = 1$ you can understand μ as modeling a random draw $\omega \in \Omega$. So the extra assumption essentially gives an isomorphism from the abstract $(\Omega, \mathcal{F}, \mu)$ to a physical random procedure.

So a solution can be made on the abstract structure $(\Omega, \mathcal{F}, \mu)$ or the physical random draws it models.