Homework 4

Due March 6, 2017

Exercise 1 (Convex projection). *Let C be a convex subset of $L_2(\Omega, \mathcal{F}, P)$ (so that $X, Y \in C$ and $a \in [0,1]$ implies $aX + (1-a)Y \in C$). Suppose also that C is closed. Let $Y \in L_2(\Omega, \mathcal{F}, P)$. Generalize the Projection Theorem by showing that there exists an almost surely unique member of C, denoted $\mathcal{P}_C Y$, such that*

$$\|Y - \mathcal{P}_C Y\|_2 = \inf_{X \in C} \|Y - X\|_2. \tag{1}$$

Also show that $\mathcal{P}_C Y$ is characterized by the condition that $\mathcal{P}_C Y \in C$ and

$$\langle Y - \mathcal{P}_C Y, X - \mathcal{P}_C Y \rangle \leq 0 \text{ for all } X \in C. \tag{2}$$

Exercise 2. *Let $X_1, X_2, \ldots$ be an infinite sequence of real random variables on $(\Omega, \mathcal{A})$ such that $\mathcal{A} := \sigma\langle X_n \colon n \geq 1 \rangle$. Let P and Q be probability measures on $(\Omega, \mathcal{A})$ with the following properties: under P the $X_n \overset{iid}{\sim} \mathcal{N}(0,1)$, while under Q the $X_n \overset{iid}{\sim} \mathcal{N}(\theta_n, 1)$. The goal of this exercise is to show that*

$$Q \perp P \Longleftrightarrow \lim_n \tau_n = \infty \tag{3}$$

$$Q \ll P \Longleftrightarrow \lim_n \tau_n < \infty \tag{4}$$

where $\tau_n := \sum_{k=1}^n \theta_k^2$.

(a) Let $\mathcal{A}_n := \sigma\langle X_1, \ldots, X_n \rangle$ and let $P_n = P|_{\mathcal{A}_n}$ and $Q_n = Q|_{\mathcal{A}_n}$ be the restrictions of P and Q to $\mathcal{A}_n$. Show that

$$\frac{dQ_n}{dP_n} = e^{S_n}$$

serves as a density of Q_n with respect to P_n over $\mathcal{A}_n$ where $S_n := Y_1 + \cdots + Y_n$ with $Y_k := \theta_k X_k - \theta_k^2/2$.

(b) What is the distribution of S_n under P and under Q. Draw a rough sketch of these two distributions for $\tau_n^2 = 25$.

(c) Suppose $\lim_n \tau_n = \infty$ and show that $Q \perp P$. Hint: Find $\mathcal{A}$-sets $A_1, A_2, \ldots$ such that $\lim_n P(A_n^c) = \lim_n Q(A_n) = 0$ and invoke an exercise from STAT235A.

(d) Suppose $\tau := \lim_n \tau_n < \infty$. Show that under P, $S_n \xrightarrow{P} S$ where S is some random variable with $S \sim \mathcal{N}(-\frac{1}{2}\tau, \tau)$.

(e) Suppose $\tau := \lim_n \tau_n < \infty$. Show that $\frac{dQ_n}{dP_n} \xrightarrow{L_1(P)} e^S$.

(f) Suppose $\tau := \lim_n \tau_n < \infty$. Show that $Q \ll P$ with $\frac{dQ}{dP} := e^S$. Hint: first show that $Q(A) = \int_A e^{S_n} dP$ for all $A \in \mathcal{A}_m$ and all $m \leq n$.

(f) Show (3) and (4).