Homework 1

Due Wednesday, January 25, 2016

Exercise 1. *For $n = 1, 2, \ldots$ let X_n be a random variable uniformly distributed over the $n+1$ points k/n for $k = 0, 1, \ldots, n$. Show that as $n \to \infty$, $X_n \xrightarrow{\mathcal{D}} X$ where $X \sim$ Uniform$(0, 1)$.*

Exercise 2. *Let $\delta(x) := e^{-e^{-x}} e^{-x}$ be the density (with respect to Lebesque measure on $\mathbb{R}$) of the random variable X with. Show $\phi_X(t) = \Gamma(1-it)$ and $F_X(x) = e^{-e^{-x}}$ for $t, x \in \mathbb{R}$ where ϕ_X and F_X are the characteristic function and cdf for X, respectively. Hint: for ϕ_X use the substitution principle from Lecture 12 on generating functions and moments.*

Exercise 3. *Let $Y_1, Y_2, \ldots$ be iid standard exponential random variables with density $\delta(x) := e^{-x}$ (with respect to Lebesque measure on the non-negative real numbers). For each n set $X_n := max(Y_1, Y_2, \ldots, Y_n)$. Show that as $n \to \infty$, $X_n - \log(n) \xrightarrow{\mathcal{D}} X$, where X is a random variable with distribution function $F(x) = e^{-e^{-x}}$ for $x \in \mathbb{R}$.*

Exercise 4. *Show that a sequence X_n of integer-valued random variables converges in distribution to a random variable X if and only if X is an integer-valued random variable and for every $k \in \mathbb{Z}$*

$$\lim_{n \to \infty} P(X_n = k) = P(X = k).$$

Exercise 5. *Let $X_n \sim$ Binomial(n, p_n). Show that if $np_n \to \lambda > 0$ then $X_n \xrightarrow{\mathcal{D}} X$ where $X \sim$ Poisson(λ) has density $\delta(k) := \frac{\lambda^k}{k!} e^{-\lambda}$ with respect to counting measure on $\{0, 1, 2, \ldots\}$.*

Exercise 6. *For $n = 1, 2, \ldots$ let G_n be a random variable with with density $\delta(k) := (1-p)^{k-1} p$ with respect to counting measure on $\{1, 2, 3, \ldots\}$ where $p \in (0, 1)$ is a parameter. Show that if $\mu_n := E(G_n) \to \infty$ as $n \to \infty$, then $G_n/\mu_n \xrightarrow{\mathcal{D}} X$ where X is a standard exponential random variable.*