Homework 4

Due Tuesday, November 1, 2016

Exercise 1. *Jim and John were twin brothers who lived their life as gardeners. When they passed away Jim was sent to heaven and John—being the evil twin—was sent to hell.*

In heaven Jim was given a bed of flowers arranged on a line segment, say $(0,1]$, with one flower planted at each real number in $(0,1]$. He was given a watering can and instructions to water each flower infinitely often (any amount of water will do) or else it would die. To make it challenging the amount of water coming out of the spout reduces with time so that Jim can only water a region of width $1/n$ during the n^{th} minute (where $n = 1, 2, \dots$ recorded the minutes of a stopwatch).

Jim, in hell, was given the exact same setup as John but his watering can only waters a region of width $1/n^2$ at the n^{th} minute.

When n reaches infinity both the Devil and God will pick a flower from their respective gardens. Show that if Jim is smart he can move back and fourth across his garden so that God's flower will be alive with probability 1. Also show that no matter what John does the Devil's flower will be dead with probability 1.

Exercise 2. *Let $X_1, X_2, \dots$ denote the sequence of independent coin flip random variables defined in Lecture 1. In particular, let $(\Omega, \mathcal{F}, P)$ represent the uniform probability measure on $\Omega := (0,1]$ where $\mathcal{F} := \mathcal{B}((0,1])$ and $X_k\colon \Omega \to \{0,1\}$ satisfies*

$$X_k = \begin{cases} 0 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2. \end{cases}$$

Show that $U := \sum_{k=1}^{\infty} 2^{-k} X_k$ is uniformly distributed on $[0,1]$. Hint: show

$$P[U \le x] = \begin{cases} x & \text{when } x \in [0,1]; \\ 1 & \text{when } x > 1; \\ 0 & \text{when } x < 0. \end{cases}$$

for all $x \in \mathbb{R}$ by analyzing $P[U_n \le x]$ as $n \to \infty$ where $U_n := \sum_{k=1}^{n} 2^{-k} X_k$.

Exercise 3. *Let $(\Omega, \mathcal{F}, P)$ and $X_1, X_2, \dots$ denote the same coin flip random variables described in the previous exercise. Let $\theta \in (0, 1/2)$ and define $W := (1-\theta)\sum_{k=1}^{\infty} \theta^{k-1} X_k$. Describe $P(W \le x)$ as a function of x and show that the probability measure Q, defined as $Q(B) := P(W \in B)$ for $B \in \mathcal{B}((0,1])$, is singular with respect to P (i.e. $P \perp Q$). Hint: Start by noticing that if $X_1 = 0$ then $W \le \theta$ and if $X_1 = 1$ then $W \ge 1 - \theta$.*

Exercise 4. *Let $(\Omega, \mathcal{F})$ be a measure space and let $f_0, f_1, \dots$ be an infinite sequence of $\mathcal{F}$-measurable functions mapping Ω to $\mathbb{R}$. Show that the radius R of convergence of the random power series $\sum_{k=0}^{\infty} f_k x^k$ is an $\mathcal{F}$-measurable function of Ω.*

Exercise 5. *Give an example of two measurable spaces $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$, a ⓜ$\mathcal{F}_1/\mathcal{F}_2$ mapping $f : \Omega_1 \to \Omega_2$, and an event $B \in \mathcal{F}_1$ such that $f(B) \notin \mathcal{F}_2$.*