Homework 1

Due October 4, 2016

Exercise 1. *For* $\omega \in (0,1]$ *let* $s_n(\omega) = \sum_{k=1}^n r_k(\omega)$ *where*

$$r_k(\omega) := \begin{cases} 1 & \text{if the } k^{th} \text{ binary (non-terminating) digit of } \omega \text{ is } 1 \\ -1 & \text{if the } k^{th} \text{ binary (non-terminating) digit of } \omega \text{ is } 0 \end{cases}$$

denotes the Rademacher maps defined in Lecture 1. Using just calculus show that

$$\int_0^1 e^{ts_n(\omega)}\, d\omega = \Big(\frac{e^t + e^{-t}}{2}\Big)^n \tag{1}$$

for each $t \in \mathbb{R}$*. Hint: It may help to use the fact that* s_n *is constant over the dyadic intervals* $\big\{\big(\frac{k-1}{2^n}, \frac{k}{2^n}\big] \colon k \in \{0, 1, 2, \ldots, 2^n\}\big\}$ *and that the dyadic intervals are in one-to-one correspondence with* $\{-1, 1\}^n$

Exercise 2. *Show that*

$$P\big[|s_n/n| \geq \epsilon\big] \leq 2e^{-n\epsilon^2/2}$$

for each $\epsilon > 0$*.*

Hint: Use (1) in conjunction with the inequality $(e^x + e^{-x})/2 \leq \exp(x^2/2)$ *which holds (why?) for all* $x \in \mathbb{R}$*.*

Definition 1 (Semi-ring with unit). *A collection of events* $\mathcal{A} \subset 2^\Omega$ *is called a semi-ring with unit if*

1. $\Omega \in \mathcal{A}$
2. $A, B \in \mathcal{A} \Longrightarrow A \cap B \in \mathcal{A}$
3. *If* $A \in \mathcal{A}$ *then* A^c *is a finite disjoint union of* $\mathcal{A}$*-sets*

Exercise 3. *Suppose* $\mathcal{A} \subset 2^\Omega$ *is a semi-ring with unit. Let* $\mathscr{D}$ *denote the class of finite disjoint unions of* $\mathcal{A}$*-sets. Show* $f\langle\mathcal{A}\rangle = \mathscr{D}$*. Hint: first show* $\mathscr{D}$ *is closed intersections, then complements.*

Exercise 4. *Show that* $\mathcal{B}_0^{(0,1]}$*, defined in the first section of the notes, is a field and coincides with* $f\langle(a,b] : 0 \leq a \leq b \leq 1\rangle$*.*

Exercise 5. *Let* $\Omega = \mathbb{R}$*. Show that* $f\langle(-\infty, a] : -\infty < a < \infty\rangle$ *is the the set of finite (possibly empty) disjoint unions of intervals of the form* $(-\infty, b]$*,* (a, ∞) *and* $(a, b]$ *for finite* $a < b$*. Hint: Modify the generators to get a semi-ring.*

Exercise 6. *Let* $\Omega = \mathbb{R}$*,* $\mathcal{C} = \{(-\infty, a] : a \in \mathbb{R}\}$ *and* $\mathcal{F} = \sigma\langle\mathcal{C}\rangle$*. For any* $x \in \mathbb{R}$*, use good sets to show*

$$A \in \mathcal{F} \Longrightarrow A + x \in \mathcal{F}$$

where $A + x := \{a + x : a \in A\}$*.*