Lecture 18: L_p spaces of r.v.s

Just as in the previous lecture we will be fixing a probability space $(\Omega, \mathcal{F}, P)$ and consider the collection of r.v.s defined on that space.

In particular:

Assumption for the remainder of this lecture:

Suppose $X, Y, X_1, X_2, \dots$ are r.v.s all defined on the same probability space

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$(\Omega, \mathcal{F}, P) \xrightarrow{Y} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$
$$\vdots$$

L_p spaces $(p \geq 1)$

Definition: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $p \geq 1$. The L_p space of r.v.s defined on $(\Omega, \mathcal{F}, P)$ is defined as

$$\{ X : \Omega \to \mathbb{R} \text{ s.t. } X \in \mathcal{F}/\mathcal{B}(\mathbb{R}) \ \& \ E|X|^p < \infty \}$$

and denoted $L_p(\Omega, \mathcal{F}, P) \equiv L_p(P) \equiv L_p$.

Remark: We work with random variables but most of the following results can be extended to the set of random vectors all mapping into $\mathbb{R}^d$.

We will be interested in the metric & geometric properties of L_p & interpreting some classic functional analysis results from a probabilistic perspective.

Example:

Let W_t be Brownian Motion so that

$$(\Omega, \mathcal{F}, P) \xrightarrow{W(\omega)} (C[0,\infty), \mathcal{B}(C[0,\infty)))$$

Since for each fixed $t \in [0,\infty)$ W_t is a r.v. defined on $(\Omega, \mathcal{F}, P)$ we can consider the stochastic process $(W_t : t \in [0,\infty))$ as a collection of r.v.s indexed by t

$$\{ W_t : t \in [0,\infty) \} \subset L_2(\Omega, \mathcal{F}, P)$$

Definition:

For $X, Y \in L_p(\Omega, \mathcal{F}, P)$ define

$$\|X\|_p := (E|X|^p)^{1/p}$$
$$d_p(X,Y) := \|X - Y\|_p$$

Theorem (Hölder)

For any two r.v.s X & Y defined on $(\Omega, \mathcal{F}, P)$ and $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$E|XY| \leq \underbrace{\|X\|_p \|Y\|_q}_{\text{could be } 0 \cdot \infty = 0 \text{ by convention}}$$

Moreover if $X, Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ then

$$|E(XY)| \leq \|X\|_p \|Y\|_q$$

Proof: We already proved this in Lecture 11.

③

Theorem:

$1 \le p < q \Rightarrow L_q(\Omega, \mathcal{F}, P) \subset L_p(\Omega, \mathcal{F}, P)$.

Proof: Like Hölder, this comes from Young's inequality: $a^{w_1} b^{w_2} \le w_1 a + w_2 b$ when $w_1, w_2, a, b > 0$ and $w_1 + w_2 = 1$.

Indeed set $w_1 = \frac{p}{q} < 1$ & $w_2 = 1 - w_1$. Then

$$X \in L_q \Longrightarrow E|X|^q = E|X|^{p\frac{q}{p}}$$
$$= E\left((|X|^p)^{w_1} 1^{w_2}\right)$$
$$\le \underbrace{w_1 E|X|^p + w_2}_{<\infty}$$

QED

Theorem: ($\|\cdot\|_p$ is a pseudo-norm)

$\forall X \in L_p(\Omega, \mathcal{F}, P)$ we have that

(i) $\|X\|_p \ge 0$

(ii) $\|X\|_p = 0 \Rightarrow X = 0$ P-a.e. ← hence its only pseudo. By "a.e. useful" Thm.

(iii) $\|cX\|_p = |c|\,\|X\|_p \quad \forall c \in \mathbb{R}$

(iv) $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$ (Minkowski's ineq)

Proof: we just need to show (iv).

$$E|X+Y|^p = E\left(|X+Y|\,|X+Y|^{p-1}\right)$$
$$\le E\left(|X|\,|X+Y|^{p-1}\right) + E\left(|Y|\,|X+Y|^{p-1}\right)$$

Now notice $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow 1 + \frac{p}{q} = p$
$\Rightarrow \frac{p}{q} = p - 1$

$$= E\left(|X|\,|X+Y|^{p/q}\right) + E\left(|X|\,|X+Y|^{p/q}\right)$$
$$\overset{\text{Hölder}}{\le} \|X\|_p \left\| |X+Y|^{p/q} \right\|_q + \|Y\|_p \left\| |X+Y|^{p/q} \right\|_q$$
$$= \left(\|X\|_p + \|Y\|_p\right) \underbrace{\left(E|X+Y|^p\right)^{1/q}}_{\text{divide this out of both sides}}$$

④

$$\therefore \underbrace{\left(E|X+Y|^p\right)^{1 - \frac{1}{q}}}_{= \|X+Y\|_p} \le \|X\|_p + \|Y\|_p$$

QED.

Remark: The previous Thm shows $d_p(X,Y)$ is a pseudo-metric on L_p.

It will also be useful to note that $\|\cdot\|_p$ is continuous w.r.t d_p.

Theorem: $X, Y \in L_p \Longrightarrow \left| \|X\|_p - \|Y\|_p \right| \le d_p(X,Y)$

Proof:

Minkowski: $\|X\|_p \le \|X - Y\|_p + \|Y\|_p$

$\|Y\|_p \le \underbrace{\|X - Y\|_p}_{= d_p(X,Y)} + \|X\|_p$

QED

L_p convergence

Here we study completness, closure & separability of L_p and prove the "L_p convergence theorem" which will be useful later.

Definition:

$X_n \xrightarrow{L_p} X$ iff $E\,|\underbrace{X_n}_{\text{technically no requirement that } X_n, X \in L_p} - X|^p \to 0$ as $n \to \infty$.

Theorem (uniqness of limits)

$$X_n \xrightarrow{L_p} X \ \& \ X_n \xrightarrow{L_p} Y \Rightarrow X = Y \ P\text{-a.e.}$$

Proof:

Note the following useful identity which follows by convexity of $|\cdot|^p$

$$\left|\frac{x+y}{2}\right|^p \le \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

$$\therefore \quad E|\underset{\pm X_n}{X-Y}|^p \le 2^p\Big(\underbrace{\tfrac{1}{2}E|X-X_n|^p + \tfrac{1}{2}E|Y-X_n|^p}_{\to 0 \text{ as } n\to\infty}\Big)$$

QED

Theorem: (Cauchy Criteria)

$X_n \xrightarrow{L_p}$ to some r.v. X iff

$$\lim_n \lim_m E|X_m - X_n|^p = 0$$

Proof:

$(\Longrightarrow)$ $$E|\underset{\pm X}{X_m - X_n}|^p \le 2^p\Big(\underbrace{E\frac{|X_m-X|^p}{2} + E\frac{|X_n-X|^p}{2}}_{\to 0 \text{ as } m,n\to\infty}\Big)$$

$(\Longleftarrow)$

$$P(|X_m - X_n| \ge \varepsilon) \le \frac{E|X_m - X_n|^p}{\varepsilon^p}$$

implies $\{X_n\}_{n\ge 1}$ is Cauchy for convergence in probability.

$\therefore$ $\exists$ r.v. X s.t. $X_n \xrightarrow{P} X$

$\therefore$ $\exists n_k$ s.t. $X_{n_k} \xrightarrow[k\to\infty]{a.e.} X$ by sub-sub-seq Thm.

$$\therefore \quad |X_n - X_{n_k}|^p \xrightarrow[k\to\infty]{a.e.} |X_n - X|^p \quad \forall n$$

by continuous mapping since $X_n - X_{n_k} \xrightarrow{a.e.} X_n - X$

Now

$$\begin{aligned} E|X_n - X|^p &\le \liminf_k E|X_n - X_{n_k}|^p, \text{ Fatou} \\ &\le \limsup_k E|X_n - X_{n_k}|^p \\ &\le \limsup_m E|X_n - X_m|^p \end{aligned}$$

Taking $\lim_n$ of both sides gives

$$X_n \xrightarrow{L_p} X.$$

QED

Theorem (L_p is Polish w.r.t d_p)

If $p \ge 1$ then $L_p(\Omega, \mathcal{F}, P)$ is a linear space which is closed & complete w.r.t d_p.

If, in addition, $\mathcal{F}$ is countably generated then $L_p(\Omega, \mathcal{F}, P)$ is separable.

Proof:

(L_p is linear): Follows by $|X+Y|^p \le 2^p\left(\frac{1}{2}|X|^p + \frac{1}{2}|Y|^p\right)$

(L_p is closed): if $X_n \in L_p$ & $X_n \xrightarrow{L_p} X$

then $$|X|^p \le 2^p\left(\frac{|X_n|^p}{2} + \frac{|X - X_n|^p}{2}\right)$$

Taking expected value of both sides gives the result.

(L_p is complete): Follows by the Cauchy Criteria thm.

(L_p is separable):

Suppose $\mathcal{F} = \sigma\langle \mathcal{A} \rangle$ where $\mathcal{A}$ is a countable collection of generators.

Let $X \in L_p$. By the structure thm of Lecture 9 $\exists$ bdd simple X_n's s.t.

$$X_n \xrightarrow{a.e.} X$$

where $X_n \in L_p$ by bddness.

Also, although not explicitly stated in the structure Thm, the X_n's satisfy $|X_n| \le |X|$

$$\therefore \ |X-X_n|^p \le 2^p\left(\frac{|X|^p}{2}+\frac{|X_n|^p}{2}\right)$$

$$\le 2^p|X|^p$$

so by the DCT we have

$$E|X_n-X|^p \to 0 \text{ as } n\to\infty.$$

Since X_n is simple, it has the form

$$X_n=\sum_{k=1}^{m} c_k I_{F_k}, \qquad F_k \in \mathcal{F}=\sigma\langle \mathcal{A}\rangle =\sigma\langle f\langle\mathcal{A}\rangle\rangle$$

Using a result in Lecture 5 we can find $\hat{F}_k \in f\langle\mathcal{A}\rangle$, $\hat{c}_k\in\mathbb{Q}$ s.t.

$$\| X_n-\hat{X}_n\|_p \le \frac{1}{n}$$

where $\hat{X}_n=\sum_{k=1}^{m}\hat{c}_k I_{\hat{F}_k}$ (hint: choose $\hat{F}_k$ so that $P(F_k\,\Delta\,\hat{F}_k)<\left[\frac{(1/n)}{2^k|c_k|}\right]^p$).

For this $\hat{X}_n$ we have

$$\|X-\hat{X}_n\|_p\to 0 \ \ \& \ \ \hat{X}_n\in L.$$

Since any field generate by a countable collection of events is countable, $f\langle\mathcal{A}\rangle$ is countable.

$\therefore$ The collection of all such approximating $\hat{X}_n$'s forms a dense countable subset of $L_p(\Omega,\mathcal{F},P)$.

QED

Recall the definition of Uniform integrability (UI) specialized to r.v.s:

$X_1,X_2,\dots$ are UI iff

$$\lim_{c\to\infty}\sup_n E\left(|X_n| I_{|X_n|\ge c}\right)=0$$

when talking about limits its understood we can drop any finite number of X_n's

Lets also recall the UI theorems we did in Lecture 10 but specialized to r.v.s

Theorem: (UI for $\lim E \equiv E\lim$)

If $X_n\xrightarrow{a.e.}X$ & the X_n's are UI then $EX_n\to EX$ & $X, X_n\in L_1$

Theorem: (UI converse)

If $X_n\xrightarrow{a.e.}X$ & $EX_n\to EX$ & $X,X_n\in L_1$ then the X_n's are UI.

Here is our L_p convergence Thm which effectively shows

$$\xrightarrow{L_p} \ \equiv\ \xrightarrow{P} + |X_n|^p\text{'s are UI}$$

Theorem: (L_p convergence Thm)

Let $X_n\in L_p$ for all n. Then the following are equivalent:

(i) $X_n\xrightarrow{L_p}X$

(ii) $X_n\xrightarrow{P}X$ and $E|X_n|^p\to E|X|^p<\infty$

(iii) $X_n\xrightarrow{P}X$ and the $|X_n|^p$'s are UI

Proof:

(i) $\Rightarrow$ (ii)

we already know $X_n \xrightarrow{P} X$ by Markov's ineq. $X \in L_p$ since L_p is closed. Finally by $\left| \|X\|_p - \|X_n\|_p \right| \leq d_p(X, X_n) \to 0$ we have

$$E|X_n|^p \to E|X|^p < \infty$$

(ii) $\Rightarrow$ (i).

Here is where we use the Probability Sandwich result proved in the last lecture.

$$0 \leq |X_n - X|^p \leq 2^p \underbrace{\left(\tfrac{1}{2}|X_n|^p + \tfrac{1}{2}|X|^p\right)}_{\downarrow P \; 2^p|X|^p =: Y} =: Y_n$$

$|X_n - X|^p \xrightarrow{P} 0$ by continuous mapping since $X_n - X \xrightarrow{P} 0$

since $Y_n, Y \in L_1$ & $EY_n \to EY$ by assumption sandwich says that $E|X_n - X|^p \to 0$.

(ii) $\Rightarrow$ (iii):

using the sub-sub-seq characterization of $\xrightarrow{P}$ one can extend the UI converse to require $\xrightarrow{P}$ instead of $\xrightarrow{a.e.}$.

$\therefore$ From (ii) we have $X_n \xrightarrow{P} X$ & by continuous mapping $|X_n|^p \xrightarrow{P} |X|^p$

Also by assump $E|X_n|^p \to E|X|^p < \infty$ so that $X_n, X \in L_p$ for suff large n

$\therefore$ The X_n's are UI by UI converse

(iii) $\Rightarrow$ (ii):

This one similarly follows from an $\xrightarrow{P}$ version of the UI theorem

QED.

Hilbert Space Geometry of $L_2(\Omega, \mathcal{F}, P)$

For L_2 Hölder gives $E(XY) \leq \|X\|_2 \|Y\|_2 < \infty$

$\therefore$ we can define an inner product on L_2 defined as

$$\langle X, Y \rangle := E(XY)$$

Much of what statistitions do basically corresponds to geometric operations w.r.t. $\langle \cdot, \cdot \rangle$. The geometry of $(L_2, \langle \cdot, \cdot \rangle)$ is the geometry of variation & co-variation:

i.e. when $E(X) = E(Y) = 0$ then

$$\langle X, Y \rangle = cov(X, Y)$$
$$\langle X, X \rangle = \|X\|_2^2 = var(X)$$
$$\|X\|_2 = sd(X).$$

Basic Properties of $\langle \cdot, \cdot \rangle$:

$\forall X, Y \in L_2(\Omega, \mathcal{F}, P)$

(1) $\langle X, X \rangle \geq 0$

(2) $\langle X, X \rangle > 0$ unless $X = 0$ P-a.e.

(3) $\langle X, Y \rangle = \langle X, Y \rangle$

(4) $\langle X, Y + \alpha Z \rangle = \langle X, Y \rangle + \alpha \langle X, Z \rangle$

(5) $|\langle X, Y \rangle| \leq \|X\|_2 \|Y\|_2$

(6) $X_n \xrightarrow{L_2} X \Rightarrow \langle X_n, Y \rangle \to \langle X, Y \rangle$

which is true since

$$|\langle X_n, Y \rangle - \langle X, Y \rangle| = |\langle X_n - X, Y \rangle|$$
$$\leq \|X_n - X\|_2 \|Y\|_2$$

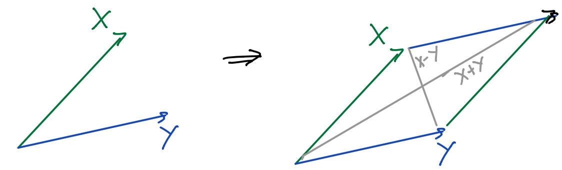
(7) $\|X + Y\|_2^2 = \|X\|_2^2 + 2\langle X, Y \rangle + \|Y\|_2^2$

(8) $\|X+Y\|_2^2 + \|X-Y\|_2^2 = 2\|X\|_2^2 + 2\|Y\|_2^2$

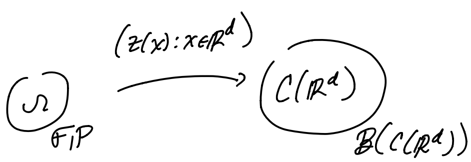
This follows by adding

$$\|X+Y\|_2^2 = \|X\|_2^2 + 2\langle X,Y\rangle + \|Y\|_2^2$$

$$\|X-Y\|_2^2 = \|X\|_2^2 - 2\langle X,Y\rangle + \|Y\|_2^2$$



To see an example of the L_2 geometry viewpoint in estimation problems suppose $Z(x)$ is a mean zero continuous Gaussian random field defined on a region $S \subset \mathbb{R}^d$.
i.e. there exists $(\Omega, \mathcal{F}, P)$ s.t.



and $Z(x)$ has Gaussian f.d.d.s & $E(Z(x))=0$ $\forall x\in S$

$\therefore$ The collection of r.v.s $Z(x)$ indexed by $x\in S$ satisfies

$$\{Z(x): x\in S\} \subset L_2(\Omega,\mathcal{F},P)$$

In random field theory we often study the following Hilbert space

$L(S) :=$ closed linear span (in L_2) of $\{Z(x): x\in S\}$

$=$ closure with L_2 limits of $\left\{\sum_{k=1}^n c_k Z(x_k) : x_k\in S, c_k\in\mathbb{R}\right\}$

$\subset L_2(\Omega,\mathcal{F},P)$

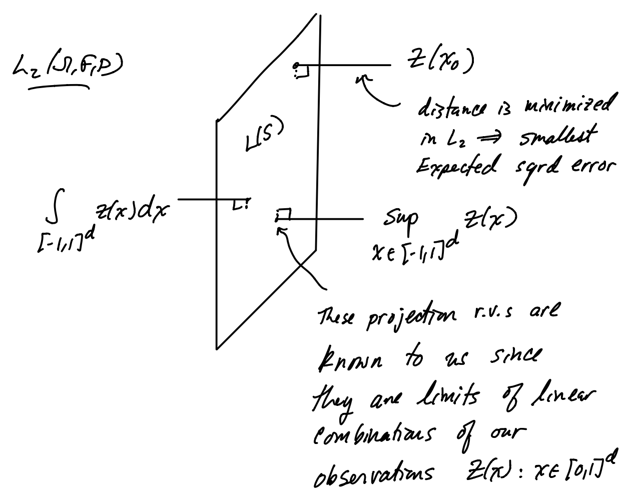
Now "Best linear prediction" (BLP) of some unobserved $L_2(\Omega,\mathcal{F},P)$ r.v. simply given by projection:

e.g. Suppose $Z(x)$ is defined on $x\in\mathbb{R}^d$ but only observed on $x\in[0,1]^d$.

If we want to predict things like

- $Z(x_0)$ for $x_0\notin[0,1]^d$
- $\int_{[-1,1]^d} Z(x)\,dx$
- $\sup_{x\in[-1,1]^d} Z(x)$

So long as these r.v.s are in L_2 the BLP is a projection $L(S)$



Definition $X \in L_2$ is orthogonal to $Y \in L_2$ iff $\langle X, Y \rangle = 0$ (denoted by $X \perp Y$).

Theorem: (Projection Thm)

Let S be a closed linear subspace of L_2 & $Y \in L_2$. Then $\exists$ a P-a.e. unique $P_S Y \in S$ s.t.

$$\|Y - P_S Y\|_2 = \inf_{X \in S} \|Y - X\|_2.$$

Moreover $P_S Y$ is characterized by the following two properties

(1) $P_S Y \in S$

(2) $\underbrace{(Y - P_S Y)}_{\text{prediction residual.}} \perp X \quad \forall X \in S$

Proof:

(Find $P_S Y$) Let $X_n \in S$ s.t.

$$\|Y - X_n\|_2 \xrightarrow{n \to \infty} \inf_{X \in S} \|Y - X\|_2$$

Now we show $\{X_n\}_{n \geq 1}$ is Cauchy with the Parallelogram Thm:

$$\|(X_n - Y) + (X_m - Y)\|_2^2 + \|\underbrace{X_n - X_m}_{(X_n-Y)-(X_m-Y)}\|_2^2$$

$$= \underbrace{2\|X_n - Y\|_2^2 + 2\|X_m - Y\|_2^2}_{=: I_{nm}}$$

where $\lim_{nm} I_{nm} = 4\left(\inf_{X \in S} \|Y - X\|_2\right)^2$

$\therefore \|X_n - X_m\|_2^2 = I_{nm} - \|\underbrace{(X_n - Y) + (X_m - Y)}_{2\left(\frac{X_n + X_m}{2} - Y\right)}\|_2^2$

$$= I_{nm} - 4\|\underbrace{\tfrac{X_n + X_m}{2}}_{\in S \text{ by linearity}} - Y\|_2^2$$

$$\leq I_{nm} - 4\left(\inf_{X \in S} \|X - Y\|_2\right)^2$$

$$\to 0 \text{ as } n, m \to \infty$$

$\therefore \{X_n\}_{n \geq 1}$ is Cauchy & by completeness

$\exists P_S Y \in L_2$ s.t. $\underbrace{X_n}_{\in S} \xrightarrow{L_2} \underbrace{P_S Y}_{\in S}$ implies, since S is closed

Also, for this $P_S Y$ we have

$$\inf_{X \in S} \|X - Y\|_2 \leq \|P_S Y \underset{\pm X_n}{} - Y\|_2$$

$$\leq \underbrace{\|P_S Y - X_n\|_2}_{\to 0} + \underbrace{\|X_n - Y\|_2}_{\to \inf_{X \in S} \|X - Y\|_2}$$

$$\therefore \inf_{X \in S} \|X - Y\|_2 = \|P_S Y - Y\|_2$$

(Show $P_S Y$ is unique P-a.e.)

Suppose $X_0 \in S$ s.t. $\|X_0 - Y\|_2 = \inf \cdots$

Again by the Parallelogram Thm

$$\|(X_0 - Y) + (P_S Y - Y)\|_2^2 + \|X_0 \underset{\pm Y}{} - P_S Y\|_2^2$$

$$= 2\underbrace{\|X_0 - Y\|_2^2}_{\inf^2} + 2\underbrace{\|P_S Y - Y\|_2^2}_{\inf^2}$$

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$\therefore \|X_0 - \mathcal{P}_S Y\|_2^2 \leq 4\inf{}^2 - \|2(\underbrace{\tfrac{X_0+\mathcal{P}_S Y}{2}}_{\in S} - Y)\|_2^2$

$\leq 4\inf{}^2 - 4\inf{}^2 = 0$

$\therefore X_0 \overset{P\text{ a.e.}}{=} \mathcal{P}_S Y.$

(<u>Show $(Y - \mathcal{P}_S Y) \perp X, \forall X \in S$</u>):

Choose $X \in S$ s.t. $X \neq 0$ a.e. (if $X=0$ then the result is true).

For $c \in \mathbb{R}$ set

$$f(c) = \|Y - (\mathcal{P}_S Y - cX)\|_2^2$$

Let $c_{min} := \underset{c\in\mathbb{R}}{\operatorname{argmin}}\, f(c)$.

Two ways to compute c_{min}

1st: $c_{min} = 0$ by minimizing properties of $\mathcal{P}_S Y$.

2nd:

$f(c) = \|Y - \mathcal{P}_S Y\|_2^2 + 2c\langle Y - \mathcal{P}_S Y, X\rangle + c^2\|X\|_2^2$

$\therefore f'(c) = 2\langle Y - \mathcal{P}_S Y, X\rangle + 2c\|X\|_2^2$

$\therefore c_{min} = -\dfrac{\langle Y - \mathcal{P}_S Y, X\rangle}{\|X\|_2^2}$ ← need $\|X\|_2^2 > 0$

$= 0$ since $c_{min} = 0$

$\therefore \langle Y - \mathcal{P}_S Y, X\rangle = 0$

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(<u>Show $W \in S$ & $(Y-W) \perp X$ $\forall X \in S \Rightarrow W = \mathcal{P}_S Y$</u>)

$\forall X \in S$ we have

$$\|X - Y\|_2^2 = \|X - W\|_2^2 + 2\underbrace{\langle \underset{\in S}{X-W}, Y-W\rangle}_{0} + \|Y-W\|_2^2$$

$\pm W$

$\therefore \inf_{X\in S}\|X-Y\|_2^2 = \Big[\underbrace{\inf_{X\in S}\|X-W\|_2^2}_{=0 \text{ since } W\in S}\Big] + \|Y-W\|_2^2$

$= \|W - Y\|_2^2$

$\therefore W = \mathcal{P}_S Y$ since $\mathcal{P}_S Y$ is the unique such r.v.

QED

The next theorem shows that to compute a projection you define coordinates aligned with the space your projecting to

Theorem: (projection in coordinates)

Suppose $X_1, X_2, \ldots \in L_2(\Omega, \mathcal{F}, P)$ are orthonormal and let

$S = \overline{\text{span}}\{X_n : n \in \mathbb{N}\} =:$ (The collection of L_2 limits of finite linear combinations of the X_n's)

Then S is a closed linear subset of L_2 & $\forall Y \in L_2(\Omega, \mathcal{F}, P)$ the following holds

$$\underbrace{\mathcal{P}_S Y}_{\text{projection of } Y \text{ onto } S} \overset{L_2}{=} \sum_{n=1}^{\infty}\langle X_n, Y\rangle X_n$$

This means $\sum_{n=1}^{N}\langle X_n, Y\rangle X_n \xrightarrow[N\to\infty]{L_2} \mathcal{P}_S Y$

Proof:

(S is closed and linear):

$W, Z \in S \Rightarrow \begin{array}{l} W_n \xrightarrow{L_2} W \\ Z_n \xrightarrow{L_2} Z \end{array}$ where W_n, Z_n is a finite linear comb of the X_n's

$$\Rightarrow \| aW_n + bZ_n - (aW+bZ)\|_2 \le |a| \underbrace{\|W_n - W\|_2 + |b| \|Z_n - Z\|_2}_{\to 0}$$

$$\Rightarrow \quad aW + bZ \in S$$

$\therefore$ S is Linear

To see that S is closed let

$$Z_n \xrightarrow{L_2} Z$$

where $Z_n \in S$. For each n let $\hat{Z}_n$ be a finite linear combination of the X_n's s.t.

$$\|\hat{Z}_n - Z_n\|_2 \le \frac{1}{n}$$

$\therefore \|Z - \hat{Z}_n\|_2 \le \|Z - Z_n\|_2 + \frac{1}{n} \to 0$ as $n \to \infty$

$\therefore Z \in S$ by definition.

($\sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$ exists as a infinite sum)

Let $S_n := \operatorname{span}\{X_1, \dots, X_n\}$

Notice that $P_{S_n} Y = \sum_{i=1}^{n} \langle X_i, Y\rangle X_i$ since

$$\langle Y - P_{S_n} Y, X_j\rangle = \langle Y, X_j\rangle - \langle X_j, Y\rangle = 0$$

$\forall j = 1, \dots, n$. Also notice that $P_{S_n} Y$ decreases length in that

$$\|Y\|_2^2 = \|Y - P_{S_n} Y\|_2^2 + \|P_{S_n} Y\|_2^2$$

$$\ge \|P_{S_n} Y\|_2^2$$

$$\therefore \quad \sum_{i=1}^{n} \langle X_i, Y\rangle^2 = \|P_{S_n} Y\|_2^2 \le \|Y\|_2^2 < \infty$$

$$\therefore \quad \sum_{i=1}^{\infty} \langle X_i, Y\rangle^2 < \infty$$

$\therefore \left\{ \sum_{i=1}^{n} \langle X_i, Y\rangle X_i \right\}_{n \ge 1}$ is Cauchy in L_2

and has a L_2 limit, denoted $\sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$.

(Show $P_S Y = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$):

Since $\underbrace{\sum_{i=1}^{n} \langle X_i, Y\rangle X_i}_{= P_{S_n} Y} \xrightarrow[n \to \infty]{L_2} \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$

we have that $\forall j \in \mathbb{N}$

$$\langle P_{S_n} Y, X_j\rangle \xrightarrow[n \to \infty]{} \left\langle \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i, X_j \right\rangle$$

$$\therefore \underbrace{\langle Y - P_{S_n} Y, X_j\rangle}_{\substack{\text{This is eventually} \\ \text{zero for large} \\ \text{enough } n}} \xrightarrow[n \to \infty]{} \underbrace{\left\langle Y - \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i, X_j \right\rangle}_{\text{This is } 0}$$

$\therefore P_S Y = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$ since $Y - P_S Y \perp X$ $\forall X \in S$.

QED.

Definition: $\{X_i : i \in \mathcal{I}\} \subset L_2(\Omega, \mathcal{F}, P)$ is an orthonormal basis (ONB) if finite linear combinations of the X_i's are dense in $L_2(\Omega, \mathcal{F}, P)$.

Theorem: (characterizing a ONB)

If $\{X_n\}_{n\geq 1} \subset L_2(\Omega,\mathcal{F},P)$ are orthonormal then the following are equivalent:

(i) $\{X_n\}_{n\geq 1}$ is a ONB

(ii) $Y \overset{\text{in } L_2}{=} \sum_{i=1}^\infty \langle X_i, Y\rangle X_i \quad \forall Y \in L_2(\Omega,\mathcal{F},P)$

(iii) $\langle Y, Z\rangle = \sum_{i=1}^\infty a_i b_i \quad \forall Y,Z \in L_2(\Omega,\mathcal{F},P)$

where $a_i := \langle X_i, Y\rangle$ & $b_i := \langle X_i, Z\rangle$.

(iv) $\|Y\|_2^2 = \sum_{i=1}^\infty a_i^2 \quad \forall Y \in L_2(\Omega,\mathcal{F},P)$

where $a_i := \langle X_i, Y\rangle$.

Proof:

(ii) $\Rightarrow$ (i): Trivial

(i) $\Rightarrow$ (ii): Almost trivial.

If $S := \overline{\text{span}}\{X_i : i \in \mathbb{N}\}$ then

$$\underset{\substack{\parallel \\ Y}}{P_S Y} = \sum_{i=1}^\infty \langle X_i, Y\rangle X_i \quad \text{by "Computing a projection Thm"}$$

Since (i) implies $S = L_2(\Omega,\mathcal{F},P)$.

(ii) $\Rightarrow$ (iii): Set $a_i = \langle X_i, Y\rangle$, $b_i = \langle X_i, Y\rangle$ and notice

$$\begin{aligned}\langle Y, Z\rangle &= \Big\langle \sum_{i=1}^\infty a_i X_i, \sum_{i=1}^\infty b_i X_i \Big\rangle \\ &= \lim_n \Big\langle \sum_{i=1}^n a_i X_i, \sum_{i=1}^\infty b_i X_i \Big\rangle \\ &= \lim_n \lim_m \underbrace{\Big\langle \sum_{i=1}^n a_i X_i, \sum_{i=1}^m b_i X_i \Big\rangle}_{\sum_{i=1}^n a_i b_i}\end{aligned}$$

(iii) $\Rightarrow$ (iv): Trivial

(iv) $\Rightarrow$ (ii):

Let $S := \overline{\text{span}}\{X_i : i \in \mathbb{N}\}$.

Since we know $P_S Y = \sum_{i=1}^\infty \langle X_i, Y\rangle X_i$ by "Projection in coordinates Thm" it will be sufficient to show

$$\|P_S Y - Y\|_2^2 = 0$$

$$\text{since } \underbrace{\|Y\|_2^2}_{\substack{\sum_{i=1}^\infty \langle X_i, Y\rangle^2 \\ \text{by (iv)}}} = \|Y - P_S Y\|_2^2 + \underbrace{\|P_S Y\|_2^2}_{\sum_{i=1}^\infty \langle X_i, Y\rangle^2}$$

We therefore have $\|Y - P_S Y\|_2^2$ as was to be shown

QED

Our last Thm on projections shows that the ordering of the ONB is irrelevant.

Theorem: (permuting coordinates)

If $\{X_n\}_{n\geq 1} \subset L_2(\Omega,\mathcal{F},P)$ is an orthonormal collection and $Y \in L_2(\Omega,\mathcal{F},P)$ then

$$\sum_{i=1}^\infty \langle X_i, Y\rangle X_i \overset{L_2}{=} \sum_{i=1}^\infty \langle X_{\pi(i)}, Y\rangle X_{\pi(i)}$$

for any permutation $\pi : \mathbb{N} \to \mathbb{N}$.

Proof:

Let $Z = \sum_{i=1}^\infty \langle X_i, Y\rangle X_i$ and

$$Z_\pi = \sum_{i=1}^\infty \langle X_{\pi(i)}, Y\rangle X_{\pi(i)}.$$

Now

$$\begin{aligned}\langle Z, Z_\pi\rangle &= \sum_{i=1}^\infty \sum_{j=1}^\infty \langle X_i, Y\rangle \langle X_{\pi(j)}, Y\rangle \underbrace{\langle X_i, X_{\pi(j)}\rangle}_{= \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{o.w.} \end{cases}} \\ &= \sum_{i=1}^\infty \langle X_i, Y\rangle^2 \\ &= \|Z\|_2^2\end{aligned}$$

$$\therefore \|Z - Z_\pi\|_2^2 = \|Z\|_2^2 + \|Z_\pi\|_2^2 - 2\langle Z, Z_\pi\rangle$$

$$= \underbrace{\sum_{i=1}^{\infty} \langle X_{\pi(i)}, Y\rangle^2 - \sum_{i=1}^{\infty} \langle X_i, Y\rangle^2}_{}$$

$=0$ since positive sums can be arbitrarily re-ordered.

QED

Remark: Notice that whenever $\mathcal{F}$ is countably generated, $L_2(\Omega, \mathcal{F}, P)$ is separable so there exists a countable dense subset $\{Y_n\}_{n\geq 1}$.

In this case one can use Gram-Schmidt to construct an ONB $\{X_n\}_{n\geq 1}$ as follows:

$$X_1 := \frac{Y_1}{\|Y_1\|_2}$$

$$X_2 := \frac{Y_2 - \langle X_1, Y_2\rangle X_1}{\|Y_2 - \langle X_1, Y_2\rangle X_1\|_2}$$

← project out X_1

← Normalize

$\vdots$

$$X_n := \frac{Y_n - \sum_{i=1}^{n-1} \langle X_i, Y_n\rangle X_i}{\|Y_n - \sum_{i=1}^{n-1}\langle X_i, Y_n\rangle X_i\|_2^2}$$

← Project out $\{X\}_{i=1}^{n-1}$

← Normalize

$\vdots$

Now each Y_n is a finite linear comb of the X_i's so $\{X_n\}_{n\geq 1}$ are dense in L_2. If we identify r.v.'s with the equivalence classes of P-a.e. modifications then $L_2(\Omega, \mathcal{F}, P)$ becomes a Hilbert Space (complete, separable, linear vector space with pos. def. inner product).

Projection Example

$\Omega = [-1, 1]$

$\mathcal{F} = \mathcal{B}([-1,1])$

$P = \frac{1}{2} \times$ Lebesgue measure on $[-1,1]$

$S = \{X \in L_2(\Omega, \mathcal{F}, P) : X(\omega) = X(-\omega) \;\; P\text{-a.e}\}$

Clearly S is linear. It is also closed since $\underbrace{X_n}_{\in S} \xrightarrow{L_2} X \Rightarrow X_n \xrightarrow{P} X$

$\Rightarrow X_{n_k} \xrightarrow{a.e.} X$ for some sub-seq n_k

$$\Rightarrow X(\omega) \overset{a.e.}{=} \lim_k X_{n_k}(\omega) \overset{a.e.}{=} \lim_k X_{n_k}(-\omega) \overset{a.e.}{=} X(-\omega)$$

For any $Y \in L_2(\Omega, \mathcal{F}, P)$ lets find $P_S Y$.

Techique: Guess the answer and show it is orthogonal to S.

Here is the Guess:

$$P_S Y \overset{?}{=} \frac{Y(\omega) + Y(-\omega)}{2}$$

To verify let $X \in S$ and notice

$$\langle Y - P_S Y, X\rangle = E[(Y - P_S Y)X]$$

$$= \int_{[-1,1]} \left(Y(\omega) - \frac{Y(\omega) + Y(-\omega)}{2}\right) X(\omega)\, dP(\omega)$$

$$= \frac{1}{2}\int_{[-1,1]} \underbrace{(Y(\omega) - Y(-\omega))X(\omega)}_{\text{odd symmetry about } 0} dP(\omega)$$

$$= 0$$

$\therefore$ Indeed $P_S Y = \frac{Y(\omega) + Y(-\omega)}{2}$.

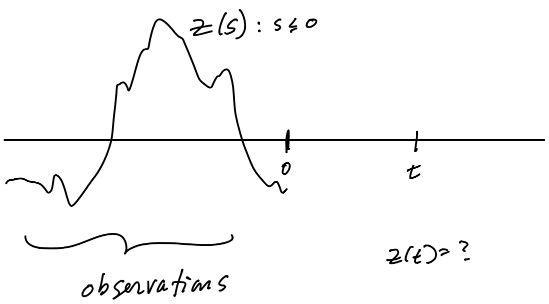
Projection application to Gaussian random field prediction

Let $\{z(t) : t \in \mathbb{R}\}$ be a Gaussian random field (GRF) s.t. $\forall\, t, s \in \mathbb{R}$

$$\left.\begin{aligned} &E(z(t)) = 0 \\ &\operatorname{cov}(z(t), z(s)) = e^{-|t-s|} \end{aligned}\right\} \text{specifies the f.d.d.s.}$$

Note part of the requirement of a GRF is that all the random variables $z(t)$, for $t \in \mathbb{R}$, must be defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Suppose we observe $z(s)$, $\forall s \le 0$ & want to predict $z(t)$ for some $t > 0$.



Now, we will see later that

$$E(z(t) \mid z(s), s \le 0) = P_s z(t)$$

To find $P_s z(t)$ lets guess that $P_s z(t) = a_t z(0)$ and prove the residuals are orthogonal to $\underbrace{\overline{\operatorname{span}}\{z(s) : s \ge 0\}}_{\text{Set of all linear } L_2 \text{ quantities from } z(s),\, s \le 0}$.

Note that by linearity & continuity of $\langle \cdot, \cdot \rangle$ w.r.t L_2 limits it is sufficient to show $\langle z(t) - a_t z(0), z(s) \rangle = 0$ $\forall s \le 0$.

For $t > 0$ & $s \le 0$ we have

$$\begin{aligned} \langle z(t) - a_t z(0), z(s) \rangle &= e^{-|t-s|} - a_t e^{-|s|} \\ &= e^{-(t-s)} - a_t e^{s} \quad \text{since } s \le 0 < t \\ &= 0 \quad \text{iff } a_t := e^{-t} \end{aligned}$$

$$\therefore\ E(z(t) \mid z(s), s \le 0) = e^{-t} z(0)$$

and

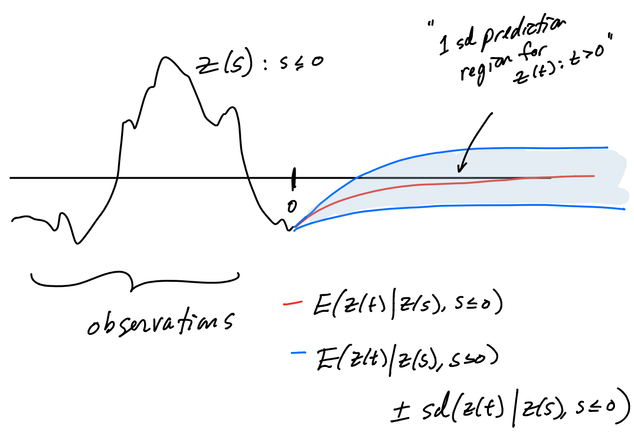
$$\begin{aligned} &\operatorname{var}(z(t) \mid z(s), s \le 0) \\ &= E\Big(\big[\underbrace{z(t) - e^{-t} z(0)}\big]^2 \,\Big|\, \underbrace{z(s), s \le 0}\Big) \end{aligned}$$

since these are orthogonal in L_2, they are uncorrelated. Since they are Gaussian they are then independent

$$= \operatorname{var}\big(\underbrace{z(t) - e^{t} z(0)}_{\text{marginal residual}}\big)$$

$$= e^{0} - 2e^{-t} e^{-|t|} + e^{2t} e^{0}$$

$$= 1 - e^{-2t}$$



Riesz representation

The Riesz representation Theorem will be important for proving the existence of $\frac{dP}{dQ}$ for two probability measures P, Q.

To motivate recall that if $X_n, X \in L_2(\Omega, \mathcal{F}, P)$ then for any $Y \in L_2(\Omega, \mathcal{F}, P)$

$$X_n \xrightarrow{L_2} X \Rightarrow \langle Y, X_n \rangle \to \langle Y, X \rangle$$

$\therefore$ The map $\langle Y, \cdot \rangle : L_2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a continuous linear functional on L_2.

Riesz says that all such functionals are in the form $\langle Y, \cdot \rangle$ for some $Y \in L_2$.

Theorem (Riesz for L_2):

Let $f: L_2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be linear and continuous (w.r.t L_2 metric).
Then $\exists$ a P-a.e. unique $Y \in L_2(\Omega, \mathcal{F}, P)$ s.t.

$$f(X) = \langle Y, X \rangle \quad \forall X \in L_2(\Omega, \mathcal{F}, P).$$

Proof:
The idea is that if such a Y did exist, then one could Gram-schmit to get a ONB:

$$\underbrace{cY}_{\text{This spans } N^\perp \text{ where } \dim N^\perp = 1}, \underbrace{X_1, X_2, \ldots}_{\text{Now this spans the null space of } f,\ N := \{X : f(X) = 0\}}$$

So lets find $Y \in N^\perp$ with the desired property.

Step 1: choose any $Y \notin N$ s.t. $\underbrace{Y \neq 0 \text{ P-a.e.}}$

if no such Y then $f(X) = 0$ $\forall X$ & setting $Y = 0$ gives the result.

Step 2: Project out N

i.e. $\underbrace{Y}_{\text{New}} := \underbrace{Y - \mathcal{P}_N Y}$

Note N is closed & linear by assumptions on f so $\mathcal{P}_N Y$ exists.

Step 3: scale Y by a constant so it satisfies $f(Y) = \|Y\|_2^2$.

Now I can project any $X \in L_2$ to span $Y := \{cY : c \in \mathbb{R}\}$ by

$$\mathcal{P}_{\text{span } Y} X = \left\langle \frac{Y}{\|Y\|_2}, X \right\rangle \frac{Y}{\|Y\|_2}$$

and is characterized by

$$\langle X - \mathcal{P}_{\text{span } Y} X, Y \rangle = 0 \quad \forall X$$

$$\| \qquad$$

$$\left\langle X - \frac{\langle Y, X \rangle}{\|Y\|_2^2} Y, Y \right\rangle$$

If it was the case that $f(X) = \langle Y, X \rangle$ then it should satisfy

$$f\left(X - \frac{f(X)}{\|Y\|_2^2} Y\right) = 0$$

Indeed

$$f\left(X - \frac{f(X)}{\|Y\|_2^2} Y\right) = f(X) - f(X) \underbrace{\frac{f(Y)}{\|Y\|_2^2}}_{=1} = 0$$

(27)

$\therefore \quad X - \dfrac{f(X)}{\|Y\|_2^2} Y \in N$ ← the null space of f

$\therefore \quad \left\langle X - \dfrac{f(X)}{\|Y\|_2^2} Y , Y \right\rangle = 0$ since we projected N out of Y.

$\therefore \quad \langle X, Y \rangle - \dfrac{f(X)}{\cancel{\|Y\|_2^2}} \cancel{\langle Y, Y \rangle} = 0$

$\therefore \quad f(X) = \langle Y, X \rangle \quad \forall X \in L_2(\Omega, \mathcal{F}, P)$.

To show uniqueness let $\tilde{Y} \in L_2$ s.t.

$$\langle \tilde{Y}, X \rangle = f(X) = \langle Y, X \rangle \quad \forall X \in L_2$$

$\therefore \quad \langle Y - \tilde{Y}, X \rangle = 0 \quad \forall X \in L_2$

$\therefore \quad \langle Y - \tilde{Y}, \underbrace{Y - \tilde{Y}}_{\in L_2} \rangle = 0$

$\therefore \quad Y = \tilde{Y}$ P-a.e.

QED