Homework 2

Due Thursday, October 13, 2016

These exercises will give you practice working with some of the regularity conditions we will encounter when interpreting conditional probability as disintegration. The main assumption will be Radon measures over Polish spaces Ω (a complete and separable metric space). The techniques developed here will easily extend to this generalized case. If you want to peak ahead and see where this will be used, take a look at the paper "Conditioning as disintegration" by Chang and Pollard (1997).

Remark: I've decided that I don't like the notation $\mathcal{B}^\Omega$ for the Borel σ-field over a metric space Ω (defined as $\sigma\langle\text{open sets}\rangle$). I'm switching to the notation $\mathcal{B}(\Omega)$ instead. I've made the switch to this new notion in the first part of the lecture notes, but some of the later sections still use it. Just FYI.

Exercise 1. *Let Ω be a metric space with distance function d. Ω is said to be* **separable** *if there exists a countable $\Omega_0 \subset \Omega$ which is dense in Ω (i.e., every point of Ω is a limit of some sequence of points of Ω_0). Show that $\mathcal{B}(\Omega) = \sigma\langle\text{open balls in } \Omega\rangle$ if Ω is separable.*

The goal of the next two exercises is to show the following theorem.

Theorem 1. *Let μ be any measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which assigns finite measure to bounded sets in $\mathcal{B}(\mathbb{R})$. Then for any $B \in \mathcal{B}(\mathbb{R})$ and $\epsilon > 0$ there exists a closed set C and an open set O such that $C \subset B \subset O$ and*

$$\mu(O - C) < \epsilon.$$

Exercise 2. *Show Theorem 1 when μ is a finite measure. Hint: Use "good sets" and the fact that $\mathcal{B}(\mathbb{R}) = \sigma\langle\text{closed sets}\rangle$. Also note that the identity $\mu(\cup_{n=1}^\infty A_k - \cup_{n=1}^\infty B_k) \le \sum_{k=1}^\infty \mu(A_k - B_k)$, when $B_k \subset A_k$ and μ is a finite measure, might come in handy.*

Exercise 3. *Show Theorem 1 for general μ. Hint: It will be sufficient (why?) to show that for any $B \in \mathcal{B}(\mathbb{R})$ and $\epsilon > 0$ there exists an open set $O \supset B$ such that $\mu(O - B) < \epsilon$. Use the previous exercise and the fact that $\mu_n(\cdot) := \mu(\cdot \cap (-n, n))$ is a finite measure.*

Exercise 4. *Give an example of a σ-finite measure μ on $\mathcal{B}(\mathbb{R})$ and a Borel set B such that*

$$\mu(B - C) = \infty = \mu(O - B)$$

for every closed $C \subset B$ and every open set $O \supset B$.

Definition 1. *Let μ be a measure $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then*

- *μ is said to be* **outer-regular** *if*

$$\mu(B) = \inf\{\mu(O) : B \subset O,\ O \text{ open}\}$$

for all $B \in \mathcal{B}(\mathbb{R})$.

- *μ is said to be* **inner-regular** *if*

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\}$$

for all $B \in \mathcal{B}(\mathbb{R})$.

- *μ is said to be a* **Radon measure** *if μ is inner-regular and finite on compact subsets of $\mathbb{R}$.*

Exercise 5. *Show that any measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which assigns finite measure to bounded sets is outer-regular.*

Exercise 6. *Show that any finite measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a Radon measure.*