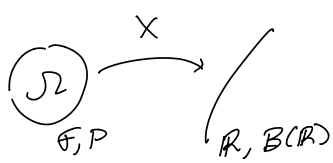
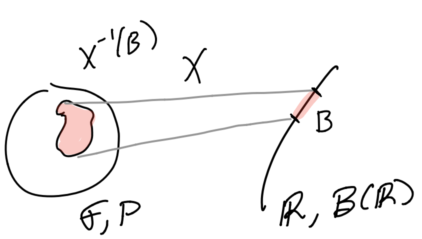
Lecture 8: Measurable functions, Random variables and distribution functions

In this lecture we will start by developing the measure theoretic notion of measurable functions & define random variables X as measurable functions $X : \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{F}, P)$ is a prob. space.



Measurability of X is required since we want $P(X \in B)$ to be defined where $B \in \mathcal{B}(\mathbb{R})$ and

$$\{X \in B\} \equiv \{\omega \in \Omega : X(\omega) \in B\} =: \underbrace{X^{-1}(B)}_{\text{pre-image of } B \text{ under } X.}$$



Measurable functions

Let $(\Omega_1, \mathcal{F}_1)$ & $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces & $f : \Omega_1 \to \Omega_2$.

<u>Def</u>: f <u>is measurable between $\mathcal{F}_1$ & $\mathcal{F}_2$</u> (written $f \text{ ⓜ } \mathcal{F}_1/\mathcal{F}_2$ for short) iff

$$f^{-1}(A) \in \mathcal{F}_1, \quad \forall A \in \mathcal{F}_2. \qquad (*)$$

<u>Note</u>: It will sometimes be convinient to write $f \text{ ⓜ } \mathcal{F}_1/\mathcal{F}_2$ when f satisfies $(*)$ even when $\mathcal{F}_1$ or $\mathcal{F}_2$ are <u>Not</u> σ-fields ... just collections of sets.

A few basic facts about $f^{-1}(A)$

(1) $f^{-1}(\Omega_2) = \Omega_1$ since f maps into Ω_2

(2) $f^{-1}(\emptyset) = \emptyset$

(3) $f^{-1}(A^c) = (f^{-1}(A))^c$

since $\omega \in f^{-1}(A^c) \iff f(\omega) \in A^c \iff f(\omega) \notin A \iff \omega \notin f^{-1}(A)$

(4) $f^{-1}(\bigcup_k A_k) = \bigcup_k f^{-1}(A_k)$ even A_k's are not disjoint

since $\omega \in f^{-1}(\bigcup_k A_k) \iff f(\omega) \in A_k$ some k $\iff \omega \in f^{-1}(A_k)$ some k $\iff \omega \in \bigcup_k f^{-1}(A_k)$

<u>Thm</u> (Generators are enough)

If $\underset{\mathcal{F}_1}{\Omega_1} \xrightarrow{f} \underset{\sigma\langle\mathcal{Q}\rangle}{\Omega_2}$ & $\mathcal{F}_1$ is a σ-field

then $f \text{ ⓜ } \mathcal{F}_1/\sigma\langle\mathcal{Q}\rangle \iff f \text{ ⓜ } \mathcal{F}_1/\mathcal{Q}$.

<u>Proof</u>:

$\Rightarrow$: trivial

$\Leftarrow$: Good sets on

$$\mathcal{G} = \{A \subset \Omega_2 : f^{-1}(A) \in \mathcal{F}_1\}.$$

$\mathcal{A} \subset \mathcal{G}$ by assumption & $\mathcal{G}$ is a σ-field by facts (1), (3), (4).

QED.

e.g. $\underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \xrightarrow{f} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}}$

f is monotone $\Rightarrow f^{-1}((-\infty, x])$ is an interval $\forall x$

show that $a, b \in f^{-1}((-\infty, x])$ & $a \le y \le b \Rightarrow \underbrace{y \in f^{-1}((-\infty, x])}_{\text{could be open or closed}}$

$$\Rightarrow f^{-1}((-\infty, x]) \in \mathcal{B}(\mathbb{R}), \forall x$$

$$\Rightarrow f \text{ m } \mathcal{B}(\mathbb{R}) / \{(-\infty, x] : x \in \mathbb{R}\}$$

$$\Leftrightarrow f \text{ m } \mathcal{B}(\mathbb{R}) / \underbrace{\sigma\langle (-\infty, x] : x \in \mathbb{R}\rangle}_{=\mathcal{B}(\mathbb{R})}$$

$\therefore$ All monotone funs are measurable.

e.g. $\underset{\mathcal{B}(\mathbb{R}^d)}{\mathbb{R}^d} \xrightarrow{f} \underset{\mathcal{B}(\mathbb{R}^d)}{\mathbb{R}^k}$

f is continuous

$$\Leftrightarrow f^{-1}(G) \text{ is open } \forall \text{ open } G \subset \mathbb{R}^k$$

$$\Rightarrow f \text{ m } \mathcal{B}(\mathbb{R}^d) / \text{opens in } \mathbb{R}^k$$

$$\Leftrightarrow f \text{ m } \mathcal{B}(\mathbb{R}^d) / \underbrace{\sigma\langle \text{opens in } \mathbb{R}^k \rangle}_{\mathcal{B}(\mathbb{R}^k)}$$

This extends to metric spaces Ω_1 & Ω_2 so that

f m $\mathcal{B}(\Omega_1)/\mathcal{B}(\Omega_2)$ for all continuous $f : \Omega_1 \to \Omega_2$

Thm (composition of m is m).

If

$$\begin{tikzcd} \underset{\mathcal{F}_1}{\Omega_1} \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & \underset{\mathcal{F}_2}{\Omega_2} \arrow[r, "g"] & \underset{\mathcal{F}_3}{\Omega_3} \end{tikzcd}$$

where f m $\mathcal{F}_1/\mathcal{F}_2$ & g m $\mathcal{F}_2/\mathcal{F}_3$

then $g \circ f$ m $\mathcal{F}_1/\mathcal{F}_3$

Proof:

If $B \in \mathcal{F}_3$ then

$$(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$$

$$= \underbrace{f^{-1}(\underbrace{g^{-1}(B)}_{\in \mathcal{F}_2})}_{\in \mathcal{F}_1}$$

QED.

e.g. if f m $\mathcal{F}/\mathcal{B}(\mathbb{R})$ then $|f|$, f^2, $\sin(f)$ ... are all m $\mathcal{F}/\mathcal{B}(\mathbb{R})$ by continuity.

Corollary (Just check the coordinates)

If $\underset{\mathcal{F}}{\Omega} \xrightarrow{f} \underset{\mathcal{B}(\mathbb{R}^n)}{\mathbb{R}^n}$ works for $\bar{\mathbb{R}}^n$, $\mathcal{B}(\bar{\mathbb{R}}^n)$ too

where $\mathcal{F}$ is a σ-field and $f(\omega) = (f_1(\omega), \dots, f_n(\omega))$

then

$$f \text{ m } \mathcal{F}/\mathcal{B}(\mathbb{R}^n) \Leftrightarrow \text{each } f_k \text{ is m } \mathcal{F}/\mathcal{B}(\mathbb{R})$$

⑤

Proof:

$\Longrightarrow$: follows since the coordinate mappings $\pi_k(x_1, \dots, x_n) := x_k$ are continuous & $f_k = \pi_k \circ f$

$\Longleftarrow$:

$$\underbrace{f^{-1}\Big(\underbrace{(a_1,b_1]\times\cdots\times(a_n,b_n]}_{\text{bdd rectangle}}\Big) = \bigcap_{k=1}^{n} \underbrace{f_k\big((a_k,b_k]\big)}_{\substack{\in \mathcal{F} \text{ since} \\ f_k \text{ ⓜ } \mathcal{F}/\mathcal{B}(\mathbb{R})}}}_{\in \mathcal{F}}$$

$\therefore$ f ⓜ $\mathcal{F}/$ bdd rectangles

$\therefore$ f ⓜ $\mathcal{F}/\underbrace{\sigma\langle \text{bdd rectangles}\rangle}_{\mathcal{B}(\mathbb{R}^n)}$.

QED

Thm (cut & paste over countable & ⓜ pieces)

If $\Omega_1 \xrightarrow{f} \Omega_2$ where Ω_1 has $\mathcal{F}_1$, Ω_2 has $\mathcal{F}_2$,

$\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-fields and

$$\Omega_1 = \bigcup_{k=1}^{\infty} A_k \quad \text{s.t. } \forall k,\ A_k \in \mathcal{F}$$

then

$$f \text{ ⓜ } \mathcal{F}_1/\mathcal{F}_2 \iff f\big|_{A_k} \text{ ⓜ } (\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$$

for all $k = 1, 2, \dots$

⑥

Proof:

$\Longrightarrow$: suppose f ⓜ $\mathcal{F}_1/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$w \in f\big|_{A_k}^{-1}(B) \iff f\big|_{A_k}(w) \in B \ \&\ w \in A_k$$
$$\iff f(w) \in B \ \&\ w \in A_k$$

$$\therefore f\big|_{A_k}^{-1}(B) = \underbrace{A_k}_{\in \mathcal{F}_1} \cap \underbrace{f^{-1}(B)}_{\in \mathcal{F}_1} \in \mathcal{F}_1$$

$\Longleftarrow$: suppose $f\big|_{A_k}$ ⓜ $(\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$\begin{aligned}
f^{-1}(B) &= f^{-1}(B) \cap \Omega_1 \\
&= f^{-1}(B) \cap \bigcup_k A_k \\
&= \bigcup_k \big(f^{-1}(B) \cap A_k\big) \\
&= \underbrace{\bigcup_k \underbrace{f\big|_{A_k}^{-1}(B)}_{\in\, \mathcal{F}_1 \cap A_k \text{ by assumption}}}_{\substack{\in \mathcal{F}_1 \text{ since } \mathcal{F}_1 \cap A_k \subset \mathcal{F}_1 \\ \& \ \mathcal{F}_1 \text{ is a } \sigma\text{-field}}}
\end{aligned}$$

QED.

Corollary: Piecewise metric continuous functions are ⓜ if the "pieces" are countable & Borel measurable.

Thm: (just check ⓜ on the range)

If $f: \Omega_1 \longrightarrow \Omega_2^\circ \subset \Omega_2$ ← metric space

and $\mathcal{F}_1$ is a σ-field on Ω_1 then

$$f \text{ ⓜ } \mathcal{F}_1/\mathcal{B}(\Omega_2) \iff f \text{ ⓜ } \mathcal{F}_1/\mathcal{B}(\Omega_2^\circ).$$

Proof: The borel restriction thm says

$$\mathcal{B}(\Omega_2^\circ) = \mathcal{B}(\Omega_2) \cap \Omega_2^\circ .$$

$\therefore$

$$\begin{aligned} f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{B}(\Omega_2^\circ) &\iff f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{B}(\Omega_2)\cap\Omega_2^\circ \\ &\iff \underbrace{f^{-1}(B\cap\Omega_2^\circ)}_{=\, f^{-1}(B)} \in \mathcal{F}_1,\ \forall B \in \mathcal{B}(\Omega_2) \\ &\iff f^{-1}(B) \in \mathcal{F}_1,\ \forall B \in \mathcal{B}(\Omega_2) \\ &\qquad \text{since } f \text{ maps into } \Omega_2^\circ \\ &\iff f \ \textcircled{m}\ \mathcal{F}_1/\mathcal{B}(\Omega_2). \end{aligned}$$

QED.

e.g. $\sin(x)\ \textcircled{m}\ \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$

$\iff \sin(x)\ \textcircled{m}\ \mathcal{B}(\mathbb{R})/\mathcal{B}([-1,1])$

Question:

is $f(x) = \begin{cases} \frac{1}{x}, & x>0 \\ -\infty, & x=0 \\ \sin(x), & x<0 \end{cases}$ $\textcircled{m}\ \mathcal{B}(\mathbb{R})/\mathcal{B}(\overline{\mathbb{R}})$

Yes since f is metric continuous on countably many measurable pieces.

Notation:

- f is "$\mathcal{F}$-measurable" if

$$(\Omega, \mathcal{F}) \xrightarrow{f\ \textcircled{m}} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$$

indicates $f\ \textcircled{m}\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

- f is "Borel measurable" if

sometimes just "measurable"

$$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \xrightarrow{f\ \textcircled{m}} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$$

- f is "Lebesgue measurable" if

$$(\mathbb{R}^d, \overline{\mathcal{B}(\mathbb{R}^d)}) \xrightarrow{f\ \textcircled{m}} (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$$

when $\overline{\mathcal{B}(\mathbb{R}^d)}$ is the completion w.r.t. Lebesgue measure.

Convention for ∞

- $\infty + x = \infty$ when $x \in (-\infty, \infty]$
- $\infty \cdot 0 = 0$
- $\infty \cdot \infty = \infty$
- $\frac{x}{\infty} = 0$ when $x \in \mathbb{R}$
- $\frac{x}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty - \infty$ are not defined.

Thm (Closure thm)

Let $(\Omega, \mathcal{F})$ be a measurable space.

(i) If $\Omega_{\mathcal{F}} \overset{f\,(m)}{\underset{g\,(m)}{\rightrightarrows}} \overline{\mathbb{R}}_{\mathcal{B}(\overline{\mathbb{R}})}$ then

$f+g$, $f\cdot g$, f/g, $\max(f,g)$, $\min(f,g)$, f^+, f^-, $|f|$ are all (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

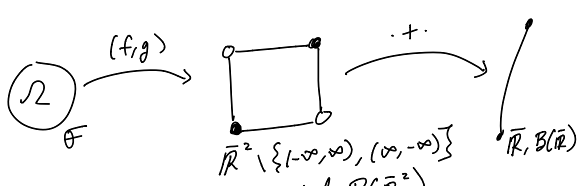
provided they are defined $\forall w \in \Omega$ ← i.e. No $\infty - \infty$, $\frac{\infty}{\infty}$ ...

(ii) If $\Omega_{\mathcal{F}} \xrightarrow{f_1\,(m)} \overline{\mathbb{R}}_{\mathcal{B}(\overline{\mathbb{R}})}$, $\vdots$, $\Omega_{\mathcal{F}} \xrightarrow{f_n\,(m)} \overline{\mathbb{R}}_{\mathcal{B}(\overline{\mathbb{R}})}$, $\vdots$

then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Proof:

(i) Just show $f+g$ ... the others are similar. Since $f(w)+g(w)$ is defined $\forall w \in \Omega$ we have



$+$ is (m) by metric continuity

f & g are (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

$\Leftrightarrow$ (f,g) is (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}}^2)$ by "just check cords"

$\Leftrightarrow$ (f,g) is (m) on the restricted space, by "just check the range"

$\therefore$ $f+g$ (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by "Composition of (m) is (m)".

To show (ii) just notice

$$\left(\sup_n f_n\right)^{-1}([-\infty, c]) = \{w : \sup_n f_n(w) \le c\}$$
$$= \bigcap_{n=1}^{\infty} \underbrace{\{w : f_n(w) \le c\}}_{\in \mathcal{F} \text{ since } f\,(m)\,\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})}$$

$\in \mathcal{F}$ since $\mathcal{F}$ is a σ-field.

Note: this will not be true for $<$

$\therefore$ $\sup_n f_n$ (m) $\mathcal{F}/\{[-\infty,c] : c \in \overline{\mathbb{R}}\}$

$\therefore$ $\sup_n f_n$ (m) $\mathcal{F}/\underbrace{\sigma\langle [-\infty,c] : c \in \overline{\mathbb{R}}\rangle}_{=\mathcal{B}(\overline{\mathbb{R}})}$.

For the others

$\inf_n f_n = -\sup_n(-f_n)$

$$\limsup_{n\to\infty} f_n = \lim_{m\to\infty} \underbrace{\sup_{n\ge m} f_n}_{\substack{\text{decreases} \\ \text{as } m\to\infty}} = \underbrace{\inf_m \underbrace{\sup_{n\ge m} f_n}_{(m)}}_{(m)}$$

$\liminf_{n\to\infty} f_n$ ... similar.

QED

e.g. Coin flip model from lecture 1.

$S_n(w) = \sum_{k=1}^{n} X_k(w)$ maps $(0,1] \to \overline{\mathbb{R}}$

Since S_n is constant over intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$

we have $S_n^{-1}([-\infty, x]) =$ finite disjoint union of dyadics $\in \mathcal{B}((0,1])$

By "generators are enough"

$$(0,1]_{\mathcal{B}((0,1])} \xrightarrow{S_n\,(m)} \overline{\mathbb{R}}_{\mathcal{B}(\overline{\mathbb{R}})} \quad \forall n$$

$\therefore$ $\limsup_n \frac{S_n}{\sqrt{2n \log\log n}}$ is (m) $\mathcal{B}((0,1])/\mathcal{B}(\overline{\mathbb{R}})$

$\therefore$ $\left\{\limsup_n \frac{S_n}{\sqrt{2n \log\log n}} = 1\right\} \in \mathcal{B}((0,1])$ since it is the pre-image of $\{1\} \in \mathcal{B}(\overline{\mathbb{R}})$.

Random variables, induced measures and c.d.f's

Def: X is a random variable if there exists a probability space $(\Omega, \mathcal{F}, P)$ where $X: \Omega \to \mathbb{R}$ s.t. $X \text{ \textcircled{m} } \mathcal{F}/\mathcal{B}(\mathbb{R})$.

X is an extended random variable if $X: \Omega \to \overline{\mathbb{R}}$ & $X \text{ \textcircled{m} } \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Picture:

$$\underset{\mathcal{F}, P}{\Omega} \xrightarrow{X \text{ \textcircled{m}}} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}}.$$

we write $X(\omega)$ instead of $f(\omega)$ to indicate $(\Omega, \mathcal{F})$ has a probability measure attached. Think of P as modeling a random draw $\omega \in \Omega$ & $X(\omega)$ as a "variable" or "label" associated with each $\omega \in \Omega$.

Since $X \text{ \textcircled{m} } \mathcal{F}/\mathcal{B}(\mathbb{R})$ it makes sense to talk about quantities like:

$$P(X=1) = P(\underbrace{X^{-1}(\underbrace{\{1\}}_{\in \mathcal{B}(\mathbb{R})})}_{\in \mathcal{F}})$$

$$P(X \le x) = P(\underbrace{X^{-1}((-\infty, x])}_{\in \mathcal{F}})$$

$$P(X \in \mathbb{R}) = P(X^{-1}(\mathbb{R})).$$

Def: If X is a random variable defined on $(\Omega, \mathcal{F}, P)$, the distribution of X (also called the induced probability measure) is a set function $PX^{-1}: \mathcal{B}(\mathbb{R}) \to [0,1]$ given by

$$PX^{-1}(B) := P(X^{-1}(B)) = P(X \in B).$$

More generally if

$$\underset{\mathcal{F}_1, \mu}{\Omega_1} \xrightarrow{f \text{ \textcircled{m}}} \underset{\mathcal{F}_2}{\Omega_2}$$

where $\mathcal{F}_1$ & $\mathcal{F}_2$ are σ-fields & $(\Omega_1, \mathcal{F}_1, \mu)$ is a measure then

$$\mu f^{-1}(F) := \mu(f^{-1}(F)), \ \forall F \in \mathcal{F}_2$$

induced measure on $(\Omega_2, \mathcal{F}_2)$.

Thm: In the setup above μf^{-1} is a measure on $(\Omega_2, \mathcal{F}_2)$

Proof: Follows immediately since pre-images & set operations commute. QED

Since $\mu f^{-1}(\Omega_2) = \mu(\Omega_1)$ we have the following facts:

- $\mu(\Omega_1) = 1 \Rightarrow \mu f^{-1}(\Omega_2) = 1$

  i.e. PX^{-1} is a probability measure

- $\mu(\Omega_1) < \infty \Rightarrow \mu f^{-1}(\Omega_2) < \infty$

- Warning:

  μ is a σ-finite measure

  $\not\Rightarrow \mu f^{-1}$ is a σ-finite measure

e.g.

$$\underset{\mathcal{B}(\mathbb{R}),\, \mathcal{L}^1}{\mathbb{R}} \xrightarrow{f \equiv 0} \Omega_2 = \{0\}, \quad \mathcal{F}_2 = \{\emptyset, \Omega_2\}$$

f is $\mathcal{B}(\mathbb{R})/\mathcal{F}_2$ measurable and $\mathcal{L}^1$ is σ-finite but $\mathcal{L}^1 f^{-1}$ is not σ-finite since

$$\mathcal{L}^1 f^{-1}(\emptyset) = 0 \text{ and}$$

$$\mathcal{L}^1 f^{-1}(\Omega_2) = \mathcal{L}^1(\mathbb{R}) = \infty.$$

Notice that even if two r.v.s X & Y are defined on different probability spaces the induceded distributions are both on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Def: If X & Y are two r.v.s s.t.

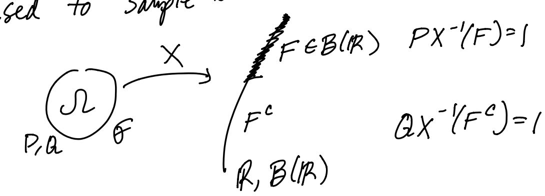
$$\underset{\mathcal{F}_1, P}{\Omega_1} \xrightarrow{X} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}} \quad \& \quad \underset{\mathcal{F}_2, Q}{\Omega_2} \xrightarrow{Y} \underset{\mathcal{B}(\mathbb{R})}{\mathbb{R}}$$

we write $X \sim Y$ or $X \overset{\mathcal{D}}{=} Y$ if $PX^{-1} = QY^{-1}$ over $\mathcal{B}(\mathbb{R})$.

In the simple statistics setup there are two possible measures P & Q, both defined on $(\Omega, \mathcal{F})$. $\omega \in \Omega$ is picket at random either from P or Q & the "observer" only gets to see the value of $X(\omega)$ (i.e. the data).

The observer then trys to figure out which P or Q ω was sampled from.

Note: using the notation from HWK3 if $PX^{-1} \perp QX^{-1}$ then the observer will know exactly which P or Q was used to sample ω



- If $X(\omega) \in F$ then ω was drawn from P.
- If $X(\omega) \in F^c$ then ω was drawn from Q.

Def: The distribution function (sometimes called cumulative distribution function .. CDF) for a random variable X defined on $(\Omega, \mathcal{F}, P)$ is the function $F: \mathbb{R} \to [0,1]$ defined as

$$F(t) = P(X \le t), \ \forall t \in \mathbb{R}.$$
$$= PX^{-1}((-\infty, t])$$

15

Thm: If X & Y are two r.v.s with c.d.f.s F_X & F_Y, respectively then

$$X \overset{\mathcal{D}}{=} Y \iff F_X(t) = F_Y(t) \ \forall t \in \mathbb{R}.$$

Proof:

$\Rightarrow$: trivial when taking $B = (-\infty, t]$ since $X^{-1}((-\infty,t]) = \{X \le t\}$

$\Leftarrow$:

$F_X(t) = F_Y(t) \ \forall t \in \mathbb{R}$

$\Rightarrow PX^{-1}((-\infty,t]) = PY^{-1}((-\infty,t]) \ \forall t \in \mathbb{R}$.

$\Rightarrow PX^{-1} = PY^{-1}$ on $\mathcal{P} = \{(-\infty,t] : t \in \mathbb{R}\}$

$\overset{\pi\text{-uniq}}{\Longrightarrow} PX^{-1} = PY^{-1}$ on $\underbrace{\sigma\langle \mathcal{P} \rangle}_{\pi\text{-sys}} = \mathcal{B}(\mathbb{R})$.

QED

Thm (Properties of c.d.f.s)

Let $F(t) = P(X \le t)$ for a r.v. X defined on $(\Omega, \mathcal{F}, P)$. Then

(I) $F(x) \le F(y)$, $\forall x \le y$

(II) $\lim_{x \downarrow y} F(x) = F(y)$

(III) $\lim_{x \to \infty} F(x) = 1$ & $\lim_{x \to -\infty} F(x) = 0$

16

Proof:

(I): clear since $A \subset B \Rightarrow PX^{-1}(A) \le PX^{-1}(B)$

(II): $\{X \le x\} \downarrow \{X \le y\}$ as $x \downarrow y$

$\therefore P(X \le x) \downarrow P(X \le y)$.

(warning: it is not true that $\{X < x\} \downarrow \{X < y\}$

e.g. take $x = \frac{1}{n}$, $y = 0$, $X \ge 0$ & $P(X=0) > 0$)

(III): Follows since $\{X \le x\} \uparrow \mathbb{R}$ & $\{X \le x\} \downarrow \emptyset$.

QED.

It turns out properties (I), (II), (III) are characterizing features of c.d.f.s i.e. if $F : \mathbb{R} \to [0,1]$ satisfies (I), (II), (III) then $\exists$ a r.v. X s.t. $F(t) = P(X \le t)$.

In fact, this will be the main tool we use to show the existance of an infinite sequence of indep. r.v.s all with a specified distribution.

Def: if $F : \mathbb{R} \to [0,1]$ satisfies (I), (II) & (III) of the above thm define

$$F^{-1}(u) := \inf\{x \in \mathbb{R} : F(x) \ge u\}$$

Fact 1: The inf in F^{-1} is attained. 17

18