Homework 5

Due March 15, 2017

Let (Ω, F, P) be a probability space and $\mathcal{A}_1$ and $\mathcal{A}_2$ sub-σ-fields of $\mathcal{F}$. Recall that $\mathcal{A}_1$ and $\mathcal{A}_2$ are **independent**, written $\mathcal{A}_1 \perp \mathcal{A}_2$, if and only if

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

for all $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Notice that this definition is also equivalent to the condition

$$E(X_1 X_2) = E(X_1)E(X_2)$$

for all $X_1 \in \mathscr{N}(\Omega, \mathcal{A}_1)$ and $X_2 \in \mathscr{N}(\Omega, \mathcal{A}_2)$ (where $\mathscr{N}(\Omega, \mathcal{A}_i)$ denotes the set of non-negative extended random variables).

Conditional independence of σ-fields is an important concept for studying Markov properties of random fields. Two σ-fields $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be **conditionally independent** given a sub-σ-field $\mathcal{B}$ of $\mathcal{F}$, written $\mathcal{A}_1 \perp_{\mathcal{B}} \mathcal{A}_2$, if and only if

$$E(X_1 X_2 | \mathcal{B}) = E(X_1 | \mathcal{B})E(X_2 | \mathcal{B})$$

for all $X_1 \in \mathscr{N}(\Omega, \mathcal{A}_1)$ and $X_2 \in \mathscr{N}(\Omega, \mathcal{A}_2)$. Also let $\mathcal{C} \vee \mathcal{D}$ denote $\sigma\langle \mathcal{C}, \mathcal{D} \rangle$ when $\mathcal{C}$ and $\mathcal{D}$ are two collections of events on Ω.

Exercise 1. *Show the following*

1. $\mathcal{A}_1 \perp_{\mathcal{B}} \mathcal{A}_2 \iff E(I_{A_1} I_{A_2} | \mathcal{B}) \stackrel{a.e.}{=} E(I_{A_1} | \mathcal{B})E(I_{A_2} | \mathcal{B})$ *for all* $A_1 \in \mathcal{A}_1$ *and* $A_2 \in \mathcal{A}_2$.
2. $\mathcal{A}_1 \perp_{\mathcal{B}} \mathcal{A}_2 \iff E(X_2 | \mathcal{A}_1 \vee \mathcal{B}) \stackrel{a.e.}{=} E(X_2 | \mathcal{B})$ *for all* $X_2 \in \mathscr{N}(\Omega, \mathcal{A}_2)$.
3. $\mathcal{A}_1 \perp \mathcal{A}_2 \iff E(X_2 | \mathcal{A}_1) \stackrel{a.e.}{=} E(X_2)$ *for all* $X_2 \in \mathscr{N}(\Omega, \mathcal{A}_2)$.
4. $\mathcal{A}_1 \perp_{\mathcal{B}} \mathcal{A}_2 \iff (\mathcal{A}_1 \vee \mathcal{B}) \perp_{\mathcal{B}} (\mathcal{A}_2 \vee \mathcal{B})$.
5. $(\mathcal{A}_1 \vee \mathcal{B}) \perp \mathcal{A}_2 \iff \mathcal{B} \perp \mathcal{A}_2$ *and* $\mathcal{A}_1 \perp_{\mathcal{B}} \mathcal{A}_2$.

Hint for 2. ($\Longrightarrow$), it suffices (why?) to consider the case where X_2 is integrable; apply π-system results for conditional expected value with $\mathcal{P} = \{A_1 \cap B \colon A_1 \in \mathcal{A}_1$ and $B \in \mathcal{B}\}$.