Homework 6

Due Tuesday, November 22, 2016

Exercise 1 (Approximating functions in $L_1(\mu)$). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show the following statements:*

1. *If $f \in L_1(\mu)$ then for each $\epsilon > 0$ there exists an integrable simple function g such that $\int |f - g| d\mu \leq \epsilon$;*

2. *If $\mathcal{F}_0$ is a field generating $\mathcal{F}$ and μ is σ-finite on $\mathcal{F}_0$, then the function g from item 1 can be taken to be of the form $g = \sum_{k=1}^{n} c_k I_{A_k}$ where each $A_k \in \mathcal{F}_0$.*

When proving item 2 in the above exercise, you can use the following claim mentioned in Lecture 5 on the Caratheodory Extension Theorem.

Claim 1. *Suppose $\mathcal{F}_0$ is a field on Ω and μ is a σ-finite measure on $(\Omega, \sigma\langle\mathcal{F}_0\rangle)$. Then for any $A \in \sigma\langle\mathcal{F}_0\rangle$ there exists $B \in \mathcal{F}_0$ such that $\mu(A \triangle B) \leq \epsilon$.*

Exercise 2. *Suppose $f \colon \mathbb{R} \to \mathbb{R}$ and $f \in L_1(\mathcal{L}^1)$. Show that*

$$\lim_{t \to 0} \int |f(x+t) - f(x)| dx = 0.$$

Hint: Use the previous exercise to find a g which approximates f and then notice that $g(\bullet + t)$ approximates $f(\bullet + t)$. Then compare $g(\bullet)$ with $g(\bullet + t)$

Exercise 3 (L^1 is complete). *Let $f_1, f_2, \ldots$ be integrable functions such that $\alpha_{m,n} := \int |f_n - f_m| d\mu$ tends to 0 as m and n tend to ∞. Show that there exists an integrable function f such that $\beta_n := \int |f - f_n| d\mu$ tends to 0 as n tends to ∞. Hint: inductively choose indices $n_k > n_{k-1}$ such that $\alpha_{m,n} \leq 2^{-k}$ for all $m, n \geq n_k$ and set $f = \sum_{k=1}^{\infty} (f_{n_k} - f_{n_{k-1}})$ with $f_{n_0} = 0$.*

Exercise 4. *Suppose that $f_1, f_2, \ldots$ and f are integrable and that $f_n \to f$ μ-a.e. Show that $\lim_n \int |f_n - f| d\mu = 0$ if and only if $\int |f_n| d\mu \to \int |f| d\mu$. Hint: for '$\Leftarrow$' study the proof of the DCT to show that $\limsup_n \int |f_n - f| d\mu \leq \int \limsup_n |f_n - f| d\mu$. In particular, show that $\int 2|f| d\mu - \int \limsup_n |f_n - f| d\mu \leq \int 2|f| d\mu - \limsup_n \int |f_n - f| d\mu$.*