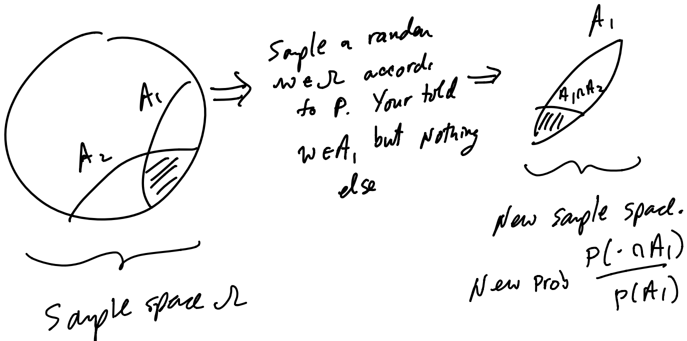
Lecture 6: Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space
sample space, σ-field, prob measure

Suppose $A_1, A_2 \in \mathcal{F}$ with $P(A_1) > 0$ & $P(A_2) > 0$.
Recall from undergrad probability

$$P(A_2|A_1) \equiv \text{prob of } A_2 \text{ given } A_1 := \frac{P(A_1 \cap A_2)}{P(A_1)}$$

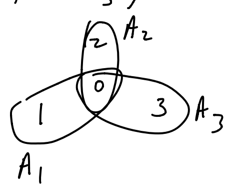


A_1 is independent of A_2 if $P(A_2|A_1) = P(A_2)$

i.e. if $P(A_1 \cap A_2) = P(A_1)P(A_2)$.

Question: How to make sense of independence among a collection of events (possibly uncountably many)? Is pairwise independence enough?

e.g. $\Omega = \{0,1,2,3\}$, $\mathcal{F} = 2^\Omega$, $P =$ unif on Ω.



$$i \neq j \Rightarrow P(\underbrace{\underbrace{A_i \cap A_j}_{=\{0\}}}_{=1/4}) = \underbrace{P(A_i)}_{1/2}\underbrace{P(A_j)}_{1/2}$$

so A_1, A_2, A_3 are pairwise independent.
But A_1, A_2, A_3 are not jointly indep:

$$P(A_1 \cap A_2 \cap A_3) = 1/4$$
$$P(A_1)P(A_2)P(A_3) = \frac{1}{8}$$

(Note: $P(A_1 | A_2 \cap A_3) = 1 \neq P(A_1)$)

e.g. Let $A_1, \ldots, A_n$ be events (i.e. $A_i \in \mathcal{F}$) s.t. $A_1 = \emptyset$. Then

$$P(A_1 \cap \cdots \cap A_n) = 0 = P(A_1) \cdots P(A_n)$$

So the full factorization criterian will not work as a def of independence either

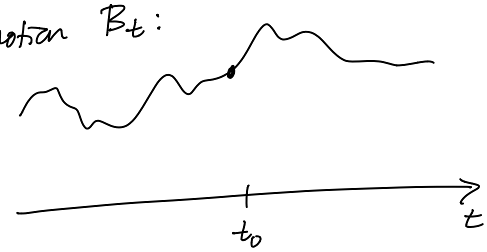
Here is the "right" def of indep for a collection of events.

Def: A collection of events $\{A_k\}_{k \in \mathcal{K}}$ are independent events iff $\forall$ finite $\mathcal{H} \subset \mathcal{K}$

$$P\left(\bigcap_{h \in \mathcal{H}} A_h\right) = \prod_{h \in \mathcal{H}} P(A_h).$$

Note: $\mathcal{K}$ is allowed to be any index set.

We will also need the notion of independent σ-fields to make sense of things like the strong markov property of Brownian motion B_t:



$\sigma\langle B_t : t < t_0 \rangle$ is indep of $\sigma\langle B_t : t > t_0 \rangle$ given $\sigma\langle B_{t_0} \rangle$.

Def: Let $\mathcal{K}$ be an arbitrary index set. $\forall k \in \mathcal{K}$, let $\mathcal{A}_k$ be a collection of events. The $\mathcal{A}_k$'s are independent collections if $\{A_k\}_{k \in \mathcal{K}}$ are indedendent events for each choice $A_k \in \mathcal{A}_k$.

Thm: Let $\mathcal{A}_k, \mathcal{B}_k$ be collections of events for each $k \in \mathcal{K}$ (arb index set). Then

(i) (subclasses):

If $\mathcal{A}_k \subset \mathcal{B}_k$ $\forall k \in \mathcal{K}$ & the $\mathcal{B}_k$'s are indep then the $\mathcal{A}_k$'s are indep.

(ii) (augmentation):

$\mathcal{A}_k$'s are indep iff $\mathcal{A}_k \cup \{\Omega\}$'s are indep.

(iii) (simplified product):

If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ all contain Ω then the $\mathcal{A}_k$'s are indep iff

$$P\Big(\bigcap_{k=1}^{n} A_k\Big) = \prod_{k=1}^{n} P(A_k)$$

$\forall A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$.

Proof:

(i): trivial

(ii): $\Leftarrow$ follows by "subclasses".

For $\Rightarrow$ choose $A_k \in \mathcal{A}_k \cup \{\Omega\}$ & finite $\mathcal{H} \subset \mathcal{K}$. Let $\mathcal{K}_0 = \{k : A_k \in \mathcal{A}_k\}$.

$\therefore$ $P\Big(\bigcap_{h \in \mathcal{H}} A_h\Big)$

$= P\Big(\bigcap_{h \in \mathcal{H} \cap \mathcal{K}_0} A_h\Big)$, $A_k = \Omega$ when $k \in \mathcal{K} - \mathcal{K}_0$

$= \prod_{h \in \mathcal{H} \cap \mathcal{K}_0} P(A_h)$, $\mathcal{A}_k$'s indep

$= \prod_{h \in \mathcal{H}} P(A_h)$, $P(A_k) = P(\Omega) = 1$ when $k \in \mathcal{K} - \mathcal{K}_0$

$\therefore$ $\mathcal{A}_k \cup \{\Omega\}$'s are indep.

(iii) $P\Big(\bigcap_{k=1}^{n} A_k\Big) = \prod_{k=1}^{n} P(A_k)$

$\Rightarrow P\Big(\bigcap_{h \in \mathcal{H}} A_h\Big) = \prod_{h \in \mathcal{H}} P(A_h)$

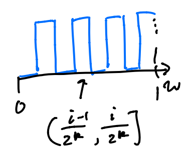
for $\mathcal{H} \subset \{1, 2, \dots, n\}$ by replacing A_k with $\Omega \in \mathcal{A}_k$. $k \notin \mathcal{H}$.

QED

e.g. Coin flip model from lecture 1:

$\Omega = (0,1]$, $\mathcal{F} = \mathcal{B}((0,1])$, P = uniform measure.

$X_k(\omega) :=$ k^{th} binary digit of ω =

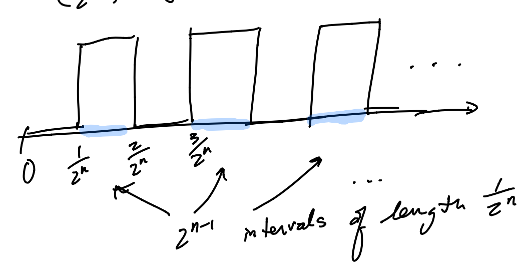


$H_k := \{\omega \in \Omega : X_k(\omega) = 1\}$

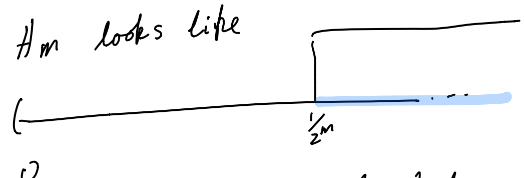
event of flipping a heads on the k^{th} toss if we want X_k to model fair coin flips.

Claim: $H_1, H_2, H_3, \dots$ are indep events.

Proof: $H_n = \left(\frac{1}{2^n}, \frac{2}{2^n}\right] \cup \left(\frac{3}{2^n}, \frac{4}{2^n}\right] \cup \left(\frac{5}{2^n}, \frac{6}{2^n}\right] \cdots$



If $m < n$ then H_m looks like



$\therefore$ $H_n \cap H_m$ = union of half the intervals which make up H_n

Let $1 \le i_1 < i_2 < \dots < i_n$ & show

$$P(H_{i_1} \cap \dots \cap H_{i_n}) = \underbrace{P(H_{i_1}) \cdots P(H_{i_n})}_{= \frac{1}{2^n}}$$

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Now $H_{i_n} \cap H_{i_{n-1}} \cap \cdots \cap H_{i_1} = \dfrac{2^{i_n-1}}{2^{n-1}}$ disjoint intervals of length $\dfrac{1}{2^{i_n}}$

2^{i_n-1} intervals of length $\frac{1}{2^{i_n}}$

reduce the # of intervals by $\frac{1}{2}$ for each further intersection

$$\therefore\ P(H_{i_n} \cap H_{i_{n-1}} \cap \cdots \cap H_{i_1}) = \frac{2^{i_n-1}}{2^{n-1}} \cdot \frac{1}{2^{i_n}} = \frac{1}{2^n}$$

as was to be shown QED.

π-generators are enough & ANOVA

At this point checking two σ-fields are indep would be a daunting task since we have no representation for general events in a σ-field.

The following thm helps this.

Thm (π-generators are enough):

Let $\mathcal{A}_k \subset \mathcal{F}$, $\forall k \in \mathcal{K}$. Then

$\mathcal{A}_k$'s are indep π-systems $\Longrightarrow \sigma\langle \mathcal{A}_k \rangle$'s are independent.

Proof: Let $\mathcal{B}_k := \mathcal{A}_k \cup \{\Omega\}$

Suppose the $\mathcal{A}_k$'s are indep π-sys

∴ the $\mathcal{B}_k$'s are indep π-sys, by augmentation

∴ $\forall$ distinct $k_1, k_2, \ldots, k_n \in \mathcal{K}$

the $\mathcal{B}_{k_1}, \mathcal{B}_{k_2}, \ldots, \mathcal{B}_{k_n}$ are indep π-sys

show $\sigma\langle \mathcal{B}_{k_1} \rangle, \mathcal{B}_{k_2}, \ldots, \mathcal{B}_{k_n}$ are indep π-sys and we will be done (by induction)

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By the simplified product criterion this is equivalent to showing

$$P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n) \quad (*)$$

$\forall B_1 \in \sigma\langle \mathcal{B}_{k_1} \rangle, B_2 \in \mathcal{B}_{k_2}, \ldots, B_n \in \mathcal{B}_{k_n}$

Fixing $B_2, \ldots, B_n$ let

$$\mathcal{L} := \{B_1 \in \mathcal{F} : (*) \text{ holds}\}$$

& show $\sigma\langle \mathcal{B}_{k_1} \rangle \subset \mathcal{L}$.

- $\mathcal{B}_{k_1} \subset \mathcal{L}$: Yes, since $\mathcal{B}_k$'s are indep.
- $\Omega \in \mathcal{L}$: Yes, since $\Omega \in \mathcal{B}_k$ $\forall k \in \mathcal{K}$.
- $B \in \mathcal{L} \Rightarrow$

$$P(B^c \cap \underbrace{B_2 \cap \cdots}_{A}) = P(B_2 \cap \cdots) - P(B \cap B_2 \cap \cdots)$$

since $P(B^c \cap A) = P(A - B \cap A)$

$$= P(\Omega \cap B_2 \cap \cdots) - P(B \cap B_2 \cap \cdots)$$

$$= P(\Omega) \cdot P(B_2) \cdots - P(B) P(B_2) \cdots$$

since $\Omega, B \in \mathcal{L}$

$$= \underbrace{[P(\Omega) - P(B)]}_{P(B^c)} P(B_2) \cdots P(B_n)$$

$$\Longrightarrow B^c \in \mathcal{L}$$

- $\underbrace{A_1, A_2, \ldots}_{\text{disjoint}} \in \mathcal{L}$

$$\Longrightarrow P\Big(\big(\bigcup_k A_k\big) \cap B_2 \cap \cdots \cap B_n\Big)$$

$$= \sum_k P(A_k \cap B_2 \cap \cdots \cap B_n)$$

$$= \sum_k P(A_k) P(B_2) \cdots P(B_n)$$

$$= P(B_2) \cdots P(B_n) \underbrace{\Big[\sum_k P(A_k)\Big]}_{P(\bigcup_k A_k)}$$

$$\Longrightarrow \bigcup_k A_k \in \mathcal{L}$$

$\therefore \mathcal{A}$ is a λ-sys & B_{k_1} is a π-sys.

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$\therefore \sigma\langle B_{k_1}\rangle \subset \mathcal{A}$.

QED.

e.g. coin flip example showed $H_1, H_2, \ldots$ are indep since $\{H_k\}$ is a π-sys for each k, $\sigma\langle H_1\rangle, \sigma\langle H_2\rangle, \ldots$ are indep σ-fields (where $\sigma\langle H_k\rangle = \{\emptyset, \Omega, H_k, H_k^c\}$)

$\therefore$ Any sequence $H_1, H_2^c, H_3, H_4^c, H_5^c, \ldots$ are indep.

tails on the 5th toss

To modivate the next thm let

A = the event $\sum_{k=1}^{n}(1-2X_{2k})=0$ for infinitely many n

B = the event $\sum_{k=1}^{n}(1-2X_{2k+1})=0$ for infinitely many n

is A indep of B?

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Thm (ANOVA): Matrix of π-systems

$$\begin{matrix} \mathcal{O}_{11} & \mathcal{O}_{12} & \mathcal{O}_{13} & \cdots \\ \mathcal{O}_{21} & \mathcal{O}_{22} & \mathcal{O}_{23} & \cdots \\ \mathcal{O}_{31} & \mathcal{O}_{32} & \mathcal{O}_{33} & \cdots \\ \vdots & & & \ddots \end{matrix}$$

Let $R_i = \sigma\langle \underbrace{\mathcal{O}_{i1}, \mathcal{O}_{i2}, \ldots}_{i\text{th Row}} \rangle$

Then

all the $\mathcal{O}_{ik}$'s are indep $\iff$ (i) R_k's are indep (ii) the $\mathcal{O}_{ik}$'s within each Row are independent

Proof:

($\Longrightarrow$) Suppose all the $\mathcal{O}_{ik}$'s are indep.

$\therefore$ (ii) clearly holds

To show (i) note

$R_k = \sigma\langle \underbrace{\mathcal{O}_{k1} \cup \mathcal{O}_{k2} \cup \cdots}_{\substack{\text{would like to} \\ \text{use } \pi\text{-generators} \\ \text{but this isn't a} \\ \pi\text{-sys}}} \rangle = \sigma\langle P_k \rangle$

where P_k = the closure of $\mathcal{O}_{k1} \cup \mathcal{O}_{k2} \cup \cdots$ under finite intersection

Clearly P_k's are π-systems.

Lets show the P_k's are indep.

Select one P_k from $\mathcal{P}_k$ and note: (9)

$$P(P_{i_1} \cap \cdots \cap P_{i_n})$$

write this as $\underbrace{(A_1 \cap \cdots)}_{P_{i_1}}^{\text{Row } i_1} \cap \overbrace{(B_1 \cap \cdots)}^{\text{Row } i_2} \cap \overbrace{(C_1 \cap \cdots)}^{\text{Row } i_3} \cap \cdots$

each event in here is from a unique $\mathcal{A}_{i_2,j}$.

merging (via "$\cap$") multiple sets from the same $\mathcal{A}_{k_j}$ if necessary ... still a $\mathcal{A}_{k_i}$ set by π-sys props.

Now,

$$P(P_{i_1} \cap \cdots \cap P_{i_n}) = P(A_1) \cdots P(B_1) \cdots P(C_1) \cdots$$
$$= P(P_{i_1}) \cdots P(P_{i_n})$$

e.g.

$$P\Big(\overbrace{\underbrace{A_1 \cap A_2}_{\substack{\text{both in}\\ \mathcal{A}_{i_1,1}}}}^{P_{i_1}} \cap \overbrace{\underbrace{B_1}_{\in \mathcal{A}_{i_2,1}} \cap \underbrace{B_2}_{\in \mathcal{A}_{i_2,2}}}^{P_{i_2}}\Big)$$

$$= P(\underbrace{A_1 \cap A_2}_{\substack{\in \mathcal{A}_{i_1,1} \text{ by}\\ \pi\text{-sys}}})\, P(B_1)\, P(B_2)$$ since $\mathcal{A}_{i_1,1}, \mathcal{A}_{i_2,1}$ & $\mathcal{A}_{i_2,2}$ are indep

$$= P(A_1 \cap A_2)\, P(B_1 \cap B_2)$$ since $\mathcal{A}_{i_2,1}$ & $\mathcal{A}_{i_2,2}$ are indep

$$= P(P_{i_1})\, P(P_{i_2})$$

$\therefore$ $\mathcal{P}_k$'s are indep π-sys.

$\therefore$ The σ-fields $\mathcal{R}_k := \sigma\langle \mathcal{P}_k \rangle$ are independent by π-generators.

(10)

($\Leftarrow$)

Suppose the row σ-fields $\mathcal{R}_k$ are indep & the $\mathcal{A}_{k_i}$'s within each Row are indep.

Let $\mathcal{H}$ be a finite set of (Row, col) index tuples

For each $(i,k) \in \mathcal{H}$ choose one $A_{ik} \in \mathcal{A}_{ik}$.

$$\therefore P\Big(\bigcap_{(i,k)\in\mathcal{H}} A_{ik}\Big) = P\Big(\bigcap_{\substack{\text{Rows } i\\ \text{in } \mathcal{H}}} \overbrace{\bigcap_{\substack{\text{Cols } k\\ \text{s.t. } (i,k)\in\mathcal{H}}} A_{ik}}^{\in \mathcal{R}_i}\Big)$$

$$\overset{R_i\text{'s indep}}{=} \prod_{\substack{\text{Rows } i\\ \text{in } \mathcal{H}}} P\Big(\bigcap_{\substack{\text{Cols } k\\ \text{s.t. } (i,k)\in\mathcal{H}}} A_{ik}\Big)$$

$$\overset{\text{within Row indep}}{=} \prod_{\substack{\text{Rows } i\\ \text{in } \mathcal{H}}} \prod_{\substack{\text{Cols } k\\ \text{s.t. } (i,k)\in\mathcal{H}}} P(A_{ik})$$

$$= \prod_{(i,k)\in\mathcal{H}} P(A_{ik})$$

QED

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Kolmogorov's 0-1 law

Let $\mathcal{A}_1, \mathcal{A}_2, \dots$ be a sequence of collections of $\mathcal{F}$-sets (i.e. $\mathcal{A}_k \subset \mathcal{F}$)

Def: The tail σ-field of the $\mathcal{A}_k$'s is defined as

$$\mathcal{T} := \bigcap_{m=1}^{\infty} \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \dots \rangle$$

$$= \{ A \in \mathcal{F} : A \in \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \dots \rangle \ \forall m \}$$

e.g. Let $A =$ the abnormal numbers in $(0,1]$ from Lecture 1.

Let $\varepsilon_n \searrow 0$ as $n \to \infty$

$$\therefore A = \{ w : \tfrac{S_n(w)}{n} \not\to 0 \text{ as } n \to \infty \}$$

$$\subset \{ w : \left| \tfrac{S_n(w)}{n} \right| > \varepsilon_n \text{ for infinitely many } n \}$$

$$= \bigcap_k \bigcup_{n \geq k} \underbrace{\{ w : \left| \tfrac{S_n(w)}{n} \right| > \varepsilon_n \}}_{A_n}$$

$$= \cancel{\Big(\bigcup_{n\geq 1} A_n\Big) \cap \Big(\bigcup_{n \geq 2} A_n\Big)} \cap \Big(\bigcup_{n\geq 3} A_n\Big) \cap \dots$$

Can drop the first m-1 terms

These get smaller

$$= \bigcap_{k=m}^{\infty} \bigcup_{n \geq k} A_n \quad \text{for any } m$$

$$\in \sigma\langle \{A_m\}, \{A_{m+1}\}, \dots \rangle$$

for any m

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$\therefore A \in \mathcal{T}$ The tail σ-field generated by $\{A_1\}, \{A_2\}, \dots$

Thm (Kolmogorov's 0-1 law)

If $\mathcal{A}_1, \mathcal{A}_2, \dots$ are indep π-systems then $\forall A \in \mathcal{T}$, $P(A) = 0$ or $P(A) = 1$.

Proof:

$\mathcal{A}_1, \dots, \mathcal{A}_{m-1}, \mathcal{A}_m, \mathcal{A}_{m+1} \dots$ are indep π-sys.

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \dots \sigma\langle \mathcal{A}_{m-1} \rangle, \sigma\langle \mathcal{A}_m, \mathcal{A}_{m+1}, \dots \rangle$ are indep π-sys by anova.

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \dots, \sigma\langle \mathcal{A}_{m-1} \rangle, \mathcal{T}$ are indep π-sys by subclasses.

holds for all m

$\therefore \sigma\langle \mathcal{A}_1 \rangle, \sigma\langle \mathcal{A}_2 \rangle, \dots, \mathcal{T}$ are indep π-sys by the finite selection requirement of the def of indep.

$\therefore \sigma\langle \mathcal{A}_1, \mathcal{A}_2, \dots \rangle, \mathcal{T}$ are indep π-sys by Anova.

$\therefore \mathcal{T}, \mathcal{T}$ are indep π-sys by subclasses

$\therefore \forall A \in \mathcal{T}, \ P(A \cap A) = P(A)P(A)$

$\therefore P(A) = 0$ or 1.

QED

This implies (from the previous example) that either

$$P\left(\begin{array}{c}\text{Abnormal numbers in} \\ [0,1]\end{array}\right) = \begin{cases} 1 & \text{or} \\ 0 & \end{cases}$$

we showed it is zero.

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