Homework 7

Due Thursday, December 8, 2016

Exercise 1 (Sterling's formula for the Gamma function). *The Gamma function is defined by the equality* $\Gamma(r+1) := \int_0^\infty y^r e^{-y} dy$ *for* $r \in (0, \infty)$.

1. *Use the change of variable* $z = (y-r)/\sqrt{r}$ *to show that*

$$\rho_r := \frac{\Gamma(r+1)}{r^r e^{-r}\sqrt{r}} = \int_{-\sqrt{r}}^{\infty} e^{-r\phi(z/\sqrt{r})} dz$$

   where $\phi(u) := u - \log(1+u)$.

2. *Show that* $e^{-r\phi(z/\sqrt{r})} \to e^{-z^2/2}$ *as* $r \to \infty$ *for each* z.

3. *Show that there exists non-zero constants* $a, b \in \mathbb{R}$ *such that*

$$\phi(u) \geq \min(au^2, |bu|)$$

   for all $u \in (-1, \infty)$. *Hint: Use Taylor's theorem with remainder to bound* $\phi(u) \geq au^2$ *for* $-1 < u \leq 1$. *Then use convexity to bound* $\phi(u) \geq bu$ *for* $u \geq 1$.

4. *Conclude that there exists an integrable* g *and a sufficiently large* $r_0 > 0$ *such that*

$$\sup_{r > r_0} |e^{-r\phi(z/\sqrt{r})} I_{(-\sqrt{r}, \infty)}(z)| \leq g(z)$$

5. *Finally use the DCT to deduce that*

$$\lim_{r\to\infty} \frac{\Gamma(r+1)}{r^r e^{-r}\sqrt{r}} = \int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}.$$

Exercise 2. *Suppose* X *is a random variable with a t-distribution on* $\nu > 0$ *degrees of freedom. Then* X *has density* f_ν *(with respect to Lebesque measure) given by*

$$f_\nu(x) := \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)} \frac{1}{(1 + x^2/\nu)^{(\nu+1)/2}}$$

Show that the law of the random variable X *converges, in norm* $\|\cdot\|_{TV}$, *to a standard Gaussian distribution as* $\nu \to \infty$.

Exercise 3 (Conditions for $d\mu/d\rho = 1/(d\rho/d\mu)$**).** *Suppose that* ρ *is a measure with density* δ *with respect to* μ . *Show that:*

1. μ *has density* $1/\delta$ *with respect to* ρ *if and only if* $0 < \delta < \infty$ μ*-a.e.;*

2. *If* μ *is* σ*-finite and* $\delta < \infty$ μ*-a.e. then* ρ *is* σ*-finite.*

3. *If* μ *is* σ*-finite and* μ *has some density, say* f, *with respect to* ρ *then* $f = 1/\delta$ ρ*-a.e and* μ*-a.e..*

Hint for item 1: first find $\int_\bullet 1/\delta d\rho$.