Homework 2

Due Monday, February 6, 2016

Exercise 1. *Let P and Q be probability measures on $\mathbb{R}^d$ with finite second moments (i.e. if $X \sim P$ and $Y \sim Q$ then $E|X|^2 < \infty$ and $E|Y|^2 < \infty$). The L_2 Wasserstein distance d_W between P and Q is defined as*

$$d_W^2(P,Q) := \inf_{\mathcal{L}(X,Y)\in\Pi(P,Q)} E|X-Y|^2$$

where $\Pi(P,Q)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals given by P and Q. In particular, if X and Y are two d-dimensional random vectors, defined on the same probability space, then the distribution of the $2d$-dimensional random vector (X,Y) is in $\Pi(P,Q)$ if and only if $X \sim P$ and $Y \sim Q$.

Show that there exists d-dimensional random vectors X^* and Y^*, definied on the same probability space, such that $\mathcal{L}(X^*,Y^*) \in \Pi(P,Q)$ and

$$d_W^2(P,Q) := E|X^* - Y^*|^2.$$

Exercise 2. *Suppose $X_{i,n}$, for $n \in \mathbb{N}$ and $i \leq n$, forms a triangular array of independent random variables which satisfy the following conditions*

1. $EX_{i,n} = 0$ *for all n and $i \leq n$;*
2. $\sum_{i=1}^n EX_{i,n}^2 = 1$ *for all n;*
3. $\sum_{i=1}^n E(X_{i,n}^2 I_{\{|X_{i,n}|\geq\delta\}}) \to 0$ *as $n \to \infty$ for all $\delta > 0$.*

Use Lindebergs method to show

$$\sum_{i=1}^n X_{i,n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Exercise 3. *Give a counterexample to the conjecture: If $X_1, X_2, \ldots$ are independent random variables with $EX_n = 0$ and $EX_n^2 = 1$ for all n, then $(X_1 + \cdots + X_n)/\sqrt{n} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.*

Hint: *Consider X_n to take values $-v_n, 0, v_n$ with probabilities $p_n/2, 1-p_n, p_n/2$.*

Exercise 4. *Let $Z_1, Z_2, \ldots$ be iid random variables with density $\delta(z)$ with respect to Lebesque measure. Think of each Z_n recording the time it takes a random person to run a mile. Now let*

$$X_n := \begin{cases} 1 & \text{if } Z_n > \max(Z_1,\ldots,Z_{n-1}) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, X_n is a 1 if the n^{th} random person breaks a speed record. Show

$$\frac{S_n - E(S_n)}{\sqrt{var(S_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

where $S_n := X_1 + \cdots + X_n$. Hint: *You can use the fact that*

$$P(Z_n > Z_1, \ldots, Z_n > Z_{n-1}) = \int_{\mathbb{R}} P(z > Z_1, \ldots, z > Z_{n-1})\delta(z)dz.$$