

Lecture 6 : Independence

①

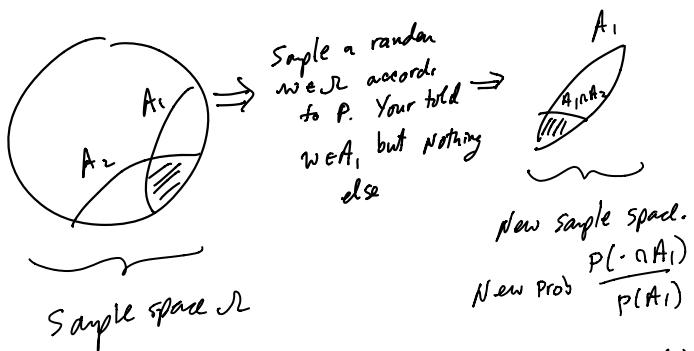
Let (Ω, \mathcal{F}, P) be a probability space

sample space
↑
 \mathcal{F} -field
prob measure

Suppose $A_1, A_2 \in \mathcal{F}$ with $P(A_1) > 0$ & $P(A_2) > 0$.

Recall from undergrad probability

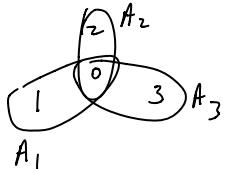
$$P(A_2|A_1) = \frac{\text{prob of } A_2 \text{ given } A_1}{\text{prob of } A_1} := \frac{P(A_1 \cap A_2)}{P(A_1)}$$



A_1 is independent of A_2 if $P(A_2|A_1) = P(A_2)$.
i.e. if $P(A_1 \cap A_2) = P(A_1)P(A_2)$.

Question: How to make sense of independent among a collection of events (possibly uncountably many)? Is pairwise independent enough?

e.g. $\Omega = \{0, 1, 2, 3\}$, $\mathcal{F} = 2^\Omega$, $P = \text{unif on } \Omega$.



$$i \neq j \Rightarrow P(A_i \cap A_j) = \underbrace{P(A_i)}_{=\{0\}} \underbrace{P(A_j)}_{=\{1\}} = \frac{1}{4}$$

so A_1, A_2, A_3 are pairwise independent.

But A_1, A_2, A_3 are not jointly indep:

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4}$$

$$P(A_1)P(A_2)P(A_3) = \frac{1}{8}$$

$$(\text{Note: } P(A_1|A_2 \cap A_3) = 1 \neq P(A_1))$$

e.g. Let A_1, \dots, A_n be events (i.e. $A_i \in \mathcal{F}$) ②

s.t. $A_1 = \emptyset$. Then

$$P(A_1 \cap \dots \cap A_n) = 0 = P(A_1) \cdots P(A_n)$$

so the full factorization criterian will not work as a def of independence either

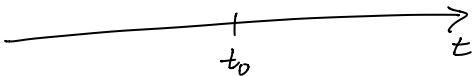
Here is the "right" def of indep for a collection of events.

Def: A collection of events $\{A_h\}_{h \in K}$ are independent events iff \forall finite $H \subset K$

$$P(\bigcap_{h \in H} A_h) = \prod_{h \in H} P(A_h).$$

Note: K is allowed to be any index set.

We will also need the notion of independent σ -fields to make sense of things like the strong markov property of Brownian motion B_t :



$\sigma(B_t : t < t_0)$ is indep of $\sigma(B_t : t > t_0)$ given $\sigma(B_{t_0})$.

Def: Let K be an arbitrary index set. $\forall k \in K$, let \mathcal{A}_k be a collection of events.

The \mathcal{A}_k 's are independent collections $\{A_k\}_{k \in K}$ are independent events for each choice $A_k \in \mathcal{A}_k$.

Thm: Let $\mathcal{A}_k, \mathcal{B}_k$ be collections of events for each $k \in K$ (arb index set). Then (3)

(i) (subclasses):

If $\mathcal{A}_k \subset \mathcal{B}_k \forall k \in K$ & the \mathcal{B}_k 's are indep then the \mathcal{A}_k 's are indep.

(ii) (augmentation):

\mathcal{A}_k 's are indep iff $\mathcal{A}_k \cup \{\mathcal{J}\}$'s are indep.

(iii) (simplified product):

If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ all contain \mathcal{J} then the \mathcal{A}_k 's are indep iff

$$P\left(\bigcap_{k=1}^n \mathcal{A}_k\right) = \prod_{k=1}^n P(\mathcal{A}_k)$$

$\forall A_1 \in \mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$.

Proof:

(i): trivial

(ii): \Leftarrow follows by "subclasses".

For \Rightarrow choose $A_k \in \mathcal{A}_k \cup \{\mathcal{J}\}$ & finite $H \subset K$. Let $K_0 = \{k : A_k \in \mathcal{A}_k\}$.

$$\therefore P\left(\bigcap_{h \in H} A_h\right)$$

$$= P\left(\bigcap_{h \in H \cap K_0} A_h\right), A_h = \mathcal{J} \text{ when } h \in K - K_0$$

$$= \prod_{h \in H \cap K_0} P(A_h), \mathcal{A}_k \text{'s indep}$$

$$= \prod_{h \in H} P(A_h), P(A_h) = P(\mathcal{J}) = 1 \text{ when } h \in K - K_0$$

$\therefore \mathcal{A}_k \cup \{\mathcal{J}\}$'s are indep.

(4)

$$(iii) P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

$$\Rightarrow P\left(\bigcap_{h \in H} A_h\right) = \prod_{h \in H} P(A_h)$$

for $H \subset \{1, 2, \dots, n\}$ by
replacing A_k with $\mathcal{J} \in \mathcal{A}_k$.
 $k \notin H$.

QED

e.g. Coin flip Model from lecture 1:

$\mathcal{J} = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, P = uniform measure.

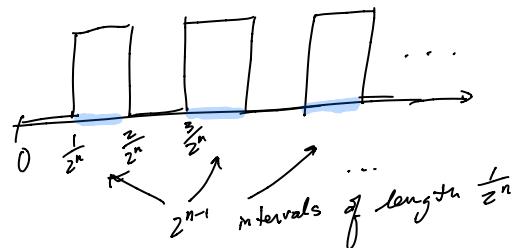
$$X_k(w) := k^{\text{th}} \text{ binary digit of } w = \begin{array}{c} \boxed{ } \\ \boxed{ } \end{array} \quad \begin{array}{c} \uparrow \\ \text{at } w \end{array}$$

$$H_k := \{w \in \mathcal{J} : X_k(w) = 1\} \quad \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$$

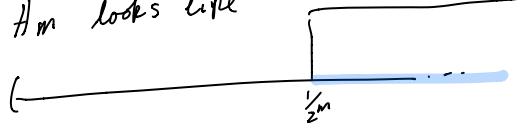
↑ event of flipping a heads on the k^{th} toss if we want X_k to model fair coin flips.

Claim: H_1, H_2, H_3, \dots are indep events.

Proof: $H_n = \left(\frac{1}{2^n}, \frac{2}{2^n}\right] \cup \left(\frac{3}{2^n}, \frac{4}{2^n}\right] \cup \left(\frac{5}{2^n}, \frac{6}{2^n}\right] \dots$



If $m < n$ then H_m looks like



$\therefore H_n \cap H_m = \text{union of half the intervals which make up } H_n$

Let $1 \leq i_1 < i_2 < \dots < i_n$ & show

$$P(H_{i_1} \cap \dots \cap H_{i_n}) = \underbrace{P(H_{i_1}) \dots P(H_{i_n})}_{= \frac{1}{2^n}}$$

Now $H_{i_n} \cap H_{i_{n-1}} \cap \dots \cap H_{i_1} = \frac{2^{i_n-1}}{2^{n-1}}$ disjoint intervals of length $\frac{1}{2^n}$ (5)

\uparrow \uparrow \uparrow
 2^{i_n-1} intervals of length $\frac{1}{2^n}$
 reduce the # of intervals by $\frac{1}{2}$ for each further intersection

$$\therefore P(H_{i_n} \cap H_{i_{n-1}} \cap \dots \cap H_{i_1}) = \frac{2^{i_n-1}}{2^{n-1}} \cdot \frac{1}{2^{i_n}} = \frac{1}{2^n}$$

as was to be shown QED.

π -generators are enough & ANOVA

At this point checking two σ -fields are indep would be a daunting task since we have no representation for general events in a σ -field.

The following thm helps this.

Thm (π -generators are enough):

Let $\mathcal{Q}_k \subset \mathcal{F}$, $k \in K$. Then

\mathcal{Q}_k 's are indep π -systems

$\Rightarrow \sigma(\mathcal{Q}_k)$'s are independent

Proof: Let $B_k := \mathcal{Q}_k \cup \{\emptyset\}$

Suppose the \mathcal{Q}_k 's are indep π -sys

\therefore the B_k 's are indep π -sys, by augmentation

$\therefore \forall$ distinct $k_1, k_2, \dots, k_n \in K$

the $B_{k_1}, B_{k_2}, \dots, B_{k_n}$ are indep π -sys

Show $\sigma(B_{k_1}), B_{k_2}, \dots, B_{k_n}$ are indep π -sys

and we will be done (by induction)

By the simplified product criterion (6)
this is equivalent to showing

$$P(B_1 \cap \dots \cap B_n) = P(B_1) \dots P(B_n) \quad (*)$$

$$\forall B_1 \in \sigma(B_{k_1}), B_2 \in \sigma(B_{k_2}), \dots, B_n \in \sigma(B_{k_n})$$

Fixing B_2, \dots, B_n let

$$\mathcal{Y} := \{B_1 \in \mathcal{F} : (*) \text{ holds}\}$$

& show $\sigma(B_{k_1}) \subset \mathcal{Y}$.

• $B_{k_1} \in \mathcal{Y}$: yes, since B_k 's are indep.

• $\emptyset \in \mathcal{Y}$: yes, since $\emptyset \in B_k \forall k \in K$.

• $B \in \mathcal{Y} \Rightarrow$

$$P(B^c \cap \underbrace{B_2 \cap \dots}_{A}) = P(B_2 \cap \dots) - P(B \cap B_2 \cap \dots)$$

since $P(B^c \cap A) = P(A - B \cap A)$

$$= P(\emptyset \cap B_2 \cap \dots) - P(B \cap B_2 \cap \dots)$$

$$= P(\emptyset) \cdot P(B_2) \dots - P(B) P(B_2) \dots$$

since $\emptyset, B \in \mathcal{Y}$

$$= \underbrace{[P(\emptyset) - P(B)]}_{P(B^c)} \underbrace{P(B_2) \dots P(B_n)}$$

$$\Rightarrow B^c \in \mathcal{Y}$$

• $\underbrace{A_1, A_2, \dots}_{\text{disjoint}} \in \mathcal{Y}$

disjoint

$$\Rightarrow P((\bigcup_k A_k) \cap B_2 \cap \dots \cap B_n)$$

$$= \sum_k P(A_k \cap B_2 \cap \dots \cap B_n)$$

$$= \sum_k P(A_k) P(B_2) \dots P(B_n)$$

$$= P(B_2) \dots P(B_n) \left[\sum_k P(A_k) \right]$$

$$\Rightarrow \bigcup_k A_k \in \mathcal{Y}$$

$$P(\bigcup_k A_k)$$

$\therefore \mathcal{M}$ is a λ -sys & B_{k_i} is a π -sys. (7)

$\therefore \sigma\langle B_{k_i} \rangle \subset \mathcal{M}$. QED.

e.g. coin flip example showed

H_1, H_2, \dots are indep

since $\{H_p\}$ is $\subset \pi$ -sys for each p ,

$\sigma\langle H_1 \rangle, \sigma\langle H_2 \rangle, \dots$ are indep

σ -fields (where $\sigma\langle H_p \rangle = \{\emptyset, \Omega, H_p, H_p^c\}$)

\therefore Any segment $H_1, H_2^c, H_3, H_4^c, H_5^c, \dots$ are indep.

\uparrow
tails
in the
n-th toss

To motivate the next thm let

$A =$ the event $\sum_{k=1}^n (1 - z_{2k}) = 0$

for infinitely many n

$B =$ the event $\sum_{k=1}^n (1 - z_{2k+1}) = 0$

for infinitely many n

is A indep of B ?

Thm (ANORA): Matrix of π -systems (8)

$$\begin{matrix} \mathcal{O}_{11} & \mathcal{O}_{12} & \mathcal{O}_{13} & \cdots \\ \mathcal{O}_{21} & \mathcal{O}_{22} & \mathcal{O}_{23} & \cdots \\ \mathcal{O}_{31} & \mathcal{O}_{32} & \mathcal{O}_{33} & \cdots \\ \vdots & & & \ddots \end{matrix}$$

Let $R_i = \sigma\langle \underbrace{\mathcal{O}_{i1}, \mathcal{O}_{i2}, \dots}_{i\text{-th row}} \rangle$

Then

all the \mathcal{O}_{ik} 's are indep \iff (i) R_k 's are indep
indep \iff (ii) the \mathcal{O}_{ik} 's within
each row are independent

Proof:

\Rightarrow Suppose all the \mathcal{O}_{ik} 's are indep.

\therefore (ii) clearly holds

To show (i) note

$$R_k = \sigma\langle \underbrace{\mathcal{O}_{k1} \cup \mathcal{O}_{k2} \cup \dots}_\text{would like to use \pi-generators} \rangle = \sigma\langle P_k \rangle$$

but this isn't a π -sys

where $P_k =$ the closure of $\mathcal{O}_{k1} \cup \mathcal{O}_{k2} \cup \dots$
under finite intersection

Clearly P_k 's are π -systems.

Let's show the P_k 's are indep.

Select one P_k from P_k and note:

(9)

$$P(P_{i_1} \cap \dots \cap P_{i_n}) = P(P_{i_1}) \dots P(P_{i_n})$$

write this as $A_1 \cap \dots \cap A_n$

where each A_j is a unique \mathcal{O}_{ik} (merging sets from the same \mathcal{O}_{ik} by intersection ... still a \mathcal{O}_{ik} -set by π -sys). Use independence of \mathcal{O}_{ik} 's to factor then group for each row

e.g.

$$\begin{aligned} & P(A_1 \cap A_2 \cap A_3 \cap A_4) \\ & \quad \text{both in } \mathcal{O}_{i_2,1} \text{ & } \mathcal{O}_{i_2,2} \\ & \quad \mathcal{O}_{i_1,1} \\ &= P(A_1 \cap A_2) P(A_3) P(A_4) \\ & \quad \text{in } \mathcal{O}_{i_1,1} \text{ by } \pi\text{-sys} \quad \text{since } \mathcal{O}_{i_1,1}, \mathcal{O}_{i_2,1} \text{ &} \\ & \quad \mathcal{O}_{i_2,2} \text{ are indep} \\ &= P(A_1 \cap A_2) P(A_3 \cap A_4) \text{ since } \\ & \quad \mathcal{O}_{i_2,1} \text{ & } \mathcal{O}_{i_2,2} \text{ are indep} \\ &= P(P_{i_1}) P(P_{i_2}) \end{aligned}$$

$\therefore P_k$'s are indep π -sys.

\therefore The σ -fields $R_k := \sigma\langle P_k \rangle$ are independent by π -generators.

(\Leftarrow)

(10)

Suppose the row σ -fields R_k are indep & the \mathcal{O}_{ik} 's within each row are indep.

Let \mathcal{H} be a finite set of (Row, col)

index tuples

For each $(i, k) \in \mathcal{H}$ choose one

$A_{ik} \in \mathcal{O}_{ik}$.

$$\therefore P(\bigcap_{(i, k) \in \mathcal{H}} A_{ik}) = P\left(\bigcap_{\substack{\text{Rows } i \\ \text{in } \mathcal{H}}} \bigcap_{\substack{\text{cols } k \\ \text{s.t. } (i, k) \in \mathcal{H}}} A_{ik}\right)$$

$$\stackrel{R_i \text{ is indep}}{=} \prod_{\substack{\text{Rows } i \\ \text{in } \mathcal{H}}} P\left(\bigcap_{\substack{\text{cols } k \\ \text{s.t. } (i, k) \in \mathcal{H}}} A_{ik}\right)$$

$$\stackrel{R_i \text{ is indep}}{=} \prod_{\substack{\text{Rows } i \\ \text{in } \mathcal{H}}} \prod_{\substack{\text{cols } k \\ \text{s.t. } (i, k) \in \mathcal{H}}} P(A_{ik})$$

$$= \prod_{(i, k) \in \mathcal{H}} P(A_{ik})$$

QED

Kolmogorov's 0-1 law

(11)

Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots$ be a sequence of collections of \mathcal{F} -sets (i.e. $\mathcal{Q}_k \subset \mathcal{F}$)

Dfn: The tail σ-field of the \mathcal{Q}_k 's is defined as

$$\begin{aligned}\Sigma &:= \bigcap_{m=1}^{\infty} \sigma(\mathcal{Q}_m, \mathcal{Q}_{m+1}, \dots) \\ &= \{A \in \mathcal{F} : A \in \sigma(\mathcal{Q}_m, \mathcal{Q}_{m+1}, \dots)^{\text{fin}}\}\end{aligned}$$

e.g. Let $A =$ the abnormal numbers n [0,1] from lecture 1.

Let $\varepsilon_n \searrow 0$ as $n \rightarrow \infty$

$$\therefore A = \{w : \frac{s_n(w)}{n} \not\rightarrow 0 \text{ as } n \rightarrow \infty\}$$

$$\subset \{w : \left| \frac{s_n(w)}{n} \right| > \varepsilon_n \text{ for infinitely many } n\}$$

$$= \bigcap_k \bigcup_{n \geq k} \underbrace{\{w : \left| \frac{s_n(w)}{n} \right| > \varepsilon_n\}}_{A_n}$$

$$= \left(\bigcup_{n \geq 1} A_n \right) \cap \left(\bigcup_{n \geq 2} A_n \right) \cap \left(\bigcup_{n \geq 3} A_n \right) \cap \dots$$

These get smaller

$$= \bigcap_{k=m}^{\infty} \bigcup_{n \geq k} A_n \quad \text{for any } m$$

$$\in \sigma(\{A_m\}, \{A_{m+1}\}, \dots)$$

for any m

$\therefore A \in \Sigma$ The tail σ-field generated by $\{\mathcal{A}_1\}, \{\mathcal{A}_2\}, \dots$

(12)

Thm (Kolmogorov's 0-1 law)

If $\mathcal{Q}_1, \mathcal{Q}_2, \dots$ are indep π-systems then $\forall A \in \Sigma, P(A) = 0$ or $P(A) = 1$.

Prof:

$\mathcal{Q}_1, \dots, \mathcal{Q}_{m-1}, \mathcal{Q}_m, \mathcal{Q}_{m+1}, \dots$ are indep π-sys.

$\therefore \sigma(\mathcal{Q}_1), \dots, \sigma(\mathcal{Q}_{m-1}), \sigma(\mathcal{Q}_m, \mathcal{Q}_{m+1}, \dots)$ are indep π-sys by anova.

$\therefore \sigma(\mathcal{Q}_1), \dots, \sigma(\mathcal{Q}_{m-1}), \Sigma$ are indep π-sys by subclasses.

$\therefore \sigma(\mathcal{Q}_1), \sigma(\mathcal{Q}_2), \dots, \Sigma$ are indep π-sys by the finite selection requirement of the def of indep.

$\therefore \sigma(\mathcal{Q}_1, \mathcal{Q}_2, \dots), \Sigma$ are indep π-sys by Anova.

$\therefore \Sigma, \Sigma$ are indep π-sys by subclasses

$\therefore \forall A \in \Sigma, P(A \cap A) = P(A)P(A)$

$\therefore P(A) = 0$ or 1 .

QED

This implies (from the previous example) that either

$$P(\text{Abnormal numbers } n) = \begin{cases} 1 & \text{or} \\ 0 & \end{cases}$$

↑
we showed
if β zero-