Homework 3

Due Feburary 22, 2015

Exercise 1 (Metrizing $\xrightarrow{P}$). *Let $\mathfrak{R}$ be the space of real-valued random variables on $(\Omega, \mathcal{F}, P)$. Let $d\colon \mathfrak{R} \times \mathfrak{R} \to [0,1]$ be defined by $d(X,Y) := E(|X-Y| \wedge 1)$. Show the following*

- *d is a pseudo metric on $\mathfrak{R}$*
- *$X_n \xrightarrow{P} X$ if and only if $d(X_n, X) \to 0$*
- *$\mathfrak{R}$ is speparable if $\mathcal{F}$ is countably generated (Hint: use Theorem 58 (part 1), Theorem 27 and Exercise 9 from the class notes).*
- *$\mathfrak{R}$ is complete.*

Exercise 2. *Suppose $X_n \xrightarrow{ae} X$. Show that for every $\epsilon > 0$, $\lim_m P\big[\sup_{n \geq m} |X_n - X| > \epsilon\big] = 0$.*

Definition 1. *X_n is said to* **converge almost uniformly to** *X, written $X_n \xrightarrow{au} X$, if for every $\epsilon > 0$ there exists a measubale U_ϵ such that $P[U_\epsilon] \geq 1 - \epsilon$ and $X_n(\omega) \to X(\omega)$ uniformly for all $\omega \in U_\epsilon$.*

Exercise 3 (Egoroff's Theorem). *Show that $X_n \xrightarrow{ae} X$ if and only if $X_n \xrightarrow{au} X$. Hint: if $X_n \xrightarrow{ae} X$ then there exists a subsequence n_k such that $P(\sup_{n \geq n_k} |X_n - X| > 1/k) < 1/k^2$.*

Exercise 4. *Let $X_1, X_2, \ldots$ be an infinite sequence of square integrable random variables, all defined on the same probability space, such that $E(X_i) = 0$ and*

$$E(X_i X_j) = cov(X_i, X_j) = \begin{cases} a + b, & \text{if } i = j \\ b. & \text{if } i \neq j \end{cases}$$

for all i, j. Put $S_n := X_1 + X_2 + \cdots + X_n$.

1. *Show the above conditions on X_i are satisfies when $X_i = Y_i + Z$ where the Y_i's and Z are L_2 random variables with zero means, variances given by $E(Y_i^2) = a$ and $E(Z^2) = b$ and covariances $E(Y_i Y_j) = 0$ when $i \neq j$ and $E(Z Y_i) = 0$.*
2. *Show $\|S_n/n - S_m/m\|_2^2 = (1/m - 1/n)a$ when $m \leq n$. Hint: first show $\langle S_m/m, S_n/n \rangle = a/n + b$.*
3. *Show that as n tends to infinity, S_n/n converges in L_2 to some random variable, say Z. Find the mean and variance of Z.*
4. *Show that S_n/n converges to Z with probability one. Hint: use the method of subsequences. First put $A_k = S_{k^2}/k^2$, show that $\|A_k - Z\|_2^2 = a/k^2$ and deduce that A_k converges a.e. to Z as $k \to \infty$. Then show that the S_n/n's have the same limit a.e. as the A_k's.*
5. *Show that the X_i's necessarily have the representation specified in item 1 above.*