

Homework 7

Due Thursday, December 8, 2016

Exercise 1 (Sterling's formula for the Gamma function).

The Gamma function is defined by the equality $\Gamma(r+1) := \int_0^\infty y^r e^{-y} dy$ for $r \in (0, \infty)$.

1. Use the change of variable $z = (y-r)/\sqrt{r}$ to show that

$$\rho_r := \frac{\Gamma(r+1)}{r^r e^{-r} \sqrt{r}} = \int_{-\sqrt{r}}^\infty e^{-r\phi(z/\sqrt{r})} dz$$

where $\phi(u) := u - \log(1+u)$.

2. Show that $e^{-r\phi(z/\sqrt{r})} \rightarrow e^{-z^2/2}$ as $r \rightarrow \infty$ for each z .
3. Show that there exists positive constants a, b such that

$$\phi(u) \geq \min(au^2, b|u|)$$

for all $u \in (-1, \infty)$. Hint: Use Taylor's theorem with remainder to bound $\phi(u) \geq au^2$ for $-1 < u \leq 1$. Then use convexity to bound $\phi(u) \geq bu$ for $u \geq 1$.

4. Conclude that there exists an integrable g and a sufficiently large $r_0 > 0$ such that

$$\sup_{r > r_0} |e^{-r\phi(z/\sqrt{r})} I_{(-\sqrt{r}, \infty)}(z)| \leq g(z)$$

5. Finally use the DCT to deduce that

$$\lim_{r \rightarrow \infty} \frac{\Gamma(r+1)}{r^r e^{-r} \sqrt{r}} = \int_{-\infty}^\infty e^{-z^2/2} dz = \sqrt{2\pi}.$$

Exercise 2. Suppose X is a random variable with a t -distribution on $\nu > 0$ degrees of freedom. Then X has density f_ν (with respect to Lebesgue measure) given by

$$f_\nu(x) := \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)} \frac{1}{(1+x^2/\nu)^{(\nu+1)/2}}$$

Show that the law of the random variable X converges, in norm $\|\cdot\|_{TV}$, to a standard Gaussian distribution as $\nu \rightarrow \infty$.

Exercise 3 (Conditions for $d\mu/d\rho = 1/(d\rho/d\mu)$). Suppose that ρ is a measure with density δ with respect to μ . Show that:

1. μ has density $1/\delta$ with respect to ρ if and only if $0 < \delta < \infty$ μ -a.e.;
2. If μ is σ -finite and $\delta < \infty$ μ -a.e. then ρ is σ -finite.
3. If μ is σ -finite and μ has some density, say f , with respect to ρ then $f = 1/\delta$ ρ -a.e and μ -a.e..

Hint for item 1: first find $\int_\bullet 1/\delta d\rho$.