Table of Contents

- Thm: If $(\Omega, \mathcal{F}, \mu)$ is σ-finite then

$$\left.\begin{array}{l} A = \bigcup_{i \in \mathcal{I}} B_i \\ \mathcal{I} \text{ uncountable} \\ B_i \in \mathcal{F} \\ \mu(B_i) > 0 \end{array}\right\} \Longrightarrow A \notin \mathcal{F}$$

- Def: Borel and Lebesgue measurable sets
- Thm: Why we need σ-fields.

Lecture 6: Independent events

- Def: Independence for events & collections of events.
- Thm: Subclasses, Augmentation, simplified product
- Thm: π-generators
- Thm: ANOVA
- Thm: First Borel-Cantelli lemma (FBCL)

$$\sum_n P(A_n) < \infty \Rightarrow P(A_n \text{ i.o.n}) = 0$$

- Def: $\{A_n \text{ i.o.n}\}$ & $\{A_n \text{ a.a.n}\}$
- Thm: Second Borel-Cantelli lemma (SBCL)

$$\sum_n P(A_n) = \infty \underset{A_n\text{'s indep}}{\Longleftrightarrow} P(A_n \text{ i.o.n}) = 1$$

- Thm: Fatou

$$P(A_n \text{ a.a.n}) \le \liminf_n P(A_n) \le \limsup_n P(A_n) \le P(A_n \text{ i.o.n})$$

- Thm: Erdős-Renyi's extension of SBCL
- Thm: Kolmogorov's 0-1 law & tail σ-fields
- Application: Hewitt-Savage 0-1 law for coin flips

Lecture 7: Maximal inequalities & the law of the iterated log for coin flips.

- Thm: Symmetric Random Walk Max ineq.

$$P\left(\max_{k \le n} S_k \ge a\right) \le 2P(S_n \ge a)$$

- Thm: Kolmogorov's Max ineq.

$$P\left(\max_{k \le n} |S_k| \ge a\right) \le \frac{1}{a^2} \sum_{k=1}^{n} E(R_k^2)$$

- Application: Series with random signs
- Thm: Law of the iterated log for coin flips

Lecture 8: Measurable functions, Random variables and CDF's.

- Def: measurable functions
- Thm: Generators are enough
- Thm: composition of ⓜ is ⓜ
- Corollary: Just check the coordinates
- Thm: Cut & paste over countable & ⓜ pieces
- Thm: Just check Borel ⓜ on the range
- Conventions for ∞
- Thm: Closure
- Def: $\underbrace{\text{random variables}}_{\text{r.v.}}$ & extended r.v.s

- Def: Induced measure & the dist of a r.v.
- Thm: μf^{-1} is a measure
- Def: $X \stackrel{\mathcal{D}}{=} Y$
- Def: The c.d.f F_X for a r.v. X
- Thm: $X \stackrel{\mathcal{D}}{=} Y \iff F_X(t) = F_Y(t) \;\forall t \in \mathbb{R}$
- Thm: properties of c.d.f's
- Def: The inverse c.d.f.
- Lemma: switching formula
- Lemma: c.d.f sandwich
- Thm: c.d.f representation

Lecture 9: σ-fields generated by functions, the structure Thm & random variables

- Def: $\sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I} \rangle$ σ-field on the range of f_i
- Thm: $\sigma\langle f, \mathcal{F} \rangle = f^{-1}(\mathcal{F})$
- Thm: Generators are enough
- Def: The product σ-field $\bigotimes_{i \in \mathcal{I}} \mathcal{F}_i$
- Thm: Just check the coordinates
- Def: Simple function: ⓜ & finite range
- Thm: Simple functions are $\sum_{i=1}^{n} c_i I_{A_i}$ ⓜ

- Def: $\mathcal{M}_s(\Omega, \mathcal{F}) \equiv$ non-neg simple funs

  $\mathcal{M}(\Omega, \mathcal{F}) \equiv$ non-neg ⓜ $\mathcal{F}/\mathcal{B}(\bar{\mathbb{R}})$ funs
- Thm: Structure theorem
- Thm: What it means to be ⓜ w.r.t $\sigma\langle f_1, \ldots, f_n \rangle$:

  Y ⓜ $\sigma\langle X_1, X_2, \ldots \rangle \iff Y = g(X_1, X_2, \ldots)$ for g ⓜ $\mathcal{B}(\mathbb{R}^\infty)$
- Def: Independence for r.v.s
- Thm: ANOVA for r.v.s
- Thm: Existance of indep copies $X_1, X_2, \ldots$
- Thm: 0-1 law for r.v.s
- Def: Symmetric function $f: \mathbb{R}^\infty \to \mathbb{R}$
- Thm: Hewitt-Savage for r.v.s
- Example: Bond percolation
- Def: r.v.s being purely atomic, purely singular, purely abs cont, & of pure type.
- Thm: Jessen-Wintner law of pure types.

7

Lecture 10: Integration

- Def: $\int_\Omega f\,d\mu$
- Thm: Simple 3, Little 3, Big 3.
- Thm: a.e. useful facts
- Side facts for $\int_\Omega f\,d\mu$
- Def: $\mathcal{Q}^+(\Omega,\mathcal{F},\mu)$, $\mathcal{Q}^-(\Omega,\mathcal{F},\mu)$, $\mathcal{Q}(\Omega,\mathcal{F},\mu)$, $L_1(\Omega,\mathcal{F},\mu)$
- Lebesgue integration vrs Riemann integration
- Fat Cantor set
- Def: uniformly integrable (UI)
- Thm: Dilitation critearian for UI
- Thm: Fatou

  $f_n \geq 0$ a.e. $\Rightarrow \int_\Omega \liminf_n f_n\,d\mu \leq \liminf_n \int_\Omega f_n\,d\mu$
- Thm: DCT, BCT

  $$\left.\begin{matrix} f_n \xrightarrow{\mu\text{-a.e.}} f \\ \sup_n |f_n| \in L_1 \end{matrix}\right\} \Rightarrow \int_\Omega \lim_n f_n\,d\mu = \lim_n \int_\Omega f_n\,d\mu$$
- Def: UI: $\lim_{c\to\infty} \sup_n \int_{|f_n|\geq c} |f_n|\,d\mu = 0$
- Thm: Dilatation criterion for UI
- Thm: UI & UI converse
- Thm: $\frac{d}{dt}\int = \int \frac{d}{dt}$
- Thm: change of variables

8

Lecture 11: Expectation and densities

- Def: $E(X) = \int_\Omega X^+ dP - \int_\Omega X^- dP$
- Example: using probabilities for Graph Theory proofs.
- Thm: Jensens ineq: $\varphi(EX) \leq E\,\varphi(X)$ ← convex
- Thm: what to expect of the max:

  sub-gussian tails of the MGF $\Rightarrow E(\max_{1\leq i\leq n} X_i) \lesssim \sqrt{\log n}$
- Thm: $E(XY) = E(X)E(Y)$ if $X, Y \in L_1$ & X indep Y.
- Def: var, sd, cov
- Thm: Hölder
- Thm: $X \geq 0 \Rightarrow E(X) = \int_0^\infty P(X\geq t)\,dt$
- Thm: cdf's have at most countably many jumps
- Thm: Markov, Chebyshev, chernoff inequalities
- Thm: Weierstrauss Approx
- Thm: Hoeffding's ineq.
- Thm: SLLN for bdd r.v.s
- Thm: Paley-Zygmund lower bound
- Thm: Kolmogorov's max ineq:

  $P(\max_{1\leq i\leq n} |S_i| \geq \alpha) \leq \frac{var(S_n)}{\alpha^2}$
- Thm: Etemadi's max ineq:

  $P(\max_{1\leq i\leq n} |S_i| \geq \alpha) \leq 3 \max_{1\leq i\leq n} P(|S_i| \geq \alpha)$

(9)

- Def: $\int_\bullet f\,d\mu$
- Thm: $\int_\bullet f\,d\mu$ is σ-additive
- Def: δ is a density of ν w.r.t. μ
- Thm: If $f, g \in Q(\Omega, \mathcal{F}, \mu)$ then

$$\left.\begin{array}{l} f \in L_1 \text{ or } g \in L_1 \\ \text{or } \mu\ \sigma\text{-finite} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \int_\bullet f\,d\mu \le \int_\bullet g\,\mu \text{ on } \mathcal{F} \\ \quad \text{iff} \\ \quad f \le g \ \ \mu\text{-a.e.} \end{array}\right.$$

- Corollary: uniqness of densities
- Thm: slap in the densitity:

$$d\nu = \delta\,d\mu$$

- Thm: chain rule: $\frac{d\rho}{d\mu} = \frac{d\rho}{d\nu}\frac{d\nu}{d\mu}$
- Thm: Probabilists world view

$$\begin{array}{l} \mu \text{ non-trivial} \\ \sigma\text{-finite} \end{array} \Rightarrow \begin{array}{l} \exists\ 0 < \delta < \infty,\ P \text{ s.t.} \\ d\mu = \delta\,dP \end{array}$$

- Def: Joint Gaussian
- Thm: Glavenko-Cantelli
- Thm: Scheffe

(10)

Lecture 12: Generating functions and Moments

- Def: MGF $M_X(t) = E(e^{tX})$
  CGF $G_X(z) = E(e^{zX})$
  CF $\phi_X(t) = E(e^{itX})$
- Def: A°, $\bar{A}$, ∂A
- Def: $\mathcal{D}_X := \{u + iv \in \mathbb{C} : E(e^{uX}) < \infty\}$
- Thm: $\mathrm{Re}\,\mathcal{D}_X$ is an interval & $0 \in \mathcal{D}_X$.
- Thm: G_X is analytic over $\mathcal{D}_X^\circ$
- Thm: $\phi_X(t) = G_X(it)$
  $M_X(t) = G_X(t)$
- Thm: If $g(z)$ is analytic on $\mathcal{D}_X^\circ$ and

$$\left.\begin{array}{l} \phi_X(t) = g(it) \\ \quad \text{or} \\ M_X(t) = g(t) \text{ on} \\ (-\varepsilon, \varepsilon) \subset \mathrm{Re}\,\mathcal{D}_X^\circ \end{array}\right\} \Rightarrow \begin{array}{l} g(z) = G_X(z) \\ \text{on } \mathcal{D}_X^\circ. \end{array}$$

- Corollary: Moments from M_X
- Thm: Moments from ϕ_X

11

Lecture 13: Separating classes

- Example: Wiener Measure
- Def: Polish space (complete & separable metric space)
- Def: $C(\Omega)$, $C_b(\Omega)$, $C_c(\Omega)$, $Lip(\Omega)$, $C^k(\mathbb{R}^d)$.
- Def:
  - Γ is a separating class for $(\Omega, \mathcal{B}(\Omega))$
  - Γ separates $\mathcal{P} \equiv$ a collection of prob measures on $(\Omega, \mathcal{B}(\Omega))$
  - Γ separates a collection of r.v.s mapping into $(\Omega, \mathcal{B}(\Omega))$
- Facts about convolution
- Def: locally compact
- Thm: If $(\Omega, \mathcal{B}(\Omega))$ is polish then
  (i) $Lip(\Omega) \cap C_b(\Omega)$ is a separating class
  (ii) $C_c(\Omega)$ is separating class if Ω is locally compact.
  (iii) $C_c^\infty(\mathbb{R}^d)$ is a separating class
- Thm: if X & Y are random vectors in $\mathbb{R}^d$ then $\phi_X(k) = \phi_Y(k)$ $\forall k \in \mathbb{R}^d$ implies $X \overset{\mathcal{D}}{=} Y$.
- Corollary: $k \cdot X \overset{\mathcal{D}}{=} k \cdot Y$ $\forall k \in \mathbb{R}^d$
  $$\Updownarrow$$
  $$X \overset{\mathcal{D}}{=} Y$$
- Corollary: $M_X(t)$ characterizes the dist of X if $M_X(t)$ is finite on a non-empty open interval.
- Thm: Moments characterize prob distributions for bdd r.v.s

12

- Def: finite coordinate projections $\pi_{t_1,\dots,t_n}(f) = (f_{t_1}, \dots, f_{t_n})$
- Thm: $\sigma\langle \pi_{t_1,\dots,t_n} : t_1,\dots,t_n \in \mathbb{R} \rangle = \mathcal{B}(C[0,1])$.
- Def: The finite dimensional distributions of a stochastic process.
- Thm: The finite dimensional distributions of a stochastic process on $C[0,1]$ characterize the measure.

Lecture 14: Convergence in distribution

- Def: $P_n \xrightarrow{\mathcal{D}} P$ and $X_n \xrightarrow{\mathcal{D}} X$
- Thm: Portmanteau I
  $P_n \xrightarrow{\mathcal{D}} P \iff \int_\Omega f\,dP_n \to \int_\Omega f\,dP$ $\forall f \in Lip(\Omega) \cap C_b(\Omega)$
  $\iff \dots$
- Def: $C_F := \{x \in \mathbb{R} : F \text{ is continuous at } x\}$
- Thm: Portmanteau II
  $X_n \xrightarrow{\mathcal{D}} X \iff F_n(x) \to F(x)$ $\forall x \in C_F$
  $\iff \dots$
- Thm: Uniqness of $\xrightarrow{\mathcal{D}}$ limits.
- Thm: sub-sub-seq check for $\xrightarrow{\mathcal{D}}$
- Thm: Skorokhod Representation thm
  $$X_n \xrightarrow{\mathcal{D}} X \iff \begin{array}{ccc} X_n^* & \xrightarrow{a.e.} & X^* \\ \shortparallel \mathcal{D} & & \shortparallel \mathcal{D} \\ X_n & & X \end{array}$$
- Thm: Continuous mapping for $\xrightarrow{\mathcal{D}}$
- Thm: UI extension for $\xrightarrow{\mathcal{D}}$

13

- Thm: The delta method
- Def: $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{a.e.} X$
- Thm: Slutzky:

$$X_n \xrightarrow{\mathcal{D}} X \;\&\; d(X_n, Y_n) \xrightarrow{P} 0 \Rightarrow Y_n \xrightarrow{\mathcal{D}} Y$$

- Corollary:

$$X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X$$

- Corollary: $X_n \xrightarrow{\mathcal{D}} c \iff X_n \xrightarrow{P} c$ constant
- Corollary:

$$\left.\begin{array}{l} X_n \xrightarrow{\mathcal{D}} X \\ Y_n \xrightarrow{P} c \end{array}\right\} \Rightarrow (X_n, Y_n) \xrightarrow{\mathcal{D}} (X, c)$$

- Def: $\mathcal{P}$ is tight iff $\sup_{P \in \mathcal{P}} P(K^c) < \varepsilon$ for some compact K.
- Def: $|X_n| = O_p(1)$ means $\{X_n\}_{n \geq 1}$ is tight
- Thm: Prohorov: For polish spaces

$\mathcal{P}$ is tight $\iff$ $\forall P_1, \ldots \in \mathcal{P}$ $\exists n_k$ s.t. $P_{n_k} \xrightarrow{\mathcal{D}} P$ (might not live in $\mathcal{P}$.)

- Def: Relatively compact w.r.t $\xrightarrow{\mathcal{D}}$
- Thm: Portmanteau III:

$X_n \xrightarrow{\mathcal{D}} X \iff \{X_n\}_{n\geq}$ is tight & $\pi_{t_1, \ldots, t_n}(X_n) \xrightarrow{\mathcal{D}} \pi_{t_1, \ldots, t_n}(X)$

for X_n taking values in $C([0,1])$.

14

- Thm: Portmanteau IV:

$X_n \xrightarrow{\mathcal{D}} X \iff$ $\{X_n\}_{n \geq 1}$ is tight & $Eg(X_n) \to Eg(X)$ $\forall g \in \Gamma \subset C_b(\Omega)$ which separates

X_n r.v.s taking values in polish Ω.

- Thm: Portmanteau V:

Different Γ which satisfy

$X_n \xrightarrow{\mathcal{D}} X \iff Ef(X_n) \to Ef(X)$ $\forall f \in \Gamma$

- Corollary: characteristic functions for $\xrightarrow{\mathcal{D}}$
- Corollary: Cramér-Wald device.

Lecture 15 : The Central Limit Theorem

- Example: Lindeberg's Method
- Thm: The Central Limit Theorem
- Corollary:

$$\left.\begin{array}{l} Z \overset{\mathcal{D}}{=} \frac{Z_1}{\sqrt{n}} + \cdots + \frac{Z_n}{\sqrt{n}} \\ \text{where } E(Z^2) < \infty \;\& \\ Z_1, \ldots, Z_n \overset{iid}{\sim} Z \end{array}\right\} \Rightarrow Z \sim N(0, \sigma^2)$$

- Thm: CLT for random vectors
- Thm: Lindeberg - Feller Sufficiency
- Thm: Lindeberg - Feller Necessity
- Lemma: Cauchy criterion for $\xrightarrow{a.e.}$

- Thm: $\sum_{p=1}^{\infty} EX_p^2 < \infty \Rightarrow P(\sum_{p=1}^{\infty} X_p < \infty) = 1$
  indep $EX_n = 0$
- Thm: Kolmogorov's Three series Thm.
- Application to series with random sign
- Thm: Law of rare events
- Thm: Rates for law of Rare Events
- Def: A coupling of P & Q
- Lemma: TV & coupling

Lecture 16: Brownian Motion & Brownian Bridge

- Thm: f.d.d. of scaled random walk converges.
- Thm: Brownian motion on $[0,1]$ exists & is unique.
- Example: $\frac{1}{\sqrt{n}} \max_{1 \le k \le n} S_k \xrightarrow{\mathcal{D}} |Z|$
- Extending BM to $[0,\infty)$
- Thm: independent increments
- Thm: non-differentiability
- Brownian Bridge.

Lecture 17: Convergence a.e. & in Prob

- Thm: Uniqueness of $\xrightarrow{a.e.}$ & $\xrightarrow{P}$ limits
- Thm: i.o. characterization of $\xrightarrow{a.e.}$
- Corollary: $X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X$
- Thm: $X_n(\omega)$ monotonic in n implies
  $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{a.e.} X$
- Thm: Cauchy criterion for $\xrightarrow{a.e.}$
- Thm: sub-sub-seq for $\xrightarrow{P}$
- Thm: continuous mapping thm
- Thm: $\xrightarrow{P}$ sandwich
- Def: $X_n = o_p(Y_n) \Leftrightarrow \frac{X_n}{Y_n} \xrightarrow{P} 0$
  $X_n = O_p(X_n) \Leftrightarrow \{\frac{X_n}{Y_n}\}$ is tight
- Thm: SLLN for bdd r.v.
- Thm: SLLN for L_2
- Thm: Kolomogorov's SLLN

Lecture 18: L_p spaces

17

- Def: $L_p(\Omega, \mathcal{F}, P) \equiv L_p(P) \equiv L_p$

  $\|X\|_p = (E|X|^p)^{1/p}$

  $d_p(X,Y) = \|X-Y\|_p$

- Thm: Hölder
- Thm: $1 \le p < q \Rightarrow L_q \subset L_p$
- Thm: $\|\cdot\|_p$ is a pseudo-norm
- Thm: $\|\cdot\|_p$ is continuous

  $X,Y \in L_p \Rightarrow \big| \|X\|_p - \|Y\|_p \big| \le d_p(X,Y)$

- Def: $X_n \xrightarrow{L_p} X$
- Thm: uniqness of $\xrightarrow{L_p}$ limits
- Thm: Cauchy criteria
- Thm: L_p is Polish w.r.t d_p
- Recall UI & UI convers
- Thm: L_p convergence thm:

  if $X_n \in L_p$ $\forall p$ then

  $$X_n \xrightarrow{L_p} X$$
  $$\Updownarrow$$
  $$X_n \xrightarrow{P} X \ \& \ E|X_n|^p \to E|X|^p < \infty$$
  $$\Updownarrow$$
  $$X_n \xrightarrow{P} X \ \& \ |X_n|^p\text{'s are UI}$$

- Def: $\langle X,Y \rangle := E(XY)$
- Basic properties of $\langle \cdot, \cdot \rangle$
- Def: $X \perp Y$

18

- Thm: Projection in L_2
- Thm: Projection in coordinates
- Def: An orthonormal basis (ONB)
- Thm: Characterizing a ONB
- Thm: Permuting indicies in L_2 sums
- Projection application to Gaussian random fields
- Def: continuous linear functional on L_2
- Thm: Riesz for L_2

Lecture 19: Radon-Nikodym Derivatives and Lebesgue decomposition

- Def: For measures μ, ν on $(\Omega, \mathcal{A})$

  $\nu \perp \mu$ & $\nu \ll \mu$

- Thm: Radon-Nikodym

  If $\nu \ll \mu$, both σ-finite, then $\frac{d\nu}{d\mu}$ exists and μ-unique.

- Thm: Improved Radon-Nikodym

  If $\nu \ll \mu$, μ σ-finite, then $\frac{d\nu}{d\mu}$ exists and μ-unique.

- Def: $\nu \lll \mu$ means $\nu \ll \mu$ & μ is σ-finite
- Thm: Properties of Radon-Nikodym derivatives.

(19)

- Thm: Lebesgue Decomposition

$$Q = \underbrace{Q_{<<}}_{Q_{<<} << P} + \underbrace{Q_{\perp}}_{Q_{\perp} \perp P}$$

Lecture 20: Conditional expected value w.r.t. a sub σ-field

- Thm: Existance of $E(X|\mathcal{Q})$ satisfying: $E(X|\mathcal{Q}) \in Q(\Omega, \mathcal{Q}, P)$
  $$\int_A X dP = \int_A E(X|\mathcal{Q}) dP \quad \forall A \in \mathcal{Q}$$
- Thm: Smoothing Properties of $E(X|\mathcal{Q})$
- Thm: Expected value properties of $E(X|\mathcal{Q})$
- Lemma: $E(I_A X|\mathcal{Q}) \overset{P\text{-a.e.}}{=} I_A E(X|\mathcal{Q})$
- Thm: π-generators are enough
- Def: $E(X|Y_1, Y_2, \dots) = E(X|\sigma\langle Y_1, Y_2, \dots\rangle)$
- Corollary: $E(X|Y_1 \dots) = g(Y_1, \dots)$

(20)

Lecture 21: Martingales

- Def: A filtration $\{\mathcal{F}_n\}_{n\geq 1}$
- Def: $\{X_n\}_{n\geq 1}$ adapted to a filtration
- Def: martingale, submartingale, supermartingale
- Def: The natural filtration
- Thm: X_n a subM w.r.t. $\mathcal{F}_n$
  $$\Updownarrow$$
  $$E(X_m|\mathcal{F}_n) \overset{a.e.}{\geq} X_n \quad \forall m > n$$
- Corollary: X_n a subM $\Rightarrow E(X_m) \geq E(X_n)$ $\forall m > n$
- Polya's urn
- Thm: transformation of Martingales
- Example: strategy against a bad game
- Example: Smoothing martingale
- Example: Martingale difference seq.
- Example: Lebesgue decomp supM.
- Def: stopping time $\{\tau = n\} \in \mathcal{F}_n$ $\forall n \in \mathbb{N}$
- Thm: The X_n values determine when to stop under the natural filtration.

21

- Thm: $\sigma \wedge \tau$ is a ST
- Def: The stopped σ-field $\mathcal{F}_\tau$
- Thm: equiv def of $\mathcal{F}_\tau$
- Thm: X_τ is (m) $\mathcal{F}_\tau$ & $\mathcal{F}_\tau$ is a σ-field.
- Thm: $\sigma \le \tau \Rightarrow \mathcal{F}_\sigma \subset \mathcal{F}_\tau$

Lecture 22: Martingale Convergence

- Def: upcrossing
- Thm: Doob's upcrossing ineq.

$$E(\underbrace{U_{0c}}_{\substack{\text{upcrossings} \\ \text{of } [0,c]}}) \le \frac{E(X_n) - E(X_1)}{c}, \quad X_n \text{ subM}$$

- Corollary:

$$E(U_{ab}) \le \frac{E(X_n-a)^+ - E(X_1-a)^+}{b-a} \le \frac{E(X_n^+) + a^-}{b-a}$$

- Thm: $E(X_n^+)$ bdd $\Rightarrow$ a.e. conv for subM
- Thm: X_n^+ UI $\Rightarrow E(X_n^+)$ bdd
- Thm: Lévy's smoothing Martingale
- Def: • closer on the right
  • nearest closer

22

- Thm: $\exists$ closer $\Leftrightarrow$ X_n^+'s are UI, for subM
- Thm: $\exists$ closer $\Rightarrow$ a.e. convergence, for subM
- Thm: $\frac{dQ_n^a}{dP} \xrightarrow{P\text{-a.e.}} \frac{dQ_\infty^a}{dP_\infty}$ as $n \to \infty$
- Thm: subM L_p conv Thm for $1 \le p < \infty$
- Thm: checking $|X_n|^p$ UI for $X_n \ge 0$ subM.