Homework 6

Due March 22, 2017

Let $X_1, X_2, \dots$ be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and let Q be another probabilty measure on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_n := \sigma\langle X_1, \dots, X_n \rangle$, $\mathcal{F}_\infty := \sigma\langle X_n \colon n \in \mathbb{N}\rangle$ and

$$P_n = P\big|_{\mathcal{F}_n} \text{ and } Q_n = Q\big|_{\mathcal{F}_n}$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Let $Q_n = Q_n^a + Q_n^s$ be the Lebesque decomposition of Q_n with respect to P_n and

$$\ell_n := \frac{dQ_n^a}{dP_n} \text{ for all } n \in \mathbb{N} \cup \{\infty\}.$$

Exercise 1. *Show that*

$$\begin{aligned} Q_\infty \ll P_\infty &\iff E(\ell_\infty) = 1; \\ Q_\infty \perp P_\infty &\iff E(\ell_\infty) = 0 \end{aligned}$$

Exercise 2. *Show that the process* $(\sqrt{\ell_n})_{n\in\mathbb{N}}$ *is a super-martingale, is UI and*

$$E(\sqrt{\ell_n}) \downarrow E(\sqrt{\ell_\infty})$$

as $n \to \infty$.

Exercise 3. *Show that*

$$\begin{aligned} Q_\infty \perp P_\infty &\iff \lim_n E(\sqrt{\ell_n}) = 0 \\ &\iff \lim_n H(Q_n, P_n) = \sqrt{2} \end{aligned}$$

where $H(\cdot,\cdot)$ *is the Hellinger distance on probability measures defined as*

$$H(Q_n, P_n) = \left[\int_\Omega (\sqrt{q_n} - \sqrt{p_n}\,)^2 d\mu\right]^{1/2}$$

where $q_n = \frac{dQ_n}{d\mu}$, $p_n = \frac{dP_n}{d\mu}$ *and* μ *is any measure such that* $Q_n, P_n \ll \mu$ *(*H *is invariant to which* μ *you choose).*
Hint: For the second iff show that $\int_\Omega \sqrt{q_n}\sqrt{p_n} d\mu = \int_\Omega \sqrt{\ell_n} dP_n$.

Exercise 4. *Show that if* $Q_\infty \ll P_\infty$ *then* $E(\ell_\infty | \mathcal{F}_n) \overset{a.e.}{=} \ell_n$.

Exercise 5. *Show that*

$$\begin{aligned} &Q_\infty \ll P_\infty \\ &\quad \iff Q_n \ll P_n \text{ for all } n \in \mathbb{N} \text{ and the } \ell_n\text{'s are UI} \\ &\quad \iff Q_n \ll P_n \text{ for all } n \in \mathbb{N} \text{ and the } \sqrt{\ell_n}\text{'s converge in } L_2 \end{aligned}$$