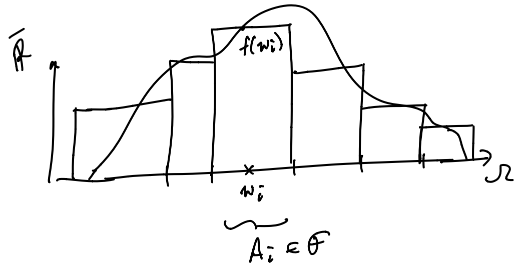
Lecture 10: Integration and expected value

In this lecture we will define

$$\int_\Omega f(\omega)\,d\mu(\omega)$$

where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f:\Omega \to \overline{\mathbb{R}}$ s.t. f (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

The notation $\int_\Omega f(\omega)\,d\mu(\omega)$ is extremely suggestive of Riemann integration

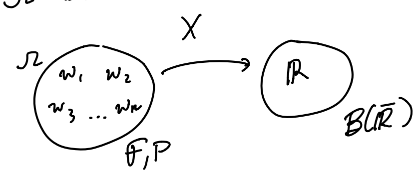


where one might guess

$$\int_\Omega f(\omega)\,d\mu(\omega) \approx \underbrace{\sum_i f(\omega_i)\,\mu(A_i)}_{}$$

area of block i when width of A_i is measured with μ

To see the connection with expected value suppose Ω has n members:



In this case we would want the definition of "expected value of X", denoted $E(X)$, to, at the very least, satisfy:

$$E(X) = \left\{ \begin{array}{l} \text{the weighted average of the} \\ \text{numbers } \{X(w_1), X(w_2), \dots, X(w_n)\} \\ \text{with weights } P(\{w_i\}). \end{array} \right.$$

$$= \sum_{i=1}^n X(w_i)\, P(\underbrace{\{w_i\}}_{\text{partition}}) \quad \text{(measure)}$$

$$= \int_\Omega X(w)\,dP(w).$$

Assumption: For the rest of this lecture suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space

Game plan:

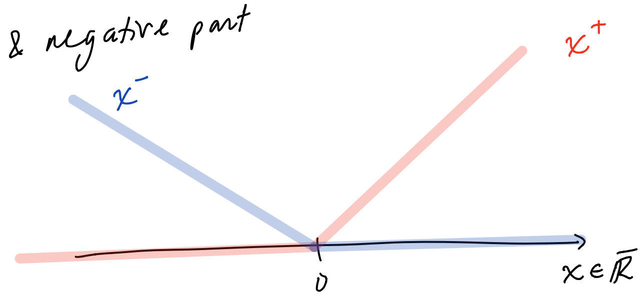
Step 1: Define $\int_\Omega f\,d\mu$ for $f \in \mathcal{M}_s(\Omega, \mathcal{F})$ ← Non-negative simple functions.

Step 2: extend to $f \in \mathcal{M}(\Omega, \mathcal{F})$ ← Non-negative measurable functions.

Step 3: extend to some, but not all, f (m) $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by

$$\int_\Omega f\,d\mu := \int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu$$

where $(\cdot)^+, (\cdot)^-$ denotes the positive part & negative part



so that $|x| = x^+ + x^-$.

Remark: Although this construction seems tedious & annoying, the method of construction is general & broadly applicable. For example, the same game plan is use for defining Itô's integral

$$\int_0^t \underbrace{X(s)}_{}\, d\underbrace{B(s)}_{}$$

Brownian Motion

A stochastic process.

Step 1

Def: If $f \in \mathcal{M}_s(\Omega, \mathcal{F})$ has the form

$f = \sum_{i=1}^n c_i I_{A_i}$ then $\int_\Omega f(w)\,d\mu(w) := \sum_{i=1}^n c_i\, \mu(A_i)$

$c_i \in [0, \infty]$, $A_i \in \mathcal{F}$

Note: use $\int_\Omega f\,d\mu$ as shorthand for $\int_\Omega f(w)\,d\mu(w)$

Here are the four basic properties of $\int_\Omega f\,d\mu$ we will show at each step in the game plan:

Thm (simple 3)

(0) $\int_\Omega f\,d\mu$ is well defined over $\mathcal{M}_s(\Omega,\mathcal{F})$

(1) Monotonicity:

If $f,g\in\mathcal{M}_s(\Omega,\mathcal{F})$ & $f(\omega)\le g(\omega)\ \forall\omega\in\Omega$ then
$$\int_\Omega f\,d\mu \le \int_\Omega g\,d\mu.$$

(2) linearity:

If $f,g\in\mathcal{M}_s(\Omega,\mathcal{F})$ & $\alpha,\beta\in[0,\infty]$ then
$$\int_\Omega \alpha f+\beta g\,d\mu = \alpha\int_\Omega f\,d\mu+\beta\int_\Omega g\,d\mu$$

(3) Continuity from below (CFB):

If $f_n(\omega)\uparrow f(\omega)$ as $n\to\infty$ for all $\omega\in\Omega$ where $f_n,f\in\mathcal{M}_s(\Omega,\mathcal{F})$ then
$$\int_\Omega f_n\,d\mu \uparrow \int_\Omega f\,d\mu.$$

Proof:

Suppose $f=\sum_{i=1}^n c_i I_{A_i}$, $g=\sum_{k=1}^m d_k I_{B_k}$ both in $\mathcal{M}_s(\Omega,\mathcal{F})$

$\mathcal{F}$-sets which partition Ω

$\therefore\ f=\sum_{ik} c_{ik} I_{A_i\cap B_k}$ where $c_{ik}:=c_i$

$g=\sum_{ik} d_{ik} I_{A_i\cap B_k}$ where $d_{ik}:=d_k$

a finer partition of Ω.

To show (0) & (1) it is sufficient to show
$$f\le g \Rightarrow \sum_{i=1}^n c_i\mu(A_i)\le\sum_{k=1}^m d_k\mu(B_k)$$

$$f\le g \Rightarrow \sum_{ik} c_{ik} I_{A_i\cap B_k}\le \sum_{ik} d_{ik} I_{A_i\cap B_k}$$

exactly one term is non-zero (assuming $A_i\ne\emptyset$ & $B_k\ne\emptyset$).

$\Rightarrow\ c_{ik}\le d_{ik}\quad\forall i,k$

$$\Rightarrow \underbrace{\sum_{ik} c_{ik}\mu(A_i\cap B_k)}_{=\sum_i c_i\mu(A_i)} \le \underbrace{\sum_{ik} d_{ik}\mu(A_i\cap B_k)}_{=\sum_k d_k\mu(B_k)}$$

by additivity of μ.

For (2)

$$\int_\Omega \alpha f+\beta g\,d\mu = \int_\Omega \sum_{ik}(\alpha c_{ik}+\beta d_{ik})I_{A_i\cap B_k}\,d\mu$$
$$=\sum_{ik}(\alpha c_{ik}+\beta d_{ik})\mu(A_i\cap B_k)$$

$\vdots$ use additivity of μ & linearity of $\sum_{ik}$

$$=\alpha\int_\Omega f\,d\mu+\beta\int_\Omega g\,d\mu$$

For (3)

Suppose $\underbrace{f_n\uparrow f}_{\text{all in }\mathcal{M}_s(\Omega,\mathcal{F})}$.

Notice $\int_\Omega f_n\,d\mu\uparrow$ by (1) so just show
$$\lim_n\int_\Omega f_n\,d\mu=\int_\Omega f\,d\mu.$$

Case 1: $f=cI_A$ for $c\in(0,\infty]$.

Let $0<b<c$ so that
$$bI_{\{f_n\ge b\}}\le f_n\le f=cI_A.$$

Now integrate each term & use (1) to get
$$b\mu(f_n\ge b)\le\int_\Omega f_n\,d\mu\le\int_\Omega f\,d\mu=c\mu(A)$$

$\therefore$ $$\underbrace{b \lim_n \mu(f_n \geq b)}_{=\mu(A)} \leq \lim_n \int_\Omega f_n d\mu \leq \int_\Omega f\, d\mu = c\mu(A)$$ 5

$=\mu(A)$ by CFB since $0<b<c$ & $f_n \uparrow f$ implies $\{f_n \geq b\} \uparrow \{f=c\}=A$ as $n \to \infty$

Now take the limit as $b \uparrow c$ to get

$$\underbrace{c\mu(A) \leq \lim_n \int_\Omega f_n d\mu \leq \int_\Omega f\, d\mu}_{\therefore \text{ these are all equal.}} = c\mu(A)$$

Case 2: $f=\sum_{i=1}^{n} c_i I_{A_i}$, where A_i's partition Ω.

Fix $k \in \{1,2,\dots,m\}$

$\therefore$ Now $f_n I_{A_k} \uparrow f I_{A_k}$ so that Case 1 applies (since $f I_{A_k} = c_k I_{A_k}$) to give

$$\int_\Omega f_n I_{A_k} d\mu \uparrow \int_\Omega f I_{A_k} d\mu$$

Now sum over $k=1,\dots,m$ to get.

$$\int_\Omega f_n \underbrace{\sum_{k=1}^{m} I_{A_k}}_{=1} d\mu \uparrow \int_\Omega f \underbrace{\sum_{k=1}^{m} I_{A_k}}_{=1} d\mu$$

QED

Step 2

Recall the structure theorem:

If $f \in \mathcal{M}(\Omega,\mathcal{F})$ then $\exists f_n \in \mathcal{M}_s(\Omega,\mathcal{F})$ s.t.

$$f_n \uparrow f$$

Def: if $f \in \mathcal{M}(\Omega,\mathcal{F})$ define

$$\int_\Omega f\, d\mu := \lim_n \uparrow \int_\Omega f_n\, d\mu$$

$f_n \in \mathcal{M}_s$ s.t. $f_n \uparrow f$.

6

Thm (little 3)

Statements (0)–(3) in "simple 3" hold when $\mathcal{M}_s(\Omega,\mathcal{F})$ is replaced with $\mathcal{M}(\Omega,\mathcal{F})$.

Proof:

To show (0) & (1), i.e. $\int_\Omega f d\mu$ is well defined & monotonic, start by assuming

$$\underbrace{f \leq g}_{\text{both in } \mathcal{M}(\Omega,\mathcal{F})}$$

$\therefore$ $\exists f_n, g_n \in \mathcal{M}_s(\Omega,\mathcal{F})$ s.t.

$$\lim_n \uparrow f_n = f \leq g = \lim_n \uparrow g_n$$

Notice the following "trick"

$$\begin{aligned}\lim_m \uparrow f_n \wedge g_m &= f_n \wedge (\lim_m \uparrow g_m) \\ &= f_n \wedge g \\ &= f_n \quad \text{since } f_n \leq f \leq g\end{aligned}$$

$$\begin{aligned}\therefore \int f_n d\mu &= \int_\Omega \lim_m \uparrow f_n \wedge g_m\, d\mu \\ &= \lim_m \uparrow \int_\Omega f_n \wedge g_m\, d\mu \quad \text{by "simple 3"} \\ &\leq \lim_m \uparrow \int_\Omega g_m\, d\mu\end{aligned}$$

Now take a limit as $n \to \infty$ to get

$$\lim_n \uparrow \int_\Omega f_n d\mu \leq \lim_m \uparrow \int_\Omega g_m d\mu.$$

This shows (0) & (1).

The proof of (2), i.e. that

$$\int_\Omega \alpha f + \beta g\, d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g\, d\mu$$

when $\alpha, \beta \in [0,\infty]$ is easy (using the fact that $\alpha f + \beta g = \lim_n \uparrow (\alpha f_n + \beta g_n)$).

For (3):

Show $\underbrace{f_n \uparrow f}_{\text{all in } \mathcal{M}(\Omega,\mathcal{F})} \implies \int_\Omega f_n \, d\mu \uparrow \int_\Omega f \, d\mu$

Suppose $f_n \uparrow f$ & let $f_n = \lim_m \uparrow \underbrace{\phi_{nm}}_{\in \mathcal{M}_s(\Omega,\mathcal{F})}$ so that

$$\boxed{\begin{array}{ccccccc} \phi_{11} & \le & \phi_{12} & \le & \cdots & \le & \phi_{1n} \\ \vdots & & & & & & \vdots \\ \phi_{k1} & \le & \phi_{k2} & \le & \cdots & \le & \phi_{kn} \\ \vdots & & & & & & \vdots \\ \phi_{n1} & \le & \phi_{n2} & \le & \cdots & \le & \phi_{nn} \end{array}} \begin{array}{ccc} \le & \longrightarrow & f_1 \\ & & \rotatebox{90}{$\le$} \ \vdots \\ \le & \longrightarrow & f_k \\ & & \rotatebox{90}{$\le$} \ \vdots \\ \le & \longrightarrow & f_n \end{array}$$

define $\phi_n := \max_{1 \le i,j \le n} \phi_{ij} \in \mathcal{M}_s(\Omega,\mathcal{F})$

Now $\phi_{kn} \le \phi_n \le f_n \le f, \quad \forall k \le n \quad (\ast)$

Taking limits as $n \to \infty$ in $(\ast)$ gives

$$f_k = \lim_n \uparrow \phi_{kn} \le \lim_n \uparrow \phi_n \le \lim_n \uparrow f_n \le f$$

Taking limits as $k \to \infty$

$$f = \lim_k \uparrow \lim_n \uparrow \phi_{kn} = \lim_n \uparrow \phi_n = \lim_n \uparrow f_n = f$$

$\in \mathcal{M}_s(\Omega,\mathcal{F})$

$\therefore$ $f = \lim_n \uparrow \phi_n$ where $\phi_n \in \mathcal{M}_s(\Omega,\mathcal{F})$ so

$$\int f \, d\mu := \lim_n \uparrow \int_\Omega \phi_n \, d\mu \quad \text{by def.}$$

Now just show $\lim_n \int_\Omega \phi_n \, d\mu = \lim_n \int_\Omega f_n \, d\mu$

Instead of taking limits in $(\ast)$ first, integrate to get

$$\int_\Omega \phi_{kn} \, d\mu \le \int_\Omega \phi_n \, d\mu \le \int_\Omega f_n \, d\mu, \quad \forall k \le n$$

Now let $n \to \infty$ for

$$\int_\Omega f_k \, d\mu \le \lim_n \int_\Omega \phi_n \, d\mu \le \lim_n \int f_n \, d\mu$$

where $\int_\Omega f_k \, d\mu = \lim_n \uparrow \int_\Omega \phi_{kn} \, d\mu$ by def.

Finally let $k \to \infty$ to give

$$\lim_k \int_\Omega f_k \, d\mu = \lim_n \int_\Omega f_n \, d\mu.$$

QED

Before we move to step 3 we need some useful facts.

Def: $f = g$ μ-a.e. means $\mu(\{f \ne g\}) = 0$

Thm (a.e. useful facts)

(i) $f \in \mathcal{M}(\Omega,\mathcal{F})$ & $\int_\Omega f \, d\mu < \infty \implies f < \infty$ μ-a.e.

(ii) If $f \in \mathcal{M}(\Omega,\mathcal{F})$ then

$$\int_\Omega f \, d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}$$

(iii) If $f, g \in \mathcal{M}(\Omega,\mathcal{F})$ and $f = g$ μ-a.e. then $\int_\Omega f \, d\mu = \int_\Omega g \, d\mu$.

which implies I can change f on μ-null sets without changing $\int_\Omega f \, d\mu$.

Proof:

For (i) notice that $f\in\mathcal{M}(\Omega,\mathcal{F})$ implies

$$\infty I_{\{f=\infty\}} \le f$$

using our convention that $\infty\cdot 0=0$

$$\therefore \int_\Omega f\,d\mu<\infty \overset{\text{little 3}}{\Longrightarrow} \infty\,\mu(f=\infty)\le\int_\Omega f\,d\mu<\infty$$

$$\Longrightarrow \underbrace{\mu(f=\infty)=0}_{\text{i.e. } f<\infty\ \mu\text{-a.e.}}$$

For (ii) suppose $f\in\mathcal{M}(\Omega,\mathcal{F})$.

$$\int f\,d\mu=0 \iff \int f I_{\{f\ge\frac{1}{n}\}}\,d\mu=0,\ \forall n$$

$\left\{\begin{array}{l}\text{The direction} \Leftarrow \text{follows since}\\ \{f\ge\frac{1}{n}\}\uparrow\{f>0\}\\ \therefore f I_{\{f\ge\frac{1}{n}\}}\uparrow f I_{\{f>0\}}=f\\ \therefore \int_\Omega f I_{\{f\ge\frac{1}{n}\}}\,d\mu\uparrow\int_\Omega f\,d\mu\end{array}\right.$

$$\iff \mu(f\ge\tfrac{1}{n})=0\quad\forall n$$

$\left\{\begin{array}{l}\text{since } \frac{1}{n}I_{\{f\ge\frac{1}{n}\}}\le f I_{\{f\ge\frac{1}{n}\}}\le\infty I_{\{f\ge\frac{1}{n}\}}\\ \therefore \frac{1}{n}\mu(f\ge\frac{1}{n})\le\int_\Omega f I_{\{f\ge\frac{1}{n}\}}\,d\mu\le\infty\,\mu(f\ge\frac{1}{n})\end{array}\right.$

$$\iff \mu(f>0)=0$$

$\left\{\begin{array}{l}\text{since } \mu(f\ge\frac{1}{n})\uparrow\mu(f>0)\\ \text{by CFB}\end{array}\right.$

$$\iff f=0\ \mu\text{-a.e.}$$

For (iii) suppose $f,g\in\mathcal{M}(\Omega,\mathcal{F})$ & $f=g$ μ-a.e.

$$\therefore \int_\Omega f\,d\mu \overset{\ell 3}{=} \int_\Omega f I_{\{f=g\}}\,d\mu+\underbrace{\int_\Omega f I_{\{f\ne g\}}\,d\mu}_{=0\text{ by (ii)}}$$

$$=\int_\Omega g I_{\{f=g\}}\,d\mu+\underbrace{\int_\Omega g I_{\{f\ne g\}}\,d\mu}_{=0\text{ by (ii)}}$$

$$=\int_\Omega g\,d\mu.$$

QED.

Step 3

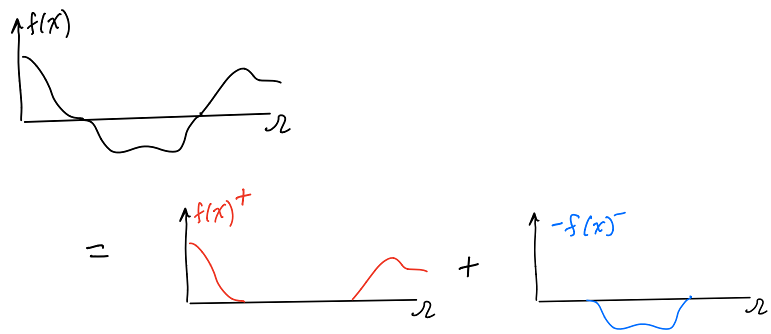
For $x\in\overline{\mathbb{R}}$ let

$$x^+:=\begin{cases}x & \text{if } x\ge 0\\ 0 & \text{o.w.}\end{cases}\quad \&\quad x^-=\begin{cases}0 & \text{if } x\ge 0\\ |x| & \text{if } x<0.\end{cases}$$

Now for any $f:\Omega\to\overline{\mathbb{R}}$ s.t. f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ we have

- $f^+, f^-\in\mathcal{M}(\Omega,\mathcal{F})$ by composition of ⓜ is ⓜ
- $f=f^+-f^-$
- $|f|=f^++f^-$.

Picture:



Def: If $f:\Omega\to\overline{\mathbb{R}}$ s.t. f ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ and either $\int_\Omega f^+\,d\mu<\infty$ or $\int_\Omega f^-\,d\mu<\infty$ then define

$$\int_\Omega f\,d\mu:=\int_\Omega f^+\,d\mu-\int_\Omega f^-\,d\mu.$$

Notation:

$$\mathcal{Q}^+(\Omega,\mathcal{F},\mu):=\left\{f:\Omega\to\overline{\mathbb{R}}\ \middle|\ f\text{ ⓜ }\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})\ \&\ \int_\Omega f^+\,d\mu<\infty\right\}$$

$$\mathcal{Q}^-(\Omega,\mathcal{F},\mu):=\left\{f:\Omega\to\overline{\mathbb{R}}\ \middle|\ f\text{ ⓜ }\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})\ \&\ \int_\Omega f^-\,d\mu<\infty\right\}$$

$$\mathcal{Q}(\Omega,\mathcal{F},\mu):=\mathcal{Q}^+(\Omega,\mathcal{F},\mu)\cup\mathcal{Q}^-(\Omega,\mathcal{F},\mu)$$

$$L_1(\Omega,\mathcal{F},\mu):=\mathcal{Q}^+(\Omega,\mathcal{F},\mu)\cap\mathcal{Q}^-(\Omega,\mathcal{F},\mu)$$

$Q^+ \equiv$ quasi-integrable from above

$Q^- \equiv$ quasi-integrable from below

$Q \equiv$ quasi-integrable

$L_1 \equiv$ integrable.

Thm (Big 3):

(1) If $f, g \in Q(\Omega, \mathcal{F}, \mu)$ then

$$f \le g \ \mu\text{-a.e.} \Rightarrow \int f\,d\mu \le \int g\,d\mu$$

(2)

[a] $f \in \mathcal{M}(\Omega, \mathcal{F}, \mu)$ & $\alpha \in [0, \infty]$

$$\Rightarrow \int_\Omega \alpha f\,d\mu = \alpha \int_\Omega f\,d\mu$$

[b] $f \in Q(\Omega, \mathcal{F}, \mu)$ & $\alpha \in \mathbb{R}$

$$\Rightarrow \alpha f \in Q(\Omega, \mathcal{F}, \mu) \text{ and } \int_\Omega \alpha f\,d\mu = \alpha \int_\Omega f\,d\mu$$

[c] $f, g \in Q^+(\Omega, \mathcal{F}, \mu)$ or $f, g \in Q^-(\Omega, \mathcal{F}, \mu)$

$$\Rightarrow f + g \in Q(\Omega, \mathcal{F}, \mu) \text{ and } \int_\Omega f+g\,d\mu = \int_\Omega f\,d\mu + \int_\Omega g\,d\mu$$

(3) If $f_1, f_2, \dots \in \mathcal{M}(\Omega, \mathcal{F})$ then

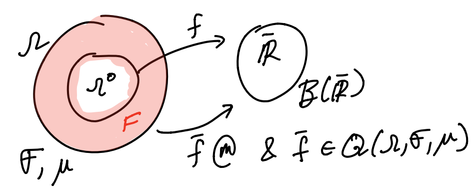
$$\lim_n \uparrow f_n = f \ \mu\text{-a.e.} \Rightarrow \lim_n \uparrow \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu$$

The only difference from little 3 is this μ-a.e.

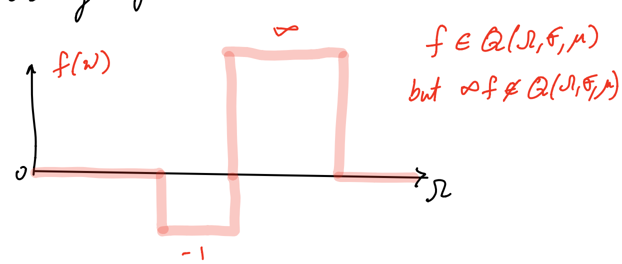
Remark:

- In (2)[c] it could happen that
$$f(\omega) + g(\omega) = \infty - \infty$$
but since $\mu(\{f = \infty\} \cap \{g = -\infty\}) = 0$ we can modify f & g to be defined everywhere.
In fact this allows us to define $\int_\Omega f\,d\mu$ for all functions $f : \Omega^\circ \to \mathbb{R}$ s.t.
  - $\exists F \in \mathcal{F}$ s.t. $(\Omega^\circ)^c \subset F$ & $\mu(F) = 0$
  - $\bar{f}(\omega) = \begin{cases} f(\omega) & \omega \in F^c \\ 0 & \omega \in F \end{cases} \in Q(\Omega, \mathcal{F}, \mu)$.

Picture:



- In (2)[b] the restriction $\alpha \in \mathbb{R}$ is necessary. e.g.



Proof:

For (1)

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14