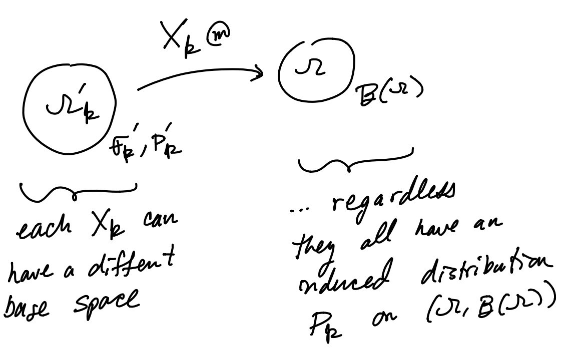
Lecture 14: Convergence in distribution and the Central limit theorem

Convergence in distribution is probably the most important notion of a limit of r.v.s $X_1, X_2, \ldots$ or a sequence of probability measures $P_1, P_2, \ldots$ on $(\Omega, \mathcal{B}(\Omega))$.

Just as in last lecture we will always assume $(\Omega, \mathcal{B}(\Omega))$ is Polish w.r.t. metric d.

Let $P, P_1, P_2, \ldots$ be probability measures on $(\Omega, \mathcal{B}(\Omega))$ and/or $X, X_1, X_2, \ldots$ a sequence of (m) maps from some prob. space into $(\Omega, \mathcal{B}(\Omega))$



Definition:

$P_n \xrightarrow{\mathcal{D}} P$ iff $\forall f \in C_b(\Omega)$, $\int_\Omega f\, dP_n \longrightarrow \int_\Omega f\, dP$.

$X_n \xrightarrow{\mathcal{D}} X$ iff $\forall f \in C_b(\Omega)$, $E f(X_n) \longrightarrow E f(X)$

called "convergence in distribution." or "weak convergence".

Remark: This notion of convergence is equiv to weak-* convergence in functional analysis. Its easier to formally see the connection when $P_1, P_2, \ldots, P$ have densities $\nu_1, \nu_2, \ldots, \nu$ w.r.t some base measure μ on $(\Omega, \mathcal{B}(\Omega))$, i.e.

$$dP_n = \nu_n d\mu \quad \& \quad dP = \nu d\mu$$

so that

$$P_n \xrightarrow{\mathcal{D}} P$$
$$\Updownarrow$$
$$\int f \nu_n d\mu \to \int f \nu d\mu, \ \forall f \in C_b(\Omega)$$

Remark: You can loosely interpret $X_n \xrightarrow{\mathcal{D}} X$ as meaning that for large n, m both X_n and X_m resemble random draws from X but that X_n & X_m are unrelated...

Warning: This is only a loose interpretation since it is possible that $\exists A \in \mathcal{B}(\Omega)$ s.t.

$$P(X_n \in A) \not\to P(X \in A)$$

Most of the examples of $\xrightarrow{\mathcal{D}}$ we will work with come from the central limit theorem ... which effectively says:

If $X_1, X_2, \ldots$ are independent r.v.s (all defined on a common $(\Omega', \mathcal{F}', P')$ with

$$E X_n = 0 \quad \& \quad \mathrm{var}(X_n) = \sigma^2 < \infty$$

then $\sqrt{n}\,\bar{X}_n = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} Z \sim N(0, \sigma^2)$.

We will derive this near the end of this lecture. However a good (but somewhat degenerate) example which helps interpret future results is as follows.

Example: $\Omega = \mathbb{R}$, $P(X_n = \frac{1}{n}) = 1$, $P(X=0)=1$.

$$\therefore \quad \underbrace{Ef(X_n)}_{f(\frac{1}{n})} \xrightarrow{n\to\infty} \underbrace{Ef(X)}_{f(0)} \quad \forall f \in C_b(\mathbb{R})$$

so $X_n \xrightarrow{\mathcal{D}} X$ but notice

$$\underbrace{P(X_n \le 0)}_{=0} \not\to \underbrace{P(X \le 0)}_{=1}$$

} mass can magically appear on the boundaries of closed sets

$$\underbrace{P(X_n > 0)}_{=1} \not\to \underbrace{P(X > 0)}_{=0}$$

} mass can magically disappear on the boundaries of open sets

Definition: $\forall A \subset \Omega$ define

$\bar{A} :=$ closure of A (w.r.t. the polish metric d)

$A^\circ :=$ open interior of A (all $x \in A$ s.t. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$)

$\partial A :=$ boundary of $A := \bar{A} - A^\circ$.

Here are some Portmanteau (French for coat hanger) results which give $\xrightarrow{\mathcal{D}}$ equivalence.

Theorem (Portmanteau I):
Let $P_1, P_2, \ldots, P$ be probability measures on a Polish space $(\Omega, \mathcal{B}(\Omega))$.
Then the following are equivalent.

(a) $P_n \xrightarrow{\mathcal{D}} P$

(b) $\int_\Omega f\,dP_n \to \int_\Omega f\,dP$, $\forall f \in \mathrm{lip}(\Omega) \cap C_b(\Omega)$

Possible magically appearing mass

(c) $\limsup_n P_n(F) \le P(F)$, $\forall$ closed $F \subset \Omega$.

Possible magically disappearing mass

(d) $P(G) \le \liminf_n P_n(G)$, $\forall$ open $G \subset \Omega$.

(e) $\lim_n P_n(A) = P(A)$, $\forall A \in \mathcal{B}(\Omega)$ s.t. $P(\partial A) = 0$

Proof:

(a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): Let $F \subset \Omega$ be closed. As in the proof of the separating class Thm let

$$f_\varepsilon(\omega) = \left(1 - \frac{d(\omega, F)}{\varepsilon}\right)^+$$

so that $f_\varepsilon \in \mathrm{Lip}(\Omega) \cap C_b(\Omega)$ and $\forall$ prob Q on $(\Omega, \mathcal{B}(\Omega))$

$$\int_\Omega I_F\,dQ \le \int_\Omega f_\varepsilon\,dQ \le \int_\Omega I_{F^\varepsilon}\,dQ \xrightarrow{\varepsilon\to 0} Q(F). \quad (*)$$

$$\therefore \limsup_n P_n(F) = \limsup_n \int_\Omega I_F\,dP_n$$

$$\le \limsup_n \int_\Omega f_\varepsilon\,dP_n, \text{ by } (*)$$

$$= \int_\Omega f_\varepsilon\,dP, \quad f_\varepsilon \in \mathrm{Lip}(\Omega) \cap C_b(\Omega)$$

$$\xrightarrow{\varepsilon \to 0} P(F), \text{ by } (*)$$

(c) $\Leftrightarrow$ (d): Take complements of (c)

(c) & (d) $\Rightarrow$ (e): Suppose $P(\partial A) = 0$

$\therefore \; 0 = P(\bar{A} - A^\circ) = P(\bar{A}) - P(A^\circ)$

by nested set subtraction props of P

and $P(A^\circ) \le \liminf_n P_n(A^\circ)$, by (c)

$\le \limsup_n P_n(\bar{A})$, $\quad$ inf $\le$ sup & $P(A^\circ) \le P(\bar{A})$

$\le P(\bar{A})$, by (d)

$\liminf_n P_n(A)$ & $\limsup_n P_n(A)$ sandwiched in here.

Since $P(A^\circ) \subset P(A) \subset P(\bar{A})$ & $P(\bar{A}) - P(A^\circ) = 0$

$$\lim_n P_n(A) = P(A)$$

(e) $\Rightarrow$ (a):

Let $f \in C_b(\Omega)$ & show $\int_\Omega f\,dP_n \to \int_\Omega f\,dP$.

Adjust f by adding a constant and re-scaling we can assume w.l.g that

$$0 < f < 1.$$

Recall Thm from lecture 11 that says

r.v. $X \geq 0 \Rightarrow E(X) = \int_0^\infty P(X>t)\,dt$

This applies to f so that

(*) $\int_\Omega f\,dP_n = \int_0^1 P_n(f>t)\,dt$

$\downarrow ?$ as $n\to\infty$

(**) $\int_\Omega f\,dP = \int_0^1 P(f>t)\,dt.$

Moreover continuity implies

$$\{f>t\} = f^{-1}((t,\infty)) = \text{open} = \{f>t\}^\circ$$

$$\{f\geq t\} = (f^{-1}((-\infty,t)))^c = \text{closed} = \overline{\{f>t\}}$$

$\therefore$ $\partial\{f>t\} = \{f\geq t\} - \{f>t\} = \underbrace{\{f=t\}}$

has non-zero P mass for at most countably many t

$\therefore$ (e) implies

$$P_n(f>t) \xrightarrow{n\to\infty} P(f>t)$$

for $\mathcal{L}^1$-a.e. t

$\therefore$ DCT implies

$$\int_0^1 P_n(f>t)\,dt \xrightarrow{n\to\infty} \int_0^1 P(f>t)\,dt$$

$\Vert$(*) $\qquad\qquad$ $\Vert$(**)

$$\int_\Omega f\,dP_n \qquad\qquad \int_\Omega f\,dP$$

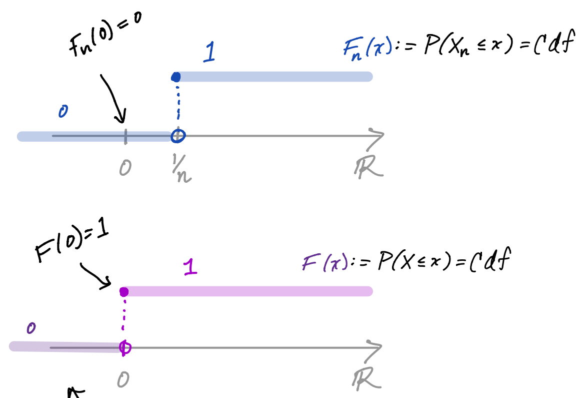
QED

The next result covers the special case of univariate real valued r.v.s.

Recall earlier example

$$X_n = \tfrac{1}{n} \xrightarrow{\mathcal{D}} X = 0$$

Note $F_n(x) \to F(x)$ $\forall x \neq 0$. Here is the picture



The problem is that $A = (-\infty, 0)$ is s.t. $P(X \in \partial A) \neq 0$

Define $C_F := \{x \in \mathbb{R} : F \text{ is continuous at } x\}$

so that $x \in C_F \Leftrightarrow 0 = F(x) - F(x-)$ $\leftarrow$ F is always right cont.

$\Leftrightarrow 0 = P(X = x)$

$\Leftrightarrow PX^{-1}(\partial(-\infty, x]) = 0$

Theorem (Portmanteau II):

Let $X_1, X_2, \ldots, X$ be real-valued r.v.s with cdfs $F_1, F_2, \ldots, F$. Then the following are equivalent

(i) $X_n \xrightarrow{\mathcal{D}} X$

(ii) $F_n(x) \to F(x)$, $\forall x \in C_F$

(iii) $F_n^{-1}(u) \to F^{-1}(u)$, $\forall x \in C_{F^{-1}}$

the left-continuous quasi-inverse defined on $u \in (0,1)$ in lecture 8

Proof:

Let $P_1, P_2, \ldots, P$ be the induced measures of $X_1, X_2, \ldots, X$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

((i) ⇒ (ii))

$$X_n \xrightarrow{\mathcal{D}} X \iff P_n \xrightarrow{\mathcal{D}} P$$

$$\Rightarrow P_n((-\infty,x]) \to P((-\infty,x]) \quad \forall x \in C_F$$

since $x \in C_F \Rightarrow P(\partial(-\infty,x]) = 0$

$$\Rightarrow F_n(x) \to F(x) \quad \forall x \in C_F$$

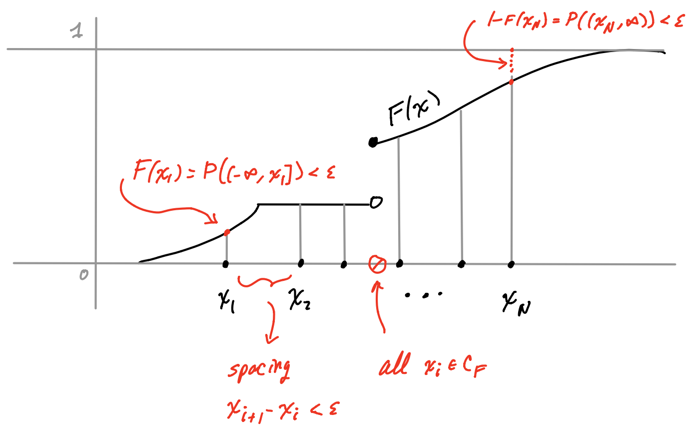
((ii) ⇒ (i))

For the other direction suppose $F_n(x) \to F(x)$ $\forall x \in C_F$.

Using Portmanteau I it will be sufficient to show

$$\int_{\mathbb{R}} f\,dP_n \to \int_{\mathbb{R}} f\,dP, \quad \forall f \in Lip(\mathbb{R}) \cap C_b(\mathbb{R}).$$

Now for $\varepsilon > 0$ choose $x_1, \dots, x_N$ points s.t.



These exist since the results in Lecture 11 & 8 imply C_F^c is at most countable, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

Let
$A_1 = (-\infty, x_1]$
$A_2 = (x_1, x_2]$
$\vdots$
$A_N = (x_{N-1}, x_N]$
$A_{N+1} = (x_N, \infty)$

Note that by design $P(\partial A_i) = 0$ so $P_n(A_i) \to P(A_i)$

Now let $f \in Lip(\mathbb{R}) \cap C_b(\mathbb{R})$.
By rescaling we can suppose w.l.g that

$$0 \le f \le 1.$$

$$\therefore \limsup_n \int_{A_1} f\,dP_n \le \limsup_n \int_{A_1} 1\,dP_n$$

$$= \int_{A_1} 1\,P \quad \text{since } P(\partial A_1) = 0$$

$$\le \varepsilon + \underbrace{\int_{A_1} f\,dP}_{\text{positive}}$$

Similarly

$$\limsup_n \int_{A_{N+1}} f\,dP_n \le \varepsilon + \int_{A_{N+1}} f\,dP$$

Also

$$\limsup_n \int_{A_2 \cup \dots \cup A_N} f\,dP_n$$

$$= \limsup_n \sum_{i=2}^{N} \int_{A_i} f\,dP_n$$

$$\le \limsup_n \sum_{i=2}^{N} (f(x_i) + c\varepsilon) \int_{A_i} 1\,dP_n$$

since $|f(x) - f(x_i)| \le c\varepsilon$ for $x \in A_i$ & $2 \le i \le N$

$$= \sum_{i=2}^{N} (f(x_i) + c\varepsilon) \int_{A_i} dP$$

since $P_n(A_i) \to P(A_i)$

$$= \sum_{i=2}^{N} \int_{A_i} f(x_i)\,dP + c\varepsilon \int_{\mathbb{R}} dP$$

$$\le \sum_{i=2}^{N} \int_{A_i} f\,dP + 2c\varepsilon$$

$$= \int_{A_2 \cup \dots \cup A_N} f\,dP + 2c\varepsilon$$

Putting everything together & letting $\varepsilon \to 0$

$$\limsup_n \int_{\mathbb{R}} f\,dP_n \le \int_{\mathbb{R}} f\,dP$$

Replacing f with $1-f$ above gives

$$\int_{\mathbb{R}} f\,dP_n \to \int_{\mathbb{R}} f\,dP$$

$\big((ii) \Rightarrow (iii)\big)$

We show that $\forall u \in (0,1)$

$$F^{-1}(u) \underset{(*)}{\le} \liminf F_n^{-1}(u) \le \limsup F_n^{-1}(u) \underset{(**)}{\le} F^{-1}(u+)$$

To show $(*)$ suppose not.

Now we can choose $x \in C_F$ s.t.

$$\liminf_n F_n^{-1}(u) < x < F^{-1}(u).$$

By the switching formula $F(x) < u$ so that

$$F_n(x) \to F(x) < u$$

$\therefore$ $F_n(x) < u$ $\forall$ large n

$\therefore$ $x < F_n^{-1}(u)$ $\forall$ large n by "switching" again

which contradicts $\liminf_n F_n^{-1}(u) < x$.

To show $(**)$ use the same trick & suppose not.

$\therefore$ $\exists x \in C_F$ s.t.

$$\underbrace{F^{-1}(u+) < x}_{} < \limsup F_n^{-1}(u)$$

$\therefore$ $u < F(x-) = F(x)$

by switching again

$\therefore$ $u < F_n(x)$ $\forall$ large n

$\therefore$ $F_n^{-1}(u) < x$ $\forall$ large n

$\therefore$ contradiction

$\big((iii) \Rightarrow (ii)\big)$ Similar.

QED.

Theorem: (uniqueness of $\xrightarrow{\mathcal{D}}$ limits)

Let $P, Q, P_1, P_2, \ldots$ be probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. If $P_n \xrightarrow{\mathcal{D}} P$ and $P_n \xrightarrow{\mathcal{D}} Q$ then $P = Q$.

Proof:

If $P_n \xrightarrow{\mathcal{D}} P$ & $P_n \xrightarrow{\mathcal{D}} Q$ then $\forall f \in C_b(\Omega)$

$$\lim_n \int_\Omega f\,dP_n = \underbrace{\int_\Omega f\,dQ = \int_\Omega f\,dP}$$

since $Lip(\Omega) \cap C_b(\Omega) \subset C_b(\Omega)$ is a separating class this implies $P = Q$.

QED

Theorem: (sub-sub-seq check for $\xrightarrow{\mathcal{D}}$)

Let $P, P_1, P_2, \ldots$ be a collection of Probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. If $\forall$ sub-seq n_k, $\exists$ a sub-sub-seq n_{k_j} s.t. $P_{n_{k_j}} \xrightarrow{\mathcal{D}} P$ as $j \to \infty$ then

$$P_n \xrightarrow{\mathcal{D}} P \text{ as } n \to \infty.$$

Proof:

Suppose not.

$\therefore$ $\exists f \in C_b(\Omega)$ s.t.

$$\int_\Omega f\,dP_n \not\to \int_\Omega f\,dP$$

$\therefore$ $\exists n_k$ & $\varepsilon > 0$ s.t.

$$\left| \int_\Omega f\,dP_{n_k} - \int_\Omega f\,dP \right| \ge \varepsilon > 0, \ \forall k.$$

Now by assumption we can choose n_{k_j} so that $\int_\Omega f\,dP_{n_{k_j}} \longrightarrow \int_\Omega f\,dP$.

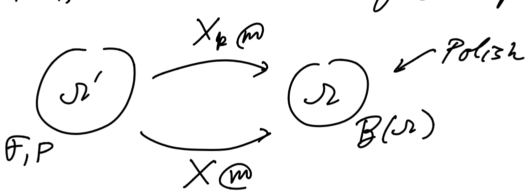
$\therefore$ contradiction

QED

The next theorem deals with the notion of convergence in probability. We will cover this in a later lecture but give some brief notation for it now, along with almost everywhere convergence.

Definition: ($\xrightarrow{P}$ & $\xrightarrow{a.e.}$)

Let $X, X_1, \ldots$ be a collection of ⓜ maps



Then

$X_n \xrightarrow{P} X$ iff $\forall \varepsilon > 0$, $P(d(X_n, X) \geq \varepsilon) = 0$

$X_n \xrightarrow{a.e.} X$ iff $P(\lim_n d(X_n, X) = 0) = 1$.

Remark:

$$X_n \xrightarrow{P} X \iff d(X_n, X) \xrightarrow{P} 0$$

$$X_n \xrightarrow{a.e.} X \iff d(X_n, X) \xrightarrow{a.e.} 0$$

Remark: In lecture 6 we remarked that SLLN $\Rightarrow$ WLLN. Indeed

$$X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X$$

since

$$\limsup_n P(d(X_n, X) \geq \varepsilon) \overset{\text{Fatou}}{\leq} P(d(X_n, X) \geq \varepsilon \text{ i.o.}_n)$$

$$\leq \underbrace{P(d(X_n, X) \not\to 0)}_{=0}$$

Theorem: (Slutzky)

Let $X, X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be collections of generalized r.v.s taking values in a Polish space Ω (with σ-field $\mathcal{B}(\Omega)$) all defined on the same probability space.

Then

$$X_n \xrightarrow{\mathcal{D}} X \;\&\; d(X_n, Y_n) \xrightarrow{P} 0 \Rightarrow Y_n \xrightarrow{\mathcal{D}} X.$$

Proof:

We use Portmanteau I and show

$$\limsup_n P(Y_n \in F) \leq P(X \in F)$$

for all closed $F \subset \Omega$.

First note $\forall \varepsilon > 0$

$$\{Y_n \in F\} \subset \{d(X_n, Y_n) \geq \varepsilon\} \cup \{X_n \in F^\varepsilon\}$$

$$\therefore \limsup_n P(Y_n \in F)$$

$$\leq \underbrace{\limsup_n P(d(X_n, Y_n) \geq \varepsilon)}_{\substack{=0 \text{ by} \\ \text{assumption}}} + \limsup_n P(X_n \in F^\varepsilon)$$

$$\leq \limsup_n P(X_n \in \overline{F^\varepsilon})$$

$$\leq P(X \in \overline{F^\varepsilon}) \text{ by Portmanteau I}$$

Since $\overline{F^\varepsilon} \downarrow F$ as $\varepsilon \downarrow 0$

$$P(X \in \overline{F^\varepsilon}) \downarrow P(X \in F)$$

as $\varepsilon \downarrow 0$ by CFA.

$$\therefore \limsup_n P(Y_n \in F) \leq P(X \in F)$$

as was to be shown

QED

Slutkzy's Theorem gives us a nice corrolary that relates $\xrightarrow{\mathcal{D}}$, $\xrightarrow{P}$ and a.e. convergence.

Corollary:

$$X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X.$$

Warning: The reverse implications do not hold in general.

The "moving spike" showed $\exists X_n$ & X s.t.

$$X_n \xrightarrow{P} X \text{ but } X_n \not\xrightarrow{a.e.} X$$

To see why $\xrightarrow{\mathcal{D}} \not\Rightarrow \xrightarrow{P}$ consider

$$X = \begin{cases} -1 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$$

and $X_n := (-1)^n X$ so that $X_n \overset{\mathcal{D}}{=} X$

but $P(|X_n - X| \geq \varepsilon) = 1$ when n is odd.

$$\therefore \quad X_n \xrightarrow{\mathcal{D}} X \text{ but } X_n \not\xrightarrow{P} X.$$

Skorokhod Representation

Theorem: (Skorokhod Representation theorem)

Let $P, P_1, P_2, \ldots$ be probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$ s.t.

$$P_n \xrightarrow{\mathcal{D}} P.$$

Then there exist a probability space $(\Omega^*, \mathcal{F}^*, P^*)$ & maps $X^*, X_1^*, \ldots$

s.t.

$$\begin{tikzcd} \Omega^* \arrow[r, bend left, "X_k^*(\omega)"] \arrow[r, bend right, "X^*(\omega)"'] & \Omega \end{tikzcd}$$

$\mathcal{F}^*, P^*$ $\qquad$ $\mathcal{B}(\Omega)$

s.t. $\mathcal{L}(X^*) = P$, $\mathcal{L}(X_n^*) = P_n$ and

$$\lim_{n\to\infty} X_n^*(\omega) = X^*(\omega)$$

for all $\omega \in \Omega^*$.

Notation: In the above theorem I'm using $\mathcal{L}(X^*)$, $\mathcal{L}(X_n^*)$ as a shorthand for $P^*(X^*)^{-1}$ & $P^*(X_n^*)^{-1}$.

i.e. the induced distributions on $(\Omega, \mathcal{B}(\Omega))$.

I'll use this notation, when convenient, throughout the rest of the course.

Skorokhods Representation theorem is really hand and effectively says

$$X_n \xrightarrow{\mathcal{D}} X \Rightarrow \exists X^*, X_n^* \text{ defined on a common probability space s.t. } \begin{array}{ccc} X_n^* & \xrightarrow{a.e.} & X^* \\ \| \mathcal{D} & & \| \mathcal{D} \\ X_n & & X \end{array}$$

The proof of the general Skorokhod's result is nasty and doesn't really provide any insight. We will just show the case $\Omega = \mathbb{R}$ and cite the general result in Billingsley's book "Convergence of Prob. Meas."

Proof of Skorokhod's Rep when $\Omega = \mathbb{R}$:

Define c.d.f.s

$$F_n(x) = P_n((-\infty, x])$$

$$F(x) = P((-\infty, x]).$$

By Portmanteau II

$$F_n^{-1}(u) \longrightarrow F^{-1}(u) \quad \forall u \in C_{F^{-1}} \qquad (*)$$

Let $\Omega^* = (0,1)$, $\mathcal{F}^* = \mathcal{B}((0,1))$, $P^* = \mathcal{L}'$ uniform measure on $(0,1)$.

and set

$$X_n^*(u) := F_n^{-1}(u)$$

$$X^*(u) := F^{-1}(u)$$

$\forall u \in \Omega^*$.

$$\therefore \quad X_n^*(u) \to X^*(u), \ \forall u \in C_{F^{-1}} \text{ by } (*).$$

Finally notice that $C_{F^{-1}}^c$ must be countable since

$$C_{F^{-1}}^c = \bigcup_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \ \bigcup_{\substack{0<a<b<1 \\ a,b \in \mathbb{Q}}} \underbrace{\{ u \in [a,b] : F^{-1}(u+) - F^{-1}(u-) > \varepsilon \}}$$

Since F^{-1} is monotonic on $[a,b]$ the sum of the jumps of size exceeding ε must be no greater than $F^{-1}(b) - F^{-1}(a)$ ... hence this set is finite.

$$\therefore P^*(C_{F^{-1}}) = 1$$

$$\therefore X_n^* \xrightarrow{a.e.} X^*$$

QED

The following result demonstraits Skorokhod's usefullness.

Theorem: (continuous mapping Thm for $\xrightarrow{\mathcal{D}}$)

Let $X, X_1, X_2, \ldots$ be generalized r.v.s taking values in a Polish space Ω (with σ-field $\mathcal{B}(\Omega)$) s.t. $X_n \xrightarrow{\mathcal{D}} X$.

If $g : \Omega \to \mathbb{R}$ satisfies

$$P(g \text{ is continuous @ } X) = 1$$

then

$$g(X_n) \xrightarrow{\mathcal{D}} g(X).$$

Proof:

By Skorokhods Rep Thm $\exists X_n^*, X^*$ defined on $(\Omega^*, \mathcal{F}^*, P^*)$ s.t.

$$\lim_{n\to\infty} X_n^*(\omega) = X^*(\omega) \quad \forall \omega \in \Omega^*$$

where $X_n \overset{\mathcal{D}}{=} X_n^*$, $X \overset{\mathcal{D}}{=} X^*$.

Let $A := \{\omega \in \Omega^* : g \text{ is continuous at } X^*(\omega)\}$

$$\therefore \ \omega \in A \Rightarrow \lim_{n\to\infty} g(X_n^*(\omega)) = g(X^*(\omega))$$

Since $X \overset{\mathcal{D}}{=} X^*$, $P^*(A) = 1$.

$$\therefore \quad g(X_n^*) \xrightarrow{a.e.} g(X^*)$$

$$\therefore \quad g(X_n^*) \xrightarrow{\mathcal{D}} g(X^*) \text{ by corollary above}$$

$$\overset{\mathcal{D}}{\|} \qquad\qquad \overset{\mathcal{D}}{\|}$$

$$g(X_n) \qquad\qquad g(X)$$

QED

Skorokhods Representation Theorem also gives us extensions of Fatou, DCT, .. which apply to the case $X_n \xrightarrow{\mathcal{D}} X$.

Here is an example.

Theorem: (UI extension for $\xrightarrow{\mathcal{D}}$)

If $X_n \xrightarrow{\mathcal{D}} X$ and the X_n's are UI then $E|X_n| \to E|X| < \infty$ and

$$E(X_n) \to E(X) < \infty.$$

Proof:

By skorokhod $\exists X_n^*, X^*$ s.t.

$$X_n \overset{\mathcal{D}}{=} X_n^* \xrightarrow{a.e.} X^* \overset{\mathcal{D}}{=} X$$

since $X_n \overset{\mathcal{D}}{=} X_n^*$ the X_n^*'s are also UI.

$\therefore$ by old UI Theorem we have

$$E|X_n^*| \to E|X^*| < \infty \text{ and}$$

$$E(X_n^*) \to E(X^*) < \infty$$

but again since $X_n^* \overset{\mathcal{D}}{=} X_n$ & $X^* \overset{\mathcal{D}}{=} X$

$$E|X_n| \to E|X| < \infty \text{ and}$$

$$E(X_n) \to E(X) < \infty.$$

QED.

Finally lets use Skorokhods Rep Thm to prove the "Delta-method"

Theorem: (Delta method)

Let $Z, X_1, X_2, \ldots$ be random d-dimensional real vectors s.t.

$$c_n(X_n - x_0) \xrightarrow{\mathcal{D}} Z \qquad (*)$$

where $0 < c_n \to \infty$ as $n \to \infty$ & $x_0 \in \mathbb{R}$.

If $g: \mathbb{R}^d \to \mathbb{R}^d$ is differentiable @ x_0 then

$$c_n(g(X_n) - g(x_0)) \xrightarrow{\mathcal{D}} Dg(x_0) Z$$

Remark: $g: \mathbb{R}^d \to \mathbb{R}^d$ differentiable @ x_0 means that

$$\lim_{v \to 0} \frac{g(x_0+v) - g(x_0) - Dg(x_0)v}{|v|} = 0$$

as $\varepsilon \to 0$ where $Dg(x_0) = \left(\frac{\partial}{\partial x_1} g, \ldots, \frac{\partial}{\partial x_d} g\right)\Big|_{x = x_0}$

Remark: The intuitive way to understand this theorem is that $(*)$ suggest

$$X_n \overset{\mathcal{D}}{\approx} x_0 + \underbrace{\frac{Z}{c_n}}_{\to 0 \text{ as } n \to \infty}$$

so $g(X_n) \overset{\mathcal{D}}{\approx} g(x_0 + \frac{Z}{c_n}) \overset{\mathcal{D}}{\approx} g(x_0) + Dg(x_0)\frac{Z}{c_n}$

Proof:

Lets prove this for $d = 1$ ... the case $d > 0$ is similar.

we can use $(*)$ along with skorokhod's Rep to get X_n^* and Z^* s.t.

$$c_n(X_n^* - x_0) \xrightarrow{a.e.} Z^*.$$

Now

define this to be $g'(x_0)$ when $X_n^*(\omega) - x_0 = 0$

$$c_n(g(X_n^*) - g(x_0)) = \left(\frac{g(X_n^*) - g(x_0)}{X_n^* - x_0}\right) c_n(X_n^* - x_0)$$

Let $n \to \infty$ so that $X_n^* - x_0 \xrightarrow{a.e.} 0$ which implies

$$\frac{g(X_n^*) - g(x_0)}{X_n^* - x_0} \xrightarrow{a.e.} g'(x_0) \text{ and } c_n(X_n^* - x_0) \xrightarrow{a.e.} Z^*$$

$$\therefore c_n(g(X_n^*) - g(x_0)) \xrightarrow{a.e.} g'(x_0) Z^*$$

$$\therefore c_n(g(X_n) - g(x_0)) \xrightarrow{\mathcal{D}} g'(x_0) Z \qquad \text{QED.}$$

Tightness and Porhorov's Thm

An extremly usefal fact, when working with a sequence of real numbers $x_1, x_2, \ldots$ is that if $\{x_n\}_{n\geq 1}$ is bounded then there exists a sub-sequence n_k and a real number x s.t. $x_{n_k} \xrightarrow{k\to\infty} x$

We would like to have something similar for a sequence $P_1, P_2, \ldots$ of probability measures on a Polish.

The problem is to find the right generalization of "boundedness" to guarantee the existance of a probability measure P & a sub-sequence n_k s.t.

$$P_{n_k} \xrightarrow{\mathcal{D}} P.$$

It turns out the right definition is our old friend "tightness" from the homeworks last lecture.

Definition:

Let $\mathcal{P}$ be a collection of Probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. Then

$\mathcal{P}$ is tight iff $\forall \varepsilon$ $\exists$ a compact $K \subset \Omega$ s.t.

$$\sup_{P\in\mathcal{P}} P(K^c) < \varepsilon$$

The following result shows this is the "right" definition of boundedness.

Theorem: (Prohorov)

Let $\mathcal{P}$ be a collection of Probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. Then

$$\mathcal{P} \text{ is tight}$$
$$\Updownarrow$$

$*$ $\left\{ \begin{array}{l} \forall P_1, P_2, \ldots \in \mathcal{P} \text{ there exists a sub-seq } n_k \\ \text{and a prob measure } P \text{ on } (\Omega, \mathcal{B}(\Omega)) \\ \text{s.t. } \quad P_{n_k} \xrightarrow{\mathcal{D}} P \end{array} \right.$

not necessarily in $\mathcal{P}$.

Note: If $\mathcal{P}$ satisfies $*$ then $\mathcal{P}$ is said to be relatively compact w.r.t. $\xrightarrow{\mathcal{D}}$.

Proof: This proof is a bit nasty and doesn't really provide much probabalistic intuition so we will skip it and simply cite Billingsley's book on convergence of Probability measures.

Our main use of Prohorov's Thm will be through the following consequence

Theorem: (tightness + separating class)

Let $P, P_1, P_2, \ldots$ be probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$.

If $\{P_n\}_{n\geq 1}$ is tight and there exists a separating family $\Gamma \subset C_b(\Omega)$ s.t.

$$\int_\Omega f\,dP_n \longrightarrow \int_\Omega f\,dP \quad \forall f \in \Gamma$$

Then $P_n \xrightarrow{\mathcal{D}} P$.

Proof:

Assume $P_n \overset{\mathcal{D}}{\not\to} P$.

$\therefore \exists f_0 \in C_b(\Omega)$ & a sub-seq n_k s.t.

$$\left| \int_\Omega f_0 dP_{n_k} - \int_\Omega f_0 dP \right| > \varepsilon \quad \forall k \in \mathbb{N}. \quad (a)$$

If $\{P_n\}$ is tight Prohorov's Theorem implies there exists a sub-sub-seq n_{k_j} and a measure Q on $(\Omega, \mathcal{B}(\Omega))$ s.t.

$$P_{n_{k_j}} \xrightarrow{\mathcal{D}} Q \quad \text{as } j \to \infty$$

$$\therefore \int_\Omega f \, dP_{n_{k_j}} \xrightarrow{j\to\infty} \int_\Omega f \, dQ \quad \forall f \in C_b(\Omega) \quad (b)$$

Now by assumption we have

$$\int_\Omega f \, dP_{n_{k_j}} \xrightarrow{j\to\infty} \int_\Omega f \, dP \quad \forall f \in \Gamma \quad (c)$$

Combining (b) and (c) gives

$$P = Q$$

Since Γ is a separating class & $\Gamma \subset C_b(\Omega)$.

Now (a) and (b) contradict eachother.

QED

We get imediate utility out of this result for another Portmanteau Theorem

Theorem: (Portmanteau III)

Let $P, P_1, P_2, \ldots$ be probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. Then

$$P_n \xrightarrow{\mathcal{D}} P \quad \text{iff} \quad \int_\Omega f dP_n \to \int_\Omega f dP \quad \forall f \in \Gamma$$

whenever

(I) $\Gamma = C_c(\Omega)$ & Ω is locally compact

(II) $\Gamma = C_c^\infty(\Omega)$ & $\Omega = \mathbb{R}^d$

(III) $\Gamma = \{ e^{i x \cdot k} : k \in \mathbb{R}^d \}$ & $\Omega = \mathbb{R}^d$ & $\sup_n \int_{\mathbb{R}^2} |x|^2 dP_n(x) < \infty$

(IV) $\Gamma = \{\text{monomials}\}$ & Ω is a compact subset of $\mathbb{R}^d$

Theorem: (Portmanteau IV)

Let $P, P_1, P_2, \dots$ be probability measures on $(C[0,1], \mathcal{B}(C[0,1]))$.

Then the following are equivalent:

(A) $P_n \xrightarrow{\mathcal{D}} P$

(B) $P\pi_{t_1,\dots t_m}^{-1} \xrightarrow{\mathcal{D}} P\pi_{t_1,\dots,t_m}^{-1} \quad \forall t_1, \dots t_m \in [0,1]$

The central limit Theorem

Kolmogorov's 3 series Theorem

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