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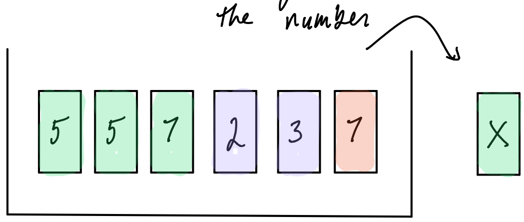
Lecture 20: Conditional expected value with respect to a sub-σ-field

Lets start with a modivation.

Consider a box with numbered tickets which are colored.

I pick one at random & show you the color but not the number



Before you know the color your best guess for X is

$$E(X) = \int X\,dP = \left\{ \begin{array}{l} \text{average of the} \\ \text{ticket numbers} \\ \text{in the box.} \end{array} \right.$$

After I tell you the color is green your new best guess for X is

$$E(X \mid \boxed{?}) = \left\{ \begin{array}{l} \text{average green} \\ \text{ticket number} \end{array} \right.$$

$$= \frac{5+5+7}{3} = \frac{22}{3}$$

If you wanted to automate this prediction you could pre-compute

$$E(X \mid \boxed{?}) \qquad E(X \mid \boxed{?}) \qquad E(X \mid \boxed{?})$$

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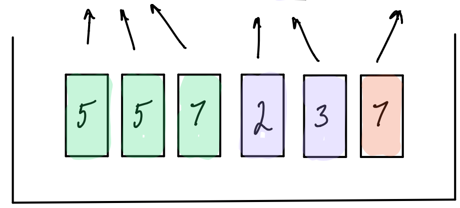
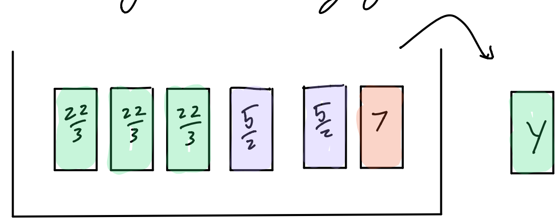
This can be thought of as a map from ticket to prediction value

$$\begin{array}{ccc} \frac{22}{3} & \frac{5}{2} & 7 \\ \| & \| & \| \\ E(X \mid \boxed{?}) & E(X \mid \boxed{?}) & E(X \mid \boxed{?}) \end{array}$$



or as a granular smoothing of X



where $Y = E(X \mid \text{color})$.

Notice two key facts about $Y = E(X \mid \text{color})$.

(i) The collection of events that we can place bets on for Y is less than for X

(ii) If A is an event that corresponds to a bet on Y, i.e. $A = \{Y = \frac{22}{3}\} \cup \{Y = 7\}$, then

$$\underbrace{\int_A X\,dP}_{\frac{5+5+7+7}{6}} = \underbrace{\int_A Y\,dP}_{\frac{22}{3}\cdot\frac{1}{2} + 7\cdot\frac{1}{6}}$$

Now make the correspondence

Ω = the collection of tickets

$\mathcal{F}$ = the possible bets on all tickets

$\mathcal{A}$ = the possible bets on color

For $w \in \Omega$, $X(w)$ = ticket #

$E(X|\mathcal{A}) = Y$ maps $w \in \Omega \longmapsto$ ave X $\forall$ tickets with the same color as w

and we have

$E(X|\mathcal{A})$ is $\mathcal{A}$-measurable and

$$\int_A X dP = \int_A E(X|\mathcal{A}) dP \quad \forall A \in \mathcal{A}.$$

Theorem: (existence of $E(X|\mathcal{A})$)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in \mathcal{Q}(\Omega, \mathcal{F}, P)$ be a possibly extended r.v.

If $\mathcal{A} \subset \mathcal{F}$ is a σ-field then $\exists$ a P-unique extended r.v. $E(X|\mathcal{A}) \in \mathcal{Q}(\Omega, \mathcal{F}, P)$ such that

(i) $E(X|\mathcal{A})$ is $\mathcal{A}$-measurable $\leftarrow$ More granular than X

(ii) $\int_A X dP = \int_A E(X|\mathcal{A}) dP \quad \forall A \in \mathcal{A}$

Proof:

Start by assuming $X \geq 0$.

Let $v(\cdot) = \int_\cdot X dP$ be a measure on $(\Omega, \mathcal{A})$.

and $\bar{P}(\cdot) = P(\cdot)$ but only defined over $(\Omega, \mathcal{A})$.

Now to show (ii) we want a $E(X|\mathcal{A})$ @ $\mathcal{A}$

$$v(A) = \int_A E(X|\mathcal{A}) \, d\bar{P}$$

So look for this as $dv/d\bar{P}$

replaced P with $\bar{P}$ since $E(X|\mathcal{A})$ is supposed to be @ $\mathcal{A}$

Notice $v \lll \bar{P}$ since

$$\bar{P}(A) = 0, \ A \in \mathcal{A} \Rightarrow I_A X = 0 \ \ P\text{-a.e.}$$
$$\Rightarrow v(A) = \int_\Omega I_A X dP = 0$$

$\therefore$ By the Radon-Nikodym Thm $\exists$ a $\bar{P}$-unique $dv/d\bar{P} \in \mathcal{M}(\Omega, \mathcal{A})$ s.t. $\forall A \in \mathcal{A}$

$$\int_A \frac{dv}{d\bar{P}} d\bar{P} = v(A) = \int_A X dP$$

So setting $E(X|\mathcal{A}) := \frac{dv}{d\bar{P}} \in \mathcal{M}(\Omega, \mathcal{A})$ we have (ii). To see why $E(X|\mathcal{A}) \in \mathcal{Q}(\Omega, \mathcal{F}, P)$

$$\forall A \in \mathcal{A}, \ \int_A E(X|\mathcal{A}) d\bar{P} = \int_A E(X|\mathcal{A}) dP$$

This follows by change of variables

$$(\Omega, \mathcal{F}, P) \xrightarrow{id @} (\Omega, \mathcal{A}, \bar{P}) \xrightarrow{E(X|\mathcal{A}) @} (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})), \qquad \bar{P} = P\, id^{-1}$$

which says

$$\int_A E(X|\mathcal{A}) d\bar{P} = \int_{id^{-1}(A)} E(X|\mathcal{A}) \circ id \, dP$$
$$= \int_A E(X|\mathcal{A}) dP$$

Now just suppose $X \in \mathcal{Q}(\Omega, \mathcal{F}, P)$.

Assume $X \in \mathcal{Q}^+(\Omega, \mathcal{F}, P)$ w.l.g.

$\therefore v(\cdot) := \int_\cdot X^+ dP$ is a finite measure on $\mathcal{A}$

$\therefore E(X^+|\mathcal{A}) := \frac{dv}{d\bar{P}} \in L_1(\Omega, \mathcal{A}, \bar{P})$ by Thm "props of RND."

$\therefore E(X^+|\mathcal{A}) \in L_1(\Omega, \mathcal{F}, P)$, change of variables

$$\therefore\ E(X|\mathcal{A}) := \underbrace{E(X^+|\mathcal{A})}_{\in L_1(\Omega,\mathcal{F},P)} \overset{\text{P-a.e. defined}}{-} \underbrace{E(X^-|\mathcal{A})}_{\in \mathcal{M}(\Omega,\mathcal{F})} \in \mathcal{Q}(\Omega,\mathcal{F},P)$$

and $\forall A \in \mathcal{A}$

$$\int_A E(X|\mathcal{A})\,dP = \int_A \underbrace{E(X^+|\mathcal{A})}_{\in L_1}\,dP - \int_A E(X^-|\mathcal{A})\,dP$$

by Big 3

$$= \int_A X^+ dP - \int_A X^- dP$$

$$= \int_A X\,dP$$

This establishes (i) & (ii) & $E(X|\mathcal{A}) \in \mathcal{Q}(\Omega,\mathcal{F},P)$.

For uniqueness suppose $\widetilde{E(X|\mathcal{A})}$ is another version.

Now by Thm on indefinite integrals in Lecture 11

$$\int_\cdot \widetilde{E(X|\mathcal{A})}\,dP \overset{(ii)}{=} \int_\cdot E(X|\mathcal{A})\,dP$$

$$\Rightarrow \widetilde{E(X|\mathcal{A})} = E(X|\mathcal{A}) \quad P\text{-a.e}$$

QED

Remark: By construction we have

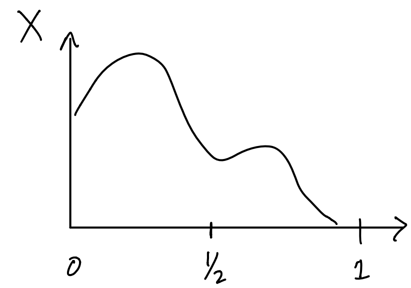
$X \in \mathcal{Q}^+(\Omega,\mathcal{F},P) \Rightarrow E(X|\mathcal{A}) \in \mathcal{Q}^+(\Omega,\mathcal{A},P)$

$X \in \mathcal{Q}^-(\Omega,\mathcal{F},P) \Rightarrow E(X|\mathcal{A}) \in \mathcal{Q}^-(\Omega,\mathcal{A},P)$.

Remark: It is useful to think of $E(X|\mathcal{A})$ as the weighted average of X over the "smallest $\mathcal{A}$-set", i.e. a smoothing or granulation of X, or a projection of X onto the space of $\mathcal{A}$-measurable functions.

Example:

$$(\Omega,\mathcal{F},P) = \big((0,1],\ \mathcal{B}((0,1]),\ P\big)$$



$$\mathcal{A} = \{\emptyset, \Omega, (0,\tfrac12], (\tfrac12,1]\}$$

Guess at $E(X|\mathcal{A})$ & show it has the correct properties

$$E(X|\mathcal{A})(\omega) := \begin{cases} \dfrac{E(I_{(0,\frac12]}X)}{P((0,\frac12])} & \text{if } \omega \in (0,\tfrac12] \\[2ex] \dfrac{E(I_{(\frac12,1]}X)}{P((\frac12,1])} & \text{if } \omega \in (\tfrac12,1] \end{cases}$$

$E(X|\mathcal{A})$ is $\mathcal{A}$-measurable (it's a simple function w.r.t $\mathcal{A}$)

$E(X|\mathcal{A}) \in \mathcal{Q}(\Omega,\mathcal{F},P)$

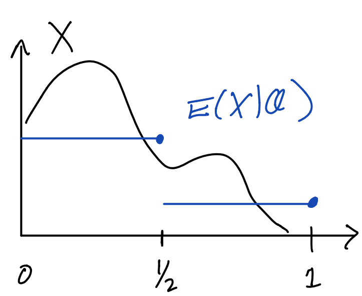
Also if $A = (0,\tfrac12]$

$$\int_A E(X|\mathcal{A})\,dP = \frac{E(I_{(0,\frac12]}X)}{\cancel{P((0,\frac12])}}\cancel{P(A)}$$

$$= \int_A X\,dP$$

and similarly for $A = \emptyset, \Omega$ or $(\tfrac12,1]$.

$\therefore$ indeed this is $E(X|\mathcal{A})$ by P-uniqueness



Example

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in \mathcal{Q}(\Omega, \mathcal{F}, P)$. Suppose

$$\{\phi, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$$

is an increasing sequence of sub-σ-fields

Then

$$\underset{\underset{E(X)}{\parallel}}{E(X|\mathcal{F}_0)}, E(X|\mathcal{F}_1), \ldots, \underset{\underset{X}{\parallel}}{E(X|\mathcal{F})}$$

$\xrightarrow{\hspace{3em}}$
increasing
resolution approx
to X

Example:

This example shows how to understand $E(X|\mathcal{A})$ as a projection when $X \in L_2(\Omega, \mathcal{F}, P)$.

Define

$$S := \{Y \in L_2(\Omega, \mathcal{F}, P) : Y \text{ is } \mathcal{A}\text{-measurable}\}$$

Notice that S a closed linear subspace of $L_2(\Omega, \mathcal{F}, P)$ by the Closure thm.

The projection $\mathcal{P}_S X$ satisfies

$$X - \mathcal{P}_S X \perp W \quad \forall W \in S$$

$$\therefore \quad E((X - \mathcal{P}_S X)W) = 0 \quad \forall W \in S$$

$$\therefore \quad E(XW) = E(\mathcal{P}_S X W) \quad \forall W \in S$$

Given $A \in \mathcal{A}$, set $W = I_A \in S$ so that

$$\int_A X dP = \int_A \mathcal{P}_S X dP$$

Since $\mathcal{P}_S X \in S \subset L_2(\Omega, \mathcal{F}, P)$ we have

$$E(X|\mathcal{A}) = \mathcal{P}_S X. \quad P\text{-a.e.}$$

Example

$\Omega = [-1, 1]$

$\mathcal{F} = \mathcal{B}([-1,1])$

$dP = \delta(x)dx$

Define $Y(w) = |w|$ on $w \in \Omega$ and

$$\mathcal{A} = \sigma\langle Y \rangle = Y^{-1}(\mathcal{F}) \subset \mathcal{F}$$

Pull back
of a single
map