Lecture 2: Classes of sets & the "Good sets" technique.

Part of the goal of this class is to get exposure to the measure theory that underpins probability theory.

This means that we will have to endure a fair amount of technical definitions etc before we can get to the good stuff.

Fix some set Ω, the sample space.

Let $2^{\Omega} :=$ power set of Ω.
$\quad = $ set of all subsets of Ω.

Def: If $\mathcal{F} \subset 2^{\Omega}$ is non-empty and
$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

Then $\mathcal{F}$ is a...

field if $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

σ-field if $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

λ-system if $\underbrace{A_1, A_2, \ldots}_{\text{disjoint}} \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

Note: Many textbooks replace the non-empty requirement by $\Omega \in \mathcal{F}$ or $\emptyset \in \mathcal{F}$ but closure under complementation makes them all equivalent definitions

Def: $\mathcal{M} \subset 2^{\Omega}$ is a monotone class
$$A_1, A_2 \ldots \in \mathcal{M} \ \& \ A_n \uparrow A \Rightarrow A \in \mathcal{M} \ \&$$
$$A_1, A_2, \ldots \in \mathcal{M} \ \& \ A_n \downarrow A \Rightarrow A \in \mathcal{M}$$

where $A_n \uparrow A$ means $\begin{cases} A_1 \subset A_2 \subset \cdots \\ A = \bigcup_{k=1}^{\infty} A_k \end{cases}$

and $A_n \downarrow A$ means $\begin{cases} A_1 \supset A_2 \supset \cdots \\ A = \bigcap_{k=1}^{\infty} A_k \end{cases}$.

Note: we will sometimes write $\lim_n \uparrow A_n$ & $\lim_n \downarrow A_n$ for the monotonic limits above.

Def: $\mathcal{P} \subset 2^{\Omega}$ is a π-system if
$$A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}.$$

Overview of important Results.

1) $\sigma = \lambda + \pi = f + \mathcal{M}$ — a class that is a field & a monotonic

2) $\sigma\langle\pi\rangle = \lambda\langle\pi\rangle$ (Dynkin's π-λ Thm)

3) $\sigma\langle f\rangle = \mathcal{M}\langle f\rangle$ (Halmo's monotone class thm)

4) if probs $P = Q$ on π then $P = Q$ on $\sigma\langle\pi\rangle$.

The monotonone class generated by a field. useful for Caratheodory Thm

Lets look at some examples first.

Examples and observations

e.g. $\left(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}\right] \uparrow \left(\frac{1}{2}, \frac{1}{4}\right)$ since $\left(\frac{1}{2}, \frac{1}{4}\right) = \bigcup_{n=1}^{\infty} \underbrace{\left(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}\right]}_{\text{increase as sets.}}$

$\left(\frac{1}{4} - \frac{1}{n}, \frac{1}{4} + \frac{1}{n}\right] \downarrow \left\{\frac{1}{4}\right\} \ldots$

e.g. $\Omega = (0, 1]$

$\mathcal{F}$ = Finite disjoint unions of intervals $(a, b] \subset \Omega$

$\mathcal{F}$ is a field but not a σ-field since

$\left(\frac{1}{2}, \frac{1}{4}\right) \notin \mathcal{F}$ but $\left(\frac{1}{2}, \frac{1}{4}\right) = \bigcup_{n=1}^{\infty} \underbrace{\left(\frac{1}{4}, \frac{1}{2} - \frac{1}{n}\right]}_{\in \mathcal{F}}$

e.g. 2^{Ω} satisfies the conditions of all 5 defs.

... so they are not vacuous.

Note: fields & σ-fields are closed under

$A \cap B = (A^c \cup B^c)^c$

$A - B = A \cap B^c$

$A \Delta B = (A - B) \cup (B - A)$

Note: λ-systems are not necessarily closed under intersection.

e.g. $\Omega = \{1, 2, 3, 4\}$

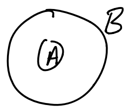
$\mathcal{F} = \{\{1,2\}, \{2,3\}, \{1,4\}, \{3,4\}, \emptyset, \Omega\}$

is a λ-sys but $\{2\} = \{1,2\} \cap \{2,3\} \notin \mathcal{F}$.

Note: λ-systems are closed under nested set subtraction.

i.e. if $\mathcal{F}$ is a λ-system then

$$A, B \in \mathcal{F} \ \& \ A \subset B \implies B - A = B \cap A^c = (\underbrace{B^c \cup A}_{\text{disjoint}})^c \in \mathcal{F}.$$



Note: For any $A_1, A_2, \dots \in 2^\Omega$

$$\bigcup_{k=1}^{n} A_k \uparrow \bigcup_{k=1}^{\infty} A_k$$

$$\bigcap_{k=1}^{n} A_k \downarrow \bigcap_{k=1}^{\infty} A_k.$$

$\sigma = \lambda + \pi = f + m$

Thm: If $\mathcal{F} \subset 2^\Omega$ then

$\mathcal{F}$ is a σ-field $\iff$ $\mathcal{F}$ is a λ-system & a π-system

$\iff$ $\mathcal{F}$ is a field & a monotone class.

Proof:

$(\sigma \Rightarrow \lambda+\pi)$: Trivial

$(\sigma \Leftarrow \lambda+\pi)$: Suppose $\mathcal{F}$ is a λ-sys & a π-sys.

All we need to show is that $\mathcal{F}$ is closed under countable non-disjoint unions.

We use a trick that we'll use later.

Let $A_1, A_2, \dots \in \mathcal{F}$.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n - \underbrace{(A_1 \cup \dots \cup A_{n-1})}_{\text{only add in unique points} \implies \text{these are disjoint.}}$$

$$= \bigcup_{n=1}^{\infty} A_n \cap A_1^c \cap \dots \cap A_{n-1}^c \in \mathcal{F}.$$

in $\mathcal{F}$ by λ-sys.

in $\mathcal{F}$ by π-sys.

$(\sigma \Rightarrow f+m)$: Suppose $\mathcal{F}$ is a σ-field

Clearly $\mathcal{F}$ is a field.

Let $A_1, A_2, \dots \in \mathcal{F}$

$$A_n \uparrow A \implies A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \text{ by } \sigma\text{-field props}$$

$$A_n \downarrow A \implies A = \bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$$

$(\sigma \Leftarrow f+m)$: All we need to show is

$$A_1, A_2, \dots \in \mathcal{F} \implies \underbrace{\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}}$$

follows by $\bigcup_{n=1}^{N} A_n \uparrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ by m props.

QED.

Generators

Let $\mathcal{C} \subset 2^\Omega$ & define

$$\sigma\langle\mathcal{C}\rangle = \sigma\text{-field generated by } \mathcal{C} := \bigcap_{\substack{\mathcal{F} \text{ is a } \sigma\text{-field} \\ \mathcal{C} \subset \mathcal{F}}} \mathcal{F}$$

Define $f\langle\mathcal{C}\rangle$, $\lambda\langle\mathcal{C}\rangle$, $m\langle\mathcal{C}\rangle$ similarly.

Thm: $\sigma\langle\mathcal{C}\rangle$ is a σ-field containing $\mathcal{C}$.

Proof:

- $\mathcal{C} \subset \sigma\langle\mathcal{C}\rangle$ since $\exists$ a σ-field $\mathcal{F}$ s.t. $\mathcal{C} \subset \mathcal{F}$ i.e. $\mathcal{F} = 2^\Omega$
- $\Omega \in \mathcal{F}$, $\forall$ σ-fields $\mathcal{F}$ so $\sigma\langle\mathcal{C}\rangle$ is not empty
- $A \in \sigma\langle\mathcal{C}\rangle \implies A \in \mathcal{F}$, $\forall$ such $\mathcal{F}$
  $\implies A^c \in \mathcal{F}$, $\forall$ such $\mathcal{F}$
  $\implies A^c \in \sigma\langle\mathcal{C}\rangle$.
- countable additivity is similar QED.

Same goes for $\lambda\langle\mathcal{C}\rangle$, $f\langle\mathcal{C}\rangle$ & $\mathcal{M}\langle\mathcal{C}\rangle$. 5

e.g. $\mathcal{B}_0^{(0,1]}$ = finite disjoint unions of sets $(a,b] \subset (0,1]$.

$\mathcal{B}^{(0,1]} = \sigma\langle \mathcal{B}_0^{(0,1]}\rangle$ is called the Borel σ-field of $(0,1]$

$\mathcal{B}^{(0,1]}$ is very rich. It contains all closed, open, one point, countable sets.

Also note, $\underbrace{N}_{\text{set of Normal numbers.}} \in \mathcal{B}^{(0,1]}$

To see why:

$$\omega \in N \iff \lim_n \frac{S_n(\omega)}{n} = 0$$

$$\iff \forall k \, \exists m \text{ s.t. } \forall n \geq m \quad \left| \frac{S_n(\omega)}{n} \right| < \frac{1}{k}$$

$$\iff \omega \in \bigcap_k \bigcup_m \bigcap_{n \geq m} \underbrace{\left\{ \left| \frac{S_n}{n} \right| < \frac{1}{k} \right\}}_{\in \mathcal{B}_0^{(0,1]}}$$

Important Fact: There is no simple recipe for general $A \in \sigma\langle\mathcal{C}\rangle$, i.e. $\exists A \in \sigma\langle\mathcal{C}\rangle$ s.t.

$$A \neq \bigcup_{n=1}^{\infty} G_n$$

for any choice of $G_n \in \mathcal{C}$.

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This makes it hard to prove things about $\sigma\langle\mathcal{C}\rangle$. Here is the main tool you can use.

Suppose you want to show each $A \in \sigma\langle\mathcal{C}\rangle$ satisfies some property.

Let $\mathcal{G} \subset 2^{\Omega}$ be all sets that have this property ("the good sets").

Thm: If $\underbrace{\mathcal{C} \subset \mathcal{G}}_{\text{The generators are "good"}}$ & $\underbrace{\mathcal{G} \text{ is a } \sigma\text{-field}}_{\text{The good sets have enough closure properties}}$

then $\underbrace{\sigma\langle\mathcal{C}\rangle \subset \mathcal{G}}_{\text{everything in } \sigma\langle\mathcal{C}\rangle \text{ is "good".}}$

Proof: $\sigma\langle\mathcal{C}\rangle = \bigcap_{\left\{ \substack{\mathcal{F} \text{ is a } \sigma\text{-field} \\ \mathcal{C} \subset \mathcal{F}} \right\}} \mathcal{F}$

$\mathcal{G}$ is one of these.

QED.

Lets finish with an easy example of this technique.

Next time we will use it to full effect to prove Dynkin's π-λ Thm.

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Thm: Let Ω be a sample space

$\Omega_0 \subset \Omega$

$\mathcal{C} \subset 2^{\Omega}$.

a σ-field of Ω_0 sets

Then $\sigma\langle \mathcal{C} \cap \Omega_0 \rangle = \sigma\langle \mathcal{C} \rangle \cap \Omega_0$.

where $\mathcal{F} \cap \Omega_0 := \{ F \cap \Omega_0 : F \in \mathcal{F} \}$

Proof:

$\underline{\sigma\langle \mathcal{C} \cap \Omega_0 \rangle \subset \sigma\langle \mathcal{C} \rangle \cap \Omega_0}$:

Follows by good sets since

$$\underbrace{\mathcal{C} \cap \Omega_0}_{\text{generators}} \subset \underbrace{\sigma\langle \mathcal{C} \rangle \cap \Omega_0}_{\text{good sets.}}$$

good sets. Form a σ-field by an exercise

$\therefore \sigma\langle \mathcal{C} \cap \Omega_0 \rangle \subset \sigma\langle \mathcal{C} \rangle \cap \Omega_0$ by good sets.

$\underline{\sigma\langle \mathcal{C} \rangle \cap \Omega_0 \subset \sigma\langle \mathcal{C} \cap \Omega_0 \rangle}$:

Let $\mathcal{G} \subset 2^{\Omega}$ include all sets s.t.

$\mathcal{G} \cap \Omega_0 \subset \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$, i.e. $G \in \mathcal{G}$ iff $G \cap \Omega_0 \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$

Clearly $\mathcal{C} \subset \mathcal{G}$. Now just show $\mathcal{G}$ is a σ-field.

✓ $\Omega \in \mathcal{G}$ since $\sigma\langle \mathcal{C} \cap \Omega_0 \rangle$ is a σ-field on Ω_0 & $\Omega \cap \Omega_0 = \Omega_0 \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$

✓ $A \in \mathcal{G} \Rightarrow A \cap \Omega_0 \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$

$\Rightarrow \underbrace{A^c \cap \Omega_0}_{\text{Complement of } A \text{ in } \Omega_0} \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$

$\Rightarrow A^c \in \mathcal{G}$

✓ $A_1, A_2, \ldots \in \mathcal{G} \Rightarrow A_k \cap \Omega_0 \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle, \forall k$

$\Rightarrow \left[\bigcup_{k=1}^{\infty} A_k \right] \cap \Omega_0 \in \sigma\langle \mathcal{C} \cap \Omega_0 \rangle$ ← can add these

$\Rightarrow \bigcup_{k=1}^{\infty} A_n \in \mathcal{G}$ Q.E.D.

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