

Lecture 9: σ -fields generated by functions. The structure thm. (1)
Applications to R.V.s

σ -fields generated by functions or r.v.s are extremely useful for cleaning up & generalizing some of the stuff we did for the coin flip model & also allow us to define conditional expected value etc.

e.g. in previous lectures we said things like $\{s_n - s_p > c\}$ is indep of $\{s_p > c\}$ $\in \sigma\langle H_1, \dots, H_p \rangle$

... while true it is a bit annoying & implicitly due to facts like:

$$\{s_p > c\} = \bigcup_{\substack{r_1, \dots, r_p \in \{-1, 1\} \\ \text{s.t. } r_1 + \dots + r_p = c}} \{R_1 = r_1\} \cap \dots \cap \{R_p = r_p\} \in \sigma\langle H_1, \dots, H_p \rangle$$

↑
countable

which are not very generalizable.

e.g. Recall "just check the coords": $\vec{f} = (f_1, \dots, f_d)$

$$(\mathcal{R}, \mathcal{F}) \xrightarrow{\vec{f} @} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \quad \text{iff} \quad (\mathcal{R}, \mathcal{F}) \xrightarrow{f_i @} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \quad \forall i$$

appears to use $\mathcal{B}(\mathbb{R}^d)$ as the natural σ -field on $\mathbb{R}^d = \mathbb{R} \times \dots \times \mathbb{R}$. What about when f_i maps into $(\mathcal{R}_i, \mathcal{F}_i)$... what is the σ -field on $\mathcal{R}_1 \times \dots \times \mathcal{R}_n$?

e.g. it would be nice if a r.v. Y (2)

which satisfied $\{Y \leq c\} \in \sigma\langle H_1, \dots, H_p \rangle$

it could be shown to be a function of R_1, \dots, R_p i.e. $\exists g @ \mathcal{B}(\mathbb{R}^p)/\mathcal{B}(\mathbb{R})$
s.t. $Y = g(R_1, \dots, R_p)$

e.g. we want to extend the notion of independence to non-discrete R.V.s, i.e. if B_t is a Brownian motion conclude that

$B_t, t < t_0$ is indep of B_{t_0} given B_{t_0} .

Basic definition: $\sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I} \rangle$

Let \mathcal{I} be a general index set (any cardinality allowed).

Let $(\mathcal{R}_i, \mathcal{F}_i)$ be a measurable space, $i \in \mathcal{I}$.

Let $f_i : \mathcal{R} \rightarrow \mathcal{R}_i, \forall i \in \mathcal{I}$

$$(\mathcal{R}, \mathcal{F}) \xrightarrow{f_i @} (\mathcal{R}_i, \mathcal{F}_i) \quad \vdots \quad f_k \xrightarrow{} (\mathcal{R}_k, \mathcal{F}_k)$$

Dif: $\sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I} \rangle$

$$= \sigma\langle f_i : i \in \mathcal{I} \rangle$$

when \mathcal{F}_i is implicit

$$:= \bigcap_{\sigma\text{-fields } \mathcal{G} \text{ on } \mathcal{R} \text{ s.t. } f_i @ \mathcal{G}/\mathcal{F}_i, \forall i \in \mathcal{I}} \mathcal{G}$$

= smallest σ -field on \mathcal{R} making all the f_i 's measurable.

Thm: $\sigma\langle f_i, \mathcal{F}_i \rangle = f_i^{-1}(\mathcal{F}_i)$ (3)

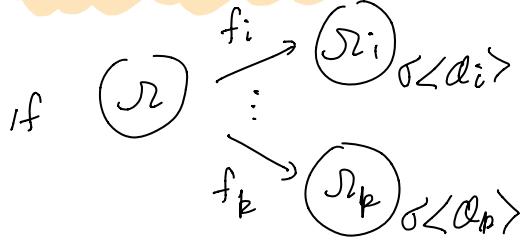
$\underbrace{\phantom{f_i^{-1}(\mathcal{Q}_k) \subset \sigma\langle f_i, \sigma\langle \mathcal{Q}_i \rangle : i \in \mathcal{I} \rangle}}$
the pull backs
of each $F \in \mathcal{F}_i$

Warning: This only works for the σ -field generated by a single function.

Proof:

This follows easily by "good sets" & the fact that $f_i^{-1}(\mathcal{F}_i)$ is a σ -field.
QED.

Thm (Generators are enough).



then

$$\sigma\langle f_i, \sigma\langle \mathcal{Q}_i \rangle : i \in \mathcal{I} \rangle = \sigma\langle f_i^{-1}(\mathcal{Q}_i) : i \in \mathcal{I} \rangle$$

Proof:

To show \supset notice that clearly

$$f_k @ \sigma\langle f_i^{-1}(\mathcal{Q}_i) : i \in \mathcal{I} \rangle / \mathcal{Q}_k, \forall k \in \mathcal{I}$$

\therefore "check @ on generators" implies

$$f_k @ \sigma\langle f_i^{-1}(\mathcal{Q}_i) : i \in \mathcal{I} \rangle / \sigma\langle \mathcal{Q}_k \rangle, \forall k \in \mathcal{I}$$

$\therefore \sigma\langle f_i^{-1}(\mathcal{Q}_i) : i \in \mathcal{I} \rangle$ is a σ -M in the def of $\sigma\langle f_i, \sigma\langle \mathcal{Q}_i \rangle : i \in \mathcal{I} \rangle$.

To show \supset notice that clearly (4)

$f_k^{-1}(\mathcal{Q}_k) \subset \sigma\langle f_i, \sigma\langle \mathcal{Q}_i \rangle : i \in \mathcal{I} \rangle, \forall k \in \mathcal{I}$

$\therefore \underbrace{\sigma\langle f_k^{-1}(\mathcal{Q}_k) : k \in \mathcal{I} \rangle} \subset \sigma\langle f_i, \sigma\langle \mathcal{Q}_i \rangle : i \in \mathcal{I} \rangle$

since this is the "smallest" σ -field containing $f_k^{-1}(\mathcal{Q}_k), \forall k \in \mathcal{I}$

QED.

Note: This trivially implies

$$\begin{aligned} \sigma\langle f_i, \mathcal{F}_i : i \in \mathcal{I} \rangle &= \sigma\langle f_i^{-1}(\mathcal{F}_i) : i \in \mathcal{I} \rangle \\ &= \sigma\langle \sigma\langle \mathcal{F}_i \rangle : i \in \mathcal{I} \rangle \end{aligned}$$

since \mathcal{F}_i generates itself & $f_i^{-1}(\mathcal{F}_i) = \sigma\langle \mathcal{F}_i \rangle$

Product σ -field

Now we can define the natural "product σ -field" on $\Omega_1 \times \dots \times \Omega_n \times \dots$ using the "coordinate projections"

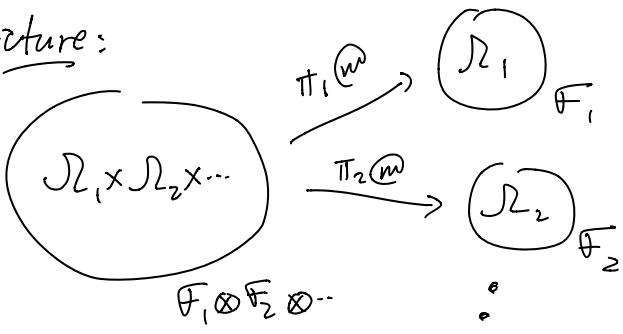
$$\pi_k(w) = w_k$$

Def: Let $(\Omega_i, \mathcal{F}_i)$ be a measurable space $\forall i \in \mathcal{I}$. Define

$$\bigotimes_{i \in \mathcal{I}} \mathcal{F}_i := \sigma\langle \pi_i, \mathcal{F}_i : i \in \mathcal{I} \rangle$$

a σ -field on $\Omega = \prod_{i \in \mathcal{I}} \Omega_i$.

Picture:

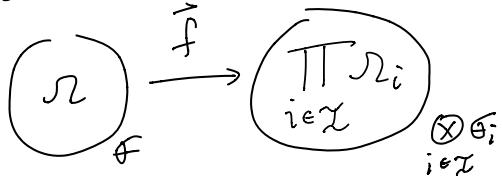


Thm (just check the coordinates)

(5)

Suppose $f_i: \Omega \rightarrow \Omega_i$ where (Ω, \mathcal{F}) and $(\Omega_i, \mathcal{F}_i)$ are measurable spaces $\forall i \in \mathbb{Z}$.

Define the vector map $\vec{f}(\omega) = (f_i(\omega))_{i \in \mathbb{Z}}$



Then $\vec{f}(\omega) \in \mathbb{F} / \bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i \Leftrightarrow f_i(\omega) \in \mathcal{F}_i \quad \forall i \in \mathbb{Z}$.

Proof:

Notice that

$$\bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i := \sigma \langle \Pi_i, \mathcal{F}_i : i \in \mathbb{Z} \rangle \\ = \sigma \langle \Pi_i^{-1}(\mathcal{F}_i) : i \in \mathbb{Z} \rangle$$

∴

$$\vec{f}(\omega) \in \mathbb{F} / \bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i \Leftrightarrow \vec{f}(\omega) \in \mathbb{F} / \left\{ \Pi_i^{-1}(\mathcal{F}_i) : \mathcal{F}_i \in \mathbb{F}_i, i \in \mathbb{Z} \right\}$$

by "generators are enough"

$$\Leftrightarrow \underbrace{\vec{f}^{-1}(\Pi_i^{-1}(\mathcal{F}_i))}_{\in \mathbb{F}, \forall \mathcal{F}_i \in \mathbb{F}_i, i \in \mathbb{Z}} \in \mathbb{F}$$

$$= (\Pi_i \circ \vec{f})^{-1}(\mathcal{F}_i)$$

$$= f_i^{-1}(\mathcal{F}_i)$$

$$\Leftrightarrow f_i(\omega) \in \mathcal{F}_i \quad \forall i \in \mathbb{Z}.$$

QED

Remark: $\bigotimes_{k=1}^d \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^d)$

and $\bigotimes_{k=1}^\infty \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^\infty)$ where $\mathcal{B}(\mathbb{R}^\infty)$ is defined with metric

$$d((x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty) := \sum_{k=1}^\infty 2^{-k} (|x_k - y_k| \wedge 1)$$

Remark: The previous Thm implies

$$\vec{f}^{-1} \left(\bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i \right) = \sigma \langle \vec{f}, \bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i \rangle = \sigma \langle f_i, \mathcal{F}_i : i \in \mathbb{Z} \rangle$$

single map pullback by "good sets" since $\sigma \langle \vec{f}, \bigotimes_{i \in \mathbb{Z}} \mathcal{F}_i \rangle$ makes each $f_i(\omega)$ and $\sigma \langle f_i, \mathcal{F}_i : i \in \mathbb{Z} \rangle$ makes $\vec{f}(\omega)$

What it means for Y to be

(6) $\sigma \langle X_1, \dots, X_n \rangle / \mathcal{B}(\mathbb{R})$ & the structure Thm

To motivate this result consider

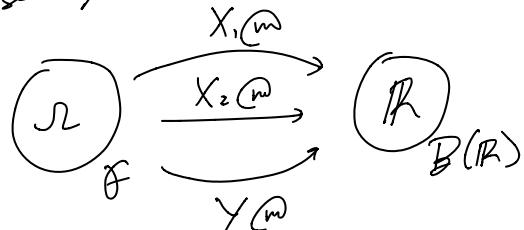
$$\Omega = (0, 1]$$

$$\mathcal{F} = \mathcal{B}((0, 1])$$

$$X_1(\omega) = I_{(0, \frac{1}{2})}(\omega)$$

$$X_2(\omega) = I_{(\frac{1}{2}, 1]}(\omega)$$

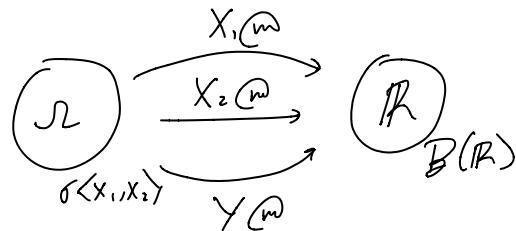
Suppose $Y: \Omega \rightarrow \mathbb{R}$ is another r.v. on Ω



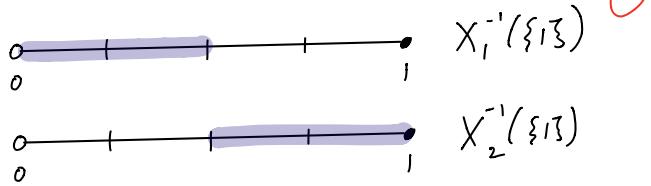
which additionally satisfies

$$Y \in \sigma \langle X_1, X_2 \rangle / \mathcal{B}(\mathbb{R})$$

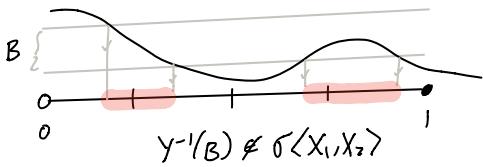
so that



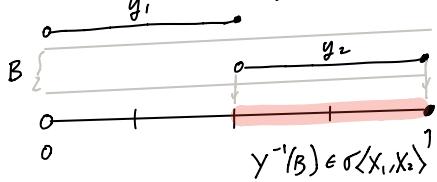
Notice that $\sigma(X_1, X_2)$ contains \emptyset , \mathcal{I} & $\textcircled{7}$



$\therefore Y$ can't look like



In fact Y must only look like



$$\begin{aligned} \text{i.e. } Y(w) &= y_1 I_{\{X_1 = 1\}}(w) + y_2 I_{\{X_2 = 1\}}(w) \\ &= y_1 I_{\{\xi_1\}}(X_1(w)) + y_2 I_{\{\xi_2\}}(X_2(w)) \\ &= g(X_1, X_2) \\ &\curvearrowleft g \text{ is } \cap B(R)/B(R) \end{aligned}$$

This holds in complete generality.

e.g. Y, X_1, X_2, \dots are r.v.s on (Ω, \mathcal{F}, P) . Then

$$Y \in \sigma(X_1, X_2, \dots) \Leftrightarrow Y = g(X_1, X_2, \dots) \text{ where } g \in B(R^\infty)/B(R)$$

↑ also extends to uncountable collections

$X_i, i \in \mathbb{Z}$.

To prove this we need an important theorem $\textcircled{8}$
which is also used for defining $\int f(w) d\mu(w)$
when $f \in \mathcal{F}/B(\mathbb{R})$.

Def: $f: \Omega \rightarrow \mathbb{R}$ is a simple function if $\text{range}(f)$ is a finite set & $f \in \mathcal{F}/B(\mathbb{R})$.

Thm:

Suppose $f: \Omega \rightarrow \mathbb{R}$ is $\cap \mathcal{F}/B(\mathbb{R})$ where (Ω, \mathcal{F}) is a measurable space. Then

f is a simple function iff $f = \sum_{k=1}^n c_k I_{A_k}$
where $n < \infty$, $c_k \in \mathbb{R}$, $A_1, A_2, \dots, A_n \in \mathcal{F}$ are
disjoint & $\Omega = \bigcup_{k=1}^n A_k$.

Proof:

\Leftarrow : Clearly $f: \Omega \rightarrow \mathbb{R}$ & the range of f is finite. To see why $f \in \mathcal{F}/B(\mathbb{R})$
let $B \in B(\mathbb{R})$ and note:

$$f^{-1}(B) = \bigcup_{\substack{k \text{ s.t.} \\ c_k \in B}} A_k \in \mathcal{F}$$

since $A_k \in \mathcal{F}$

$\therefore f$ is simple

\Rightarrow : Suppose f is simple.
Let $\underbrace{\{c_1, c_2, \dots, c_n\}}_{\text{unique}} = \text{range}(f)$

$$\text{Define } A_k := \{w : f(w) = c_k\}.$$

$\therefore A_k$'s are disjoint since c_k 's are unique.

$A_k \in \mathcal{F}$ since $f \in \mathcal{F}/B(\mathbb{R})$ & $\{c_k\} \in B(\mathbb{R})$

$$\Omega = f^{-1}(\{c_1, \dots, c_n\}) = \bigcup_{k=1}^n A_k$$

QED.

Def: Let (Ω, \mathcal{F}) be a measurable space. (9)

Let

$\eta_S :=$ all non-negative simple functions on Ω

$\eta :=$ all non-negative $\mathbb{B}(\mathbb{R})$ functions on Ω .

Thm (Structure theorem)

Let (Ω, \mathcal{F}) be a measurable space &

$f: \Omega \rightarrow \mathbb{R}$. For each $n=1, 2, \dots$ define

$$f_n(\omega) := \begin{cases} \lfloor 2^n f(\omega) \rfloor 2^{-n} & \text{if } -n \leq f(\omega) \leq n \\ n & \text{if } f(\omega) \geq n \\ -n & \text{if } f(\omega) \leq -n \end{cases}$$

Then

(i) If $f \in \mathbb{B}(\mathbb{R})$ then

$$\underbrace{f_n(\omega)}_{\text{bdd \& simple}} \rightarrow f(\omega) \text{ as } n \rightarrow \infty, \forall \omega \in \Omega$$

(ii) If $f \in \mathbb{B}(\mathbb{R})$ and bdd then

$$\sup_{n \in \mathbb{N}} |\underbrace{f_n(\omega)}_{\text{bdd \& simple}} - f(\omega)| \rightarrow 0 \text{ as } n \rightarrow \infty, \forall \omega \in \Omega$$

(iii) If $f \in \eta$ then

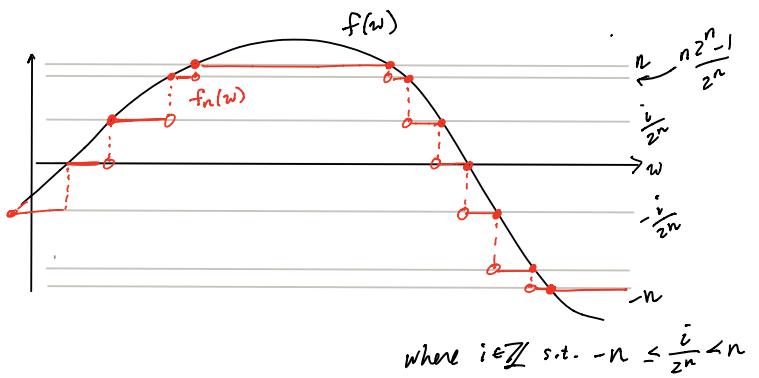
$$\underbrace{f_n(\omega)}_{\text{bdd \& in } \eta_S} \uparrow f(\omega) \text{ as } n \rightarrow \infty, \forall \omega \in \Omega$$

Proof:

f_n is clearly bdd.

If $f \in \mathbb{B}(\mathbb{R})$ then f_n is a simple function since f_n has finite range and $f_n \in \mathbb{B}(\mathbb{R})$ by cut & paste & composition of $\mathbb{B}(\mathbb{R})$.

Here is the picture: (10)



$$\text{where } i \in \mathbb{Z} \text{ s.t. } -n \leq \frac{i}{2^n} < n$$

Notice:

• if $f(w) < \infty$ then for large n

$$|f(w) - f_n(w)| \leq \frac{1}{2^n}$$

• if $f(w) = \infty$ then

$$n = f_n(w) \rightarrow f(w) \text{ as } n \rightarrow \infty$$

• if $f(w) = -\infty$ then

$$-n = f_n(w) \rightarrow f(w) \text{ as } n \rightarrow \infty$$

\therefore (i) & (ii) holds.

Finally if $f \in \eta$ then the fact that $\left\{ \frac{i}{2^n} : i \in \mathbb{Z} \right\} \subset \left\{ \frac{i}{2^{n+1}} : i \in \mathbb{Z} \right\}$ implies

$$f_n(\omega) \leq f_{n+1}(\omega)$$

which proves (iii). QED

Now we have the tools to prove implications of $Y \subseteq \sigma(X_1, X_2, \dots)$.

Thm (Characterizing σ functions of $\mathbb{B}(\mathbb{R})$ functions)

Let Y, X_1, X_2, \dots be r.v.s defined on a measurable space (Ω, \mathcal{F}) . Then the following statements are equivalent:

(i) $Y \in \sigma(X_1, X_2, \dots) / \mathbb{B}(\mathbb{R})$

(ii) There exists a $g: \mathbb{R}^\infty \rightarrow \mathbb{R}$ s.t.

$g \in \mathbb{B}(\mathbb{R}^\infty) / \mathbb{B}(\mathbb{R})$ and

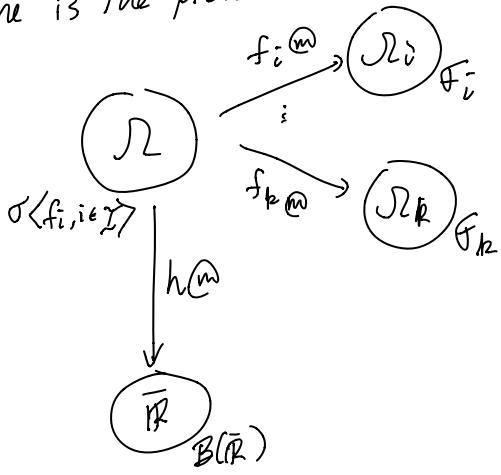
$$Y = g(X_1, X_2, \dots)$$

Note the fully general measure theoretic
thm also holds & reads: (11)

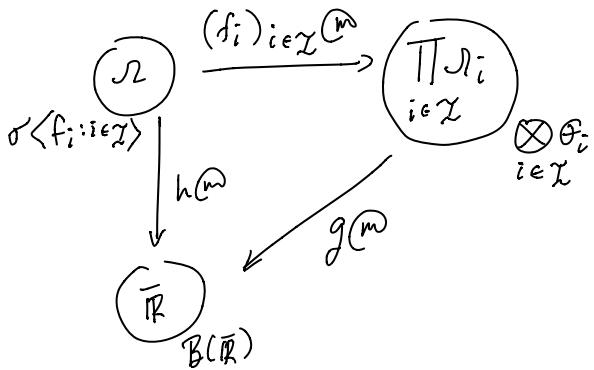
Thm: Let (Ω, \mathcal{F}) be a measurable space
and \mathcal{I} be an arbitrary index set.
For each $i \in \mathcal{I}$ suppose $f_i: \Omega \rightarrow \mathcal{F}_i$
where $(\mathcal{F}_i, \mathcal{B}_i)$ is a measure space &
 $f_i(\omega) \in \mathcal{F}_i$. Let $h: \Omega \rightarrow \bar{\mathbb{R}}$ be
 $h(\omega) = f_i(\omega)$. Then the following are equiv:
(i) $h \in \sigma(f_i: i \in \mathcal{I})$

(ii) there exists a function $g: \prod_{i \in \mathcal{I}} \mathcal{F}_i \rightarrow \bar{\mathbb{R}}$ s.t.
 $g \in \sigma(\bigotimes_{i \in \mathcal{I}} \mathcal{F}_i / \mathcal{B}(\bar{\mathbb{R}}))$ s.t.
 $h(\omega) = g((f_i(\omega))_{i \in \mathcal{I}})$.

Here is the picture



iff $\exists g$ s.t. the following commutes



The proofs are exactly the same so
lets do the less general one. (12)

Proof:

(ii) \Rightarrow (i): follows simply by composition
of (i) functions is (ii).

(i) \Rightarrow (ii): Suppose $Y \in \sigma(X_1, X_2, \dots) / \mathcal{B}(\bar{\mathbb{R}})$.

Case 1: Y is a simple function

$$\therefore Y(w) = \sum_{k=1}^n c_k \mathbf{1}_{A_k}(w) \text{ where } A_k \in \sigma(X_1, X_2, \dots)$$

Let $\vec{X}(w) := (X_1(w), X_2(w), \dots)$ & recall that
 $\vec{X} \in \sigma(X_1, X_2, \dots) / \mathcal{B}(\bar{\mathbb{R}}^\infty)$ by "just check the coords"

$$A_k \in \sigma(X_1, X_2, \dots) = \vec{X}^{-1}(\mathcal{B}(\bar{\mathbb{R}}^\infty))$$

\therefore each $A_k = \vec{X}^{-1}(B_k)$ for $B_k \in \mathcal{B}(\bar{\mathbb{R}}^\infty)$.

$$\text{Now } Y(w) = \sum_{k=1}^n c_k \mathbf{1}_{\vec{X}^{-1}(B_k)}(w)$$

$$= \sum_{k=1}^n c_k \mathbf{1}_{B_k}(\vec{X}(w))$$

$$= \underbrace{\sum_{k=1}^n c_k \mathbf{1}_{B_k}(X_1, X_2, \dots)}$$

$= g$ which is clearly
 $\in \mathcal{B}(\bar{\mathbb{R}}^\infty) / \mathcal{B}(\bar{\mathbb{R}})$.

Case 2: Y is not simple (... but $\in \sigma(X_1, X_2, \dots) / \mathcal{B}(\bar{\mathbb{R}})$)

By the structure thm \exists simple $Y_n \in \sigma(X_1, X_2, \dots)$

s.t. $Y_n(w) \rightarrow Y(w)$ as $n \rightarrow \infty$ & $w \in \Omega$.

For each n , case 1 applies to Y_n

$\therefore \exists g_n \in \mathcal{B}(\bar{\mathbb{R}}^\infty) / \mathcal{B}(\bar{\mathbb{R}})$ s.t.

$$Y_n(w) = g_n(X_1(w), X_2(w), \dots)$$

Now it is tempting to try

(13)

$$Y(w) = \lim_n Y_n(w) = \lim_n g_n(X_1(w), X_2(w), \dots)$$

and set this
to g .

However $g_n(X_1, X_2, \dots)$ is only guaranteed to have a limit (to Y) when (X_1, X_2, \dots) is in the range of (X_1, X_2, \dots) .

This is solved by setting

$$g(\vec{x}) := \begin{cases} \lim_n g_n(\vec{x}) & \text{if } \vec{x} \in A \\ 0 & \text{if } \vec{x} \in A^c \end{cases}$$

$$\text{where } A := \left\{ \vec{x} \in \mathbb{R}^\infty : \limsup_n g_n(\vec{x}) = \liminf_n g_n(\vec{x}) \right\}$$

$\in B(\mathbb{R}^\infty)$ by closure thm & that $g_n \in B(\mathbb{R}^\infty)/B(\mathbb{R})$.

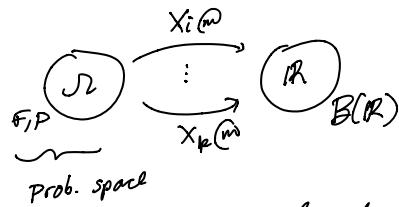
$$\text{Now } Y(w) = g(X_1(w), X_2(w), \dots)$$

QED

Independent R.V.s

(14)

Now, with the exception of expected values we have the full theory of random variables at our disposal. Here are the extensions of independence of events to independence of random variables.



where $i \in \mathbb{Z}$ a general index set -

Def: The r.v.s X_i for $i \in \mathbb{Z}$ are independent if $\sigma\langle X_i \rangle$ for $i \in \mathbb{Z}$ are independent σ -fields.

Thm (ANOVA):

Matrix of r.v.s (all defined on (Ω, \mathcal{F}, P))

$$\begin{matrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \vdots & & & \ddots \end{matrix}$$

Then the r.v.s $\{X_{ik}\}_{i,k}$ are indep

if and only if

(i) the r.v.s within each row are indep. &

(ii) the rows $R_i = \sigma\langle X_{i1}, X_{i2}, \dots \rangle$

are indep.

Proof: Just like dd ANOVA ...

noting that $\sigma\langle X_i \rangle$ are π -systems and

$$\sigma\langle X_{i1}, X_{i2}, \dots \rangle = \sigma\langle X_{i1}^{-1}(B(\mathbb{R})), X_{i2}^{-1}(B(\mathbb{R})), \dots \rangle$$

$$= \sigma\langle \sigma\langle X_{i1} \rangle, \sigma\langle X_{i2} \rangle, \dots \rangle$$

QED

Thm (existence of indep X_1, X_2, \dots)

(15)

Let P_1, P_2, \dots be prob measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
Then there exists a single prob space $(\mathcal{S}, \mathcal{F}, P)$
and r.v.s Y_i all defined on \mathcal{S} s.t.

- (i) $P_{Y_i^{-1}}(B) = P_i(B)$, $\forall B \in \mathcal{B}(\mathbb{R})$, $i = 1, 2, \dots$
- (ii) Y_1, Y_2, \dots are independent.

Proof:

Let $(\mathcal{S}, \mathcal{F}, P)$ be our old friend: "Borel's conflict model on $\mathcal{S} = [0, 1]$ ".

Let $X_k(w) = k^{\text{th}}$ binary digit of w & re-arrange them in an infinite matrix:

$$\begin{matrix} X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \vdots & \vdots & \ddots & \end{matrix} \quad \left. \begin{array}{l} \text{all indep since} \\ \sigma(X_{ik}) = \sigma(X_{j,k}) \end{array} \right\}$$

For each row i define

$$U_i(w) := \sum_{k=1}^{\infty} \frac{1}{2^k} X_{ik}(w).$$

Notice $U_i = g(X_{i1}, X_{i2}, X_{i3}, \dots)$

$$\text{where } g(\vec{x}) = \limsup_n \sum_{k=1}^n \frac{1}{2^k} \pi_k(\vec{x})$$

is $(\mathbb{R}) \mathcal{B}(\mathbb{R}^\infty) / \mathcal{B}(\bar{\mathbb{R}})$ by closure &
the fact that $\mathcal{B}(\mathbb{R}^\infty)$ makes each π_n \mathbb{R} .

$$\therefore U_i \in \sigma(\sigma(X_{i1}, X_{i2}, \dots)) / \mathcal{B}(\bar{\mathbb{R}})$$

$$\therefore \sigma(U_i) \subset \sigma(X_{i1}, X_{i2}, \dots)$$

These are indep by Axiom A

$\therefore U_1, U_2, \dots$ are independent by
sub-class Thm & each is uniform on $[0, 1]$
by Hwk 4.

Now set $Y_i := F_i^{-1}(U_i)$

where we are allowed
to modify U_i so it
is uniform on $[0, 1]$. (16)

where $F_i(x) := P_i((-\infty, x])$.

Switching lemma shows Y_i is a r.v. s.t.

$$P_{Y_i^{-1}} = P_i \text{ on } \mathcal{B}(\mathbb{R}).$$

To show the Y_i 's are independent
notice that Y_i is a function of U_i
so that $Y_i \in \sigma(U_i)$ which implies

$$\sigma(Y_i) \subset \sigma(U_i).$$

indep.

QED

Thm (Kolmogorov's 0-1 law for R.V.s)

If X_1, X_2, \dots are indep r.v.s on $(\mathcal{S}, \mathcal{F}, P)$ then
all events in $\Sigma := \bigcap_{n=1}^{\infty} \sigma(X_1, X_2, \dots, X_n)$ have

prob 0 or 1. Moreover, if Y is another
r.v. on $(\mathcal{S}, \mathcal{F}, P)$ which is $\Sigma / \mathcal{B}(\mathbb{R})$ then
 $\exists c \in \mathbb{R}$ s.t. $P(Y=c)=1$.

i.e. Y is constant
with prob 1.

Proof: Notice that

$$\Sigma = \bigcap_{n=1}^{\infty} \underbrace{\sigma}_{\text{independent }} \underbrace{\sigma(X_1), \sigma(X_2), \dots}_{\mathcal{F}-\text{systems.}}$$

$\therefore \forall A \in \Sigma, P(A) = 1 \text{ or } 0$ by "old 0-1 law".

Suppose $Y \in \Sigma$.

$$\therefore \{Y \leq c\} \in \Sigma, \forall c \in \mathbb{R}$$

$$\therefore P(Y \leq c) = 0 \text{ or } 1$$

But since $P(Y \leq c)$ is monotonic in c right continuous, $\lim_{c \rightarrow \infty} P(Y \leq c) = 1$ and $\lim_{c \rightarrow -\infty} P(Y \leq c) = 0$, there must exist a $c_0 \in \mathbb{R}$ s.t.

$$P(Y \leq c_0) = 1 \quad \& \quad P(Y < c_0) = 0$$

$$\therefore P(Y = c_0) = P(Y \leq c_0) - P(Y < c_0) = 1.$$

QED

e.g. Let X_1, X_2, \dots be indep r.v.s on (Ω, \mathcal{F}, P) .

Let $S_n = X_1 + \dots + X_n$.

Suppose a_n is any sequence of real numbers s.t. $\lim_n a_n = \infty$.

$$\text{Now } \limsup_n \frac{S_n}{a_n} = \limsup_n \frac{X_m + X_{m+1} + \dots + X_n}{a_n} \text{ for any } m$$

$\in \bigcap_{m=1}^{\infty} \sigma(X_m, X_{m+1}, \dots)$
tail σ -field of
indep r.v.s

$$\therefore \exists c \text{ s.t. } P\left(\limsup_n \frac{S_n}{a_n} = c\right) = 1.$$

In the special case $X_i = \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$

we have:

$$a_n := \frac{1}{n} \xrightarrow{\text{sum}} c = 0$$

$$a_n := \frac{1}{\sqrt{n}} \xrightarrow{\text{clt}} c = \infty$$

$$a_n := \frac{1}{\sqrt{2n \log \log n}} \xrightarrow{\text{LLN}} c = 1$$

In general, events of the form

$\left\{ \sum_{k=1}^{\infty} X_k = c \right\}$ are **not** tail events since the value of $\sum_k X_k$ depends on X_i for example.

However events of the form

$$\left\{ \sum_{k=1}^{\infty} X_k = \pm \infty \right\} \text{ or}$$

$$\left\{ \sum_{k=1}^{\infty} X_k \text{ converges} \right\}$$

are tail events, and therefore have probability 0 or 1 when the X_k 's are indep.

Kolmogorov's 3-series thm gives necessary & sufficient conditions when $P\left(\sum_{k=1}^{\infty} X_k \text{ converges}\right) = 1$ or 0 for independent X_k 's.

but we need integration before we can state it.

We can still get something out of this observation.

We used Kolmogorov's maximal inequality to show

$$P\left(\sum_{n=1}^{\infty} \frac{R_n}{n} \text{ converges}\right) = 1$$

where $R_n = \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$ are indep.

$\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}}$ was left unsolved but at

least we can now conclude

$$P\left(\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}} \text{ converges}\right) = 0 \text{ or } 1.$$

(19)

There is also a nice extension to the Hewitt-Savage 0-1 law.

Def: $f: \mathbb{R}^\infty \rightarrow \mathbb{R}$ is a symmetric function if $f \in \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$ and $f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots)$ whenever π is a permutation of N that permutes a finite number of coordinates (i.e. $\exists N$ s.t. $\pi(n)=n$ for all $n \geq N$).

Thm (Hewitt-Savage)

If X_1, X_2, \dots are iid r.v.s on (Ω, \mathcal{F}, P) then $f(X_1, X_2, \dots)$ is constant with probability 1 whenever f is a symmetric function.

Moreover, any event $A \in \mathcal{F}$, which has the form

$$(*) \quad I_A(w) = f(X_1(w), X_2(w), \dots)$$

for a symmetric function f , satisfies $P(A) = 0$ or 1.

Sketch of Proof:

If A satisfies $(*)$ then

$$A \in \sigma\langle X_1, X_2, \dots \rangle$$

$$= \sigma\left\langle \bigcup_{m=1}^{\infty} \sigma\langle X_1, \dots, X_m \rangle \right\rangle$$

This is a field.

Approximate A with $A_n \in \sigma\langle X_1, \dots, X_{m_n} \rangle$

s.t. $P(A \Delta A_n) \rightarrow 0$ as $n \rightarrow \infty$ (20)

Now notice 3 key facts

$$1) \quad A = A^{\pi_n}$$

$$2) \quad P(A_n) = P(A_n^{\pi_n})$$

$$3) \quad A_n \text{ is indep of } A_n^{\pi_n}$$

with an appropriately chosen π_n

$$\text{where } I_{A^{\pi_n}} = f(X_{\pi_n(1)}, X_{\pi_n(2)}, \dots)$$

$$\text{and } I_{A_n^{\pi_n}} = f_n(X_{\pi_n(1)}, \dots, X_{\pi_n(m_n)})$$

$$\text{where } I_{A_n} = f_n(X_1, \dots, X_{m_n})$$

existence since

$$A_n \in \sigma\langle X_1, \dots, X_{m_n} \rangle$$

QED

Recall that Hewitt-Savage 0-1 law is useful for showing things like

$$P(|S_n| \geq a_n \text{ i.o.}) = 0 \text{ or } 1$$

when $S_n = X_1 + \dots + X_n$ is a random walk

\nearrow iid \searrow

a_n



Notice that Hennitt-Savage applies (21) to more events than Kolmogorov's 0-1 law since any fail event is automatically a "symmetric event". However Hennitt-Savage requires more assumptions (that the X_i are iid).

Lets have a little more fun before we move on to integration.

Law of Pure types

In HWk 4 you studied the random series $W = (1-\theta) \sum_{n=1}^{\infty} \theta^{n-1} X_n$

where $X_n = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$ are iid.

Notice there are 3 cases

$\theta = 0 \Rightarrow W$ is concentrated on 0.

$0 < \theta < \frac{1}{2} \Rightarrow W$ is concentrated on $B \in \mathcal{B}([0,1])$ with $\mathbb{J}'(B) = 0$.

$\theta = \frac{1}{2} \Rightarrow W$ is uniform on $[0,1]$

These are "pure types"

Def: Let X be a r.v. Then

(a) X is purely atomic if

\exists a countable $B \subset \mathbb{R}$ s.t.

$$P(X^{-1}(B)) = 1$$

(b) X is purely singular if (22)

$$P(X^{-1}(\{x\}) = 0 \quad \forall x \in \mathbb{R} \quad \& \quad \exists B \in \mathcal{B}(\mathbb{R})$$

$$\text{s.t. } P(X^{-1}(B)) = 1 \quad \& \quad \mathbb{J}'(B) = 0$$

(c) X is purely absolutely continuous

$$\text{if } \forall B \in \mathcal{B}(\mathbb{R}), \quad \mathbb{J}'(B) = 0 \Rightarrow P(X^{-1}(B)) = 0.$$

(d) X is of pure type if X is purely atomic, purely singular or purely absolutely continuous.

e.g. Let U_1 & U_2 be two indep uniform r.v.s on (Ω, \mathcal{F}, P) . Then

$$X = \frac{1}{2} I_{\{U_1 \leq \frac{1}{2}\}} + U_2 I_{\{U_1 > \frac{1}{2}\}}$$

is not of pure type.

Thm (Jessen-Wintner law of pure types)

Let X_1, X_2, \dots be independent r.v.s defined on a probability space (Ω, \mathcal{F}, P) . Suppose each X_n takes its values in a countable set $C \subset \mathbb{R}$ and $\sum_{n=1}^{\infty} X_n$ converges (with prob 1) to a finite limit X .

Then X is of pure type.

Proof:

By changing the X_n 's to 0 on a set of P -prob 0 we may assume

$$X(w) = \sum_{n=1}^{\infty} X_n(w), \quad \forall w \in \Omega.$$

By assumption \exists a countable $C \subset \mathbb{R}$ s.t.

$$X_n(w) \in C, \quad \forall w \in \Omega \text{ and } \forall n \in \mathbb{N}.$$

$$\text{Let } G := \left\{ n_1 x_1 + \dots + n_k x_k : k \geq 1, x_i \in C, n_i \in \mathbb{N} \right\}$$

Notice that $\mathcal{C} \subset G$, G is countable (23) and G is closed under addition, subtraction and $G = -G$.

We will show that $\forall B \in \mathcal{B}(\mathbb{R})$,

$$P(X \in B+G) = 0 \text{ or } 1.$$

$$\text{where } B+G := \{b+g : b \in B, g \in G\}$$

$$= \bigcup_{g \in G} (B+g) \in \mathcal{B}(\mathbb{R}).$$

Now suppose $x, y \in \mathbb{R}$ s.t. $x-y \in G$,

then

$$\begin{aligned} x \in B+G &\Leftrightarrow x = b+g, \text{ for } b \in B, g \in G \\ &\Leftrightarrow y = b + \underbrace{g-x+y}_{\in G}, \text{ for } b \in B, g \in G \\ &\Leftrightarrow y \in B+G \end{aligned}$$

Since

$$X(w) - \sum_{n=m}^{\infty} X_n(w) = \sum_{n=1}^{m-1} X_n(w) \in G$$

one has

$$\begin{aligned} \{X \in B+G\} &= \left\{ \sum_{n=m}^{\infty} X_n \in B+G \right\} \\ &\in \sigma(X_m, X_{m+1}, \dots) \\ &\quad \text{↑ holds } \forall m \end{aligned}$$

By Kolmogorov's 0-1 law

$$P(X \in B+G) = 1 \text{ or } 0.$$

which holds for any $B \in \mathcal{B}(\mathbb{R})$.

Case 1: $P(X \in B+G) = 1$ for some countable set $B \in \mathcal{B}(\mathbb{R})$. (24)

$\therefore P_{X^{-1}}$ is purely atomic.

Case 2: $P(X \in B+G) = 0$, \forall countable $B \in \mathcal{B}(\mathbb{R})$ but $\exists B' \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{I}'(B') = 0$ and $P(X \in B'+G) = 1$.

Notice that

$$\mathcal{I}'(B'+G) = \mathcal{I}'\left(\bigcup_{g \in G} (B'+g)\right) \leq \sum_{g \in G} \mathcal{I}'(B'+g) = 0.$$

$\therefore P_{X^{-1}}$ is purely singular.

Case 3: $P(X \in B+G) = 0$, \forall countable $B \in \mathcal{B}(\mathbb{R})$ and $P(X \in B'+G) = 0$, $\forall B' \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{I}'(B') = 0$.

Now if $\mathcal{I}'(B) = 0$ then $\mathcal{I}'(B-G) = 0$ and therefore

$$P(X \in B) = P(X \in (B-G)+G) = 0$$

$\therefore P_{X^{-1}}$ is purely abs continuous.

QED.

e.g. we know $P\left(\sum_{n=1}^{\infty} \frac{R_n}{\sqrt{n}} \text{ converges}\right) = 1$ or 0. iid Rademacher R.V.s
if you can show it is 1 then the limit can not be of "mixed type".