# Homework 5

## Due Tuesday, November 8, 2016

We are now moving beyond our old friend, the binary digit coin flips, to more general random variable theory. As a short farewell, take a look at the paper titled *Statistical independence in probability, analysis and number theory*, by Mark Kac. Mark was a probabilist working in the 1950s who wrote the famous article *Can One Hear the Shape of a Drum* and developed much of probabilistic number theory with Erdös. In that paper, Mark develops a lot of the same theory for Rademacher functions we developed but at a time before probability theory was a major branch of mathematics. Reading this article, you can tell that Mark was being very careful to frame the probabilistic statements in measure theoretic terms.

In Homework 2 we introduced the notion of inner-regularity of a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. When working with a probability measure $P$ on a general metric space $\Omega$ it is somewhat more common to use the term *tight* as a synonym for *inner-regular*. In particular, a probability measure $P$ on $(\Omega, \mathcal{B}(\Omega))$ is **tight**, equivalently **inner-regular**, if $P(B) = \sup\{P(K) : K \subset B,\, K \text{ compact}\}$ for all $B \in \mathcal{B}(\Omega)$ where a set $K \subset \Omega$ is compact if each open cover of $K$ has a finite sub-cover. Here is another definition of a tight probability measure, equivalent for probability measures, that is more convenient for our use case.

**Definition 1.** *Suppose $\Omega$ is a metric space. A probability measure $P$ on $(\Omega, \mathcal{B}(\Omega))$ is* **tight** *if for every $\epsilon > 0$ there exits a compact $K \subset \Omega$ such that $P(K) > 1 - \epsilon$.*

The fact that this an equivalent definition follows, by similar arguments done for Exercise 6 (in Homework), since an extension of Theorem 1 (in Homework 2) holds for probability measures on $(\Omega, \mathcal{B}(\Omega))$. This new definition can now be extended to a collection of probability measures.

**Definition 2.** *Suppose $\Omega$ is a metric space and $\mathscr{P}$ is a collection of probability measures on $(\Omega, \mathcal{B}(\Omega))$. $\mathscr{P}$ is said to be* **tight** *if for every $\epsilon > 0$ there exits a compact $K \subset \Omega$ such that $P(K) > 1 - \epsilon$ for all $P \in \mathscr{P}$.*

The significance of tightness is seen in Prohorov's Theorem which says that when $\Omega$ is separable and complete (i.e. a Polish space) $\mathscr{P}$ is tight if and only if $\mathscr{P}$ is relatively compact (in the sense that for every sequence of elements in $\mathscr{P}$ there exists a subsequence which converges in distribution to a probability measure, not necessarily in $\mathscr{P}$). We will get into this later. For now, you should think of a sequence of probability measures $P_1, P_2, \ldots$ as being **tight** if there is no "mass escaping to infinity". The following exercises explores this this intuition.

**Definition 3.** *Suppose $X$ and $Y$ are two random variables where $X$ is defined on $(\Omega, F, P)$ and $Y$ is defined on $(\Omega', \mathcal{F}', Q)$. $Y$ is said to be* **stochastically larger** *than $X$ if $P[X \le x] \ge Q[Y \le x]$ for all $x \in \mathbb{R}$.*

**Exercise 1.** *Suppose $X$ and $Y$ are random variables and that $Y$ is stochastically larger than $X$. Show there exists random variables $X^*$ and $Y^*$ defined on a common probability space $(\Omega, \mathcal{F}, P)$ such that $X^* \sim X$, $Y^* \sim Y$ and $X^*(\omega) \le Y^*(\omega)$ for all $\omega \in \Omega$.*

Let $\mathcal{I}$ be an arbitrary index set and let $X_i, i \in \mathcal{I}$ be a family of random variables where each $X_i$ is defined on a probability space $(\Omega_i, \mathcal{F}_i, P_i)$. Let $F_i(x) := P_i(X_i \le x)$ be the distribution function of $X_i$. The $X_i$'s are said to be **stochastically dominated** by a random variable $X$ if $X$ is stochastically larger than $|X_i|$ for each $i \in \mathcal{I}$. The $X_i$'s are said to be **pointwise dominated** by $X$ if all the random variables $X, X_i$, for $i \in \mathcal{I}$, are defined on the same probability space and $|X_i(\omega)| \le X(\omega)$ for each $w \in \Omega$ and for each $i \in \mathcal{I}$.

**Exercise 2.** *Let $X_i, i \in \mathcal{I}$, be a family of random variables where each $X_i$ is defined on a probability space $(\Omega_i, \mathcal{F}_i, P_i)$. Show that the following are equivalent*

1. *The collection of induced measures $\left\{P_i X_i^{-1}\right\}_{i \in \mathcal{I}}$ are tight;*
2. *The $X_i$'s are stochastically dominated by some random variable;*
3. $\lim_{x \to -\infty} \sup_{i \in \mathcal{I}} F_i(x) = 0$ *and* $\lim_{y \to +\infty} \inf_{i \in \mathcal{I}} F_i(y) = 1$;
4. *There exists random variables $X_i^*$, $i \in \mathcal{I}$, all defined on a common probability space such that $X_i^* \sim X_i$ for each $i \in \mathcal{I}$ and the $X_i^*$'s are pointwise dominated by some random variable.*