# Lecture 8: Measurable functions, Random variables and distribution functions

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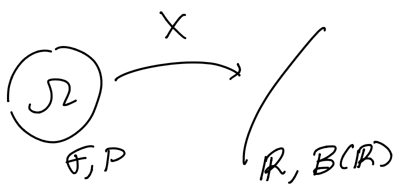
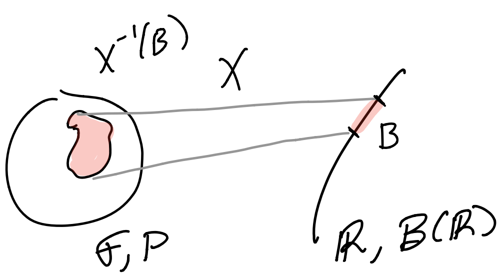
In this lecture we will start by developing the measure theoretic notion of measurable functions & define random variables $X$ as measurable functions $X : \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{F}, P)$ is a prob. space.




Measurability of $X$ is required since we want $P(X \in B)$ to be defined where $B \in \mathcal{B}(\mathbb{R})$ and

$$\{X \in B\} \equiv \{\omega \in \Omega : X(\omega) \in B\} =: \underbrace{X^{-1}(B)}_{\text{pre-image of } B \text{ under } X.}$$

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## Measurable functions

Let $(\Omega_1, \mathcal{F}_1)$ & $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces & $f : \Omega_1 \to \Omega_2$.

Def: $f$ is measurable between $\mathcal{F}_1$ & $\mathcal{F}_2$ (written $f \text{ (m) } \mathcal{F}_1/\mathcal{F}_2$ for short) iff

$$f^{-1}(A) \in \mathcal{F}_1, \quad \forall A \in \mathcal{F}_2. \qquad (*)$$

Note: It will sometimes be convinent to write $f \text{ (m) } \mathcal{F}_1/\mathcal{F}_2$ when $f$ satisfies $(*)$ even when $\mathcal{F}_1$ or $\mathcal{F}_2$ are Not $\sigma$-fields ... just collections of sets.

A few basic facts about $f^{-1}(A)$

(1) $f^{-1}(\Omega_2) = \Omega_1$ since $f$ maps into $\Omega_2$

(2) $f^{-1}(\emptyset) = \emptyset$

(3) $f^{-1}(A^c) = (f^{-1}(A))^c$

since $\omega \in f^{-1}(A^c) \iff f(\omega) \in A^c$
$\iff f(\omega) \notin A$
$\iff \omega \notin f^{-1}(A)$

(4) $f^{-1}(\bigcup_k A_k) = \bigcup_k f^{-1}(A_k)$ even $A_k$'s are not disjoint

since $\omega \in f^{-1}(\bigcup_k A_k) \iff f(\omega) \in A_k$ some $k$
$\iff \omega \in f^{-1}(A_k)$ some $k$
$\iff \omega \in \bigcup_k f^{-1}(A_k)$

Thm (Generators are enough)

If $\underset{\mathcal{F}_1}{\Omega_1} \xrightarrow{f} \underset{\sigma\langle\mathcal{A}\rangle}{\Omega_2}$ & $\mathcal{F}_1$ is a $\sigma$-field

then $f \text{ (m) } \mathcal{F}_1/\sigma\langle\mathcal{A}\rangle \iff f \text{ (m) } \mathcal{F}_1/\mathcal{A}$.

Proof:

$\Rightarrow$: trivial

$\Leftarrow$: Good sets on
$$\mathcal{G} = \{A \subset \Omega_2 : f^{-1}(A) \in \mathcal{F}_1\}.$$
$\mathcal{A} \subset \mathcal{G}$ by assumption & $\mathcal{G}$ is a $\sigma$-field by facts (1), (3), (4).

QED

e.g. $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \xrightarrow{f} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$f$ is monotone $\Rightarrow f^{-1}((-\infty,x])$ is an interval $\forall x$

show that $a,b \in f^{-1}((-\infty,x])$ & $a \le y \le b$ $\Rightarrow y \in \underbrace{f^{-1}((-\infty,x])}_{\text{could be open or closed}}$

$\Rightarrow f^{-1}((-\infty,x]) \in \mathcal{B}(\mathbb{R}), \forall x$

$\Rightarrow f$ (m) $\mathcal{B}(\mathbb{R}) / \{(-\infty,x] : x \in \mathbb{R}\}$

$\Leftrightarrow f$ (m) $\mathcal{B}(\mathbb{R}) / \underbrace{\sigma\langle (-\infty,x] : x \in \mathbb{R}\rangle}_{=\mathcal{B}(\mathbb{R})}$

$\therefore$ All monotone funs are measurable.

e.g. $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \xrightarrow{f} (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$

$f$ is continuous

$\Leftrightarrow f^{-1}(G)$ is open $\forall$ open $G \subset \mathbb{R}^k$

$\Rightarrow f$ (m) $\mathcal{B}(\mathbb{R}^d)$/opens in $\mathbb{R}^k$

$\Leftrightarrow f$ (m) $\mathcal{B}(\mathbb{R}^d) / \underbrace{\sigma\langle \text{opens in } \mathbb{R}^k \rangle}_{\mathcal{B}(\mathbb{R}^k)}$

This extends to metric spaces $\Omega_1$ & $\Omega_2$ so that $f$ (m) $\mathcal{B}(\Omega_1)/\mathcal{B}(\Omega_2)$ for all metric continuous $f : \Omega_1 \to \Omega_2$



**Thm** (composition of (m) is (m)).

If
$$\begin{tikzcd} (\Omega_1, \mathcal{F}_1) \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & (\Omega_2, \mathcal{F}_2) \arrow[r, "g"] & (\Omega_3, \mathcal{F}_3) \end{tikzcd}$$
where $f$ (m) $\mathcal{F}_1/\mathcal{F}_2$ & $g$ (m) $\mathcal{F}_2/\mathcal{F}_3$

then $g \circ f$ (m) $\mathcal{F}_1/\mathcal{F}_3$

Proof:

If $B \in \mathcal{F}_3$ then
$$(g \circ f)^{-1}(B) = f^{-1} \circ g^{-1}(B)$$
$$= \underbrace{f^{-1}(\underbrace{g^{-1}(B)}_{\in \mathcal{F}_2})}_{\in \mathcal{F}_1}$$
QED.

e.g. if $f$ (m) $\mathcal{F}/\mathcal{B}(\mathbb{R})$ then $|f|$, $f^2$, $\sin(f)$ ... are all (m) $\mathcal{F}/\mathcal{B}(\mathbb{R})$ by continuity.

Corollary (Just check the coordinates)

If $(\Omega, \mathcal{F}) \xrightarrow{f} (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$

works for $\bar{\mathbb{R}}^n$, $\mathcal{B}(\bar{\mathbb{R}}^n)$ too

where $\mathcal{F}$ is a $\sigma$-field and $f(\omega) = (f_1(\omega), \dots, f_n(\omega))$

then
$$f \text{ (m) } \mathcal{F}/\mathcal{B}(\mathbb{R}^n) \Leftrightarrow \text{each } f_k \text{ is (m) } \mathcal{F}/\mathcal{B}(\mathbb{R})$$

Proof:

$\Longrightarrow$: follows since the coordinate mappings $\pi_k(x_1,\dots,x_n) := x_k$ are continuous & $f_k = \pi_k \circ f$

$\Longleftarrow$:

$$f^{-1}\big(\underbrace{(a_1,b_1]\times\cdots\times(a_n,b_n]}_{\text{rectangle}}\big) = \underbrace{\bigcap_{k=1}^{n} \underbrace{f_k^{-1}((a_k,b_k])}_{\in \mathcal{F} \text{ since } f_k \textcircled{m} \mathcal{F}/\mathcal{B}(\mathbb{R})}}_{\in \mathcal{F}}$$

$\therefore$ $f \textcircled{m} \mathcal{F}/\text{rectangles}$

$\therefore$ $f \textcircled{m} \mathcal{F}/\underbrace{\sigma\langle \text{rectangles}\rangle}_{\mathcal{B}(\mathbb{R}^n)}$.

QED.

Thm (cut & paste over countable & $\textcircled{m}$ pieces)

If $\underset{\mathcal{F}_1}{\Omega_1} \xrightarrow{f} \underset{\mathcal{F}_2}{\Omega_2}$ where $\mathcal{F}_1$ & $\mathcal{F}_2$ are $\sigma$-fields and

$$\Omega_1 = \bigcup_{k=1}^{\infty} A_k \quad \text{s.t. } \forall k,\ A_k \in \mathcal{F}_1$$

then

$$f \textcircled{m} \mathcal{F}_1/\mathcal{F}_2 \iff f|_{A_k} \textcircled{m} (\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$$

for all $k = 1, 2, \dots$



Proof:

$\Longrightarrow$: suppose $f \textcircled{m} \mathcal{F}_1/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$\omega \in f|_{A_k}^{-1}(B) \iff f|_{A_k}(\omega) \in B \;\&\; \omega \in A_k$$
$$\iff f(\omega) \in B \;\&\; \omega \in A_k$$

$$\therefore\ f|_{A_k}^{-1}(B) = \underbrace{A_k}_{\in \mathcal{F}_1} \cap \underbrace{f^{-1}(B)}_{\in \mathcal{F}_1} \in \mathcal{F}_1$$

$\Longleftarrow$: suppose $f|_{A_k} \textcircled{m} (\mathcal{F}_1 \cap A_k)/\mathcal{F}_2$. Let $B \in \mathcal{F}_2$.

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B) \cap \Omega_1 \\ &= f^{-1}(B) \cap \bigcup_k A_k \\ &= \bigcup_k \big(f^{-1}(B) \cap A_k\big) \\ &= \underbrace{\bigcup_k \underbrace{f|_{A_k}^{-1}(B)}_{\in \mathcal{F}_1 \cap A_k \text{ by assumption}}}_{\in \mathcal{F}_1 \text{ since } \mathcal{F}_1 \cap A_k \subset \mathcal{F}_1 \;\&\; \mathcal{F}_1 \text{ is a } \sigma\text{-field}} \end{aligned}$$

QED.

Corollary: Piecewise metric continuous functions are $\textcircled{m}$ if the "pieces" are countable & Borel measurable.

Thm: (just check $\textcircled{m}$ on the range)

If $f: \Omega_1 \to \Omega_2^{\circ} \subset \Omega_2$ ← metric space

and $\mathcal{F}_1$ is a $\sigma$-field on $\Omega_1$ then

$$f \textcircled{m} \mathcal{F}_1/\mathcal{B}(\Omega_2) \iff f \textcircled{m} \mathcal{F}_1/\mathcal{B}(\Omega_2^{\circ}).$$

**Proof:** The borel restriction thm says

$$B(\Omega_2^\circ) = B(\Omega_2)\cap\Omega_2^\circ.$$

$\therefore$

$$f \text{ ⓜ } \mathcal{F}_1/B(\Omega_2^\circ) \iff f \text{ ⓜ } \mathcal{F}_1/B(\Omega_2)\cap\Omega_2^\circ$$

$$\iff \underbrace{f^{-1}(B\cap\Omega_2^\circ)}_{=} \in \mathcal{F}_1,\ \forall B\in B(\Omega_2)$$

$$\iff f^{-1}(B)\in\mathcal{F}_1,\ \forall B\in B(\Omega_2)$$

since $f$ maps into $\Omega_2^\circ$

$$\iff f \text{ ⓜ } \mathcal{F}_1/B(\Omega_2).$$

QED.

e.g. $\sin(x) \text{ ⓜ } B(\mathbb{R})/B(\mathbb{R})$

$\iff \sin(x) \text{ ⓜ } B(\mathbb{R})/B([-1,1])$

Question:

is $f(x)=\begin{cases}\frac{1}{x}, & x>0\\ -\infty, & x=0\\ \sin(x), & x<0\end{cases}$ ⓜ $B(\mathbb{R})/B(\bar{\mathbb{R}})$

Yes since $f$ is metric continuous on countably many measurable pieces.



**Notation:**

- $f$ is "$\mathcal{F}$-measurable" if

$$(\Omega, \mathcal{F}) \xrightarrow{f \text{ ⓜ}} (\bar{\mathbb{R}}, B(\bar{\mathbb{R}}))$$

indicates $f \text{ ⓜ } \mathcal{F}/B(\bar{\mathbb{R}})$

- $f$ is "Borel measurable" if

$$(\mathbb{R}^d, B(\mathbb{R}^d)) \xrightarrow{f \text{ ⓜ}} (\bar{\mathbb{R}}, B(\bar{\mathbb{R}}))$$

sometimes just "measurable"

- $f$ is "Lebesgue measurable" if

$$(\mathbb{R}^d, \overline{B(\mathbb{R}^d)}) \xrightarrow{f \text{ ⓜ}} (\bar{\mathbb{R}}, B(\bar{\mathbb{R}}))$$

when $\overline{B(\mathbb{R}^d)}$ is the completion w.r.t. Lebesgue measure.

**Convention for $\infty$**

- $\infty + x = \infty$ when $x\in(-\infty,\infty]$
- $\infty\cdot 0 = 0$
- $\infty\cdot\infty=\infty$
- $\frac{x}{\infty} = 0$ when $x\in\mathbb{R}$
- $\frac{x}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty-\infty$ are <u>not defined</u>.

## Thm (closure thm)

Let $(\Omega, \mathcal{F})$ be a measurable space.

(i) if $\Omega \underset{g\,Ⓜ}{\overset{f\,Ⓜ}{\rightrightarrows}} \overline{\mathbb{R}}$, with $\mathcal{F}$ on $\Omega$ and $\mathcal{B}(\overline{\mathbb{R}})$ on $\overline{\mathbb{R}}$, then

$f+g,\ f\cdot g,\ f/g,\ \max(f,g),\ \min(f,g),$
$f^+,\ f^-,\ |f|$ are all Ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

provided they are defined $\forall \omega \in \Omega$

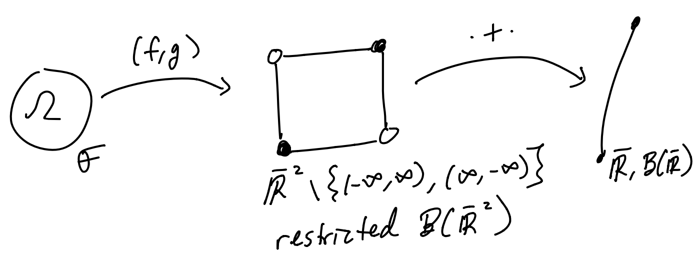
i.e. No $\infty - \infty$, $\frac{\infty}{\infty}$ ...

(ii) if $\Omega \xrightarrow{f_1\,Ⓜ} \overline{\mathbb{R}}, \ \dots,\ \Omega \xrightarrow{f_n\,Ⓜ} \overline{\mathbb{R}}, \ \dots$ with $\mathcal{F}$ on $\Omega$ and $\mathcal{B}(\overline{\mathbb{R}})$ on $\overline{\mathbb{R}}$

then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are Ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Proof:

(i) Just show $f+g$ ... the others are similar.
Since $f(\omega)+g(\omega)$ is defined $\forall \omega \in \Omega$ we have




+ is Ⓜ by metric continuity

$f$ & $g$ are Ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$

$\Leftrightarrow (f,g)$ is Ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}}^2)$ by "just check cords"

$\Leftrightarrow (f,g)$ is Ⓜ on the restricted space, by "just check the range"

$\therefore f+g$ Ⓜ $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$ by "Composition of Ⓜ is Ⓜ".



To show (ii) just notice

$$\left(\sup_n f_n\right)^{-1}([-\infty, c]) = \{\omega : \sup_n f_n(\omega) \le c\}$$
$$= \bigcap_{n=1}^{\infty} \underbrace{\{\omega : f_n(\omega) \le c\}}_{\in \mathcal{F} \text{ since } f\,Ⓜ\,\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})}$$

Note: this will not be true for $<$

$\in \mathcal{F}$ since $\mathcal{F}$ is a $\sigma$-field.

$\therefore \sup_n f_n$ Ⓜ $\mathcal{F}/\{[-\infty, c] : c \in \overline{\mathbb{R}}\}$

$\therefore \sup_n f_n$ Ⓜ $\mathcal{F}/\underbrace{\sigma\langle [-\infty, c] : c \in \overline{\mathbb{R}} \rangle}_{= \mathcal{B}(\overline{\mathbb{R}})}$.

For the others

$\inf_n f_n = -\sup_n (-f_n)$

$$\limsup_{n\to\infty} f_n = \lim_{m\to\infty} \underbrace{\sup_{n\ge m} f_n}_{\text{decreases as } m\to\infty} = \underbrace{\inf_m \underbrace{\sup_{n\ge m} f_n}_{Ⓜ}}_{Ⓜ}$$

$\liminf_{n\to\infty} f_n$ ... similar.

QED

e.g. Coin flip model from lecture 1.

$S_n(\omega) = \sum_{k=1}^{n} R_k(\omega)$ maps $(0,1] \to \overline{\mathbb{R}}$

Since $S_n$ is constant over intervals $\left(\frac{i-1}{2^n}, \frac{i}{2^n}\right]$
we have $S_n^{-1}([-\infty, x]) =$ finite disjoint union of dyatics $\in \mathcal{B}((0,1])$

By "generators are enough"

$(0,1] \xrightarrow{S_n\,Ⓜ} \overline{\mathbb{R}}$ with $\mathcal{B}((0,1])$ and $\mathcal{B}(\overline{\mathbb{R}})$, $\forall n$

$\therefore \limsup_n \frac{S_n}{\sqrt{2n \log\log n}}$ is Ⓜ $\mathcal{B}((0,1])/\mathcal{B}(\overline{\mathbb{R}})$

$\therefore \left\{\limsup_n \frac{S_n}{\sqrt{2n \log\log n}} = 1\right\} \in \mathcal{B}((0,1])$ since it is the pre-image of $\{1\} \in \mathcal{B}(\overline{\mathbb{R}})$.

## Random variables, induced measures and c.d.f.s

**Def:** $X$ is a *random variable* if there exists a probability space $(\Omega, \mathcal{F}, P)$ where $X: \Omega \to \mathbb{R}$ s.t. $X \ (m)\ \mathcal{F}/\mathcal{B}(\mathbb{R})$.

$X$ is an *extended random variable* if $X: \Omega \to \overline{\mathbb{R}}$ & $X \ (m)\ \mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})$.

Picture:

$$(\Omega, \mathcal{F}, P) \xrightarrow{X \ (m)} (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

we write $X(\omega)$ instead of $f(\omega)$ to indicate $(\Omega, \mathcal{F})$ has a probability measure attached. Think of $P$ as modeling a random draw $\omega \in \Omega$ & $X(\omega)$ as a "variable" or "label" associated with each $\omega \in \Omega$.

Since $X \ (m)\ \mathcal{F}/\mathcal{B}(\mathbb{R})$ it makes sense to talk about quantities like:

$$P(X=1) = P(\underbrace{X^{-1}(\underbrace{\{1\}}_{\in \mathcal{B}(\mathbb{R})})}_{\in \mathcal{F}})$$

$$P(X \le x) = P(\underbrace{X^{-1}((-\infty, x])}_{\in \mathcal{F}})$$

$$P(X \in \mathbb{Q}) = P(X^{-1}(\mathbb{Q})).$$



**Def:** If $X$ is a random variable defined on $(\Omega, \mathcal{F}, P)$, the *distribution* of $X$ (also called the *induced probability measure*) is a set function $PX^{-1}: \mathcal{B}(\mathbb{R}) \to [0,1]$ given by

$$PX^{-1}(B) := P(X^{-1}(B)) = P(X \in B).$$

More generally if

$$(\Omega_1, \mathcal{F}_1, \mu) \xrightarrow{f \ (m)} (\Omega_2, \mathcal{F}_2)$$

where $\mathcal{F}_1$ & $\mathcal{F}_2$ are $\sigma$-fields & $(\Omega_1, \mathcal{F}_1, \mu)$ is a measure then

$$\mu f^{-1}(F) := \mu(f^{-1}(F)), \ \forall F \in \mathcal{F}_2$$

$\uparrow$ *induced measure* on $(\Omega_2, \mathcal{F}_2)$.

**Thm:** In the setup above $\mu f^{-1}$ is a measure on $(\Omega_2, \mathcal{F}_2)$

**Proof:** Follows immediately since pre-images & set operations commute. QED

Since $\mu f^{-1}(\Omega_2) = \mu(\Omega_1)$ we have the following facts:

- $\mu(\Omega_1) = 1 \Rightarrow \mu f^{-1}(\Omega_2) = 1$

  i.e. $PX^{-1}$ is a probability measure

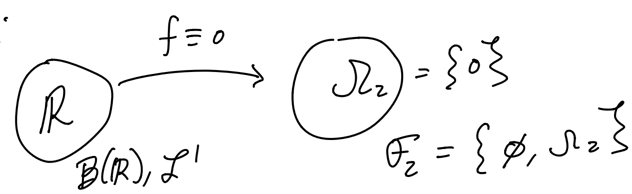
- $\mu(\Omega_1) < \infty \Rightarrow \mu f^{-1}(\Omega_2) < \infty$

- warning:

  $\mu$ is a $\sigma$-finite measure

  $\not\Rightarrow \mu f^{-1}$ is a $\sigma$-finite measure

e.g.



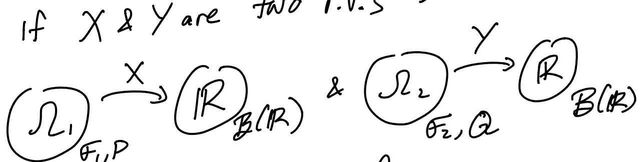

$f$ is $\mathcal{B}(\mathbb{R})/\mathcal{F}_2$ measurable and $\mathcal{L}^1$ is $\sigma$-finite but $\mathcal{L}^1 f^{-1}$ is not $\sigma$-finite since

$$\mathcal{L}^1 f^{-1}(\emptyset) = 0 \quad \text{and}$$

$$\mathcal{L}^1 f^{-1}(\Omega_2) = \mathcal{L}^1(\mathbb{R}) = \infty.$$

Notice that even if two r.v.s $X$ & $Y$ are defined on different probability spaces the induced distributions are both on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Def: If $X$ & $Y$ are two r.v.s s.t.



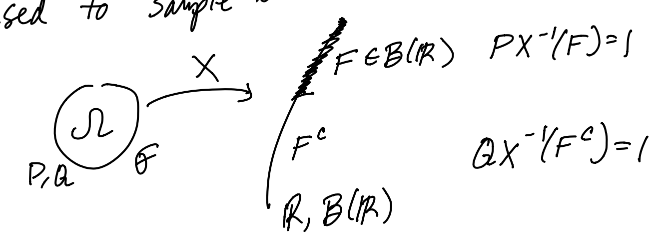

we write $X \sim Y$ or $X \overset{\mathcal{D}}{=} Y$ if $PX^{-1} = QY^{-1}$ over $\mathcal{B}(\mathbb{R})$.



In the simple statistics setup there are two possible measures $P$ & $Q$, both defined on $(\Omega, \mathcal{F})$. $\omega \in \Omega$ is picked at random either from $P$ or $Q$ & the "observer" only gets to see the value of $X(\omega)$ (i.e. the data).

The observer then trys to figure out which $P$ or $Q$ $\omega$ was sampled from.

Note: using the notation from HWK3 if $PX^{-1} \perp QX^{-1}$ then the observer will know exactly which $P$ or $Q$ was used to sample $\omega$




- If $X(\omega) \in F$ then $\omega$ was drawn from $P$.
- If $X(\omega) \in F^c$ then $\omega$ was drawn from $Q$.

Def: The distribution function (sometimes called cumulative distribution function .. c.d.f) for a random variable $X$ defined on $(\Omega, \mathcal{F}, P)$ is the function $F: \mathbb{R} \to [0,1]$ defined as

$$F(t) = P(X \le t), \quad \forall t \in \mathbb{R}.$$
$$= PX^{-1}((-\infty, t])$$

**Thm:** If $X$ & $Y$ are two r.v.s with c.d.f's $F_X$ & $F_Y$, respectively, then

$$X \overset{\mathcal{D}}{=} Y \iff F_X(t) = F_Y(t) \quad \forall t \in \mathbb{R}.$$

**Proof:**

$\Rightarrow$: trivial when taking $B = (-\infty, t]$ since $X^{-1}((-\infty,t]) = \{X \le t\}$

$\Leftarrow$:

$F_X(t) = F_Y(t) \quad \forall t \in \mathbb{R}$

$$\Rightarrow PX^{-1}((-\infty,t]) = PY^{-1}((-\infty,t]) \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow PX^{-1} = PY^{-1} \text{ on } \mathcal{P} = \{(-\infty,t] : t \in \mathbb{R}\}$$

$$\overset{\pi\text{-uniq}}{\Longrightarrow} PX^{-1} = PY^{-1} \text{ on } \sigma\langle \underbrace{\mathcal{P}}_{\pi\text{-sys}} \rangle = \mathcal{B}(\mathbb{R}).$$

QED

**Thm** (Properties of c.d.f.s)
Let $F(t) = P(X \le t)$ for a r.v. $X$ defined on $(\Omega, \mathcal{F}, P)$. Then

(I) $F(x) \le F(y), \quad \forall x \le y$

(II) $\lim_{x \downarrow y} F(x) = F(y)$

(III) $\lim_{x \to \infty} F(x) = 1$ & $\lim_{x \to -\infty} F(x) = 0$



**Proof:**

(I): clear since $A \subset B \Rightarrow PX^{-1}(A) \le PX^{-1}(B)$

(II):

$\{X \le x\} \downarrow \{X \le y\}$ as $x \downarrow y$

which follows since if $x_n \downarrow y$ as $n \to \infty$ then clearly $\{X \le y\} \subset \bigcap_{n=1}^{\infty} \{X \le x_n\}$ &
$\omega \in \bigcap_{n=1}^{\infty} \{X \le x_n\} \Rightarrow X(\omega) \le x_n \quad \forall n$
$\Rightarrow X(\omega) \le \lim_n x_n = y$

$\therefore P(X \le x) \downarrow P(X \le y)$

warning: It is not true that $\{X < x\} \downarrow \{X < y\}$ as $x \downarrow y$
since $X(\omega) < x_n \Rightarrow X(\omega) \le \lim x_n = y$, $X(\omega) < x_n \not\Rightarrow X(\omega) < \lim x_n = y$
e.g. take $X(\omega) = 0$, $x_n = \frac{1}{n}$, $y = 0$

(III): Follows since

$$\{X \le x\} \uparrow \mathbb{R} \text{ \& } \{X \le x\} \downarrow \emptyset.$$

QED.

It turns out properties (I), (II), (III) are are characterizing properties of c.d.f.s i.e. if $F: \mathbb{R} \to [0,1]$ satisfies (I), (II), (III) then $\exists$ a r.v. $X$ s.t. $F(t) = P(X \le t)$.
In fact, this will be the main tool we use to show the existance of an infinite sequence of indep. r.v.s all with a specified distribution.

**Def:** if $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) of the above thm

define

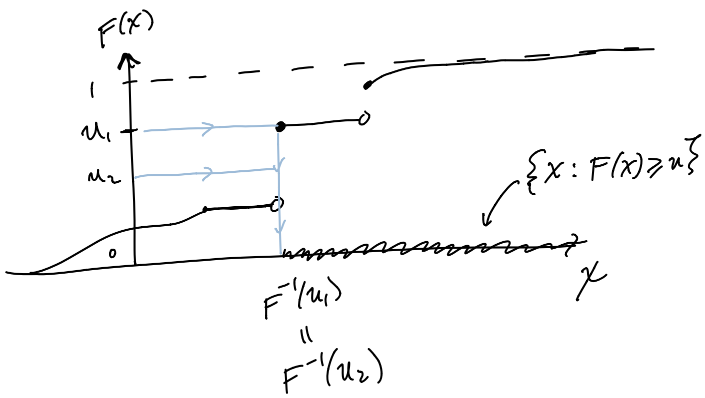
$$F^{-1}(u) := \inf\{x\in\mathbb{R} : F(x)\geq u\}$$

often called the <u>inverse c.d.f</u> or the <u>quantile function</u>.

for $u\in(0,1)$.

<u>Note:</u> $F^{-1}:(0,1)\to\mathbb{R}$ is well defined since $\{x : F(x)\geq u\}\neq\emptyset$ when $0<u<1$ by (III) & (I).

<u>Picture:</u>




There are two important facts about $F$ & $F^{-1}$ which are useful to remember.

Suppose $U$ is a r.v. which is uniform on $(0,1)$, i.e. $\exists\, (\Omega,\mathcal{F},P)$ & $\Omega_{\mathcal{F},P}\xrightarrow{U(\omega)}(0,1)_{\mathcal{B}((0,1))}$

s.t. $P(U\in B)=\mathcal{L}^1(B)$.



<u>For example</u>, digit coin flip model:

$$\Omega=(0,1],\quad \mathcal{F}=\mathcal{B}((0,1]),\quad P=\mathcal{L}^1\big|_{(0,1]}$$

where

$$U(\omega)=\begin{cases}\omega & \text{if } \omega<1\\ \tfrac12 & \text{if } \omega=\tfrac12\end{cases}$$

is (m) by cut & paste &

$$P(U\in B)=P(B)=\mathcal{L}^1(B)$$

since $P(U=\tfrac12)=0$

Now the two important facts are ... if $F$ is the c.d.f of a r.v. $X$ then

(a) $F^{-1}(U)\overset{\mathcal{D}}{=}X$

(b) $F(X)\overset{\mathcal{D}}{=}U$ when $F$ is continuous.

<u>Lemma (switching formula):</u>

If $F:\mathbb{R}\to[0,1]$ satisfies (I), (II) & (III) above then

(*) $F(x)\geq u \iff x\geq F^{-1}(u)$

$\forall u\in(0,1)$ & $x\in\mathbb{R}$.

<u>Proof:</u>

A useful restatement of (*) is simply that

(**) $\{x : F(x)\geq u\}=[F^{-1}(u),\infty)$.

To show (**) notice that

$\{x : F(x)\geq u\}$ must be an interval of the form $(F^{-1}(u),\infty)$ or $[F^{-1}(u),\infty)$ since

$$F^{-1}(u)\overset{\text{def}}{=}\inf\{x : F(x)\geq u\}$$

and

$$x'\in\{x : F(x)\geq u\}\overset{\text{by (I)}}{\Longrightarrow}\left\{x''\in\{x : F(x)\geq u\}\ \forall x''>x'.\right.$$

$\therefore$ to show $\{x : F(x) \geq u\} = [F^{-1}(u), \infty)$ just prove the inf of the LHS is attached.

Let $x_k \in \{x : F(x) \geq u\}$ s.t. $x_k \downarrow \underbrace{F^{-1}(u)}_{= \inf \text{ of LHS}}$

$$\therefore \quad F(\lim_k x_k) \overset{(II)}{=} \lim_k F(x_k) \geq u$$

since $x_k$ is in $\{x : F(x) \geq u\}$

$$\therefore \quad F^{-1}(u) = \lim_k x_k \in \{x : F(x) \geq u\}$$

so the inf is attached.

(Note we used III implicitly to show $F^{-1}(u)$ is well defined and a r.v.)

QED

## Lemma (c.d.f sandwich)

If $F : \mathbb{R} \to [0,1]$ satisfies (I), (II) & (III)

then $\forall u \in (0,1)$

$$F(F^{-1}(u)-) \leq u \leq F(F^{-1}(u)).$$

**Proof:**

$u \leq F(F^{-1}(u))$ holds since $F^{-1}(u) \in \{x : F(x) \geq u\}$.

The contrapositive of the switching formula is

$$F(x) < u \iff x < F^{-1}(u)$$

$$\therefore \quad F(F^{-1}(u)-) = \lim_{x \uparrow F^{-1}(u)} \underbrace{F(x)}_{< u} \leq u$$

these satisfy $x < F^{-1}(u)$ so $F(x) < u$ by switch

QED

Maybe the best way to remember these

$$\{x : F(x) \geq u\} = [F^{-1}(u), \infty)$$

$$F(F^{-1}(u)-) \leq u \leq \underbrace{F(F^{-1}(u)+)}_{= F(F^{-1}(u)) \text{ by (II)}}$$



## Thm (c.d.f representation)

If $F : \mathbb{R} \to [0,1]$ satisfies (I), (II) & (III)

then $\exists$ a r.v. $X$ on a prob space $(\Omega, \mathcal{F}, P)$ s.t.

(1) $\quad P(X \leq x) = F(x) \quad \forall x$

and, morover, any r.v. $\mathcal{U}$ which is uniformly distributed on $(0,1)$ satisfies

(2) $\quad F^{-1}(\mathcal{U}) \overset{\mathcal{D}}{=} X$

(3) $\quad P(F(X) \leq u) \leq u$ for all $u \in (0,1)$.

If, in addition, $F$ is continuous then

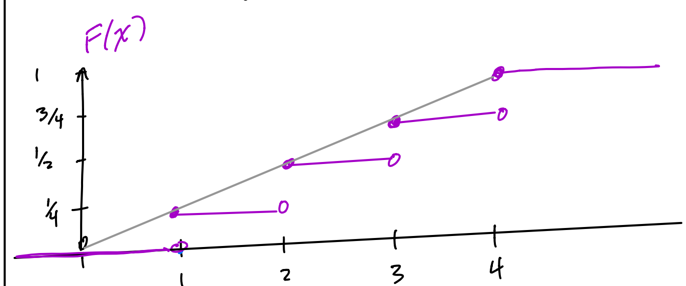
(4) $\quad F(X) \overset{\mathcal{D}}{=} \mathcal{U}$.

Here is the picture of why we don't always get (4).

Suppose $P(X = i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$

So $X$ is uniformly distributed on $\{1, 2, \ldots, 5\}$

The c.d.f of $X$ is




$\therefore$ $F(X)$ assigns $\frac{1}{4}$ prob to $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

i.e. $P(F(X) \leq 0.25) = 0.25$

but $P(F(X) \leq 0.3) = 0.25 \leq 0.3$

Proof:

Let $\mathcal{U}$ be a r.v. uniform on $(0,1)$.

$$\therefore \quad P(F^{-1}(\mathcal{U}) \leq x) = P(\mathcal{U} \leq F(x))$$

by switching lemma

$$= \mathcal{J}^1((0, F(x)))$$

$\mathcal{U}$ is uniform

$$= F(x)$$

Now set $X := F^{-1}(\mathcal{U})$ to get (1) & (2) using $\pi$-uniquess.

For (3),

$$P(F(X) \leq u) = P(F(F^{-1}(\mathcal{U})) \leq u)$$

since $X \overset{d}{=} F^{-1}(\mathcal{U})$

$$\leq P(\mathcal{U} \leq u)$$

since $\mathcal{U} \leq F(F^{-1}(\mathcal{U})) \leq u$

by c.d.f sandwich

$$= u, \quad \forall u \in (0,1)$$

If $F$ is continuous c.d.f sandwich gives $F(F^{-1}(\mathcal{U})-) = \mathcal{U} = F(F^{-1}(\mathcal{U}))$ so

$$P(F(X) \leq u) = P(\mathcal{U} \leq u)$$

which shows (4)

QED