Homework 3

Due Tuesday, October 25, 2016

The first exercise in this homework set establishes an extension to the Second Borel-Cantelli Lemma for non-independent events. Turns out there are a variety of such extensions. For example, take a look at the 1959 paper titled *On Cantor's series with convergent* $\sum \frac{1}{q_n}$ by Erdös and Rényi. In an earlier piece of work, *On the application of the Borel-Cantelli lemma* (1952), Chung and Erdös proved a somewhat related extension. It is interesting (to me) that they remark:

> The following theorem furnishes an alternative method which may be of fairly general applicability. On the other hand it does not seem to apply to the law of the iterated logarithm, etc.

In a couple of lectures we will prove the law of the iterated logarithm for our coin flip model and see just how fine an analysis is required to get the proof to go through.

Exercise 1. *Let* $A_1, A_2, \ldots$ *be* $\mathcal{F}$*-sets. Show that* $P(A_n \text{ i.o.}_n) = 1$ *if and only if* $\sum_{n=1}^\infty P(A_n|A)$ *diverges for every* $\mathcal{F}$*-set* A *of nonzero probability. Hint: show* $P(A_n \text{ i.o.}_n) < 1 \iff \sum_{n=1}^\infty P(A_n|A) < \infty$ *for some* $\mathcal{F}$*-set* A *with* $P(A) > 0$.

The next exercise explores the notion of singularity and absolute continuity for measures. Turn out this is one of the main tools in random field theory (and spatial statistics) for determining when two random field models (i.e. the measures P and Q associated with the random functions) are different enough that an observer can consistently figure out which model a particular realization came from. Obtaining these results can be highly non-trivial and somewhat surprising. For example, in a 2004 paper titled, *Inconsistent estimation and asymptotically equal interpolations in model- based geostatistics*, Zhang proved that certain (commonly used) random field models are effectively indistinguishable when observing one realization in dimension $d \leq 3$. In dimension $d > 4$ the opposite is true (see my 2010 paper titled, *On the consistent separation of scale and variance for gaussian random fields*). The case $d = 4$ is still open.

Exercise 2. *Let* P *and* Q *be probability measures on a* σ*-field* $\mathcal{F}$ *of subsets of a sample space* Ω.

- P *and* Q *are said to be* **singular**, *denoted* $P \perp Q$, *if and only if there exists a set* $F \in \mathcal{F}$ *such that*
$$P(F^c) = 0 = Q(F).$$
- P *is said to be* **absolutely continuous with respect to** Q, *denoted* $P \ll Q$, *if and only if*
$$P(F) = 0 \textit{ for every } \mathcal{F}\textit{-set } F \textit{ for which } Q(F) = 0.$$

Show that

$$P \perp Q \iff \left[\begin{array}{c} \textit{there exists } \mathcal{F}\textit{-sets } F_1, F_2, \ldots \textit{ such that} \\ P(F_n^c) \to 0 \textit{ and } Q(F_n) \to 0 \textit{ as } n \to \infty \end{array} \right]$$

and

$$P \ll Q \iff \lim_{\delta \downarrow 0} \Big(\sup\{P(F) \colon F \in \mathcal{F} \textit{ with } Q(F) \leq \delta\} \Big) = 0.$$