

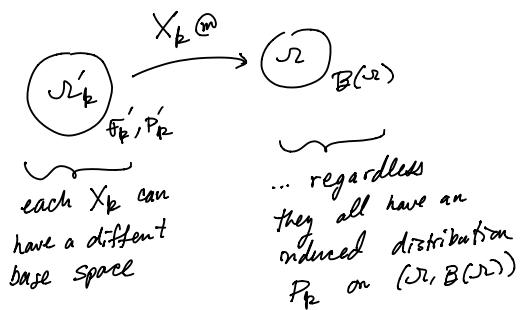
Lecture 14: Convergence in distribution

(1)

Convergence in distribution is probably the most important notion of a limit of r.v.s X_1, X_2, \dots or a sequence of probability measures P_1, P_2, \dots on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Just as in last lecture we will always assume $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is Polish w.r.t. metric d .

Let P, P_1, P_2, \dots be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and/or X, X_1, X_2, \dots a sequence of \mathbb{R} maps from some prob. space into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$



Definition:

$P_n \xrightarrow{\mathcal{D}} P$ iff $\forall f \in C_b(\mathbb{R})$, $\int_{\mathbb{R}} f dP_n \rightarrow \int_{\mathbb{R}} f dP$.

$X_n \xrightarrow{\mathcal{D}} X$ iff $\forall f \in C_b(\mathbb{R})$, $E f(X_n) \rightarrow E f(X)$



Called "convergence in distribution" or "weak convergence".

Remark: This notion of convergence is equiv to weak-* convergence in functional analysis. It's easier to formally see the connection

when P_1, P_2, \dots, P have densities v_1, v_2, \dots, v w.r.t. some base measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e.

$$dP_n = v_n d\mu \quad \& \quad dP = v d\mu$$

so that $P_n \xrightarrow{\mathcal{D}} P$

$$\int f v_n d\mu \rightarrow \int f v d\mu, \forall f \in C_b(\mathbb{R})$$

Remark: You can loosely interpret $X_n \xrightarrow{\mathcal{D}} X$ as meaning that for large n, m both X_n and X_m resemble random draws from X but that X_n & X_m are unrelated...

Warning: This is only a loose interpretation since it is possible that $\exists A \in \mathcal{B}(\mathbb{R})$ s.t.

$$P(X_n \in A) \not\rightarrow P(X \in A)$$

Most of the examples of $\xrightarrow{\mathcal{D}}$ we will work with come from the central limit theorem ... which effectively says:

If X_1, X_2, \dots are independent r.v.s (all defined on a common $(\mathbb{R}', \mathcal{F}', P')$) with

$$E X_n = 0 \quad \& \quad \text{var}(X_n) = \sigma^2 < \infty$$

then $\sqrt{n} \bar{X}_n = \frac{X_1 + \dots + X_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} Z \sim N(0, \sigma^2)$.

We will derive this near the end of this lecture. However a good (but somewhat degenerate) example which helps interpret future results is as follows.

Example: $\Omega = \mathbb{R}$, $P(X_n = \frac{1}{n}) = 1$, $P(X = 0) = 1$. (3)

$$\therefore \underbrace{\mathbb{E} f(X_n)}_{f(\frac{1}{n})} \xrightarrow{n \rightarrow \infty} \underbrace{\mathbb{E} f(X)}_{f(0)} \text{ if } f \in C_b(\mathbb{R})$$

so $X_n \xrightarrow{d} X$ but notice

$$\begin{aligned} P(X_n \leq 0) &\xrightarrow{=} P(X \leq 0) & \left. \begin{array}{l} \text{mass can} \\ \text{magically} \\ \text{appear on the} \\ \text{boundaries of} \\ \text{closed sets} \end{array} \right\} \\ &= 1 & \end{aligned}$$

$$\begin{aligned} P(X_n > 0) &\xrightarrow{=} P(X > 0) & \left. \begin{array}{l} \text{mass can} \\ \text{magically} \\ \text{disappear on} \\ \text{the boundaries} \\ \text{of open sets} \end{array} \right\} \\ &= 0 & \end{aligned}$$

Definition: $\forall A \subset \Omega$ define

$\bar{A} :=$ closure of A (w.r.t. the Polish metric d)

$A^\circ :=$ open interior of A (all $x \in A$ s.t. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A$)

$\partial A :=$ boundary of $A := \bar{A} - A^\circ$.

Here are some Portmanteau (French for coat hanger) results which give \xrightarrow{d} equivalence.

Theorem (Portmanteau I):

Let P_1, P_2, \dots, P be probability measures

on a Polish space $(\Omega, \mathcal{B}(\Omega))$.

Then the following are equivalent.

$$(a) P_n \xrightarrow{d} P$$

$$(b) \int_\Omega f dP_n \rightarrow \int_\Omega f dP, \text{ if } f \in \text{Lip}(\Omega) \cap C_b(\Omega)$$

$$(c) \limsup_n P_n(F) \leq P(F), \text{ if closed } F \subset \Omega.$$

↑ Possible magically appearing mass

$$(d) P(G) \leq \liminf_n P_n(G), \text{ if open } G \subset \Omega.$$

↑ Possible magically disappearing mass

$$(e) \lim_n P_n(A) = P(A), \forall A \in \mathcal{B}(\Omega) \text{ s.t. } P(\partial A) = 0$$

Proof:

(a) \Rightarrow (b): Trivial.

(b) \Rightarrow (c): Let $F \subset \Omega$ be closed. As in the proof of the separating class Thm let

$$f_\varepsilon(w) = (1 - \frac{d(w, F)}{\varepsilon})^+$$

so that $f_\varepsilon \in \text{Lip}(\Omega) \cap C_b(\Omega)$ and $\forall \text{prob } Q \text{ on } (\Omega, \mathcal{B}(\Omega))$

$$\int_\Omega I_F dQ \leq \int_\Omega f_\varepsilon dQ \leq \int_\Omega I_{F^\varepsilon} dQ \xrightarrow{\varepsilon \downarrow 0} Q(F). \quad (*)$$

$$\therefore \limsup_n P_n(F) = \limsup_n \int_\Omega I_F dP_n$$

$$\leq \limsup_n \int_\Omega f_\varepsilon dP_n, \text{ by } (*)$$

$$= \int_\Omega f_\varepsilon dP, \quad f_\varepsilon \in \text{Lip}(\Omega) \cap C_b(\Omega)$$

$$\leq \int_\Omega I_{F^\varepsilon} dP$$

$$\xrightarrow{\varepsilon \downarrow 0} P(F), \text{ by } (*)$$

(c) \Leftrightarrow (d): Take complements of (c)

(c) $\&$ (d) \Rightarrow (e): Suppose $P(\partial A) = 0$

$$\therefore 0 = P(\bar{A} - A^\circ) = P(\bar{A}) - P(A^\circ)$$

↑
by nested set subtraction
props of P

$$\text{and } P(A^\circ) \leq \liminf_n P_n(A^\circ), \quad \text{by (c)}$$

$$\begin{aligned} &\leq \limsup_n P_n(\bar{A}), \quad \text{int } \leq \sup \bar{A} \\ &\leq P(\bar{A}), \quad \text{by (d)} \\ &\text{& } \limsup_n P_n(A) \text{ sandwiched in here.} \end{aligned}$$

Since $P(A^\circ) \subset P(A) \subset P(\bar{A})$ & $P(\bar{A}) - P(A^\circ) = 0$

$$\lim_n P_n(A) = P(A)$$

(4)

(e) \Rightarrow (a):

$$\text{Let } f \in C_b(\mathbb{R}) \text{ & show } \int_{\mathbb{R}} f dP_n \rightarrow \int_{\mathbb{R}} f dP.$$

Adjust f by adding a constant and re-scaling we can assume w.l.o.g. that

$$0 < f < 1.$$

Recall Thm from lecture 11 that says

$$\text{r.v. } X \geq 0 \Rightarrow E(X) = \int_0^\infty P(X > t) dt$$

This applies to f so that

$$(*) \quad \int_{\mathbb{R}} f dP_n = \int_0^1 P_n(f > t) dt \quad \downarrow ? \text{ as } n \rightarrow \infty$$

$$(**) \quad \int_{\mathbb{R}} f dP = \int_0^1 P(f > t) dt.$$

Moreover continuity implies

$$\begin{aligned} \{f > t\} &= f^{-1}((t, \infty)) = \text{open} = \{f > t\}^o \\ \{f \geq t\} &= (f^{-1}([-\infty, t]))^c = \text{closed} = \overline{\{f > t\}} \end{aligned}$$

$$\therefore \partial \{f > t\} = \{f \geq t\} - \{f > t\} = \underbrace{\{f = t\}}_{\substack{\text{has non-zero} \\ \text{P mass for at} \\ \text{most countably} \\ \text{many } t}}$$

\therefore (e) implies

$$P_n(f > t) \xrightarrow{n \rightarrow \infty} P(f > t)$$

for $\mathbb{P}\text{-a.e. } t$

\therefore DCT implies

$$\int_0^1 P_n(f > t) dt \xrightarrow{n \rightarrow \infty} \int_0^1 P(f > t) dt$$

|| (*)

$$\int_{\mathbb{R}} f dP_n$$

$$\int_{\mathbb{R}} f dP$$

QED

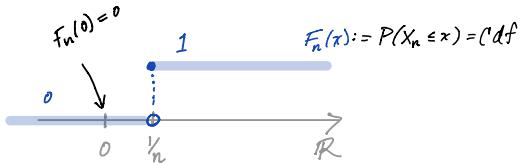
(5)

The next result covers the special case of univariate real valued r.v.s. (6)

Recall earlier example

$$X_n = \frac{1}{n} \xrightarrow{\mathcal{D}} X = 0$$

Note $F_n(x) \rightarrow F(x)$ $\forall x \neq 0$. Here is the picture



The problem is
that $A = (-\infty, 0]$
has $P X^{-1}(A) = P(X=0) \neq 0$

Define $C_F := \{x \in \mathbb{R}: F \text{ is continuous at } x\}$

so that $x \in C_F \iff 0 = F(x) - F(x^-) \quad \begin{matrix} F \text{ is always} \\ \text{right cont.} \end{matrix}$
 $\iff 0 = P(X=x)$
 $\iff P X^{-1}(\partial(-\infty, x]) = 0$

Theorem (Portmanteau II):

Let X_1, X_2, \dots, X be real-valued r.v.s with cdfs F_1, F_2, \dots, F . Then the following are equivalent

(i) $X_n \xrightarrow{\mathcal{D}} X$

(ii) $F_n(x) \rightarrow F(x), \forall x \in C_F$

(iii) $F_n^{-1}(u) \rightarrow F^{-1}(u), \forall u \in C_F^{-1}$

→ the left-continuous quasi-inverse defined on $u \in C_F$ in lecture 8

Proof:

Let P_1, P_2, \dots, P be the induced measures of X_1, X_2, \dots, X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(i) \Rightarrow (ii)

$$\begin{aligned} X_n \xrightarrow{\mathcal{P}} X &\Leftrightarrow P_n \xrightarrow{\mathcal{P}} P \\ &\Rightarrow P_n((-\infty, x]) \rightarrow P((-\infty, x]) \quad \forall x \in C_F \\ &\text{since } x \in C_F \Rightarrow P(\partial(-\infty, x]) = 0 \\ &\Rightarrow F_n(x) \rightarrow F(x) \quad \forall x \in C_F \end{aligned}$$

(ii) \Rightarrow (i)

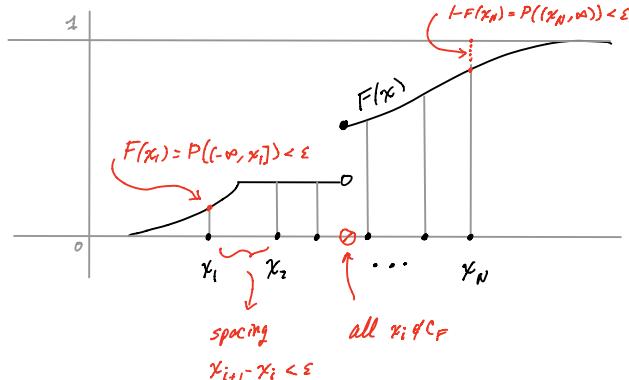
For the other direction suppose

$$F_n(x) \rightarrow F(x) \quad \forall x \in C_F.$$

Using Portmanteau I it will be sufficient to show

$$\int f dP_n \rightarrow \int f dP, \quad \forall f \in \text{Lip}(\mathbb{R}) \cap C_b(\mathbb{R}).$$

For $\epsilon > 0$ choose x_1, \dots, x_N points s.t.



These exist since the results in Lecture 17 & 18 imply C_F^c is at most countable, $\lim_{x \rightarrow -\infty} F(x) = 0$

$$\text{and } \lim_{x \rightarrow \infty} F(x) = 1.$$

$$\text{Let } A_1 = (-\infty, x_1]$$

$$A_2 = (x_1, x_2]$$

 \vdots

$$A_N = (x_{N-1}, x_N]$$

$$A_{N+1} = (x_N, \infty)$$

Note that by
design $P(\partial A_i) = 0$
so $P_n(A_i) \rightarrow P(A_i)$

Now let $f \in \text{Lip}(\mathbb{R}) \cap C_b(\mathbb{R})$.

By rescaling we can suppose w.l.o.g. that

$$0 \leq f \leq 1.$$

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$$\begin{aligned} \therefore \limsup_n \int_A f dP_n &\leq \limsup_n \int_A 1 dP_n \\ &= \limsup_n F_n(x_i) \\ &= F(x) \quad \text{since } x_i \in C_F \\ &\leq \epsilon \\ &\leq \epsilon + \int_A f dP \end{aligned}$$

Similarly

$$\limsup_n \int_{A_{N+1}} f dP_n \leq \epsilon + \int_{A_{N+1}} f dP$$

Also

$$\begin{aligned} \limsup_n \int_{A_2 \cup \dots \cup A_N} f dP_n &= \limsup_n \sum_{i=2}^N \int_{A_i} f dP_n \\ &\leq \limsup_n \sum_{i=2}^N (f(x_i) + c\epsilon) \int_{A_i} 1 dP_n \\ &\quad \text{since } |f(x) - f(x_i)| \leq c\epsilon \\ &\quad \text{for } x \in A_i \text{ and } 2 \leq i \leq N \end{aligned}$$

$$= \sum_{i=2}^N (f(x_i) + c\epsilon) \int_{A_i} f dP$$

$$\text{Since } P_n(A_i) = F_n(x_i) - F_n(x_{i-1}) \rightarrow P(A_i)$$

$$= \sum_{i=2}^N \int_{A_i} f(x_i) dP + c\epsilon \int_{A_1} f dP$$

$$\leq \sum_{i=2}^N \int_{A_i} f dP + 2c\epsilon$$

$$= \int_{A_2 \cup \dots \cup A_N} f dP + 2c\epsilon$$

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Putting everything together & letting $\varepsilon \rightarrow 0$ (9)

$$\limsup_n \int_{\Omega} f dP_n \leq \int_{\Omega} f dP$$

Replacing f with $1-f$ above gives

$$\int_{\Omega} f dP_n \rightarrow \int_{\Omega} f dP$$

$(iii) \Rightarrow (ii)$

We show that $\forall u \in (0,1)$

$$F^{-1}(u) \leq \liminf_n F_n^{-1}(u) \leq \limsup_n F_n^{-1}(u) \leq F^{-1}(u+) \quad (\dagger\dagger)$$

To show (\dagger) suppose not.

Now we can choose $x \in C_F$ s.t.

$$\liminf_n F_n^{-1}(u) < x < F^{-1}(u).$$

By the switching formula $F(x) < u$ so that

$$F_n(x) \rightarrow F(x) < u$$

$\therefore F_n(x) < u$ & large n

$\therefore x < F_n^{-1}(u)$ & large n by "switching" again
which contradicts $\liminf_n F_n^{-1}(u) < x$.

To show $(\dagger\dagger)$ use the same trick &

Suppose not.

$\therefore \exists x \in C_F$ s.t.

$$F^{-1}(u+) < x < \limsup_n F_n^{-1}(u)$$

$$\therefore u < F(x-) = F(x)$$

by switching again $(F(x-) \leq u \Leftrightarrow x \leq F^{-1}(u))$
 $\therefore F(x-) > u \Leftrightarrow x > F^{-1}(u+)$

$\therefore u < F_n(x)$ & large n

$\therefore F_n^{-1}(u) < x$ & large n

\therefore contradiction

$((iii) \Rightarrow (ii))$ similar.

QED.

Theorem: (uniqueness of \xrightarrow{P} limits) (10)

Let P, Q, P_1, P_2, \dots be probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$. If $P_n \xrightarrow{P} P$ and $P_n \xrightarrow{Q} Q$ then $P=Q$.

Proof:

If $P_n \xrightarrow{P} P$ & $P_n \xrightarrow{Q} Q$ then $\forall f \in C_b(\Omega)$

$$\lim_n \int_{\Omega} f dP_n = \int_{\Omega} f dQ = \int_{\Omega} f dP$$

since $Lip(\Omega) \cap C_b(\Omega) \subset C_b(\Omega)$
is a separating class
this implies $P=Q$.

QED

Theorem: (Sub-sub-seg check for \xrightarrow{P})

Let P, P_1, P_2, \dots be a collection of probability measures on a Polish $(\Omega, \mathcal{B}(\Omega))$.

If \forall sub-seg n_k , \exists a sub-sub-seg n_{k_j} :

s.t. $P_{n_{k_j}} \xrightarrow{P} P$ as $j \rightarrow \infty$ then

$$P_n \xrightarrow{P} P \text{ as } n \rightarrow \infty.$$

Proof: Suppose not. Then $\exists f \in C_b(\Omega)$ s.t.

$$\int_{\Omega} f dP_n \not\rightarrow \int_{\Omega} f dP.$$

$\therefore \exists n_k$ & $\varepsilon > 0$ s.t.

$$\left| \int_{\Omega} f dP_{n_k} - \int_{\Omega} f dP \right| \geq \varepsilon > 0, \forall k.$$

Now by assumption we can choose n_{k_j} s.t.

$$\int_{\Omega} f dP_{n_{k_j}} \rightarrow \int_{\Omega} f dP.$$

\therefore contradiction

QED

Skorokhod Representation

(11)

Skorokhod's representation theorem is really handy and effectively says

$$X_n \xrightarrow{d} X \Rightarrow \exists X^*, X_n^* \text{ defined on a common probability space}$$

s.t. $X_n^* \xrightarrow{\text{a.e.}} X^*$
 $\parallel \vartheta \quad \parallel \vartheta$
 $X_n \quad X$

Here is the full version of the theorem.

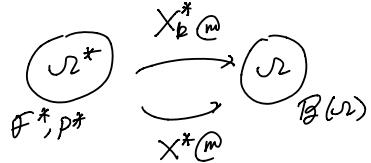
Theorem: (Skorokhod Representation Theorem)

Let P, P_1, P_2, \dots be probability measures on a Polish $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$ s.t.

$$P_n \xrightarrow{d} P.$$

Then there exist a probability space $(\mathcal{D}^*, \mathcal{F}^*, P^*)$ & maps X^*, X_1^*, \dots

s.t.



s.t. $\mathcal{L}(X^*) = P$, $\mathcal{L}(X_n^*) = P_n$ and

$$\lim_{n \rightarrow \infty} X_n^*(u) = X^*(u)$$

for all $u \in \mathcal{D}^*$.

Notation: In the above theorem

I'm using $\mathcal{L}(X^*)$, $\mathcal{L}(X_n^*)$ as a shorthand for $P^*(X^*)$ & $P^*(X_n^*)$

i.e. the induced distributions on $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$.

The proof of the general skorokhod's result is nasty and doesn't really provide any insight. We will just show the case $\mathcal{D} = \mathbb{R}$ and cite the general result in Billingsley's book "Convergence of Prob. Meas." (12)

Proof of Skorokhod's Rep when $\mathcal{D} = \mathbb{R}$:

Define c.d.f.s

$$F_n(x) = P_n((-\infty, x])$$

$$F(x) = P((-\infty, x]).$$

By Portmanteau II

$$F_n^{-1}(u) \rightarrow F^{-1}(u) \quad \forall u \in C_{F^{-1}} \quad (*)$$

Let $\mathcal{D}^* = (0, 1)$, $\mathcal{F}^* = \mathcal{B}((0, 1))$, $P^* = \underbrace{\mathcal{L}^1}_{\text{uniform measure on } (0, 1)}$,

and set

$$X_n^*(u) := F_n^{-1}(u)$$

$$X^*(u) := F^{-1}(u)$$

$\forall u \in \mathcal{D}^*$.

$\therefore X_n^*(u) \rightarrow X^*(u)$, $\forall u \in C_{F^{-1}}$ by $(*)$.

Finally notice that $C_{F^{-1}}^c$ must be countable since

$$C_{F^{-1}}^c \subset \bigcup_{\epsilon > 0} \bigcup_{\substack{0 < a < b < 1 \\ a, b \in \mathbb{Q}}} \left\{ u \in [a, b] : F^{-1}(u+) - F^{-1}(u-) > \epsilon \right\}$$

Since F^{-1} is monotonic on $[a, b]$ the sum of the jumps of size exceeding ϵ must be no greater than $F^{-1}(b) - F^{-1}(a)$... hence this set is finite.

$$\therefore P^*(C_{F^{-1}}^c) = 1$$

$$\therefore X_n^* \xrightarrow{\text{a.e.}} X^*$$

QED

The following result demonstrates Skorohod's usefulness.

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Theorem: (continuous mapping Thm for $\overset{D}{\rightarrow}$)

Let X, X_1, X_2, \dots be generalized r.v.s taking values in a Polish space \mathcal{R} (with σ -field $B(\mathcal{R})$) s.t. $X_n \xrightarrow{D} X$.

If $g: \mathcal{R} \rightarrow \mathbb{R}$ satisfies

$$P(g \text{ is continuous at } X) = 1$$

then

$$g(X_n) \xrightarrow{D} g(X).$$

Proof:

By Skorohod's Rep Thm $\exists X_n^*, X^*$ defined on $(\Omega^*, \mathcal{F}^*, P^*)$ s.t.

$$\lim_{n \rightarrow \infty} X_n^*(w) = X^*(w) \quad \forall w \in \Omega^*$$

$$\text{where } X_n = X_n^*, \quad X = X^*.$$

Let $A := \{w \in \Omega^* : g \text{ is continuous at } X^*(w)\}$

$$\therefore w \in A \Rightarrow \lim_{n \rightarrow \infty} g(X_n^*(w)) = g(X^*(w))$$

$$\text{Since } X = X^*, \quad P(A) = 1.$$

$$\therefore g(X_n^*) \xrightarrow{a.e.} g(X^*)$$

$$\therefore g(X_n^*) \xrightarrow{D} g(X^*) \quad \text{by corollary above}$$

$$\parallel D \qquad \parallel D$$

$$g(X_n) \qquad g(X)$$

QED

Skorohod's Representation Theorem also gives us extensions of Fatou, ... which apply to the case $X_n \xrightarrow{D} X$.

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Here is an example.

Theorem: (UI extension for $\overset{D}{\rightarrow}$)

If $X_n \xrightarrow{D} X$ and the X_n 's are UI

$$E|X_n| \rightarrow E|X| < \infty \text{ and}$$

$$E(X_n) \rightarrow E(X) < \infty.$$

Proof:

By Skorohod $\exists X_n^*, X^*$ s.t.

$$X_n = X_n^* \xrightarrow{a.e.} X^* = X$$

Since $X_n = X_n^*$ the X_n^* 's are also UI.

\therefore by old UI theorem we have

$$E|X_n^*| \rightarrow E|X^*| < \infty \text{ and}$$

$$E(X_n^*) \rightarrow E(X^*) < \infty$$

but again since $X_n^* = X_n$ & $X^* = X$

$$E|X_n| \rightarrow E|X| < \infty \text{ and}$$

$$E(X_n) \rightarrow E(X) < \infty.$$

QED

Skorokhod's Rep Thm also gives us the Delta method

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Theorem: (Delta method)

Let Z, X_1, X_2, \dots be random d-dimensional real vectors s.t.

$$c_n(X_n - x_0) \xrightarrow{\mathcal{D}} Z \quad (*)$$

where $0 < c_n \rightarrow \infty$ as $n \rightarrow \infty$ & $x_0 \in \mathbb{R}^d$.

If $g: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is differentiable ∂x_0 then

$$c_n(g(X_n) - g(x_0)) \xrightarrow{\mathcal{D}} Dg(x_0)Z$$

Remark: $g: \mathbb{R}^d \rightarrow \mathbb{R}^d$ differentiable ∂x_0 means that

$$\lim_{v \rightarrow 0} \frac{g(x_0+v) - g(x_0) - Dg(x_0)v}{\|v\|} = 0$$

as $v \rightarrow 0$ where $Dg(x_0) = \left(\frac{\partial}{\partial x_1} g, \dots, \frac{\partial}{\partial x_d} g \right) \Big|_{x=x_0}$

Remark: The intuitive way to understand this theorem is that $(*)$ suggest

$$X_n \xrightarrow{\mathcal{D}} x_0 + \underbrace{\frac{Z}{c_n}}_{\rightarrow 0 \text{ as } n \rightarrow \infty}$$

$$\text{so } g(X_n) \xrightarrow{\mathcal{D}} g(x_0 + \frac{Z}{c_n}) \approx g(x_0) + Dg(x_0) \frac{Z}{c_n}$$

Proof:

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Let's prove this for $d=1$... the case $d>0$ is similar. we can use $(*)$ along with Skorokhod's Rep to get X_n^* and Z^* s.t.

$$c_n(X_n^* - x_0) \xrightarrow{a.e.} Z^*$$

Now

define this to be $g'(x_0)$
when $X_n^* - x_0 = 0$

$$c_n(g(X_n^*) - g(x_0))$$

$$= c_n(X_n^* - x_0) \cdot \underbrace{\left(\frac{g(X_n^*) - g(x_0)}{X_n^* - x_0} \right)}_{\xrightarrow{a.e.} g'(x_0)}$$

$$\xrightarrow{a.e.} Z^* \quad \xrightarrow{a.e.} g'(x_0) \text{ since}$$

by Skorokhod $\lim_n (X_n^* - x_0) \xrightarrow{a.e.} \lim_n \frac{Z^*}{c_n} = 0$

$$\therefore c_n(g(X_n^*) - g(x_0)) \xrightarrow{a.e.} g'(x_0) Z^*$$

$$\therefore c_n(g(X_n) - g(x_0)) \xrightarrow{\mathcal{D}} g'(x_0) Z$$

QED

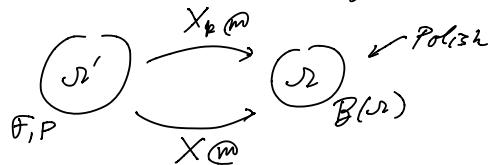
Slutsky

(17)

The next theorem deals with the notion of convergence in probability. We will cover this in a later lecture but give some brief notation for it now, along with almost everywhere convergence.

Definition: (\xrightarrow{P} & $\xrightarrow{a.e.}$)

Let X, X_1, \dots be a collection of ω maps



Then

$X_n \xrightarrow{P} X$ iff $\forall \varepsilon > 0, P(d(X_n, X) \geq \varepsilon) \xrightarrow{n \rightarrow \infty} 0$

$X_n \xrightarrow{a.e.} X$ iff $P(\lim_n d(X_n, X) = 0) = 1$.

Remark:

$$X_n \xrightarrow{P} X \Leftrightarrow d(X_n, X) \xrightarrow{P} 0$$

$$X_n \xrightarrow{a.e.} X \Leftrightarrow d(X_n, X) \xrightarrow{a.e.} 0$$

Remark: In lecture 6 we remarked that $SLN \Rightarrow WLNN$. Indeed

$$X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X$$

since

$$\begin{aligned} \limsup_n P(d(X_n, X) \geq \varepsilon) &\stackrel{\text{Fact}}{\leq} P(d(X_n, X) \geq \varepsilon \text{ i.o.}) \\ &\leq \underbrace{P(d(X_n, X) \neq 0)}_{=0} \end{aligned}$$

Theorem: (Slutsky)

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Let X, X_1, X_2, \dots and Y_1, Y_2, \dots be collections of generalized r.v.s taking values in a Polish space \mathcal{R} (with σ -field $B(\mathcal{R})$) all defined on the same probability space.

Then

$$X_n \xrightarrow{P} X \& d(X_n, Y_n) \xrightarrow{P} 0 \Rightarrow Y_n \xrightarrow{P} X.$$

Proof:

We use Portmanteau I and show

$$\limsup_n P(Y_n \in F) \leq P(X \in F)$$

for all closed $F \subset \mathcal{R}$.

First note $\forall \varepsilon > 0$

$$\{Y_n \in F\} \subset \{d(X_n, Y_n) \geq \varepsilon\} \cup \{X_n \in F^\varepsilon\}$$

$$\therefore \limsup_n P(Y_n \in F)$$

$$\leq \limsup_n P(d(X_n, Y_n) \geq \varepsilon) + \limsup_n P(X_n \in F^\varepsilon)$$

$\underbrace{\quad}_{=0 \text{ by assumption}}$

$$\leq \limsup_n P(X_n \in F^\varepsilon)$$

$$\leq P(X \in F^\varepsilon) \text{ by Portmanteau I}$$

since $F^\varepsilon \downarrow F$ as $\varepsilon \downarrow 0$

$$P(X \in F^\varepsilon) \downarrow P(X \in F)$$

as $\varepsilon \downarrow 0$ by CFA.

$$\therefore \limsup_n P(Y_n \in F) \leq P(X \in F)$$

as was to be shown

QED.

Slutsky's theorem gives us a nice corollary that relates $\xrightarrow{a.e.}$, \xrightarrow{P} and a.e. convergence.

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Corollary:

$$X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X.$$

↑
use $X \xrightarrow{\mathcal{D}} X$
 $d(X_n, X) \xrightarrow{P} 0$

Warning: The reverse implications do not hold in general.

The "moving spike" showed $\exists X_n \neq X$ s.t.

$$X_n \xrightarrow{P} X \text{ but } X_n \not\xrightarrow{a.e.} X$$

To see why $\xrightarrow{\mathcal{D}} \not\Rightarrow \xrightarrow{P}$ consider

$$X = \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$$

and $X_n := (-1)^n X$ so that $X_n \xrightarrow{\mathcal{D}} X$ but $P(|X_n - X| \geq \varepsilon) = 1$ when n is odd.

$$\therefore X_n \xrightarrow{\mathcal{D}} X \text{ but } X_n \not\xrightarrow{P} X.$$

However, the following corollary is one special case where the reverse implication does hold.

Corollary: If X_1, X_2, \dots are Ω -valued r.v.s where $(\Omega, \mathcal{B}(\Omega))$ is Polish all defined on the same prob. space then

$$X_n \xrightarrow{\mathcal{D}} c \Leftrightarrow X_n \xrightarrow{P} c$$

Proof: If $X_n \xrightarrow{\mathcal{D}} c$ then

$$\limsup_n P(X_n \in \underbrace{B_\varepsilon(c)}_{\text{closed}}^c) \leq P(c \in B_\varepsilon(c)^c) = 0$$

QED.

Here is another useful corollary of Slutsky's thm

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Corollary: Suppose X, X_n, Y_n are r.v.s

taking values in a polish space $(\Omega, \mathcal{B}(\Omega))$ all defined on the same probability space.

If $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} c \in \Omega$

then $(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, c)$.

Proof:

Suppose $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} c \in \Omega$.

$$\therefore (X_n, c) \xrightarrow{\mathcal{D}} (X, c) \quad (*)$$

since $f(x, y) \in C_b(\Omega \times \Omega)$ implies $f(x, c) \in C_b(\Omega)$ so that $E f(X_n, c) \rightarrow E f(X, c)$.

Let \tilde{d} be the product dist on $\Omega \times \Omega$ so that

$$\tilde{d}((X_n, c), (X, c)) = d(c, Y_n) \xrightarrow{P} 0. \quad (**)$$

Now $(*)$ and $(**)$ implies

$$(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, c)$$

by Slutsky's theorem.

QED

Now under the same assumptions as in the previous corollary we can use the continuous mapping theorem to obtain:

$$X_n + Y_n \xrightarrow{\mathcal{D}} X + c$$

$$X_n Y_n \xrightarrow{\mathcal{D}} Xc$$

$$\frac{X_n}{Y_n} \xrightarrow{\mathcal{D}} \frac{X}{c} \text{ provided } c \neq 0$$

Tightness and Prohorov's Thm

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An extremely useful fact, when working with a sequence of real numbers x_1, x_2, \dots is that if $\{x_n\}_{n \geq 1}$ is bounded then there exists a sub-sequence n_k and a real number x s.t. $x_{n_k} \xrightarrow{k \rightarrow \infty} x$

We would like to have something similar for a sequence P_1, P_2, \dots of probability measures on a Polish space. The problem is to find the right generalization of "boundedness" to guarantee the existence of a probability measure P & a sub-sequence n_k s.t.

$$P_{n_k} \xrightarrow{\mathcal{D}} P.$$

It turns out the right definition is our old friend "tightness" from the homeworks last lecture.

Definition:

Let \mathcal{P} be a collection of Probability measures on a Polish $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then

\mathcal{P} is tight iff $\forall \varepsilon > 0 \exists$ a compact $K \subset \mathcal{X}$ s.t.

$$\sup_{P \in \mathcal{P}} P(K^c) < \varepsilon$$

Notation: when $\{X_n\}_{n \geq 1}$ are random vectors taking values in \mathbb{R}^d there is a fancy notational shorthand:

$$|X_n| = \mathcal{O}_p(1) \text{ means } \{X_n\}_{n \geq 1} \text{ is tight}$$

Prohorov's Thm shows tightness is the right definition of "boundedness".

Theorem: (Prohorov)

Let \mathcal{P} be a collection of Probability measures on a Polish $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Then

\mathcal{P} is tight



$\left\{ \begin{array}{l} \text{if } P_1, P_2, \dots \in \mathcal{P} \text{ there exists a sub-seq } n_k \\ \text{and a prob measure } P \text{ on } (\mathcal{X}, \mathcal{B}(\mathcal{X})) \\ \text{s.t. } P_{n_k} \xrightarrow{\mathcal{D}} P \\ \text{not necessarily in } \mathcal{P}. \end{array} \right.$

Note: If \mathcal{P} satisfies (*) then \mathcal{P} is

said to be relatively compact w.r.t. $\xrightarrow{\mathcal{D}}$.

Proof: This proof is a bit nasty and doesn't really provide much probabilistic intuition so we will skip it and simply cite Billingsley's book on convergence of Probability measures.

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Here is an example of how one uses Prohorov.

Theorem: (Portmanteau III)

Let X_1, X_2, \dots, X be r.v.s taking values in $C[0,1]$ w.r.t $\mathcal{B}(C[0,1])$. Then

The following are equivalent:

$$(A) X_n \xrightarrow{\mathcal{D}} X.$$

$$(B) \{X_n\}_{n \geq 1} \text{ is tight and } \forall t_1, \dots, t_m \in [0,1]$$

$$\pi_{t_1, \dots, t_m}(X_n) \xrightarrow{\mathcal{D}} \pi_{t_1, \dots, t_m}(X).$$

Proof:

(A) \Rightarrow (B):

If $X_n \xrightarrow{\mathcal{D}} X$ the sub-sub check for $\xrightarrow{\mathcal{D}}$
implies $\{X_n\}_{n \geq 1}$ is relatively compact.

$\therefore \{X_n\}_{n \geq 1}$ is tight by Prohorov.

$$\text{Also } \pi_{t_1, \dots, t_m}(X_n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \pi_{t_1, \dots, t_m}(X)$$

by the continuous mapping theorem.

(B) \Rightarrow (A): Assume

$$(*) \pi(X_n) \rightarrow \pi(X), \quad \forall \pi \in \{\pi_{t_1, \dots, t_m} : t_i \in [0,1]\}$$

$$(**) \{X_n\}_{n \geq 1} \text{ is tight.}$$

$$\text{show } X_n \xrightarrow{\mathcal{D}} X.$$

By "sub-sub-seg check" it is sufficient to

$$\text{show } \forall n_p \exists n_{p_j} \text{ s.t. } X_{n_{p_j}} \xrightarrow[j \rightarrow \infty]{\mathcal{D}} X.$$

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For a given n_p use $(**)$ with Prohorov to find n_{p_j} and Y s.t.

$$(***) X_{n_{p_j}} \xrightarrow[j \rightarrow \infty]{\mathcal{D}} Y$$

By the continuous mapping thm

$$\pi(X_{n_{p_j}}) \xrightarrow{\mathcal{D}} \pi(Y), \quad \forall \pi = \pi_{t_1, \dots, t_m}$$

$$\downarrow \text{by } (*)$$

$$\pi(X)$$

$$\therefore \pi(X) = \pi(Y), \quad \forall \pi = \pi_{t_1, \dots, t_m}$$

by uniqueness of limits.

$\therefore X \stackrel{\mathcal{D}}{=} Y$ since the f.d.d characterize
probs on $(C[0,1], \mathcal{B}(C[0,1]))$.

$$\therefore X_{n_{p_j}} \xrightarrow[j \rightarrow \infty]{\mathcal{D}} X \quad \text{by } (***)$$

Since n_p was arb the "sub-sub check" Thm

$$\text{gives } X_n \xrightarrow{\mathcal{D}} X.$$

QED

Here is another consequence of Prohorov.

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Theorem: (Portmanteau IV)

Let X_1, X_2, \dots, X be r.v.s taking values in a Polish $(\Omega, \mathcal{B}(\Omega))$. Then the following are equivalent:

$$(i) \quad X_n \xrightarrow{\mathcal{D}} X.$$

(ii) $\{X_n\}_{n \geq 1}$ is tight and there exists a separating family Γ for $(\Omega, \mathcal{B}(\Omega))$ s.t. $\Gamma \subset C_b(\Omega)$ and

$$Eg(X_n) \rightarrow Eg(X) \quad \forall g \in \Gamma. \quad (*)$$

Proof:

(i) \Rightarrow (ii): follows immediately since (i) implies $\{X_n\}_{n \geq 1}$ is tight by Prohorov and trivially (*) holds by def of $\xrightarrow{\mathcal{D}}$.

(ii) \Rightarrow (i):

Let n_p be an arb sub-seq. By "sub-sub-seq check" it will be sufficient to show

$$\exists n_{p_j} \text{ s.t. } X_{n_{p_j}} \xrightarrow{\mathcal{D}} X \text{ as } j \rightarrow \infty.$$

Now by tightness & Prohorov's Thm

$\exists Y \& \exists n_{p_j}$ s.t.

$$X_{n_{p_j}} \xrightarrow{j \rightarrow \infty} Y \quad (**)$$

$$\therefore E f(X_{n_{p_j}}) \xrightarrow{j \rightarrow \infty} E f(Y) \quad \forall f \in C_b(\Omega).$$

Since $\Gamma \subset C_b(\Omega)$ we therefore have

$$Eg(X_{n_{p_j}}) \xrightarrow{j \rightarrow \infty} Eg(Y) \quad \forall g \in \Gamma$$

by (ii) \downarrow

$$Eg(X)$$

$$\therefore Eg(X) = Eg(Y) \quad \forall g \in \Gamma$$

$\therefore X \xrightarrow{\mathcal{D}} Y$ since Γ separates

$$\therefore X_{n_{p_j}} \xrightarrow{j \rightarrow \infty} X \text{ by } (**)$$

as was to be shown.

QED.

Now we can use our results on separating classes in Lecture 13 to get the following.

Theorem: (Portmanteau II)

Let X_1, X_2, \dots, X be r.v.s taking values in a Polish $(\mathcal{D}, \mathcal{B}(\mathcal{D}))$. Then

$$X_n \xrightarrow{\theta} X \text{ iff } E f(X_n) \rightarrow E f(X) \quad \forall f \in \Gamma$$

whenever

(I) $\Gamma = C_c(\mathcal{D})$ and \mathcal{D} is locally compact

(II) $\Gamma = C_c^\infty(\mathcal{D})$ and $\mathcal{D} = \mathbb{R}^d$

(III) $\Gamma = \{\text{monomials}\}$ and \mathcal{D} is a compact subset of \mathbb{R}^d

(IV) $\Gamma = \{e^{ix \cdot k} : k \in \mathbb{Z}^d\}$ and $\mathcal{D} = \mathbb{R}^d$ and $|X_n| = \mathcal{O}_p(1)$.

Proof:

Notice that all the above function classes Γ form separating classes by the results in Lecture 13.

\therefore Portmanteau II implies we just need to show tightness of $\{X_n\}_{n \geq 1}$ each case.

For (I) and (II) choose compact K s.t.

$$P(X \in K) > 1 - \varepsilon_2$$

existence since \mathcal{D} is Polish so that

$$P(X \in A) = \sup \{P(X \in K) : \text{compact } K \subset \mathcal{D}\}$$

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By similar techniques as in Lecture 13 we can find an approximation

$f \in \Gamma$ to I_K s.t.

$$I_K \geq f \quad \& \quad \left| E f(x) - \underbrace{E I_K(x)}_{= P(X \in K)} \right| \leq \frac{\varepsilon}{2}$$

$$\therefore P(X_n \in K) \geq E f(X_n) \xrightarrow{n \rightarrow \infty} E f(x) \quad \text{within } \varepsilon_2 \text{ of } P(X \in K) > 1 - \varepsilon_2$$

$\therefore P(X_n \in K) \geq 1 - \varepsilon \quad \forall \text{ suff large } n.$

$\therefore \{X_n\}_{n \geq 1}$ is tight.

For (III): $\{X_n\}_{n \geq 1}$ is tight since \mathcal{D} is compact.

For (IV): Trivial since $|X_n| = \mathcal{O}_p(1)$ means

$\{X_n\}_{n \geq 1}$ is tight

QED

Notice that part (IV) of Portmanteau II can be rephrased in terms of characteristic functions

Corollary: (characteristic functions for $\xrightarrow{\theta}$)

Suppose X_1, X_2, \dots, X are random vectors with characteristic functions $\phi_1, \phi_2, \dots, \phi$.

Then $X_n \xrightarrow{\theta} X$

\Updownarrow

$$\begin{aligned} & \left(\phi_n(k) \rightarrow \phi(k) \quad \forall k \in \mathbb{Z}^d \right) \\ & \text{and } |X_n| = \mathcal{O}_p(1) \end{aligned} \quad (*)$$

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Proof:

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This follows immediately from
Portmanteau I since $\phi_n(k) \rightarrow \phi(k)$ $\forall k \in \mathbb{R}^d$
means $E f(X_n) \rightarrow E f(X)$ for all
 $f \in \{x \mapsto e^{ix \cdot k} : k \in \mathbb{R}^d\}$.

GFD

This means that whenever $\phi_n(k) \rightarrow \phi(k)$

$\forall k \in \mathbb{R}^d$ it is sufficient to simply show

$|X_n| = \mathcal{O}_p(1)$ for establishing
 $X_n \xrightarrow{d} X$.

Many of the conditions for $X_n \xrightarrow{d} X$

we learn as an undergrad are

simply conditions for establishing $(*)$.

Here are a few examples:

$$\left(\phi_n(k) \rightarrow \phi(k) \quad \forall k \in \mathbb{R}^d \right) \Rightarrow (*)$$

and $\sup_n E|X_n| < \infty$

Proof: Use Markov's Thm

$$\begin{aligned} P(|X_n| \geq N) &\leq \frac{E|X_n|}{N} \\ &\leq \frac{\sup_n E|X_n|}{N} < \infty \end{aligned}$$

$\therefore \forall \varepsilon > 0$ one can choose N large enough s.t.

$$P(X_n \notin \overline{B_n}(0)) < \varepsilon$$

ball of radius N

$$\therefore |X_n| = \mathcal{O}_p(1).$$

$$\left(\phi_n(k) \rightarrow \phi(k) \quad \forall k \in \mathbb{R}^d \right) \Rightarrow (*)$$

$\phi(k)$ is continuous at $k=0$

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Proof:

See Achim Klenke's book
"Probability theory" p. 309

$$\left(M_n(k) \rightarrow M(k) \quad \forall k \in \mathbb{R}^d \right)$$

& $M(k) < \infty$ in a neighborhood of $k=0$

Proof: By analyzing the Taylor series of M_n & ϕ_n one can show $\phi_n(k) \rightarrow \phi(k)$ if k and $\phi(k)$ is continuous at $k=0$.
Hence the previous result applies.

The next condition is both sufficient and necessary ... its called The

Cramér-Wald device.:

$$\left(\langle k, X_n \rangle \xrightarrow{d} \langle k, X \rangle \quad \forall k \in \mathbb{R}^d \right) \Leftrightarrow (*)$$

Proof:

(\Leftarrow)

use the continuous mapping Thm.

\Leftrightarrow

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write $X_n = (X_{n,1}, \dots, X_{n,d})$ and

suppose $\langle b, X_n \rangle \xrightarrow{\mathcal{D}} \langle b, x \rangle$.

$$\therefore |\langle b, X_n \rangle| = \mathcal{O}_p(1) \text{ if}$$

by the "char fun check corollary"

$$\therefore |X_{n,i}| = \mathcal{O}_p(1) \quad \forall i=1, 2, \dots, d$$

\therefore Given $\varepsilon > 0$ we can find $N_1, \dots, N_d > 0$

$$\text{s.t. } P(X_{n,i} \notin [-N_i, N_i]) < \frac{\varepsilon}{d}$$

for $i=1, \dots, d$ and hence

$$\begin{aligned} P(X_n \notin \prod_{i=1}^d [-N_i, N_i]) \\ \leq \sum_{i=1}^d P(X_{n,i} \notin [-N_i, N_i]) \\ \leq \varepsilon \end{aligned}$$

$$\therefore |X_n| = \mathcal{O}_p(1)$$

To finish notice $\langle b, X_n \rangle \xrightarrow{\mathcal{D}} \langle b, x \rangle$

implies

$$\underbrace{E e^{i \langle b, X_n \rangle}}_{= \phi_n(b)} \rightarrow \underbrace{E e^{i \langle b, x \rangle}}_{\phi(b)}$$

by the continuous mapping thm.

\square