Homework 3

Due Thursday, October 20, 2016

Exercise 1. *Let* $A_1, A_2, \ldots$ *be* $\mathcal{F}$*-sets. Show that* $P(A_n \text{ i.o.}_n) = 1$ *if and only if* $\sum_{n=1}^{\infty} P(A_n|A)$ *diverges for every* $\mathcal{F}$*-set* A *of nonzero probability. Hint: show* $P(A_n \text{ i.o.}_n) < 1 \iff \sum_{n=1}^{\infty} P(A_n|A) < \infty$ *for some* $\mathcal{F}$*-set* A *with* $P(A) > 0$.

Exercise 2. *Let* P *and* Q *be probability measures on a* σ*-field* $\mathcal{F}$ *of subsets of a sample space* Ω.

- P *and* Q *are said to be* **singular**, *denoted* $P \perp Q$, *if and only if there exists a set* $F \in \mathcal{F}$ *such that*

$$P(F^c) = 0 = Q(F).$$

- P *is said to be* **absolutely continuous with respect to** Q, *denoted* $P \ll Q$, *if and only if*

$$P(F) = 0 \textit{ for every } \mathcal{F}\textit{-set } F \textit{ for which } Q(F) = 0.$$

Show that

$$P \perp Q \iff \begin{bmatrix} \textit{there exists } \mathcal{F}\textit{-sets } F_1, F_2, \ldots \textit{ such that} \\ P(F_n^c) \to 0 \textit{ and } Q(F_n) \to 0 \textit{ as } n \to \infty \end{bmatrix}$$

and

$$P \ll Q \iff \lim_{\delta \downarrow 0} \Big(\sup\{P(F) \colon F \in \mathcal{F} \textit{ with } Q(F) \le \delta\} \Big) = 0.$$