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## Lecture 1: Borel's normal numbers

- Borel's normal number theorem
- Def: Normal and Abnormal numbers of  $[0,1]$
- Def: Finitely additive probability model
- Def: Field
- WLLN & SLLN for the coin flip binary digit model

## Lecture 2: Classes of sets & Good sets

- Def:  $\sigma$ -field,  $\lambda$ -system,  $\pi$ -system  
Monotone class,  $A_n \uparrow A$ ,  $A_n \downarrow A$
- Def:  $\sigma\langle Q \rangle$ ,  $\lambda\langle Q \rangle$ ,  $M\langle Q \rangle$
- Thm:  $\sigma = \lambda + \pi = M + f$
- Thm: Good sets
- Thm:  $\sigma\langle Q \cap \mathcal{D}_0 \rangle = \sigma\langle Q \rangle \cap \mathcal{D}_0$

## Lecture 3: Dynkin's $\pi$ - $\lambda$ and Borel $\sigma$ -fields

- Thm: Dynkin's  $\pi$ - $\lambda$
- Def:  $B(\mathbb{R}) = \sigma\langle \text{open sets} \rangle$ ,  $\mathbb{R}$  metric space
- Def:  $B(\mathbb{R}^d), B(\bar{\mathbb{R}}^d)$
- Thm:  $B(\mathcal{D}_0) = B(\mathbb{R}) \cap \mathcal{D}_0$  for  $\mathcal{D}_0 \subset \mathbb{R}$   
 $\uparrow$   
metric space
- Thm:  $B(\mathbb{R}) = \sigma\langle \text{open balls} \rangle$  when  $\mathbb{R}$  is a separable metric space.

## Lecture 4: Measures

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- Def: Measures, Probability measures on fields
- Def: Measurable space  $(\mathbb{R}, \mathcal{F})$
- Def: Measure Space  $(\mathbb{R}, \mathcal{F}, \mu)$   
Probability space  $(\mathbb{R}, \mathcal{F}, P)$
- Def: finite,  $\sigma$ -finite,  $\sigma$ -finite over  $\mathcal{Q} \subset \mathcal{F}$
- Thm: Basic properties of  $\mu$  &  $P$
- Def: continuity from above and below (CFA, CFB)  
and continuity from above @  $\infty$ .
- Thm: Uniqueness for measures & probabilities.
- Thm:  $CFA \Leftrightarrow CFB \Leftrightarrow CFA @ \emptyset \Leftrightarrow \sigma\text{-additive}$ .
- Def:  $\mu$ -null,  $\mu$ -negligible, complete  $(\mathbb{R}, \mathcal{F}, \mu)$   
and the completion of  $(\mathbb{R}, \mathcal{F}, \mu)$ .
- Thm: Characterizing the completion

## Lecture 5: Carathéodory & Lebesgue measure.

### Lecture 6: Independent events

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- Def: Independence for events & collections of events.
- Thm: Subclasses, Augmentation, simplified product
- Thm:  $\pi$ -generators
- Thm: ANOVA
- Thm: First Borel-Cantelli lemma (FBCL)
- Def:  $\{A_n \text{ i.o.}\}$  &  $\{A_n \text{ a.a.}\}$
- Thm: Second Borel-Cantelli lemma (SBCL)
- Thm: Fatou
- Thm: Erdős-Renyi's extension of SBCL
- Thm: Kolmogorov's 0-1 law & tail  $\sigma$ -fields
- Thm: Hewitt-Savage 0-1 law for coin flips

### Lecture 7: Maximal inequalities & the law of the iterated log for coin flips.

### Lecture 8: Measurable functions, Random variables and CDF's.

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### Lecture 9: $\sigma$ -fields generated by functions, the structure Thm & random variables

## Lecture 10: Integration

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- Def:  $\int_a^b f d\mu$
- Thm: Simple 3, Little 3, Big 3.
- Thm: a.e. useful facts
- Side facts for  $\int_a^b f d\mu$
- Def:  $\mathcal{Q}^+(\mathcal{I}, \mathcal{F}, \mu), \mathcal{Q}^-(\mathcal{I}, \mathcal{F}, \mu), \mathcal{Q}(\mathcal{I}, \mathcal{F}, \mu), L_1(\mathcal{I}, \mathcal{F}, \mu)$
- Lebesgue integration vrs Riemann integration
- Fat Cantor set
- Def: uniformly integrable (UI)
- Thm: Diliatation criterian for UI
- Thm: Fatou, DCT, BCT, UI
- Thm:  $\frac{d}{dt} \int = \int \frac{d}{dt}$
- Thm: Change of variables

## Lecture 11: Expectation and densities

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- Thm: Markov, Chebyshov, Hoeffding inequalities
- $X \& Y$  are indep  $\Rightarrow E(XY) = E(X)E(Y)$
- Thm: Exchangeable R.V.s as a mixture of independent induced measures
- Application: Poyla's urn
- Thm: Kolmogorov's Maximal inequality
- Thm: 3-series
- Def: Density of  $\mu$  w.r.t.  $\nu$
- Thm: Densities are unique a.e.
- Thm: TV norm & Sheffé.
- Application: Schonbergs result that  $e^{-\|h-g\|^2_\nu}$  is positive definite on Hilbert spaces.
- Application: Bernstein Polynomials & Wasserstein uniform approx
- Application: likelihood ratios  $\frac{dP_X}{dQ_X}$
- Application: Poisson point process &  $\frac{dP}{dQ}$
- Application: size biased sampling.

## Lecture 12: Characteristic functions, MGF's and separating classes.

Lecture ?: Fabini & disintegration

- Coupling & the Wasserstein metric

Lecture ?: Convergence in distribution

- Stein's method
- $\Rightarrow$  in  $C[0,1]$  & Wiener measure.

• Random fields.

Lecture ?: Convergence a.e. &

- Kolmogorov's 3LN
- Ergodic theory.
- Glavanteo Cantelli

Lecture ?: Convergence in probability &  $l_p$

- Projections.
- $l_p$  spaces
- $o_p$  &  $O_p$

Lecture ?: Radon-Nikodym derivative.

Lecture ?: Conditional expectation w.r.t.  
a  $\sigma$ -field

Lecture ?: Martingales

- Martingal CLT

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- Random walks
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