Lecture 22: Martingale convergence

Under mild conditions subMs converge a.e.

Doobs upcrossing inequality is the key to the proof.

Here is a motivation in terms of how to check a sequence of numbers $x_1, x_2, x_3, \dots$ converges.

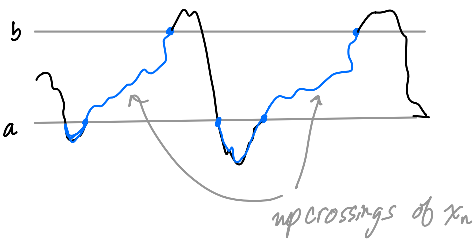
x_n does not converge

iff $\liminf_n x_n < \limsup_n x_n$

iff $\exists a,b \in \mathbb{Q}$ s.t. $a<b$

$x_n > b$ & $x_n < a$ infinitely often.

i.e.



iff $\exists a,b \in \mathbb{Q}$ s.t. x_n has infinitely many upcrossings of $[a,b]$

Doobs upcrossings

Let $(X_1, X_2, \dots, X_n)$ be adapted to the filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$.

Fix $-\infty < a < b < \infty$ and define

$$\left.\begin{aligned} \alpha_1 &= \min\left(\{k \geq 1 : X_k \leq a\} \cup \{n\}\right) \\ \beta_1 &= \min\left(\{k > \alpha_1 : X_k \geq b\} \cup \{n\}\right) \end{aligned}\right\} \text{upcrossing of } [a,b]$$

$$\left.\begin{aligned} \alpha_2 &= \min\left(\{k > \beta_1 : X_k \leq a\} \cup \{n\}\right) \\ \beta_2 &= \min\left(\{k > \alpha_2 : X_k \geq b\} \cup \{n\}\right) \end{aligned}\right\} \text{upcrossing of } [a,b]$$

$$\vdots$$

The number of upcrossings of $[a,b]$, denoted $U_{a,b}$, is defined as

can't be more than $n/2$ terms

$$U_{a,b} := \sum_{j=1}^{n} I_{\{X_{\alpha_j} \leq a,\ X_{\beta_j} \geq b\}}$$

Proposition:

$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are stopping times w.r.t $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ and U_{ab} is $\mathcal{F}_n$-measurable

Proof:

we've already shown that α_1 is a ST.

To show β_1 is a stopping time notice that

- If $1 \leq m < n$ then

$$\begin{aligned} \{\beta_1 = m\} &= \{\alpha_1 < m,\ \beta_1 = m\} \\ &= \bigcup_{j=1}^{m-1} \{\alpha_1 = j,\ \beta_1 = m\} \\ &= \bigcup_{j=1}^{m-1} \{\underbrace{\alpha_1 = j}_{\in \mathcal{F}_j \subset \mathcal{F}_m},\ \underbrace{X_{j+1} < b, \dots, X_{m-1} < b,\ X_m \geq b}_{\text{in } \mathcal{F}_m}\} \end{aligned}$$

$\in \mathcal{F}_m$

- If $m = n$ then

$$\{\beta_1 = n\} = \{\beta_1 < n\}^c = \left(\bigcup_{j=1}^{n-1} \{\beta_1 = j\}\right)^c \in \mathcal{F}_n$$

An induction argument shows $\alpha_2, \beta_2, \dots, \alpha_n, \beta_n$ are ST.

$\therefore$ X_{α_j} is $\mathcal{F}_{\alpha_j}$-measurable &

X_{β_j} is $\mathcal{F}_{\beta_j}$-measurable

Since $\mathcal{F}_{\alpha_j}$ & $\mathcal{F}_{\beta_j}$ are sub σ-fields of $\mathcal{F}_n$

U_{ab} is $\mathcal{F}_n$-measurable. QED.

Theorem: (Doob's upcrossing ineq)

Suppose $(X_1, X_2, \dots, X_n)$ is a non-negative subM w.r.t filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$. Let $c>0$ & $\mathcal{U}_{0,c}$ denote the upcrossings of $[0,c]$. Then

$$E(\mathcal{U}_{0,c}) \le \frac{E(X_n)-E(X_1)}{c}.$$

Proof: The idea is to write $X_n - X_1$ as a telescoping sum:

$$\begin{aligned} X_n - X_1 = \; & (\overbrace{X_{\beta_n}}^{\text{must be } X_n} - X_{\alpha_n}) + (X_{\alpha_n} - X_{\beta_{n-1}}) \\ & + (X_{\beta_{n-1}} - X_{\alpha_{n-1}}) + (X_{\alpha_{n-1}} - X_{\beta_{n-2}}) \\ & + \\ & \vdots \\ & + (X_{\beta_2} - X_{\alpha_2}) + (X_{\alpha_2} - X_{\beta_1}) \\ & + (X_{\beta_1} - X_{\alpha_1}) + (X_{\alpha_1} - \underbrace{X_1}_{\text{set } \beta_0 = 1}) \end{aligned}$$

Notice that

$$\begin{aligned} E(\text{a red term}) &= E(X_{\alpha_j} - X_{\beta_{j-1}}) \\ &= E\big(E(X_{\alpha_j} - X_{\beta_{j-1}} \mid \mathcal{F}_{\beta_{j-1}})\big) \\ &= E\Big(\underbrace{E(X_{\alpha_j} \mid \mathcal{F}_{\beta_{j-1}})}_{\text{i.e. } \geq X_{\beta_{j-1}} \text{ by the Finite optimal Sampling Thm}} - X_{\beta_{j-1}}\Big) \\ &\geq 0 \end{aligned}$$

Moreover

$$(X_{\beta_j} - X_{\alpha_j}) \begin{cases} > c & \text{if } (X_{\alpha_j}, X_{\beta_j}) \text{ an upcrossing} \\ 0 & \text{if } \beta_j = \alpha_j = n \\ > 0 & \text{if } \beta_j = n \text{ but } \alpha_j < n \\ & \text{since } X_{\alpha_j} = 0 \text{ \& } X_{\beta_j} \geq 0 \text{ by non-negativity} \end{cases}$$

$\therefore$ The sum of all blue terms $\geq c\,\mathcal{U}_{0,c}$

$$\therefore \; E(X_n - X_1) \geq E(\underbrace{\text{blues}}_{\geq c\,\mathcal{U}_{0,c}}) + E(\underbrace{\text{reds}}_{\geq 0}) \geq c\,E(\mathcal{U}_{0,c}).$$

QED

Corollary:

If $(X_1, \dots, X_n)$ is a subM w.r.t. $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ and $-\infty < a < b < \infty$ then

$$E(\mathcal{U}_{a,b}) \leq \frac{E(X_n - a)^+ - E(X_1 - a)^+}{b-a} \leq \frac{E(X_n^+) + a^-}{b-a}.$$

Proof:

For the first inequality:

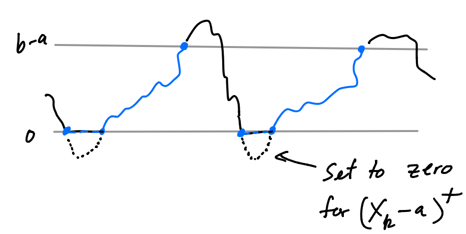
upcrossings of $[a,b]$ in $(X_1, \dots, X_n)$

$\|$

upcrossings of $[0, b-a]$ in $(X_1 - a, \dots, X_n - a)$

$\|$

upcrossings of $[0, b-a]$ in $((X_1 - a)^+, \dots, (X_n - a)^+)$



By Doobs result we therefore have

$$E(\mathcal{U}_{0,b-a}) \leq \frac{E(X_n - a)^+ - E(X_1 - a)^+}{b-a}$$

For the second inequality just notice

$$E(X_n-a)^+ - E(X_1-a)^+ \le E(X_n-a)^+$$
$$= E(X_n+(-a))^+$$
$$\le E(X_n^+ + (-a)^+)$$

since $(\cdot)^+$ is convex

$$= E(X_n^+)+a^-$$

QED

a.e. convergence of martingales

For this section let

$$\mathcal{F}_\infty := \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle$$

for a filtration $(\mathcal{F}_n)_{n\ge 1}$.

Theorem: ($E(X_n^+)$ bdd $\Rightarrow$ a.e. conv)

Let $(X_n)_{n\ge1}$ be a subM w.r.t. filtration $(\mathcal{F}_n)_{n\ge1}$. If $\sup_n E(X_n^+) < \infty$ then $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{P\text{-a.e.}} X_\infty$$

Proof:

For $-\infty < a < b < \infty$ let $U^n_{a,b}$ denote the number of upcrossings of $[a,b]$ from $(X_1,\dots,X_n)$, with $U^\infty_{a,b}$ for $(X_n)_{n\ge1}$.

We must have that

$$0 \le U^n_{a,b} \uparrow U^\infty_{a,b} \quad \text{as } n\to\infty$$

$$\therefore\ E(U^\infty_{a,b}) \overset{\text{Big 3}}{=} \lim_n E(U^n_{a,b})$$
$$\le \lim_n \frac{E(X_n^+)+a^-}{b-a}$$
$$= \frac{\sup_n E(X_n^+)+a^-}{b-a} < \infty$$

$\therefore\ U^\infty_{a,b} < \infty$ P-a.e.

Now

$$P(\liminf_n X_n < \limsup_n X_n)$$
$$\le P\Big(\bigcup_{\substack{a<b\\ a,b\in\mathbb{Q}}} \{U^\infty_{a,b} = \infty\}\Big)$$

must $\exists a,b$ with an infinite # of upcrossings.

$= 0$ since $U^\infty_{a,b} < \infty$ P-a.e.

$$\therefore\ X_n \xrightarrow{P\text{-a.e.}} \underbrace{\limsup_n X_n}_{\text{must be m } \mathcal{F}_\infty \text{ by closure Thm.}} =: X_\infty$$

To see why $X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$

since the limit exists

$$E|X_\infty| = E|\liminf_n X_n| = E(\liminf_n |X_n|)$$
$$\overset{\text{Fatou}}{\le} \liminf_n E|X_n|$$
$$\le \sup_n E|X_n|$$
$$= \sup_n E(2X_n^+ - X_n)$$
$$\le \sup_n \big(2E(X_n^+) - \underbrace{E(X_1)}_{<\infty}\big)$$

since $E(X_1) \le E(X_n)$ by subM

$$< \infty$$

QED

Remark: Note the analog to monotonic sequences when $\sup_n E(X_n^+) < \infty$ plays the roll of $\sup_n x_n^+ < \infty$

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Another bdd type condition is that the X_n^+'s are UI.

Proposition: ($(X_n^+)_{n\geq 1}$ UI $\Rightarrow E(X_n^+)$ bdd)

If $X_1, X_2, \ldots \in L_1(\Omega, \mathcal{F}, P)$ and $(X_n^+)_{n\geq 1}$ are UI then $\sup_n E(X_n^+) < \infty$.

Proof:

$$\sup_n E(X_n^+) \leq \underbrace{\underbrace{\sup_n E(X_n^+ I_{X_n^+ \geq c})}_{\xrightarrow{c\to\infty} 0 \text{ by definition of UI}} + \underbrace{\sup_n E(X_n^+ I_{X_n^+ < c})}_{< c}}_{\text{so for a large enough } c \text{ this is } < \infty.}$$

QED.

Here is an application of the above proposition which comes in handy for subsequent results.

Theorem: (Lévy's Smoothing Martingale)

If $X \in L_1(\Omega, \mathcal{F}, P)$ & $(\mathcal{F}_n)_{n\geq 1}$ is a filtration Then

$$E(X|\mathcal{F}_n) \xrightarrow{n\to\infty} E(X|\mathcal{F}_\infty) \quad \text{a.e. \& in } L_1$$

where $\mathcal{F}_\infty = \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \ldots \rangle$.

Proof:

Let $X_n := E(X|\mathcal{F}_n)$.

We first show the X_n's are UI.

Since $|X_n| = |E(X|\mathcal{F}_n)| \leq E(|X| \,|\, \mathcal{F}_n)$ it will be sufficient to show the $E(|X||\mathcal{F}_n)$'s are UI.

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i.e. show

$$\lim_{c\to\infty} \sup_n \underbrace{\int_{E(|X||\mathcal{F}_n)\geq c} E(|X|\,|\mathcal{F}_n)\, dP}_{} = 0$$

$$\| $$

$$\int_{E(|X||\mathcal{F}_n)\geq c} |X|\, dP \quad \text{since } \{E(|X||\mathcal{F}_n) \geq c\} \text{ is a } \mathcal{F}_n\text{-set}$$

$$\| $$

$$\tilde{P}\big(E(|X||\mathcal{F}_n) \geq c\big) \cdot \underbrace{\int_\Omega |X| dP}_{\text{finite}}$$

where $\tilde{P}(\cdot) = \dfrac{\int_\cdot |X| dP}{\int_\Omega |X| dP}$ is a prob measure.

By Markov's ineq

$$\tilde{P}\big(E(|X||\mathcal{F}_n) \geq c\big) \leq \frac{E\{E(|X||\mathcal{F}_n)\}}{c} = \frac{E|X|}{c} \xrightarrow[c\to\infty]{} 0$$

$\therefore$ the X_n's are indeed UI.

$\therefore$ the X_n^+'s are UI so the $E(X_n^+)$'s bdd

$\therefore$ $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty \quad \text{by subM} \xrightarrow{a.e.} \text{Thm}$$

$\therefore$ $X_n \xrightarrow{L_1} X_\infty$ by the L_p convergence thm since the X_n's are UI

To finish we show $X_\infty \overset{a.e.}{=} E(X|\mathcal{F}_\infty)$.

In particular show:

(i) X_∞ is $\mathcal{F}_\infty$-measurable ✓

(ii) $X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ ✓

(iii) Integrates like X over $\mathcal{F}_\infty$-sets.
i.e. $E(X I_A) = E(X_\infty I_A)\ \forall A \in \mathcal{F}_\infty$

For (iii) notice that $\forall A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$

$$E(XI_A) \overset{(*)}{=} E(X_n I_A) \quad \forall \text{ large } n$$

since $X_n = E(X|\mathcal{F}_n)$

$\therefore \forall A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n \leftarrow$ a field generating $\mathcal{F}_\infty$

$$E(X_\infty I_A) = E(\lim_n X_n I_A) \quad \text{since } X_n \xrightarrow{a.e.} X_\infty$$

$$= \lim_n E(X_n I_A) \quad \text{by UI cond. for } \lim\int = \int\lim$$

$$\overset{(*)}{=} E(XI_A) \quad \text{by } (*)$$

To show $E(X_\infty I_A) = E(XI_A)\ \forall A \in \mathcal{F}_\infty$

Let

$$\mathcal{G} = \{A \in \mathcal{F} : E(X_\infty I_A) = E(XI_A)\}$$

By (*) we have

field & $\therefore$ a π-sys $\rightarrow$ $\bigcup_{n=1}^{\infty} \mathcal{F}_n \subset \mathcal{G} \leftarrow$ a λ-system

$\therefore \quad \lambda\langle \bigcup_{n=1}^{\infty} \mathcal{F}_n \rangle \subset \mathcal{G}$

$\quad \parallel$ Dynkin's π-λ

$\sigma\langle \bigcup_{n=1}^{\infty} \mathcal{F}_n \rangle$

$\therefore E(X_\infty I_A) = E(XI_A)\ \forall A \in \mathcal{F}_\infty$

by "Good sets".

QED

Another way to quantify "bddness" of subMs is with "closers".

Definition: A pair $(X_\bullet, \mathcal{F}_\bullet)$ (r.v., sub σ-field of $\mathcal{F}$) **closes a subM $(X_n)_{n\geq 1}$ on the right** if

$$X_1, X_2, \dots, X_n, \dots\ X_\bullet$$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n, \dots, \mathcal{F}_\bullet$,

i.e if

$$E(X_\bullet | \mathcal{F}_n) \overset{P\text{-a.e.}}{\geq} X_n \quad \forall n \in \mathbb{N}.$$

Definition: $(X_0, \mathcal{F}_0)$ is a **nearest closer** of $(X_n)_{n\geq 1}$ if

$$X_1, X_2, \dots\ X_0, X_\bullet$$

is a subM w.r.t $\mathcal{F}_1, \mathcal{F}_2, \dots \mathcal{F}_0, \mathcal{F}_\bullet$

for every closer $(X_\bullet, \mathcal{F}_\bullet)$ of $(X_n)_{n\geq 1}$.

The existance of a closer is equivalent to the UI condition:

Theorem: ($\exists$ closer $\Leftrightarrow$ X_n^+'s are UI)

If $(X_n)_{n\geq 1}$ is a subM w.r.t filt $(\mathcal{F}_n)_{n\geq 1}$ then

$$\exists \text{ a closer for } (X_n)_{n\geq 1} \Leftrightarrow (X_n^+)_{n\geq 1} \text{ are UI}$$

Proof:

$(\Longrightarrow)$: Suppose $(X_\bullet, \mathcal{F}_\bullet)$ closes $(X_n)_{n\geq 1}$.

$\therefore (X_\bullet^+, \mathcal{F}_\bullet)$ closes the subM $(X_n^+)_{n\geq 1}$

by Trans of Ms.

$$\therefore \underbrace{E(X_\bullet^+ | \mathcal{F}_n)}_{\text{These are UI by proof of Lévy's thm}} \overset{a.e.}{\geq} \underbrace{X_n^+}_{\therefore \text{ these are too.}} \quad \forall n \in \mathbb{N}$$

($\Leftarrow$) Suppose $(X_n^+)_{n\geq 1}$ are UI. 11

$\therefore \exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty$$

we show that $(X_\infty, \mathcal{F}_\infty)$ is a closer.

Case 1: $X_n \geq c > -\infty \quad \forall n \in \mathbb{N}$.

$\therefore$ the X_n^-'s are UI & hence the X_n's are UI.

$\therefore X_n \xrightarrow{L_1} X_\infty$ by the L_p-convergence Thm.

Now to show $E(X_\infty | \mathcal{F}_n) \overset{a.e.}{\geq} X_n$ Let $A \in \mathcal{F}_n$ so that

$$\int_A X_n \, dP \overset{\text{subM}}{\leq} \int_A E(X_{n+m} | \mathcal{F}_n) \, dP$$

$$= \int_A X_{n+m} \, dP$$

$$\xrightarrow{m\to\infty} \int_A X_\infty \, dP \quad \text{since } X_{n+m} I_A \xrightarrow{L_1} X_\infty I_A$$

$$= \int_A E(X_\infty | \mathcal{F}_n) \, dP$$

$\therefore X_n \overset{a.e.}{\leq} E(X_\infty | \mathcal{F}_n)$ by our results on indefinite integrals.

$\therefore (X_\infty, \mathcal{F}_\infty)$ closes $(X_n)_{n\geq 1}$.

Case 2: $X_n \in L_1(\Omega, \mathcal{F}, P)$.

Now case 1 applies to $\underbrace{X_n \vee c}$ where $c > -\infty$.

a subM since $\max(X_n, c)$ is a subM, also UI and bdd below. Also $X_n \vee c \xrightarrow{L_1} X_\infty \vee c$

$\therefore E(X_\infty \vee c | \mathcal{F}_n) \overset{a.e.}{\geq} X_n \vee c$

$\therefore E(X_\infty | \mathcal{F}_n) \overset{a.e.}{\geq} X_n$ by taking limits as $c \to -\infty$ by MCT for $E(\cdot | \mathcal{F}_n)$

QED

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Now we get the "closer" condition for a.e. convergence almost as a corollary.

Theorem: ($\exists$ closer $\Rightarrow$ a.e. convergence)

If the subM $(X_n)_{n\geq 1}$ has a closer then $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ $= \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle$

s.t.

$$X_n \xrightarrow{a.e.} X_\infty$$

and $(X_\infty, \mathcal{F}_\infty)$ is the nearest closer.

Proof:

By the previous result

$\exists$ closer $\Leftrightarrow (X_n^+)$'s are UI

$\Rightarrow E(X_n^+)$'s are bdd

$\Rightarrow \exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty.$$

The proof of the previous Thm also establishes that $(X_\infty, \mathcal{F}_\infty)$ closes $(X_n)_{n\geq 1}$.

Now we just show $(X_\infty, \mathcal{F}_\infty)$ is the <u>nearest</u> closer.

Let $(X_\bullet, \mathcal{F}_\bullet)$ be some closer.

We need to show

$$X_1, X_2, \dots X_\infty, X_\bullet \quad \text{is a subM}$$

w.r.t filtration $\mathcal{F}_1, \mathcal{F}_2, \dots \mathcal{F}_\infty, \mathcal{F}_\bullet$

it's a filtration since $\bigcup_{i=1}^\infty \mathcal{F}_i \subset \mathcal{F}_\bullet$, hence $\underbrace{\sigma\langle \bigcup_{i=1}^\infty \mathcal{F}_i \rangle}_{= \mathcal{F}_\infty} \subset \mathcal{F}_\bullet$

Notice it is sufficient to show

$$(*) \quad E(X_\bullet | \mathcal{F}_\infty) \overset{P\text{-a.e.}}{\geq} X_\infty$$

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To show (*) use Lévy's smoothing result as follows

$$\begin{array}{ccc} E(X_\circ \mid \mathcal{F}_n) & \overset{a.e.}{\geq} & X_n \quad \text{since } (X_\circ, \mathcal{F}_\circ) \text{ is a closer} \\ \big\downarrow{\scriptstyle a.e.} & & {\scriptstyle a.e.}\big\downarrow \\ E(X_\circ \mid \mathcal{F}_\infty) & \overset{a.e.}{\geq} & X_\infty \end{array}$$

as was to be shown.

QED

Likelihood ratio Example from last Lecture

Two models P & Q for a random ω generating an infinite sequence of r.v.s $X = (X_1, X_2, \ldots)$

$$(\Omega, \mathcal{F}, P \text{ or } Q) \xrightarrow{(X_1, X_2, \ldots)(\omega)} (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$$

where

P & Q are distinguishable (a.e.) from one sample of X $\iff PX^{-1} \perp QX^{-1}$

$\iff Q_\infty \perp P_\infty$ where $\mathcal{F}_\infty := \sigma\langle X_i : i \geq 1 \rangle$, $Q_\infty := Q|_{\mathcal{F}_\infty}$

$\iff \dfrac{dQ^a_\infty}{dP_\infty} = 0$

$\overset{?}{\iff} \dfrac{dQ^a_n}{dP_n} \xrightarrow{a.e.} 0$ where $\mathcal{F}_n := \sigma\langle X_1, \ldots, X_n \rangle$, $Q_n := Q|_{\mathcal{F}_n}$

where $\dfrac{dQ^a_n}{dP_n}$ & $\dfrac{dQ^a_\infty}{dP_\infty}$ represents the finite & infinite data likelihood ratio.

Recall that we showed $\dfrac{dQ^a_n}{dP_n}$ is a supM wrt $(\mathcal{F}_n)_{n\geq 1}$ under P

Theorem: $\dfrac{dQ^a_n}{dP_n} \xrightarrow{P\text{-a.e.}} \dfrac{dQ^a_\infty}{dP_\infty}$ as $n \to \infty$.

Proof:

$\dfrac{dQ^a_n}{dP_n}$ is a supM and non-neg by construction

$\therefore \left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ is a non-positive subM

$\therefore (0, \mathcal{F}_\infty)$ closes $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ on the right:

i.e. $-\dfrac{dQ^a_1}{dP_1}, -\dfrac{dQ^a_2}{dP_2}, \ldots, 0$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\infty$ because

$$0 = E(0 \mid \mathcal{F}_n) \geq -\frac{dQ^a_n}{dP_n} \quad \forall n \in \mathbb{N}.$$

By the Closer Thm $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$\dfrac{dQ^a_n}{dP_n} \xrightarrow{a.e.} X_\infty$ & $(-X_\infty, \mathcal{F}_\infty)$ is the nearest closer of $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$.

note $X_\infty \overset{a.e.}{\geq} 0$

We show $X_\infty \overset{a.e.}{=} \dfrac{dQ^a_\infty}{dP_\infty}$

(show $X_\infty \overset{a.e.}{\leq} \dfrac{dQ^a_\infty}{dP_\infty}$)

Use that $-X_\infty$ closes $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ on the right.

Since X_∞ closes we have

$$E(X_\infty \mid \mathcal{F}_n) \overset{P\text{-a.e.}}{\leq} \frac{dQ^a_n}{dP_n}$$

$$\therefore \int_A X_\infty \, dP \leq \int_A \frac{dQ^a_n}{dP_n} dP \leq Q_\infty(A) \quad \forall A \in \bigcup_{k=1}^\infty \mathcal{F}_k$$

Since $\bigcup_{k=1}^{\infty} \mathcal{F}_k$ is a field generating $\mathcal{F}_\infty$

& $X_\infty \overset{a.e.}{\geq} 0$ we can apply "Good sets" to show

$$\underbrace{\int_A X_\infty \, dP}_{\text{a finite measure}} \leq Q_\infty(A) \quad \forall A \in \mathcal{F}_\infty$$

but $\frac{dQ_\infty^a}{dP_\infty}$ is the P-largest (by Leb Decomp) such, so we have

$$X_\infty \overset{P\text{-a.e.}}{\leq} \frac{dQ_\infty^a}{dP_\infty}$$

($\underline{\text{show } X_\infty} \overset{a.e.}{\geq} \frac{dQ_\infty^a}{dP_\infty}$)

Use that $-\frac{dQ_\infty^a}{dP_\infty}$ closes $\left(-\frac{dQ_n^a}{dP_n}\right)_{n\geq 1}$ on the right.

Indeed this follows from the same method used to show $\left(\frac{dQ_n^a}{dP_n}\right)_{n\geq 1}$ is a supM.

nearest closer

i.e. $-\frac{dQ_1^a}{dP_1}, -\frac{dQ_2^a}{dP_2}, \ldots, -X_\infty, -\frac{dQ_\infty^a}{dP_\infty}, 0$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\infty, \mathcal{F}_\infty, \mathcal{F}_\infty$

$$\therefore \quad E\left(-\frac{dQ_\infty^a}{dP_\infty} \,\middle|\, \mathcal{F}_\infty\right) \overset{a.e.}{\geq} -X_\infty$$

$$\overset{a.e.}{=} -\frac{dQ_\infty^a}{dP_\infty}$$

QED

Example: Polyas Urn

L_p convergence

Theorem: (subM L_p convergence Thm)

Suppose $1 \le p < \infty$ and $(X_n)_{n \ge 1}$ is a subM wrt filtration $(\mathcal{F}_n)_{n \ge 1}$.

If $|X_n|^p$ is UI then $\exists X_\infty \in L_p(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \longrightarrow X_\infty \quad P\text{-a.e. \& in } L_p$$

where $(X_\infty, \mathcal{F}_\infty)$ is the nearest closer of $(X_n)_{n \ge 1}$

$= \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle$

Theorem: (Checking $|X_n|^p$ UI for $X_n \ge 0$ subM)

If $(X_n)_{n \ge 1}$ forms a non-neg subM & $p > 1$, then

$$X_n^p \text{ are UI} \iff \sup_n E(X_n^p) < \infty$$

$$\iff E\left(\sup_n X_n^p\right) < \infty$$