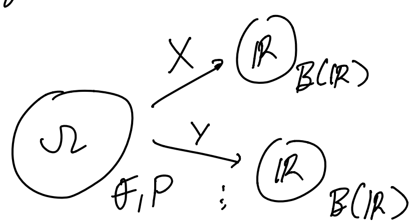
Lecture 18: L_p spaces of r.v.s

Just as in the previous lecture we will be fixing a probability space $(\Omega, \mathcal{F}, P)$ and consider the collection of r.v.s defined on that space.

In particular:

Assumption for the remainder of this lecture:

Suppose $X, Y, X_1, X_2, \dots$ are r.v.s all defined on the same probability space



L_p spaces $(p \geq 1)$

Definition: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $p \geq 1$.
The **L_p space of r.v.s** defined on $(\Omega, \mathcal{F}, P)$ is defined as

$$\{ X : \Omega \to \mathbb{R} \text{ s.t. } X \text{ (m) } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ \& } E|X|^p < \infty \}$$

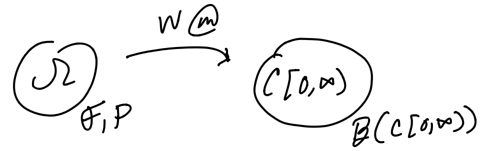
and denoted $L_p(\Omega, \mathcal{F}, P) \equiv L_p(P) \equiv L_p$.

Remark: We work with random variables but most of the following results can be extended to the set of random vectors all mapping into $\mathbb{R}^d$.

We will be interested in the metric & geometric properties of L_p & interpreting some classic functional analysis results from a probabilistic perspective.

Example:

Let W_t be Brownian Motion so that



Since for each fixed $t \in [0, \infty)$ W_t is a r.v. defined on $(\Omega, \mathcal{F}, P)$ we can consider the stochastic process $(W_t : t \in [0,\infty))$ as a collection of r.v.s indexed by t

$$\{ W_t : t \in [0, \infty) \} \subset L_2(\Omega, \mathcal{F}, P)$$

Definition:

For $X, Y \in L_p(\Omega, \mathcal{F}, P)$ define

$$\|X\|_p := (E|X|^p)^{1/p}$$

$$d_p(X,Y) := \|X - Y\|_p$$

Theorem (Hölder)

For any two r.v.s X & Y defined on $(\Omega, \mathcal{F}, P)$ and $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$E|XY| \leq \underbrace{\|X\|_p \|Y\|_q}$$

could be $0 \cdot \infty = 0$ by convention

Moreover if $X, Y \in \mathcal{Q}(\Omega, \mathcal{F}, P)$ then

$$|E(XY)| \leq \|X\|_p \|Y\|_q$$

Proof: We already proved this in Lecture 11.

Theorem:

$$1 \le p < q \implies L_q(\Omega, \mathcal{F}, P) \subset L_p(\Omega, \mathcal{F}, P).$$

Proof: Like Hölder, this comes from Young's inequality: $a^{w_1} b^{w_2} \le w_1 a + w_2 b$ when $w_1, w_2, a, b > 0$ and $w_1 + w_2 = 1$.

Indeed set $w_1 = \frac{p}{q} < 1$ & $w_2 = 1 - w_1$. Then

$$X \in L_q \implies E|X|^q = E|X|^{p\frac{q}{p}}$$
$$= E\left(\left(|X|^p\right)^{w_1} 1^{w_2}\right)$$
$$\le \underbrace{w_1 E|X|^p + w_2}_{<\infty}$$

QED.

Theorem: ($\|\cdot\|_p$ is a pseudo-norm)

$\forall X \in L_p(\Omega, \mathcal{F}, P)$ we have that

(i) $\|X\|_p \ge 0$

(ii) $\|X\|_p = 0 \implies X = 0$ P-a.e. — hence its only pseudo. By "a.e. useful" Thm.

(iii) $\|cX\|_p = |c|\,\|X\|_p \quad \forall c \in \mathbb{R}$

(iv) $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$ (Minkowski's ineq)

Proof:

we just need to show (iv).

$$E|X+Y|^p = E\left(|X+Y|\,|X+Y|^{p-1}\right)$$
$$\le E\left(|X|\,|X+Y|^{p-1}\right) + E\left(|Y|\,|X+Y|^{p-1}\right)$$

Now notice $\frac{1}{p} + \frac{1}{q} = 1 \implies 1 + \frac{p}{q} = p$
$\implies \frac{p}{q} = p - 1$

$$= E\left(|X|\,|X+Y|^{p/q}\right) + E\left(|X|\,|X+Y|^{p/q}\right)$$
$$\overset{\text{Hölder}}{\le} \|X\|_p \,\big\| |X+Y|^{p/q} \big\|_q + \|Y\|_p \,\big\| |X+Y|^{p/q} \big\|_q$$
$$= \left(\|X\|_p + \|Y\|_p\right) \underbrace{\left(E|X+Y|^p\right)^{1/q}}_{\text{divide this out of both sides}}$$

$$\therefore \underbrace{\left(E|X+Y|^p\right)^{1-\frac{1}{q}}}_{=\|X+Y\|_p} \le \|X\|_p + \|Y\|_p$$

QED.

Remark: The previous Thm show's $d_p(X,Y)$ is a pseudo-metric on L_p.

It will also be useful to note that $\|\cdot\|_p$ is continuous w.r.t d_p.

Theorem: $X, Y \in L_p \implies \big|\|X\|_p - \|Y\|_p\big| \le d_p(X,Y)$

Proof:

Minkowski: $\|X\|_p \le \|X-Y\|_p + \|Y\|_p$

$$\|Y\|_p \le \underbrace{\|X-Y\|_p}_{=d_p(X,Y)} + \|X\|_p$$

QED

L_p convergence

Here we study completeness, closure & separability of L_p and prove the "L_p convergence theorem" which will be useful later.

Definition:

$$X_n \xrightarrow{L_p} X \text{ iff } E\underbrace{|X_n - X|^p}_{\text{technically no requirement that } X_n, X \in L_p} \to 0 \text{ as } n \to \infty.$$

Theorem (uniqueness of limits)

$$X_n \xrightarrow{L_p} X \ \& \ X_n \xrightarrow{L_p} Y \Rightarrow X = Y \ P\text{-a.e.}$$

Proof:

Note the following useful identity which follows by convexity of $|\cdot|^p$

$$\left|\frac{x+y}{2}\right|^p \le \frac{1}{2}|x|^p + \frac{1}{2}|y|^p$$

$$\therefore \quad E|X-Y|^p \underset{\pm X_n}{\le} 2^p \Big(\underbrace{\tfrac{1}{2}E|X-X_n|^p + \tfrac{1}{2}E|Y-X_n|^p}_{\to 0 \text{ as } n\to\infty} \Big)$$

QED

Theorem: (Cauchy Criteria)

$X_n \xrightarrow{L_p}$ to some r.v. X iff

$$\lim_n \lim_m E|X_m - X_n|^p = 0$$

Proof:

$(\Longrightarrow)$
$$E|X_m - X_n|^p \underset{\pm X}{\le} 2^p \Big(\underbrace{E\left|\frac{X_m - X}{2}\right|^p + E\left|\frac{X_n - X}{2}\right|^p}_{\to 0 \text{ as } m,n\to\infty} \Big)$$

$(\Longleftarrow)$

$$P(|X_m - X_n| \ge \varepsilon) \le \frac{E|X_m - X_n|^p}{\varepsilon^p}$$

implies $\{X_n\}_{n\ge 1}$ is Cauchy for convergence in probability.

$\therefore \exists$ r.v. X s.t. $X_n \xrightarrow{P} X$

$\therefore \exists n_k$ s.t. $X_{n_k} \xrightarrow[k\to\infty]{a.e.} X$ by sub-sub-seq Thm.

$\therefore \quad |X_n - X_{n_k}|^p \xrightarrow[k\to\infty]{a.e.} |X_n - X|^p \ \forall n$

by continuous mapping since $X_n - X_{n_k} \xrightarrow{a.e.} X_n - X$

Now

$$E|X_n - X|^p \le \liminf_k E|X_n - X_{n_k}|^p, \text{ Fatou}$$
$$\le \limsup_k E|X_n - X_{n_k}|^p$$
$$\le \limsup_m E|X_n - X_m|^p$$

Taking $\lim_n$ of both sides gives

$$X_n \xrightarrow{L_p} X.$$

QED

Theorem (L_p is Polish w.r.t d_p)

If $p \ge 1$ then $L_p(\Omega, \mathcal{F}, P)$ is a linear space which is closed & complete w.r.t d_p.

If, in addition, $\mathcal{F}$ is countably generated then $L_p(\Omega, \mathcal{F}, P)$ is separable.

Proof:

(L_p is Linear): Follows by $|X+Y|^p \le 2^p(\frac{1}{2}|X|^p + \frac{1}{2}|Y|^p)$

(L_p is closed): if $X_n \in L_p$ & $X_n \xrightarrow{L_p} X$

then $$|X|^p \le 2^p \left(\frac{|X_n|^p}{2} + \frac{|X - X_n|^p}{2} \right)$$

Taking expected value of both sides gives the result.

(L_p is complete): Follows by the Cauchy Criteria thm.

(L_p is separable):

Suppose $\mathcal{F} = \sigma\langle \mathcal{Q} \rangle$ where $\mathcal{Q}$ is a countable collection of generators.

Let $X \in L_p$. By the structure thm of Lecture 9 $\exists$ bdd simple X_n's s.t.

$$X_n \xrightarrow{a.e.} X$$

where $X_n \in L_p$ by bddness.

7

Also, although not explicitly stated in the structure Thm, the X_n's satisfy $|X_n| \le |X|$

$$\therefore \ |X - X_n|^p \le 2^p\left(\frac{|X|^p}{2} + \frac{|X_n|^p}{2}\right)$$

$$\le 2^p |X|^p$$

so by the DCT we have

$$E|X_n - X|^p \to 0 \text{ as } n \to \infty.$$

Since X_n is simple, it has the form

$$X_n = \sum_{k=1}^m c_k I_{F_k}, \quad F_k \in \mathcal{F} = \sigma\langle \mathcal{Q} \rangle = \sigma\langle f\langle \mathcal{Q}\rangle\rangle$$

Using a result in Lecture 5 we can find $\hat{F}_k \in f\langle \mathcal{Q}\rangle$, $\hat{c}_k \in \mathbb{Q}$ s.t.

$$\| X_n - \hat{X}_n \|_p \le \frac{1}{n}$$

where $\hat{X}_n = \sum_{k=1}^m \hat{c}_k I_{\hat{F}_k}$ (hint: choose $\hat{F}_k$ so that $P(F_k \triangle \hat{F}_k) < \left[\frac{(1/n)}{2^k |c_k|}\right]^p$).

For this $\hat{X}_n$ we have

$$\| X - \hat{X}_n \|_p \to 0 \ \& \ \hat{X}_n \in L.$$

Since any field generate by a countable collection of events is countable, $f\langle \mathcal{Q}\rangle$ is countable.

$\therefore$ The collection of all such approximating $\hat{X}_n$'s forms a dense countable subset of $L_p(\Omega, \mathcal{F}, P)$.

QED

8

Recall the definition of Uniform integrability (UI) specialized to r.v.s:

$X_1, X_2, \dots$ are UI iff

$$\lim_{c \to \infty} \sup_n E\left(|X_n| I_{|X_n| \ge c}\right) = 0$$

when talking about limits its understood we can drop any finite number of X_n's

Lets also recall the UI theorems we did in Lecture 10 but specialized to r.v.s

Theorem: (UI for $\lim E = E \lim$)

If $X_n \xrightarrow{a.e.} X$ & the X_n's are UI then $EX_n \to EX$ & $X, X_n \in L_1$

Theorem: (UI converse)

If $X_n \xrightarrow{a.e.} X$ & $EX_n \to EX$ & $X, X_n \in L_1$ then the X_n's are UI.

Here is our L_p convergence Thm which effectively shows

$$\xrightarrow{L_p} \ \equiv \ \xrightarrow{P} + |X_n|^p\text{'s are UI}$$

Theorem: (L_p convergence Thm)

Let $X_n \in L_p$ for all n. Then the following are equivalent:

(i) $X_n \xrightarrow{L_p} X$

(ii) $X_n \xrightarrow{P} X$ and $E|X_n|^p \to E|X|^p < \infty$

(iii) $X_n \xrightarrow{P} X$ and the $|X_n|^p$'s are UI

Proof:

(i) $\Rightarrow$ (ii)

we already know $X_n \xrightarrow{P} X$ by Markov's ineq. $X \in L_p$ since L_p is closed. Finally by $\big|\|X\|_p - \|X_n\|_p\big| \le d_p(X, X_n) \to 0$ we have

$$E|X_n|^p \to E|X|^p < \infty$$

(ii) $\Rightarrow$ (i).

Here is where we use the Probability Sandwich result proved in the last lecture.

$$0 \le |X_n - X|^p \le \underbrace{2^p\left(\tfrac{1}{2}|X_n|^p + \tfrac{1}{2}|X|^p\right)}_{\downarrow P \;\; 2^p|X|^p \;=: Y} =: Y_n$$

$|X_n - X|^p \xrightarrow{P} 0$ by continuous mapping since $X_n - X \xrightarrow{P} 0$

since $Y_n, Y \in L_1$ & $EY_n \to EY$ by assumption sandwich says that $E|X_n - X|^p \to 0$.

(ii) $\Rightarrow$ (iii):

using the sub-sub-seq characterization of $\xrightarrow{P}$ one can extend the UI converse to require $\xrightarrow{P}$ instead of $\xrightarrow{a.e.}$.

$\therefore$ From (ii) we have $X_n \xrightarrow{P} X$ & by continuous mapping $|X_n|^p \xrightarrow{P} |X|^p$

Also by assump $E|X_n|^p \to E|X|^p < \infty$ so that $X_n, X \in L_p$ for suff large n

$\therefore$ The X_n's are UI by UI converse

(iii) $\Rightarrow$ (ii):

This one similarly follows from an $\xrightarrow{P}$ version of the UI theorem

QED.

Hilbert Space Geometry of $L_2(\Omega, \mathcal{F}, P)$

For L_2 Hölder gives $E(XY) \le \|X\|_2 \|Y\|_2 < \infty$

$\therefore$ we can define an inner product on L_2 defined as

$$\langle X, Y \rangle := E(XY)$$

Much of what statisticians do basically corresponds to geometric operations w.r.t. $\langle \cdot, \cdot \rangle$. The geometry of $(L_2, \langle \cdot, \cdot \rangle)$ is the geometry of variation & co-variation:

i.e. when $E(X) = E(Y) = 0$ then

$$\langle X, Y \rangle = \mathrm{cov}(X, Y)$$
$$\langle X, X \rangle = \|X\|_2^2 = \mathrm{var}(X)$$
$$\|X\|_2 = sd(X).$$

Basic Properties of $\langle \cdot, \cdot \rangle$:

$\forall X, Y \in L_2(\Omega, \mathcal{F}, P)$

(1) $\langle X, X \rangle \ge 0$

(2) $\langle X, X \rangle > 0$ unless $X = 0$ P-a.e.

(3) $\langle X, Y \rangle = \langle X, Y \rangle$

(4) $\langle X, Y + \alpha Z \rangle = \langle X, Y \rangle + \alpha \langle X, Z \rangle$

(5) $|\langle X, Y \rangle| \le \|X\|_2 \|Y\|_2$

(6) $X_n \xrightarrow{L_2} X \Rightarrow \langle X_n, Y \rangle \to \langle X, Y \rangle$

which is true since

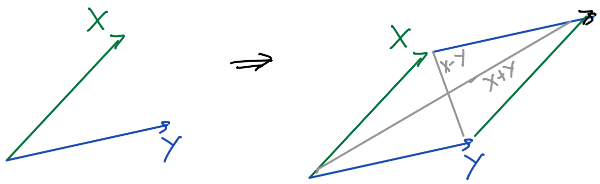
$$|\langle X_n, Y \rangle - \langle X, Y \rangle| = |\langle X_n - X, Y \rangle| \le \|X_n - X\|_2 \|Y\|_2$$

(7) $\|X + Y\|_2^2 = \|X\|_2^2 + 2\langle X, Y \rangle + \|Y\|_2^2$

(8) $\|X+Y\|_2^2 + \|X-Y\|_2^2 = 2\|X\|_2^2 + 2\|Y\|_2^2$

this follows by adding

$$\|X+Y\|_2^2 = \|X\|_2^2 + 2\langle X,Y\rangle + \|Y\|_2^2$$
$$\|X-Y\|_2^2 = \|X\|_2^2 - 2\langle X,Y\rangle + \|Y\|_2^2$$



To see an example of the L_2 geometry viewpoint in estimation problems suppose $Z(x)$ is a mean zero continuous Gaussian random field defined on a region $S \subset \mathbb{R}^d$.

i.e. there exists $(\Omega, \mathcal{F}, P)$ s.t.

$$\underset{\mathcal{F},P}{\Omega} \xrightarrow{(Z(x):\, x\in\mathbb{R}^d)} \underset{\mathcal{B}(C(\mathbb{R}^d))}{C(\mathbb{R}^d)}$$

and $Z(x)$ has Gaussian f.d.d.s & $E(Z(x))=0$ $\forall x\in S$

$\therefore$ The collection of r.v.s $Z(x)$ indexed by $x\in S$ satisfies

$$\{Z(x):\, x\in S\} \subset L_2(\Omega,\mathcal{F},P)$$

In random field theory we often study the following Hilbert space

$$\begin{aligned} L(S) &:= \text{closed linear span (in } L_2) \text{ of } \{Z(x): x\in S\} \\ &= \text{closure with } L_2 \text{ limits of } \Big\{\sum_{k=1}^{n} c_k Z(x_k):\, x_k\in S,\, c_k\in\mathbb{R}\Big\} \\ &\subset L_2(\Omega,\mathcal{F},P) \end{aligned}$$

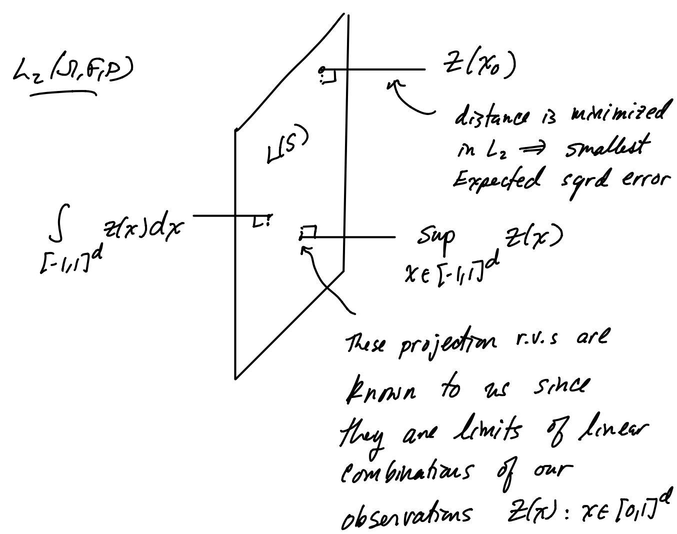
Now "Best linear prediction" (BLP) of some unobserved $L_2(\Omega,\mathcal{F},P)$ r.v. simply given by projection:

e.g. Suppose $Z(x)$ is defined on $x\in\mathbb{R}^d$ but only observed on $x\in[0,1]^d$.

If we want to predict things like

- $Z(x_0)$ for $x_0\notin[0,1]^d$
- $\int_{[-1,1]^d} Z(x)\,dx$
- $\sup_{x\in[-1,1]^d} Z(x)$

So long as these r.v.s are in L_2 the BLP is a projection $L(S)$



Definition $X \in L_2$ is orthogonal to $Y \in L_2$ iff $\langle X, Y \rangle = 0$ (denoted by $X \perp Y$).

Theorem: (Projection Thm)

Let S be a closed linear subspace of L_2 & $Y \in L_2$. Then $\exists$ a P-a.e. unique $P_S Y \in S$ s.t.

$$\|Y - P_S Y\|_2 = \inf_{X \in S} \|Y - X\|_2.$$

Moreover $P_S Y$ is characterized by the following two properties

(1) $P_S Y \in S$

(2) $\underbrace{(Y - P_S Y)}_{\text{prediction residual.}} \perp X \quad \forall X \in S$

Proof:

(Find $P_S Y$) Let $X_n \in S$ s.t.

$$\|Y - X_n\|_2 \xrightarrow{n \to \infty} \inf_{X \in S} \|Y - X\|_2$$

Now we show $\{X_n\}_{n \geq 1}$ is Cauchy with the Parallelogram Thm:

$$\|(X_n - Y) + (X_m - Y)\|_2^2 + \|\underbrace{X_n - X_m}_{(X_n - Y) - (X_m - Y)}\|_2^2$$

$$= \underbrace{2\|X_n - Y\|_2^2 + 2\|X_m - Y\|_2^2}_{=: I_{nm}}$$

where $\lim_{nm} I_{nm} = 4\left(\inf_{X \in S} \|Y - X\|_2\right)^2$

$$\therefore \|X_n - X_m\|_2^2 = I_{nm} - \|\underbrace{(X_n - Y) + (X_m - Y)}_{2\left(\frac{X_n + X_m}{2} - Y\right)}\|_2^2$$

$$= I_{nm} - 4\Big\|\underbrace{\tfrac{X_n + X_m}{2}}_{\in S \text{ by linearity}} - Y\Big\|_2^2$$

$$\leq I_{nm} - 4\left(\inf_{X \in S} \|X - Y\|_2\right)^2$$

$$\longrightarrow 0 \quad \text{as } n, m \to \infty$$

$\therefore \{X_n\}_{n \geq 1}$ is Cauchy & by completeness

$\exists P_S Y \in L_2$ s.t. $\underbrace{X_n}_{\in S} \xrightarrow{L_2} \underbrace{P_S Y}_{\in S}$ implies since S is closed

Also, for this $P_S Y$ we have

$$\inf_{X \in S} \|X - Y\|_2 \leq \|P_S Y \underset{\pm X_n}{-} Y\|_2$$

$$\leq \underbrace{\|P_S Y - X_n\|_2}_{\to 0} + \underbrace{\|X_n - Y\|_2}_{\to \inf_{X \in S} \|X - Y\|_2}$$

$$\therefore \inf_{X \in S} \|X - Y\|_2 = \|P_S Y - Y\|_2$$

(Show $P_S Y$ is unique P-a.e.)

Suppose $X_0 \in S$ s.t. $\|X_0 - Y\|_2 = \inf \cdots$

Again by the Parallelogram Thm

$$\|(X_0 - Y) + (P_S Y - Y)\|_2^2 + \|X_0 \underset{\pm Y}{-} P_S Y\|_2^2$$

$$= 2\underbrace{\|X_0 - Y\|_2^2}_{\inf^2} + 2\underbrace{\|P_S Y - Y\|_2^2}_{\inf^2}$$

15

$$\therefore \; \|X_0 - P_S Y\|_2^2 \le 4\inf{}^2 - \Big\|2\Big(\underbrace{\tfrac{X_0 + P_S Y}{2}}_{\in S} - Y\Big)\Big\|_2^2$$

$$\le 4\inf{}^2 - 4\inf{}^2 = 0$$

$$\therefore \; X_0 \overset{P.a.e.}{=} P_S Y.$$

(Show $(Y - P_S Y) \perp X, \; \forall X \in S$):

Choose $X \in S$ s.t. $X \neq 0$ a.e. (if $X = 0$ then the result is true).

For $c \in \mathbb{R}$ set

$$f(c) = \| Y - (P_S Y - cX)\|_2^2$$

Let $c_{min} := \underset{c \in \mathbb{R}}{\operatorname{argmin}} f(c)$.

Two ways to compute c_{min}

1st: $c_{min} = 0$ by minimizing properties of $P_S Y$.

2nd:

$$f(c) = \|Y - P_S Y\|_2^2 + 2c\langle Y - P_S Y, X\rangle + c^2 \|X\|_2^2$$

$$\therefore \; f'(c) = 2\langle Y - P_S Y, X\rangle + 2c\|X\|_2^2$$

$$\therefore \; c_{min} = -\frac{\langle Y - P_S Y, X\rangle}{\|X\|_2^2}$$ ← need $\|X\|_2^2 > 0$

$$= 0 \quad \text{since } c_{min} = 0$$

$$\therefore \; \langle Y - P_S Y, X\rangle = 0$$

16

(Show $W \in S$ & $(Y - W) \perp X \; \forall X \in S \Rightarrow W = P_S Y$)

$\forall X \in S$ we have

$$\|X - Y\|_2^2 = \|X - W\|_2^2 + 2\underbrace{\langle \underbrace{X - W}_{\in S}, Y - W\rangle}_{0} + \|Y - W\|_2^2$$

$\pm W$

$$\therefore \; \inf_{X \in S} \|X - Y\|_2^2 = \Big[\underbrace{\inf_{X \in S} \|X - W\|_2^2}_{=0 \text{ since } W \in S}\Big] + \|Y - W\|_2^2$$

$$= \|W - Y\|_2^2$$

$\therefore$ $W = P_S Y$ since $P_S Y$ is the unique such r.v.

QED

The next theorem shows that to compute a projection you define coordinates aligned with the space your projecting to

Theorem: (projection in coordinates)

Suppose $X_1, X_2, \ldots \in L_2(\Omega, \mathcal{F}, P)$ are orthonormal and let

$$S = \overline{\operatorname{span}}\{X_n : n \in \mathbb{N}\} =: \left(\begin{array}{l}\text{The collection of } L_2 \\ \text{limits of finite} \\ \text{linear combinations} \\ \text{of the } X_n\text{'s}\end{array}\right)$$

Then S is a closed linear subset of L_2 & $\forall Y \in L_2(\Omega, \mathcal{F}, P)$ the following holds

$$\underbrace{P_S Y}_{\text{projection of } Y \text{ onto } S} \overset{L_2}{=} \sum_{n=1}^{\infty} \langle X_n, Y\rangle X_n$$

This means

$$\sum_{n=1}^{N} \langle X_n, Y\rangle X_n \xrightarrow[N \to \infty]{L_2} P_S Y$$

Proof:

(S is closed and linear):

$W, Z \in S \Rightarrow \begin{array}{l} W_n \xrightarrow{L_2} W \\ Z_n \xrightarrow{L_2} Z \end{array}$ where W_n, Z_n is a finite linear comb of the X_n's

$$\Rightarrow \| aW_n + bZ_n - (aW+bZ) \|_2 \le |a| \, \underbrace{\|W_n - W\|_2 + |b| \, \|Z_n - Z\|_2}_{\to 0}$$

$$\Rightarrow aW + bZ \in S$$

$\therefore$ S is linear

To see that S is closed let

$$Z_n \xrightarrow{L_2} Z$$

where $Z_n \in S$. For each n let $\hat{Z}_n$ be a finite linear combination of the X_n's s.t.

$$\| \hat{Z}_n - Z_n \|_2 \le \frac{1}{n}$$

$\therefore \| Z - \hat{Z}_n \|_2 \le \| Z - Z_n \|_2 + \frac{1}{n} \to 0$ as $n \to \infty$

$\therefore Z \in S$ by definition.

($\sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i$ exists as a infinite sum)

Let $S_n := \operatorname{span}\{X_1, \dots, X_n\}$

Notice that $P_{S_n} Y = \sum_{i=1}^{n} \langle X_i, Y \rangle X_i$ since

$$\langle Y - P_{S_n} Y, X_j \rangle = \langle Y, X_j \rangle - \langle X_j, Y \rangle = 0$$

$\forall j = 1, \dots, n$. Also notice that $P_{S_n} Y$ decreases length in that

$$\|Y\|_2^2 = \|Y - P_{S_n} Y\|_2^2 + \|P_{S_n} Y\|_2^2 \ge \|P_{S_n} Y\|_2^2$$

$$\therefore \sum_{i=1}^{n} \langle X_i, Y \rangle^2 = \|P_{S_n} Y\|_2^2 \le \|Y\|_2^2 < \infty$$

$$\therefore \sum_{i=1}^{\infty} \langle X_i, Y \rangle^2 < \infty$$

$\therefore \left\{ \sum_{i=1}^{n} \langle X_i, Y \rangle X_i \right\}_{n \ge 1}$ is Cauchy in L_2 and has a L_2 limit, denoted $\sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i$.

(Show $P_S Y = \sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i$):

Since $\underbrace{\sum_{i=1}^{n} \langle X_i, Y \rangle X_i}_{= P_{S_n} Y} \xrightarrow[n \to \infty]{L_2} \sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i$

we have that $\forall j \in \mathbb{N}$

$$\langle P_{S_n} Y, X_j \rangle \xrightarrow[n \to \infty]{} \left\langle \sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i, X_j \right\rangle$$

$$\therefore \underbrace{\langle Y - P_{S_n} Y, X_j \rangle}_{\text{This is eventually zero for large enough } n} \xrightarrow[n \to \infty]{} \underbrace{\left\langle Y - \sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i, X_j \right\rangle}_{\text{This is } 0}$$

$\therefore P_S Y = \sum_{i=1}^{\infty} \langle X_i, Y \rangle X_i$ since $Y - P_S Y \perp X$ $\forall X \in S$.

QED.

Definition: $\{X_i : i \in \mathcal{I}\} \subset L_2(\Omega, \mathcal{F}, P)$ is an orthonormal basis (ONB) if finite linear combinations of the X_i's are dense in $L_2(\Omega, \mathcal{F}, P)$.

Theorem: (characterizing a ONB)

If $\{X_n\}_{n\geq 1} \subset L_2(\Omega,\mathcal{F},P)$ are orthonormal then the following are equivalent:

(i) $\{X_n\}_{n\geq 1}$ is a ONB

(ii) $\underset{\text{in } L_2}{Y} = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i \quad \forall Y \in L_2(\Omega,\mathcal{F},P)$

(iii) $\langle Y, Z\rangle = \sum_{i=1}^{\infty} a_i b_i \quad \forall Y,Z \in L_2(\Omega,\mathcal{F},P)$

where $a_i := \langle X_i, Y\rangle$ & $b_i := \langle X_i, Z\rangle$.

(iv) $\|Y\|_2^2 = \sum_{i=1}^{\infty} a_i^2 \quad \forall Y \in L_2(\Omega,\mathcal{F},P)$

where $a_i := \langle X_i, Y\rangle$.

Proof:

(ii) $\Rightarrow$ (i): Trivial

(i) $\Rightarrow$ (ii): Almost trivial.

If $S := \text{span}\{X_i : i \in \mathbb{N}\}$ then

$$\underset{\overset{\parallel}{Y}}{\mathcal{P}_S Y} = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$$

by "Computing a projection Thm"

Since (i) implies $S = L_2(\Omega,\mathcal{F},P)$.

(ii) $\Rightarrow$ (iii): Set $a_i = \langle X_i, Y\rangle$, $b_i = \langle X_i, Y\rangle$ and notice

$$\langle Y, Z\rangle = \left\langle \sum_{i=1}^{\infty} a_i X_i, \sum_{i=1}^{\infty} b_i X_i \right\rangle$$

$$= \lim_n \left\langle \sum_{i=1}^{n} a_i X_i, \sum_{i=1}^{\infty} b_i X_i \right\rangle$$

$$= \lim_n \lim_m \underbrace{\left\langle \sum_{i=1}^{n} a_i X_i, \sum_{i=1}^{m} b_i X_i \right\rangle}_{\sum_{i=1}^{n} a_i b_i}$$

(iii) $\Rightarrow$ (iv): Trivial

(iv) $\Rightarrow$ (ii):

Let $S := \text{span}\{X_i : i \in \mathbb{N}\}$.

Since we know $\mathcal{P}_S Y = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$ by "Projection in coordinates Thm" it will be sufficient to show

$$\|\mathcal{P}_S Y - Y\|_2^2 = 0$$

Since $\underbrace{\|Y\|_2^2}_{\sum_{i=1}^{\infty}\langle X_i, Y\rangle^2 \text{ by (iv)}} = \|Y - \mathcal{P}_S Y\|_2^2 + \underbrace{\|\mathcal{P}_S Y\|_2^2}_{\sum_{i=1}^{\infty}\langle X_i, Y\rangle^2}$

we therefore have $\|Y - \mathcal{P}_S Y\|_2^2$ as was to be shown

QED

Our last Thm on projections shows that the ordering of the ONB is irrelevant.

Theorem: (Permuting coordinates)

If $\{X_n\}_{n\geq 1} \subset L_2(\Omega,\mathcal{F},P)$ is an orthonormal collection and $Y \in L_2(\Omega,\mathcal{F},P)$ then

$$\sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i \overset{L_2}{=} \sum_{i=1}^{\infty} \langle X_{\pi(i)}, Y\rangle X_{\pi(i)}$$

for any permutation $\pi : \mathbb{N} \to \mathbb{N}$.

Proof:

Let $Z = \sum_{i=1}^{\infty} \langle X_i, Y\rangle X_i$ and

$$Z_\pi = \sum_{i=1}^{\infty} \langle X_{\pi(i)}, Y\rangle X_{\pi(i)}.$$

Now

$$\langle Z, Z_\pi\rangle = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle X_i, Y\rangle \langle X_{\pi(j)}, Y\rangle \underbrace{\langle X_i, X_{\pi(j)}\rangle}_{= \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{o.w.} \end{cases}}$$

$$= \sum_{i=1}^{\infty} \langle X_i, Y\rangle^2$$

$$= \|Z\|_2^2$$

$$\therefore\ \|Z - Z_\pi\|_2^2 = \|Z\|_2^2 + \|Z_\pi\|_2^2 - 2\langle Z, Z_\pi\rangle$$

$$= \underbrace{\sum_{i=1}^\infty \langle X_{\pi(i)}, Y\rangle^2 - \sum_{i=1}^\infty \langle X_i, Y\rangle^2}$$

$= 0$ since positive sums can be arbitrarily re-ordered.

QED

Remark: Notice that whenever $\mathcal{F}$ is countably generated, $L_2(\Omega, \mathcal{F}, P)$ is separable so there exists a countable dense subset $\{Y_n\}_{n\geq 1}$.

In this case one can use Gram-Schmidt to construct an ONB $\{X_n\}_{n\geq 1}$ as follows:

$$X_1 := \frac{Y_1}{\|Y_1\|_2}$$

$$X_2 := \frac{Y_2 - \langle X_1, Y_2\rangle X_1}{\|Y_2 - \langle X_1, Y_2\rangle X_1\|_2}$$

$\leftarrow$ project out X_1

$\leftarrow$ Normalize

$\vdots$

$$X_n := \frac{Y_n - \sum_{i=1}^{n-1} \langle X_i, Y_n\rangle X_i}{\|Y_n - \sum_{i=1}^{n-1} \langle X_i, Y_n\rangle X_i\|_2}$$

$\leftarrow$ Project out $\{X\}_{i=1}^{n-1}$

$\leftarrow$ Normalize

$\vdots$

Now each Y_n is a finite linear comb of the X_i's so $\{X_n\}_{n\geq 1}$ are dense in L_2. If we identify r.v.'s with the equivalence classes of P-a.e. modifications then $L_2(\Omega, \mathcal{F}, P)$ becomes a Hilbert Space (complete, separable, linear vector space with pos. def. inner product).

Projection Example

$\Omega = [-1, 1]$

$\mathcal{F} = \mathcal{B}([-1,1])$

$P = \frac{1}{2} \times$ Lebesgue measure on $[-1,1]$

$S = \{X \in L_2(\Omega, \mathcal{F}, P) : X(\omega) = X(-\omega) \ P\text{-a.e}\}$

Clearly S is linear. It is also closed since $\underbrace{X_n}_{\in S} \xrightarrow{L_2} X \Rightarrow X_n \xrightarrow{P} X$

$\Rightarrow X_{n_k} \xrightarrow{a.e.} X$ for some sub-seq n_k

$$\Rightarrow X(\omega) \overset{a.e.}{=} \lim_k X_{n_k}(\omega) \overset{a.e.}{=} \lim_k X_{n_k}(-\omega) \overset{a.e.}{=} X(-\omega)$$

For any $Y \in L_2(\Omega, \mathcal{F}, P)$ lets find $\mathcal{P}_S Y$.

Technique: Guess the answer and show it is orthogonal to S.

Here is the Guess:

$$\mathcal{P}_S Y \overset{?}{=} \frac{Y(\omega) + Y(-\omega)}{2}$$

To verify let $X \in S$ and notice

$$\langle Y - \mathcal{P}_S Y, X\rangle = E[(Y - \mathcal{P}_S Y)X]$$

$$= \int_{[-1,1]} \left(Y(\omega) - \frac{Y(\omega) + Y(-\omega)}{2}\right) X(\omega)\, dP(\omega)$$

$$= \frac{1}{2}\int_{[-1,1]} \underbrace{(Y(\omega) - Y(-\omega))X(\omega)}_{\text{odd symmetry about } 0}\, dP(\omega)$$

$= 0$

$\therefore$ Indeed $\mathcal{P}_S Y = \frac{Y(\omega) + Y(-\omega)}{2}$.

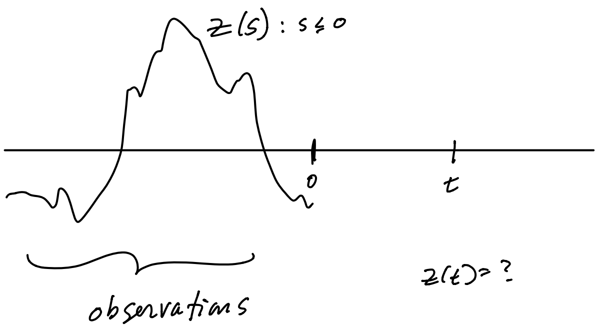
Projection application to Gaussian random field prediction

Let $\{Z(t) : t \in \mathbb{R}\}$ be a Gaussian random field (GRF) s.t. $\forall\, t, s \in \mathbb{R}$

$$\left.\begin{aligned} &E(Z(t)) = 0 \\ &\mathrm{cov}(Z(t), Z(s)) = e^{-|t-s|} \end{aligned}\right\} \text{specifies the f.d.d.s.}$$

Note part of the requirement of a GRF is that all the random variables $Z(t)$, for $t \in \mathbb{R}$, must be defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Suppose we observe $Z(s)$, $\forall s \le 0$ & want to predict $Z(t)$ for some $t > 0$.



Now, we will see later that

$$E(Z(t) \mid Z(s),\ s \le 0) = P_s Z(t)$$

To find $P_s Z(t)$ lets guess that $P_s Z(t) = a_t Z(0)$ and prove the residuals are orthognal to $\overline{\mathrm{span}}\{Z(s) : s \ge 0\}$.

Note that by linearity & continuity of $\langle \cdot, \cdot \rangle$ w.r.t L_2 limits it is sufficient to show $\langle Z(t) - a_t Z(0), Z(s) \rangle = 0$ $\forall s \le 0$.

For $t > 0$ & $s \le 0$ we have

$$\begin{aligned} \langle Z(t) - a_t Z(0), Z(s) \rangle &= e^{-|t-s|} - a_t e^{-|s|} \\ &= e^{-(t-s)} - a_t e^{s} \quad \text{since } s \le 0 < t \\ &= 0 \quad \text{iff } a_t := e^{-t} \end{aligned}$$

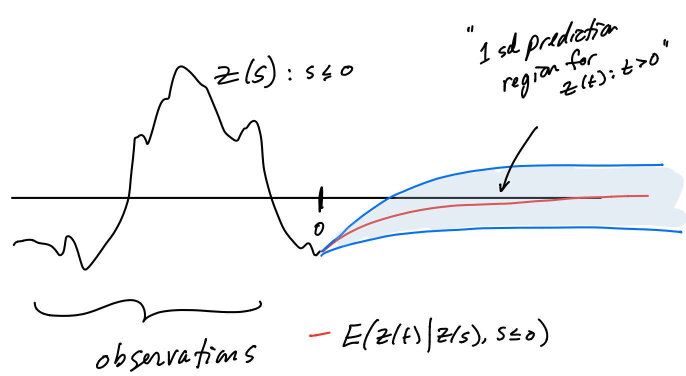
$\therefore\ E(Z(t) \mid Z(s),\ s \le 0) = e^{-t} Z(0)$

and

$$\begin{aligned} \mathrm{var}(Z(t) \mid Z(s), s \le 0) &= E\Big(\underbrace{[Z(t) - e^{-t} Z(0)]^2}\ \Big|\ \underbrace{Z(s), s \le 0}\Big) \end{aligned}$$

Since these are orthognal in L_2, they are uncorrelated. Since they are Gaussian they are then independent

$$\begin{aligned} &= \mathrm{var}\big(\underbrace{Z(t) - e^{t} Z(0)}_{\text{marginal residual}}\big) \\ &= e^{0} - 2e^{-t} e^{-|t|} + e^{2t} e^{0} \\ &= 1 - e^{-2t} \end{aligned}$$



— $E(Z(t) \mid Z(s), s \le 0)$

— $E(Z(t) \mid Z(s), s \le 0) \pm sd(Z(t) \mid Z(s), s \le 0)$

Riesz representation

The Riesz representation Theorem will be important for proving the existance of $\frac{dP}{dQ}$ for two probability measures P, Q.

To modivate recall that if $X_n, X \in L_2(\Omega, \mathcal{F}, P)$ then

$$X_n \xrightarrow{L_2} X \implies \langle Y, X_n \rangle \longrightarrow \langle Y, X \rangle$$

i.e. The map $\langle Y, \cdot \rangle : L_2(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ is a continuous linear functional on L_2.

Riesz says that all such functionals are in the form $\langle Y, \cdot \rangle$ for some $Y \in L_2$.

Theorem (Riesz for L_2):

Let $f: L_2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be linear and continuous. Then $\exists$ a P-a.e. unique $Y \in L_2(\Omega, \mathcal{F}, P)$ s.t.

$$f(X) = \langle Y, X \rangle \quad \forall X \in L_2(\Omega, \mathcal{F}, P)$$