

Homework 3

Due February 22, 2015

Exercise 1 (Metrizing \xrightarrow{P}). Let \mathfrak{R} be the space of real-valued random variables on (Ω, \mathcal{F}, P) . Let $d: \mathfrak{R} \times \mathfrak{R} \rightarrow [0, 1]$ be defined by $d(X, Y) := E(|X - Y| \wedge 1)$. Show the following

- d is a pseudo metric on \mathfrak{R}
- $X_n \xrightarrow{P} X$ if and only if $d(X_n, X) \rightarrow 0$
- \mathfrak{R} is separable if \mathcal{F} is countably generated (Hint: use Exercise 1 and Claim 1 from last quarters 235A Homework 6).
- \mathfrak{R} is complete.

Exercise 2. Suppose $X_n \xrightarrow{ae} X$. Show that for every $\epsilon > 0$, $\lim_m P[\sup_{n \geq m} |X_n - X| > \epsilon] = 0$.

Definition 1. X_n is said to **converge almost uniformly to** X , written $X_n \xrightarrow{au} X$, if for every $\epsilon > 0$ there exists a measurable U_ϵ such that $P[U_\epsilon] \geq 1 - \epsilon$ and $X_n(\omega) \rightarrow X(\omega)$ uniformly for all $\omega \in U_\epsilon$.

Exercise 3 (Egoroff's Theorem). Show that $X_n \xrightarrow{ae} X$ if and only if $X_n \xrightarrow{au} X$. Hint: if $X_n \xrightarrow{ae} X$ then there exists a subsequence n_k such that $P(\sup_{n \geq n_k} |X_n - X| > 1/k) < 1/k^2$.

Exercise 4. Let X_1, X_2, \dots be an infinite sequence of square integrable random variables, all defined on the same probability space, such that $E(X_i) = 0$ and

$$E(X_i X_j) = \text{cov}(X_i, X_j) = \begin{cases} a + b, & \text{if } i = j \\ b, & \text{if } i \neq j \end{cases}$$

for all i, j . Put $S_n := X_1 + X_2 + \dots + X_n$.

1. Show the above conditions on X_i are satisfied when $X_i = Y_i + Z$ where the Y_i 's and Z are L_2 random variables with zero means, variances given by $E(Y_i^2) = a$ and $E(Z^2) = b$ and covariances $E(Y_i Y_j) = 0$ when $i \neq j$ and $E(Z Y_i) = 0$.
2. Show $\|S_n/n - S_m/m\|_2^2 = (1/m - 1/n)a$ when $m \leq n$. Hint: first show $\langle S_m/m, S_n/n \rangle = a/n + b$.
3. Show that as n tends to infinity, S_n/n converges in L_2 to some random variable, say Z . Find the mean and variance of Z .
4. Show that S_n/n converges to Z with probability one. Hint: use the method of subsequences. First put $A_k = S_{k^2}/k^2$, show that $\|A_k - Z\|_2^2 = a/k^2$ and deduce that A_k converges a.e. to Z as $k \rightarrow \infty$. Then show that the S_n/n 's have the same limit a.e. as the A_k 's.
5. Show that the X_i 's necessarily have the representation specified in item 1 above.