Lecture 13
Separating classes

Often probabilist study random quantities X which, instead of mapping into $\mathbb{R}$, map into more complicated spaces.

E.g. X = random tree

X = random continuous function

X = random finite set

$\vdots$

In this case it doesn't make sense to talk about $E(X) = \int_\Omega x \, dP$ since our definition of integration requires X maps into $\bar{\mathbb{R}}$.

Moreover, characterizing the distribution of X, i.e. PX^{-1}, using densities is difficult especially if there is no canonical measure on the range space of X like Lebesgue measure to compare densities w.r.t.

However, it is often the case that the range space of X has a natural metric space associated with it.

metric space

$$\underset{\mathcal{F}, P}{\Omega'} \xrightarrow{X(\omega)} \underset{\mathcal{B}(\Omega)}{\Omega}$$

Since $\mathcal{B}(\Omega)$ is the Borel σ-field we have available to us the notion of continuity for functions mapping $\Omega \to \mathbb{R}$

$$\underset{\mathcal{F}, P}{\Omega'} \xrightarrow{X(\omega)} \underset{\mathcal{B}(\Omega)}{\Omega} \xrightarrow{f(\omega)} \mathbb{R} \Rightarrow$$

$f(X)$ is a r.v. and $Ef(X)$ is defined.

Now to study properties of X a general approach is to study how $Ef(X)$ behaves for "nice" functions f, such as those which are continuous and bdd.

This will be a general approach used in many of the future lectures.

In this lecture we will use this approach for characterizing the distribution of X: i.e. find some nice function classes Γ s.t.

$$Ef(X) = Ef(Y), \ \forall f \in \Gamma \implies X \overset{d}{=} Y.$$

or $\int_\Omega f \, dP = \int_\Omega f \, dQ, \ \forall f \in \Gamma \implies P = Q$ on $\mathcal{B}(\Omega)$

There is basically one important theorem in this lecture. The imediate utility of this theorem will be to establish the following result:

If X and Y are random d-dimensional random vectors such that $Ee^{iX\cdot k} = Ee^{iY\cdot k}$ $\forall k \in \mathbb{R}^d$ then $X \overset{d}{=} Y$.

The characteristic functions for X and Y.

However the main use of this result will be for establishing the existance of the Wiener measure (also called the Wiener measure or Brownian motion).

Wiener Measure

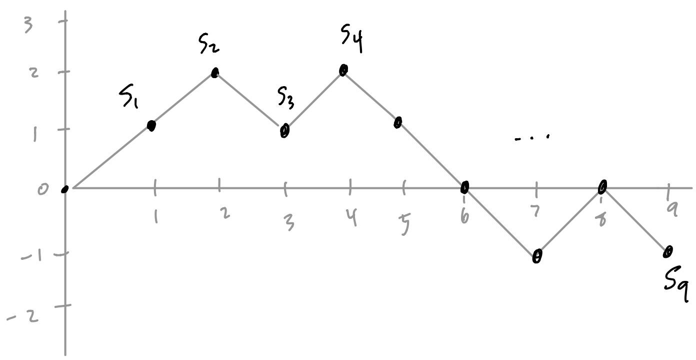
The Wiener measure is probably one of the most important objects in advanced probability.

I find it interesting in that you need some "high powered" theory to rigourously show its existance, yet you can easily give a non-rigourous description of its properties.

Recall in lecture 6 we discussed the 1-d random walk $S_0, S_1, \ldots$ defined by

$$S_n = \sum_{k=1}^{n} R_k, \quad S_0 = 0$$

where $R_1, R_2, \ldots$ are independent Rademacher r.v.s s.t. $P(R_k = 1) = P(R_k = -1) = \frac{1}{2}$.



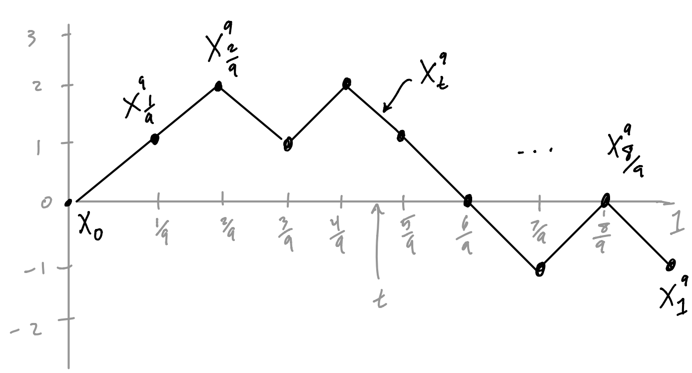
Notice that $E(S_n) = \sum_{k=1}^{n} E(R_k) = 0$

$$\mathrm{var}(S_n) = \sum_{k=1}^{n} \mathrm{var}(R_k) = n$$

$$E(R_k^2) = 1$$

Now for each n re-scale the x-axis by $\frac{1}{n}$

define X_t^n on $[0,1]$ by linear interp of $X_{i/n}^n = S_i$



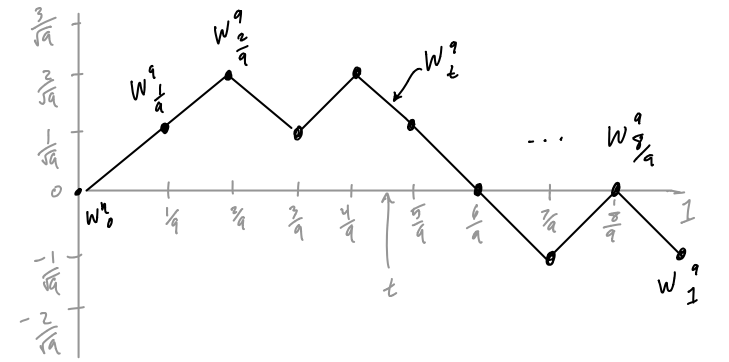
Notice X_t^n is continuous on $[0,1]$, i.e. $X_t^n \in C[0,1]$, and

$$E(X_t^n) = 0 \qquad \mathrm{var}(X_{\frac{i}{n}}^n) = n.$$

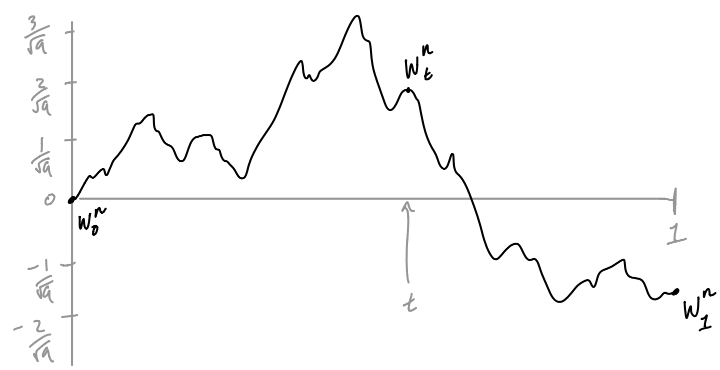
If we want a limit we need to re-scale X_t^n so the variability explode... try re-scaling so $\mathrm{var}(X_1^n) = 1$.

So re-scale the y-axis by $\frac{1}{\sqrt{n}}$ and

define $W_t^n := \frac{1}{\sqrt{n}} X_t^n$



For large n, W_t^n looks like this:



Clearly $W_t^n \in C[0,1]$ so, in some sense, we constructed n measures on $C[0,1]$ with sup-norm metric used for $\mathcal{B}(C[0,1])$.

<u>Question 1</u>: Why is

$$(\Omega, \mathcal{F}) \xrightarrow{W^n} (C[0,1], \mathcal{B}(C[0,1]))$$

measurable?

This is easy to see by writing

$$W_t^n(\omega) = W_t^n(R_1(\omega), \dots, R_n(\omega))$$

$$\begin{tikzcd}
\underset{\mathcal{F}, P}{\Omega} \arrow[r, "(R_1,\dots,R_n)\ ⓜ"] \arrow[rr, bend right, "W_t^n\ ⓜ"'] & \underset{\mathcal{B}(\mathbb{R}^n)}{\mathbb{R}^n} \arrow[r, "\substack{\text{continuous} \\ \text{w.r.t. sup-norm}}"] & \underset{\mathcal{B}(C[0,1])}{C[0,1]}
\end{tikzcd}$$

Question 2: Does there exist a limit

$$W_t \overset{?}{=} \lim_{n\to} W_t^n$$

Definitely not to a fixed function in $C[0,1]$ since $\mathrm{var}(W_t^n) = 1$, $\forall n$.

However we will later show that W_t^n have a random limit W_t (in distribution):

$$(*) \qquad E f(W_\cdot^n) \overset{n\to\infty}{\longrightarrow} E f(W_\cdot)$$

for all continuous and bdd $f : C[0,1] \to \mathbb{R}$.

This W_t is Brownian motion on $[0,1]$.

To show this we will show

(i) $\{W_\cdot^n\}_{n\in\mathbb{N}}$ is tight

(ii) For any fixed $t_1, \dots, t_m$

$$E\, f(W_{t_1}^n, \dots, W_{t_m}^n) \to E f(W_{t_1}, \dots, W_{t_m})$$

$\forall$ continuous and bdd $f : \mathbb{R}^m \to \mathbb{R}$.

The fact that (i) & (ii) are sufficient uses the main theorem in this lecture.

Question 3: To show (ii) how do we know what $E f(W_{t_1}, \dots, W_{t_m})$ is?

Since $E(W_t^n) = 0$ the CLT will imply

$$(W_{t_1}, \dots, W_{t_m}) \overset{\mathcal{D}}{=} N_m(0, \Sigma)$$

Also note that for integers $j < i < q < p$

$$W_{\frac{i}{n}}^n = \frac{1}{\sqrt{n}} S_i$$

$$W_{i/n}^n - W_{j/n}^n = \frac{1}{n}(S_i - S_j) = \frac{1}{n} \sum_{k=j+1}^{i} R_k$$

$W_{i/n}^n - W_{j/n}^n$ indep of $W_{\frac{p}{n}}^n - W_{q/n}^n$

So that

$$\mathrm{var}(W_{i/n}^n) = \frac{1}{n} \mathrm{var}(S_i) = \frac{i}{n}$$

$$\mathrm{var}(W_{i/n}^n - W_{j/n}^n) = \frac{1}{n} \mathrm{var}(S_i - S_j) = \frac{i-j}{n}$$

$$\mathrm{cov}(W_{i/n}^n - W_{j/n}^n,\ W_{\frac{p}{n}}^n - W_{q/n}^n) = 0$$

This suggest that

$$\mathrm{var}(W_t^n) = t$$

$$\mathrm{var}(W_t^n - W_s^n) = |t - s|$$

$W_t^n - W_s^n$ indep of $W_x^n - W_y^n$

when $0 \le t < s < x < y \le 1$.

These are enough to determine Σ and therefore speculate the value of

$$E f(W_{t_1}, \dots, W_{t_m})$$

Some definitions

Definition:
If Ω is a complete and separable metric space then Ω is called a Polish space.

In some sense Polish spaces are the most general spaces we will need to work with in the remainder of the class. One can generalize results to non-polish spaces but I don't think the extra complexity is worth it.

One nice thing about Polish spaces is that

$$\mathcal{B}(\Omega) = \sigma\langle \text{open balls} \rangle$$

and any probability measure on $(\Omega, \mathcal{B}(\Omega))$ is automatically a Radon measure and therefore satisfies

$$P(A) = \sup\{P(K) : \text{compact } K \subset A\}$$

proof is similar to what was done in HwK 2.

Note: I'll often write "Let $(\Omega, \mathcal{B}(\Omega))$ be Polish..." which means Ω is a polish space with metric d and $\mathcal{B}(\Omega)$ is the Borel σ-field w.r.t d.

It will also be useful to define notation for some common function spaces when Ω is a metric space.

$C(\Omega) := \{\text{continuous maps } f : \Omega \to \mathbb{R}\}$

$C_b(\Omega) := \{\text{bdd and continuous maps } f : \Omega \to \mathbb{R}\}$

$C_c(\Omega) := \{\text{compactly supported continuous maps } f : \Omega \to \mathbb{R}\}$

$Lip_K(\Omega) := \{f : \Omega \to \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq K\, d(x,y)\ \forall x,y \in \Omega\}$

$C^k(\mathbb{R}^d) := \{k\text{-times differentiable maps } f : \mathbb{R}^d \to \mathbb{R}\}$

Definition:
Let $(\Omega, \mathcal{B}(\Omega))$ be Polish and $\Gamma \subset \underbrace{\mathcal{Q}}_{\text{quasi-integrable}}(\Omega, \mathcal{B}(\Omega))$

- Γ is a separating class for $(\Omega, \mathcal{B}(\Omega))$ if
$$\int_\Omega f\,dP = \int_\Omega f\,dQ,\ \forall f \in \Gamma \implies P = Q$$
for all probability measures P, Q on $(\Omega, \mathcal{B}(\Omega))$.

- If $\mathcal{P}$ is a collection of probability measures on $(\Omega, \mathcal{B}(\Omega))$, Γ separates $\mathcal{P}$ if
$$\int_\Omega f\,dP = \int_\Omega f\,dQ,\ \forall f \in \Gamma \implies P = Q$$
$\forall P, Q \in \mathcal{P}$

- If $\mathcal{P}$ is a collection of random variables mapping into $(\Omega, \mathcal{B}(\Omega))$, Γ separates $\mathcal{P}$ if
$$E f(X) = E f(Y),\ \forall f \in \Gamma \implies X \overset{\mathcal{D}}{=} Y$$
$\forall X, Y \in \mathcal{P}$.

Before we prove the main result of this section lets briefly discuss convolutions.

Convolution

working with functions $f \in C(\mathbb{R}^d)$ can be annoying since they are not regular enough to do thing like Taylor approximations, etc...

To get access to these tools we often smooth f by convolving with a member of $C_0^\infty(\mathbb{R}^d)$.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ s.t. φ has support in $[-1,1]^d$ and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. Define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$$ also satisfies $\int \varphi_\varepsilon(x)dx = 1$

$$\varphi_\varepsilon * f(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y) f(y)\, dy.$$

Now $\varphi_\varepsilon * f$ are very nice in that:

- $\varphi_\varepsilon * f \in C^\infty(\mathbb{R}^d)$
- $\varphi_\varepsilon * f \in C_c^\infty(\mathbb{R}^d)$ if $f \in C_c(\mathbb{R}^d)$
- $\partial(\varphi_\varepsilon * f) = (\partial\varphi_\varepsilon) * f$

Moreover, as $\varepsilon \to 0$, $\varphi_\varepsilon * f$ approximates f in a way that works well with integrals:

- $\varphi_\varepsilon * f \to f$ uniformly on compacts
- $\varphi_\varepsilon * f \to f$ uniformly if $f \in C_c(\mathbb{R}^d)$

Convolutions also have an important probabilistic interpretation:

If $\varphi(x) \geq 1$ then φ is the density of some random vector Z.

In this case

$$\varphi * f(x) = E f(x - Z)$$
$$\varphi_\varepsilon * f(x) = E f(x - \varepsilon Z)$$

and if $f(x)$ is also a density for some r.v. X defined on the same probability space as Z then

$$\varphi * f \text{ is the density of } X + Z.$$

So, if you want to study a r.v. X which is not very well behaved you can often study $X + \varepsilon Z$ for a very nice Z & limit $\varepsilon \to 0$.

Remark: One can use probability theory to construct a $\varphi \in C_0^\infty([-1,1])$ which is a probability density. In a previous lecture you showed

$$\mathcal{U} = \sum_{k=1}^\infty 2^{-k} X_k \sim \text{Unif}(0,1).$$

Now let $\mathcal{U}_1, \mathcal{U}_2, \dots$ be independent copies of $\mathcal{U}$ and define

$$Z = \sum_{k=1}^\infty 2^{-k}\mathcal{U}_k$$

The density φ of Z is the infinite convolution of the densities of $2^{-k}\mathcal{U}_k$ & satisfies $\varphi \in C_0^\infty([-1,1])$

these have compactly supported densities

Main theorem for separating classes

we can already construct a bunch of separating classes using indicators of generating sets. Indeed suppose P & Q are two measures on $(\Omega, \mathcal{B}(\Omega))$ and

$\mathcal{P} \subset \mathcal{B}(\Omega)$ is a π-system s.t.

$$\mathcal{B}(\Omega) = \sigma\langle \mathcal{P} \rangle.$$

By π-uniqueness of prob. measures we have

$$P(A) = Q(A),\ \forall \in \mathcal{P} \Rightarrow P = Q$$

$\therefore$

$$\int_\Omega I_A\, dP = \int_\Omega I_A\, dQ,\ \forall A \in \mathcal{P} \Rightarrow P = Q$$

$\therefore$ $\Gamma = \{ I_A : A \in \mathcal{P} \}$ is a separating class for $(\Omega, \mathcal{B}(\Omega))$.

e.g. since $\mathcal{B}(\Omega) = \sigma\langle \underbrace{\text{closed sets}}_{\text{forms a } \pi\text{-system}} \rangle$

$\Gamma := \{ I_C : \text{closed } C \subset \Omega \}$ is a separating class.

The key, however, is to work with a separating class that is either

"small" $\Rightarrow$ don't need to check $\int_\Omega f\,dP = \int_\Omega f\,dQ$ for too many $f \in \Gamma$

or

"regular" $\Rightarrow$ the functions $f \in \Gamma$ are well behaved and easier to control.

The following result illustrates both

Theorem 1

Suppose $(\Omega, \mathcal{B}(\Omega))$ is Polish.

Then each of the following function classes is a separating class over $(\Omega, \mathcal{B}(\Omega))$

(i) $Lip_1(\Omega) \cap C_b(\Omega)$, always

(ii) $Lip_1(\Omega) \cap C_c(\Omega)$, if Ω is locally compact
i.e. $\forall x \in \Omega\ \exists$ open $\mathcal{U}$ s.t. $x \in \mathcal{U}$ & $\bar{\mathcal{U}}$ is compact

(iii) $C_c^\infty(\Omega)$, if $\Omega = \mathbb{R}^d$.

(iv) $\{ e^{i x \cdot k} \text{ s.t. } k \in \mathbb{R}^d \}$, if $\Omega = \mathbb{R}^d$

(v) $\{ \underbrace{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}}_{\text{monomials}} \text{ s.t. } p_1, \dots p_n \in \mathbb{Z}^+ \}$, if Ω is a compact subset of $\mathbb{R}^d$.

Proof:

Suppose P and Q are two probability measures on $(\Omega, \mathcal{B}(\Omega))$ s.t.

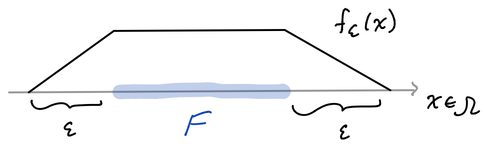
$$(*) \quad \int_\Omega f\,dP = \int_\Omega f\,dQ \quad \forall f \in \Gamma$$

For (i) Suppose $\Gamma := Lip_1(\Omega) \cap C_b(\Omega)$.

If we show $P(F) = Q(F)$, $\forall$ closed $F \subset \Omega$, we are done (by π-uniqueness).

Let F be closed & $\varepsilon > 0$.

Define $f_\varepsilon(x) := \left(1 - \frac{d(x,F)}{\varepsilon}\right)^+$



Notice that f_ε is bdd and Lipschitz continuous since

$$|f_\varepsilon(x) - f_\varepsilon(y)| \le \left| \frac{d(x,F)}{\varepsilon} - \frac{d(y,F)}{\varepsilon} \right|$$

since $|(1-z)^+ - (1-w)^+| \le |z-w|$

$$\le \frac{d(x,y)}{\varepsilon}$$

left as an exercise.

$\therefore$ $f_\varepsilon \in \Gamma$, $\forall \varepsilon > 0$.

Moreover if $F^\varepsilon := \{y : d(y,F) < \varepsilon\}$ then

$$I_F(x) \le f_\varepsilon(x) \le I_{F^\varepsilon}(x)$$

since $x \in F \Rightarrow f_\varepsilon(x) = 1$

since $x \notin F^\varepsilon \Rightarrow f_\varepsilon(x) = 0$

$$\therefore \quad P(F) \le \int_\Omega f_\varepsilon \, dP = \int_\Omega f_\varepsilon \, dQ \le Q(F^\varepsilon)$$

since $f_\varepsilon \in \Gamma$ by (*)

Since F is closed $\underbrace{F^\varepsilon \downarrow F}$ as $\varepsilon \to 0$

because $x \in \bigcap_{\varepsilon \in \mathbb{Q}} F^\varepsilon \Rightarrow d(x,F) = 0 \Rightarrow x \in F$

$$\therefore \quad P(F) \le \lim_{\varepsilon \downarrow 0} Q(F^\varepsilon) = Q(F)$$

Similarly one obtains $Q(F) \le P(F)$

$\therefore$ $P(F) = Q(F)$ $\forall$ closed $F \subset \Omega$ as was to be shown.

QED

For (ii)

Suppose $\Gamma := Lip_1(\Omega) \cap C_b(\Omega)$ and Ω is locally compact.

Since Ω is Polish and P, Q are probabilities

$$P(A) = \sup\{P(K) : \text{compact } K \subset A\}$$

$$Q(A) = \sup\{Q(K) : \text{compact } K \subset A\}$$

$\therefore$ we just need to show $P = Q$ on compacts.

Let K be compact and again write

$$f_\varepsilon(x) := \left(1 - \frac{d(x,K)}{\varepsilon}\right)^+.$$

The same proof as in (i) shows

$$P(K) = Q(K).$$

Now we just show $f_\varepsilon \in \Gamma$, i.e. that

$$\operatorname{supp} f_\varepsilon := \{x : d(x,K) < \varepsilon\} =: K^\varepsilon$$

is contained in a compact set.

Start by writing

$$K \subset \bigcup_{x \in K} \mathcal{U}_x$$

where $\mathcal{U}_x$ is an open neighborhood of x s.t. $\overline{\mathcal{U}}_x$ is compact... using our assumption for (ii).

Since K is compact $\exists x_1, \dots, x_n$ s.t.

$$K \subset \bigcup_{i=1}^n \mathcal{U}_{x_i} =: \underbrace{\mathcal{U}}_{\text{open}}$$

Now for all sufficiently small $\varepsilon < 0$

$$\underbrace{K \subset K^\varepsilon \subset \mathcal{U}}_{\substack{\text{since } \mathcal{U}^c \cap K = \emptyset \\ \text{and } \mathcal{U}^c \text{ is closed} \\ \text{implies } d(K, \mathcal{U}^c) > 0}} \subset \underbrace{\bigcup_{i=1}^n \overline{\mathcal{U}}_{x_i}}_{\text{compact}}$$

QED

For (iii)

Suppose $\Omega = \mathbb{R}^d$ & $\Gamma = C_c^\infty(\mathbb{R}^d)$.

Now we just need to show

$$\int_\Omega f\,dP = \int_\Omega f\,dQ \quad \forall f \in C_c(\mathbb{R}^d)$$

which will imply $P = Q$ by (ii)

Here is where we use convolutions.
Fix $f \in C_c(\mathbb{R}^d)$ and simply notice that when $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ as in previous section then

$$\int_\Omega f\,dP = \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon * f\,dP \quad \text{by uniform approx since } f \in C_c(\mathbb{R}^d)$$

$$= \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon * f\,dQ \quad \text{by assumption}$$

$$= \int_\Omega f\,dQ \quad \text{again by uniform}$$

QED

For (iv) and (v)

We use the Stone-Weierstrass thm:

> If $f \in C([-N,N]^d)$ then f is uniformly approximated by polynomials and finite linear combinations of $\sin(x\cdot k)$ & $\cos(x\cdot k)$ for $k \in \frac{\pi}{N}\mathbb{Z}^d$.

For (iv) we can use the technique in (iii) and just show that $\forall f \in C_c(\mathbb{R}^d)$ $\exists f_1, f_2, \ldots \in$ finite linear span of $\sin(x\cdot k), \cos(x\cdot k)$

s.t. $\displaystyle\int_\Omega f_n\,dP \xrightarrow{n\to\infty} \int_\Omega f\,dP$

(*) $\displaystyle\int_\Omega f_n\,dQ \xrightarrow{n\to\infty} \int_\Omega f\,dQ$

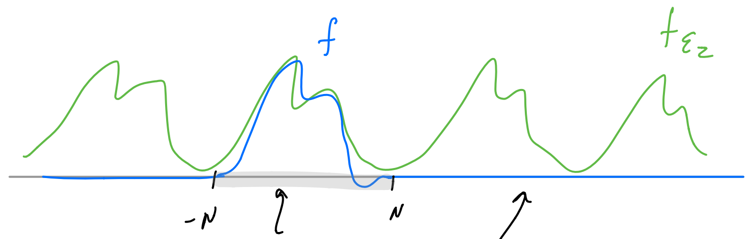
Note $\forall \varepsilon_1 > 0$ $\exists N > 0$ s.t.

$$P(\mathbb{R}^d \setminus [-N,N]^d) < \varepsilon_1$$
$$Q(\mathbb{R}^d \setminus [-N,N]^d) < \varepsilon_1$$
$$\text{support of } f \subset [-N,N]^d.$$

Also note $\forall \varepsilon_2 > 0$ $\exists f_{\varepsilon_2} \in$ finite span $\{\sin(x\cdot k), \cos(x\cdot k)\}_{k \in \frac{\pi}{N}\mathbb{Z}^d}$

s.t. $\displaystyle\sup_{x \in [-N,N]^d} |f_{\varepsilon_2}(x) - f(x)| < \varepsilon_2$

since f_{ε_2} is $2N$-periodic we have



In here $|f - f_{\varepsilon_2}| < \varepsilon_2$

out here $|f_{\varepsilon_2}| \leq \|f\|_\infty + \varepsilon_2$

$\therefore$

$$\left| \int_\Omega f_{\varepsilon_2}\,dP - \int_\Omega f\,dP \right|$$

$$\leq \int_{[-N,N]^d} |f_{\varepsilon_2} - f|\,dP + \int_{\mathbb{R}^d \setminus [-N,N]^d} |f_{\varepsilon_2}|\,dP$$

$$\leq \varepsilon_2 P([-N,N]^d) + (\|f\|_\infty + \varepsilon_2)\varepsilon_1$$

$$\leq \underbrace{\varepsilon_2 + (\|f\|_\infty + \varepsilon_2)\varepsilon_1}_{\to 0 \text{ as } \varepsilon_1, \varepsilon_2 \to 0}$$

Similar approximation holds for Q.

This gives (*) as was to be shown.

Finally for the monomials $m(v)$ the same proof works except that the term

$$\underbrace{\int_{\mathbb{R}^d\setminus[-N,N]^d} |f_{\varepsilon_2}|\,dP}$$ could be infinite $\forall N$ & ε_2

... but if P and Q have compact support, this term is zero for large enough N.

QED

Application: Multivariate charateristic functions

In Lecture 12 we defined the characteristic function for a univariate r.v. X

$$\phi_X(k) := E e^{ikX} \quad \text{for } k\in\mathbb{R}.$$

The advantage for studying properties of ϕ_X vrs PX^{-1} is that ϕ_X is a regular function $\mathbb{R}\to\mathbb{C}$ whereas working with PX^{-1} is hard since it maps $\mathcal{B}(\mathbb{R})\to\mathbb{R}$.

The characteristic function for a random vector X

$$\underset{\mathcal{F},P}{\Omega} \xrightarrow{X(\omega)} \mathbb{R}^d$$

is similarly defined

$$\phi_X(k) = E\left(e^{ik\cdot X}\right) \quad \forall k\in\mathbb{R}^d$$

The following theorem shows that $\phi_X(\cdot)$ encodes all the information about the randomness in X.

Theorem 2: Let X & Y be random vectors taking values in $\mathbb{R}^d$ with characteristic functions ϕ_X & ϕ_Y. If $\phi_X(k)=\phi_Y(k)$ $\forall k\in\mathbb{R}^d$ then

$$X \overset{\mathcal{D}}{=} Y.$$

Proof:

$\phi_X(k)=\phi_Y(k)$ $\forall k\in\mathbb{R}^d$ implies

$$\begin{aligned} E\cos(k\cdot X) &= \mathrm{Real}\,\phi_X(k) \\ &= \mathrm{Real}\,\phi_Y(k) \\ &= E\cos(k\cdot Y) \end{aligned}$$

and

$$\begin{aligned} E\sin(k\cdot X) &= \mathrm{Imag}\,\phi_X(k) \\ &= \mathrm{Imag}\,\phi_Y(k) \\ &= E\sin(k\cdot Y) \end{aligned}$$

$\therefore$ $Ef(X)=Ef(Y)$ $\forall f\in\Gamma$ where Γ is the separating class over $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ given in Theorem 1.

$\therefore$ $X\overset{\mathcal{D}}{=}Y$. QED.

This also gives the following corollary:

Corollary 1: Let X & Y be random vectors taking values in $\mathbb{R}^d$. Then

$$X\overset{\mathcal{D}}{=}Y \iff k\cdot X \overset{\mathcal{D}}{=} k\cdot Y \quad \forall k\in\mathbb{R}^d.$$

Application: Moments characterizing X

Also in lecture 12 we saw that the moment generating function

$$M_X(t) := E e^{tX}$$

only characterizes the distribution of X when $M_X(t)$ is finite on a non-empty open interval.

We also noticed a slightly weaker result that the moments of X, i.e. EX^n, characterize the distribution of X only when $M_X(t)$ is finite on a non-empty open interval containing 0.

We get an illustration of this and an extension to bdd multivariate r.v.s via separating classes

Theorem 3:
Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be real random variables which are bounded.
Then

$$X \overset{\mathcal{D}}{=} Y \iff E(X_1^{\alpha_1} \cdots X_d^{\alpha_d}) = E(Y_1^{\alpha_1} \cdots Y_d^{\alpha_d}) \quad \forall\, \alpha_1, \dots, \alpha_d \in \{0,1,2,\dots\}$$

Proof: Similar to Proof of Theorem 2 using $\Gamma := \{\text{monomials}\}$ from Theorem 1.

The Spaces $\mathbb{R}^\infty$ and $C([0,1])$

Here we show that the finite dimensional projections form separating classes for $\mathbb{R}^\infty$ and $C([0,1])$.