

Lecture 10: Integration and expected value

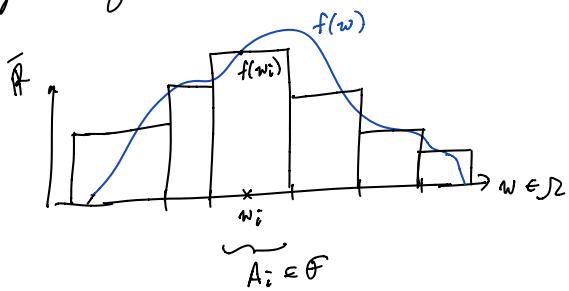
(1)

In this lecture we will define

$$\int_{\Omega} f(w) d\mu(w)$$

where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f: \Omega \rightarrow \bar{\mathbb{R}}$ s.t. $f \in \mathcal{F}/B(\bar{\mathbb{R}})$.

The notation $\int_{\Omega} f(w) d\mu(w)$ is extremely suggestive of Riemann integration

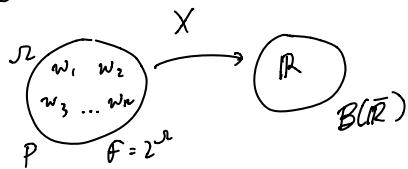


where one might guess

$$\int_{\Omega} f(w) d\mu(w) \approx \sum_i f(w_i) \mu(A_i)$$

area of block i : when
width of A_i is measured
with μ

To see the connection with expected value suppose Ω has n members:



In this case we would want the definition of "expected value of X ", denoted $E(X)$, to, at the very least, satisfy:

$$E(X) = \left\{ \begin{array}{l} \text{the weighted average of the} \\ \text{numbers } \{X(w_1), X(w_2), \dots, X(w_n)\} \\ \text{with weights } p(\{w_i\}). \end{array} \right\}$$

$$= \sum_{i=1}^n X(w_i) P(\{w_i\})$$

partition
measure

$$= \int_{\Omega} X(w) dP(w).$$

Assumption: For the rest of this lecture suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space

(2)

$\int_{\Omega} f d\mu$ as shorthand for $\int_{\Omega} f(w) d\mu(w)$

Game plan:

Step 1: Define $\int_{\Omega} f d\mu$ for $f \in \mathcal{N}_s(\Omega, \mathcal{F})$

Non-negative simple functions.

Step 2: extend to $f \in \mathcal{N}(\Omega, \mathcal{F})$

Non-negative measurable functions.

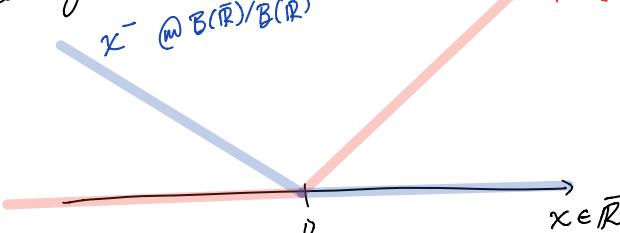
Step 3: extend to some, but not all, $f \in \mathcal{F}/B(\bar{\mathbb{R}})$ by

$$\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$$

where $(\cdot)^+$, $(\cdot)^-$ denotes the positive part & negative part

$$x^- @ B(\bar{\mathbb{R}})/B(\bar{\mathbb{R}})$$

$$x^+ @ B(\bar{\mathbb{R}})/B(\bar{\mathbb{R}})$$



$$\text{so that } |x| = x^+ + x^- \text{ & } x = x^+ - x^-$$

Remark: Although this construction seems tedious & annoying, the method of construction is general & broadly applicable. For example, the same game plan is use for defining

$$\int_0^t f(s) dB(s)$$

Brownian motion

$f: \mathbb{R} \rightarrow \mathbb{R}$.

Step 1

Def: If $f \in \mathcal{N}_s(\Omega, \mathcal{F})$ has the form

$$f = \sum_{i=1}^n c_i I_{A_i} \quad \text{then} \quad \int_{\Omega} f(w) d\mu(w) := \sum_{i=1}^n c_i \mu(A_i)$$

$c_i \in [0, \infty]$
forms a measurable partition

Here are the four basic properties
of $\int f d\mu$ we will show at each step
in the game plan:

(3)

Thm (simple 3)

(0) $\int f d\mu$ is well defined over $\mathcal{H}_s(\Omega, \mathcal{F})$

(1) Monotonicity:

If $f, g \in \mathcal{H}_s(\Omega, \mathcal{F})$ & $f(w) \leq g(w) \forall w \in \Omega$ then

$$\int f d\mu \leq \int g d\mu.$$

(2) Linearity:

If $f, g \in \mathcal{H}_s(\Omega, \mathcal{F})$ & $\alpha, \beta \in [0, \infty]$ then

$$\int_{\Omega} (\alpha f + \beta g) d\mu = \alpha \int_{\Omega} f d\mu + \beta \int_{\Omega} g d\mu$$

(3) Continuity from below (CFB):

If $f_n(w) \uparrow f(w)$ as $n \rightarrow \infty$ for all $w \in \Omega$

where $f_n, f \in \mathcal{H}_s(\Omega, \mathcal{F})$ then

$$\int_{\Omega} f_n d\mu \uparrow \int_{\Omega} f d\mu.$$

Proof:

Suppose $f = \sum_{i=1}^n c_i I_{A_i}$, $g = \sum_{k=1}^m d_k I_{B_k}$ both in $\mathcal{H}_s(\Omega, \mathcal{F})$

\uparrow \mathcal{F} -sets which partition Ω

$$\therefore f = \sum_{i,k} c_{ik} I_{A_i \cap B_k} \text{ where } c_{ik} = c_i$$

$$g = \sum_{i,k} d_{ik} I_{A_i \cap B_k} \text{ where } d_{ik} = d_k$$

\uparrow a finer partition of Ω .

To show (0) & (1) it is sufficient to show

$$f \leq g \Rightarrow \sum_{i=1}^n c_i \mu(A_i) \leq \sum_{k=1}^m d_k \mu(B_k)$$

(4)

$$f \leq g \Rightarrow \sum_{i \neq k} c_{ik} I_{A_i \cap B_k} \leq \sum_{i \neq k} d_{ik} I_{A_i \cap B_k}$$

exactly one term is
non-zero (assuming $A_i \neq \emptyset$
& $B_k \neq \emptyset$).

$$\Rightarrow c_{ik} \leq d_{ik} \quad \forall i, k$$

$$\Rightarrow \underbrace{\sum_{i \neq k} c_{ik} \mu(A_i \cap B_k)}_{= \sum_i c_i \mu(A_i)} \leq \underbrace{\sum_{i \neq k} d_{ik} \mu(A_i \cap B_k)}_{= \sum_k d_k \mu(B_k)}$$

by additivity of μ .

For (2)

$$\begin{aligned} \int_{\Omega} (\alpha f + \beta g) d\mu &= \int_{\Omega} \sum_{i,k} (\alpha c_{ik} + \beta d_{ik}) I_{A_i \cap B_k} d\mu \\ &= \sum_{i,k} (\alpha c_{ik} + \beta d_{ik}) \mu(A_i \cap B_k) \\ &\because \text{use additivity of } \mu \text{ &} \\ &\text{linearity of } \sum_{i,k} \\ &= \alpha \int_{\Omega} f d\mu + \beta \int_{\Omega} g d\mu \end{aligned}$$

For (3)

Suppose $f_n \uparrow f$.
 \uparrow all in $\mathcal{H}_s(\Omega, \mathcal{F})$

Notice $\int f_n d\mu \uparrow$ by (1) so just show

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Case 1: $f = c I_A$ for $c \in (0, \infty]$. \uparrow The case $c=0$ is trivial.

Let $0 < b < c$ so that

$$b I_{\{f_n \geq b\}} \leq f_n \leq f = c I_A.$$

Now integrate each term & use (1) to get

$$b \mu(f_n \geq b) \leq \int f_n d\mu = \int f d\mu = c \mu(A)$$

$$\begin{aligned} b \lim_n \mu(f_n \geq b) &\leq \lim_n \int_R f_n d\mu = \int_R f d\mu = c\mu(A) \quad (5) \\ &= \mu(A) \text{ by CFB since } 0 < b < c \text{ &} \\ f_n \uparrow f \text{ implies } \{f_n \geq b\} &\uparrow \{f = c\} = A \\ \text{as } n \rightarrow \infty \end{aligned}$$

Now take the limit as $b \uparrow c$ to get

$$c\mu(A) \leq \lim_n \int_R f_n d\mu \leq \int_R f d\mu = c\mu(A)$$

$\therefore \text{these are all equal}$

Case 2: $f = \sum_{i=1}^n c_i I_{A_i}$, where A_i 's partition R .

Fix $k \in \{1, 2, \dots, m\}$

\therefore Now $f_n I_{A_k} \uparrow f I_{A_k}$ so that
Case 1 applies (since $f I_{A_k} = c_k I_{A_k}$)

to give $\int_R f_n I_{A_k} d\mu \uparrow \int_R f I_{A_k} d\mu$

Now sum over $k = 1, \dots, m$ to get.

$$\int_R f_n \underbrace{\sum_{k=1}^m I_{A_k}}_{=1} d\mu \uparrow \int_R f \underbrace{\sum_{k=1}^m I_{A_k}}_{=1} d\mu \quad \text{QED}$$

Step 2

Recall the structure theorem:
if $f \in \mathcal{H}(R, \mathbb{F})$ then $\exists f_n \in \mathcal{H}_s(R, \mathbb{F})$ s.t.

$$f_n \uparrow f$$

Def: if $f \in \mathcal{H}(R, \mathbb{F})$ define

$$\int_R f d\mu := \lim_n \int_R f_n d\mu$$

$f_n \in \mathcal{H}_s$ s.t.
 $f_n \uparrow f$

Thm (little 3)

(6)

Statements (0) - (3) in "simple 3" hold when $\mathcal{H}_s(R, \mathbb{F})$ is replaced with $\mathcal{H}(R, \mathbb{F})$.

Proof:

To show (0) & (1), i.e. $\int_R f d\mu$ is well defined & monotonic, start by assuming

$$\begin{array}{c} f \leq g \\ \text{both in } \mathcal{H}(R, \mathbb{F}) \end{array}$$

$\therefore \exists f_n, g_n \in \mathcal{H}_s(R, \mathbb{F})$ s.t.

$$\lim_n f_n = f \leq g = \lim_n g_n$$

Notice the following "trick"

$$\begin{aligned} \lim_m \int_R f_n \wedge g_m d\mu &= f_n \wedge (\lim_m g_m) \\ &= f_n \wedge g \\ &= f_n \quad \text{since } f_n \leq f \leq g \end{aligned}$$

$$\therefore \int_R f_n d\mu = \int_R \lim_m f_n \wedge g_m d\mu$$

$$= \lim_m \int_R f_n \wedge g_m d\mu \quad \text{by "simple 3"}$$

$$\leq \lim_m \int_R g_m d\mu$$

Now take a limit as $n \rightarrow \infty$ to get

$$\lim_n \int_R f_n d\mu = \lim_m \int_R g_m d\mu.$$

This shows (0) & (1).

The proof of (2), i.e. that

$$\int_R \alpha f + \beta g d\mu = \alpha \int_R f d\mu + \beta \int_R g d\mu$$

when $\alpha, \beta \in [0, \infty]$ is easy (using the fact that $\alpha f + \beta g = \lim_n (\alpha f_n + \beta g_n)$ which implies

$$\int_R \alpha f + \beta g d\mu = \lim_n \int_R \alpha f_n + \beta g_n d\mu \quad \text{by def.}$$

For (3):

$$\text{Show } \underbrace{f_n \uparrow f}_{\text{all in } \mathcal{H}(R, \mathcal{F})} \Rightarrow \int_R f_n d\mu \uparrow \int_R f d\mu$$

Suppose $f_n \uparrow f$ & let $f_n = \lim_m^{\uparrow} \phi_{nm}$
so that $\epsilon \mathcal{H}_s(R, \mathcal{F})$

$$\begin{array}{ccccccc} \phi_{11} & \leq & \phi_{12} & \leq \dots & \leq & \phi_{1n} & \leq \rightarrow f_1 \\ : & & : & & : & & : \\ \phi_{k1} & \leq & \phi_{k2} & \leq \dots & \leq & \phi_{kn} & \leq \rightarrow f_k \\ : & & : & & : & & : \\ \phi_{n1} & \leq & \phi_{n2} & \leq \dots & \leq & \phi_{nn} & \leq \rightarrow f_n \end{array}$$

$$\text{define } \phi_n := \max_{1 \leq i, j \leq n} \phi_{ij} \in \mathcal{H}_s(R, \mathcal{F})$$

$$\text{Now } \phi_{kn} \leq \phi_n \leq f_n \leq f, \quad \forall k \leq n \quad (\star)$$

Taking limits as $n \rightarrow \infty$ in (\star) gives

$$f_k = \lim_n^{\uparrow} \phi_{kn} \leq \lim_n^{\uparrow} \phi_n \leq \lim_n^{\uparrow} f_n \leq f$$

Taking limits as $k \rightarrow \infty$

$$f = \lim_k^{\uparrow} f_k = \lim_n^{\uparrow} \phi_n = \lim_n^{\uparrow} f_n = f$$

$\epsilon \mathcal{H}_s(R, \mathcal{F})$

$\therefore f = \lim_n^{\uparrow} \phi_n$ where $\phi_n \in \mathcal{H}_s(R, \mathcal{F})$ so

$$\int_R f d\mu := \lim_n^{\uparrow} \int_R \phi_n d\mu \quad \text{by def.}$$

$$\text{Now just show } \lim_n \int_R \phi_n d\mu = \lim_n \int_R f_n d\mu$$

(7)

Instead of taking limits in (\star) first, integrate to get

$$\int_R \phi_{kn} d\mu \leq \int_R \phi_n d\mu \leq \int_R f_n d\mu, \quad \forall k \leq n$$

Now let $n \rightarrow \infty$ for

$$\int_R f_k d\mu \leq \lim_n \int_R \phi_n d\mu \leq \lim_n \int_R f_n d\mu$$

where $\int_R f_k d\mu = \lim_n \int_R \phi_{kn} d\mu$ by def.

Finally let $k \rightarrow \infty$ to give

$$\lim_k \int_R f_k d\mu = \lim_n \int_R \phi_n d\mu.$$

(8) ED

Before we move to Step 3 we need some useful facts.

Def: $f=g$ μ -a.e. means $\mu(f \neq g) = 0$

$f \leq g$ μ -a.e. means $\mu(f \neq g) = 0$

Thm (a.e. integral facts)

(i) $f \in \mathcal{H}(R, \mathcal{F})$ & $\int_R f d\mu < \infty \Rightarrow f < \infty \mu$ -a.e.

(ii) If $f \in \mathcal{H}(R, \mathcal{F})$ then

$$\int_R f d\mu = 0 \Leftrightarrow f = 0 \mu\text{-a.e.}$$

(iii) If $f, g \in \mathcal{H}(R, \mathcal{F})$ and $f=g$ μ -a.e.

$$\text{then } \int_R f d\mu = \int_R g d\mu.$$

which implies
I can change f
on μ -null sets without
changing $\int_R f d\mu$.

Proof:

For (i) Notice that $f \in \mathcal{H}(\Omega, \mathcal{F})$ implies

$$\int_{\Omega} I_{\{f=\infty\}} d\mu \leq f$$

using our convention that $\infty \cdot 0 = 0$

$$\begin{aligned} \int_{\Omega} f d\mu < \infty &\stackrel{\text{little } 3}{\Rightarrow} \infty \mu(f=\infty) \leq \int_{\Omega} f d\mu < \infty \\ &\Rightarrow \underbrace{\mu(f=\infty)}_{\text{i.e. } f < \infty \text{ } \mu\text{-a.e.}} = 0 \end{aligned}$$

For (ii) suppose $f \in \mathcal{H}(\Omega, \mathcal{F})$.

$$\begin{aligned} \int_{\Omega} f d\mu = 0 &\Leftrightarrow \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu = 0, \quad \forall n \\ &\left\{ \begin{array}{l} \text{the direction } \Leftarrow \text{ follows since} \\ \{f \geq \frac{1}{n}\} \uparrow \{f > 0\} \\ \therefore f I_{\{f \geq \frac{1}{n}\}} \uparrow f I_{\{f > 0\}} = f \\ \therefore \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu \uparrow \int_{\Omega} f d\mu \end{array} \right. \\ &\Leftrightarrow \mu(f \geq \frac{1}{n}) = 0 \quad \forall n \\ &\left\{ \begin{array}{l} \text{since } \frac{1}{n} I_{\{f \geq \frac{1}{n}\}} \leq f I_{\{f \geq \frac{1}{n}\}} \leq n I_{\{f \geq \frac{1}{n}\}} \\ \therefore \frac{1}{n} \mu(f \geq \frac{1}{n}) \leq \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu \leq n \mu(f \geq \frac{1}{n}) \end{array} \right. \\ &\Leftrightarrow \mu(f > 0) = 0 \\ &\left\{ \begin{array}{l} \text{since } \mu(f \geq \frac{1}{n}) \uparrow \mu(f > 0) \\ \text{by CFB} \end{array} \right. \\ &\Leftrightarrow f = 0 \text{ } \mu\text{-a.e.} \end{aligned}$$

For (iii) suppose $f, g \in \mathcal{H}(\Omega, \mathcal{F})$ & $f = g \mu\text{-a.e.}$

$$\begin{aligned} \int_{\Omega} f d\mu &\stackrel{3}{=} \int_{\Omega} f I_{\{f=g\}} d\mu + \underbrace{\int_{\Omega} f I_{\{f \neq g\}} d\mu}_{=0 \text{ by (ii)}} \\ &= \int_{\Omega} g I_{\{f=g\}} d\mu + \underbrace{\int_{\Omega} g I_{\{f \neq g\}} d\mu}_{=0 \text{ by (ii)}} \\ &= \int_{\Omega} g d\mu. \quad \underline{\text{QED.}} \end{aligned}$$

Step 3

Recall

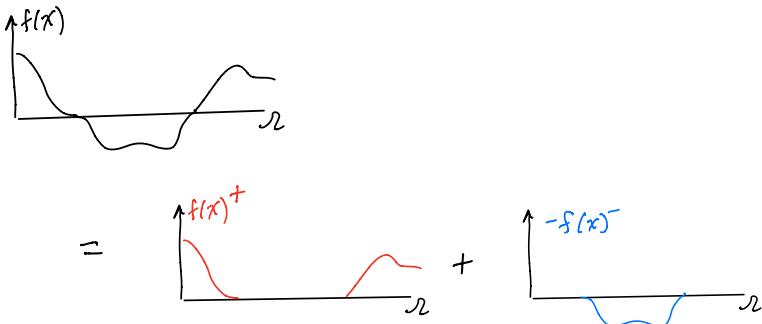
$$x^+ := \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{o.w.} \end{cases} \quad x^- := \begin{cases} 0 & \text{if } x \geq 0 \\ |x| & \text{if } x < 0. \end{cases}$$

For any $f: \Omega \rightarrow \bar{\mathbb{R}}$ s.t. $f \in \mathcal{H}/\mathcal{B}(\bar{\mathbb{R}})$

we have

- $f^+, f^- \in \mathcal{H}(\Omega, \mathcal{F})$ by composition of \mathcal{H} isom
- $f = f^+ - f^-$
- $|f| = f^+ + f^-$.

Picture:



Def: If $f: \Omega \rightarrow \bar{\mathbb{R}}$ s.t. $f \in \mathcal{H}/\mathcal{B}(\bar{\mathbb{R}})$ and either $\int_{\Omega} f^+ d\mu < \infty$ or $\int_{\Omega} f^- d\mu < \infty$ then

define

$$\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu.$$

Notation:

$$\mathcal{Q}^+(\Omega, \mathcal{F}, \mu) := \left\{ f: \Omega \rightarrow \bar{\mathbb{R}} \mid f \in \mathcal{H}/\mathcal{B}(\bar{\mathbb{R}}) \text{ and } \int_{\Omega} f^+ d\mu < \infty \right\}$$

$$\mathcal{Q}^-(\Omega, \mathcal{F}, \mu) := \left\{ f: \Omega \rightarrow \bar{\mathbb{R}} \mid f \in \mathcal{H}/\mathcal{B}(\bar{\mathbb{R}}) \text{ and } \int_{\Omega} f^- d\mu < \infty \right\}$$

$$\mathcal{Q}(\Omega, \mathcal{F}, \mu) := \mathcal{Q}^+(\Omega, \mathcal{F}, \mu) \cup \mathcal{Q}^-(\Omega, \mathcal{F}, \mu)$$

$$\mathcal{L}_1(\Omega, \mathcal{F}, \mu) := \mathcal{Q}^+(\Omega, \mathcal{F}, \mu) \cap \mathcal{Q}^-(\Omega, \mathcal{F}, \mu)$$

(10)

\mathcal{Q}^+ = quasi-integrable from above

\mathcal{Q}^- = quasi-integrable from below

\mathcal{Q} = quasi-integrable

L_1 = integrable.

Thm (Big 3):

(1) If $f, g \in \mathcal{Q}(\Omega, \mathcal{F}, \mu)$ then

$$f \leq g \text{ } \mu\text{-a.e.} \Rightarrow \int f d\mu = \int g d\mu$$

(2)

[a] $f \in \mathcal{H}(\Omega, \mathcal{F}, \mu)$ & $\alpha \in [0, \infty]$

$$\Rightarrow \int_{\Omega} \alpha f d\mu = \alpha \int_{\Omega} f d\mu$$

[b] $f \in \mathcal{Q}(\Omega, \mathcal{F}, \mu)$ & $\alpha \in \mathbb{R}$

$$\Rightarrow \alpha f \in \mathcal{Q}(\Omega, \mathcal{F}, \mu) \text{ and} \\ \int_{\Omega} \alpha f d\mu = \alpha \int_{\Omega} f d\mu$$

[c] $f, g \in \mathcal{Q}^+(\Omega, \mathcal{F}, \mu)$ or $f, g \in \mathcal{Q}^-(\Omega, \mathcal{F}, \mu)$

$$\Rightarrow f+g \in \mathcal{Q}(\Omega, \mathcal{F}, \mu) \text{ and} \\ \int_{\Omega} f+g d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu$$

(3) if $f_1, f_2, \dots \in \mathcal{H}(\Omega, \mathcal{F})$ then

$$\lim_n f_n = f \text{ } \mu\text{-a.e.} \Rightarrow \lim_n \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$$

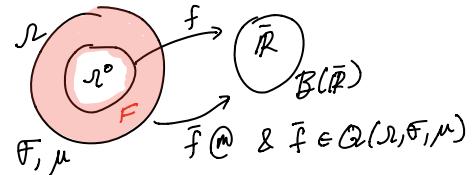
The only difference
from little 3 is this
 $\mu\text{-a.e.}$

Remark:

• In (2)[c] it could happen that

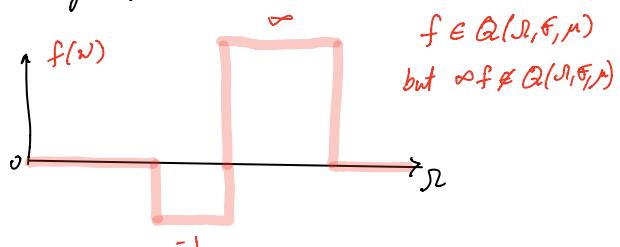
- $f(w) + g(w) = \infty - \infty$
- but since $\mu(\{f = \infty\} \cap \{g = -\infty\}) = 0$ we can modify $f \geq g$ to be defined everywhere.
- In fact this allows us to define $\int_{\Omega} f d\mu$ for all functions $f: \Omega^0 \rightarrow \mathbb{R}$ s.t.
- * $\exists F \in \mathcal{F}$ s.t. $(\Omega^0)^c \subset F$ & $\mu(F) = 0$
- * $\bar{f}(w) = \begin{cases} f(w) & w \in F^c \\ 0 & w \in F \end{cases} \in \mathcal{Q}(\Omega, \mathcal{F}, \mu)$.

Picture:



• In (2)[b] the restriction $\alpha \in \mathbb{R}$ is

necessary. e.g.



Proof:

For (1)

(13)

(14)