

Homework 2

Due Monday, February 6, 2016

Exercise 1. Let P and Q be probability measures on \mathbb{R}^d with finite second moments (i.e. if $X \sim P$ and $Y \sim Q$ then $E|X|^2 < \infty$ and $E|Y|^2 < \infty$). The L_2 Wasserstein distance d_W between P and Q is defined as

$$d_W^2(P, Q) := \inf_{\mathcal{L}(X, Y) \in \Pi(P, Q)} E|X - Y|^2$$

where $\Pi(P, Q)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals given by P and Q . In particular, if X and Y are two d -dimensional random vectors, defined on the same probability space, then the distribution of the $2d$ -dimensional random vector (X, Y) is in $\Pi(P, Q)$ if and only if $X \sim P$ and $Y \sim Q$.

Show that there exists d -dimensional random vectors X^* and Y^* , defined on the same probability space, such that $\mathcal{L}(X^*, Y^*) \in \Pi(P, Q)$ and

$$d_W^2(P, Q) := E|X^* - Y^*|^2.$$

Exercise 2. Suppose $X_{i,n}$, for $n \in \mathbb{N}$ and $i \leq n$, forms a triangular array of independent random variables which satisfy the following conditions

1. $EX_{i,n} = 0$ for all n and $i \leq n$;
2. $\sum_{i=1}^n EX_{i,n}^2 = 1$ for all n ;
3. $\sum_{i=1}^n E(X_{i,n}^2 I_{\{|X_{i,n}| \geq \delta\}}) \rightarrow 0$ for all $\delta \geq 0$ and all $n \in \mathbb{N}$.

Use Lindeberg's method to show

$$\sum_{i=1}^n X_{i,n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Exercise 3. Give a counterexample to the conjecture: If X_1, X_2, \dots are independent random variables with $EX_n = 0$ and $EX_n^2 = 1$ for all n , then $S_n/\sqrt{n} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$.

Hint: Consider X_n to take values $-v_n, 0, v_n$ with probabilities $p_n/2, 1 - p_n, p_n/2$.

Exercise 4. Let Z_1, Z_2, \dots be iid random variables with density $\delta(z)$ with respect to Lebesgue measure. Think of each Z_n recording the time it takes a random person to run a mile. Now let

$$X_n := \begin{cases} 1 & \text{if } Z_n > \max(Z_1, \dots, Z_{n-1}) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, X_n is a 1 if the n^{th} random person breaks a speed record. Show

$$\frac{S_n - EX_n}{\sqrt{\text{var}(S_n)}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Hint: You can use the fact that

$$P(Z_n > Z_1, \dots, Z_n > Z_{n-1}) = \int_{\mathbb{R}} P(z > Z_1, \dots, z > Z_{n-1}) \delta(z) dz.$$