

## Lecture 10: Integration and expected value

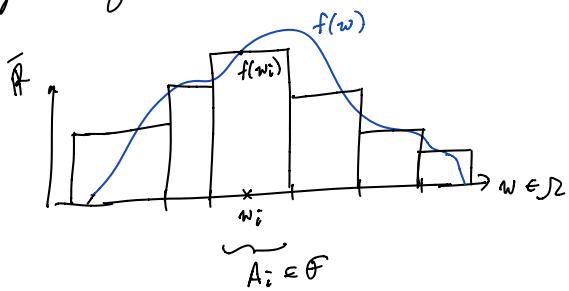
(1)

In this lecture we will define

$$\int_{\Omega} f(w) d\mu(w)$$

where  $(\Omega, \mathcal{F}, \mu)$  is a measure space and  $f: \Omega \rightarrow \bar{\mathbb{R}}$  s.t.  $f \in \mathcal{F}/B(\bar{\mathbb{R}})$ .

The notation  $\int_{\Omega} f(w) d\mu(w)$  is extremely suggestive of Riemann integration

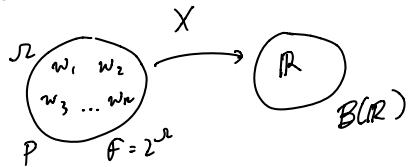


where one might guess

$$\int_{\Omega} f(w) d\mu(w) \approx \sum_i f(w_i) \mu(A_i)$$

area of block  $i$ : when  
width of  $A_i$  is measured  
with  $\mu$

To see the connection with expected value suppose  $\Omega$  has  $n$  members:



In this case we would want the definition of "expected value of  $X$ ", denoted  $E(X)$ , to, at the very least, satisfy:

$$E(X) = \left\{ \begin{array}{l} \text{the weighted average of the} \\ \text{numbers } \{X(w_1), X(w_2), \dots, X(w_n)\} \\ \text{with weights } P(\{w_i\}). \end{array} \right\}$$

$$= \sum_{i=1}^n X(w_i) P(\{w_i\})$$

partition  
measure

$$= \int_{\Omega} X(w) dP(w).$$

Assumption: For the rest of this lecture suppose  $(\Omega, \mathcal{F}, \mu)$  is a measure space

(2)

$\int_{\Omega} f d\mu$  as shorthand for  $\int_{\Omega} f(w) d\mu(w)$

Game plan:

Step 1: Define  $\int_{\Omega} f d\mu$  for  $f \in \mathcal{N}_s(\Omega, \mathcal{F})$

Non-negative simple functions.

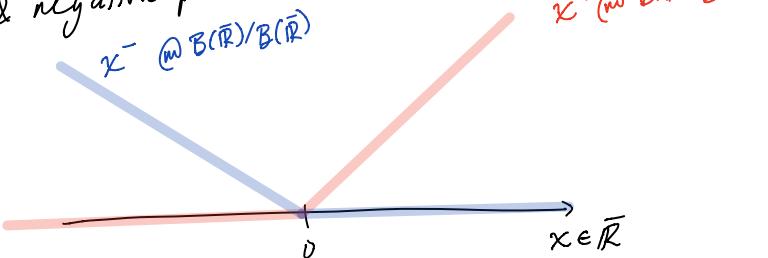
Step 2: extend to  $f \in \mathcal{N}(\Omega, \mathcal{F})$

Non-negative measurable functions.

Step 3: extend to some, but not all,  $f \in \mathcal{F}/B(\bar{\mathbb{R}})$  by

$$\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$$

where  $(\cdot)^+$ ,  $(\cdot)^-$  denotes the positive part & negative part



so that  $|x| = x^+ + x^-$  &  $x = x^+ - x^-$

Remark: Although this construction seems tedious & annoying, the method of construction is general & broadly applicable. For example, the same game plan is use for defining

$$\int_0^t f(s) dB(s)$$

Brownian motion

$f: \mathbb{R} \rightarrow \mathbb{R}$ .

Step 1

Def: If  $f \in \mathcal{N}_s(\Omega, \mathcal{F})$  has the form

$$f = \sum_{i=1}^n c_i I_{A_i} \quad \text{then} \quad \int_{\Omega} f(w) d\mu(w) := \sum_{i=1}^n c_i \mu(A_i)$$

$c_i \in [0, \infty]$   
forms a measurable partition

Here are the four basic properties  
of  $\int f d\mu$  we will show at each step  
in the game plan:

Thm (simple 3)

(0)  $\int f d\mu$  is well defined over  $\mathcal{H}_s(\Omega, \mathcal{F})$

(1) Monotonicity:

If  $f, g \in \mathcal{H}_s(\Omega, \mathcal{F})$  &  $f(w) \leq g(w) \forall w \in \Omega$  then

$$\int f d\mu \leq \int g d\mu.$$

(2) Linearity:

If  $f, g \in \mathcal{H}_s(\Omega, \mathcal{F})$  &  $\alpha, \beta \in [0, \infty]$  then

$$\int_{\Omega} (\alpha f + \beta g) d\mu = \alpha \int_{\Omega} f d\mu + \beta \int_{\Omega} g d\mu$$

(3) Continuity from below (CFB):

If  $f_n(w) \uparrow f(w)$  as  $n \rightarrow \infty$  for all  $w \in \Omega$

where  $f_n, f \in \mathcal{H}_s(\Omega, \mathcal{F})$  then

$$\int_{\Omega} f_n d\mu \uparrow \int_{\Omega} f d\mu.$$

Proof:

Suppose  $f = \sum_{i=1}^n c_i I_{A_i}$ ,  $g = \sum_{k=1}^m d_k I_{B_k}$  both in  $\mathcal{H}_s(\Omega, \mathcal{F})$

$\uparrow$   $\mathcal{F}$ -sets which partition  $\Omega$

$$\therefore f = \sum_{i,k} c_{ik} I_{A_i \cap B_k} \text{ where } c_{ik} = c_i$$

$$g = \sum_{i,k} d_{ik} I_{A_i \cap B_k} \text{ where } d_{ik} = d_k$$

$\uparrow$  a finer partition of  $\Omega$ .

To show (0) & (1) it is sufficient to show

$$f \leq g \Rightarrow \sum_{i=1}^n c_i \mu(A_i) \leq \sum_{k=1}^m d_k \mu(B_k)$$

(3)

$$f \leq g \Rightarrow \sum_{i \neq k} c_{ik} I_{A_i \cap B_k} \leq \sum_{i \neq k} d_{ik} I_{A_i \cap B_k} \quad (4)$$

exactly one term is  
non-zero (assuming  $A_i \neq \emptyset$   
&  $B_k \neq \emptyset$ ).

$$\Rightarrow c_{ik} \leq d_{ik} \quad \forall i, k$$

$$\Rightarrow \underbrace{\sum_{ik} c_{ik} \mu(A_i \cap B_k)}_{= \sum_i c_i \mu(A_i)} \leq \underbrace{\sum_{ik} d_{ik} \mu(A_i \cap B_k)}_{= \sum_k d_k \mu(B_k)}$$

by additivity of  $\mu$ .

For (2)

$$\begin{aligned} \int_{\Omega} (\alpha f + \beta g) d\mu &= \int_{\Omega} \sum_{ik} (\alpha c_{ik} + \beta d_{ik}) I_{A_i \cap B_k} d\mu \\ &= \sum_{ik} (\alpha c_{ik} + \beta d_{ik}) \mu(A_i \cap B_k) \\ &\because \text{use additivity of } \mu \text{ &} \\ &\text{linearity of } \sum_{ik} \\ &= \alpha \int_{\Omega} f d\mu + \beta \int_{\Omega} g d\mu \end{aligned}$$

For (3)

Suppose  $f_n \uparrow f$ .  
 $\uparrow$  all in  $\mathcal{H}_s(\Omega, \mathcal{F})$

Notice  $\int f_n d\mu \uparrow$  by (1) so just show

$$\lim_n \int f_n d\mu = \int f d\mu.$$

Case 1:  $f = c I_A$  for  $c \in (0, \infty]$ .  $\uparrow$  The case  $c=0$  is trivial.

Let  $0 < b < c$  so that

$$b I_{\{f_n \geq b\}} \leq f_n \leq f = c I_A.$$

Now integrate each term & use (1) to get

$$b \mu(f_n \geq b) \leq \int f_n d\mu = \int f d\mu = c \mu(A)$$

$$\begin{aligned} b \lim_n \mu(f_n \geq b) &\leq \lim_n \int_R f_n d\mu = \int_R f d\mu = c\mu(A) \quad (5) \\ &= \mu(A) \text{ by CFB since } 0 < b < c \text{ &} \\ f_n \uparrow f \text{ implies } \{f_n \geq b\} &\uparrow \{f = c\} = A \\ \text{as } n \rightarrow \infty \end{aligned}$$

Now take the limit as  $b \uparrow c$  to get

$$c\mu(A) \leq \lim_n \int_R f_n d\mu \leq \int_R f d\mu = c\mu(A)$$

$\therefore \text{these are all equal}$

Case 2:  $f = \sum_{i=1}^n c_i I_{A_i}$ , where  $A_i$ 's partition  $R$ .

Fix  $k \in \{1, 2, \dots, m\}$

$\therefore$  Now  $f_n I_{A_k} \uparrow f I_{A_k}$  so that  
Case 1 applies (since  $f I_{A_k} = c_k I_{A_k}$ )

to give  $\int_R f_n I_{A_k} d\mu \uparrow \int_R f I_{A_k} d\mu$

Now sum over  $k = 1, \dots, m$  to get.

$$\int_R f_n \underbrace{\sum_{k=1}^m I_{A_k}}_{=1} d\mu \uparrow \int_R f \underbrace{\sum_{k=1}^m I_{A_k}}_{=1} d\mu \quad \text{QED}$$

## Step 2

Recall the structure theorem:

If  $f \in \mathcal{N}(R, \mathbb{F})$  then  $\exists f_n \in \mathcal{N}_s(R, \mathbb{F})$  s.t.

$$f_n \uparrow f$$

Def: If  $f \in \mathcal{N}(R, \mathbb{F})$  define

$$\int_R f d\mu := \lim_n \int_R f_n d\mu$$

$f_n \in \mathcal{N}_s$  s.t.  
 $f_n \uparrow f$

## Thm (little 3)

(6)

Statements (0) - (3) in "simple 3" hold when  $\mathcal{N}_s(R, \mathbb{F})$  is replaced with  $\mathcal{N}(R, \mathbb{F})$ .

Proof:

To show (0) & (1), i.e.  $\int_R f d\mu$  is well defined & monotonic, start by assuming

$$\begin{array}{c} f \leq g \\ \text{both in } \mathcal{N}(R, \mathbb{F}) \end{array}$$

$\therefore \exists f_n, g_n \in \mathcal{N}_s(R, \mathbb{F})$  s.t.

$$\lim_n f_n = f \leq g = \lim_n g_n$$

Notice the following "trick"

$$\begin{aligned} \lim_m \uparrow f_n \wedge g_m &= f_n \wedge (\lim_m \uparrow g_m) \\ &= f_n \wedge g \\ &= f_n \quad \text{since } f_n \leq f \leq g \end{aligned}$$

$$\therefore \int_R f_n d\mu = \int_R \lim_m \uparrow f_n \wedge g_m d\mu$$

$$= \lim_m \uparrow \int_R f_n \wedge g_m d\mu \quad \text{by "simple 3"}$$

$$\leq \lim_m \uparrow \int_R g_m d\mu$$

Now take a limit as  $n \rightarrow \infty$  to get

$$\lim_n \uparrow \int_R f_n d\mu = \lim_m \uparrow \int_R g_m d\mu.$$

This shows (0) & (1).

The proof of (2), i.e. that

$$\int_R \alpha f + \beta g d\mu = \alpha \int_R f d\mu + \beta \int_R g d\mu$$

when  $\alpha, \beta \in [0, \infty]$  is easy (using the fact that  $\alpha f + \beta g = \lim_n \uparrow (\alpha f_n + \beta g_n)$  which implies

$$\int_R \alpha f + \beta g d\mu = \lim_n \uparrow \int_R \alpha f_n + \beta g_n d\mu \quad \text{by def.}$$

For (3):

$$\text{Show } \underbrace{f_n \uparrow f}_{\text{all in } \mathcal{H}(R, \mathcal{F})} \Rightarrow \int_R f_n d\mu \uparrow \int_R f d\mu \quad (7)$$

Suppose  $f_n \uparrow f$  & let  $f_n = \lim_m \uparrow \phi_{nm}$   
so that  $\in \mathcal{H}_s(R, \mathcal{F})$

$$\begin{array}{ccccccc} \phi_{11} & \leq & \phi_{12} & \leq \cdots & \leq & \phi_{1n} & \leq \rightarrow f_1 \\ : & & : & & : & & : \\ \phi_{k1} & \leq & \phi_{k2} & \leq \cdots & \leq & \phi_{kn} & \leq \rightarrow f_k \\ : & & : & & : & & : \\ \phi_{n1} & \leq & \phi_{n2} & \leq \cdots & \leq & \phi_{nn} & \leq \rightarrow f_n \end{array}$$

$$\text{define } \phi_n := \max_{1 \leq i, j \leq n} \phi_{ij} \in \mathcal{H}_s(R, \mathcal{F})$$

$$\text{Now } \phi_{kn} \leq \phi_n \leq f_n \leq f, \quad \forall k \leq n \quad (\star)$$

Taking limits as  $n \rightarrow \infty$  in  $(\star)$  gives

$$f_k = \lim_n \uparrow \phi_{kn} \leq \lim_n \uparrow \phi_n \leq \lim_n \uparrow f_n \leq f$$

Taking limits as  $k \rightarrow \infty$

$$f = \lim_n \uparrow f_k \leq \lim_n \uparrow \phi_n \leq \lim_n \uparrow f_n = f$$

$$\in \mathcal{H}_s(R, \mathcal{F})$$

$$\therefore f = \lim_n \uparrow \phi_n \text{ where } \phi_n \in \mathcal{H}_s(R, \mathcal{F}) \text{ so}$$

$$\int_R f d\mu := \lim_n \int_R \phi_n d\mu \quad \text{by def.}$$

$$\text{Now just show } \lim_n \int_R \phi_n d\mu = \lim_n \int_R f_n d\mu$$

Instead of taking limits in  $(\star)$  first, integrate to get

$$\int_R \phi_{kn} d\mu \leq \int_R \phi_n d\mu \leq \int_R f_n d\mu, \quad \forall k \leq n$$

Now let  $n \rightarrow \infty$  for

$$\int_R f_k d\mu \leq \lim_n \int_R \phi_n d\mu \leq \lim_n \int_R f_n d\mu$$

where  $\int_R f_k d\mu = \lim_n \int_R \phi_{kn} d\mu$  by def.

Finally let  $k \rightarrow \infty$  to give

$$\lim_k \int_R f_k d\mu = \lim_n \int_R \phi_n d\mu.$$

QED

Before we move to Step 3 we need some useful facts.

Def:  $f=g$   $\mu$ -a.e. means  $\mu(f \neq g) = 0$

$f \leq g$   $\mu$ -a.e. means  $\mu(f \neq g) = 0$

Thm (a.e. useful facts)

(i)  $f \in \mathcal{H}(R, \mathcal{F})$  &  $\int_R f d\mu < \infty \Rightarrow f < \infty \mu$ -a.e.

(ii) If  $f \in \mathcal{H}(R, \mathcal{F})$  then

$$\int_R f d\mu = 0 \Leftrightarrow f = 0 \mu\text{-a.e.}$$

(iii) If  $f, g \in \mathcal{H}(R, \mathcal{F})$  and  $f=g$   $\mu$ -a.e.

$$\text{then } \int_R f d\mu = \int_R g d\mu.$$

which implies  
I can change f  
on  $\mu$ -null sets without  
changing  $\int_R f d\mu$ .

Proof:

For (i) Notice that  $f \in \mathcal{M}(\Omega, \mathcal{F})$  implies  
 $\int_{\Omega} I_{\{f=\infty\}} d\mu \leq f$  falls under the "indicate what you want" trick

using our convention that  $\infty \cdot 0 = 0$

$$\begin{aligned} \int_{\Omega} f d\mu < \infty &\stackrel{\text{little } 3}{\Rightarrow} \infty \mu(f=\infty) \leq \int_{\Omega} f d\mu < \infty \\ &\Rightarrow \underbrace{\mu(f=\infty)}_{\text{i.e. } f < \infty \text{ } \mu\text{-a.e.}} = 0 \end{aligned}$$

For (ii) suppose  $f \in \mathcal{M}(\Omega, \mathcal{F})$ .

$$\begin{aligned} \int_{\Omega} f d\mu = 0 &\Leftrightarrow \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu = 0, \quad \forall n \\ &\left\{ \begin{array}{l} \text{the direction } \Leftarrow \text{ follows since} \\ \{f \geq \frac{1}{n}\} \uparrow \{f > 0\} \\ \therefore f I_{\{f \geq \frac{1}{n}\}} \uparrow f I_{\{f > 0\}} = f \\ \therefore \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu \uparrow \int_{\Omega} f d\mu \end{array} \right. \\ &\Leftrightarrow \mu(f > 0) = 0 \quad \forall n \\ &\left\{ \begin{array}{l} \text{since } \frac{1}{n} I_{\{f \geq \frac{1}{n}\}} \leq f I_{\{f \geq \frac{1}{n}\}} \leq n I_{\{f \geq \frac{1}{n}\}} \\ \therefore \frac{1}{n} \mu(f > \frac{1}{n}) \leq \int_{\Omega} f I_{\{f \geq \frac{1}{n}\}} d\mu \leq n \mu(f > \frac{1}{n}) \end{array} \right. \\ &\Leftrightarrow \mu(f > 0) = 0 \\ &\left\{ \begin{array}{l} \text{since } \mu(f > \frac{1}{n}) \uparrow \mu(f > 0) \\ \text{by CFB} \end{array} \right. \\ &\Leftrightarrow f = 0 \text{ } \mu\text{-a.e.} \end{aligned}$$

For (iii) suppose  $f, g \in \mathcal{M}(\Omega, \mathcal{F})$  &  $f = g \mu\text{-a.e.}$

$$\begin{aligned} \int_{\Omega} f d\mu &\stackrel{3}{=} \int_{\Omega} f I_{\{f=g\}} d\mu + \underbrace{\int_{\Omega} f I_{\{f \neq g\}} d\mu}_{=0 \text{ by (ii)}} \\ &= \int_{\Omega} g I_{\{f=g\}} d\mu + \underbrace{\int_{\Omega} g I_{\{f \neq g\}} d\mu}_{=0 \text{ by (ii)}} \\ &= \int_{\Omega} g d\mu. \quad \underline{\text{QED.}} \end{aligned}$$

### Step 3

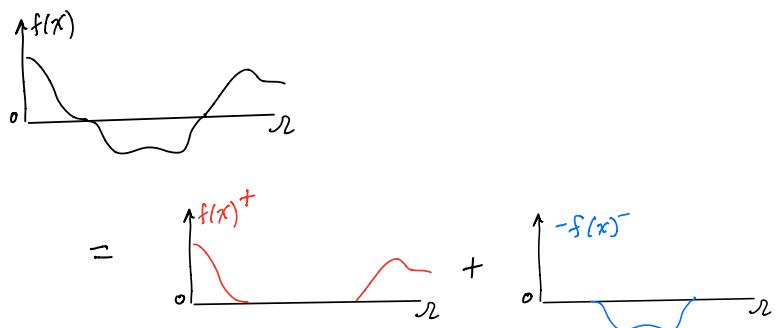
Recall

$$x^+ := \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{o.w.} \end{cases} \quad \& \quad x^- := \begin{cases} 0 & \text{if } x \geq 0 \\ |x| & \text{if } x < 0. \end{cases}$$

For any  $f: \Omega \rightarrow \bar{\mathbb{R}}$  s.t.  $f \in \mathcal{M}(\Omega, \mathcal{F})$  we have

- $f^+, f^- \in \mathcal{M}(\Omega, \mathcal{F})$  by composition of  $\mathcal{M}$  is  $\mathcal{M}$
- $f = f^+ - f^-$
- $|f| = f^+ + f^-$ .

Picture:



Def: If  $f: \Omega \rightarrow \bar{\mathbb{R}}$  s.t.  $f \in \mathcal{M}(\Omega, \mathcal{F})$  and either  $\int_{\Omega} f^+ d\mu < \infty$  or  $\int_{\Omega} f^- d\mu < \infty$  then

define  $\int_{\Omega} f d\mu := \int_{\Omega} f^+ d\mu - \int_{\Omega} f^- d\mu$ .

Notation:

$$\mathcal{Q}^+(\Omega, \mathcal{F}, \mu) := \left\{ f: \Omega \rightarrow \bar{\mathbb{R}} \mid f \in \mathcal{M}(\Omega, \mathcal{F}) \text{ and } \int_{\Omega} f^+ d\mu < \infty \right\}$$

$$\mathcal{Q}^-(\Omega, \mathcal{F}, \mu) := \left\{ f: \Omega \rightarrow \bar{\mathbb{R}} \mid f \in \mathcal{M}(\Omega, \mathcal{F}) \text{ and } \int_{\Omega} f^- d\mu < \infty \right\}$$

$$\mathcal{Q}(\Omega, \mathcal{F}, \mu) := \mathcal{Q}^+(\Omega, \mathcal{F}, \mu) \cup \mathcal{Q}^-(\Omega, \mathcal{F}, \mu)$$

$$\mathcal{L}_1(\Omega, \mathcal{F}, \mu) := \mathcal{Q}^+(\Omega, \mathcal{F}, \mu) \cap \mathcal{Q}^-(\Omega, \mathcal{F}, \mu)$$

(10)

$\mathcal{Q}^+$  = quasi-integrable from above

$\mathcal{Q}^-$  = quasi-integrable from below

$\mathcal{Q}$  = quasi-integrable

$L_1$  = integrable.

(11)

Thm (Big 3):

(1) If  $f, g \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$  then  
 $f \leq g \mu\text{-a.e.} \Rightarrow \int f d\mu = \int g d\mu$

(2)
 

- [a]  $f \in \mathcal{N}(\mathbb{R}, \mathcal{F}, \mu)$  &  $\alpha \in [0, \infty]$   
 $\Rightarrow \int_{\mathbb{R}} \alpha f d\mu = \alpha \int_{\mathbb{R}} f d\mu$

[b]  $f \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$  &  $\alpha \in \mathbb{R}$   
 $\Rightarrow \alpha f \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$  and  
 $\int_{\mathbb{R}} \alpha f d\mu = \alpha \int_{\mathbb{R}} f d\mu$

[c]  $f, g \in \mathcal{Q}^+(\mathbb{R}, \mathcal{F}, \mu)$  or  $f, g \in \mathcal{Q}^-(\mathbb{R}, \mathcal{F}, \mu)$   
 $\Rightarrow f+g \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$  and  
 $\int_{\mathbb{R}} f+g d\mu = \int_{\mathbb{R}} f d\mu + \int_{\mathbb{R}} g d\mu$

(3) if  $f_1, f_2, \dots \in \mathcal{N}(\mathbb{R}, \mathcal{F})$  then

$$\lim_n f_n = f \mu\text{-a.e.} \Rightarrow \lim_n \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} f d\mu$$

The only difference  
from little 3 is the  
 $\mu\text{-a.e.}$

Remark:

• In (2)[c] it could happen that

$$f(w) + g(w) = \infty - \infty$$

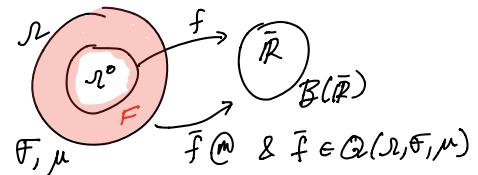
but since  $\mu(\{f = \infty\} \cap \{g = -\infty\}) = 0$  we  
can modify  $f \& g$  to be defined everywhere.

In fact this allows us to define  $\int_{\mathbb{R}} f d\mu$   
for all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  s.t.  $\mathbb{R} \subset \mathbb{R}$  and

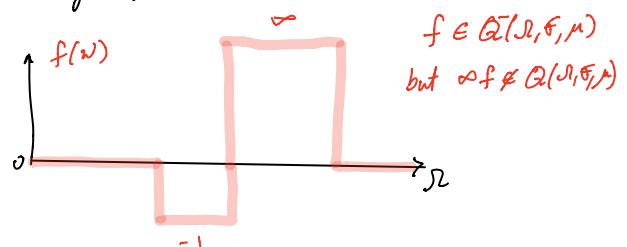
\*  $\exists F \in \mathcal{F}$  s.t.  $(\mathbb{R})^c \subset F$  &  $\mu(F) = 0$

\*  $\bar{f}(w) = \begin{cases} f(w) & w \in F^c \\ 0 & w \in F \end{cases} \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$ .

Picture:



• In (2)[b] the restriction  $\alpha \in \mathbb{R}$  is  
necessary. e.g.



Proof:

For (1)

suppose  $f, g \in \mathcal{Q}(\mathbb{R}, \mathcal{F}, \mu)$  &  $f \leq g \mu\text{-a.e.}$

Modify  $f \& g$  on a  $\mu$ -null set so that  $f = g$  everywhere.

$$\therefore f^+ - f^- \leq g^+ - g^-$$

$$\therefore f^+ \leq g^+ \text{ and } f^- \geq g^-$$

To see why notice that if not there is a contradiction.

$$\begin{array}{ccc}
 f^-(w) < g^-(w) & \text{or} & f^+(w) > g^+(w) \\
 g^-(w) > 0 & \Downarrow g^-(w) = 0 & f^+(w) > 0 \quad \Downarrow f^+(w) = 0 \\
 f(w) \leq g(w) < 0 & f^-(w) < 0 & 0 < f(w) \leq g(w) \quad \Downarrow g^+(w) < 0 \\
 f^-(w) \geq g^-(w) & \Downarrow \text{Contradiction} & f^+(w) \leq g^+(w) \quad \Downarrow \text{Contradiction} \\
 \Downarrow \text{Contradiction} & & \Downarrow \text{Contradiction}
 \end{array}$$

Now  $f^+ \leq g^+$  and  $f^- \geq g^-$  implies

$$\int_R f^+ d\mu \leq \int_R g^+ d\mu \quad \text{by little 3.}$$

$$\int_R f^- d\mu \geq \int_R g^- d\mu.$$

(13)

∴ side fact:

$$g \in Q^+ \text{ & } f \leq g \Rightarrow \int_R f^+ d\mu < \infty \Rightarrow f \in Q^+$$

$$f \in Q^- \text{ & } f \leq g \Rightarrow \int_R g^- d\mu < \infty \Rightarrow g \in Q^-$$

$$\therefore \underbrace{\int_R f^+ d\mu - \int_R f^- d\mu}_{=: \int_R f d\mu} \leq \underbrace{\int_R g^+ d\mu - \int_R g^- d\mu}_{=: \int_R g d\mu}$$

For (2)[a]. This is just little (3)

For (2)[b]. Suppose  $f \in Q(R, \mathcal{F}, \mu)$  &  $\alpha \in R$ .

Case 1:  $\alpha \in (-\infty, 0)$

$$\therefore \int_R (\alpha f)^+ d\mu = \int_R |\alpha| f^- d\mu = |\alpha| \int_R f^- d\mu < \infty$$

or

$$\int_R (\alpha f)^- d\mu = \int_R |\alpha| f^+ d\mu = |\alpha| \int_R f^+ d\mu < \infty$$

∴  $\alpha f \in Q(R, \mathcal{F}, \mu)$  and

$$\begin{aligned} \int_R \alpha f d\mu &= \int_R (\alpha f)^+ d\mu - \int_R (\alpha f)^- d\mu \\ &= |\alpha| \left[ \int_R f^- d\mu - \int_R f^+ d\mu \right] = \alpha \int_R f d\mu \end{aligned}$$

Case 2 is similar.

Side fact:

$$f \in Q^\pm \text{ & } \alpha \in (-\infty, 0) \Rightarrow \alpha f \in Q^\mp$$

$$f \in Q^\pm \text{ & } \alpha \in (0, \infty) \Rightarrow \alpha f \in Q^\mp$$

For (2)[c]:

Suppose  $f, g \in Q^+(R, \mathcal{F}, \mu)$  or  $f, g \in Q^-(R, \mathcal{F}, \mu)$ .

Show  $f+g \in Q(R, \mathcal{F}, \mu)$  and

$$\int_R (f+g) d\mu = \int_R f d\mu + \int_R g d\mu$$

(14)

Notice that if  $a, b \in \bar{R}$  s.t.  $a+b$  is defined  
then  $(a+b)^+ - (a+b)^- = a+b = a^+ - a^- + b^+ - b^-$   
so that  $(a+b)^+ + a^- + b^- = (a+b)^- + a^+ + b^+$ .

**Warning!** Be careful moving terms to the other side.  
Here it is ok since:

$$(a+b) = \infty \Rightarrow a^+ \text{ or } b^+ \text{ is } \infty \Rightarrow RHS = LHS = \infty$$

$$(a+b) = -\infty \Rightarrow a^- \text{ or } b^- \text{ is } \infty \Rightarrow RHS = LHS = \infty$$

Now (\*) implies that

$$\underbrace{(f+g)^+ + f^- + g^-}_{\in \eta(R, \mathcal{F})} = \underbrace{(f+g)^- + f^+ + g^+}_{\in \eta(R, \mathcal{F})}$$

∴ little 3 implies

$$\int_R (f+g)^+ d\mu + \int_R f^- d\mu + \int_R g^- d\mu$$

$$(*) = \int_R (f+g)^- d\mu + \int_R f^+ d\mu + \int_R g^+ d\mu$$

case 1:  $f, g \in Q^-(R, \mathcal{F}, \mu)$ .

The idea is to show  $\int_R (f+g)^- d\mu < \infty$  so one can move it, along with  $\int_R f^- d\mu$  &  $\int_R g^- d\mu$ , to the opposite side in (\*). both finite since  $f, g \in Q^-$ .

Indeed  $(f+g)^- \leq f^- + g^-$  by convexity.

∴  $\int_R (f+g)^- d\mu \leq \int_R f^- d\mu + \int_R g^- d\mu < \infty$  by little 3.

Now move the three finite terms in (\*) to get

$$\int_R (f+g) d\mu = \int_R f d\mu + \int_R g d\mu.$$

Case 2:  $f, g \in Q^+(R, \mathcal{F}, \mu)$  is similar.

For (3)

Suppose  $f_1, f_2, \dots \in \mathcal{N}(\mathbb{R}, \mathcal{F})$  and

$$\lim_n f_n = f \quad \mu\text{-a.e.}$$

all the  $f_n$ 's and  $f$  on a  $\mu$ -null set  
(note: countable unions of  $\mu$ -nulls is  $\mu$ -null)  
so that

$0 \leq f_n \uparrow f$  everywhere.

Now (3) follows directly by little (3).

QED

Corollary to Big 3:

If  $f \in Q(\mathbb{R}, \mathcal{F}, \mu)$  then

$$|\int_{\mathbb{R}} f d\mu| \leq \int_{\mathbb{R}} |f| d\mu.$$

If  $f \in \mathcal{F}/B(\mathbb{R})$  and  $\int_{\mathbb{R}} |f| d\mu < \infty$  then  $f \in L_1(\mathbb{R}, \mathcal{F}, \mu)$

and if  $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$  also then

$$\int_{\mathbb{R}} \alpha f + \beta g d\mu = \alpha \int_{\mathbb{R}} f d\mu + \beta \int_{\mathbb{R}} g d\mu$$

$\forall \alpha, \beta \in \mathbb{R}$ .

Proof:

Suppose  $f \in Q(\mathbb{R}, \mathcal{F}, \mu)$ .

$$\therefore -|f| \leq f \leq |f|$$

$$\in \mathcal{Q}^+ \quad \in \mathcal{Q}^-$$

$$\therefore - \int_{\mathbb{R}} |f| d\mu \leq \int_{\mathbb{R}} f d\mu \leq \int_{\mathbb{R}} |f| d\mu$$

$$\text{Big 3 (1) \& (2)} \quad \text{Big 3 (1)}$$

$$\therefore |\int_{\mathbb{R}} f d\mu| \leq \int_{\mathbb{R}} |f| d\mu.$$

$$\text{Also } \int_{\mathbb{R}} |f| d\mu = \int_{-\infty}^{\infty} f^+ d\mu + \int_{-\infty}^{\infty} f^- d\mu \Rightarrow f \in \mathcal{Q}^+ \cap \mathcal{Q}^- = L_1$$

Finally  $f, g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$  &  $\alpha, \beta \in \mathbb{R}$

$$\Rightarrow \alpha f, \alpha g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$$

$$\Rightarrow \int_{\mathbb{R}} \alpha f + \beta g d\mu \stackrel{B3}{=} \int_{\mathbb{R}} \alpha f d\mu + \int_{\mathbb{R}} \beta g d\mu$$

$$\stackrel{B3}{=} \alpha \int_{\mathbb{R}} f d\mu + \beta \int_{\mathbb{R}} g d\mu$$

QED

(15)

Using the linear part of Big (3)

(16)

An application typically looks like this:

$$\dots = \int_{\mathbb{R}} \alpha f + \beta g d\mu$$

You've got to  
a point where  
this is well defined  
i.e.  $\alpha f + \beta g \in Q(\mathbb{R}, \mathcal{F}, \mu)$

$$= \int_{\mathbb{R}} \alpha f d\mu + \int_{\mathbb{R}} \beta g d\mu$$

You can make this "move"  
if the terms on the right  
are defined & their sum  
isn't  $+\infty$  or  $-\infty + \infty$ .

$$\begin{aligned} \text{e.g. } -\infty &< \int_{\mathbb{R}} \alpha f d\mu \quad \& \quad \int_{\mathbb{R}} \beta g d\mu = \infty \\ \Rightarrow \int_{\mathbb{R}} (\alpha f)^+ d\mu &< \infty \quad \& \quad \int_{\mathbb{R}} (\beta g)^- d\mu < \infty \\ \Rightarrow \alpha f, \beta g &\in Q^-(\mathbb{R}, \mathcal{F}, \mu) \\ \Rightarrow \text{Big 3(2) applies} \end{aligned}$$

Now you can make this "move"  
if  $\alpha \in \mathbb{R}$  or  $f \geq 0$  or  $f \leq 0$ .

e.g. Suppose  $f \leq 0$  so that  $-f \in \mathcal{N}(\mathbb{R}, \mathcal{F})$

$$\begin{aligned} \therefore \int_{\mathbb{R}} -\infty f d\mu &= \infty \int_{\mathbb{R}} -f d\mu \quad \text{by Big 3(2)[a]} \\ &= \infty (-1) \int_{\mathbb{R}} f d\mu \quad \text{by Big 3(2)[b]} \end{aligned}$$

Notation

In the literature the following are all synonymous:

$$\int_{\mathbb{R}} f d\mu \equiv \int_{\mathbb{R}} f d\mu \equiv \int_{\mathbb{R}} f(w) d\mu(w) \equiv \int_{\mathbb{R}} f(w) \mu(dw)$$

This one  
annoys me  
for some reason.

If  $(\mathbb{R}, \mathcal{F}, \mu) = (\mathbb{R}, \overline{B(\mathbb{R})}, \overline{\mathcal{L}'})$  then

$$\int_{\mathbb{R}} f d\mu \equiv \int_{\mathbb{R}} f(x) dx \equiv \text{"Lebesgue integral"}$$

Counting measure and infinite series

Notice that  $\int_{\mathbb{R}} f d\mu$  is flexible enough  
to unify integration theory with part of  
(but not all) infinite series theory.

Let  $\mathbb{Z} = \mathbb{N} := \{1, 2, 3, \dots\}$

$$F = 2^{\mathbb{Z}}$$

$\lambda$  = counting measure

Any  $f(k)$  mapping  $\mathbb{N}$  to  $\bar{\mathbb{R}}$  is  $\text{convergent}$  (17)

$$\text{Claim: } \int_N f(k) d\lambda(k) = \sum_{k=1}^{\infty} f(k)$$

whenever  $\sum_{k=1}^{\infty} f^+(k) < \infty$  or  $\sum_{k=1}^{\infty} f^-(k) < \infty$ .

Proof: For any fixed  $N \in \mathbb{N}$

$$\begin{aligned} f_N(k) &:= \begin{cases} f(k) & \text{for } 1 \leq k \leq N \\ 0 & \text{for } k > N \end{cases} \\ &= f(1) I_{\{\{1\}\}}(k) + \cdots + f(N) I_{\{\{N\}\}}(k) \\ &\quad \underbrace{\text{has the form } \sum_{i=1}^N c_i I_{A_i}(k)} \end{aligned}$$

with similar def for  $f_N^+(k), f_N^-(k)$ .

Notice  $f_N^+, f_N^- \in \mathcal{Q}(\mathbb{N}, \mathcal{F})$ .

$$\begin{aligned} \therefore \int_N f_N^{\pm}(k) d\lambda(k) &\stackrel{\text{def}}{=} \sum_{i=1}^N f^{\pm}(i) \underbrace{\lambda(\{\{i\}\})}_{=1 \text{ by counting measure}} \\ &= \sum_{k=1}^N f^{\pm}(k) \\ \therefore \lim_{N \rightarrow \infty} \int_N f_N^{\pm}(k) d\lambda(k) &= \sum_{k=1}^{\infty} f^{\pm}(k) \\ &\text{II B3(3) since } f_N^{\pm} \uparrow \end{aligned}$$

$$\int_N \lim_{N \rightarrow \infty} f_N^{\pm}(k) d\lambda(k)$$

$$\int_N f^{\pm}(k) d\lambda(k)$$

$$\therefore \text{if } \sum_{k=1}^{\infty} f^+(k) < \infty \text{ or } \sum_{k=1}^{\infty} f^-(k) < \infty$$

then  $f \in \mathcal{Q}(\mathbb{N}, \mathcal{F}, \mu)$  &

$$\begin{aligned} \int_N f(k) d\lambda(k) &:= \int_N f^+(k) d\lambda(k) - \int_N f^-(k) d\lambda(k) \\ &= \sum_{k=1}^{\infty} [f^+(k) - f^-(k)] \\ &\quad \underbrace{f(k)}_{\text{QED}} \end{aligned}$$

Warning!  $\int_N f(k) d\lambda(k)$  isn't defined (18)

for some convergent series  $\sum_{k=1}^{\infty} f(k) < \infty$ .

e.g.  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} < \infty$  but

$$\int_N \left( \frac{(-1)^k}{k} \right)^+ d\lambda(k) = \sum_{k=1}^{\infty} \frac{1}{2k} = \infty \quad \text{Same for } \int_N \left( \frac{(-1)^k}{k} \right)^- d\lambda(k)$$

$\therefore \left( \frac{(-1)^k}{k} \right)^+ \notin \mathcal{Q}(\mathbb{N}, \mathcal{F}, \lambda)$  so  $\int_N \left( \frac{(-1)^k}{k} \right)^+ d\lambda(k)$  is not defined. The reason is that our  $\int_N f d\mu$  does not allow infinite cancellation.

### Lebesgue integration vrs. Riemann Integration.

Suppose  $[a, b]$  is a bdd interval of  $\mathbb{R}$  and

$$f: [a, b] \rightarrow \mathbb{R}$$

Let

$$\underline{R}(f) := \sup \left\{ \sum_{A \in \Pi} (\inf_{x \in A} f(x)) \cdot J'(A) : \Pi \text{ is a partition of } [a, b] \right\}$$

$$\bar{R}(f) := \inf \left\{ \sum_{A \in \Pi} (\sup_{x \in A} f(x)) \cdot J'(A) : \Pi \text{ is a partition of } [a, b] \right\}$$

Def: The Riemann integral of  $f$  exists if

$$\underline{R}(f) = \bar{R}(f) < \infty$$

In which case, the common value, denoted  $R(f)$  is the Riemann integral of  $f$ .

Thm (Lebesgue) If  $[a, b]$  is a bdd subinterval

of  $\mathbb{R}$  and  $f: [a, b] \rightarrow \mathbb{R}$  then  $R(f)$  exists iff

- $f$  is bounded and

- $J'(\{x \in [a, b] : f \text{ is discontinuous at } x\}) = 0$ .

Moreover, if  $R(f)$  exists then

$$R(f) = \int_{[a, b]} f(x) dx. \quad \text{Lebesgue integral.}$$

Proof: Note ... there is no measurability assumption on  $f$ . Proof left as an exercise.

Note:

We can use this fact to compute the Lebesgue integral via the fundamental Thm of calculus:

$f'$  is continuous &  $a, b \in \mathbb{R}$

$$\Rightarrow \underbrace{\int_a^b f'(x) dx}_{\text{Lebesgue integral}} = R(f') = f(b) - f(a)$$

Important advantage of  $\int f dm$  vs  $R(f)$

The Riemann integral of  $f$  is not invariant to changing  $f$  on a set of measure 0  
... but Lebesgue integration is.

Here are some examples to illustrate this & that Lebesgue integration is non-trivially more general than Riemann integration (for bdd  $[a, b]$ ).

example 1:  $f(x) = 0$  on  $x \in [0, 1]$

$$\therefore \text{trivially } R(f) = 0 = \int_0^1 f(x) dx.$$

example 2:  $f(x) = I_{\mathbb{Q}}(x)$  on  $x \in [0, 1]$

since  $f$  is discontinuous at all  $x \in [0, 1]$ , which has non-zero Lebesgue measure,

$R(f)$  does not exist.

but

$$\int_0^1 f(x) dx = 0.$$

example 3:  $f(x) = I_C(x)$  on  $x \in [0, 1]$  where  $C$  is the Cantor set.

Now

$$C = \left\{ x \in [0, 1] : f \text{ is discontinuous at } x \right\}$$

$$\mathcal{L}(C) = 0 \text{ and } f \text{ is bdd}$$

$$\therefore R(f) \text{ exists and equals } \int_0^1 f(x) dx = 0$$

(19)

At this point one might conjecture that for any Lebesgue integrable  $f: [a, b] \rightarrow \mathbb{R}$  one can modify  $f$  on a  $\mathcal{L}'$ -null set to get a Riemann integrable  $f$ . The next example shows this is not true (implying that  $\int f dm$  is non-trivially more general than  $R(f)$ ). (20)

example 4:  $f(x) = I_V(x)$  on  $x \in [0, 1]$  where  $V$  is the fat Cantor set

(this set is constructed by removing proportion  $\frac{1}{3^n}$  at step  $n$  instead of proportion  $\frac{1}{3}$  used to construct the Cantor set).

In this case

$$V = \left\{ x \in [0, 1] : f \text{ is discontinuous at } x \right\}$$

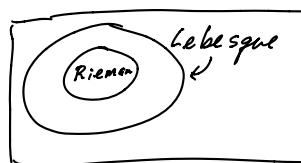
and

$$\mathcal{L}'(V) > 0.$$

$\therefore R(f)$  does not exist ... yet  $\int_0^1 f(x) dx$  does (and is non-zero).

Also notice that  $f$  can't be modified on a  $\mathcal{L}'$ -null set to get a Riemann integrable function.

Therefore we have the following picture

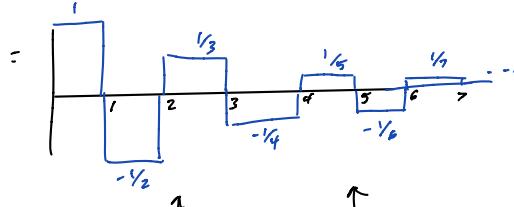


functions on bdd intervals of  $\mathbb{R}$

The story is different for functions defined on non-bdd subintervals of  $\mathbb{R}$ . (21)

In particular, there do exist improper Riemann integrals which are not Lebesgue integrable.

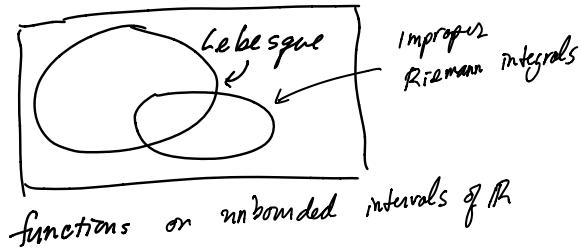
$$\text{e.g. } f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} I_{[n-1, n)}(x)$$



Has a finite  
improper Riemann  
integral

... but this  
isn't Lebesgue  
integrable.

so that ...



### Integration to the limit

For this section let  $(\Omega, \mathcal{F}, \mu)$  be a measure space and  $f_1, f_2, \dots, f$  be  $\mathcal{C}(\mathcal{F}/\mathcal{B}(\mathbb{R}))$  mapping  $\Omega$  into  $\bar{\mathbb{R}}$ .

### Fatou's lemma

If  $f_n \geq 0$   $\mu$ -a.e. then

$$\int_{\Omega} \underbrace{\liminf_{n \rightarrow \infty} f_n}_{\text{(m) by closure}} d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

Note: the fact that we don't have the "sup version" is related to the fact that general measures are not guaranteed to be continuous from above (... & if we did then  $\lim_n \int f_n d\mu$  would always equal  $\int \lim_n f_n d\mu$ ). (22)

Proof:

$$\begin{aligned} \text{LHS} &= \int_{\Omega} \limsup_{q \rightarrow \infty} \inf_{n \geq q} f_n d\mu \\ &= \limsup_{q \rightarrow \infty} \int_{\Omega} \inf_{n \geq q} f_n d\mu \quad \text{By Big 3.} \\ &\quad \underbrace{\qquad\qquad\qquad}_{\text{smaller than } \int f_n d\mu \text{ for } n \geq q \text{ by Big 3}} \\ &\leq \limsup_{q \rightarrow \infty} \inf_{n \geq q} \int_{\Omega} f_n d\mu \\ &= \liminf_{n \rightarrow \infty} \int_{\Omega} f_n d\mu \end{aligned}$$

QED.

The following theorems give conditions when

$$\lim_n \int = \int \lim_n$$

and can all be thought of as conditions for extending Fatou's lemma to include the limsup upper bounds.

### Thm (DCT)

- If (a)  $f_n \rightarrow f$   $\mu$ -a.e. as  $n \rightarrow \infty$   
 (b)  $\sup_n \|f_n\| \leq g \in L_1(\Omega, \mathcal{F}, \mu)$   
 $\qquad\qquad\qquad \mu\text{-a.e.}$

then (A)  $f_n, f \in L_1(\Omega, \mathcal{F}, \mu)$

$$(B) \quad \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu$$

Proof:

(23)

(b) implies that  $f_n, f, \liminf_n f_n$  &  $\limsup_n f_n$  are all in  $L_1(\Omega, \mathcal{F}, \mu)$

$\therefore (B)$  is true

since  $|\liminf_n f_n| \leq \liminf_n |f_n| \leq g$

To show (A) notice

$$\begin{aligned} \int_R g d\mu + \int_R f d\mu &\stackrel{\text{Big 3}}{=} \int_R g + f d\mu \\ &= \int_R \liminf_n (g + f_n) d\mu \\ &\geq 0 \text{ } \mu\text{-a.e. since} \\ &-g \leq f_n \leq g \text{ } \mu\text{-a.e.} \\ &\stackrel{\text{Fatou}}{\leq} \liminf_n \int_R g + f_n d\mu \\ &\stackrel{\text{Big 3}}{=} \int_R g d\mu + \liminf_n \int_R f_n d\mu \end{aligned}$$

These cancel since  $\int_R g d\mu < \infty$

$$\therefore \int_R f d\mu \leq \liminf_n \int_R f_n d\mu$$

Side Note: this gives us an extension to Fatou:  
 $-g \leq f_n \text{ & } g \in L^1$   
 $\Rightarrow \int_R \liminf f_n d\mu \leq \liminf_n \int_R f_n d\mu$

Now all we need is  $\limsup_n \int_R f_n d\mu \leq \int_R f d\mu$ .

$$\begin{aligned} \int_R g d\mu - \int_R f d\mu &\stackrel{\text{Big 3}}{=} \int_R \liminf_n (g - f_n) d\mu \\ &\geq 0 \text{ } \mu\text{-a.e.} \end{aligned}$$

$$\stackrel{\text{Fatou}}{\leq} \liminf_n \int_R g - f_n d\mu$$

$$\stackrel{\text{Big 3}}{=} \int_R g d\mu - \limsup_n \int_R f_n d\mu$$

$\therefore \limsup_n \int_R f_n d\mu \leq \int_R f d\mu. \quad QED$

Corollary (BCT)

(24)

If  $f_n \rightarrow f$   $\mu$ -a.e. &  $\exists B \in \mathbb{R}$  s.t.  $|f_n| \leq B$   
 $\mu$ -a.e.  $f_n$  and  $\mu(\Omega) < \infty$  then  
 $f \in L_1(\Omega, \mathcal{F}, \mu)$  and

$$\lim_n \int_R f_n d\mu = \int_R f d\mu.$$

Def:  $f_1, f_2, \dots$  are uniformly integrable

(UI) if

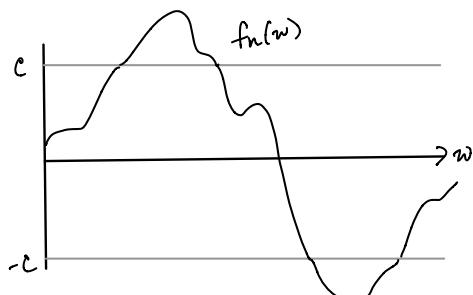
$$\lim_{c \rightarrow \infty} \sup_n \int_R |f_n| I_{\{|f_n| \geq c\}} d\mu = 0.$$

This looks strange until one notices that it allows one to control

$$\int_R |f_n - f_n^{(c)}| d\mu$$

where  $f_n^{(c)}$  is the "clamped f at c":

$$f_n^{(c)}(w) := f_n I_{\{|f_n| < c\}}$$



$$\therefore \int_R |f_n - f_n^{(c)}| d\mu = \int_R |f_n| I_{\{|f_n| \geq c\}} d\mu$$

The original definition of UI is a bit clumsy to work with. The following theorem gives a more manageable criterion.

### Thm (Dilatation criterion for UI)

If  $\exists \varepsilon > 0$  s.t.

$$\sup_n \int_{\mathbb{R}} |f_n|^{1+\varepsilon} d\mu < \infty$$

then the  $f_n$ 's are UI.

Proof:

$$\begin{aligned} \int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c\}} d\mu &\stackrel{\text{Bog}}{\leq} \int_{\mathbb{R}} |f_n| \left(\frac{|f_n|}{c}\right)^\varepsilon I_{\{|f_n| \geq c\}} d\mu \\ &\geq 1 \text{ on } \{|f_n| \geq c\} \\ &\leq \frac{1}{c^\varepsilon} \int_{\mathbb{R}} |f_n|^{1+\varepsilon} d\mu \end{aligned}$$

Take  $\sup_n$  then  $\lim_{c \rightarrow \infty}$  to get UI.  
QED.

### Thm (UI condition for $\lim \int = \int \lim$ )

If (a)  $f_n \rightarrow f$   $\mu$ -a.e.

(b) the  $f_n$ 's are UI

(c)  $\mu(\mathbb{R}) < \infty$

then (A)  $f_n, f \in L_1(\mathbb{R}, \mathcal{F}, \mu)$

$$(B) \lim_n \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} f d\mu$$

Proof:

To prove (A) Notice

$$\begin{aligned} \int_{\mathbb{R}} |f_n| d\mu &= \underbrace{\int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c\}} d\mu}_{\leq \sup_n \int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c\}} d\mu} + \underbrace{\int_{\mathbb{R}} |f_n| I_{\{|f_n| < c\}} d\mu}_{\leq c\mu(\mathbb{R}) < \infty} \\ &\rightarrow 0 \text{ as } c \rightarrow \infty \end{aligned}$$

$$\therefore \int_{\mathbb{R}} |f_n| d\mu < \infty \text{ so } f_n \in L_1(\mathbb{R}, \mathcal{F}, \mu)$$

$f \in L_1(\mathbb{R}, \mathcal{F}, \mu)$  since

$$\begin{aligned} \int_{\mathbb{R}} |f| d\mu &= \int_{\mathbb{R}} \liminf_n |f_n| d\mu \\ &\leq \liminf_n \int_{\mathbb{R}} |f_n| d\mu \text{ by Fatou} \\ &\leq \sup_n \int_{\mathbb{R}} |f_n| I_{\{|f_n|=c_k\}} d\mu + c\mu(\mathbb{R}) \\ &< \infty \text{ since } T \text{ is finite for large enough } c \end{aligned}$$

Now to show (B) notice

Fact:  $\exists c_1, c_2, \dots$  s.t.  $\mu(|f|=c_k) = 0$  and

$$\lim_{k \rightarrow \infty} c_k = \infty.$$

This follows by a Thm in Lecture 5 which states that  $\mu$  can not assign non-zero mass to uncountably many disjoint sets in  $\mathcal{F}$ . Since  $\{|f|=c\}$  forms disjoint sets for different  $c \in \mathbb{R}$ ,  $\mu(|f|=c) > 0$  for countably many  $c \in \mathbb{R}$ .

$$\begin{aligned} \therefore \limsup_n \left| \int_{\mathbb{R}} f_n d\mu - \int_{\mathbb{R}} f d\mu \right| &\stackrel{\text{I}}{=} \int_{\mathbb{R}} f_n^{(c_k)} d\mu \pm \int_{\mathbb{R}} f^{(c_k)} d\mu \\ &\leq \underbrace{\limsup_n \left| \int_{\mathbb{R}} (f_n - f_n^{(c_k)}) d\mu \right|}_{\text{I}} \\ &\quad + \underbrace{\limsup_n \left| \int_{\mathbb{R}} f_n^{(c_k)} d\mu - \int_{\mathbb{R}} f^{(c_k)} d\mu \right|}_{\text{II}} + \underbrace{\left| \int_{\mathbb{R}} (f - f^{(c_k)}) d\mu \right|}_{\text{III}} \end{aligned}$$

where

$$\text{term I} \leq \sup_n \int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c_k\}} d\mu \xrightarrow{k \rightarrow \infty} 0, \text{ since } c_k \xrightarrow{k \rightarrow \infty} \infty.$$

term II = 0  $\hookrightarrow$  To see this notice that

$$I_{\{|f_n| < c\}} \xrightarrow{n \rightarrow \infty} I_{\{|f| < c\}} \mu\text{-a.e. whenever } \mu(|f|=c)=0.$$

$$\therefore f_n^{(c_k)} \xrightarrow{n \rightarrow \infty} f^{(c_k)} \mu\text{-a.e. if } \mu(|f|=c)=0.$$

$$\therefore \text{BCT applies} \Rightarrow \int_{\mathbb{R}} f_n^{(c_k)} d\mu \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} f^{(c_k)} d\mu$$

$$\text{term III} \leq \int_{\Omega} |f| I_{\{|f| \geq c_k\}} d\mu$$

$$\xrightarrow{k \rightarrow \infty} 0 \quad \mu(|f| = \infty)$$

$$\text{since } |f| I_{\{|f| \geq c_k\}} \uparrow |f| I_{\{|f| = \infty\}}$$

Since  $f \in L_1(\Omega, \mathcal{F}, \mu)$  we have  $\mu(|f| = \infty) = 0$

by (i) of "useful facts".

$\therefore$  term III  $\rightarrow 0$  as  $k \rightarrow \infty$ .

Now we have

$$\limsup_n \left| \int_{\Omega} f_n d\mu - \int_{\Omega} f d\mu \right| = 0.$$

QED

Turns out there is also a converse to the UI theorem above. This converse will be used to show the "big"  $L_p$  convergence theorem which partly states that  $E|X_n - X|^p \rightarrow 0$  is equivalent to  $X_n \xrightarrow{P} X$  and  $|X_n|^p$ 's are UI.

### Thm (UI converse)

If (a)  $f_n \rightarrow f$   $\mu$ -a.e.

(b)  $f_n, f \in L_1(\Omega, \mathcal{F}, \mu)$

(c)  $\mu(\Omega) < \infty$

$$(d) \lim_n \int_{\Omega} |f_n| d\mu = \int_{\Omega} |f| d\mu$$

then the  $f_n$ 's are uniformly integrable.

(27)

Proof: Suppose  $c > 0$  s.t.  $\mu(|f| = c) = 0$ . (28)

Notice that

$$\begin{aligned} & \left| \int_{\Omega} |f_n| I_{\{|f_n| \geq c\}} d\mu - \int_{\Omega} |f| I_{\{|f| \geq c\}} d\mu \right| \\ & \quad \uparrow \downarrow \pm \int_{\Omega} |f_n| d\mu \pm \int_{\Omega} |f| d\mu \\ & \leq \underbrace{\left| \int_{\Omega} |f_n| I_{\{|f_n| < c\}} d\mu - \int_{\Omega} |f| I_{\{|f| < c\}} d\mu \right|}_{=: I} \\ & \quad + \underbrace{\left| \int_{\Omega} |f_n| d\mu - \int_{\Omega} |f| d\mu \right|}_{=: II} \end{aligned}$$

$$\text{Now } I = \left| \int_{\Omega} |f_n|^{(c)} d\mu - \int_{\Omega} |f|^{(c)} d\mu \right| \xrightarrow{n \rightarrow \infty} 0$$

by BCT since  $\mu(|f| = c) = 0$  and therefore  $|f_n|^{(c)} \xrightarrow{n \rightarrow \infty} |f|^{(c)}$   $\mu$ -a.e.

Also II  $\xrightarrow{n \rightarrow \infty} 0$  by assumption.

$\therefore$   $\forall c > 0$  s.t.  $\mu(|f| = c) = 0$  one has

$$(*) \quad \int_{\Omega} |f_n| I_{\{|f_n| \geq c\}} d\mu \xrightarrow{n \rightarrow \infty} \int_{\Omega} |f| I_{\{|f| \geq c\}} d\mu$$

Also note that if  $c_1, c_2, \dots$  satisfies  $c_k \rightarrow \infty$  &  $\mu(|f| = c_k) = 0$  for them

$$(**) \quad \int_{\Omega} |f| I_{\{|f| \geq c_k\}} d\mu \xrightarrow{k \rightarrow \infty} \mu(|f| = \infty) = 0$$

by Little 3 since  $|f| I_{\{|f| \geq c_k\}} \uparrow |f| I_{\{|f| = \infty\}}$  and  $f \in L_1(\Omega, \mathcal{F}, \mu)$ .

To finish show  $\forall \varepsilon > 0$ ,  $\exists c_0 > 0$  s.t.

$$\sup_n \int_{\Omega} |f_n| I_{\{|f_n| \geq c\}} d\mu < \varepsilon, \quad \forall c > c_0.$$

From (\*\*)  $\exists c_0$  s.t.  $\mu(|f| = c_0) = 0$  and

$$\int_{\Omega} |f| I_{\{|f| \geq c_0\}} d\mu < \frac{\varepsilon}{2}$$

From (\*)  $\exists N$  s.t.

(29)

$$\int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c_0\}} d\mu < \varepsilon, \quad \forall n \geq N$$

Now choose  $c_1 > c_0$  so that

$$\int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c_1\}} d\mu < \varepsilon, \quad \forall n \geq N$$

These  $\downarrow$  as  $c_1 \uparrow$  for all  $n$ !

This  $c_1$  satisfies

$$c \geq c_1 \Rightarrow \sup_n \int_{\mathbb{R}} |f_n| I_{\{|f_n| \geq c\}} d\mu \leq \varepsilon$$

$\downarrow \varepsilon$

$\therefore |f_n|'$ s are UI

$\therefore f_n$ 's are UI

QED.

We will need to differentiate under the integral when working with moment generating functions, etc. Here are sufficient conditions for reference:

Thm ( $\frac{d}{dt} \int f_t d\mu = \int \frac{d}{dt} f_t d\mu$ )

Suppose  $a < b$  are real numbers &  $\forall t \in (a, b)$   
 $f_t \in L_1(\mathbb{R}, \mathcal{F}, \mu)$ . Let  $t_0 \in (a, b)$ . If  $\exists \mathbb{R}_0 \in \mathcal{F}$  s.t.

(a)  $\mu(\mathbb{R}_0^c) = 0$

(b)  $\frac{d}{dt} f_t(w) \Big|_{t=t_0}$  exists  $\forall w \in \mathbb{R}_0$

(c)  $\sup_{\substack{t \in N \\ t \neq t_0}} \left| \frac{f_t(w) - f_{t_0}(w)}{t - t_0} \right| \leq g(w) \quad \forall w \in \mathbb{R}_0$

for some  $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$  &  
open  $N \subset (a, b)$  containing  $t_0$

Then

(A)  $\frac{d}{dt} f_t \Big|_{t=t_0} \in L_1(\mathbb{R}, \mathcal{F}, \mu)$

(B)  $\frac{d}{dt} \int_{\mathbb{R}} f_t d\mu \Big|_{t=t_0}$  exists

(C)  $\frac{d}{dt} \int_{\mathbb{R}} f_t d\mu \Big|_{t=t_0} = \int_{\mathbb{R}} \frac{d}{dt} f_t \Big|_{t=t_0} d\mu$

The proof is left as an exercise.

(30)

However it should be noted that the mean value thm implies that (b) & (c) can be replaced with the stronger conditions:

(b')  $\frac{d}{dt} f_t(w)$  exists  $\forall w \in \mathbb{R}_0$   
neighborhood  $N$  of  $t_0$

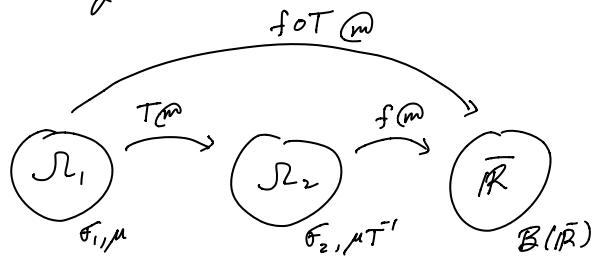
(c')  $\sup_{t \in N} \left| \frac{d}{dt} f_t(w) \right| \leq g(w)$  for  
some  $g \in L_1(\mathbb{R}, \mathcal{F}, \mu)$ .

**Warning!** You need to have  $\frac{d}{dt} f_t(w)$   
exist on  $(w, t) \in \mathbb{R}_0 \times N$  where  $\mu(\mathbb{R}_0^c) = 0$   
and  $N$  is an open neighborhood of  $t_0$ .

e.g.  $\mathbb{R} = (0, 1)$ ,  $\mathcal{F} = \mathcal{B}((0, 1))$ ,  $\mu = \mathcal{J}^1$  and  
 $f_t(w) = I_{(0, t]}(w)$ . Now  $\frac{d}{dt} f_t(w) = 0$  when  $t \neq w$   
but  $\int \frac{d}{dt} f_t(w) dw = 0 \neq 1 = \int f_t(w) dw$

### Change of Variables

The last theorem of this lecture covers an extremely useful theorem for the following setup:



The following thm says you can integrate  $f$  w.r.t.  $\mu T^{-1}$  or  $f \circ T$  w.r.t.  $\mu$  ... both give the same answer

### Thm (Change of Variables)

In the set up above

- (a)  $f \in Q^+(\Omega_2, \mathcal{F}_2, \mu T^{-1}) \Leftrightarrow f \circ T \in Q^+(\Omega_1, \mathcal{F}_1, \mu)$
- (b)  $f \in Q^-(\Omega_2, \mathcal{F}_2, \mu T^{-1}) \Leftrightarrow f \circ T \in Q^-(\Omega_1, \mathcal{F}_1, \mu)$
- (c) if any of the four statements in (a), (b) hold

then  $\int_{\Omega_1} f \circ T d\mu = \int_{\Omega_2} f d\mu T^{-1}$ .

Proof:

This is proved using the "1-2-3 argument", i.e.

Let  $\mathcal{Y}$  denote the  $f$ 's which satisfy

- (a), (b) & (c).

Step 1: show  $\eta_s(\Omega_2, \mathcal{F}_2) \subset \mathcal{Y}$

Step 2: show  $\eta(\Omega_2, \mathcal{F}_2) \subset \mathcal{Y}$

Step 3: show  $f \in \mathcal{Y}$  whenever  $f: \Omega_2 \rightarrow \bar{\mathbb{R}}$ .

For step 1, suppose  $f \in \eta_s(\Omega_2, \mathcal{F}_2)$ .

Clearly  $f \in Q^-(\Omega_2, \mathcal{F}_2, \mu)$  &  $f \circ T \in Q^-(\Omega_1, \mathcal{F}_1, \mu)$   
so (b) holds.

Now the following two integrals are defined:

$$\int_{\Omega_2} f d\mu T^{-1} = \int_{\Omega_2} \sum_{i=1}^n c_i I_{A_i} d\mu T^{-1} = \sum_{i=1}^n c_i \mu T^{-1}(A_i)$$

$$\int_{\Omega_1} f \circ T d\mu = \int_{\Omega_1} \sum_{i=1}^n c_i I_{A_i} \circ T d\mu = \sum_{i=1}^n c_i \mu(T^{-1}(A_i))$$

$\therefore$  (c) holds.

To show (a) notice that

$$f \in Q^+(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$

$$\Leftrightarrow \int_{\Omega_2} f d\mu T^{-1} < \infty \quad \text{since } f^+ = f$$

$\quad \quad \quad \text{II by (b)}$

$$\Leftrightarrow \int_{\Omega_1} f \circ T d\mu < \infty$$

$$\Leftrightarrow f \circ T \in Q^+(\Omega_1, \mathcal{F}_1, \mu)$$

$\therefore \eta_s(\Omega_2, \mathcal{F}_2) \subset \mathcal{Y}$ .

(31)

For step 2:

Again, any  $f \in \eta(\Omega_2, \mathcal{F}_2)$  satisfies (b). Also

$$\int_{\Omega_2} f d\mu T^{-1} = \int_{\Omega_2} \lim_{n \rightarrow \infty} f_n d\mu T^{-1}$$

$$= \lim_{n \rightarrow \infty} \int_{\Omega_2} f_n d\mu T^{-1} \quad \text{by Big 3}$$

$$= \lim_{n \rightarrow \infty} \int_{\Omega_1} f_n \circ T d\mu \quad \text{by step 1}$$

$$= \int_{\Omega_1} f \circ T d\mu \quad \text{by Big 3.}$$

$\therefore$  (c) holds

Now (a) holds by similar reasoning as in  
Step 2.

$$\therefore \eta(\Omega_2, \mathcal{F}_2) \subset \mathcal{Y}.$$

For step 3. Let  $f: \Omega_2 \rightarrow \bar{\mathbb{R}}$ .

To show (a) & (b) notice that

$$f \in Q^+(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$

$$\Leftrightarrow \int_{\Omega_2} f^+ d\mu T^{-1} < \infty \quad \text{or} \quad \int_{\Omega_2} f^- d\mu T^{-1} < \infty$$

$$\quad \quad \quad \text{II step 2} \quad \quad \quad \text{II step 2}$$

$$\Leftrightarrow \int_{\Omega_1} f^+ \circ T d\mu < \infty \quad \text{or} \quad \int_{\Omega_1} f^- \circ T d\mu < \infty$$

$$\Leftrightarrow f \circ T \in Q^+(\Omega_1, \mathcal{F}_1, \mu).$$

Also note that (c) follows since

$$f^+, f^- \in \eta(\Omega_2, \mathcal{F}_2) \subset \mathcal{Y}. \quad \text{QED}$$

Side fact: For any  $A \in \mathcal{F}_2$

$$f \in Q^+(\Omega_2, \mathcal{F}_2, \mu T^{-1}) \Rightarrow f I_A \in Q^+(\Omega_2, \mathcal{F}_2, \mu T^{-1})$$

since  $f^+ I_A \leq f^+$

and  $f^- I_A \leq f^-$ .

$\therefore$  If (a) or (b) hold &  $A \in \mathcal{F}_2$  then

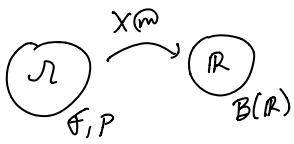
$$\underbrace{\int_A f d\mu T^{-1}}_{= \int_{\Omega_2} f I_A d\mu} = \underbrace{\int_{T^{-1}(A)} f \circ T d\mu}_{= \int_{\Omega_1} f \circ T d\mu}$$

$$= \int_{\Omega_2} f I_A d\mu \quad = \int_{\Omega_1} (f \circ T)(I_A \circ T) d\mu$$

(32)

e.g. Let  $X$  be a random variable  
defined on  $(\Omega, \mathcal{F}, P)$

(33)



We have mentioned that

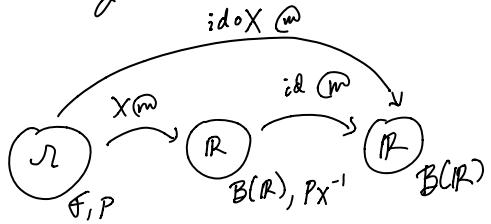
$$E(X) := \int_{\Omega} X(\omega) dP \quad (*)$$

Unfortunately this looks nothing like the  
undergrad definition

$$E(X) = \int_{\mathbb{R}} x f(x) dx \quad (**)$$

where  $f$  is "the probability density of  $X$ ".  
We will cover densities in the next lecture  
but we can halfway to (\*\*) from (\*)  
with the change of variables thm.

In particular let  $T = X$  and  $f = id$   
in the change of variables Thm to get  
the following picture:



$$\therefore E(X) = \int_{\Omega} X(\omega) dP(\omega)$$

$$= \int_{\Omega} id \circ X(\omega) dP(\omega)$$

$$= \int_{\mathbb{R}} id(x) dP_X^{-1}(x)$$

$$= \int_{\mathbb{R}} x \underbrace{dP_X^{-1}(x)}_{\text{if this step will be covered in the next lecture}}$$

$$= \int_{\mathbb{R}} x \tilde{f}(x) dx$$

by change  
of variables.

(34)