Lecture 3: Dynkin's π-λ Theorem and Borel σ-fields

Thm (Dynkins' π-λ)

$\mathcal{P}$ is a π-system $\Longrightarrow$ $\lambda\langle\mathcal{P}\rangle = \sigma\langle\mathcal{P}\rangle$.

Remark:
The most important use of Dynkin's thm is in the proof that probability measures are characterized by their values on a π-system of generators.

For example, in undergrad probability we tell students that the CDF characterizes probability distributions ... so if P & Q are probabilities on $((0,1], \mathcal{B}^{(0,1]})$ then $P = Q$ if

$$P((0,x]) = Q((0,x]) \quad \forall x \in (0,1].$$

This follows since (by a HWK)

$$\mathcal{B}^{(0,1]} = \sigma\langle\mathcal{P}\rangle$$

where $\mathcal{P} = \{(0,x] : 0 < x \le 1\}$ is a π-system.

Remark: Dynkin's π-λ thm also allows us to extend the "good sets" technique

i.e. $\underbrace{\mathcal{P}}_{\text{a } \pi\text{-system}} \subset \underbrace{\mathcal{G}}_{\text{a } \lambda\text{-system}} \Longrightarrow \lambda\langle\mathcal{P}\rangle \subset \mathcal{G} \Longrightarrow \sigma\langle\mathcal{P}\rangle \subset \mathcal{G}$

since these are equal.

This allows you to prove a little less for $\mathcal{G}$ but a little more for $\mathcal{P}$.

Remark: The proof of Dynkin's π-λ thm is an excellent example of using the "good sets" technique.

Proof of Dynkin's π-λ Thm:

show $\lambda\langle\mathcal{P}\rangle \subset \sigma\langle\mathcal{P}\rangle$: Follows immediately by good sets.

$\therefore$ just show $\sigma\langle\mathcal{P}\rangle \subset \lambda\langle\mathcal{P}\rangle$

$\therefore$ just show $\lambda\langle\mathcal{P}\rangle$ is a σ-field (by good sets)

$\therefore$ just show $\lambda\langle\mathcal{P}\rangle$ is closed under "$\cap$" (by $\sigma = \lambda + \pi$)

$\therefore$ just show $A, B \in \lambda\langle\mathcal{P}\rangle \Longrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle$

For $A \in \lambda\langle\mathcal{P}\rangle$ let

$$\mathcal{G}_A := \{B \subset \Omega : A \cap B \in \lambda\langle\mathcal{P}\rangle\}. \quad (*)$$

$\therefore$ just show $\forall A \in \lambda\langle\mathcal{P}\rangle$, $\lambda\langle\mathcal{P}\rangle \subset \mathcal{G}_A$

$\therefore$ just show $\forall A \in \lambda\langle\mathcal{P}\rangle$ $\left\{\begin{array}{l}\mathcal{P} \subset \mathcal{G}_A \text{ \&} \\ \mathcal{G}_A \text{ is a } \lambda\text{-sys}\end{array}\right\} \quad (**)$

which is sufficient by "good sets".

We will show $(**)$ first under the case $A \in \mathcal{P}$.

However first Notice

$$B \in \mathcal{G}_A \Longleftrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle \Longleftrightarrow A \in \mathcal{G}_B \quad (***)$$

Show $(**)$ when $A \in \mathcal{P}$:

- $\mathcal{P} \subset \mathcal{G}_A$ since
  $B \in \mathcal{P} \Longrightarrow A \cap B \in \mathcal{P}$, by π-sys.
  $\Longrightarrow B \in \mathcal{G}_A$, by $(*)$
- $\mathcal{G}_A$ is not $\emptyset$ since $A \in \mathcal{G}_A$.
- $\mathcal{G}_A$ is closed under complementation since $B \in \mathcal{G}_A \Longrightarrow A \cap B \in \lambda\langle\mathcal{P}\rangle$
  $\Longrightarrow \underbrace{A - A \cap B}_{= A \cap (A \cap B)^c = A \cap B^c} \in \lambda\langle\mathcal{P}\rangle$, nested set subtract.
  $\Longrightarrow B^c \in \mathcal{G}_A$

- $\mathcal{G}_A$ is closed under countable disjoint union since

$$\underbrace{B_1, B_2, \ldots}_{\text{disjoint}} \in \mathcal{G}_A \Rightarrow A \cap \underbrace{\bigcup_{k=1}^{\infty} B_k}_{} \in \mathcal{G}_A$$

$$= \bigcup_{k=1}^{\infty} (B_k \cap A) \text{ where}$$

$B_k \cap A$ are disjoint members of $\mathcal{G}_A$

Show (**) for general $A \in \lambda\langle \mathcal{P} \rangle$

- $\mathcal{P} \subset \mathcal{G}_A$ since

$$B \in \mathcal{P} \Rightarrow A \in \mathcal{G}_B, \text{ since (**) holds over } \mathcal{P}$$
$$\Leftrightarrow B \in \mathcal{G}_A$$

- The proof that $\mathcal{G}_A$ is a λ-sys is exactly similar as previous case.

QED

The following thm is similar to Dynkin's π-λ but for fields & monotonic classes.

Thm (Halmos's monotone class thm)

$$\mathcal{F} \text{ is a field} \Rightarrow \mathcal{M}\langle \mathcal{F} \rangle = \sigma\langle \mathcal{F} \rangle$$

Proof: exercise

Remark: This thm is used when extending a prob P on a field $\mathcal{F}$ to $\sigma\langle\mathcal{F}\rangle$ by adding monotonic limits to $\mathcal{F}$ & defining the extension to P with limits.

Borel σ-fields

Def:

If Ω is a metric space with distance $d : \Omega \times \Omega \to [0,\infty]$ then

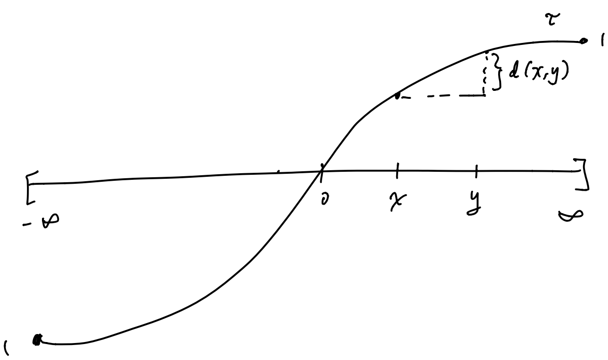
$$\mathcal{B}(\Omega) := \underbrace{\text{Borel } \sigma\text{-field}}_{\text{w.r.t } d} := \sigma\langle \underbrace{\text{open subsets of } \Omega}_{\text{w.r.t } d} \rangle$$

This defines $\mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(\bar{\mathbb{R}}^d)$, etc...

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ uses metric

$$d(x,y) = |\tau(x) - \tau(y)|$$

$$\tau(x) := \begin{cases} \frac{x}{1+|x|} & \text{for } |x| < \infty \\ -1 & \text{for } x = -\infty \\ 1 & \text{for } x = \infty \end{cases}$$



Remark: Even though $\mathcal{B}(\Omega) = \sigma\langle \text{open sets} \rangle$ there exists other generators exist & are useful for different purposes.

e.g. The FAP $((0,1], \mathcal{B}_0((0,1]), P)$ from first lecture will be extended to $((0,1], \mathcal{B}((0,1]), P)$ using $\mathcal{B}((0,1]) = \sigma\langle \mathcal{B}_0((0,1]) \rangle$... which will give Lebesgue measure on $(0,1]$.

e.g. we discussed $\mathcal{B}((0,1]) = \sigma\langle (0,x] : 0 \le x \le 1 \rangle$ is useful for proving two probability measures on $(0,1]$ are equal

Remark: It is good practice to prove a few equivalent generators for Borel σ-fields. This is typically done with "good sets" i.e.

$$\sigma\langle \mathcal{E}_1 \rangle \subset \sigma\langle \mathcal{E}_2 \rangle \text{ follows by } \mathcal{E}_1 \subset \sigma\langle \mathcal{E}_2 \rangle$$

Most are easy ... but a few can be slightly subtle:

$$\left.\begin{aligned} \mathcal{B}(\bar{\mathbb{R}}) &= \sigma\langle [-\infty, a] : a \in \mathbb{R} \rangle \\ &= \sigma\langle [-\infty, a) : a \in \mathbb{R} \rangle \\ &\neq \sigma\langle (-\infty, a) : a \in \mathbb{R} \rangle \end{aligned}\right\} \sigma\text{-fields on } \Omega = \bar{\mathbb{R}}$$

Remark: The Lebesgue σ-field of $\mathbb{R}$ extends $\mathcal{B}^{\mathbb{R}}$ using the Lebesgue measure by adding sets with "outer Lebesgue measure 0".

Thm: Suppose Ω is a metric space.

(i) $\Omega_0 \subset \Omega \implies \underbrace{\mathcal{B}(\Omega_0)}_{\text{w.r.t the induced metric on } \Omega_0} = \mathcal{B}(\Omega) \cap \Omega_0$

(ii) $\Omega_0 \subset \Omega$ & $\Omega_0 \in \mathcal{B}(\Omega)$

$$\implies \mathcal{B}(\Omega_0) = \{B : B \in \mathcal{B}(\Omega) \text{ \& } B \subset \Omega_0\}$$

Proof: see notes.

Thm: If Ω is a separable metric space then $\mathcal{B}(\Omega) = \sigma\langle \text{open balls in } \Omega \rangle$.

Proof: exercise