Lecture 23: Martingale convergence

Under mild conditions subMs converge a.e.

Doobs upcrossing inequality is the key to the proof.

Here is a motivation in terms of how to check a sequence of numbers $x_1, x_2, x_3, \dots$ converges.

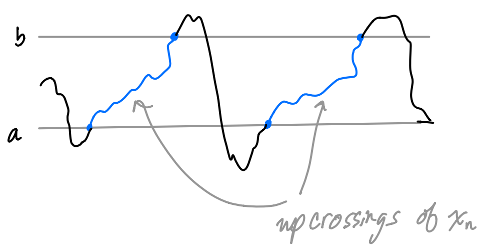
x_n does not converge

iff $\liminf_n x_n < \limsup_n x_n$

iff $\exists a, b \in \mathbb{Q}$ s.t. $a < b$

$x_n > b$ & $x_n < a$ infinitely often.

i.e.



iff $\exists a, b \in \mathbb{R}$ s.t. x_n has infinitely many upcrossings of $[a,b]$

Doobs upcrossings

Let $(X_1, X_2, \dots, X_n)$ be adapted to the filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$.

Fix $-\infty < a < b < \infty$ and define

$$\left.\begin{aligned} \alpha_1 &= \min\left(\{k \geq 1 : X_k \leq a\} \cup \{n\}\right) \\ \beta_1 &= \min\left(\{k > \alpha_1 : X_k \geq b\} \cup \{n\}\right) \end{aligned}\right\} \text{upcrossing of } [a,b]$$

$$\left.\begin{aligned} \alpha_2 &= \min\left(\{k > \beta_1 : X_k \leq a\} \cup \{n\}\right) \\ \beta_2 &= \min\left(\{k > \alpha_2 : X_k \geq b\} \cup \{n\}\right) \end{aligned}\right\} \text{upcrossing of } [a,b]$$

$$\vdots$$

The number of upcrossings of $[a,b]$, denoted $U_{a,b}$, is defined as

$$U_{a,b} := \underbrace{\sum_{j=1}^{n}}_{\text{can't be more than } n/2 \text{ terms}} I_{\{X_{\alpha_j} \leq a,\ X_{\beta_j} \geq b\}}$$

Proposition:

$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are stopping times w.r.t $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ and U_{ab} is $\mathcal{F}_n$-measurable

Proof:

we've already shown that α_1 is a ST.

To show β_1 is a stopping time notice that

- If $1 \leq m < n$ then

$$\begin{aligned} \{\beta_1 = m\} &= \{\alpha_1 < m,\ \beta_1 = m\} \\ &= \bigcup_{j=1}^{m-1} \{\alpha_1 = j,\ \beta_1 = m\} \\ &= \bigcup_{j=1}^{m-1} \{\underbrace{\alpha_1 = j}_{\in \mathcal{F}_j \subset \mathcal{F}_m},\ \underbrace{X_{j+1} < b, \dots, X_{m-1} < b,\ X_m \geq b}_{\text{in } \mathcal{F}_m}\} \\ &\in \mathcal{F}_m \end{aligned}$$

- If $m = n$ then

$$\{\beta_1 = n\} = \{\beta_1 < n\}^c = \left(\bigcup_{j=1}^{n-1} \{\beta_1 = j\}\right)^c \in \mathcal{F}_n$$

An induction argument shows $\alpha_2, \beta_2, \dots, \alpha_n, \beta_n$ are ST.

$\therefore$ X_{α_j} is $\mathcal{F}_{\alpha_j}$-measurable &

X_{β_j} is $\mathcal{F}_{\beta_j}$-measurable

Since $\mathcal{F}_{\alpha_j}$ & $\mathcal{F}_{\beta_j}$ are sub σ-fields of $\mathcal{F}_n$

U_{ab} is $\mathcal{F}_n$-measurable. QED.

Theorem: (Doob's upcrossing ineq)

Suppose $(X_1, X_2, \dots, X_n)$ is a non-negative subM w.r.t filtration $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$.

Let $c>0$ & $\mathcal{U}_{0,c}$ denote the upcrossings of $[0,c]$. Then

$$E(\mathcal{U}_{0,c}) \le \frac{E(X_n)-E(X_1)}{c}.$$

Proof: The idea is to write $X_n - X_1$ as a telescoping sum:

must be X_n

$$\begin{aligned} X_n - X_1 = {} & (X_{\beta_n} - X_{\alpha_n}) + (X_{\alpha_n} - X_{\beta_{n-1}}) \\ & + (X_{\beta_{n-1}} - X_{\alpha_{n-1}}) + (X_{\alpha_{n-1}} - X_{\beta_{n-2}}) \\ & + \\ & \vdots \\ & + (X_{\beta_2} - X_{\alpha_2}) + (X_{\alpha_2} - X_{\beta_1}) \\ & + (X_{\beta_1} - X_{\alpha_1}) \end{aligned}$$

must be X_1 since $X_1 \ge 0$

Notice that

$$\begin{aligned} E(\text{a red term}) &= E(X_{\alpha_j} - X_{\beta_{j-1}}) \\ &= E\left(E(X_{\alpha_j} - X_{\beta_{j-1}} \mid \mathcal{F}_{\beta_{j-1}})\right) \\ &= E\Big(\underbrace{E(X_{\alpha_j} \mid \mathcal{F}_{\beta_{j-1}})}_{\text{i.e. } \ge X_{\beta_{j-1}} \text{ by the Finite optional Sampling Thm}} - X_{\beta_{j-1}}\Big) \\ &\ge 0 \end{aligned}$$

Moreover

$$(X_{\beta_j} - X_{\alpha_j}) \begin{cases} > c & \text{if } (X_{\alpha_j}, X_{\beta_j}) \text{ an upcrossing} \\ 0 & \text{if } \beta_j = \alpha_j = n \\ > 0 & \text{if } \beta_j = n \text{ but } \alpha_j < n \\ & \quad \text{since } X_{\alpha_j} = 0 \text{ \& } X_{\beta_j} \ge 0 \text{ by non-negativity} \end{cases}$$

$\therefore$ The sum of all blue terms $\ge c\,\mathcal{U}_{0,c}$

$$\therefore \; E(X_n - X_1) \ge \underbrace{E(\text{blues})}_{\ge c\,\mathcal{U}_{0,c}} + \underbrace{E(\text{reds})}_{\ge 0} \ge c\,E(\mathcal{U}_{0,c}).$$

QED

Corollary:

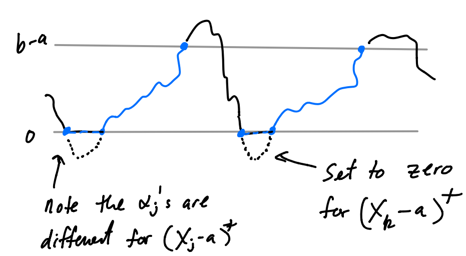
If $(X_1, \dots, X_n)$ is a subM w.r.t. $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ and $-\infty < a < b < \infty$ then

$$E(\mathcal{U}_{a,b}) \le \frac{E(X_n - a)^+ - E(X_1 - a)^+}{b-a} \le \frac{E(X_n^+) + a^-}{b-a}.$$

Proof:

For the first inequality:

$$\begin{gathered} \text{upcrossings of } [a,b] \text{ in } (X_1, \dots, X_n) \\ \| \\ \text{upcrossings of } [0, b-a] \text{ in } (X_1 - a, \dots, X_n - a) \\ \| \\ \text{upcrossings of } [0, b-a] \text{ in } ((X_1 - a)^+, \dots, (X_n - a)^+) \end{gathered}$$



By Doobs result we therefore have

$$E(\mathcal{U}_{0,b-a}) \le \frac{E(X_n - a)^+ - E(X_1 - a)^+}{b-a}$$

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For the second inequality just notice

$$E(X_n-a)^+ - E(X_1-a)^+ \leq E(X_n-a)^+$$
$$= E(X_n+(-a))^+$$
$$\leq E(X_n^+ + (-a)^+)$$

since $(\cdot)^+$ is convex

$$= E(X_n^+) + a^-$$

QED

a.e. convergence of martingales

For this section let

$$\mathcal{F}_\infty := \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle$$

for a filtration $(\mathcal{F}_n)_{n\geq 1}$.

Theorem: ($E(X_n^+)$ bdd $\Rightarrow$ a.e. conv)

Let $(X_n)_{n\geq 1}$ be a subM w.r.t. filtration $(\mathcal{F}_n)_{n\geq 1}$. If $\sup_n E(X_n^+) < \infty$ then $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{P\text{-a.e.}} X_\infty$$

Proof:

For $-\infty < a < b < \infty$ let $U_{a,b}^n$ denote the number of upcrossings of $[a,b]$ from $(X_1, \dots, X_n)$, with $U_{a,b}^\infty$ for $(X_n)_{n\geq 1}$.

We must have that

$$0 \leq U_{a,b}^n \uparrow U_{a,b}^\infty \quad \text{as } n \to \infty$$

$$\therefore \quad E(U_{a,b}^\infty) \overset{\text{Big 3}}{=} \lim_n E(U_{a,b}^n)$$
$$\leq \lim_n \frac{E(X_n^+) + a^-}{b-a}$$
$$= \frac{\sup_n E(X_n^+) + a^-}{b-a} < \infty$$

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$$\therefore \quad U_{a,b}^\infty < \infty \quad P\text{-a.e.}$$

Now

$$P\left(\liminf_n X_n < \limsup_n X_n\right)$$
$$\leq P\left(\bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{U_{a,b}^\infty = \infty\}\right)$$

must $\exists\, a,b$ with an infinite # of upcrossings.

$= 0$ since $U_{a,b}^\infty < \infty$ P-a.e

$$\therefore \quad X_n \xrightarrow{P\text{-a.e.}} \underbrace{\limsup_n X_n}_{\text{must be } \textcircled{m}\ \mathcal{F}_\infty \text{ by closure Thm.}} =: X_\infty$$

To see why $X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$

since the limit exists

$$E|X_\infty| = E|\liminf_n X_n| = E(\liminf_n |X_n|)$$
$$\overset{\text{Fatou}}{\leq} \liminf_n E|X_n|$$
$$\leq \sup_n E|X_n|$$
$$= \sup_n E(2X_n^+ - X_n)$$
$$\leq \sup_n \left(2E(X_n^+) - \underbrace{E(X_1)}_{<\infty}\right)$$

Since $E(X_1) \leq E(X_n)$ by subM

$$< \infty$$

QED

Remark: Note the analog to monotonic sequences when $\sup_n E(X_n^+) < \infty$ plays the roll of $\sup_n x_n^+ < \infty$

Another bdd type condition is that the X_n^+'s are UI.

Proposition: ($(X_n^+)_{n\geq 1}$ UI $\Rightarrow$ $E(X_n^+)$ bdd)

If $X_1, X_2, \ldots \in L_1(\Omega, \mathcal{F}, P)$ and $(X_n^+)_{n\geq 1}$ are UI then $\sup_n E(X_n^+) < \infty$.

Proof:

$$\underbrace{\sup_n E(X_n^+) \leq \underbrace{\sup_n E(X_n^+ I_{X_n^+\geq c})}_{\xrightarrow{c\to\infty} 0 \text{ by definition of UI}} + \underbrace{\sup_n E(X_n^+ I_{X_n^+ < c})}_{< c}}_{\text{so for a large enough } c \text{ this is } < \infty.}$$

QED.

Here is an application of the above proposition which comes in handy for subsequent results.

Theorem: (Lévy's Smoothing Martingale)

If $X \in L_1(\Omega, \mathcal{F}, P)$ & $(\mathcal{F}_n)_{n\geq 1}$ is a filtration Then

$$E(X|\mathcal{F}_n) \xrightarrow{n\to\infty} E(X|\mathcal{F}_\infty) \quad \text{a.e. \& in } L_1$$

where $\mathcal{F}_\infty = \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \ldots\rangle$.

Proof:
Let $X_n := E(X|\mathcal{F}_n)$.
We first show the X_n's are UI.

Since $|X_n| = |E(X|\mathcal{F}_n)| \leq E(|X||\mathcal{F}_n)$ it will be sufficient to show the $E(|X||\mathcal{F}_n)$'s are UI.

i.e. show

$$\lim_{c\to\infty} \sup_n \underbrace{\int_{E(|X||\mathcal{F}_n)\geq c} E(|X||\mathcal{F}_n)\,dP}_{} = 0$$

$$= \int_{E(|X||\mathcal{F}_n)\geq c} |X|\,dP \quad \text{since } \{E(|X||\mathcal{F}_n)\geq c\} \text{ is a } \mathcal{F}_n\text{-set}$$

$$= \tilde{P}\big(E(|X||\mathcal{F}_n)\geq c\big)\cdot \underbrace{\int_\Omega |X|\,dP}_{\text{finite}}$$

where $\tilde{P}(\cdot) = \dfrac{\int_\cdot |X|\,dP}{\int_\Omega |X|\,dP}$ is a prob measure.

By Markov's ineq

$$\tilde{P}\big(E(|X||\mathcal{F}_n)\geq c\big) \leq \frac{E\{E(|X||\mathcal{F}_n)\}}{c} = \frac{E|X|}{c} \xrightarrow[c\to\infty]{} 0$$

$\therefore$ the X_n's are indeed UI.

$\therefore$ the X_n^+'s are UI so the $E(X_n^+)$'s bdd

$\therefore$ $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty \quad \text{by subM } \xrightarrow{a.e.} \text{ Thm}$$

$\therefore$ $X_n \xrightarrow{L_1} X_\infty$ by the L_p convergence thm since the X_n's are UI

To finish we show $X_\infty \overset{a.e.}{=} E(X|\mathcal{F}_\infty)$.

In particular show:

(i) X_∞ is $\mathcal{F}_\infty$-measurable ✓

(ii) $X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ ✓

(iii) Integrates like X over $\mathcal{F}_\infty$-sets.
i.e. $E(XI_A) = E(X_\infty I_A)\ \forall A\in\mathcal{F}_\infty$

For (iii) notice that $\forall A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$

$$E(XI_A) \overset{(*)}{=} E(X_n I_A) \quad \forall \text{ large } n$$
since $X_n = E(X|\mathcal{F}_n)$

$\therefore$ $\forall A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$ ← a field generating $\mathcal{F}_\infty$

$$E(X_\infty I_A) = E(\lim_n X_n I_A) \quad \text{since } X_n \xrightarrow{a.e.} X_\infty$$

$$= \lim_n E(X_n I_A) \quad \text{by UI cond. for } \lim\int = \int\lim$$

$$\overset{(*)}{=} E(XI_A) \quad \text{by } (*)$$

To show $E(X_\infty I_A) = E(XI_A)$ $\forall A \in \mathcal{F}_\infty$

Let
$$\mathcal{G} = \{A \in \mathcal{F} : E(X_\infty I_A) = E(XI_A)\}$$

By $(*)$ we have

field & $\therefore$ a π-sys → $\bigcup_{n=1}^{\infty} \mathcal{F}_n \subset \mathcal{G}$ ← a λ-system

$\therefore$
$$\lambda\langle \bigcup_{n=1}^{\infty} \mathcal{F}_n \rangle \subset \mathcal{G}$$
$\|$ Dynkin's π-λ
$$\sigma\langle \bigcup_{n=1}^{\infty} \mathcal{F}_n \rangle$$

$\therefore$ $E(X_\infty I_A) = E(XI_A)$ $\forall A \in \mathcal{F}_\infty$ by "Good sets".

QED

Another way to quantify "bddness" of subMs is with "closers".

Definition: A pair $(X_\bullet, \mathcal{F}_\bullet)$ (r.v., sub σ-field of $\mathcal{F}$) closes a subM $(X_n)_{n\geq 1}$ on the right if

$$X_1, X_2, \ldots, X_n, \ldots X_\bullet$$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots, \mathcal{F}_\bullet$,

i.e. if
$$E(X_\bullet | \mathcal{F}_n) \overset{P\text{-a.e.}}{\geq} X_n \quad \forall n \in \mathbb{N}.$$

Definition: $(X_0, \mathcal{F}_0)$ is a nearest closer of $(X_n)_{n\geq 1}$ if

$$X_1, X_2, \ldots X_0, X_\bullet$$

is a subM w.r.t $\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_0, \mathcal{F}_\bullet$ for every closer $(X_\bullet, \mathcal{F}_\bullet)$ of $(X_n)_{n\geq 1}$.

The existance of a closer is equivalent to the UI condition:

Theorem: ($\exists$ closer $\Leftrightarrow$ X_n^+'s are UI)

If $(X_n)_{n\geq 1}$ is a subM w.r.t filt $(\mathcal{F}_n)_{n\geq 1}$ then

$$\exists \text{ a closer for } (X_n)_{n\geq 1} \Leftrightarrow (X_n^+)_{n\geq 1} \text{ are UI}$$

Proof:

($\Longrightarrow$): Suppose $(X_\bullet, \mathcal{F}_\bullet)$ closes $(X_n)_{n\geq 1}$.

$\therefore$ $(X_\bullet^+, \mathcal{F}_\bullet)$ closes the subM $(X_n^+)_{n\geq 1}$ by Trans of Ms.

$\therefore$ $\underbrace{E(X_\bullet^+|\mathcal{F}_n)}_{\text{These are UI by proof of Lévy's thm}} \overset{a.e.}{\geq} X_n^+$ $\forall n \in \mathbb{N}$ $\therefore$ these are too.

($\Leftarrow$) Suppose $(X_n^+)_{n\geq 1}$ are UI. (11)

$\therefore \exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty$$

we show that $(X_\infty, \mathcal{F}_\infty)$ is a closer.

Case 1: $X_n \geq c > -\infty \quad \forall n \in \mathbb{N}$.

$\therefore$ The X_n^-'s are UI & hence the X_n's are UI.

$\therefore X_n \xrightarrow{L_1} X_\infty$ by the L_p-convergence Thm.

Now to show $E(X_\infty|\mathcal{F}_n) \overset{a.e.}{\geq} X_n$ Let $A \in \mathcal{F}_n$ so that

$$\int_A X_n \, dP \overset{\text{subM}}{\leq} \int_A E(X_{n+m}|\mathcal{F}_n)\, dP$$

$$= \int_A X_{n+m}\, dP$$

$$\xrightarrow{m\to\infty} \int_A X_\infty \, dP \quad \text{since } X_{n+m} I_A \xrightarrow{L_1} X_\infty I_A$$

$$= \int_A E(X_\infty|\mathcal{F}_n)\, dP$$

$\therefore X_n \overset{a.e.}{\leq} E(X_\infty|\mathcal{F}_n)$ by our results on indefinite integrals.

$\therefore (X_\infty, \mathcal{F}_\infty)$ closes $(X_n)_{n\geq 1}$.

Case 2: $X_n \in L_1(\Omega, \mathcal{F}, P)$.

Now case 1 applies to $\underbrace{X_n \vee c}$ where $c > -\infty$.

a subM since $\max(X_n, c)$ is a subM, also UI and bdd below.

Also $X_n \vee c \xrightarrow{L_1} X_\infty \vee c$

$\therefore E(X_\infty \vee c|\mathcal{F}_n) \overset{a.e.}{\geq} X_n \vee c$

$\therefore E(X_\infty|\mathcal{F}_n) \overset{a.e.}{\geq} X_n$ by taking limits as $c \to -\infty$ by MCT for $E(\cdot|\mathcal{F}_n)$

QED

(12)

Now we get the "closer" condition for a.e. convergence almost as a corollary.

Theorem: ($\exists$ closer $\Rightarrow$ a.e. convergence)

If the subM $(X_n)_{n\geq 1}$ has a closer then $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t. $\quad \mathcal{F}_\infty = \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \dots \rangle$

$$X_n \xrightarrow{a.e.} X_\infty$$

and $(X_\infty, \mathcal{F}_\infty)$ is the nearest closer.

Proof:

By the previous result

$\exists$ closer $\Leftrightarrow (X_n^+)$'s are UI

$\Rightarrow E(X_n^+)$'s are bdd

$\Rightarrow \exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \xrightarrow{a.e.} X_\infty.$$

The proof of the previous Thm also establishes that $(X_\infty, \mathcal{F}_\infty)$ closes $(X_n)_{n\geq 1}$.

Now we just show $(X_\infty, \mathcal{F}_\infty)$ is the <u>nearest</u> closer.

Let $(X_\bullet, \mathcal{F}_\bullet)$ be some closer.

We need to show

$$X_1, X_2, \dots X_\infty, X_\bullet \quad \text{is a subM}$$

w.r.t filtration $\mathcal{F}_1, \mathcal{F}_2, \dots \mathcal{F}_\infty, \mathcal{F}_\bullet$

it's a filtration since $\bigcup_{i=1}^\infty \mathcal{F}_i \subset \mathcal{F}_\bullet$, hence $\underbrace{\sigma\langle \bigcup_{i=1}^\infty \mathcal{F}_i \rangle}_{=\mathcal{F}_\infty} \subset \mathcal{F}_\bullet$

Notice it is sufficient to show

(*) $\quad E(X_\bullet|\mathcal{F}_\infty) \overset{p.a.e.}{\geq} X_\infty$

To show (*) use Lévy's smoothing result as follows

$$\begin{tikzcd} E(X_\circ|\mathcal{F}_n) \arrow[r, phantom, "\overset{a.e.}{\geq}"] \arrow[d, "a.e."] & X_n \arrow[d, "a.e."'] \\ E(X_\circ|\mathcal{F}_\infty) \arrow[r, phantom, "\overset{a.e.}{\geq}"] & X_\infty \end{tikzcd}$$

since $(X_\circ, \mathcal{F}_\circ)$ is a closer

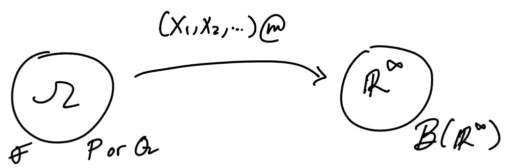
as way to be shown.

QED

Likelihood ratio Example from last Lecture

Two models P & Q for a random ω generating an infinite sequence of r.v.s

$$X = (X_1, X_2, \dots)$$



where

P & Q are distinguishable (a.e.) from one sample of X $\iff PX^{-1} \perp QX^{-1}$

$\iff Q_\infty \perp P_\infty$

where $\mathcal{F}_\infty := \sigma\langle X_i : i \geq 1\rangle$

$Q_\infty := Q|_{\mathcal{F}_\infty}$

$\iff \dfrac{dQ^a_\infty}{dP_\infty} = 0$

$\overset{?}{\iff} \dfrac{dQ^a_n}{dP_n} \xrightarrow{a.e.} 0$

where $\mathcal{F}_n := \sigma\langle X_1, \dots, X_n\rangle$

$Q_n := Q|_{\mathcal{F}_n}$

where $\dfrac{dQ^a_n}{dP_n}$ & $\dfrac{dQ^a_\infty}{dP_\infty}$ represents the finite & infinite data likelihood ratio.

Recall that we showed $\dfrac{dQ^a_n}{dP_n}$ is a supM wrt $(\mathcal{F}_n)_{n\geq 1}$ under P

Theorem: $\dfrac{dQ^a_n}{dP_n} \xrightarrow{P\text{-a.e.}} \dfrac{dQ^a_\infty}{dP_\infty}$ as $n \to \infty$.

Proof:

$\dfrac{dQ^a_n}{dP_n}$ is a supM and non-neg by construction

$\therefore \left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ is a non-positive subM

$\therefore (0, \mathcal{F}_\infty)$ closes $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ on the right:

i.e. $-\dfrac{dQ^a_1}{dP_1}, -\dfrac{dQ^a_2}{dP_2}, \dots, 0$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_\infty$ because

$$0 = E(0|\mathcal{F}_n) \geq -\frac{dQ^a_n}{dP_n} \quad \forall n \in \mathbb{N}.$$

By the closer Thm $\exists X_\infty \in L_1(\Omega, \mathcal{F}_\infty, P)$ s.t.

$\dfrac{dQ^a_n}{dP_n} \xrightarrow{a.e.} X_\infty$ & $(-X_\infty, \mathcal{F}_\infty)$ is the nearest closer of $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$.

note $X_\infty \overset{a.e.}{\geq} 0$

We show $X_\infty \overset{a.e.}{=} \dfrac{dQ^a_\infty}{dP_\infty}$

(show $X_\infty \overset{a.e.}{\leq} \dfrac{dQ^a_\infty}{dP_\infty}$)

Use that $-X_\infty$ closes $\left(-\dfrac{dQ^a_n}{dP_n}\right)_{n\geq 1}$ on the right.
Since X_∞ closes we have

$$E(X_\infty|\mathcal{F}_n) \overset{P\text{-a.e.}}{\leq} \frac{dQ^a_n}{dP_n}$$

$$\therefore \int_A X_\infty \, dP \leq \int_A \frac{dQ^a_n}{dP_n} dP \leq Q_\infty(A) \quad \forall A \in \bigcup_{k=1}^\infty \mathcal{F}_k$$

Since $\bigcup_{k=1}^{\infty} \mathcal{F}_k$ is a field generating $\mathcal{F}_\infty$ & $X_\infty \overset{a.e.}{\geq} 0$ we can apply "Good sets" to show

$$\underbrace{\int_A X_\infty \, dP}_{\text{a finite measure}} \leq Q_\infty(A) \quad \forall A \in \mathcal{F}_\infty$$

but $\frac{dQ_\infty^a}{dP_\infty}$ is the P-largest (by Leb Decomp) such, so we have

$$X_\infty \overset{p\text{-a.e.}}{\leq} \frac{dQ_\infty^a}{dP_\infty}$$

(show $X_\infty \overset{a.e.}{\geq} \frac{dQ_\infty^a}{dP_\infty}$)

Use that $-\frac{dQ_\infty^a}{dP_\infty}$ closes $\left(-\frac{dQ_n^a}{dP_n}\right)_{n\geq 1}$ on the right.

Indeed this follows from the same method used to show $\left(\frac{dQ_n^a}{dP_n}\right)_{n\geq 1}$ is a supM.

nearest closer

i.e. $-\frac{dQ_1^a}{dP_1}, -\frac{dQ_2^a}{dP_2}, \ldots, -X_\infty, -\frac{dQ_\infty^a}{dP_\infty}, 0$

is a subM w.r.t. $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\infty, \mathcal{F}_\infty, \mathcal{F}_\infty$

$$\therefore \quad E\left(-\frac{dQ_\infty^a}{dP_\infty} \,\middle|\, \mathcal{F}_\infty\right) \overset{a.e.}{\geq} -X_\infty$$

$$\overset{a.e.}{\|} \qquad \frac{dQ_\infty^a}{dP_\infty}$$

QED

L_p convergence

Theorem: (subM L_p convergence Thm)

Suppose $1 \leq p \leq \infty$ and $(X_n)_{n\geq 1}$ is a subM wrt filtration $(\mathcal{F}_n)_{n\geq 1}$.

If $|X_n|^p$ is UI then $\exists X_\infty \in L_p(\Omega, \mathcal{F}_\infty, P)$ s.t.

$$X_n \longrightarrow X_\infty \quad P\text{-a.e. \& in } L_p$$

where $(X_\infty, \mathcal{F}_\infty)$ is the nearest closer of $(X_n)_{n\geq 1}$

$= \sigma\langle \mathcal{F}_1, \mathcal{F}_2, \ldots \rangle$

Theorem: (Checking $|X_n|^p$ UI for $X_n \geq 0$ subM)

If $(X_n)_{n\geq 1}$ forms a non-neg subM & $p > 1$, then

$$X_n^p \text{ are UI} \iff \sup_n E(X_n^p) < \infty$$

$$\iff E\left(\sup_n X_n^p\right) < \infty$$