Appendix

Projections

Definition

A projection is a linear map from

$$\mathbb{C}^n \to \mathbb{C}^n \quad \text{or} \quad \mathbb{R}^n \to \mathbb{R}^n$$

which is idemotent, i.e.

$$P^2 = P$$

$\blacksquare$

Notice that if $P = P^2$ then

$$(I-P)^2 = I - 2P + P^2 = I - P$$

i.e. $I-P$ is also a projection

Also notice that for any P one has

$$x = x + Px - Px$$
$$Px + (I-P)x$$

Therefore when $P^2 = P$

$x \in null(P)$

$$\Leftrightarrow \quad Px = 0$$

$$\Rightarrow \quad x = (I-P)x$$

$$\Rightarrow \quad x \in range(I-P)$$

Also

$x \in range(I-P)$

$$\Leftrightarrow \quad x = (I-P)y \quad \text{for some } y$$

$$\Rightarrow \quad Px = \underbrace{(P-P^2)y}_{=0} \quad \text{for some } y$$

$$\Rightarrow \quad x \in null(P)$$

$$\therefore \quad null(P) = range(I-P)$$

Claim: If P is a projection on $\mathbb{C}^n$ (or $\mathbb{R}^n$) then every $x \in \mathbb{C}^n$ (or $\mathbb{R}^n$) has a unique decomposition

$$x = x_r + x_n$$

where $x_r \in range(P)$, $x_n \in null(P)$.

Proof:

First we show $null(P) \cap range(P) = \{0\}$.

Indeed

$$x \in range(P) \Rightarrow x = Py \quad \text{some } y$$

$$\Rightarrow Px = P^2 y = Py = x$$

$$\Rightarrow x = Px$$

$$\therefore \quad x \in null(P) \cap range(P)$$

$$\Rightarrow x = Px = 0$$

Now clearly every x satisfies

$$x = \underbrace{Px}_{\in range(P)} + \underbrace{(I-P)x}_{\in range(I-P) = null(P)}$$

To see why this decomposition is unique let

$$x = x_r + x_n$$

$$\therefore \quad Px = Px_r + Px_n$$

$$= x_r + 0$$

since $P^2 = P$

so that $x_r = Px$ and

$x_n = x - x_r = x - Px$ so that

$x_n = (I-P)x$.

Definition: A projection P is orthogonal if

$$\langle Px, (I-P)x \rangle = 0 \quad \forall x$$

otherwise P is oblique.

Characterizing a projection P:

(1) specify $\text{range}(P)$ & a linear space $\mathcal{L}$

s.t. $\dim(\mathcal{L}) = \dim(\text{range}(P))$

$$\text{range}(P) \cap \underbrace{\mathcal{L}^{\perp}}_{\substack{\text{null}(P) \\ = \\ \text{range}(I-P)}} = \{0\}$$

(2) require

$$x - Px \perp \mathcal{L}$$

With this characterization

$$\text{range}(P) = \mathcal{L} \iff x - Px \perp \text{range}(P)$$

$\iff$ P is a projection s.t.

$$\langle Px, (I-P)x \rangle = 0$$

$\iff$ P is a orthogonal projection

$\iff$ P is a projection that satisfies $P^H = P$

side fact

Computing a Projection in coordinates

If $\mathcal{L} = \text{range}(P)$, choose ONB $\{p_1, \dots, p_m\}$ of range(P), then

$$Px = \sum_{k=1}^{m} \langle x, p_k \rangle p_k .$$

Notice that if $p_1, \dots, p_m$ isn't orthonormal, but still a basis of range(P) one can orthonormalize by setting $R = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_m \\ | & & | \end{bmatrix}$ and note that

$$R^H R = \begin{bmatrix} \langle p_1, p_1 \rangle & \cdots & \langle p_1, p_m \rangle \\ & & \\ \langle p_m, p_1 \rangle & \cdots & \langle p_m, p_m \rangle \end{bmatrix}$$

so

$$\tilde{R} = R(R^H R)^{-1/2}$$

satisfies

$$\tilde{R}^H \tilde{R} = I$$

and therefore the colums of $\tilde{R}$ form a ONB of M and

$$P = \tilde{R}\tilde{R}^H$$
$$= R(R^H R)^{-1} R^H$$

<u>Note</u>:

If $R^H R = I$ then RR^H is a projection onto the column span of R and $(I - RR^H)$ projects <u>out</u> the colums of R

If $\mathcal{L} \neq \text{range}(P)$ choose a basis $\{p_1, \dots, p_m\}$ for range(P) and a basis $\{l_1, \dots, l_m\}$ for $\mathcal{L}$ that is biorthogonal in the sense that

$$\langle p_i, l_j \rangle = \delta_{ij}$$

Notice that if $v \in \text{range}(P)$ then we can write $v = \sum_{k=1}^{m} c_k p_k$ (since $p_1, \dots, p_m$ is a basis) and c_i's can be found by

$$\langle v, l_i \rangle = \sum_{k=1}^{m} c_k \langle p_k, l_i \rangle$$
$$= c_i$$

$$\therefore \quad v = \sum_{k=1}^{m} \langle v, l_k \rangle p_k$$

Now for general x

$$Px = \sum_{k=1}^{m} \langle x, l_k \rangle p_k \in \text{range}(P)$$

and $\langle x - Px, l_j \rangle$

$$= \langle x, l_j \rangle - \langle x, l_j \rangle$$
$$= 0 \quad \text{as required}$$

If $p_1, \dots, p_m$ & $l_1, \dots, l_m$ are not biorthogonal then form R as before and $L = \begin{bmatrix} | & & | \\ l_1 & \cdots & l_m \\ | & & | \end{bmatrix}$ so that

$$\tilde{R} = R(R^H L)^{-1}$$

has the property

$$\tilde{R}^H L = I$$

i.e. cols of $\tilde{R}$ and L are bi-orth

Therefore $P = \tilde{R} L^H$

$$= R(R^H L)^{-1} L^H$$

Projections within a general Hilbert space

Let $\mathcal{H}$ denote a finite abstract dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Def: A projection $P : \mathcal{H} \to \mathcal{H}$ (i.e. $P^2 = P$) is ortho w.r.t. $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ iff

$$\forall x, y \in \mathcal{H}, \quad \langle Px, y \rangle_{\mathcal{H}} = \langle x, Py \rangle_{\mathcal{H}}$$

claim: Let P be a proj on $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$

$$P \text{ is ortho} \iff Px = \underset{y \in \mathcal{H}}{\operatorname{argmin}} \|x - y\|_{\mathcal{H}}^2$$

$$\iff P \text{ is symmetric w.r.t. } \langle \cdot, \cdot \rangle_{\mathcal{H}}, \text{ i.e. } \langle Px, y \rangle_{\mathcal{H}} = \langle x, Py \rangle_{\mathcal{H}} \ \forall x, y \in \mathcal{H}$$

$$\iff x - Px \perp_{\mathcal{H}} \text{range}(P)$$

e.g. $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, A, \cdot \rangle_{\ell_2}$ for pos def A

$$P \text{ is ortho} \iff \langle Px, A, y \rangle = \langle x, A, Py \rangle \quad \forall x, y \in \mathbb{R}^n$$

$$\iff x^T P^T A y = x^T A P y \quad \forall x, y \in \mathbb{R}^n$$

$$\iff P^T A = AP$$

$$\iff \underbrace{(AP)^T = AP}_{\text{i.e. self-adjoint w.r.t. } A}$$

The form of this projection ... $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, A, \cdot \rangle$ when cols of R & $\tilde{R}$ are linear indep & span range(P) is as follows:

$$\underbrace{P = \underbrace{R R^H}_{\substack{\text{gets coeffs} \\ \text{w.r.t. } \langle \cdot, A, \cdot \rangle}} A}_{\substack{\text{Note this is self-adj} \\ \text{w.r.t. } A}} \quad \text{when} \quad \underbrace{R^H A R}_{\substack{\text{matrix of} \\ \langle p_i, A, p_j \rangle}} = I$$

orth w.r.t ...

or in general

$$P = R (R^H A R)^{-1} R^H A$$

$$= \Big[\underbrace{R (R^H A R)^{-1/2}}_{\substack{\equiv R \text{ when} \\ R^H A R = I}} \Big] \Big[R (R^H A R)^{-1/2} \Big]^H A$$

Indeed if

$$P^2 = P \text{ is orth w.r.t. } \langle \cdot, A, \cdot \rangle$$

$$\Updownarrow$$

$$P^2 = P \text{ and } (AP)^T = AP$$

$$\Updownarrow$$

$$P^2 = P \text{ and } x - Px \perp_A \text{range}(P)$$

$$\Downarrow$$

$$P^2 = P \text{ and } x - Px \perp_{\ell_2} A\,\text{range}(P)$$

For the other directions:

s.t. $L^T \cap \text{range}(P) = \{0\}$

(i) $P^2 = P$ and $x - Px \perp \mathcal{L}$

$$\Downarrow$$

$P^2 = P$ and $(AP)^T = AP$

with $A = L(L^H L)^{-1} L^H$

since we can choose R s.t. $R^H L = I$ (i.e. biorth) and $\text{span}(R) = \text{range}(P)$ then write $P = RL^H$ so that

$$\underbrace{AP = L(L^H L)^{-1} \underbrace{L^H R}_{I} L^H}_{\text{symmetric}}$$

$$\Updownarrow$$

$P^2 = P$ and $x - Px \perp_A \text{range}(P)$

$$\Updownarrow$$

P is an ortho projection w.r.t $\langle \cdot, A, \cdot \rangle$

Claim:

Let $P^2 = P$ and $\mathcal{L}$ be a linear space s.t. $\dim(\mathcal{L}) = \dim(\text{range}(P))$ and $\mathcal{L}^\perp \cap \text{range}(P) = \{0\}$. Then

P is oblique with $x - Px \perp \mathcal{L} \;\; \forall x$

$$\Updownarrow$$

$P = R(R^H L)^{-1} L^H$ (columns of R and L span $\text{range}(P)$ and $\mathcal{L}$, resp.)

$(L := AR) \Uparrow \quad \Downarrow (A := L(L^H L)^{-1} L^H)$

$P = R(R^H A R)^{-1} R^H A$

$$\Updownarrow$$

P is orthogonal w.r.t. $\langle \cdot, A, \cdot \rangle$

$$\Updownarrow$$

$\langle Px, A, y \rangle = \langle x, A, Py \rangle, \; \forall x, y$

i.e. AP is symmetric

Projections within the Hilbert space generated by the coordinates of $X \sim N(0,\Sigma)$

Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \sim N(0, \Sigma)$

and define Hilbert spaces

collection of random variables

$$\begin{cases} \mathcal{X} = \{ X^T a : a \in \mathbb{R}^N \} \\ \langle Z, W \rangle_{\mathcal{X}} := E(ZW) \end{cases}$$

$$\begin{cases} \mathcal{C} = \{ \Sigma a : a \in \underbrace{\mathbb{R}^N}_{\mathbb{R}^N} \} \\ \langle f, g \rangle_{\mathcal{C}} := \langle f, \Sigma^{-1} g \rangle \end{cases}$$

$$\begin{cases} \mathcal{O} = \{ a : a \in \mathbb{R}^N \} \\ \langle a, b \rangle_{\mathcal{O}} = \langle a, \Sigma, b \rangle \end{cases}$$

Let $\Sigma[:,i]$ denote the i^{th} col of Σ and $e^i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$.
$\quad$ ↖ i^{th} coordinate

The generators of $\mathcal{X}, \mathcal{C}$ an $\mathcal{O}$ can be put into 1-1 correspondence as follows.

$$X_i \longleftrightarrow \Sigma[:,i] \longleftrightarrow e^i \quad , \forall i = 1, \ldots, N$$

where for every $i,j \in \{1, \ldots, N\}$ we have

$$\langle X_i, X_j \rangle_{\mathcal{X}} = \langle \Sigma[:,i], \Sigma[:,j] \rangle_{\mathcal{C}} = \langle e^i, e^j \rangle_{\mathcal{O}}$$

Proof: By direct calculation

$$\langle X_i, X_j \rangle_{\mathcal{X}} := E(X_i X_j) = \Sigma[i,j]$$

$$\begin{aligned} \langle \Sigma[:,i], \Sigma[:,j] \rangle_{\mathcal{C}} &= \Sigma[:,i]^T \Sigma^{-1} \Sigma[:,j] \\ &= \Sigma[:,i]^T e^j \\ &= \Sigma[i,j] \end{aligned}$$

Finally

$$\begin{aligned} \langle e^i, e^j \rangle_{\mathcal{O}} &= (e^i)^T \Sigma e^j \\ &= (e^i)^T \Sigma[:,j] \\ &= \Sigma[i,j]. \end{aligned}$$

$\blacksquare$

By extending the 1-1 correspondance of the generators, using linearity, we obtain the following:

Claim: The Hilbert spaces $\mathcal{X}, \mathcal{C}$ & $\mathcal{O}$ are all isometric under the correspondence given as follows:

$$Y \in \mathcal{X} \longmapsto \underbrace{E(\underbrace{Y}_{\in \mathbb{R}}\underbrace{X}_{\in \mathbb{R}^N})}_{} \in \mathcal{C}$$

$$f \in \mathcal{C} \longmapsto \Sigma^{-1} f \in \mathcal{O}$$

$$v \in \mathcal{O} \longmapsto \sum_{i=1}^{N} X_i v_i \in \mathcal{X}$$

Notation: if $h \in \mathcal{X}$ or $h \in \mathcal{C}$ or $h \in \mathcal{O}$ and $\mathcal{H}$ is one of $\mathcal{X}, \mathcal{C}$ or $\mathcal{O}$ let

$$\mathcal{H}(h) = \begin{cases} \text{the member of } \mathcal{H} \\ \text{corresponding to } h \text{ under} \\ \text{the isometry.} \end{cases}$$

e.g.

$$Y \in \mathcal{X} \longmapsto \mathcal{E}(Y) := E(XY)$$

$$f \in \mathcal{E} \longmapsto \mathcal{A}(f) := \Sigma^{-1} f$$

$$v \in \mathcal{A} \longmapsto \mathcal{X}(v) := \sum_{i=1}^{N} X_i v_i$$

$$\begin{aligned} Y \in \mathcal{X} \longmapsto \mathcal{A}(Y) &= \mathcal{A}(\mathcal{E}(Y)) \\ &= \Sigma^{-1} \mathcal{E}(Y) \\ &= \Sigma^{-1} E(XY) \\ &= \Sigma^{-1} E(XX^T a) \quad \text{if } Y = X^T a \\ &= a \end{aligned}$$

$$\begin{aligned} f \in \mathcal{E} \longmapsto \mathcal{X}(f) &= \mathcal{X}(\mathcal{A}(f)) \\ &= \mathcal{X}(\Sigma^{-1} f) \\ &= X^T \Sigma^{-1} f \end{aligned}$$

KL expansion

Since $\mathcal{X}$ is a finite dimensional Hilbert space we can find an Orthonormal Basis (ONB)

$$Z_1, \dots, Z_N \in \mathcal{X}.$$

so that

$$\langle Z_i, Z_j \rangle_{\mathcal{X}} = \delta_{ij}$$

The isometries give ONB for $\mathcal{E}$ and $\mathcal{A}$:

$$f_i = \mathcal{E}(Z_i) = E(XZ_i) \quad \text{ONB for } \mathcal{E}$$

$$u_i = \mathcal{A}(Z_i) = \Sigma^{-1} f_i \quad \text{ONB for } \mathcal{A}$$

Since $Z_1, \dots, Z_n$ is a basis for $\mathcal{X}$ any $Y \in \mathcal{X}$ has the form

$$Y = \sum_{n=1}^{N} c_n Z_n$$

with

$$\begin{aligned} c_n &= \langle Z_n, Y \rangle_{\mathcal{X}} \quad E(Z_n Y) \\ &= \langle f_n, \mathcal{E}(Y) \rangle_{\mathcal{E}} \quad f_n^T \Sigma^{-1} E(XY) \\ &= \langle u_n, \mathcal{A}(Y) \rangle_{\mathcal{A}} \quad u_n^T E(XY) \end{aligned}$$

We can also expand the original random vector X in Z coordinates

$$X_i = \sum_{n=1}^{N} \underbrace{\langle Z_n, X_i \rangle_{\mathcal{X}}}_{E(Z_n X_i)} Z_n$$

Therefore the vector X has coordinate expansion

a vector

$$X = \sum_{n=1}^{N} \underbrace{E(Z_n X)}_{\substack{= \mathcal{C}(Z_n) \\ = f_n}} Z_n$$

So $X = \sum_{n=1}^{N} f_n Z_n$

ONB of $\mathcal{C}$ ONB of $\mathcal{Q}$

By the orthonormality of $f_1, \ldots, f_N$ we have

$$I = \left[\langle f_n, f_m \rangle_{\mathcal{C}} \right]_{n,m=1}^{N}$$

Now let

$$\mathcal{U} \Lambda \mathcal{U}^T = \left[\langle f_n, f_m \rangle_{L_2} \right]_{n,m=1}^{N}$$

$$X = \sum_{n=1}^{N} f_n Z_n$$

$$= \begin{bmatrix} | & & | \\ f_1 & \cdots & f_N \\ | & & | \end{bmatrix} \begin{bmatrix} Z_1 \\ \vdots \\ Z_N \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & & | \\ f_1 & \cdots & f_N \\ | & & | \end{bmatrix} \mathcal{U}}_{K} \underbrace{\mathcal{U}^T \begin{bmatrix} Z_1 \\ \vdots \\ Z_N \end{bmatrix}}_{W}$$

Let $K = \begin{bmatrix} | & & | \\ k_1 & \cdots & k_N \\ | & & | \end{bmatrix}$ then

$$K^T K = \left[\langle k_i, k_j \rangle_{L_2} \right]_{n,m=1}^{N}$$

$$= \mathcal{U}^T \left[\langle f_n, f_m \rangle_{L_2} \right]_{n,m=1}^{N} \mathcal{U}$$

$$= \Lambda$$

and

L_2 orthogonal vectors

$$X = \sum_{n=1}^{N} k_n W_n$$

iid $N(0,1)$

KL expansion.

(up to a re-normalization of $k_1, \ldots, k_N$)

Given a subset $\Omega \subset \{1, \ldots, N\}$ we can write $E(X | X_\Omega)$ as a projection on $\mathcal{X}$

Let $a \in \mathbb{R}^N$ and $Y = a^T X \in \mathcal{X}$ so that

$$E(Y | X_\Omega) = E(Y X_\Omega^T)(E(X_\Omega X_\Omega^T))^{-1} X_\Omega$$

$$= a^T E(X X_\Omega^T)(E(X_\Omega X_\Omega^T))^{-1} X_\Omega$$

$$= a^T \Sigma_{:,\Omega} \Sigma_{\Omega\Omega}^{-1} X_\Omega$$

By linear properties of $E(\cdot | X_\Omega)$ we can write

a linear operator on $\mathcal{X}$

$$E(Y | X_\Omega) = P^{\mathcal{X}} Y \in \mathcal{X}_\Omega$$

where $\mathcal{X}_\Omega = \text{span}\{X_i : i \in \Omega\}$.

Since $P^{\mathcal{X}} P^{\mathcal{X}} = P^{\mathcal{X}}$ so $P^{\mathcal{X}}$ is a projection on $\mathcal{X}$

Also by properties of Gaussian r.v.s

$$Y - P^x Y \perp_x X_i \text{ for } i \in \Omega$$

Since $\mathcal{X}, \mathcal{C}, \mathcal{A}$ are all isometric Hilbert spaces, the linear projection $P^x Y$ has equivalent projections P^c & P^a defined on $\mathcal{C}$ and $\mathcal{A}$ respectively.

In particular

$$Y \longmapsto P^x Y$$

$$c(Y) \longmapsto c(P^x Y) =: P^c c(Y)$$

$$a(Y) \longmapsto a(P^x Y) =: P^a a(Y)$$

in coordinates the arguments expand as

$$Y = \sum_{i=1}^{N} X_i a_i := X^T a$$

by linearity

$$c(Y) = \sum_{i=1}^{N} \underbrace{c(X_i)}_{\Sigma[:,i]} a_i = \Sigma a$$

$$a(Y) = \Sigma^{-1} c(Y) = a$$

which map to

$$P^x Y = \sum_{i \in \Omega} X_i b_i = X^T \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

$$P^c c(Y) = \Sigma \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

$$P^c a(Y) = \Sigma^{-1} c(Y) = \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

where $b_\Omega = \Sigma_{\Omega\Omega}^{-1} \Sigma_{\Omega,:} a$

So

$$X^T a \xmapsto{P^x} X^T \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

$$\Sigma a \xmapsto{P^c} \Sigma \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

$$a \xmapsto{P^a} \begin{bmatrix} b_\Omega \\ 0 \end{bmatrix}$$

$$\therefore P^a = \begin{bmatrix} I_{|\Omega|} \\ 0 \end{bmatrix} \Sigma_{\Omega,\Omega}^{-1} (\Sigma_{:,\Omega})^H$$

has N rows

and $|\Omega|$ columns

which has the form

$$P^a = R (R^H \Sigma R)^{-1} R^H \Sigma$$

with $R = \begin{bmatrix} I_{|\Omega|} \\ 0 \end{bmatrix}$ since

$$R^H \Sigma = \begin{bmatrix} I_{|\Omega|} & 0 \end{bmatrix} \begin{bmatrix} - \Sigma_{\Omega,:} - \\ - \Sigma_{\Omega^c,:} - \end{bmatrix}$$

$$= \Sigma_{\Omega,:}$$

and therefore

$$(R^H \Sigma R)^{-1} = \left(\Sigma_{\Omega,:} \begin{bmatrix} I_{|\Omega|} \\ 0 \end{bmatrix} \right)^{-1}$$

$$= \Sigma_{\Omega,\Omega}^{-1}$$

Therefore letting $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^N$

P^a is an orthogonal projection in $\mathbb{R}^N$ onto $\text{span}\{e_i : i \in \Omega\}$ w.r.t. $\langle \cdot, \cdot \rangle_a = \langle \cdot, \Sigma, \cdot \rangle$, i.e.

$$a - P^a a \perp_\Sigma e^i, \quad i \in \Omega$$

$$\Updownarrow$$

P^a is an oblique projection onto $\text{span}\{e_i : i \in \Omega\}$ w.r.t $\langle \cdot, \cdot \rangle_{L_2}$ s.t.

$$a - P^a a \perp_{L_2} \Sigma[:,i], \forall i \in \Omega$$

The isometric results for P^e follow immediately...

P^e is an orthogonal projection in $\mathbb{R}^N$ onto $\text{span}\{\Sigma[:,i] : i \in \Omega\}$ w.r.t. $\langle \cdot, \cdot \rangle_e = \langle \cdot, \Sigma^{-1} \cdot \rangle$, i.e.

$$f - P^e f \perp_{\Sigma^{-1}} \Sigma[:,i]$$

and equivalently

P^e is an oblique projection onto $\text{span}\{\Sigma[:,i] : i \in \Omega\}$ w.r.t $\langle \cdot, \cdot \rangle_{L_2}$ s.t.

$$f - P^e f \perp_{L_2} e_i, \forall i \in \Omega$$

So that

$$P^e = \underbrace{\Sigma_{:,\Omega}}_{R} \underbrace{\left(\Sigma_{\Omega,:} \begin{bmatrix} I_{|\Omega|} \\ 0 \end{bmatrix} \right)^{-1}}_{(R^H L)^{-1}} \underbrace{\begin{bmatrix} I_{|\Omega|} & 0 \end{bmatrix}}_{L^H}$$

Stopping critearion for iterative algorithms for $E(f|d)$ and sampling $P(f|d)$

Consider the GRF "smoothing" problem

$$d = f + w$$

where

$$w \sim N\left(0, \frac{N}{\Omega}\right)$$

$$f \sim N\left(0, \frac{C}{\Omega}\right)$$

$E(f|d)$ solves

$$[N^{-1} + C^{-1}]E(f|d) = N^{-1}d$$

which maximizes (for fixed d)

$$2\log P(d,f)$$
$$= -\Omega\|d-f\|_N^2 - \Omega\|f\|_C^2 \quad + \text{const}$$

The maximum log-likelihood value is

$$2\log P(d, E(f|d))$$
$$= -\Omega\|d - C(C+N)^{-1}d\|_N^2$$
$$-\Omega\|C(C+N)^{-1}d\|_C^2$$

Notice that

$$d - C(C+N)^{-1}d = ((C+N)-C)(C+N)^{-1}d$$
$$= N(C+N)^{-1}d$$

$$\therefore -\Omega\|d - C(C+N)^{-1}d\|_N^2$$
$$= -\Omega d^T(C+N)^{-1}NN^{-1}N(C+N)^{-1}d$$

and

$$-\Omega\|C(C+N)^{-1}d\|_C^2$$
$$= -\Omega d^T(C+N)^{-1}CC^{-1}C(C+N)^{-1}d$$

$$\therefore \quad 2\log P(d, E(f|d))$$
$$= -\Omega d^T(C+N)^{-1}d$$
$$= -\chi^2_{df(d)}$$

$$\therefore E(2\log P(d,E(f|d))) = -df(d)$$

$$\text{var}(2\log P(d,E(f|d))) = 2\,df(d)$$

$$\text{sd}(2\log P(d,E(f|d))) = \sqrt{2\,df(d)}$$

$$2\log P(d,f) - 2\log P(d,E(f|d))$$
$$= -\Omega\|d-f\|_N^2 - \Omega\|f\|_C^2 + \Omega d^T(C+N)^{-1}d$$
$$= -\Omega\left(\langle d-f, N^{-1}(d-f)\rangle + \langle f, C^{-1}f\rangle - \langle d, (C+N)^{-1}d\rangle\right)$$

check this

$$= -\Omega\|f - E(f|d)\|^2_{(C^{-1}+N^{-1})^{-1}}$$

So for

$$f \sim N\left(E(f|d), \mathrm{var}(f|d)\right)$$

$$2\log P(d,f) - 2\log P(d, E(f|d))$$

$$= -\|f - E(f|d)\|^2_{\mathrm{var}(f|d)}$$

$$\overset{\mathcal{D}}{=} -\chi^2_{df(f)}$$

So

$$2\log P(d,f)$$

$$= \underbrace{2\log P(d, E(f|d))}_{\substack{\text{This varies like} \\ -\chi^2_{df(d)}}} - \underbrace{\|f - E(f|d)\|^2_{\mathrm{var}(f|d)}}_{\substack{\text{This varies like} \\ -\chi^2_{df(f)}}}$$

$\approx -df(d) \pm \sqrt{2df(d)}$

over different d

$d \sim P(d)$

$\approx -df(f) \pm \sqrt{2df(f)}$

for fixed d

over $f \sim P(f|d)$

The ensemble varies like

$$-(df(d) + df(f)) \pm \sqrt{2df(d) + 2df(f)}$$

Basics of Complex Gaussian random variables

Definition:

Let $Z \sim N(\mu, \Gamma, C)$ denote a complex random vector such that $(Re(Z), Im(Z))$ is jointly Gaussian and

where $Z = X + iY$

$$\mu = E(Z) = E(X) + iE(Y)$$
$$\Gamma = E(ZZ^H)$$
$$C = E(ZZ^T)$$

First notice that for any complex or real matrix M we have

$$MZ \sim N(M\mu, M\Gamma M^H, MCM^T)$$

Now if $Z = X + iY$ $(X, Y \in \mathbb{R}^d)$ then

$$\Gamma = E(XX^T) + E(YY^T) + i(E(YX^T) - E(XY^T))$$
$$C = E(XX^T) - E(YY^T) + i(E(YX^T) + E(XY^T))$$
$$\Longrightarrow \quad C = C^T \ \& \ \Gamma = \Gamma^H$$

Also notice that

$$\frac{\Gamma + C}{2} = E(XX^T) + iE(YX^T) = E(ZX^T)$$

$$i\frac{\Gamma - C}{2} = iE(YY^T) + E(XY^T) = E(ZY^T)$$

and

$$\frac{\Gamma + \bar{C}}{2} = E(XZ^H) \qquad \frac{\Gamma - \bar{C}}{2} = E(YZ^T)$$

$$\frac{C + \bar{\Gamma}}{2} = E(XZ^T) \qquad \frac{C - \bar{\Gamma}}{2} = -E(YZ^H)$$

The matrices Γ and C arise most naturally from

$$E\left(\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}^H\right) = E\left(\begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}\begin{pmatrix} Z^H & Z^T \end{pmatrix}\right)$$

$$= \begin{pmatrix} E(ZZ^H) & E(ZZ^T) \\ \overline{E(ZZ^T)} & \overline{E(ZZ^H)} \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma & C \\ \bar{C} & \bar{\Gamma} \end{pmatrix}$$

Example

If $X \sim \underbrace{N(\mu, \Sigma)}_{\text{real Gaussian vector}}$ then

$$X \sim N(\mu, \Sigma, \Sigma).$$

Consider the Fourier transform operators $\mathcal{W}, \mathcal{U}, \mathcal{F}$

unitary

scaled by $\frac{\Delta x}{(2\pi)^{d/2}}$

and suppose Σ is circulant. Now

$$\mathcal{U}X \sim N\left(\mathcal{U}\mu, \underbrace{\mathcal{U}\Sigma\mathcal{U}^H}_{\text{diag}(\mathcal{W}\Sigma[:,1])}, \underbrace{\mathcal{U}\Sigma\mathcal{U}^T}_{\text{diag}(\mathcal{W}\Sigma[:,1])\mathcal{U}^2}\right)$$

where

$$\mathcal{U}^2 = \begin{cases} \begin{bmatrix} 1 & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} & \text{if length}(X) \text{ is even} \\ \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} & \text{if length}(X) \text{ is odd} \end{cases}$$

Note: The pattern of ones in $\mathcal{U}^2$ corresponds to the real coordinates of $\mathcal{U}X$ when X is a real random vector.

Example

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \sim N\left(0, \overbrace{\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}}^{\Gamma}, \overbrace{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}}^{C}\right)$$

$$\Gamma_{ii} = E(|W_i|^2), \qquad \Gamma_{ij} = E(W_i \overline{W_j})$$

$$C_{ii} = E(W_i^2), \qquad C_{ij} = E(W_i W_j)$$

$$C_{12} = C_{21}, \quad \Gamma_{12} = \overline{\Gamma_{21}}$$

$$\Gamma = \begin{bmatrix} E(X_1^2+Y_1^2) & E(X_1X_2+Y_1Y_2) \\ E(X_1X_2+Y_1Y_2) & E(X_2^2+Y_2^2) \end{bmatrix} + i\begin{bmatrix} 0 & E(Y_1X_2-X_1Y_2) \\ E(Y_2X_1-X_2Y_1) & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} E(X_1^2-Y_1^2) & E(X_1X_2-Y_1Y_2) \\ E(X_1X_2-Y_1Y_2) & E(X_2^2-Y_2^2) \end{bmatrix} + i\begin{bmatrix} E(2X_1Y_1) & E(Y_1X_2+X_1Y_2) \\ E(Y_2X_1+X_2Y_1) & E(2X_2Y_2) \end{bmatrix}$$

Suppose Γ_{11} & Γ_{22} are real &

Let $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(0, \begin{pmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$

$$\begin{bmatrix} E(X_1^2+Y_1^2) & E(X_1X_2+Y_1Y_2) \\ E(X_1X_2+Y_1Y_2) & E(X_2^2+Y_2^2) \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix} \qquad \begin{bmatrix} E(X_1^2-Y_1^2) & E(X_1X_2-Y_1Y_2) \\ E(X_1X_2-Y_1Y_2) & E(X_2^2-Y_2^2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$i\begin{bmatrix} 0 & E(Y_1X_2-X_1Y_2) \\ E(Y_2X_1-X_2Y_1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad i\begin{bmatrix} E(2X_1Y_1) & E(Y_1X_2+X_1Y_2) \\ E(Y_2X_1+X_2Y_1) & E(2X_2Y_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(0, \frac{1}{2}\begin{pmatrix} \Gamma_{11}+\Gamma_{22} & \Gamma_{11}-\Gamma_{22} \\ \Gamma_{11}-\Gamma_{22} & \Gamma_{11}+\Gamma_{22} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$\begin{bmatrix} E(X_1^2+Y_1^2) & E(X_1X_2+Y_1Y_2) \\ E(X_1X_2+Y_1Y_2) & E(X_2^2+Y_2^2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \Gamma_{11}+\Gamma_{22} & \Gamma_{11}-\Gamma_{22} \\ \Gamma_{11}-\Gamma_{22} & \Gamma_{11}+\Gamma_{22} \end{bmatrix} \qquad \begin{bmatrix} E(X_1^2-Y_1^2) & E(X_1X_2-Y_1Y_2) \\ E(X_1X_2-Y_1Y_2) & E(X_2^2-Y_2^2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$i\begin{bmatrix} 0 & E(Y_1X_2-X_1Y_2) \\ E(Y_2X_1-X_2Y_1) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad i\begin{bmatrix} E(2X_1Y_1) & E(Y_1X_2+X_1Y_2) \\ E(Y_2X_1+X_2Y_1) & E(2X_2Y_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ has the effect of

equalizing power:

$$E|W_1|^2 = E|W_2|^2 = \frac{\Gamma_{11}+\Gamma_{22}}{2}$$

Correlating across-mode Real & imag:

$$E(W_1 W_2^*) = \frac{\Gamma_{11}-\Gamma_{22}}{2}$$

$$E(\mathrm{Re}(W_1)\mathrm{Re}(W_2)) = \frac{\Gamma_{11}-\Gamma_{22}}{2}$$

$$E(\mathrm{Im}(W_1)\mathrm{Im}(W_2)) = \frac{\Gamma_{11}-\Gamma_{22}}{2}$$

This has a similar effect

$$\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta} & 1 \\ -e^{i\theta} & 1 \end{bmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(0, \frac{1}{2}\begin{bmatrix} \Gamma_{11}+\Gamma_{22} & \Gamma_{22}-\Gamma_{11} \\ \Gamma_{22}-\Gamma_{11} & \Gamma_{11}+\Gamma_{22} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

since

$$\frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta} & 1 \\ -e^{i\theta} & 1 \end{bmatrix}\begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\theta} & -e^{-i\theta} \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{i\theta} & 1 \\ -e^{i\theta} & 1 \end{bmatrix}\begin{bmatrix} e^{-i\theta}\Gamma_{11} & -e^{-i\theta}\Gamma_{11} \\ \Gamma_{22} & \Gamma_{22} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \Gamma_{11}+\Gamma_{22} & \Gamma_{22}-\Gamma_{11} \\ \Gamma_{22}-\Gamma_{11} & \Gamma_{11}+\Gamma_{22} \end{bmatrix}$$

To get phase correlation one can use this:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N\left(0,\ \frac{1}{2}\begin{bmatrix} \Gamma_{11}+\Gamma_{22} & e^{i\theta}(\Gamma_{11}-\Gamma_{22}) \\ e^{-i\theta}(\Gamma_{11}-\Gamma_{22}) & \Gamma_{11}+\Gamma_{22} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Since

$$\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}\begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix}\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}\begin{bmatrix} \Gamma_{11} & e^{i\theta}\Gamma_{11} \\ e^{-i\theta}\Gamma_{22} & -\Gamma_{22} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \Gamma_{11}+\Gamma_{22} & e^{i\theta}(\Gamma_{11}-\Gamma_{22}) \\ e^{-i\theta}(\Gamma_{11}-\Gamma_{22}) & \Gamma_{11}+\Gamma_{22} \end{bmatrix}$$

Also notice that $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}$ is unitary since,

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Probability density for complex Gaussians

Density change of variables gives

the density of $\begin{bmatrix} z \\ \bar z \end{bmatrix}$ w.r.t $dz\,d\bar z$

$$p'(z,\bar z) = p(x,y)\left|\frac{\partial(x,y)}{\partial(z,\bar z)}\right|$$

$$= p\left(\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)\left|\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|$$

where

$$(2\pi)^n|\Sigma|^{1/2}\, p\left(\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)$$

$$= \exp\left[-\frac12\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\begin{pmatrix} I/2 & iI/2 \\ I/2 & -iI/2 \end{pmatrix}\Sigma^{-1}\begin{pmatrix} I/2 & I/2 \\ -iI/2 & iI/2 \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right]$$

$$= \exp\left[-\frac14\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\left(\frac{\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}}{\sqrt2}\Sigma\frac{\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}}{\sqrt2}\right)^{-1}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right]$$

Now

$$\frac{\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}}{\sqrt2}\Sigma\frac{\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}}{\sqrt2}$$

$$= \frac12 E\left(\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\left[\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right]^H\right)$$

$$= \frac12 E\left(\begin{pmatrix} z \\ \bar z \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\right)$$

$$= \frac12\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}$$

so that

$$(2\pi)^n|\Sigma|^{1/2}\, p\left(\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)$$

$$= \exp\left(-\frac12\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}^{-1}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)$$

p is the density of $\begin{pmatrix} X \\ Y \end{pmatrix}$

We also have

since these are unitary

$$|\Sigma| = \left|\frac{\begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}}{\sqrt2}\Sigma\frac{\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}}{\sqrt2}\right| = \left|\frac12\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}\right|$$

$$\operatorname{logdet}\left(\left|\begin{matrix} \Gamma & C \\ \bar C & \bar\Gamma \end{matrix}\right|\right) = \operatorname{logdet}(|\Sigma|) + \operatorname{logdet}(2^{2n})$$

$$\frac{\left|\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|}{|\Sigma|^{1/2}} = \frac{\left|\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|}{\left|\frac12\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}\right|^{1/2}} = \frac{\cancel{\left|\frac{1}{\sqrt2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right|}\cancelto{1}{\left|\frac{1}{\sqrt2}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|}}{\cancel{\left|\frac12\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right|^{1/2}}\left|\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}\right|^{1/2}}$$

Therefore

$$p'(z,\bar z) = p\left(\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)\left|\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|$$

$$= \frac{\left|\frac12\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}\right|}{(2\pi)^n|\Sigma|^{1/2}}\exp\left(-\frac12\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}^{-1}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)$$

and therefore

$$p'(z,\bar z) = \frac{\left|\begin{matrix} \Gamma & C \\ \bar C & \bar\Gamma \end{matrix}\right|^{-1/2}}{(2\pi)^n}\exp\left(-\frac12\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}^{-1}\begin{pmatrix} z \\ \bar z \end{pmatrix}\right)$$

Remark: For testing

$$-\frac12\begin{pmatrix} X \\ Y \end{pmatrix}^T\Sigma^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} - \frac12\log|\Sigma| - \operatorname{logdet}(2^n)$$

$$= -\frac12\begin{pmatrix} z \\ \bar z \end{pmatrix}^H\begin{pmatrix} \Gamma & C \\ \bar C & \bar\Gamma \end{pmatrix}^{-1}\begin{pmatrix} z \\ \bar z \end{pmatrix} - \frac12\log\left|\begin{matrix} \Gamma & C \\ \bar C & \bar\Gamma \end{matrix}\right|$$

where $n = \text{length}(z) = \text{length}(X) = \text{length}(Y)$

Impulse responses for computing columns of Γ & C

Consider a real random vector

$$X \sim N(0, \Sigma).$$

In this section we explore how to use impulse responses to obtain the columns of Γ and C where

$$Z \sim N(0, \Gamma, C)$$

$$Z = \mathcal{F}X \quad (\text{or } \mathcal{U}X)$$

Fourier transform.

For simplicity suppose $N = \text{length}(X)$ is even and we are working with 1-dimensional FT $\mathcal{F}$

$$\underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N/2} \\ z_{N/2+1} \end{bmatrix}}_{\text{from rFFT}} \hspace{-2.2em}\begin{matrix}\\ \\ \\ \\ \\ \bar{z}_{N/2} \\ \vdots \\ \bar{z}_2\end{matrix} = \begin{bmatrix} 1 & & & \\ & \overset{I}{\boxed{\begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1\end{smallmatrix}}} & & \\ & & 1 & \\ & & & \underset{J}{\boxed{\begin{smallmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{smallmatrix}}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{N/2} \\ z_{N/2+1} \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{N/2} \end{bmatrix}$$

Same order of indici

$$JJ = I \quad \text{and} \quad J^T = J$$

Let

$$w = \begin{bmatrix} z_2 \\ \vdots \\ z_{N/2} \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{N/2} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} z_2 \\ \vdots \\ z_{N/2} \end{bmatrix} \quad \text{so that}$$

$$w = \begin{pmatrix} u \\ J\bar{u} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$$

Also notice that in the case that

$$u \sim N(0, \Gamma, C)$$

Therefore

$$E(ww^H) = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \Gamma & C \\ \bar{C} & \bar{\Gamma} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \Gamma & CJ \\ \bar{C} & \bar{\Gamma}J \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma & CJ \\ J\bar{C} & J\bar{\Gamma}J \end{pmatrix}$$

set

$$\varphi_k = \begin{pmatrix} e_k \\ J\bar{e}_k \end{pmatrix} = \begin{pmatrix} e_k \\ Je_k \end{pmatrix}$$

$$\psi_k = \begin{pmatrix} ie_k \\ J\overline{(ie_k)} \end{pmatrix} = i \begin{pmatrix} e_k \\ -Je_k \end{pmatrix}$$

where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$.

k^{th} spot

Then

$$E(ww^H)\varphi_k = \begin{pmatrix} \Gamma & CJ \\ J\bar{C} & J\bar{\Gamma}J \end{pmatrix} \begin{pmatrix} e_k \\ Je_k \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma e_k + C\cancelto{I}{JJ}e_k \\ J\bar{C}e_k + J\bar{\Gamma}\cancelto{I}{JJ}e_k \end{pmatrix}$$

$$= \begin{pmatrix} (\Gamma + C)e_k \\ J(\bar{C} + \bar{\Gamma})e_k \end{pmatrix}$$

This is what comes out of rFFT... the unique, complex modes.

and

$$E(ww^H)\psi_k = i \begin{pmatrix} \Gamma & CJ \\ J\bar{C} & J\bar{\Gamma}J \end{pmatrix} \begin{pmatrix} e_k \\ -Je_k \end{pmatrix}$$

$$= i \begin{pmatrix} \Gamma e_k - C\cancelto{I}{JJ}e_k \\ J\bar{C}e_k - J\bar{\Gamma}\cancelto{I}{JJ}e_k \end{pmatrix}$$

$$= \begin{pmatrix} i(\Gamma - C)e_k \\ iJ(\bar{C} - \bar{\Gamma})e_k \end{pmatrix}$$

Notice that

$$\frac{1}{2}\left(E(ww^\#)\varphi_k + iE(\omega\omega^\#)\psi_k\right)$$

$$= \frac{1}{2}\begin{pmatrix} (\Gamma + C)e_k \\ J(\bar{C} + \bar{\Gamma})e_k \end{pmatrix} + \frac{i}{2}\begin{pmatrix} i(\Gamma - C)e_k \\ iJ(\bar{C} - \bar{\Gamma})e_k \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (\Gamma + C)e_k \\ J(\bar{C} + \bar{\Gamma})e_k \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (C - \Gamma)e_k \\ J(\bar{\Gamma} - \bar{C})e_k \end{pmatrix}$$

$$\frac{1}{2}\left(E(ww^\#)\varphi_k - iE(\omega\omega^\#)\psi_k\right)$$

$$= \frac{1}{2}\begin{pmatrix} (\Gamma + C)e_k \\ J(\bar{C} + \bar{\Gamma})e_k \end{pmatrix} - \frac{i}{2}\begin{pmatrix} i(\Gamma - C)e_k \\ iJ(\bar{C} - \bar{\Gamma})e_k \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (\Gamma + C)e_k \\ J(\bar{C} + \bar{\Gamma})e_k \end{pmatrix} - \frac{1}{2}\begin{pmatrix} (C - \Gamma)e_k \\ J(\bar{\Gamma} - \bar{C})e_k \end{pmatrix}$$

$$\frac{1}{2}\left(E(ww^\#)\varphi_k + iE(\omega\omega^\#)\psi_k\right) = \begin{pmatrix} Ce_k \\ J\bar{\Gamma}e_k \end{pmatrix}$$

$$\frac{1}{2}\left(E(ww^\#)\varphi_k - iE(\omega\omega^\#)\psi_k\right) = \begin{pmatrix} \Gamma e_k \\ J\bar{C}e_k \end{pmatrix}$$

This comes out of rFFT