# Lecture 5

**Today:**

Dependence between R.V.s & conditional PMFs

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Suppose $X$ is a r.v. with PMF

| $x$ | $p(x)$ |
|---|---|
| 1 | 0.5 |
| 2 | 0.1 |
| 3 | 0.1 |
| 4 | 0.2 |
| 5 | 0.1 |
| | 1 |

What if I sampled $X$ but didn't tell you which number it was but instead told you that $X \geq 3$.

The actual value of $X$ is still random but now the possible values are 3, 4, 5 instead of 1, 2, 3, 4, 5.

The conditional PMF given $X \geq 3$, denoted $p(x \mid X \geq 3)$, characterizes the randomness in $X$ after observing $X \geq 3$.

To find $p(x \mid X \geq 3)$ just zero out $p(x)$ for $x$'s that are not observed & renormalize to 1.

original

| $x$ | $p(x)$ |
|---|---|
| 1 | 0.5 |
| 2 | 0.1 |
| 3 | 0.1 |
| 4 | 0.2 |
| 5 | 0.1 |
| | 1 |

$\longrightarrow$

after observing $X \geq 3$

| $x$ | $p(x \mid X \geq 3)$ |
|---|---|
| 1 | 0 |
| 2 | 0 |
| 3 | $0.1/0.4 = \frac{1}{4}$ |
| 4 | $0.2/0.4 = \frac{1}{2}$ |
| 5 | $0.1/0.4 = \frac{1}{4}$ |
| | $0.4/0.4$ |

$\therefore$ $P(X=4) = p(4) = 0.2$ but $P(X=4 \mid X \geq 3) = p(4 \mid X \geq 3) = \frac{1}{2}$

You can also compute the "new" expected value, variance & sd after observing $X \geq 3$:

$$\underbrace{E(X \mid X \geq 3)}_{\text{adjusted prediction after given new information that } X \geq 3} = \sum_x x\, p(x \mid X \geq 3)$$

$$= 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} \cdot 5 \cdot \frac{1}{4}$$

$$= \frac{3 + 8 + 5}{4} = \frac{16}{4}$$

$$\mathrm{var}(X \mid X \geq 3) = E(X^2 \mid X \geq 3) - \left(E(X \mid X \geq 3)\right)^2$$

$$= 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{2} + 5^2 \cdot \frac{1}{4} - \left(\frac{16}{4}\right)^2$$

$$\mathrm{sd}(X \mid X \geq 3) = \sqrt{(\quad)}$$

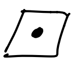
## Dependence btwn 2 r.v.s $X, Y$

Lets start with an example.

Roll two 6-sided die, one red one blue.

Let $X$ = the sum of the numbers showing for the <u>red</u> & <u>blue</u> die

$Y$ = the value of just the red die.

Now suppose your betting that the red die is , i.e. that $Y = 1$.

Suppose the Croupier rolls the die but does not initally show the result, ... just tells everyone the value of $X$, then allows bets to be adjusted.

If the Croupier says

$$X = 2$$

do you want to change your bet?

On the other hand, what if the Croupier says

$$X = 7$$

... still want to change your bet?

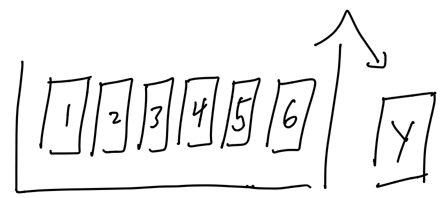
It <u>turns out</u>

If $X = 2$ then the adjusted PMF for $Y$ is $\begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$

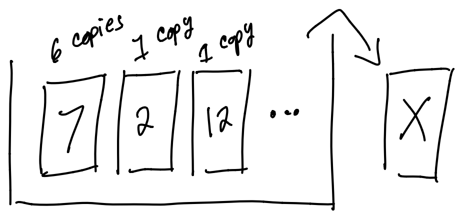
If $X = 7$ then there is no adjustment

Suppose I wanted a box model of this game.

Does this work:



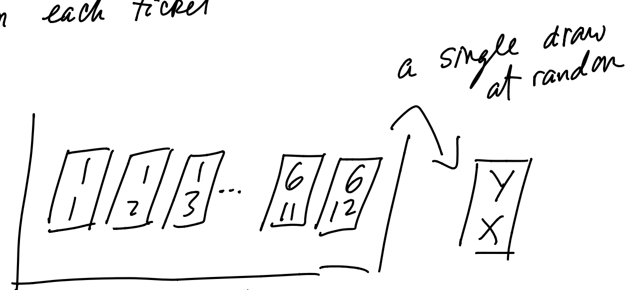

6 tickets




36 tickets

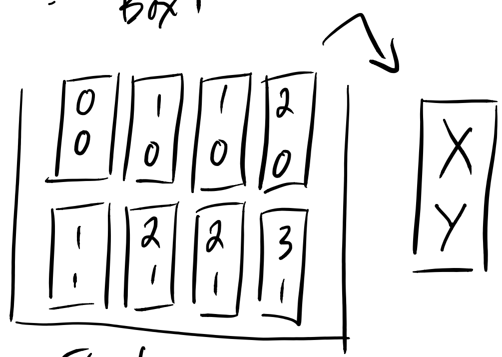
This doesn't work since given $X$ there is no adjustment for the PMF of $Y$.

To model two dependent r.v.s $X, Y$ you need a single box, but with two numbers on each ticket




36 tickets

Example : Box 1




8 tixs

Find $p_X(x)$, $p_Y(y)$ $P(Y=0)$ and $P(Y=0|X=2)$

Note: $p_X(x)$ and $p_Y(y)$ are the PMF's for X and Y computed by completely ignoring the other r.v. ... called the magibnal PMF's for X and Y

$$P(Y=0) = \frac{1}{2}$$

$$P(Y=0 | X=2) = \frac{1}{3}$$

| $x$ | $p_X(x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

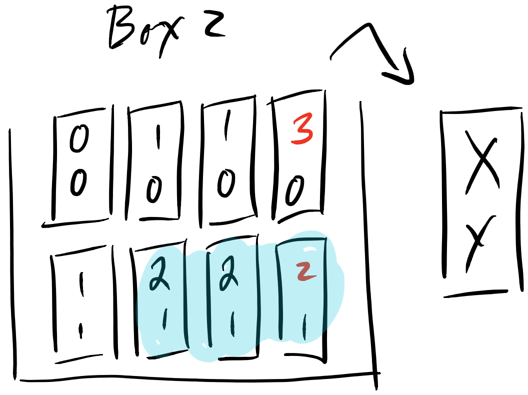
Marginal PMF for X

| $y$ | $p_Y(y)$ |
|---|---|
| 0 | 1/2 |
| 1 | 1/2 |

Marginal PMF for Y

Example :

Box 2




8 tixs

Note: The marginal PMFs are the same for Box 1 & Box 2 but

Box 1: $P(Y=0 | X=2) = \frac{1}{3}$

Box 2: $P(Y=0 | X=2) = 0$

## Joint PMF

The last example shows that marginal PMFs are not sufficient to provide a mathematical blueprint for characterizing the dependence between two r.v.s X & Y which come from one play of a random game.

To characterize a pair of r.v. $X$ & $Y$ use **joint PMF**:

$p(x,y)$ = "The probability that $X=x$ and $Y=y$ simultaneously"

= $P$(getting $\boxed{\begin{smallmatrix} x \\ y \end{smallmatrix}}$) ← in a box model for $X,Y$

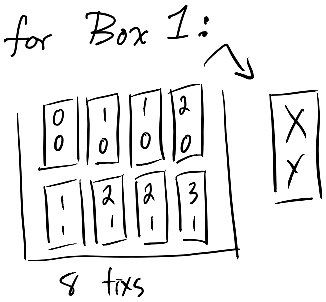
= $P(X=x \text{ and } Y=y)$

To find the probability of some event involving $X$ & $Y$ just add $p(x,y)$ over all pairs $x,y$ in the event

Note: Sometimes I will write $p(x,y)$ as $p_{X,Y}(x,y)$ if I want to emphasize that $p_{X,Y}(x,y)$ is the blueprint for $(X,Y)$ and not some other pair of r.v.s

e.g. Lets see that the joint PMF can distinguish the previous two Box models. (which have different dependence but the same *marginal* PMFs $p_X(x)$ $p_Y(y)$).

The joint PMF $p(x,y)$ for Box 1:




| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 1/8 | 2/8 |
| 3 | 0 | 1/8 |

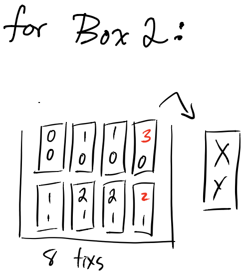
1 ← all entries should sum to 1

The $x,y$ entry is $p(x,y)$

$p(0,0) = P(X=0 \;\&\; Y=0) = 1/8$

$p(1,0) = P(X=1 \;\&\; Y=0) = \frac{2}{8}$

The joint PMF $p(x,y)$ for Box 2:




| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

So the joint PMF can tell the diffence btwn Box 1 & 2.

The joint PMF $p(x,y)$ can be used to calculate any probability involving $X$ & $Y$.

**Before:** To find the probability of some event involving $X$ just add $p_X(x)$ over all $x$ in the event.

**New:** To find the probability of some event involving $X$ & $Y$ just add $p(x,y)$ over all pairs $x,y$ in the event.

Example: From Box 2 in previous example

$$P(X=2 \text{ and } Y=1) = p(2,1) = 3/4$$

$p(x,y)$ for Box 2

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/4 |
| 2 | 0 | 3/4 |
| 3 | 1/8 | 0 |
| | | 1 |

$$P(X=1) = p(1,0) + p(1,1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

$p(x,y)$ for Box 2

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/4 |
| 2 | 0 | 3/4 |
| 3 | 1/8 | 0 |
| | | 1 |

$$P(Y=0) = p(0,0) + p(1,0) + p(2,0) + p(3,0) = \frac{1}{8} + \frac{2}{8} + 0 + \frac{1}{8} = \frac{1}{2}$$

$p(x,y)$ for Box 2

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/4 |
| 2 | 0 | 3/4 |
| 3 | 1/8 | 0 |
| | | 1 |

$$P(X=1 \text{ or } Y=1) = 0 + \frac{2}{8} + \frac{1}{8} + \frac{3}{8} + 0 = \frac{6}{8} = \frac{3}{4}$$

$p(x,y)$ for Box 2

| X \ Y | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/4 |
| 2 | 0 | 3/4 |
| 3 | 1/8 | 0 |
| | | 1 |

Notice:

$P(X=x) = p_X(x) =$ row sum $p(x,y)$ at row $x$

$P(Y=y) = p_Y(y) =$ column sum $p(x,y)$ at column $y$

Example: Find $E(X)$ & $E(Y)$ when $X,Y$ have joint PMF

| X \ Y | 0 | 1 | $p_X(x)$ |
|---|---|---|---|
| 0 | $\frac{1}{8}$ | 0 | $\frac{1}{8}$ |
| 1 | $\frac{2}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| 2 | 0 | $\frac{3}{8}$ | $\frac{3}{8}$ |
| 3 | $\frac{1}{8}$ | 0 | $\frac{1}{8}$ |
| $p_Y(y)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

$$\therefore E(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3+6+3}{8} = \frac{12}{8} = \frac{3}{4}$$

$$E(Y) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

# Modifying a PMF given information

Suppose $X, Y$ have joint PMF:

| $X \backslash Y$ | 0 | 1 | |
|---|---|---|---|
| 0 | $\frac{1}{8}$ | 0 | |
| 1 | $\frac{2}{8}$ | $\frac{1}{8}$ | |
| 2 | 0 | $\frac{3}{8}$ | |
| 3 | $\frac{1}{8}$ | 0 | |
| | | | 1 |

For a given draw of $(X,Y)$ suppose your told that $X \geq 1$ and $Y = 1$

But not the exact values of $X$ & $Y$.

Find

(i) $P(X = 2 \mid X \geq 1 \text{ and } Y = 1)$

(ii) Your best prediction for $X$, given $X \geq 1$ and $Y = 1$, denoted

$$E(X \mid X \geq 1 \text{ and } Y = 1)$$

(iii) The typical prediction error for $X$ given $X \geq 1$ and $Y = 1$, denoted

$$sd(X \mid X \geq 1 \text{ and } Y = 1)$$

The first step is to update the Joint PMF given $X \geq 1$ and $Y = 1$ by following the these two steps

step 1: Set the entries of $p(x,y)$ to zero that don't satisfy the given information.

| $X \backslash Y$ | 0 | 1 | |
|---|---|---|---|
| 0 | $\cancel{\frac{1}{8}}$ 0 | 0 | |
| 1 | $\cancel{\frac{2}{8}}$ 0 | $\frac{1}{8}$ | |
| 2 | 0 | $\frac{3}{8}$ | |
| 3 | $\cancel{\frac{1}{8}}$ 0 | 0 | |
| | | | $\cancel{1}$ $\frac{4}{8}$ |

– these are the possibilities for $X \geq 1$ and $Y = 1$

step 2: Re-normalize the entries to add to 1.

| $X \backslash Y$ | 0 | 1 | |
|---|---|---|---|
| 0 | 0 | 0 | |
| 1 | 0 | $\frac{1}{8} \cdot \frac{8}{4} = \frac{1}{4}$ | |
| 2 | 0 | $\frac{3}{8} \cdot \frac{8}{4} = \frac{3}{4}$ | |
| 3 | 0 | 0 | |
| | | | $\frac{4}{8} \cdot \frac{8}{4} = 1$ |

This is the new Joint PMF describing the probabilities for $X, Y$ given the info $X \geq 1$ and $Y = 1$

given $X \geq 1$ and $Y = 1$

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | $\frac{1}{4}$ |
| 2 | 0 | $\frac{3}{4}$ |
| 3 | 0 | 0 |

Now find

$$P(X=2 \mid X\geq 1 \text{ and } Y=1)$$

$$E(X \mid X\geq 1 \text{ and } Y=1)$$

$$sd(X \mid X\geq 1 \text{ and } Y=1)$$

as you would before

given $X\geq 1$ and $Y=1$

| $x \backslash y$ | 0 | 1 | $P_X(x \mid X\geq 1 \,\&\, Y=1)$ | $x P_X(x \mid \ldots)$ | $x^2 P_X(x \mid \ldots)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| 2 | 0 | $\frac{3}{4}$ | $\frac{3}{4}$ | $2\cdot\frac{3}{4}$ | $2^2\cdot\frac{3}{4}$ |
| 3 | 0 | 0 | 0 | 0 | 0 |
| | | | 1 | $\frac{7}{4}$ | $\frac{13}{4}$ |

$$P(X=2 \mid X\geq 1 \text{ and } Y=1) = \frac{3}{4}$$

$$E(X \mid X\geq 1 \text{ and } Y=1) = \frac{7}{4} = 1.75$$

$$var(X \mid X\geq 1 \text{ and } Y=1) = \frac{13}{4} - \left(\frac{7}{4}\right)^2$$

$$= 0.1875$$

$$sd(X \mid X\geq 1 \text{ and } Y=1) = \sqrt{0.1875}$$

$$= 0.433$$

$\therefore$ Given $X\geq 1$ and $Y=1$ we predict $X$ will be about

$$1.75 \pm 0.433.$$

e.g. Suppose $X, Y$ have the following joint PMF:

| $x \backslash y$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | .1 | 0 | .2 |
| 2 | .1 | 0 | .2 |
| 3 | 0 | .1 | .3 |
| | | | 1 |

Find $E(X \mid Y\geq 2)$.

Modified $p(x,y)$ given $Y\geq 2$

| $x \backslash y$ | 1 | 2 | 3 | |
|---|---|---|---|---|
| 1 | | 0 | $\frac{.2}{.8}$ | $\frac{2}{8} = \frac{1}{4}$ |
| 2 | | 0 | $\frac{.2}{.8}$ | $\frac{2}{8} = \frac{1}{4}$ |
| 3 | | $\frac{.1}{.8}$ | $\frac{.3}{.8}$ | $\frac{4}{8} = \frac{1}{2}$ |
| | | | | $.8/.8$ |

$\therefore$

| $x$ | $P(x \mid Y\geq 2)$ |
|---|---|
| 1 | $1/4$ |
| 2 | $1/4$ |
| 3 | $1/2$ |
| | 1 |

$$\therefore E(X \mid Y\geq 2) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{2}$$

$$= \frac{9}{4}.$$

# Independent r.v.s

The informal definition of independence for two r.v.s is that the PMF for one doesn't change when given information about the other.

Here is an example of a pair of r.v.s $X, Y$ s.t. knowing the value of $Y$ does not change the PMF of $X$:

| $X \backslash Y$ | 0 | 1 | $P_X(x)$ |
|---|---|---|---|
| 0 | .06 | .04 | 0.1 |
| 1 | .54 | .36 | 0.9 |
| | | | 1 |

given $Y=0$

| $X \backslash Y$ | 0 | 1 | $P_X(x \mid Y=0)$ |
|---|---|---|---|
| 0 | $\frac{.06}{0.6}$ | 0 | 0.1 |
| 1 | $\frac{.54}{0.6}$ | 0 | 0.9 |
| | | | $\frac{0.6}{0.6}$ |

given $Y=1$

| $X \backslash Y$ | 0 | 1 | $P_X(x \mid Y=1)$ |
|---|---|---|---|
| 0 | 0 | $\frac{.04}{0.4}$ | 0.1 |
| 1 | 0 | $\frac{.36}{0.4}$ | 0.9 |
| | | | $0.4/0.4$ |

Since $P_X(x) = P_X(x \mid Y=0) = P_X(x \mid Y=1)$ the randomness in $X$ is unaffected by $Y \Rightarrow X$ & $Y$ are independent.

But there is an easier way to check if $X$ and $Y$ are independent:

Fact:

Two r.v.s $X$ and $Y$ are independent whenever

$$\underbrace{P(x,y)}_{\text{Joint PMF for } X \,\&\, Y} = P_X(x) \cdot P_Y(y)$$

marginal PMFs for $X$ and $Y$

But since the marginal PMFs are given by the row & column sums of the joint this means:

X and Y are independent whenever each entry of the Joint PMF equals the product of the corresponding row and column sum.

Note: If $X$ and $Y$ are not independent then they are said to be dependent

Example: $X$ and $Y$, with the following Joint PMF, are independent since

| $X \backslash Y$ | 0 | 1 | $P_X(x)$ |
|---|---|---|---|
| 0 | .06 | .04 | .1 |
| 1 | .54 | .36 | .9 |
| $P_Y(y)$ | .6 | .4 | 1 |

| | | | |
|---|---|---|---|
| | .6(.1) | .4(.1) | .1 |
| | .6(.9) | .4(.9) | .9 |
| | .6 | .4 | |

The entries are the same $\Rightarrow X$ & $Y$ are indep.

Example: Are $X$ and $Y$ independent with the following PMF?

| X \ Y | 0 | 1 | |
|---|---|---|---|
| 0 | .2 | .3 | |
| 1 | .1 | .4 | |
| | | | 1 |

No $X$ and $Y$ are dependent since

$(0.3)(0.5) \neq 0.2$

| X \ Y | 0 | 1 | |
|---|---|---|---|
| 0 | 0.2 | 0.3 | 0.5 |
| 1 | 0.1 | 0.4 | |
| | 0.3 | | 1 |

Example:

Roll 2 die, one red one blue, and let $X$ be the number showing on red and $Y$ be the number showing on blue.

Here is the Joint PMF for $X$ and $Y$:

(Notice $X$ & $Y$ are indep)

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | | | | | | | 1 |

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | $P_X(x)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 2 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 3 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 4 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 5 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| 6 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/6 |
| | | | | | | | 1 |

$$P(X \geq 3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$E(X) = \sum_{n=1}^{6} n\left(\frac{1}{6}\right) = 3.5$$

$$E(X^2) = \sum_{n=1}^{6} n^2\left(\frac{1}{6}\right) = 15.1\overline{6}$$

$$\mathrm{var}(X) = 15.1\overline{6} - 3.5^2 = 2.91\overline{6}$$

Now suppose we are given that $X+Y=7$ or $11$ but not the exact values of $X$ or $Y$.

The modified Joint then becomes

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | $P_X(x \mid X+Y=7 \text{ or } 11)$ |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | $\frac{1}{36}\cdot\frac{36}{8}$ | $\frac{1}{8}$ |
| 2 | | | | | $\frac{1}{36}\cdot\frac{36}{8}$ | | $\frac{1}{8}$ |
| 3 | | | | $\frac{1}{36}\cdot\frac{36}{8}$ | | | $\frac{1}{8}$ |
| 4 | | | $\frac{1}{36}\cdot\frac{36}{8}$ | | | | $\frac{1}{8}$ |
| 5 | | $\frac{1}{36}\cdot\frac{36}{8}$ | | | | $\frac{36}{8}\cdot\frac{1}{36}$ | $\frac{2}{8} = \frac{1}{4}$ |
| 6 | $\frac{1}{36}\cdot\frac{36}{8}$ | | | | $\frac{36}{8}\cdot\frac{1}{36}$ | | $\frac{2}{8} = \frac{1}{4}$ |
| | | | | | | | $\frac{8}{36}\cdot\frac{36}{8} = 1$ |

New PMF for $X$ given $X+Y=7$ or $11$

| $x$ | $p(x)$ | $x\,p(x)$ | $x^2 p(x)$ |
|---|---|---|---|
| 1 | 1/8 | 1/8 | 1/8 |
| 2 | 1/8 | 2/8 | $2^2/8$ |
| 3 | 1/8 | 3/8 | $3^2/8$ |
| 4 | 1/8 | 4/8 | $4^2/8$ |
| 5 | 1/4 | 5/4 | $5^2/4$ |
| 6 | 1/4 | 6/4 | $6^2/4$ |
| | 1 | 4 | 19 |

$$P(X \geq 3 \mid X+Y=7 \text{ or } 11) = \frac{1}{8}+\frac{1}{8}+\frac{1}{4}+\frac{1}{4}$$
$$= \frac{1}{4}+\frac{1}{2} = 0.75$$

was 2/3

$E(X \mid X+Y=7 \text{ or } 11) = 4 \quad \leftarrow$ was 3.5

$\mathrm{var}(X \mid X+Y=7 \text{ or } 11) = 19-4^2 = 3 \quad \leftarrow$ was $2.91\overline{7}$

## Computing $E f(X,Y)$

Recall our old shortcut for computing $E(f(X))$:

$$E f(X) = \sum_{\text{all } x} f(x)\, p_X(x)$$

which is much faster than finding the new PMF for $f(X)$, call it $p_{f(X)}(y)$, and using

$$E f(X) = \sum_{\text{all } y} y\, p_{f(X)}(y).$$

A similar shortcut works with a joint PMF $p(x,y)$ for two r.v.s $X$ and $Y$

$$E f(X,Y) = \sum_{\substack{\text{all pairs}\\ x,y}} f(x,y)\, p(x,y).$$

Example:

Let $(X,Y)$ have joint PMF

| $x \backslash y$ | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | 1/8 | |
| 2 | | | | | 1/8 | | |
| 3 | | | | 1/8 | | | |
| 4 | | | 1/8 | | | | |
| 5 | | 1/8 | | | | 1/8 | |
| 6 | 1/8 | | | | 1/8 | | |
| | | | | | | | 1 |

Find $E(X+Y)$, $E(XY)$, $\mathrm{var}(XY)$.

Using our shortcut

$E(X+Y) = E(f(X,Y))$, where $f(x,y) = x+y$

$= \text{sum } f(x,y)\,p(x,y) \text{ over } x,y$

$= \text{sum } (x+y)\,p(x,y) \text{ over } x,y.$

$= (6+1)\frac{1}{8} + (5+2)\frac{1}{8} + \cdots + (1+6)\frac{1}{8}$

$\quad + (5+6)\frac{1}{8} + (6+5)\frac{1}{8}$

$= 6 \cdot 7 \cdot \frac{1}{8} + 2 \cdot 11 \cdot \frac{1}{8}$

$= 8$

$E(XY) = E(f(X,Y)), \quad f(x,y) = xy$

$= 6 \cdot 1 \cdot \frac{1}{8} + 5 \cdot 2 \cdot \frac{1}{8} + \cdots + 1 \cdot 6 \cdot \frac{1}{8}$

$\quad + 5 \cdot 6 \frac{1}{8} + 6 \cdot 5 \frac{1}{8}$

$= 14.5$

$\text{var}(XY) = E((XY)^2) - (E(XY))^2$

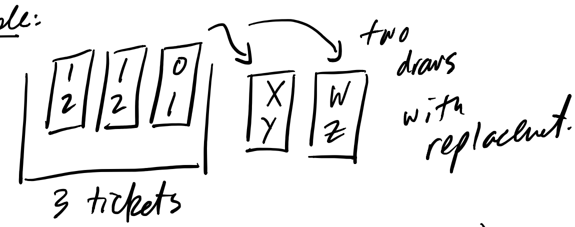
$= E(X^2Y^2) - (14.5)^2$

$= 6^2 \cdot 1^2 \cdot \frac{1}{8} + 5^2 \cdot 2^2 \cdot \frac{1}{8} + \cdots + 1^2 6^2 \frac{1}{8}$

$\quad + 5^2 6^2 \frac{1}{8} + 6^2 5^2 \frac{1}{8} - 14.5^2$

$= 295 - 14.5^2 = 84.75$

Example:




Find $E(XY)$ & $E(XZ)$

Here is the joint PMF $p_{X,Y}(x,y)$

| $y \backslash x$ | 1 | 0 | $p_Y(y)$ |
|---|---|---|---|
| 1 | 0 | 1/3 | 1/3 |
| 2 | 2/3 | 0 | 2/3 |
| $p_X(x)$ | 2/3 | 1/3 | 1 |

$\therefore E(XY) = 1 \cdot 2 \cdot \frac{2}{3} + 0 \cdot 1 \cdot \frac{1}{3} = 1$

For $E(XZ)$ we need $p_{X,Z}(x,z)$

Since $X$ & $Z$ are indep

$p_{X,Z}(x,z) = p_X(x)\,p_Z(z)$

$= p_X(x)\,\underbrace{p_Y(z)}$

since $Z$ & $Y$ are identical in terms of randomness.

$\therefore E(XZ) = 1 \cdot 1 \cdot p_X(1) \cdot p_Y(1)$

$\quad + 1 \cdot 2 \cdot p_X(1) \cdot p_Y(2)$

$\quad + 0 \cdot 1 \cdot p_X(0) \cdot p_Y(1)$

$\quad + 0 \cdot 2 \cdot p_X(0) \cdot p_Y(2)$

$= 1 \cdot 1 \cdot \frac{2}{3} \cdot \frac{1}{3} + 1 \cdot 2 \cdot \frac{2}{3} \cdot \frac{2}{3}$

$= \frac{2+8}{9} = \frac{10}{9}$