Lecture 11

①

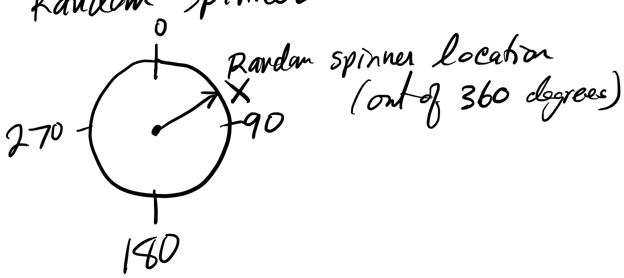
Topics:

Continuous Random variable & probability density functions (PDFs)

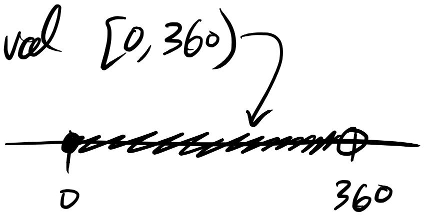
Up to now we have been studying Random variables X whose possible values are finite or discrete (the possible values are well separated "points" on the real number line).

Now we study continuous R.V.s whose possible values lie within some interval of the real number line

e.g. Random spinner



So X can be any number (to infinite precision) in the interval [0, 360)



The problem is that PMF's don't make sense for X.

②

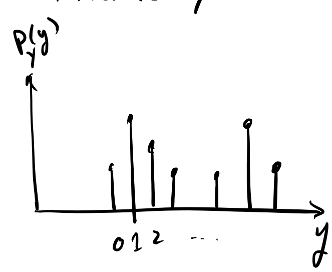
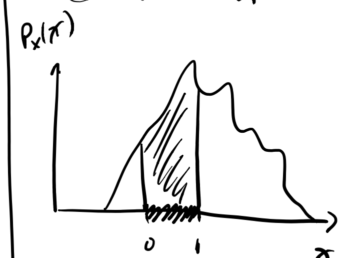
(i.e. sum $p_X(x)$ over all possible x $= \infty$)

To characterize such a R.V. X (i.e. find a blueprint) (which has infinitly precize possible values) we need something called a probability density function $p_X(x)$.

this is also a function like a PMF

The only difference with PMFs is how to compute probability

Differences btwn PMFs & PDFs

Discrete Y	Continuous X
	
$Pr(0 \le Y \le 1) = p_Y(0) + p_Y(1)$	$Pr(0 \le X \le 1) = \int_0^1 p_X(x)dx$
$E(Y) =$ sum $y\, p_Y(y)$ over y	$E(X) = \int_{-\infty}^{\infty} x\, p_X(x)dx$
$E(f(Y)) =$ sum $f(y)\, p_Y(y)$ over y	$E(f(X)) = \int_{-\infty}^{\infty} f(x) p_X(x)dx$

Similarities btwn PMFs & PDFs

The following formulas still hold for both continuous & discrete R.V.s

$$var(X) = E(X^2) - (E(X))^2$$

$$cov(X,Y) = E(XY) - E(X)E(Y)$$

$$var(aX+bY) \overset{MF1}{=} \ldots$$

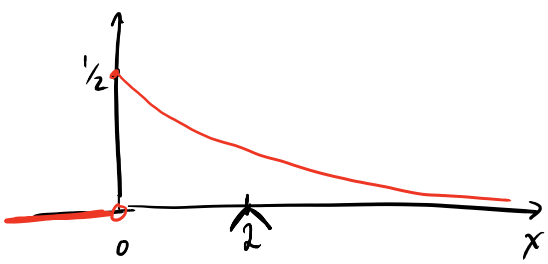
$$E(aX+bY) \overset{MF2}{=} \ldots$$

$$\vdots$$

all the other facts about E, var, cov

e.g. Let X be a continuous R.V. with PDF

$$p_X(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$



Find $P(1 \leq X \leq 2)$, $E(X)$

Note: $p_X(x) \geq 0$

Note:
$$\int_{-\infty}^{\infty} p_X(x)\,dx = \underbrace{\int_{-\infty}^{0} p_X(x)\,dx}_{=0} + \int_{0}^{\infty} \underbrace{p_X(x)}_{\frac{1}{2}e^{-x/2}}\,dx$$

$$= 0 + \int_0^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \left[-e^{-x/2}\right]_0^\infty$$

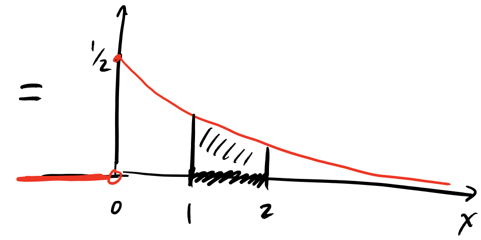
$$= 0 - (-e^{-0/2}) = 1$$

Since $\frac{d}{dx} e^{-x/2} = (-\frac{1}{2}) e^{-x/2}$

all PDF must satisfy

$$p_X(x) \geq 0 \quad \& \quad \int_{-\infty}^{\infty} p_X(x)\,dx = 1$$

$Pr(1 \leq X \leq 2) =$



$$= \int_1^2 \frac{1}{2} e^{-x/2}\,dx$$

$$= \left[-e^{-x/2}\right]_1^2$$

$$= -e^{-2/2} - (-e^{-1/2})$$

$$= -\frac{1}{e} + \frac{1}{\sqrt{e}}$$

$$= 0.2386$$

$$E(X) = \int_{-\infty}^{\infty} x\, p_X(x)\,dx$$

$$= \int_0^\infty x \underbrace{\tfrac{1}{2} e^{-x/2}\,dx}$$

$$\frac{d}{dx}(-e^{-x/2}) = \frac{1}{2} e^{-x/2}$$

$$\therefore d(-e^{-x/2}) = \frac{1}{2} e^{-x/2}\,dx$$

$$= \int_0^\infty x\, \overbrace{d(-e^{-x/2})}$$

$$= x(-e^{-x/2})\Big|_0^\infty - \int_0^\infty (-e^{-x/2})\,dx$$

$$= 0 - 0 + \int_0^\infty e^{-x/2}\,dx$$

$$= \left[-2e^{-x/2}\right]_0^\infty$$

$$= 0 - (-2e^0)$$

$$= 2$$