Lecture 3

Today:

- Using the PMF to get probabilities and the expected value of a r.v.

Example from last lecture

X = the number of flips of a fair coin it takes to get the first tails.

Let $p(x)$ be the PMF for X.

Can write $p(x)$ in many different ways.

Formulas:

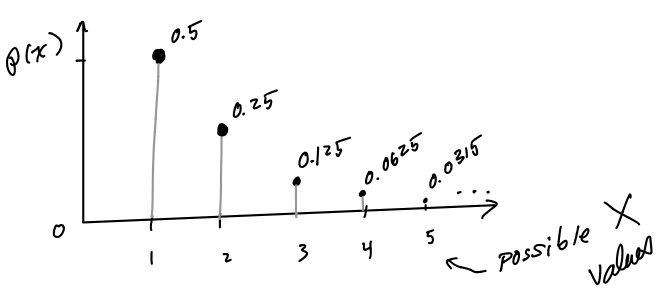
$$p(x) = \begin{cases} (1/2)^x & \text{if } x \text{ is a positive integer} \\ 0 & \text{if } x \text{ is not a positive integer.} \end{cases}$$

Tables:

x	$p(x)$
1	$(1/2)^1$
2	$(1/2)^2$
3	$(1/2)^3$
4	$(1/2)^4$
5	$(1/2)^5$
$\vdots$	

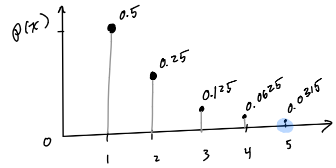
Possible values of X

Plots:



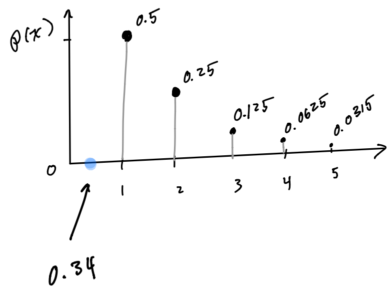
Lets use $p(x)$ to find some probabilities

$$P(X=5) \overset{\text{def}}{=} p(5) = \left(\tfrac{1}{2}\right)^5 = 0.03125$$



x	$p(x)$
1	$(1/2)^1$
2	$(1/2)^2$
3	$(1/2)^3$
4	$(1/2)^4$
5	$(1/2)^5$
$\vdots$	

$$P(X=0.34) \overset{\text{def}}{=} p(0.34) = 0$$

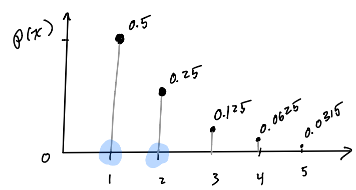


x	$p(x)$
1	$(1/2)^1$
2	$(1/2)^2$
3	$(1/2)^3$
4	$(1/2)^4$
5	$(1/2)^5$
$\vdots$	
0.34	0

$$P(X \le 2) = P(X \text{ in } \{1,2\})$$

$$= p(1) + p(2)$$

$$= \tfrac{1}{2} + \tfrac{1}{4} = 0.75$$



x	$p(x)$
1	$(1/2)^1$
2	$(1/2)^2$
3	$(1/2)^3$
4	$(1/2)^4$
5	$(1/2)^5$
$\vdots$	

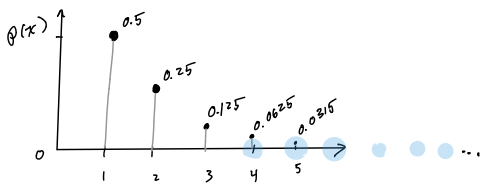
$P(X \geq 4) = P(X \text{ in } \{4,5,6,7\dots\})$

$= P(X \text{ not in } \{1,2,3\})$

Prob Rule $= 1 - P(X \text{ in } \{1,2,3\})$

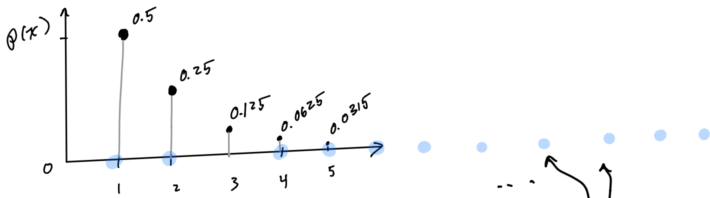
$= 1 - (p(1) + p(2) + p(3))$

$= 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} = 0.125$



x	$p(x)$
1	$(1/2)^1$
2	$(1/2)^2$
3	$(1/2)^3$
4	$(1/2)^4$
5	$(1/2)^5$
⋮	

$P(X \geq 4 \text{ or } X \leq 2)$



If X lands in here you win the bet $(X \geq 4 \text{ or } X \leq 2)$

$\therefore P(X \geq 4 \text{ or } X \leq 2) = 1 - P(X = 3)$

$= 1 - 0.125$

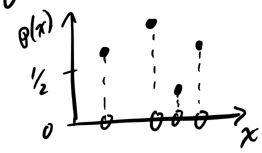
PMF $p_X(x)$ are special.
i.e. Not any function can be a PMF.

PMFs must satisfy

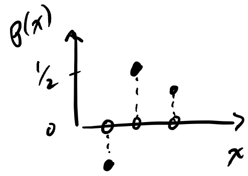
i) sum $p(x)$ over all $x = 1$

ii) $p(x) \geq 0$ for all x.

e.g.



Not a PMF violates i)



Not a PMF violates ii)

x	$p(x)$
π	$\sin^2(22)$
-1000	$\cos^2(22)$

any other x has $p(x)=0$

Yes this $p(x)$ is a PMF of some R.V. X … $p(x)$ satisfies i) & ii)

Expected Value

Imagine a RV X from a single play of a game.

If you play the game multiple times you get a list of RVs

$$X_1, X_2, X_3, \ldots$$

$$\frac{X_1 + \cdots + X_n}{n} = \text{average after } n \text{ plays}$$

There is a number you can find from a PMF which predicts the value of this average for large n, called the expected value of X and denoted $E(X)$.

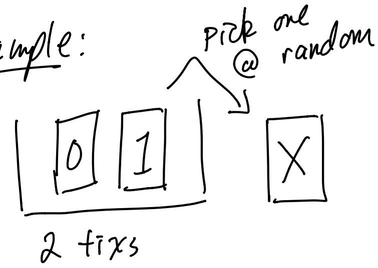
Here is the formula

$E(X) =$ sum $x\,p(x)$ over all x values where $p(x) > 0$.

$$= \sum_x x\,p(x)$$

short hand. possible value times the Prob $X = x$

Example:



Play once, record X, put ticket back,

Play again, ...

$$X_1, X_2, X_3, \ldots$$

PMF

x	$p(x)$
0	0.5
1	0.5

$$E(X) = \sum_x x\,p(x)$$

$$= 0 \cdot p(0) + 1\,p(1)$$

$$= 0 + 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}$$ long run ave.

Think of $E(X)$ as the best prediction of what a future value of X will be.

Example:

Roll a 6-sided die let X be the number showing.

$$p_X(x) = \begin{cases} 1/6 & \text{if } x = 1, 2, \dots, 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad E(X) = 1p_X(1) + 2p_X(2) + 3p_X(3) + \cdots + 6p_X(6)$$

$$= 1\left(\tfrac{1}{6}\right) + 2\left(\tfrac{1}{6}\right) + 3\left(\tfrac{1}{6}\right) + \cdots + 6\left(\tfrac{1}{6}\right)$$

$$= 3.5$$

Example:

Let X be a random draw from the following list:

20, 40, 21, 19, 20, 20.

Find $E(X)$.

x	$p(x)$	$x\,p(x)$
19	$1/6$	$19(1/6)$
20	$1/2$	$20(1/2)$
21	$1/6$	$21(1/6)$
40	$1/6$	$40(1/6)$
		$23.\overline{3}$

This is $E(X)$

same

Note: the list ave is

$$\frac{20+40+21+19+20+20}{6} = 23.\overline{3}$$