Lecture 5

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So far we have defined:

PMF $p_X(x)$, $E(X)$, $E(f(X))$, $sd(X)$, $var(X)$.

all for a single R.V. X.

Today we will discuss properties of $sd(X)$ & $var(X)$

Properties of $sd(X)$ and $var(X)$

Fact 1:

$$var(X) \underset{\substack{\uparrow \\ \text{definition}}}{=} E(X^2) - (E(X))^2 = E(X - E(X))^2$$

To understand what $sd(X)$ & $var(X)$ are telling you about a R.V. X we need to interpret $E(X-E(X))^2$.

Start inside & work out.

$X - E(X)$: This is a R.V. which represents the difference btwn what you predicted X to be (i.e. $E(X)$) & what the actual value of X is.

$(X-E(X))^2$: squared prediction error.

$E(X-E(X))^2$: this is what you expect your squared prediction error is.

$\therefore$ $var(X)$ tells you the "typical squared prediction error for X"

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$\therefore$ $sd(X) = \sqrt{var(X)}$

tells you the "typical prediction error when using $E(X)$ to predict X"

e.g. Suppose X has PMF

x	$p_X(x)$
0	$1/2$
1	$1/2$

i.e. X is 0 or 1 based on a coin flip.

$E(X) = 1/2$ ← best prediction.

The difference btwn X & your prediction will be $-1/2$ or $1/2$ with equal prob.

$\therefore$ typical prediction error should be $1/2$.

Lets check...

$$var(X) = E(X^2) - (E(X))^2$$
$$= \left(0^2 \cdot 1/2 + 1^2 \cdot 1/2\right) - (1/2)^2$$
$$= 1/2 - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

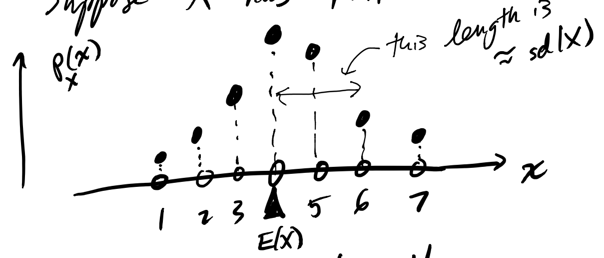
expected squared pred. error.

$$\therefore \; sd(X) = \sqrt{var(X)}$$
$$= \sqrt{1/4} = 1/2$$

Fact 2

sd(X) can be visually approximated as "half the spread of the plot of the PMF."

e.g. Suppose X has PMF



$E(X)$ = the balance point ≈ 4

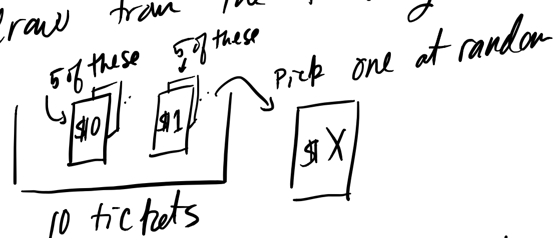
$sd(X)$ = ½ the width $\approx \frac{1}{2}(6-2) = 2$

Fact 3

If X represents the future profit of a financial investment (typically random and can be negative)

Then $E(X)$ quantifies expected profit & $sd(X)$ quantifies risk.

e.g. Suppose the future profit X of some investment can be modeled as a draw from the following box:



Quantify expected profit & $\underbrace{\text{risk}}_{}$.

i.e. how much the actual profit can vary from what you expect.

Use $E(X)$ & $sd(X)$

x	$p_X(x)$
0	½
1	½

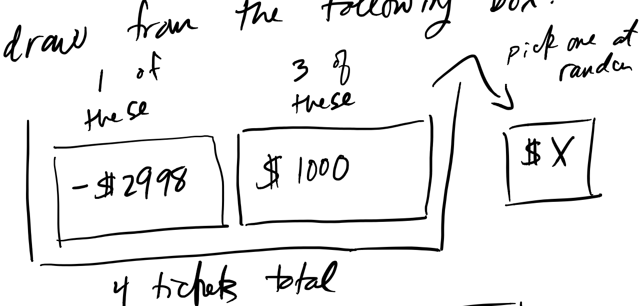
From last e.g.

expected profit $= E(X) = 0\left(\frac{5}{10}\right) + 1\left(\frac{5}{10}\right) = \frac{1}{2}$

Typical difference btwn what you expect & the actual profit $X = sd(X) = \frac{1}{2}$

quantifies risk

e.g. Suppose the future profit X of some investment can be modeled as a draw from the following box:



∴ if you happen to pick $\boxed{-\$2998}$ you loose \$2998 on that investment.

Quantify expected profit & Risk.

x	$p_X(x)$	$x\,p_X(x)$	$x^2 p_X(x)$
-2998	$\frac{1}{4}$	$-2998(\frac{1}{4})$	$2998^2(\frac{1}{4})$
1000	$\frac{3}{4}$	$1000(\frac{3}{4})$	$1000^2(\frac{3}{4})$
		$\frac{2}{4} = \frac{1}{2}$	$2{,}997{,}000$

∴ expected profit $= E(X) = \frac{1}{2}$

(Note: same as last e.g.)

Risk is quantified by

$$sd(X) = \sqrt{var(X)} = \sqrt{E(X^2) - (E(X))^2} = \sqrt{2{,}997{,}000 - \tfrac{1}{4}} = 1731.1$$

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So, for the first investment

expected profit = $\$ \frac{1}{2}$ (i.e. 50 cents)
& the actual profit deviates from
what we expect by $\pm \$ \frac{1}{2}$

For the second investment

expected profit = $\$ \frac{1}{2}$ (i.e. 50 cents)
& the actual profit deviates from
what we expect by $\pm \$ 1731.1$

$\therefore$ second investment has
the same expected profit
but is more risky!!!

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