Lecture 4

Last Lecture:

- using the PMF to find

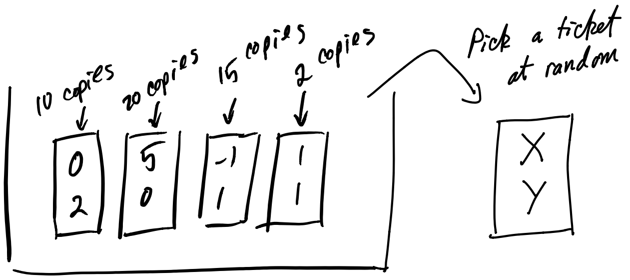
$$E(X) = \text{long run average of repeated draws from } X$$

$$= \text{best prediction of a future draw from } X$$

Today:

- More facts about $E(X)$
- $var(X)$ and $sd(X)$
- $E(f(X))$

Warmup example:



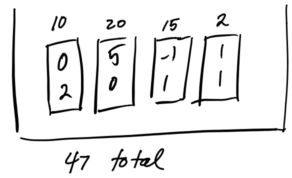
$10+20+15+2=47$ tickets total

- X and Y are both random variables.
- We can define new random variables from X and Y. e.g. Let W be defined by

$$W=(X+Y)Y$$

- Find $E(W)$

The formula:



$$E(W) = \sum w \underbrace{p(w)}_{\text{PMF for } W}$$

To find $p(w)$ start by listing the possible values that W can be

if $\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$ then $W=(X+Y)Y=(0+2)2=4$

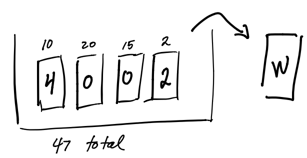
if $\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$ then $W=(X+Y)Y=(5+0)0=0$

if $\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$ then $W=(X+Y)Y=(-1+1)1=0$

if $\begin{bmatrix}X\\Y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ then $W=(X+Y)Y=(1+1)1=2$

Box Model for W

Now find $p(w)$:



w	$p(w)$	$w\,p(w)$
0	$\frac{35}{47}$	0
2	$\frac{2}{47}$	$\frac{4}{47}$
4	$\frac{10}{47}$	$\frac{40}{47}$
	1	$\frac{44}{47}$

This is $E(W)$.

Again notice the coincidence that $E(W)$ is the same as the ave of the tickets in a box model for W:

$$\frac{\overbrace{0+\cdots+0}^{35\text{ terms}} + \overbrace{4+\cdots+4}^{10\text{ terms}} + 2+2}{47} = \frac{10\cdot 4+2\cdot 2}{47}$$

$$= 0\cdot\frac{35}{47} + 4\cdot\frac{10}{47} + 2\cdot\frac{2}{47}$$

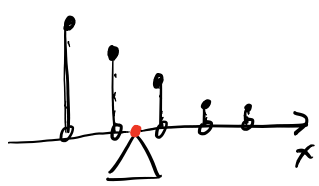
$$= \sum w\, p(w)$$

Facts about E(X)

Fact 1: when you have a Box Model for a R.V. X, then E(X) can be computed by simply averaging the ticket numbers in the Box.

Fact 2: E(X) is a number that can be interpreted as the "best" prediction of a future value of X or the long run average of multiple independent draws of X_i

Fact 3: If $p(x)$ is the PMF for X then E(X) is the point on the x-axis which "balances" the plot of $p(x)$.



● = E(X) = board fulcrum

This fact is useful for visualizing where E(X) is.

Fact 4: If the R.V. X can only be 0 or 1 then E(X) is just the probability X is 1.

x	$P_X(x)$	$x P_X(x)$
0	$1-p$	0
1	p	p
		$p = E(X)$.

Var(X) and sd(X)

Definition: Let X be a random variable. Then

$$var(X) = E(X^2) - (E(X))^2$$

$$sd(X) = \sqrt{var(X)}$$

Note: to find $E(X^2)$ you technically need to find the PMF for X^2. However there is an easy short cut.

$$E(X^2) = \sum_x x^2 \overbrace{p(x)}^{\text{The PMF for } X \text{ not } X^2}$$

Note the difference with

No square

$$E(X) = \sum x\, p(x)$$

In fact to compute $E(f(X))$

$f(x)$ is a function

The argument of f is random

You can always use the following shortcut.

$$E(f(X)) = \left(\begin{matrix}\text{sum } f(x)\, p(x) \text{ over} \\ \text{all possible } x \text{ values for } X\end{matrix}\right) = \sum_x f(x) \underbrace{p(x)}_{\text{PMF for } X \text{ not } f(X)}$$

i.e.

$$E(\cos(X)) = \left(\begin{matrix}\text{sum } \cos(x)\, p(x) \text{ over} \\ \text{all possible } x \text{ values for } X\end{matrix}\right)$$

$$E(\log(X)) = \left(\begin{matrix}\text{sum } \log(x)\, p(x) \text{ over} \\ \text{all possible } x \text{ values for } X\end{matrix}\right)$$

$$E(e^{-X^2}) = \left(\begin{matrix}\text{sum } e^{-x^2}\, p(x) \text{ over} \\ \text{all possible } x \text{ values for } X\end{matrix}\right)$$

Facts about var(X) & sd(X)

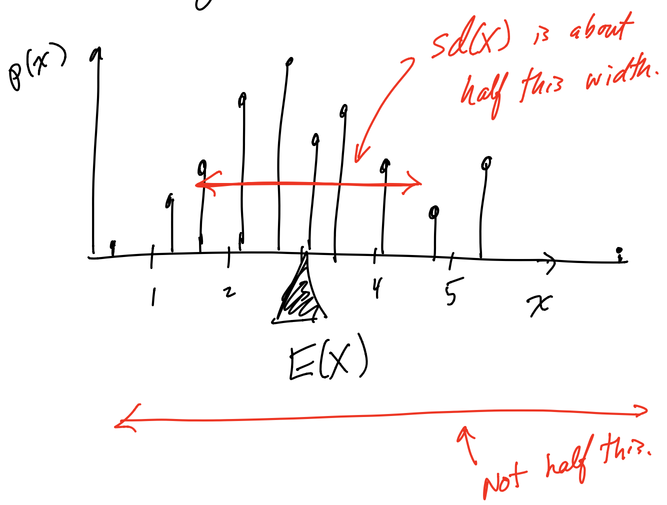
Fact 1

Recall $E(X)$ can be interpreted as the "best" prediction of a future value of X.

sd(X) gives a good guess as to how far off that prediction will be.

Fact 2:

sd(X) can be visualized as about ½ of the spread of main part of a plot of the PMF



Fact 3

There are actually two ways to compute var(X)... both give the same answer

$$\text{var}(X) = E(X^2) - (E(X))^2 \quad \checkmark \text{ easy to compute}$$

$$\text{var}(X) = E\left[(X - E(X))^2\right] \quad \checkmark \text{ easy to understand.}$$

To understand the second formula notice:

- $X - E(X)$: This is a R.V. which represents the difference btwn what you predicted X to be (i.e. $E(X)$) & what the actual value of X is.
- $(X - E(X))^2$: squared prediction error.
- $E(X - E(X))^2$: this is what you expect your squared prediction error is.

∴ var(X) tells you the "typical squared prediction error for X"

& sd(X) tells you the typical prediction error for X

Some Practice

Example: Suppose X has PMF

x	$p_X(x)$
0	$1/2$
1	$1/2$

i.e. X is 0 or 1 based on a coin flip.

$E(X) = 1/2$ ← best prediction.

The difference b/wn X & your prediction will be $-1/2$ or $1/2$ with equal prob.

∴ typical prediction error should be $1/2$.

Let's check...

$$\text{var}(X) = E(X^2) - (E(X))^2$$

$$= (0^2 \cdot 1/2 + 1^2 \cdot 1/2) - (1/2)^2$$

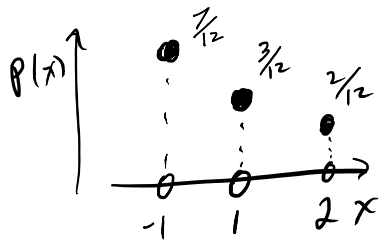
$$= 1/2 - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

expected squared pred. error.

$$sd(X) = \sqrt{\text{var}(X)}$$

$$= \sqrt{1/4} = 1/2$$

Example: Let X have PMF



Find $E(X)$ & $sd(X)$

x	$p(x)$	$xp(x)$	$x^2p(x)$
-1	$7/12$	$-7/12$	$7/12$
1	$3/12$	$3/12$	$3/12$
2	$2/12$	$4/12$	$8/12$
		0	$\frac{18}{12} = \frac{3}{2}$

$$\therefore E(X) = 0$$

$$E(X^2) = \frac{3}{2}$$

$$\text{var}(X) = \frac{3}{2} - 0^2 = \frac{3}{2}$$

$$sd(X) = \sqrt{\frac{3}{2}}$$

Example

Your friend is going to repeatedly roll a 6-sided die and gives you 2 options

Option 1:

For each roll you pay him \$1 to play the game.

Then if the roll is ⚀ or ⚅ he gives you \$4 (so total gain would be \$3 since you had to pay \$1 to get into the game).

In r.v. notation

X = amount you win on a single play

$$= \begin{cases} \$3 \leftarrow \text{happens with probability } \frac{2}{6}=\frac{1}{3} \\ -\$1 \leftarrow \text{happens with probability } \frac{2}{3} \end{cases}$$

Option 2:

Similar to option 1 but now you pay \$50 to play for a chance to win \$151 on ⚀ or ⚅

In r.v. notation

Y = amount you win on a single play

$$= \begin{cases} \$101 & \text{with prob } \frac{1}{3} \\ -\$50 & \text{with prob } \frac{2}{3} \end{cases}$$

Question 1:

How much do you expect to win on each play of the game for option 1 & 2

i.e. if you play for a long time what is the value of $\frac{\text{total \$ gain}}{\text{\# of plays}}$

i.e. find $E(X)$, $E(Y)$

Question 2:

Compute $var(X)$, $var(Y)$
$sd(Y)$, $sd(Y)$.

and use them to decided which option is preferred.

For option 1:

$$E(X) = \$3\left(\frac{1}{3}\right) + (-\$1)\left(\frac{2}{3}\right)$$

$$= 1 - \frac{2}{3} = \frac{1}{3} \approx 33 \text{ cents.}$$

$$E(X^2) = \$3^2\left(\frac{1}{3}\right) + (-\$1)^2\left(\frac{2}{3}\right)$$

$$= \frac{9}{3} + \frac{2}{3} = \frac{11}{3}$$

$$\therefore var(X) = E(X^2) - (E(X))^2$$

$$= \frac{11}{3} - \left(\frac{1}{3}\right)^2 = \frac{3 \cdot 11 - 1}{9} = \frac{32}{9}$$

$$sd(X) = \sqrt{\frac{32}{9}} \approx 1.89$$

For option 2

$$E(Y) = (\$101)\left(\frac{1}{3}\right) + (-50)\left(\frac{2}{3}\right)$$

$$= \frac{101 - 50 \cdot 2}{3} = \frac{1}{3} \approx 33 \text{ cents.}$$

Expected winnings are the same

$$E(Y^2) = (101)^2 \frac{1}{3} + (-50)^2\left(\frac{2}{3}\right)$$

$$= 5067$$

$$\therefore \text{var}(Y) = E(Y^2) - (E(Y))^2$$

$$= 5067 - \left(\frac{1}{3}\right)^2$$

$$= 5066.89$$

$$sd(Y) = \sqrt{5066.89} = 71.18$$

We can interpret these numbers as follows:

Expect X to be $\$.33$ $\pm \$1.89$

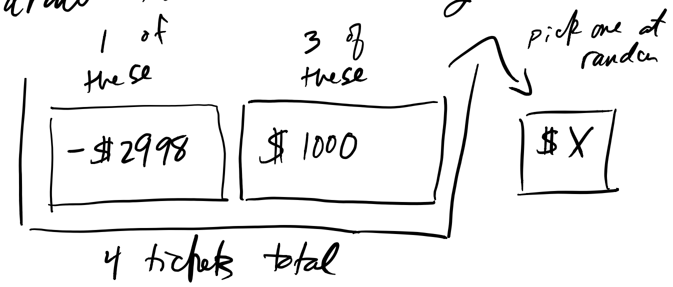
Expect Y to be $\underbrace{\$0.33}_{\text{expected winnings on each play}}$ $\underbrace{\pm \$71.18}_{\text{quantifies how much } Y \text{ typically deviates from } E(Y)}$

Expected winnings are the same but X is less variable/risky/volatile so its preferred over Y.

Variability/risk/volatility can be quantified with $\underbrace{\text{var}(X)}_{\text{called the variance of } X}$ or $\underbrace{sd(X)}_{\text{called the standard deviation of } X.}$

Example:

Suppose the future profit X of some investment can be modeled as a draw from the following box:



if you happen to pick $\boxed{-\$2998}$ you loose \$2998 on that investment.

Quantify expected profit & Risk.

x	$P_X(x)$	$x P_X(x)$	$x^2 P_X(x)$
-2998	$\frac{1}{4}$	$-2998(1/4)$	$2998^2(1/4)$
1000	$3/4$	$1000(3/4)$	$1000^2(\frac{3}{4})$
		$\frac{2}{4}=\frac{1}{2}$	$2,997,000$

$\therefore$ expected profit $= E(X) = 1/2$

Risk is quantified by

$$sd(X) = \sqrt{var(X)} = \sqrt{E(X^2) - (E(X))^2}$$
$$= \sqrt{2,997,000 - 1/4} = 1731.1$$

$\therefore$ expected profit $= \$1/2$ (i.e. 50 cents) & the actual profit deviates from what we expect by $\pm \$1731.1$