# Lecture 6



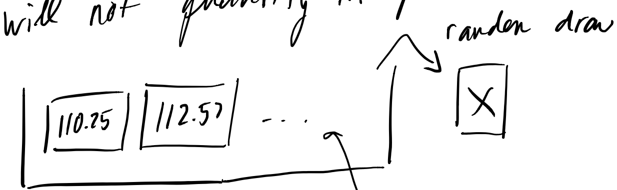
Today: Dependence btwn R.V.s

Start with an example.

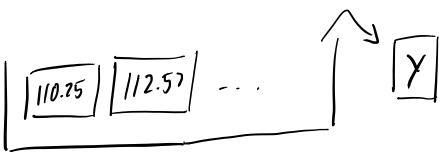
X = apple stock price (per share) today at close

Y = apple stock price tomorrow at close.

Seperate box models for X & Y will not quantify independence

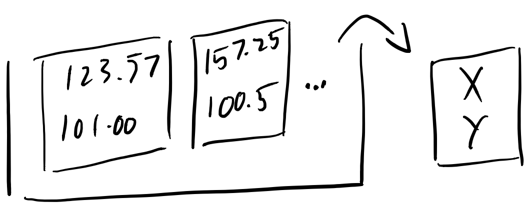



ticket #s represent possibilities for X



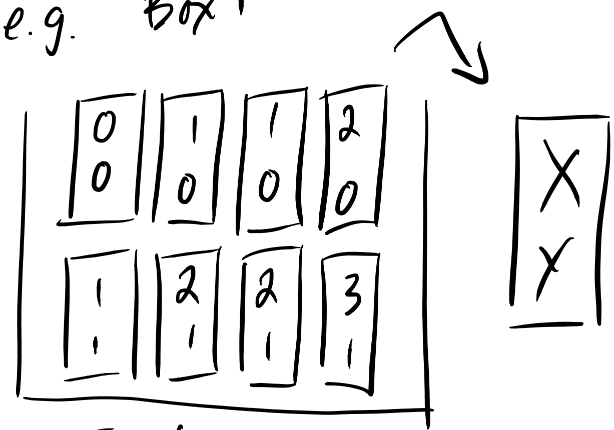

These X & Y R.V.s are not dependent so this doesn't work.

To quantify dependence btwn two R.V.s X & Y you need a single Box Model with two numbers written on each ticket






e.g. Box 1




8 tixs

$$P(Y=0) = \frac{1}{2}$$

$$P(Y=0 \mid X=2) = \frac{1}{3}$$

Find the marginal PMFs for X & Y

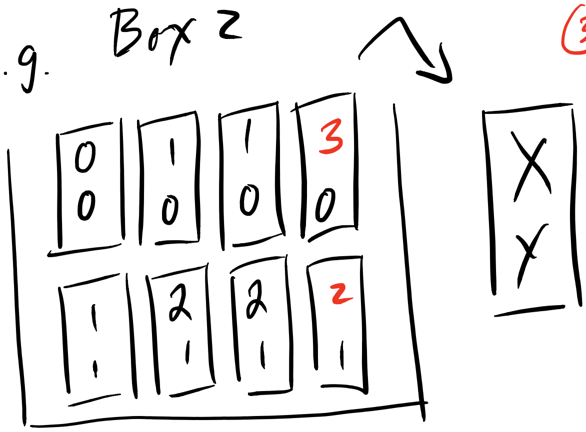
| $x$ | $P_X(x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

Marginal PMF for X

| $y$ | $P_Y(y)$ |
|---|---|
| 0 | 1/2 |
| 1 | 1/2 |

Marginal PMF for Y

e.g. Box 2




8 tixs

Note: the marginal PMFs are the same for Box 1 & Box 2.

Box 1: $P(Y=0 \mid X=2) = 1/3$

Box 2: $P(Y=0 \mid X=2) = 0$

To characterize a pair of R.V.s $(X, Y)$ use joint PMF:

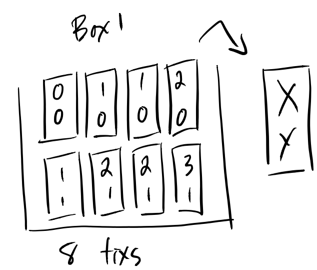
$p(x,y) =$ "the probability that $X = x$ & $Y = y$ simultaneously"

$= P(X=x \;\&\; Y=y)$

If I want to be explicit as to what pair of R.V.s $X, Y$ the joint PMF $p(x,y)$ is characterizing, will write $p(x,y)$ as $p_{X,Y}(x,y)$.

e.g. Lets see that the joint PMF can distinguish the previous two Box Models. (which have different dependence but the same marginal PMFs $p_X(x)$ $p_Y(y)$).

Joint PMF for Box 1 in table format:




$p(x,y)$ for Box 1

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 1/8 | 2/8 |
| 3 | 0 | 1/8 |

1 ← all entries should sum to 1

The $x,y$ entry is $p(x,y)$

$p(0,0) = P(X=0 \;\&\; Y=0)$

$= 1/8$

$p(1,0) = P(X=1 \;\&\; Y=0)$

$= \frac{2}{8}$

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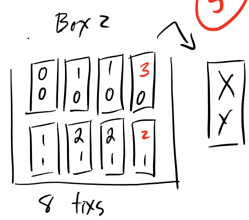


## $p(x,y)$ for Box 2

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

So the joint PMF can tell the diffence btwn Box 1 & 2.

The joint PMF $p(x,y)$ can be used to calculate any probability involving X & Y.

Before: To find the probability of some event involving X just add $P_X(x)$ over all $x$ in the event.

New: To find the probability of some event involving X & Y just add $p(x,y)$ over all pairs $x,y$ in the event.

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e.g. Box 2 from last e.g.

$p(x,y)$ for Box 2

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

$$P(X=2 \text{ and } Y=1) = p(2,1) = 3/8$$

$p(x,y)$ for Box 2

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

$$P(X=1) = P(1,0) + P(1,1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

$p(x,y)$ for Box 2

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

$$P(Y=0) = P(0,0) + P(1,0) + P(2,0) + P(3,0) = \frac{1}{8} + \frac{2}{8} + 0 + \frac{1}{8} = \frac{1}{2}$$

$p(x,y)$ for Box 2

| $x \backslash y$ | 0 | 1 |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 2/8 | 1/8 |
| 2 | 0 | 3/8 |
| 3 | 1/8 | 0 |

1

$$P(X=1 \text{ or } Y=1) = 0 + \frac{2}{8} + \frac{1}{8} + \frac{3}{8} + 0 = \frac{6}{8} = \frac{3}{4}$$

Note:

$P(X=x) = P_X(x) =$ row sum Joint PMF @ row label $x$

$P(Y=y) = P_Y(y) =$ column sum Joint PMF @ column label $y$.

marginal PMFs from Joint PMF

e.g. Find the marginal PMFs $P_X(x)$ & $P_Y(y)$ from the joint PMF for $X, Y$ & find $E(X), E(Y)$

| $X \backslash Y$ | 7 | 2 | 3 | |
|---|---|---|---|---|
| 1 | 0 | 1/9 | 2/9 | 3/9 |
| 5 | 0 | 3/9 | 0 | 3/9 |
| 7 | 2/9 | 0 | 1/9 | 3/9 |
| | 2/9 | 4/9 | 3/9 | 1 |

gives $P_X(x)$

gives $P_Y(y)$

∴

| $x$ | $P_X(x)$ | $x\,P_X(x)$ |
|---|---|---|
| 1 | 1/3 | 1/3 |
| 5 | 1/3 | 5/3 |
| 7 | 1/3 | 7/3 |
| | | 13/3 |

13/3 ← $E(X)$

| $y$ | $P_Y(y)$ | $y\,P_Y(y)$ |
|---|---|---|
| 7 | 2/9 | 14/9 |
| 2 | 4/9 | 8/9 |
| 3 | 3/9 | 9/9 |
| | | 31/9 |

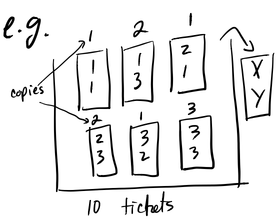
31/9 ← $E(Y)$

## Conditional Probability with Joint PMFs

Conditional prob is easy with Box Models & Joint PMF tables

- Box Models: Remove tickets
- Joint PMF table: set entries to zero & Re-normalize

e.g.

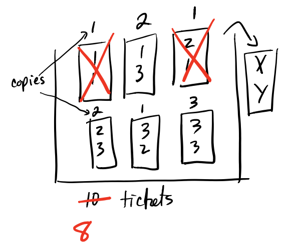



| $X \backslash Y$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1/10 | 0 | 2/10 |
| 2 | 1/10 | 0 | 2/10 |
| 3 | 0 | 1/10 | 3/10 |
| | | | 1 |

Find $P(X=2 \text{ and } Y=3 \mid \underbrace{Y \geq 2}_{\text{new information about } X,Y \text{ we are given}})$



Given $Y \geq 2$ we modify as follows




| $X \backslash Y$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\cancel{\frac{1}{10}}$ 0 | 0 | $\frac{2}{10}\cdot\frac{10}{8}$ |
| 2 | $\cancel{\frac{1}{10}}$ 0 | 0 | $\frac{2}{10}\cdot\frac{10}{8}$ |
| 3 | 0 | $\frac{1}{10}\cdot\frac{10}{8}$ | $\frac{3}{10}\cdot\frac{10}{8}$ |
| | | | ~~1~~ ~~8/10~~ 1 |

Now

$P(X=2 \text{ and } Y=3 \mid Y \geq 2)$ from old table

$= P(X=2 \text{ and } Y=3)$ from modified table

$= \begin{cases} \frac{2}{8} & \text{from Modified box} \\ \frac{2}{10}\cdot\frac{10}{8} & \text{from Modified table} \end{cases}$

## Conditional expected value sd & var

e.g.

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | .1 | .5 |
| 1 | .3 | .2 |
| | | 1 |

Find $\underbrace{E(X \mid Y=1)}$

the best prediction for $X$ after given the information that $Y=1$.