Lecture 7

Today:
- Modifying PMFs & joint PMF to account for partial information
- Conditional expected value & var.

Modifying PMFs & joint PMFs

e.g. Let X & Y be two R.V.s with joint PMF $p(x,y)$ given in the following table:

x\y	0	1	
0	.1	.5	.6
1	.2	.2	.4
			1

← Row sums

Find $E(X)$, $P(X=1)$ & $P(X=1 \mid X \neq Y)$.

For $E(X)$ & $P(X=1)$ we need the marginal PMF for X, i.e. $P_X(x)$. Summing the Rows gives

x	$P_X(x)$	$xP_X(x)$
0	.6	0
1	.4	.4
	1	.4

$\therefore E(X) = 0.4$ & $P(X=1) = 0.4$

For $P(X=1 \mid X \neq Y)$ we modify $p(x,y)$

original $p(x,y)$ $\xrightarrow{\text{given } X \neq Y}$ modified $p(x,y)$

x\y	0	1	
0	.1	.5	.6
1	.2	.2	.4
			1

$P(X=1) = 0.4$

x\y	0	1	
0	0	.5/.7	
1	.2/.7	0	2/7
			1

$P(X=1 \mid X \neq Y) = 2/7$

A similar modification works for a single R.V. X.

e.g. You take a quiz (10 questions). Let X be your score. The Professor says you got @ least 8 out of 10 correct. Find the prob $X=10$ using the following PMF (computed from your historical performance on these quizzes).

You never got below 1/2 correct

x	$P_X(x)$
5	.1
6	.2
7	.1
8	.1
9	.3
10	.2
	1

$\xrightarrow[X \geq 8]{\text{taking into account}}$

x	$P_X(x)$
5	0
6	0
7	0
8	$.1/.6 = \frac{1}{6}$
9	$.3/.6 = 1/2$
10	$.2/.6 = 1/3$
	$.6/.6 = 1$

$\therefore P(X=10 \mid X \geq 8) = 1/3$

Conditional PMF, expected value, s.d., var

After modifying a joint (or marginal) PMF you can compute expected value, s.d, var like before (but they now are adjusted to account for the extra information.

e.g. In last example, where X = quiz score,

Find $E(X \mid \underbrace{X \geq 8}) =$ best prediction of X after being told $X \geq 8$

Just a Notational Reminder that we are working the Modified PMF

$sd(X \mid X \geq 8) =$ s.d. of X after being told $X \geq 8$.

Let $P_X(x \mid X \geq 8)$ denote the modified PMF which accounts for the info $X \geq 8$.

$\therefore$

x	$P_X(x \mid X \geq 8)$	$x P_X(x \mid X \geq 8)$	$x^2 P_X(x \mid X \geq 8)$
8	$1/6$	$8/6$	$8^2/6$
9	$1/2$	$9/2$	$9^2/2$
10	$1/3$	$10/3$	$10^2/3$
	1	$9.16\bar{6}$	84.5

$$\therefore E(X \mid X \geq 8) = 9.16\bar{6}$$

$$\text{var}(X \mid X \geq 8) = E(X^2 \mid X \geq 8) - E(X \mid X \geq 8)^2$$

$$= 84.5 - (9.16\bar{6})^2$$

$$= 0.472$$

$$sd(X \mid X \geq 8) = \sqrt{0.472} = 0.687$$

$\therefore$ Your predicted score for X after being told $X \geq 8$ is:

$$9.16\bar{6} \pm \underbrace{0.687}_{\text{typical prediction error}}$$

e.g. Suppose X, Y have the following joint PMF:

$x \backslash y$	1	2	3
1	.1	0	.2
2	.1	0	.2
3	0	.1	.3

Total: 1

Find $E(X \mid Y \geq 2)$.

Modified $p(x,y)$ given $Y \geq 2$

$x \backslash y$	1	2	3	
1		0	$\frac{.2}{.8}$	$\frac{2}{8} = \frac{1}{4}$
2		0	$\frac{.2}{.8}$	$\frac{2}{8} = \frac{1}{4}$
3		$\frac{.1}{.8}$	$\frac{.3}{.8}$	$\frac{4}{8} = \frac{1}{2}$
				$.8/.8$

$\therefore$

x	$P(x \mid Y \geq 2)$
1	$1/4$
2	$1/4$
3	$1/2$
	1

$$\therefore E(X \mid Y \geq 2) = 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{4} + 3 \cdot \tfrac{1}{2}$$

$$= \frac{9}{4}.$$

Independent R.V.s

Two R.V.s X, Y are said to be **independent** whenever

$$\underbrace{P_{X,Y}(x,y)}_{\text{joint PMF}} = P_X(x) \cdot P_Y(y)$$

Marginal PMFs for X & Y.

for all possible (x,y) values for the R.V.s (X,Y).

The idea is that when X & Y are independent then obtaining information on one R.V. doesn't change the probabilities for the other.

e.g.

$x \backslash y$	0	1
0	.06	.04
1	.54	.36

Total: 1

x	$P_X(x)$
0	.1
1	.9
	1

y	$P_Y(y)$
0	.6
1	.4
	1

Are X & Y indep?

check if $p(x,y) \overset{(*)}{=} p_X(x)\,p_Y(y)$ (5)

(x,y)	$p(x,y)$	$p_X(x)\,p_Y(y)$	equal?
0, 0	.06	(.1)(.6)	Yes
0, 1	.04	(.1)(.4)	Yes
1, 0	.54	(.9)(.6)	Yes
1, 1	.36	(.9)(.4)	Yes

all these are yes so X & Y are indep.

$\therefore\ E(X \mid Y=0) = E(X)$

$E(X \mid Y=1) = E(X)$

$E(Y \mid X=0) = E(Y)$

$p_X(x \mid Y=1) = p_X(x)$

$\vdots$

e.g. Suppose X & Y have the following joint PMF

$x \backslash y$	0	1
0	.2	.3
1	.1	.4

1

are X & Y indep?

(x,y)	$p(x,y)$	$p_X(x)\,p_Y(y)$	equal?
0, 0	.2	(.5)(.3)	No
			$\vdots$

just one No implys X & Y are Not indep.

More practice (6)

e.g. Roll two die (one Red one Blue).
Let X be the result of the Red die
Let Y be the result of the Blue die.
Find $p_X(x)$, $E(X)$, $sd(X)$, $P(X \geq 3)$
$p_X(x \mid X+Y=7 \text{ or } 11)$, $E(X \mid X+Y=7 \text{ or } 11)$
$sd(X \mid X+Y=7 \text{ or } 11)$, $P(X \geq 3 \mid X+Y=7 \text{ or } 11)$

Joint PMF $p(x,y)$

$X \backslash Y$	1	2	3	4	5	6	$p_X(x)$
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
							1

$E(X) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \cdots + \frac{6}{6} = 3.5$

$var(X) = E(X^2) - (E(X))^2$

$= \frac{(1)^2}{6} + \frac{(2)^2}{6} + \frac{(3)^2}{6} + \cdots + \frac{6^2}{6} - 3.5^2$

$= 2.917$

$sd(X) = \sqrt{2.917} = 1.7$

$Pr(X \geq 3) = \frac{4}{6} = \frac{2}{3}$

Joint PMF $p(x,y)$ update after observing $X+Y=7$ or 11

$X \backslash Y$	1	2	3	4	5	6	$p_X(x \mid X+Y=7 \text{ or } 11)$
1						1/8	1/8
2					1/8		1/8
3				1/8			1/8
4			1/8				1/8
5		1/8				1/8	2/8 = 1/4
6	1/8				1/8		2/8 = 1/4
							$\frac{8}{36}\left(\frac{36}{8}\right)=1$

$\therefore E(X \mid X+Y=7 \text{ or } 11)$

$$= \frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{4}{8}+\frac{5}{4}+\frac{6}{4} = 4 \quad (\text{was } 3.5)$$

$\mathrm{var}(X \mid X+Y=7 \text{ or } 11)$

$$= \frac{(1)^2}{8}+\frac{(2)^2}{8}+\frac{(3)^2}{8}+\frac{(4)^2}{8}+\frac{(5)^2}{4}+\frac{(6)^2}{4} - 4^2$$

$$= 3 \quad (\text{was } 2.917)$$

$$sd(X \mid X+Y=7 \text{ or } 11) = \sqrt{3} = 1.732 \quad (\text{was } 1.7)$$

$$\Pr(X \geq 3 \mid X+Y=7 \text{ or } 11) = \frac{2}{8}+\frac{2}{4} = \frac{3}{4} \quad \left(\text{was } \frac{2}{3}\right)$$

So if X represents Profit.

Marginally: predicted profit is

$$\approx 3.5 \pm 1.7$$

After observing $X+Y=7$ or 11 the predicted profit is

$$\approx 4 \pm 1.732$$