Lecture 9

Today:

I) Master formula 1

II) Covariance ← The missing link for var(X+Y) when X & Y are dependent.

III) Master formula 2.

Master Formula 1

An extremely useful shortcut for computing $E(X+Y)$ and more generally $E(aX+bY)$.

From before:

$$E(X+Y) = \underbrace{\left(\begin{array}{l}\text{sum } (x+y)\,p_{X,Y}(x,y) \\ \text{over all } (x,y)\end{array}\right)}_{\text{but this is a pain}}$$

The following shortcut is much easier.

$$E(X+Y) = E(X) + E(Y)$$

Even more is true

$$E(X-Y) = E(X) - E(Y)$$

$$E(10X + \tfrac{1}{2}Y) = 10E(X) - \tfrac{1}{2}E(Y)$$

$$\vdots$$

Master Formula 1 (MF1):

$$E(aX+bY) = aE(X) + bE(Y)$$

always! These can be any non-random numbers

e.g.

X\Y	0	1	$p_X(x)$
0	1/6	0	1/6
1	2/6	2/6	4/6
2	0	1/6	1/6
$p_Y(y)$	3/6	3/6	1

$$E(X) = \frac{0}{6} + \frac{4}{6} + \frac{2}{6} = 1$$

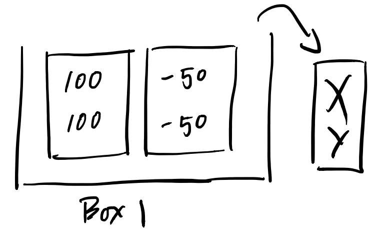
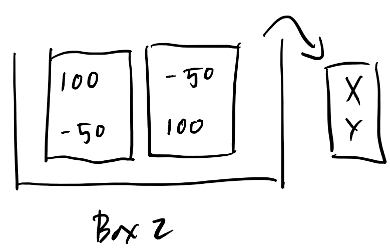
$$E(Y) = \frac{1}{2}$$

By MF1: $E(X+Y) = 1.5$

So if X & Y represent the profit of two investments the expected combined profit (i.e. $E(X+Y)$) isn't effected by dependence.

Question: Is risk or volatility of combined profit (as measured by $sd(X+Y)$) effected by dependence?

Gain some intuition with a box Model

	expected profit $E(X+Y)$	Risk $sd(X+Y)$
	\$50	$sd(X+Y) > 0$
	\$50	\$0

Both boxes have $E(X+Y) = E(X) + E(Y) = \left(\frac{100}{2} - \frac{50}{2}\right) + \left(\frac{100}{2} - \frac{50}{2}\right) = \$50$

So Box 1 is much more risky!

The reason is that the dependency in Box 1 makes X, Y move up or down in unison.

The covariance btwn X & Y, denoted $cov(X,Y)$, quantifies exactly how risk is effected by dependence.

Def:

$$cov(X,Y) = E(XY) - E(X)E(Y)$$

From Box 1 we have

$$E(XY) = 100 \cdot 100 \cdot \frac{1}{2} + (-50)(-50)\frac{1}{2}$$
$$= 6250$$

$$\therefore \quad cov(X,Y) = 6250 - (25)^2$$
$$= 5625$$

From Box 2 we have

$$E(XY) = 100(-50)\frac{1}{2} + 100(-50)\frac{1}{2}$$
$$= -5000$$

$$\therefore \quad cov(X,Y) = -5000 - (25)^2$$
$$= -5625$$

To use $cov(X,Y)$ to see the impact on risk use:

Master Formula 2 (MF2):

$$var(aX+bY) = a^2 var(X) + b^2 var(Y) + 2ab\, cov(X,Y)$$

$$\therefore \quad var(X+Y) = var(X) + var(Y) + 2\,cov(X,Y)$$

small cov ⇒ small risk
large cov ⇒ large risk

From both boxes

$$var(X) = var(Y)$$
$$= E(X^2) - (E(X))^2$$
$$= \frac{100^2}{2} + \frac{(-50)^2}{2} - (25)^2$$
$$= 5625$$

∴ From Box 1:

$$var(X+Y) = var(X) + var(Y) + 2\,cov(X,Y)$$
$$= 2(5625) + 2(5625)$$
$$= 22{,}500$$

$$sd(X+Y) = \$150$$

From Box 2

$$var(X+Y) = 2(5625) + 2(-5625)$$
$$= 0$$

$$sd(X+Y) = \$0.$$

This explains the subprime Mortgage crash & why people invest in Gold.

Mortgage-backed securities:

$$\text{total return} = X_1 + X_2 + \cdots + X_{100,000}$$

Returns on individual Mortgages. each one has a chance of defaulting.

Previously assumed X_i's are nearly independent but bank practices introduced subtle dependence... they can default in unison
$\Longrightarrow$ made total return risky.

Gold is typically a hedging investment i.e. often has negative cov with the market so adding it to a portfolio reduces risk.

e.g. Find $\text{cov}(X,Y)$, $\text{var}(X-Y)$

$X \backslash Y$	0	1	$p_X(x)$
0	1/6	0	1/6
1	2/6	2/6	4/6
2	0	1/6	1/6
$p_Y(y)$	3/6	3/6	1

$$E(X) = \frac{0}{6} + \frac{4}{6} + \frac{2}{6} = 1$$

$$E(Y) = \frac{1}{2}$$

$$E(XY) = 0\cdot 0\cdot \frac{1}{6} + 0\cdot 1\cdot 0 + 1\cdot 0\cdot \frac{2}{6} + 1\cdot 1\cdot \frac{2}{6} + 2\cdot 0\cdot 0 + 2\cdot 1\cdot \frac{1}{6}$$
$$= 4/6$$

$$\text{cov}(X,Y) = \frac{4}{6} - 1\cdot\frac{1}{2} = \frac{1}{6}$$

$$\text{var}(X-Y) = \text{var}(X + bY)$$

$a = 1$
$b = -1$

$$= \text{var}(X) + \text{var}(Y) - 2\,\text{cov}(X,Y)$$

Fact: $\text{cov}(X,X) = E(XX) - E(X)E(X)$
$$= E(X^2) - (E(X))^2$$

$\therefore$ $\text{cov}(X,X) = \text{var}(X)$

Fact: If X & Y are independent then $\text{cov}(X,Y) = 0$.

But not the other way around.

e.g. Let X, Y have joint PMF 7

$x \backslash y$	0	1
-1	0	1/3
0	1/3	0
1	0	1/3

Are X, Y independent? No

Is $cov(X,Y) = 0$?

$$cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= (-1)(1)\frac{1}{3} + 0 + (1)(1)\frac{1}{3} - (0)(2/3)$$

$$= 0.$$

So you can have $cov(X,Y) = 0$ even for dependent R.V.s

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e.g. Allocate a fund of \$100 btwn two investments with future payouts X & Y (per \$1)

Let a = amount allocated to X

b = amount allocated to Y.

if $a=30$ & $b=70$ then your betting \$30 on X & \$70 on Y & Your total payout is $\underbrace{30X + 70Y}_{aX+bY}$

Suppose $sd(X) = 25$

$sd(Y) = 10$

$cov(X,Y) = -175$

$E(X) = E(Y)$

$$sd(aX+bY)$$

$$= \sqrt{a^2 var(X) + b^2 var(Y) + 2ab\, cov(X,Y)}$$

$$= \begin{cases} \$563.47 & \text{if } a=30 \text{ \& } b=70 \\ \$1000 & \text{if } a=0 \text{ \& } b=100 \end{cases}$$