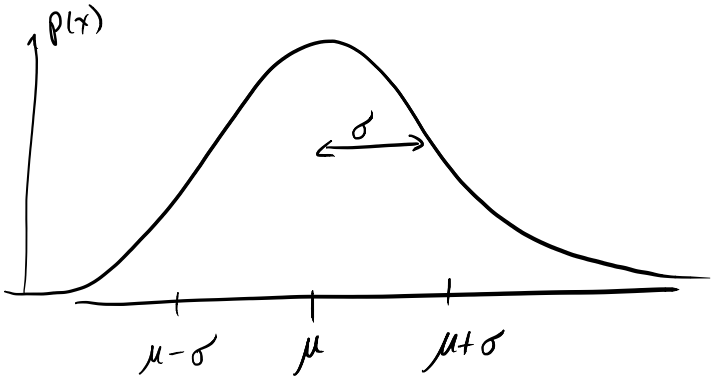
Lecture 12

Topic: Normal R.V.s

The most important R.V. (ever) is called a **Normal (or Gaussian) R.V.** with PDF that has the following form



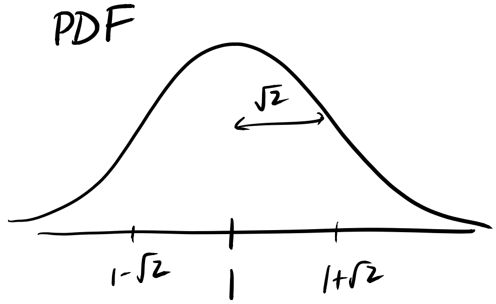
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

If X has the above PDF then we write $\underbrace{X \sim N(\mu, \sigma^2)}$

in words

"X is a Normal R.V. with $E(X) = \mu$, $var(X) = \sigma^2$"

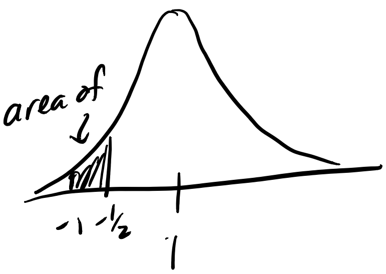
e.g. If $X \sim N(1, 2)$ then $E(X) = 1$, $var(X) = 2$ & X has PDF



Since areas under PDF's correspond to probabilities

$P(0 \le X \le 3) =$ area of

$P(-1 \le X < -\frac{1}{2}) =$ area of



Since $e^{-x^2/2}$ is nearly impossible to integrate by hand you either need to use a computer or a table.

"Table II" from the book (uploaded into Smartsite) gives some areas under the $\underbrace{N(0,1)}_{\text{called}}$ PDF. "standard Normal"

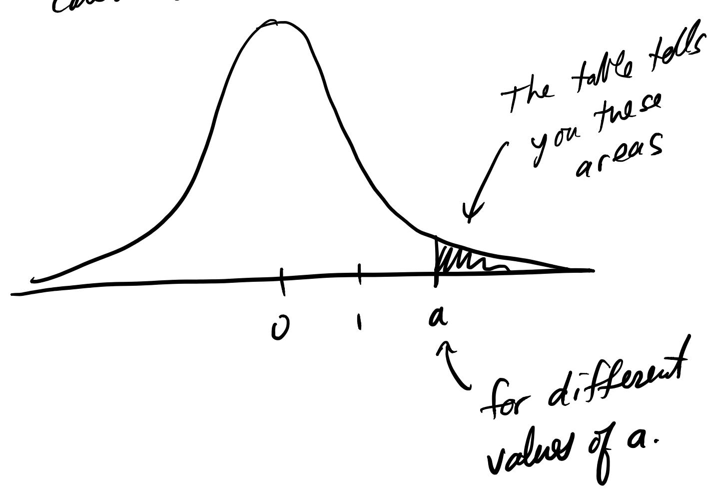
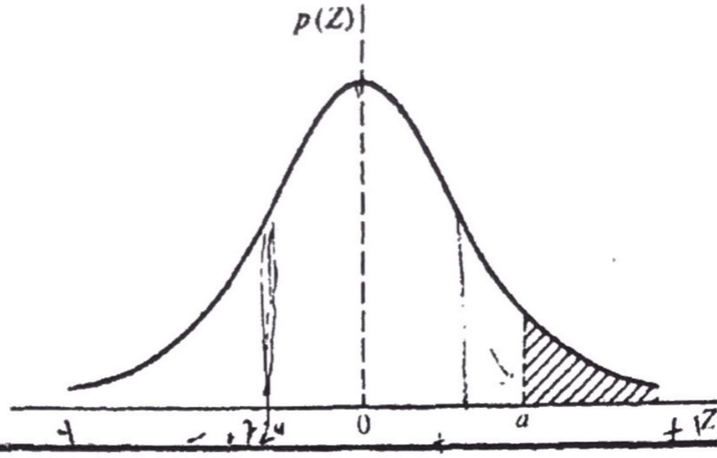


Table II PROBABILITIES FOR THE STANDARD NORMAL DISTRIBUTION
($Pr(Z \geq a)$ when $Z \sim N(0, 1)$)



a	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010

3.50	.0002326
4.00	.0000317
4.50	.0000034
5.00	.000000287

e.g. $Z \sim N(0,1) \leftarrow$ standard Normal.

$P(Z > 2.24) =$ area of

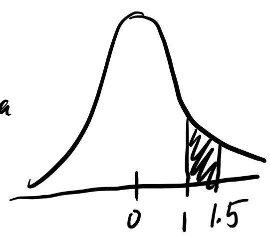
$=$ table @ $a = 2.24$
(row lable 2.2 & col lable 0.04)

$= 0.0125$

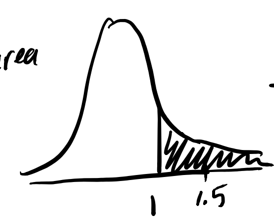
By manipulating areas you can find areas not of the form

e.g. $Z \sim N(0,1)$

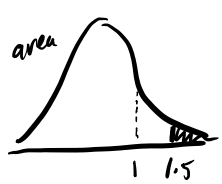
$P(1 < Z \leq 1.5) =$ area



$=$ area



$-$ area



$= (\text{table} @ a = 1.00) - (\text{table} @ a = 1.5)$

$= 0.1587 - 0.0668$

$= 0.0919$

Similar techniques will allow you to find any basic probability for $Z \sim N(0,1)$.

what about $X \sim N(\mu, \sigma^2)$ when $\mu \neq 0$ or $\sigma^2 \neq 1$?

You need to use the following facts:

Fact 1: if $X \sim N(\mu, \sigma^2)$ then $aX + b \sim N(a\mu + b, a^2\sigma^2)$

i.e. any linear function of a Normal R.V. is also Normal.

Fact 2: if X & Y are independent R.V.s s.t.

$X \sim N(\mu_1, \sigma_1^2)$

$Y \sim N(\mu_2, \sigma_2^2)$

then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

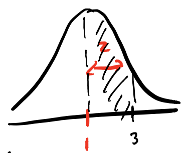
i.e. sum of two indep. Normals is Normal

Fact 3: if $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - E(X)}{\text{Var}(X)} = \frac{X - \mu}{\sigma} \sim N(0,1)$$

i.e. The Z-score of a Normal is standard Normal.

e.g. $X \sim N(1,4)$. (5)

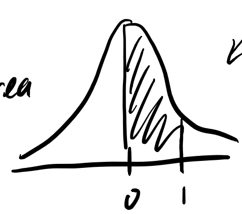
Find $P(1 \le X \le 3)$ = 

Can't use table II right away since X isn't standard Normal. We need to convert X to Z-scores first.

By Fact 3 $\frac{X-1}{2} = Z \sim N(0,1)$.

$$\underbrace{P(1 \le X \le 3)}_{\text{convert this event in terms of } Z.} = P\left(\frac{1-1}{2} \le \frac{X-1}{2} \le \frac{3-1}{2}\right)$$

$$= \underbrace{P(0 \le Z \le 1)}_{\text{Now we can use table II.}}$$

= area 

$= (\text{table @ } a=0) - (\text{table @ } a=1.00)$

$= \frac{1}{2}$ ← by symmetry $- (\text{table @ } a=1.00)$

$= \frac{1}{2} - 0.1587$

$= 0.3413$

e.g. $Y \sim N(-5,7)$ (6)

Find $P(Y < -2)$

Since $\frac{Y-(-5)}{\sqrt{7}} = Z \sim N(0,1)$ we have

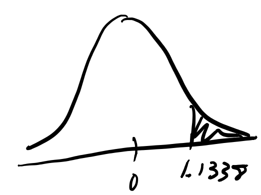
$$P(Y<-2) = P\left(\frac{Y-(-5)}{\sqrt{7}} < \frac{-2-(-5)}{\sqrt{7}}\right)$$

$$= P\left(Z < \frac{3}{\sqrt{7}}\right)$$

$$= P(Z < 1.1338)$$

= 

= 

− 

$= 1 - (\text{table @ } a=1.1338)$

$\approx 1 - 0.1292$

$= 0.8708$.