

Lecture 3: Gaussian random field spectral densities

Recall from last time:

If $f(x)$ is a mean zero stationary (real) GRF on \mathbb{R}^d with auto cov fun $C(x)$ and spectral density $C_{\mathbf{k}}^{ff}$ then

$$1) E(f_{\mathbf{k}} \bar{f}_{\mathbf{k}'}) = C_{\mathbf{k}}^{ff} \delta_{\mathbf{k}-\mathbf{k}'} \\ = (2\pi)^{d/2} C_{\mathbf{k}} \delta_{\mathbf{k}-\mathbf{k}'}$$

2) there exists a real, unit variance, white noise GRF $w(x)$ s.t.

$$f(x) \stackrel{D}{=} \int_{\mathbb{R}^d} e^{ix \cdot \mathbf{k}} \sqrt{C_{\mathbf{k}}^{ff}} w_{\mathbf{k}} \frac{d\mathbf{k}}{(2\pi)^{d/2}}$$

$$\text{i.e. } f_{\mathbf{k}} = \sqrt{C_{\mathbf{k}}^{ff}} w_{\mathbf{k}}$$

↑ Not necessarily rigorous

①

Smoothness of the sample paths of $f(x)$ are determined by how fast $C_{\mathbf{k}}^{ff} \rightarrow 0$ as $|\mathbf{k}| \rightarrow \infty$.

②

This follows since

$$f(x) \cong \sum_{\mathbf{k}} \left(\sqrt{C_{\mathbf{k}}^{ff}} \frac{w_{\mathbf{k}}}{\sqrt{2\pi}} e^{i\mathbf{x} \cdot \mathbf{k}} \right)$$

Coefs have magnitude $\sqrt{C_{\mathbf{k}}^{ff}}$ has wavelength
on the order $\sqrt{\frac{2\pi}{|k|}}$ so large
 $|k|$ wavenumber $|k|$
yields wiggly functions of x

From now on we will focus on isotropic GRFs on \mathbb{R}^d so the auto cov & spectral density can be written $C(|x|)$ & $C_{|\mathbf{k}|}^{ff}$.

Behavior of $C(|x|)$ & $C_{|\mathbf{k}|}^{ff}$
when restricting $f(x)$ to a lower dimension

Let $\{f(x) : x \in \mathbb{R}^3\}$ be a mean zero isotropic GRF & for $y = (y_1, y_2) \in \mathbb{R}^2$ define $g(y) := f(y_1, y_2, 0)$.
Let $C^g(|x|) : \mathbb{R}^2 \rightarrow \mathbb{R}$ & $C^f(|x|) : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the auto covariance functions for g & f .

Fact 1) $g(y)$ is an isotropic GRF on \mathbb{R}^2 ③

Fact 2) The radial profile of the autocov for f & g are the same:

$$C^f(r) = C^g(r) \quad \forall r \in \mathbb{R}^+$$

Fact 3) the radial profile of the spectral densities are different

$$C_{|\mathbf{k}|}^{ff} = \int_{\mathbb{R}^3} e^{i\mathbf{x} \cdot \mathbf{k}} C^f(|\mathbf{x}|) \frac{d^3x}{(2\pi)^3}, \quad \mathbf{k} \in \mathbb{R}^3$$

$$C_{|\mathbf{k}|}^{gg} = \int_{\mathbb{R}^3} e^{i\mathbf{y} \cdot \mathbf{k}} C^g(|\mathbf{y}|) \frac{d^3y}{(2\pi)^3}, \quad \mathbf{k} \in \mathbb{R}^3$$

Restricting isotropic $\{f(x) : x \in \mathbb{R}^3\}$
to the sphere $S^2 \subset \mathbb{R}^3$.

Notation:

• [1] = "Mathematical methods for physicists" Arfken, Weber.

• $\hat{n} \in S^2 \subset \mathbb{R}^3$ where $\hat{n} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$

• $\theta \in [0, \pi]$, polar angle p. 123 [1]

• $\varphi \in [0, 2\pi]$, azimuth

• $J_\nu(r)$: Bessel fun of the first kind

• $j_\ell(r) := \sqrt{\frac{2\pi}{r}} J_{\ell+1/2}(r)$, p. 726 [1]

• $P_\ell(x) := \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell$, p. 767 [1]

= Legendre polys on $[-1, 1]$

$$\star \int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \quad p. 757 [1]$$

• $P_e^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_e(x), \quad 0 \leq m \leq l$ ④
 = associated Legendre functions

$$P_e^{-m}(x) := (-1)^m \frac{(l-m)!}{(l+m)!} P_e^m(x) \quad p. 772 [1]$$

$$Y_{em}(\hat{n}) = Y_{em}(\theta, \varphi)$$

$$= (-1)^m \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_e^m(\cos \theta) e^{im\varphi}$$

= spherical harmonics, $l \geq 0, -l \leq m \leq l$

These are orthonormal on the sphere:

$$\int_{S^2} Y_{em}(\hat{n}) \overline{Y_{e'm'}(\hat{n})} dS^2(\hat{n}) \quad \text{spherical area element}$$

$$= \int_0^{\pi} \int_0^{2\pi} Y_{em}(\theta, \varphi) \overline{Y_{e'm'}(\theta, \varphi)} \sin \theta d\theta d\varphi$$

$$= \delta_{ee'} \delta_{mm'}$$

useful identities:

$$Y_{e'm}^*(\hat{n}) = (-1)^m Y_{e,-m}(\hat{n}), \quad 796 [1]$$

• (Addition Thm 797 [1])

$$P_e(\cos \gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{em}(\hat{n}_1) \overline{Y_{em}(\hat{n}_2)}$$

$$\hat{n}_1 \cdot \hat{n}_2 = \cos \gamma$$

• (Rayleigh egn 769 [1], Jacobi-Argee egn 687 [1])

$e^{ix \cdot \mathbf{k}} = e^{i|\mathbf{x}| |\mathbf{k}| \cos \gamma}$, γ angle btwn \mathbf{x} & \mathbf{k} $\in \mathbb{R}^3$

$$\Rightarrow \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(|\mathbf{x}| |\mathbf{k}|) P_e(\cos \gamma)$$

$$= J_0(|\mathbf{x}| |\mathbf{k}|) + 2 \sum_{\ell=1}^{\infty} i^\ell J_\ell(|\mathbf{x}| |\mathbf{k}|) \cos(\ell \gamma)$$

$$\therefore e^{ix \cdot \mathbf{k}} = \sum_{\ell,m} i^\ell 4\pi j_\ell(|\mathbf{x}| |\mathbf{k}|) Y_{em}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \overline{Y_{em}\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right)}$$

(*)

Thm: Let $\{f(\mathbf{x}): \mathbf{x} \in \mathbb{R}^3\}$ be a (5)
real mean zero isotropic GRF with a.c.f. $C^f(|\mathbf{x}|)$.

Then $\forall \hat{\mathbf{n}} \in S^2 \subset \mathbb{R}^3$

$$f(\hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{\mathbf{n}})$$

where

i) $a_{lm} = (-1)^m a_{l,-m}^*$

ii) a_{lm} 's are jointly complex Gaussian

iii) $E(a_{lm}) = 0$

iv) $E(a_{lm} \overline{a_{l'm'}}) = \delta_{ll'} \delta_{mm'} C_l^{ff}$

v) $C_l^{ff} := \frac{2}{\pi} \int_0^\infty j_l^2(r) C_r^{ff} r^2 dr$

where $l=0, 1, \dots$
 δ is called
the spherical
spectral density

where $r \in [0, \infty)$ &
so C_r^{ff} is the radial
profile of the
 \mathbb{R}^3 spectral density

vi) $\text{Cov}(f(\hat{\mathbf{n}}_1), f(\hat{\mathbf{n}}_2))$

$$= C^f(\sqrt{2 - 2 \cos \theta_{12}})$$

$$= \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) C_l^{ff}$$

$= \cos \theta_{12}$

where θ_{12} is the angle
btwn $\hat{\mathbf{n}}_1$ & $\hat{\mathbf{n}}_2$.

Proof:

(6)

$$f(\hat{\mathbf{n}}) = \int_{\mathbb{R}^3} e^{i\hat{\mathbf{n}} \cdot \mathbf{k}} \sqrt{C_{|\mathbf{k}|}^{ff}} W_{\mathbf{k}} \frac{d\mathbf{k}}{(2\pi)^3/2}$$

$$\begin{aligned} &\stackrel{\text{by } (*)}{=} \int_{\mathbb{R}^3} \left[\sum_{lm} i^l 4\pi j_l(|\mathbf{k}|) Y_{lm}(\hat{\mathbf{n}}) \overline{Y_{lm}(\frac{\mathbf{k}}{|\mathbf{k}|})} \right] \sqrt{C_{|\mathbf{k}|}^{ff}} W_{\mathbf{k}} \frac{d\mathbf{k}}{(2\pi)^3/2} \\ &= \sum_{lm} Y_{lm}(\hat{\mathbf{n}}) \underbrace{\left[i^l 4\pi \int_{\mathbb{R}^3} j_l(|\mathbf{k}|) Y_{lm}(\frac{\mathbf{k}}{|\mathbf{k}|}) \sqrt{C_{|\mathbf{k}|}^{ff}} W_{\mathbf{k}} \frac{d\mathbf{k}}{(2\pi)^3} \right]}_{=: a_{lm}} \end{aligned}$$

where

$$E(a_{lm} \overline{a_{l'm'}})$$

$$= (4\pi)^2 \int_{\mathbb{R}^3} j_l(|\mathbf{k}|) j_{l'}(|\mathbf{k}|) \overline{Y_{lm}(\frac{\mathbf{k}}{|\mathbf{k}|})} Y_{l'm'}(\frac{\mathbf{k}}{|\mathbf{k}|}) C_{|\mathbf{k}|}^{ff} \frac{d\mathbf{k}}{(2\pi)^3}$$

$$= (4\pi)^2 \int_0^\infty j_l(r) j_{l'}(r) C_r^{ff} \left[\int_0^\pi \int_0^{2\pi} \overline{Y_{lm}(0,\theta)} Y_{l'm'}(0,\theta) \sin \theta d\theta d\phi \right] \frac{r^2 dr}{(2\pi)^3}$$

$$= \delta_{ll'} \delta_{mm'} (4\pi)^2 \int_0^\infty j_l^2(r) C_r^{ff} \frac{r^2 dr}{(2\pi)^3}$$

This shows i) - v)

To show vi)

$$\text{Cov}(f(\hat{\mathbf{n}}_1), f(\hat{\mathbf{n}}_2))$$

$$= C^f(|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2|)$$

$$= C^f(\sqrt{|\hat{\mathbf{n}}_1|^2 + |\hat{\mathbf{n}}_2|^2 - 2 \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2})$$

$$= C^f(\sqrt{2 - 2 \cos \theta_{12}})$$

Finally

⑦

$$\text{Cov}(f(\hat{n}_1), f(\hat{n}_2))$$

$$= \sum_{em} \sum_{e'm'} E(a_{em} \bar{a}_{e'm'}) Y_{em}(\hat{n}_1) \overline{Y_{e'm'}(\hat{n}_2)}$$

$$= \sum_{em} C_e^{ff} Y_{em}(\hat{n}_1) \overline{Y_{em}(\hat{n}_2)}$$

$$= \sum_e C_e^{ff} \frac{(2l+1)}{4\pi} P_e(\cos \theta_{12})$$

by the Addition Thm

