Lecture 3: Gaussian random field spectral densities

Recall from last time:

If $f(x)$ is a mean zero stationary (& real) GRF on $\mathbb{R}^d$ with auto cov fun $C(x)$ and spectral density C_k^{ff} then

1) $E\left(f_k \bar{f}_{k'}\right) = C_k^{ff}\, \delta_{k-k'}$

$$= (2\pi)^{d/2} C_k\, \delta_{k-k'}$$

2) there exists a real, unit variance, white noise GRF $w(x)$ s.t.

$$f(x) \overset{\mathcal{D}}{=} \int_{\mathbb{R}^d} e^{ix\cdot k} \sqrt{C_k^{ff}}\, W_k \frac{dk}{(2\pi)^{d/2}}$$

i.e. $f_k = \sqrt{C_k^{ff}}\, W_k$

Not necessarily rigourous

Smoothness of the sample paths of $f(x)$ are determined by how fast $C_k^{ff} \to 0$ as $|k| \to \infty$.

This follows since

$$f(x) \cong \sum_k \underbrace{\left(\sqrt{C_k^{ff}}\, W_k \tfrac{dk}{(2\pi)^{d/2}}\right)}_{\text{Coeffs are } \sim \sqrt{C_k^{ff}} \frac{\sqrt{dk}}{(2\pi)^{d/2}}} \underbrace{e^{ix\cdot k}}_{\text{has wavelength } \frac{2\pi}{|k|}}$$

so large wavenumber $|k|$ yields wiggly functions of x

From now on we will focus on Isotropic GRFs on $\mathbb{R}^d$ so the auto cov & spectral density can be written $C(|x|)$ & $C_{|k|}^{ff}$.

Behavior of $C(|x|)$ & $C_{|k|}^{ff}$ when restricting $f(x)$ to a lower dimension

Let $\{f(x) : x \in \mathbb{R}^3\}$ be a mean zero isotropic GRF & for $y = (y_1, y_2) \in \mathbb{R}^2$ define $g(y) := f(y_1, y_2, 0)$.

Let $C^g(|x|) : \mathbb{R}^2 \to \mathbb{R}$ & $C^f(|x|) : \mathbb{R}^3 \to \mathbb{R}$ be the auto covariance functions for g & f.

Fact 1) $g(y)$ is an isotropic GRF on $\mathbb{R}^2$

Fact 2) The radial profile of the autocov for f & g are the same:

$$C^f(r) = C^g(r) \quad \forall r \in \mathbb{R}^+$$

Fact 3) the radial profile of the spectral densities are different

$$C^{ff}_{|k|} = \int_{\mathbb{R}^3} e^{ix\cdot k}\, C^f(|x|)\, \frac{d^3x}{(2\pi)^3}, \quad k \in \mathbb{R}^3$$

$$\updownarrow \text{same}$$

$$C^{gg}_{|k'|} = \int_{\mathbb{R}^2} e^{iy\cdot k'}\, C^g(|y|)\, \frac{d^2y}{(2\pi)^2}, \quad k' \in \mathbb{R}^2$$

Restricting isotropic $\{f(x) : x \in \mathbb{R}^3\}$ to the sphere $S^2 \subset \mathbb{R}^3$

Notation:

- [1] = "Mathematical methods for physicists" ARFKen, Weber
- $\hat{n} \in S^2 \subset \mathbb{R}^3$ where $\hat{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$
- $\theta \in [0,\pi]$, polar angle
- $\varphi \in [0, 2\pi]$, azimuth

p.123 [1]

- $J_\nu(r) :=$ Bessel fun of the first kind
- $j_\ell(r) := \sqrt{\frac{2\pi}{2r}}\, J_{\ell+1/2}(r)$, p. 726 [1]
- $P_\ell(x) := \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$, p. 767 [1]

  = Legendre polys on $[-1,1]$

  $* \int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2\delta_{mn}}{2n+1}$ p.757 [1]

- $P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$, $0 \le m \le \ell$

  $\equiv$ associated Legendre functions p. 772 [1]

- $P_\ell^{-m}(x) := (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$
- $Y_{\ell m}(\hat{n}) \equiv Y_{\ell m}(\theta,\varphi)$

  $= (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$

  = sherical Harmonics, $\ell \ge 0, -\ell \le m \le \ell$

These are orthonormal on the shpere:

$$\int_{S^2} Y_{\ell m}(\hat{n}) \overline{Y_{\ell' m'}(\hat{n})}\, \underbrace{dS^2(\hat{n})}_{\text{sherical area element}}$$

$$= \int_0^{2\pi}\int_0^{\pi} Y_{\ell m}(\theta,\varphi)\overline{Y_{\ell' m'}(\theta,\varphi)} \sin\theta\, d\theta\, d\varphi$$

$$= \delta_{\ell\ell'}\delta_{mm'}$$

useful identities:

- $Y^*_{\ell m}(\hat{n}) = (-1)^m Y_{\ell,-m}(\hat{n})$, 796 [1]
- (Addition Thm 797 [1])

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}_1)\overline{Y_{\ell m}(\hat{n}_2)}$$

$$\hat{n}_1 \cdot \hat{n}_2 = \cos\gamma$$

- (Rayleigh eqn 769 [1])

$$e^{ix\cdot k} = e^{i|x||k|\cos\gamma}, \quad \gamma \text{ angle btwn } x \,\&\, k \in \mathbb{R}^3$$

$$= \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(|x|\cdot|k|)\, P_\ell(\cos\gamma)$$

$$\therefore\ e^{ix\cdot k} \underset{(*)}{=} \sum_{\ell m} i^\ell 4\pi\, j_\ell(|x|\cdot|k|)\, Y_{\ell m}\!\left(\tfrac{x}{|x|}\right) \overline{Y_{\ell m}\!\left(\tfrac{k}{|k|}\right)}$$

Thm: Let $\{f(x) : x \in \mathbb{R}^3\}$ be a real mean zero isotropic GRF with a.c.f. $C^f(|x|)$.

Then $\forall \hat{n} \in S^2 \subset \mathbb{R}^3$

$$f(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$$

where

i) $a_{\ell m} = (-1)^m a_{\ell,-m}$

ii) $a_{\ell m}$'s are jointly complex Gaussian

iii) $E(a_{\ell m}) = 0$

iv) $E(a_{\ell m} \overline{a_{\ell' m'}}) = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{ff}$

v) $C_\ell^{ff} := \frac{2}{\pi} \int_0^\infty j_\ell^2(r)\, C_r^{ff}\, r^2 dr$

where $\ell = 0, 1, \ldots$ & is called the spherical spectral density

where $r \in [0,\infty)$ & so C_r^{ff} is the radial profile of the $\mathbb{R}^3$ spectral density

vi) $\operatorname{cov}(f(\hat{n}_1), f(\hat{n}_2))$

$$= C^f\left(\sqrt{2 - 2\cos\theta_{12}}\right)$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\underbrace{\hat{n}_1 \cdot \hat{n}_2}_{=\cos\theta_{12}}) C_\ell^{ff}$$

where θ_{12} is the angle btwn $\hat{n}_1$ & $\hat{n}_2$.

Proof:

$$f(\hat{n}) = \int_{\mathbb{R}^3} e^{i\hat{n}\cdot k} \sqrt{C_k^{ff}}\, W_k \frac{dk}{(2\pi)^{d/2}}$$

$$\overset{\text{by }(*)}{=} \int_{\mathbb{R}^3} \left[\sum_{\ell m} i^\ell 4\pi j_\ell(|k|) Y_{\ell m}(\hat{n}) \overline{Y_{\ell m}\left(\tfrac{k}{|k|}\right)}\right] \sqrt{C_{|k|}^{ff}}\, W_k \frac{dk}{(2\pi)^{d/2}}$$

$$= \sum_{\ell m} Y_{\ell m}(\hat{n}) \underbrace{\left[i^\ell 4\pi \int_{\mathbb{R}^3} j_\ell(|k|) \overline{Y_{\ell m}\left(\tfrac{k}{|k|}\right)} \sqrt{C_{|k|}^{ff}}\, W_k \frac{dk}{(2\pi)^{d/2}}\right]}_{=:\,a_{\ell m}}$$

where

$E(a_{\ell m} \overline{a_{\ell' m'}})$

$$= (4\pi)^2 \int_{\mathbb{R}^3} j_\ell(|k|)\, j_{\ell'}(|k|)\, \overline{Y_{\ell m}\left(\tfrac{k}{|k|}\right)} Y_{\ell' m'}\left(\tfrac{k}{|k|}\right) C_{|k|}^{ff} \frac{dk}{(2\pi)^3}$$

$$= (4\pi)^2 \int_0^\infty j_\ell(r) j_{\ell'}(r) C_r^{ff} \underbrace{\left[\int_0^\pi \int_0^{2\pi} \overline{Y_{\ell m}(\theta,\varphi)} Y_{\ell' m'}(\theta,\varphi) \sin\theta\, d\theta\, d\varphi\right]}_{=\delta_{\ell\ell'}\delta_{mm'}} \frac{r^2 dr}{(2\pi)^3}$$

$$= (4\pi)^2 \int_0^\infty j_\ell^2(r)\, C_r^{ff} \frac{r^2 dr}{(2\pi)^3}$$

This shows i) – v)

To show vi)

$\operatorname{cov}(f(\hat{n}_1), f(\hat{n}_2))$

$$= C^f(|\hat{n}_1 - \hat{n}_2|)$$

$$= C^f\left(\sqrt{|\hat{n}_1|^2 + |\hat{n}_2|^2 - 2\hat{n}_1 \cdot \hat{n}_2}\right)$$

$$= C^f\left(\sqrt{2 - 2\cos\theta_{12}}\right)$$

Finally

$\mathrm{cov}(f(\hat{n}_1), f(\hat{n}_2))$

$$= \sum_{\ell m} \sum_{\ell' m'} E(a_{\ell m} \overline{a_{\ell' m'}}) Y_{\ell m}(\hat{n}_1) \overline{Y_{\ell' m'}(\hat{n}_2)}$$

$$= \sum_{\ell m} C_\ell^{ff} Y_{\ell m}(\hat{n}_1) \overline{Y_{\ell' m'}(\hat{n}_2)}$$

$$= \sum_{\ell} C_\ell^{ff} \frac{(2\ell+1)}{4\pi} P_\ell(\cos\theta_{12})$$

by the Addition Thm

$\blacksquare$