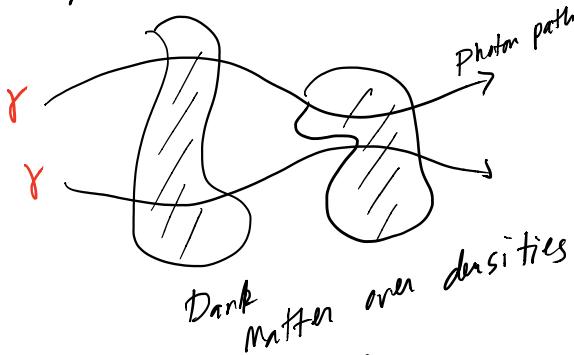


# Lecture 11

Topics:

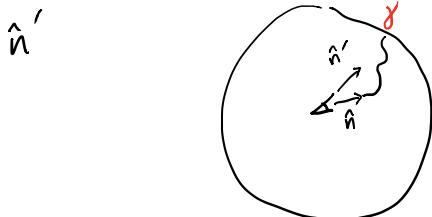
- I) Lensing
- II) Polarization
- III) the quadratic estimate
- IV) The lensing bispectrum
- V) Bayesian lensing.

As a CMB photon travels towards us, it passes through/near clumps of dark matter. Since dark matter imposes a gravitational influence on the photon, its path is distorted



This distortion is called  
Gravitational lensing

The basic effect is that when we observe CMB photons (from a blackbody spectrum) in direction  $\hat{n}$ , they actually emitted from a distorted direction



(1)

∴ on our observed temperature denoted  $\tilde{T}(\hat{n})$  is actually a warped/lensed version of the temp  $T(\hat{n})$  on the surface of last scattering:

$$\tilde{T}(\hat{n}) = T(\hat{n}')$$

where  $\hat{n}'$  varies with  $\hat{n}$ .

## Flat sky

For the remainder of this lecture we focus on the flat sky approximation. i.e. the observed lensed temperature field (without noise) takes the form

$$\tilde{T}(x) = T(x'), \quad x, x' \in \mathbb{R}^2$$

↑  
temp on the last scattering surface

## The lensing Potential

under some simplified assumptions on the geometry of the photon path as well as on the nature of the dark matter the lensing of the CMB photons are characterized by a lensing (or gravitational) potential

$$\phi(x) = \text{a weighted line of sight integral over the 3-d gravitational potential.}$$

where  $\nabla \phi(x)$  gives the photon displacement due to lensing &

$\Delta\phi(x) \approx$  projected matter density. (3)

$$\therefore \tilde{T}(x) = T(x + \nabla\phi(x))$$

Back of the envelope calculations (2 data) establish that

$$\sqrt{E|\nabla\phi(x)|^2} \approx 2.7 \text{ arcmins}$$

typical size  
of the lensing  
displacement

But that  $\phi(x)$  is coherent (or correlated) over large spatial scales on the order of degrees.

Moreover, since the lensing power is strongest at the midpoint btwn us & the surface of last scattering the mass distribution is relatively smooth & diffuse.

Along with the Einstein principle one can therefore model  $\phi(x)$  as a isotropic Gaussian random field with

$$E(\phi(x)) = 0 \quad \leftarrow \text{since only the fluctuations in } \phi(x) \text{ generate lensing.}$$

$$E(\phi_k \phi_{k'}^\dagger) = C_k \delta_{k-k'}$$

Moreover we can assume

$$E(\phi_k T_{k'}) = 0 \quad \forall k, k' \in \mathbb{R}^2$$

i.e.  $\phi$  &  $T$  are essentially uncorrelated.

Is  $\tilde{T}$  non-isotropic or non-Gaussian? (4)

There are two ways to view the impact of lensing.

- 1) if one considers  $\phi(x)$  fixed,  
i.e. conditioning on  $\phi(x)$ , then

$\tilde{T}(x)$  is a non-stationary Gaussian random field since

$$\begin{aligned} (\tilde{T}(x_1), \dots, \tilde{T}(x_n))^T \\ = (T(x_1 + \nabla\phi(x_1)), \dots, T(x_n + \nabla\phi(x_n)))^T \\ \sim N(T_{\text{CMB}}, \Sigma) \\ \approx 2.7 \text{ K} \end{aligned}$$

where  $\Sigma_{ij} := C^T(|x_i - x_j + \nabla\phi(x_i) - \nabla\phi(x_j)|)$   
= non-stationary cov fun.

- 2) If, in contrast, one marginalizes over  $\phi(x)$  then  $T(x)$  becomes isotropic again but is now Non-Gaussian. We will see this later by analyzing the Bispectrum of  $T$  (which is zero for isotropic GRFs).

Show pictures of simulations  
of  $\tilde{T}$ ,  $T$  &  $\phi$

The impact of lensing on  $C_{1k1}^{\tilde{T}\tilde{T}}$  when  
marginalizing over  $\phi(x)$  (5)

Modeling  $\phi(x)$  as a isotropic GRF one can  
compute the flat sky spectral density of  $\tilde{T}$ ,  
even though  $\tilde{T}$  is non-Gaussian (so  $C_{1k1}^{\tilde{T}\tilde{T}}$  will  
not completely characterize the distribution  
of  $\tilde{T}$ ).

First take a Fourier decomposition of  $T(x)$ :

$$T(x) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{x} \cdot \mathbf{k}} T_{\mathbf{k}}$$

This is my shorthand for  $\frac{d\mathbf{k}}{(2\pi)^d} d\mathbf{k}$   
where  $d=2$  in this case.

$$\therefore \tilde{T}(x) = T(x + \nabla\phi(x))$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i(x + \nabla\phi(x)) \cdot \mathbf{k}} T_{\mathbf{k}}$$

$$\therefore E(\tilde{T}(x)\tilde{T}(y)) = E \int \frac{d\mathbf{k} d\mathbf{k}'}{(2\pi)^d} e^{i(x + \nabla\phi(x)) \cdot \mathbf{k} - i(y + \nabla\phi(y)) \cdot \mathbf{k}'} \\ \times T_{\mathbf{k}} T_{\mathbf{k}'}^*$$

$$= \int \frac{d\mathbf{k} d\mathbf{k}'}{(2\pi)^d} e^{ix \cdot \mathbf{k} - iy \cdot \mathbf{k}'} E(e^{i(\nabla\phi(x) \cdot \mathbf{k} - \nabla\phi(y) \cdot \mathbf{k}')} T_{\mathbf{k}} T_{\mathbf{k}'}^*)$$

$$\text{Now } E(e^{i\nabla\phi(x) \cdot \mathbf{k} - i\nabla\phi(y) \cdot \mathbf{k}'} T_{\mathbf{k}} T_{\mathbf{k}'}^*)$$

$$= E \left[ e^{i(\nabla\phi(x) \cdot \mathbf{k} - \nabla\phi(y) \cdot \mathbf{k}')} \right] \delta_{\mathbf{k}-\mathbf{k}'} C_{1k1}^{TT}$$

$$= E \left[ e^{i(\nabla\phi(x) - \nabla\phi(y)) \cdot \mathbf{k}} \right] \delta_{\mathbf{k}-\mathbf{k}'} C_{1k1}^{TT}$$

has the form  $E(e^{iX \cdot \mathbf{k}})$  for  
a gaussian vector  $X \sim N(0, \Sigma)$   
which, by characteristic functions, equals  
 $\exp(-\frac{1}{2} \mathbf{k}^T \Sigma \mathbf{k})$

The covariance matrix of  $\nabla\phi(x) - \nabla\phi(y)$   
is given by (6)

$$E \left[ (\nabla\phi(x) - \nabla\phi(y)) (\nabla\phi(x) - \nabla\phi(y))^T \right] \\ = 2 \left( \Sigma^{\nabla\phi}(0) - \Sigma^{\nabla\phi}(x-y) \right)$$

where  $\Sigma^{\nabla\phi}(x-y) := E(\nabla\phi(x) \nabla\phi(y)^T)$ .

$\therefore E(\tilde{T}(x)\tilde{T}(y))$  is given by

$$\left( \int \frac{d\mathbf{k} d\mathbf{k}'}{(2\pi)^d} e^{i(x \cdot \mathbf{k} - y \cdot \mathbf{k} + \mathbf{k}^T [\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)] \mathbf{k})} \delta_{\mathbf{k}-\mathbf{k}'} C_{1k1}^{TT} \right)$$

$$= \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i(x-y) \cdot \mathbf{k}} e^{\mathbf{k}^T [\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)] \mathbf{k}} C_{1k1}^{TT}$$

$$= C^{\tilde{T}}(x-y)$$

so, indeed,  $\tilde{T}$  stationary

In fact isotropy can be shown by analyzing  
how  $\Sigma^{\nabla\phi}(R(x-y))$  behaves for rotations  $R$ .

Expanding  $e^{\mathbf{k}^T [\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)] \mathbf{k}}$  to  
first order gives

$$E(\tilde{T}(x)\tilde{T}(0))$$

$$\approx \int \frac{d\mathbf{k}}{(2\pi)^d} e^{ix \cdot \mathbf{k}} \left[ 1 + \mathbf{k}^T [\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)] \mathbf{k} \right] C_{1k1}^{TT}$$

$$= C^T(1|x|) - \sigma_R^2 \int \frac{d\mathbf{k}}{(2\pi)^d} e^{ix \cdot \mathbf{k}} |\mathbf{k}|^2 C_{1k1}^{TT}$$

$$+ \sum_{i,j=1}^2 \Sigma_{ij}^{\nabla\phi}(x) \int \frac{d\mathbf{k}}{(2\pi)^d} e^{ix \cdot \mathbf{k}} k_i k_j C_{1k1}^{TT}$$

$$= C^T(1|x|) - \sigma_R^2 \int \frac{d\mathbf{k}}{(2\pi)^d} e^{ix \cdot \mathbf{k}} |\mathbf{k}|^2 C_{1k1}^{TT}$$

$$+ \sum_{i,j=1}^2 \left[ -\frac{\partial^2 C^{\nabla\phi}(x)}{\partial x_i \partial x_j} \right] \left[ -\frac{\partial^2 C^T(x)}{\partial x_i \partial x_j} \right]$$

where  $\sigma_R^2 = E\left(\frac{\partial \phi(x)}{\partial x_i}\right)^2 = \frac{1}{2} E|\nabla\phi(x)|^2 \sim 3 \times 10^{-7}$   
so that the typical deflection size  $\sqrt{E|\nabla\phi(x)|^2}$   
is about 2.7 arcmins.

Notice that the third term is a sum (7) of pixelwise products  $A(x)B(x)$ .

Recall how to take the Fourier transform of such functions

$$\int dx e^{-ix \cdot l} A(x)B(x) = \int dx dk e^{-ix \cdot l} A(x) e^{ikx}$$

$$= \int dk B_{-k} \int dx e^{-ix(k+l)} A(x)$$

$$= \int dk A_{k+l} B_{-k}$$

Now taking the Fourier transform of

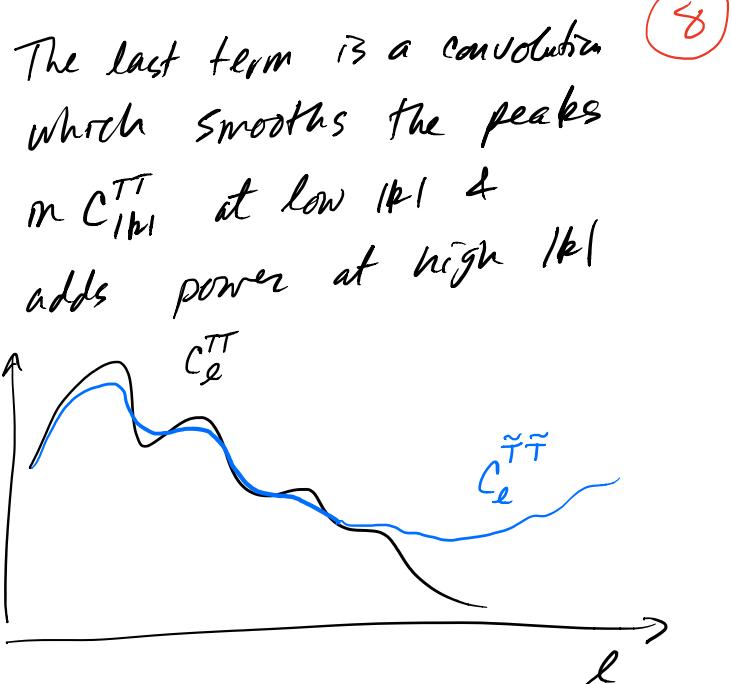
$$C^{\tilde{T}}(|x|) = E(\tilde{T}(x)\tilde{T}(0))$$

gives  $\frac{1}{(2\pi)^d} \int dk C^{\tilde{T}\tilde{T}}_{|k|}$  so that

$$C^{\tilde{T}\tilde{T}}_{|k|} = (1 - \sigma_R^2 |k|^2) C^T_{|k|} + \sum_{i,j=1}^d \int dk' (k'_i + k_i)(k'_j + k_j) C^{\phi\phi}_{|k+k'|} \times (-k'_i)(-k'_j) C^T_{|k'|}$$

$$= (1 - \sigma_R^2 |k|^2) C^T_{|k|} + \int dk' [(k+k') \cdot k]^2 C^{\phi\phi}_{|k+k'|} C^T_{|k'|}$$

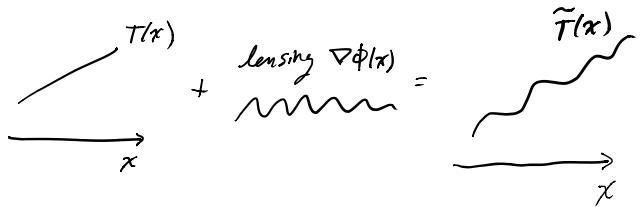
$$= (1 - \sigma_R^2 |k|^2) C^T_{|k|} + \int dk' [(k-k') \cdot k]^2 C^{\phi\phi}_{|k-k'|} C^T_{|k'|}$$



Intuitively, the "peak smoothing" is due to  $C_e^{\tilde{T}\tilde{T}}$  being thought of as a "average" of magnified or de-magnified  $C^T$  due to lensing.

The "added power" in  $C_e^{\tilde{T}\tilde{T}}$  can be explained by the fact that at extremely small scales  $T(x)$  is approximately linear &  $\nabla\phi(x)$  induces small ripples in this linear structure.

1-d analog at small scales



# The quadratic estimate for $T(x)$

(9)

Consider the case where we are treating  $\phi$  as fixed so that  $\tilde{T}(x)$  is Gaussian but non-stationary.

Recall Bochner's theorem which states that any stationary RF  $f(x)$  has uncorrelated Fourier coefficients:

$$E(f_k f_{k'}^*) = C_{kk'}^{ff} \delta_{k-k'}$$

(assuming  $E(f(x)) = 0$ ).

This condition is not only necessary but also sufficient.

$\therefore \tilde{T}(x)$  must have correlated Fourier coefficients (due exclusively to  $\phi$ ).

To approximate this correlation (as a function of  $\phi(x)$ ) Taylor expand  $\tilde{T}(x)$  as follows

$$\begin{aligned} \tilde{T}(x) &= T(x + \nabla \phi(x)) \\ &= T(x) + \underbrace{\nabla T(x) \cdot \nabla \phi(x)}_{\text{isotropic}} + \underbrace{\mathcal{O}(\phi)}_{\text{non-stationary}} \end{aligned}$$

Therefore

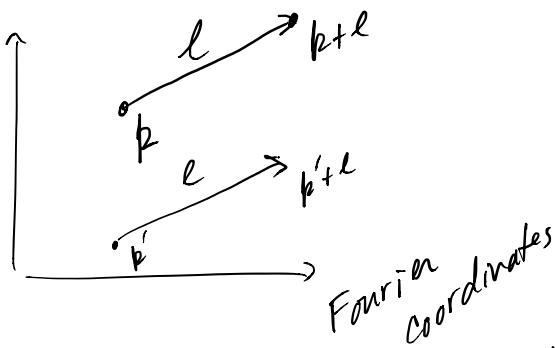
$$\begin{aligned} \tilde{T}_k &= T_k + \sum_{l=1}^{\infty} \int dk' i(k'+k) T_{k'+k} i(-k') \phi_{-k'}^* \\ &\quad + \mathcal{O}(\phi^2) \\ &= T_k + \int dk' (k'+k) \cdot k' T_{k'+k} \phi_{-k'}^* + \mathcal{O}(\phi^2) \end{aligned}$$

Let's look at the covariance btwn  $\tilde{T}_{k+l}$  &  $\tilde{T}_k$ :

(10)

$$\begin{aligned} E(\tilde{T}_{k+l} \tilde{T}_k^*) &= E(T_{k+l} T_k^*) \\ &\quad + E(T_{k+l} \int dk' (k'+k) \cdot k' T_{k'+k} \phi_{-k'}^*) \\ &\quad + E(T_k^* \int dk' (k'+k+l) \cdot k' T_{k'+k+l} \phi_{-k'}) \\ &\quad + \mathcal{O}(\phi^2) \\ &= \delta_{k,l} C_k^{TT} \quad k'=l \\ &\quad + \int dk' (k'+k) \cdot k' \phi_{-k'}^* f_{k+l-k'-k} C_{k+l}^{TT} \\ &\quad + \int dk' (k'+k+l) \cdot k' \phi_{-k'}^* f_{-k+k'+k+l} C_k^{TT} \quad k'=-l \\ &\quad + \mathcal{O}(\phi^2) \\ &= \delta_{k,l} C_k^{TT} + \frac{1}{2\pi} (k+l) \cdot l \phi_l C_{k+l}^{TT} \\ &\quad + \frac{1}{2\pi} (k)(-l) \phi_l C_k^{TT} \\ &\quad + \mathcal{O}(\phi^2) \\ &= \delta_{k,l} C_k^{TT} + \underbrace{\frac{l \cdot [(k+l)C_{k+l}^{TT} - k C_k^{TT}]}{2\pi}}_{\substack{\uparrow \\ \text{This term} \\ \text{is } 0 \text{ when} \\ l \neq 0}} \phi_l + \mathcal{O}(\phi^2) \\ &\quad \underbrace{\text{Call this } f_{k,l}.}_{\substack{\text{known if } C_k^{TT} \\ \text{is known}}} \end{aligned}$$

$$\therefore E(\tilde{T}_{k+l}\tilde{T}_k^*) = f_{k+l} \phi_l + \mathcal{O}(\phi^2) \quad (11)$$



A naive estimate of  $\phi_l$ , at a fixed  $l \in \mathbb{R}^2$  is then

$$\hat{\phi}_l = \frac{1}{N} \sum_{j=1}^N \frac{\tilde{T}_{k_j+l}\tilde{T}_{k_j}^*}{f_{k_j+l}}$$

when  $\tilde{T}_k$  is observed at frequencies  $k_1, \dots, k_N$  &  $k_1+l, \dots, k_N+l$ . Note  $\hat{\phi}_l$  is unbiased up to  $\mathcal{O}(\phi^2)$ .

$$E(\hat{\phi}_l) = \phi_l + \mathcal{O}(\phi^2)$$

↑

Fixing  $\phi$   
& considering  
 $T_k$  as random

A much better estimate can be found by doing an approximate inverse weighted average estimate ...

$$\hat{\phi}_l = \sum_{j=1}^N w_{k_j+l} \frac{\tilde{T}_{k_j+l}\tilde{T}_{k_j}^*}{f_{k_j+l}}$$

weights

s.t.

$$w_{k_j+l} \propto \frac{1}{\text{var}(\tilde{T}_{k_j+l}\tilde{T}_{k_j}^* / f_{k_j+l})} = \frac{|f_{k_j+l}|^2}{\text{var}(\tilde{T}_{k_j+l}\tilde{T}_{k_j}^*)}$$

$$E(\hat{\phi}_l) = \phi_l + \mathcal{O}(\phi^2).$$

Working on the continuum we look for

$w_{k+l}$  s.t.

$$\hat{\phi}_l = \int dk w_{k+l} \frac{\tilde{T}_{k+l}\tilde{T}_k^*}{f_{k+l}}$$

To find  $w_{k+l}$  we first need to compute  $\text{var}(\tilde{T}_{k+l}\tilde{T}_k^*)$  considering both  $\tilde{T}$  &  $\phi$  random!

$$= E(\tilde{T}_{k+l}\tilde{T}_k^* \tilde{T}_{k+l}^* \tilde{T}_k) - E(\tilde{T}_{k+l}\tilde{T}_k^*) E(\tilde{T}_{k+l}^* \tilde{T}_k)$$

approximate this by writing them these terms cancel

$$= E(\tilde{T}_{k+l}\tilde{T}_k^*) E(\tilde{T}_k^* \tilde{T}_k) + E(\tilde{T}_{k+l}\tilde{T}_k^*) E(\tilde{T}_k^* \tilde{T}_{k+l}^*)$$

$$= \int_0^2 C_{k+l}^{TT} \int_0^2 C_k^{TT}$$

$$+ \int_{2k+l}^2 C_{k+l}^{TT} \int_{2k+l}^2 C_k^{TT}$$

only non-zero when  $2k+l$  so we ignore it

$$= \int_0^2 C_{k+l}^{TT} C_k^{TT}$$

(12)

Now replacing  $w_{k,e}$  with

$$\frac{Ae^{2\pi|f_{k,e}|^2}}{\text{var}(\tilde{T}_{k+e}\tilde{T}_k^*)} = \frac{Ae^{2\pi|f_{k,e}|^2}}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}}$$

The yet to be determined normalizing constant

we have

$$\hat{\phi}_e = \int dk w_{k,e} \frac{\tilde{T}_{k+e} \tilde{T}_k^*}{f_{k,e}}$$

$$= A_e 2\pi \int dk \frac{|f_{k,e}|^2}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}} \frac{\tilde{T}_{k+e} \tilde{T}_k^*}{f_{k,e}}$$

$$= A_e 2\pi \int dk \frac{f_{k,e}^*}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}} \tilde{T}_{k+e} \tilde{T}_k^*$$

$$= A_e \int dk \frac{l \cdot [l/(k+e)C_{k+e}^{\tilde{T}\tilde{T}} - k C_k^{\tilde{T}\tilde{T}}]}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}} \tilde{T}_{k+e} \tilde{T}_k^*$$

where  $A_e$  is defined by setting

$$E(\hat{\phi}_e) = \phi_e + \mathcal{O}(\phi_e)$$

i.e.

$$\frac{E(\hat{\phi}_e)}{\phi_e} = 1 + \mathcal{O}(\phi_e)$$

$$\frac{A_e 2\pi \int dk \frac{|f_{k,e}|^2}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}} [\phi_e + \mathcal{O}(\phi_e)]}{\phi_e}$$

(13)

(14)

$$A_e := \left[ 2\pi \int dk \frac{|f_{k,e}|^2}{C_{k+e}^{\tilde{T}\tilde{T}} C_k^{\tilde{T}\tilde{T}}} \right]^{-1}$$

Note: when one has data with noise  $d(x) = \tilde{T}(x) + n(x)$

Then

$$\hat{\phi}_e = A_e l \int dk \frac{(k+e)C_{k+e}^{\tilde{T}\tilde{T}} - k C_k^{\tilde{T}\tilde{T}}}{C_{k+e}^{dd} C_k^{dd}} d_{k+e} d_k^*$$

where

$$A_e := \left[ 2\pi \int dk \frac{|f_{k,e}|^2}{C_{k+e}^{dd} C_k^{dd}} \right]^{-1}$$

$$C_k^{dd} := C_k^{\tilde{T}\tilde{T}} + C_k^{nn}$$

# A fast formula for $\hat{\phi}_e$

(15)

First Note that

$$\begin{aligned} & \int d\mathbf{k} \frac{(\mathbf{k} + \mathbf{l}) C_{\mathbf{k} + \mathbf{l}}^T}{C_{\mathbf{k} + \mathbf{l}}^{dd} C_{\mathbf{k}}^{dd}} d_{\mathbf{k} + \mathbf{l}} d_{\mathbf{k}}^* \\ &= \int d\mathbf{k} \left[ \frac{(\mathbf{k} + \mathbf{l}) C_{\mathbf{k} + \mathbf{l}}^T d_{\mathbf{k} + \mathbf{l}}}{C_{\mathbf{k} + \mathbf{l}}^{dd}} \right] \left[ \frac{d_{-\mathbf{k}}}{C_{-\mathbf{k}}^{dd}} \right] \\ &:= -i \left[ i(\mathbf{k} + \mathbf{l}) \right] \mathcal{L}_{\mathbf{k} + \mathbf{l}} \mathcal{H}_{-\mathbf{k}} \\ &= -i \int dx e^{i\mathbf{l} \cdot \mathbf{x}} \nabla \mathcal{L}(x) \mathcal{H}(x) \end{aligned}$$

And

$$\begin{aligned} & \int d\mathbf{k} \frac{-\mathbf{k} C_{\mathbf{k}}^T}{C_{\mathbf{k} + \mathbf{l}}^{ad} C_{\mathbf{k}}^{dd}} d_{\mathbf{k} + \mathbf{l}} d_{\mathbf{k}}^* \\ &= -i \int dx e^{-i\mathbf{l} \cdot \mathbf{x}} \nabla f(x) \mathcal{H}(x) \end{aligned}$$

$$\therefore \hat{\phi}_e = -i A_e l \cdot \int dx e^{-i\mathbf{l} \cdot \mathbf{x}} \underbrace{\nabla f(x)}_{\substack{\text{pointwise} \\ \text{pixel mult}}} \mathcal{H}(x)$$

$\curvearrowleft$  Fourier transform

So the est  $\hat{\phi}_e$  can be constructed as follows (16)

$$\begin{aligned} d(x) &\xrightarrow[\substack{\text{IFFT} \\ \text{gradient}}]{\substack{\text{FFT} \\ \text{pointwise} \\ \text{mult.}}} \left\{ \begin{array}{l} \mathcal{L}_k := \frac{C_k^T}{C_k^{dd}} d_k \equiv \text{Filter} \\ d \text{ to est } T \end{array} \right. \\ &\quad \left. \mathcal{H}_k := \frac{d_k}{C_k^{dd}} = \text{High pass filter} \right. \\ &\xrightarrow[\substack{\text{FFT} \\ \text{pointwise} \\ \text{mult.}}]{\nabla \mathcal{L}(x) \mathcal{H}(x)} -i A_e l \cdot \int dt e^{i\mathbf{x} \cdot \mathbf{t}} \nabla \mathcal{L}(t) \mathcal{H}(t) \end{aligned}$$

## Interpretation of the fast formula

In 1-d

$$T(x) + \nabla \phi(x) = \tilde{T}(x)$$

$\therefore$  it makes sense to look for  
a low-pass filter  $\mathcal{L}(x)$  of the data  
& a high-pass filter  $\mathcal{H}(x)$

$$\tilde{T}(x) = \mathcal{L}(x) / \mathcal{H}(x) + \mathcal{H}(x)$$

s.t.  $\mathcal{L}(x)$  estimates  $T(x)$  &  
 $\mathcal{H}(x)$  estimates  $\nabla \phi(x)$ .

However there is a problem ...

since the magnitude of  $H(x)$  is dependent on  $\nabla T(x)$  as is illustrated in

$$\frac{T(x)}{\text{---}} + \frac{\nabla \phi(x)}{\text{----}} = \frac{\tilde{T}(x)}{\text{---}}$$

so

$$\frac{\tilde{T}(x)}{\text{---}} = \frac{\mathcal{Z}(x)}{\text{---}} + \frac{H(x) \approx \nabla T(x) \nabla \phi(x)}{\text{---}}$$

but the  
magnitude of  
this is  
attenuated  
by  $\nabla T(x)$ .

this still  
approximates  
 $T(x)$

To solve this notice that

$$\nabla \mathcal{Z}(x) H(x) \approx \nabla T(x)^2 \nabla \phi(x)$$

$$\text{so } \int dx e^{ix \cdot k} \nabla \mathcal{Z}(x) H(x)$$

Should approximate  $\phi(x)$  up to a normalizing constant depending on  $C_{kk}^{TT}$  &  $C_{kk}^{dd}$ . This is exactly what  $\mathcal{Z}$  &  $H$  are in  $\hat{\phi}_e$ :

$$\mathcal{Z}_k := \frac{C_{kk}^{TT}}{C_{kk}^{dd}} d_k = \begin{matrix} \text{Filter} \\ \text{d to} \\ \text{est T} \end{matrix}$$

$$H_k := \frac{d_k}{C_{kk}^{dd}} = \begin{matrix} \text{High} \\ \text{pass filter} \end{matrix}$$

The Frequentist properties of  $\hat{\phi}_e$

(18) One of the amazing things about  $\hat{\phi}_e$  is the fact that one can get a good understanding of  $C_{kk}^{\hat{\phi}\hat{\phi}}$  marginalizing over both  $T, \phi$  &  $n$ .

The variance of  $\hat{\phi}_e$ , conditional on  $\phi_e$ , is approximated by studying

$$\hat{\phi}_e^0 := A_e^{-1} \int dk \frac{(k+l) C_{k+l}^{TT} - k C_k^{TT}}{C_{k+l}^{dd} C_k^{dd}} d_{k+l}^0 d_k^0$$

$$\text{where } d_k^0 = T_k + n_k$$

$\uparrow$   
so  $d_k^0$  has no lensing signal

$$\mathbb{E}(\hat{\phi}_e^0) \approx 0$$

Now approximate the error distribution

$$\hat{\phi}_e - \phi_e \sim \hat{\phi}_e^0$$

Since

$$\mathbb{E}(\hat{\phi}_e^0 \hat{\phi}_{e'}^0) =$$

$$2\pi^2 A_e A_{e'} \int dk dk' \frac{f_{ke}^* f_{ke'}^*}{C_{k+l}^{dd} C_k^{dd} C_{k'+l'}^{dd} C_{k'}^{dd}}$$

$$\times \mathbb{E}(d_{k+l}^0 d_k^0 d_{k'+l'}^0 d_{k'}^0)$$

Where using Gaussian wkt's then  
one has

(19)

$$\begin{aligned} E(\hat{\phi}_{\ell+\ell'}^{\circ} \bar{\phi}_{\ell}^{\circ} \bar{\phi}_{\ell'+\ell}^{\circ} \hat{\phi}_{\ell'}^{\circ}) \\ = \cancel{\int_{\ell} \int_{\ell'} C_{\ell+\ell'}^{dd} C_{\ell'}^{dd}} + \int_{\ell+\ell'+\ell} \int_{\ell+\ell'+\ell} C_{\ell+\ell'}^{dd} C_{\ell'}^{dd} \\ \text{since we can either assume } \ell=0 \text{ or } \ell' \neq 0 \\ + \int_{\ell-\ell'+\ell-\ell'} \int_{\ell-\ell'} C_{\ell+\ell'}^{dd} C_{\ell'}^{dd} \\ = 2 \int_{\ell-\ell'} \int_{\ell-\ell'} C_{\ell+\ell'}^{dd} C_{\ell'}^{dd} \end{aligned}$$

since  $\ell+\ell'+\ell = \ell+\ell'+\ell'=0 \Rightarrow \ell=\ell' \& \ell=\ell'$   
 $\& \ell-\ell'+\ell-\ell' = \ell-\ell'=0 \Rightarrow \ell=\ell'$

Plugging this back into  $E(\hat{\phi}_{\ell}^{\circ} \bar{\phi}_{\ell'}^{\circ})$

gives

$$\begin{aligned} E(\hat{\phi}_{\ell}^{\circ} \bar{\phi}_{\ell'}^{\circ}) &= 2\pi^2 A_e^2 \int_{\ell-\ell'} \\ &\times \int d\ell \frac{|f_{\ell, \ell'}|^2}{(C_{\ell+\ell'}^{dd} C_{\ell'}^{dd})^2} 2 C_{\ell+\ell'}^{dd} C_{\ell'}^{dd} \end{aligned}$$

Using the approximation  $C_{\ell}^{dd} \approx C_{\ell}^{dd}$

$$E(\hat{\phi}_{\ell}^{\circ} \bar{\phi}_{\ell'}^{\circ}) \approx 2A_e^2 \int_{\ell-\ell'} \underbrace{\left[ 2\pi \int d\ell \frac{|f_{\ell, \ell'}|^2}{C_{\ell+\ell'}^{dd} C_{\ell'}^{dd}} \right]}_{= 2A_e}$$

$$= 2A_e \int_{\ell-\ell'} A_e^{-1}$$

$\therefore$  Marginalizing over  $T, \phi$  &  $n$

$$\begin{aligned} E(\hat{\phi}_{\ell}^{\circ} \bar{\phi}_{\ell'}^{\circ}) &\approx \int_{\ell-\ell'} (C_e^{\phi\phi} + 2A_e) \\ &\approx C_e^{\phi\phi} \end{aligned}$$

Estimating  $C_e^{\phi\phi}$

(20)

Given data  $d_{\ell} = \tilde{T}_{\ell} + n_{\ell}$

construct the quadratic estimate  $\hat{\phi}_{\ell}$  and the associated bandpowers

$$\sigma_{\ell}^{\hat{\phi}} = \frac{1}{N_e} \sum_{k \in \mathcal{R}_e} [|\hat{\phi}_k|^2 / s_0]$$

where  $\mathcal{R}_e$  indexes a set of frequencies in a collection of non-overlapping annuluses (with radius  $l$ ) &  $N_e$  is the number of frequencies in each  $\mathcal{R}_e$ .

Now set  $\hat{C}_e^{\phi\phi}$  to be

$$\hat{C}_e^{\phi\phi} := \sigma_e^{\hat{\phi}} - 2A_e \quad \text{assuming rotationally invariant } C_k^{nn}$$

so that

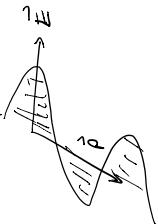
$$E(\hat{C}_e^{\phi\phi}) \approx C_e^{\phi\phi}$$

$$\text{var}(\hat{C}_e^{\phi\phi}) \approx \frac{1}{N_e} (C_e^{\phi\phi} + 2A_e)^2$$

Note: There are bias correction terms  $\mathcal{O}(C_e^{\phi\phi}) + \mathcal{O}((C_e^{\phi\phi})^2) + \dots$  the first two of which need to be accounted for when doing precision cosmology from  $\hat{C}_e^{\phi\phi}$ .

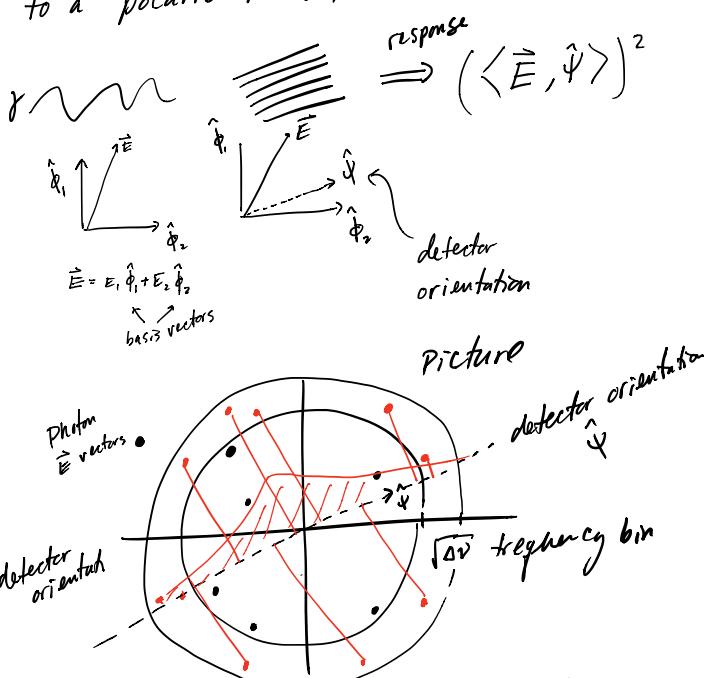
# Polarization

A photon is characterized by momentum  $\vec{p} \in \mathbb{R}^3$  & an electromagnetic vector  $\vec{E} \in \mathbb{R}^3$  describing the orientation of the propagating wave



and  $\vec{E}$  is equivalent to  $-\vec{E}$ .

If one defines  $\vec{E}$  to have magnitude equal to  $|\vec{p}|^{1/2}$  then the response to a polarization detection



The observation is the variance of  $\langle \vec{E}, \hat{\psi} \rangle$  for  $\vec{E}$  restricted to the frequency bin.

$$\begin{aligned} \text{response} &\Rightarrow T_1 = E(E_1^2) && \text{in units of } E(\vec{p})^2 = E(\vec{E}^2) \\ \text{response} &\Rightarrow T_2 = E(E_2^2) \\ \text{response} &\Rightarrow T_3 = E\left(\langle \vec{E}, \left(\frac{i_1}{\sqrt{2}}\right)^2 \rangle\right) = E\left(\frac{(E_1 + E_2)^2}{2}\right) \\ \text{response} &\Rightarrow T_4 = E\left(\langle \vec{E}, \left(\frac{-i_1}{\sqrt{2}}\right)^2 \rangle\right) = E\left(\frac{(E_1 - E_2)^2}{2}\right) \end{aligned}$$

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$$\Sigma = \begin{pmatrix} I+Q & U \\ U & I-Q \end{pmatrix}$$

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$$\therefore Q = (\Sigma_{11} - \Sigma_{22})/2$$

$$U = (2 \Sigma_{12})/2$$

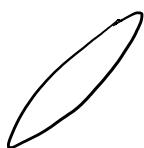
measure the ellipticity of  $\Sigma$  as follows

$$Q > 0, U = 0 \quad Q < 0, U = 0$$



$$Q = 0, U > 0$$

$$Q = 0, U < 0$$



This viewpoint of  $Q, U$  also makes clear how  $Q, U$  transform under rotations.

$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}}_{R_\theta} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

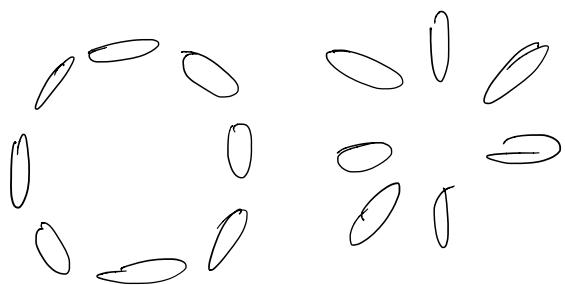
$$\therefore \Sigma' = R_\theta \Sigma R_\theta^T$$

$$= \begin{pmatrix} T + Q \cos(2\theta) - U \sin(2\theta) & U \cos(2\theta) + Q \sin(2\theta) \\ U \cos(2\theta) + Q \sin(2\theta) & T - Q \cos(2\theta) + U \sin(2\theta) \end{pmatrix}$$

$$\therefore \begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} Q \cos(2\theta) - U \sin(2\theta) \\ Q \sin(2\theta) + U \cos(2\theta) \end{pmatrix}$$

(23)

$E(\vec{n}), B(\vec{n})$  decomposition of  $\Omega(\vec{n}), \mathcal{U}(\vec{n})$



(24)