Lecture 5

Outline:

1) Simplified data model
2) Nonparametric $\hat{C}_\ell^{TT}$
3) cosmic variance & Noise bias.
4) Non parametric posterior samples.

Start by studying estimation in a simplified case: $T(\hat{n})$ observed everywhere with white noise & a beam

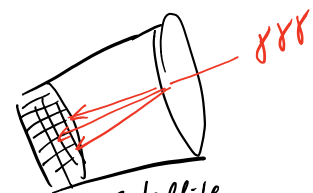
Assume the observed field $d(\hat{n})$ is measured on all $\hat{n} \in S^2$ & has the form

$$(*) \quad d(\hat{n}) = \underbrace{\varphi}_{\text{models a beam point spread function.}} * T(\hat{n}) + \underbrace{\varepsilon(\hat{n})}_{\text{white noise s.t. } E(\varepsilon_{\ell m} \bar{\varepsilon}_{\ell' m'}) = \sigma^2 \delta_{\ell \ell'} \delta_{m m'}}$$

Note: the fact that $\varepsilon(\hat{n})$ is white noise we expect statistical properties of Model $(*)$ to be similar to the case of finite pixel observations

The beam $\varphi(\hat{n})$

As the CMB pass through the detector lense they get defracted and spread out



Satellite.

The effect on the observed $T(\hat{n})$ is modeled as a convolution with a point spread function $\varphi(\hat{n})$.

on S^2

- Signal $T(\hat{n}), \hat{n} \in S^2$
- $\varphi * T(\hat{n}) = \int_{g \in SO^3} T(g\underset{\uparrow \text{North pole}}{0}) \varphi(g^{-1}\hat{n}) dg$
- $(\varphi * T)_{\ell m}$

$$= 2\pi \sqrt{\frac{4\pi}{2\ell+1}} T_{\ell m} \varphi_{\ell, 0}$$

- How do we model $\varphi(\hat{n})$ (equiv $\varphi_{\ell m}$) on S^2

on $\mathbb{R}^2$

- Signal $T(x)$, $x \in \mathbb{R}^2$
- $\varphi * T(x) = \int_{\mathbb{R}^2} T(y) \varphi(x-y) dy$
- $(\varphi * T)_k = \iint_{\mathbb{R}^2 \mathbb{R}^2} e^{-ix \cdot k} T(y) \varphi(x-y) \frac{dx\,dy}{(2\pi)^{d/2}}$

$$\overset{z = x-y}{\hookrightarrow} = \iint e^{-iz \cdot k} e^{-iy \cdot k} T(y) \varphi(z) \frac{dz\,dy}{(2\pi)^{d/2}}$$

$$= (2\pi)^{d/2} T_k \varphi_k$$

- $\varphi(x)$ modeled as

$$\frac{1}{2\pi^{d/2} \sigma^n} \exp\left(-\frac{1}{2} \frac{|x|^2}{\sigma^2}\right)$$

$$\therefore \varphi_k = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{\sigma^2}{2} |k|^2\right)$$

$$\cong \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{b^2}{8\log(2)} |k|^2\right)$$

where b = full width half max

$$\therefore (\varphi * T_k) = \exp\left(-\frac{b^2}{8\log 2} |k|^2\right) T_k$$

For $x \in \mathbb{R}^2$, $\varphi(x)$ has the property that the solution to

$$\Delta u^t(x) = \frac{d}{dt} u^t(x)$$

is expressed

$$u_k^t = \exp\left(-t\,|k|^2\right) u_k^0$$

$$= \left(\varphi * u^0\right)_k, \quad \text{for } t = \frac{b^2}{8\log(2)}.$$

On the sphere Δ is defined and has the property that the solution to

$$\Delta u^t(\hat{n}) = \frac{d}{dt} u^t(\hat{n})$$

is expressed as

doesn't depend on m

$$u_{\ell m}^t = \exp\left(-t\,\ell(\ell+1)\right) u_{\ell m}^0$$

$$= \left(\varphi * u^0\right)_{\ell m}$$

i.e. $\varphi_{\ell m} = \frac{1}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}} \exp\left(-t(\ell+1)\ell\right)$

is how we model the beam in the simplified case on S^2.

Since we assume observations of $(*)$ everywhere on S^2 we can take the spherical transform to obtain:

$$d_{\ell m} = \exp\left(-\frac{b^2}{8\log(2)}\,\ell(\ell+1)\right) T_{\ell m} + \varepsilon_{\ell m}$$

Dividing out the beam (& renaming the $d_{\ell m}$, $\varepsilon_{\ell m}$) we can assume w.l.g the data is in the form of an infinite sequence of spherical coefficients:

$$d_{\ell m} = T_{\ell m} + \varepsilon_{\ell m}$$

for $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell+1, \ldots, \ell$ s.t.

$$E\left(T_{\ell m} T^*_{\ell' m'}\right) = \delta_{\ell\ell'}\delta_{mm'} C_\ell^{TT}$$

$$E\left(\varepsilon_{\ell m} \varepsilon^*_{\ell' m'}\right) = \delta_{\ell\ell'}\delta_{mm'}\, \underset{\text{additive noise level}}{\sigma^2}\, \underset{\text{beam.}}{\exp\left(\frac{b^2}{8\log(2)}\,\ell(\ell+1)\right)}$$

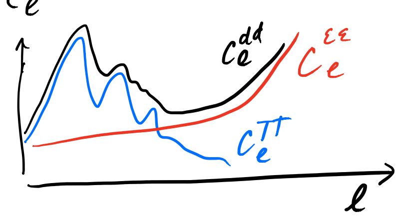
Notice

$$C_\ell^{dd} = E|d_{\ell m}|^2$$

$$= E|T_{\ell m}|^2 + 2\underbrace{E\left(T_{\ell m}\bar{\varepsilon}_{\ell m}\right)}_{=0} + E|\varepsilon_{\ell m}|^2$$

$$= C_\ell^{TT} + \sigma^2 \exp\left(\frac{b^2}{8\log(2)}\,\ell(\ell+1)\right)$$

$$= C_\ell^{dd}$$

$=$ 

Nonparametric estimates of C_ℓ^{TT}

Let $\boxed{\sigma_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}|d_{\ell m}|^2} =$ "band power"

Now
$$E(\sigma_\ell) = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} E|d_{\ell m}|^2$$
$$= C_\ell^{dd}$$
$$= C_\ell^{TT} + C_\ell^{\varepsilon\varepsilon}$$

Noise bias... but known and can be subtracted.

Now
$$var(\sigma_\ell) = E(|\sigma_\ell|^2) - (C_\ell^{dd})^2$$
$$= \frac{1}{(2\ell+1)^2}\sum_{m=-\ell}^{\ell}\sum_{m'=-\ell}^{\ell} E(d_{\ell m} d_{\ell m}^* d_{\ell m'} d_{\ell m'}^*) - (C_\ell^{dd})^2$$

where

$$E(d_{\ell m} d_{\ell m}^* d_{\ell m'} d_{\ell m'}^*) = E(d_{\ell m} d_{\ell m}^* d_{\ell m'} d_{\ell m'}^*) + E(d_{\ell m} d_{\ell m}^* d_{\ell m'} d_{\ell m'}^*) + E(d_{\ell m} d_{\ell m}^* d_{\ell m'} d_{\ell m'}^*)$$

Wicks thm by Gaussianity

$$= E|d_{\ell m}|^2 E|d_{\ell m'}|^2 + E(d_{\ell m} d_{\ell m'})E(d_{\ell m}^* d_{\ell m'}^*) + E(d_{\ell m} d_{\ell m'}^*)E(d_{\ell m}^* d_{\ell m'})$$
$$= (C_\ell^{dd})^2 + (-1)^m\delta_{m,-m'} C_\ell^{dd} (-1)^m \delta_{-m,m'} C_\ell^{dd} + \delta_{mm'} C_\ell^{dd} \delta_{m,m'} C_\ell^{dd}$$

$$\therefore\ var(\sigma_\ell) = (C_\ell^{dd})^2 + \frac{2}{(2\ell+1)^2}\sum_{m=-\ell}^{\ell}(C_\ell^{dd})^2 - (C_\ell^{dd})^2$$
$$= \frac{2}{2\ell+1}(C_\ell^{dd})^2$$

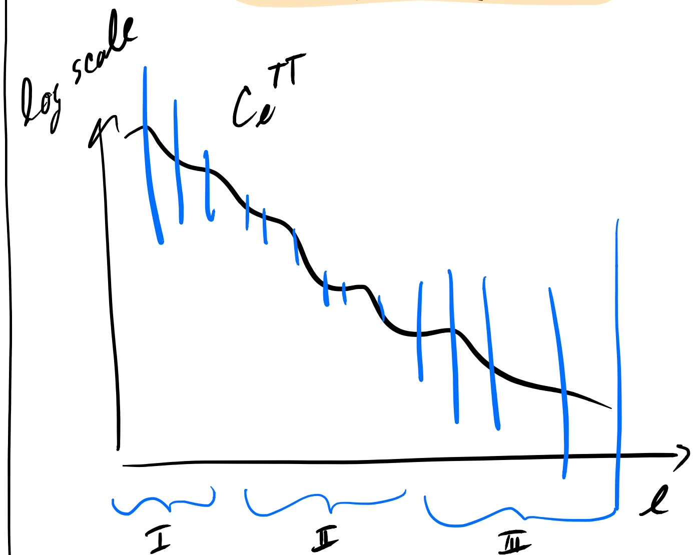
$$\therefore\ var(\sigma_\ell) = \frac{2}{2\ell+1}(C_\ell^{TT} + C_\ell^{\varepsilon\varepsilon})^2$$

Now a nonparametric estimate of C_ℓ^{TT} can be constructed as:

$$\hat{C}_\ell^{TT} := \sigma_\ell - C_\ell^{\varepsilon\varepsilon}$$

where

$$E(\hat{C}_\ell^{TT}) = C_\ell^{TT} \quad \&$$
$$sd(\hat{C}_\ell^{TT}) = \sqrt{\frac{2}{2\ell+1}}(C_\ell^{TT} + C_\ell^{\varepsilon\varepsilon})$$



Relative s.d:
$$\frac{\sqrt{\frac{2}{2\ell+1}}(C_\ell^{TT} + C_\ell^{\varepsilon\varepsilon})}{C_\ell^{TT}}$$
$$= \sqrt{\frac{2}{2\ell+1}}\left(1 + \frac{C_\ell^{\varepsilon\varepsilon}}{C_\ell^{TT}}\right)$$
$$= \begin{cases} \text{big}\left(1 + \frac{\text{small}}{\text{big}}\right) & \text{in I} \\ \text{small}(1+1) & \text{in II} \\ \text{small}\left(1 + \frac{\text{big}}{\text{small}}\right) & \text{in III} \end{cases}$$

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Cosmic variance

$P(C_\ell \mid \sigma_\ell)$

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