Statistics 250

- Technically a course on Applied & Computational Stats
- It is really a graduate topics course on statistics & the Cosmic Microwave background.
- The "topics" part means that we will be learning stuff together. Also, no exams but I'll ask you to do homeworks & present a project.

What are we going to learn?

- A basic "cartoonish" understanding of the physics
- I'm going to try & structure the course around particular statistical methods & results ... lecturing on the stat & physical theory as needed.
- We will be analyzing real data & try to re-produce as many results as possible.
- Much of what is done for the CMB is Bayesian & heavily computational.
- ... so I've decided that part of this course will be an introduction to a new programming language called Julia. I'll give tutorials along the way but you should expect to submit homework through github with IJulia notebooks.
- Random fields also play a big part in describing the CMB & we'll study that as well.
- Lastly let me remark that when analyzing the CMB, you get to be a "true Bayesian" in that many of the unknowns have "subjective" priors from other experiments. Moreover the models are physical (even the randomness).
- Next class bring laptops & we will have a group session to install Julia.

What is the CMB?

Expanding Universe

* First we need to make sure we understand that the universe is expanding & of infinite extent in all directions.

* Not really that things are moving away from eachother, rather that space itself is growing.

* Moreover, the growth rate appears to chance in time:

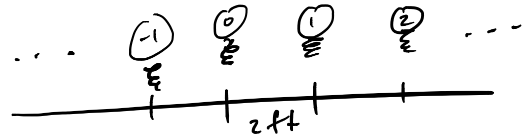
early time & very briefly { very fast growth rate a short time after the "Big bang". This period is called inflation

Now { Slower growth rate

* One consequence: even if the groth rate is constant, object appear to be accelerating away from eachother.

E.g. 40 watt light bulbs on a grid.

time = 1pm



time = 2pm

distance btwn (0) & (1) grew @ 2ft per hr.

distan btwn (0) & (2) grew @ 4ft per hr.

Other consequences

1. photons travel @ a constant speed.

∴ From the perspective of (0)

(1) looks more red than a stationary 40 watt bulb.

(2) looks even more red than (1)

⋮

etc.

2. If (0) knows the growth rate then (0) can determine which bulb a given photon came from by analyzing how red it is

3. A red photon is a snapshot of the past. More red ⇒ further back in the past it was released.

Going backwards in time

I think the best way to think about the "Big Bang" is simply by reversing time ... if space has always expanding then going backward in time space shrinks.

As space shrinks, photons, matter (& anything else) get closer together i.e. density ↑.

A some point you get to a point in the past where everything is sitting on top of eachother w/ infinite density.

This point in time is effectively the "big bang".

<u>Note</u>: The expansion of space (typically modeled with a scale factor $a(t)$) & the energy density $\rho(t)$ are dependent on one another through the Friedman equation

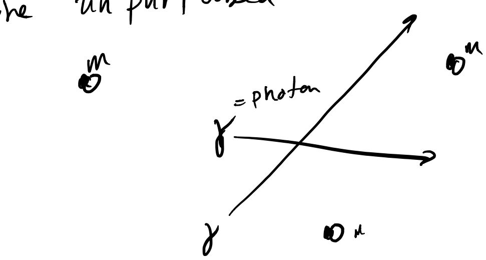
e.g. $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$

ρ = energy density
G = Gravitatinal constant
Λ = cosmological constant
K = +1, 0 or -1 (curvature)
c = speed of light

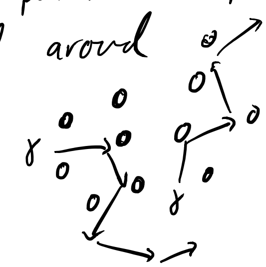
I guess this implies $\dot{a} \to \infty$ as we go backward in time.

So at the big bang everything was sitting on top of eachother but moving apart at an infinite rate.

✱ Present day there is enough space btwn matter etc that photons mostly travel in a straight line unpurturbed.



✱ Going backwards in time there was a point when photons mostly bounced around

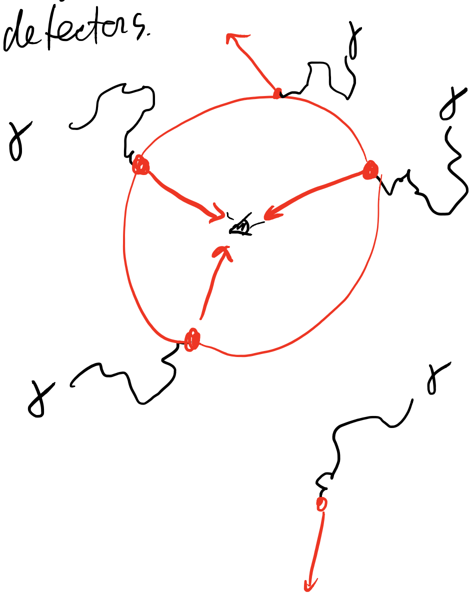


The transition between the two regemes happend in a relative instant (@ <u>recombination</u>)

The state the universe @ this instant is called the <u>last</u> <u>scattering</u> <u>surface</u>

Most of these photons are still just traveling, unperturbed, in their direction determined @ recombination

... but a select few were lucky enough to travel right to our detectors.



We know these γ came from the last scattering surface by their redshift.

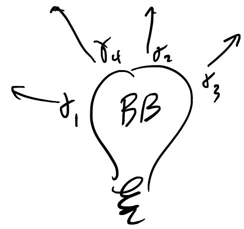
What are we measuring & what does it have to do with the surface of last scattering

Each photon γ is characterized by $(\vec{x}, \vec{p})$ $\vec{x}, \vec{p} \in \mathbb{R}^3$

$\vec{x}$: position

$\vec{p}$: momentum — characterizing the direction of travel & wavenumber $|\vec{p}|$ (i.e. wavelength $\propto \frac{1}{|\vec{p}|}$)

If you have a perfect "black body" light bulb the photons γ coming off it will have momentum drawn at random from a particular distribution



Photons	Position	momentum
γ_1	$\vec{x}$	$\vec{p}_1$
γ_2	$\vec{x}$	$\vec{p}_2$
$\vdots$	$\vdots$	$\vdots$

location of the Bulb

$$\vec{p}_i \overset{iid}{\sim} \underbrace{f(\vec{p})\,d\vec{p}}_{\text{prob measure on } \mathbb{R}^3}$$

where $f(\vec{p}) \propto \left[\exp\left(\frac{E(\vec{p})}{T(\vec{x})}\right) - 1\right]^{-1}$

$T(\vec{x})$ is the temperature of bulb.

$E(\vec{p})$ is the "energy" of a photon

$$= \sqrt{m^2 + |\vec{p}|^2} = |\vec{p}|$$

mass $= 0$

$$\therefore \; f(\vec{p}) \propto \left[\exp\left(\frac{|\vec{p}|}{T(\vec{x})}\right) - 1\right]^{-1}$$

∴ The random wavenumber $|\vec{p}|$ satisfies.

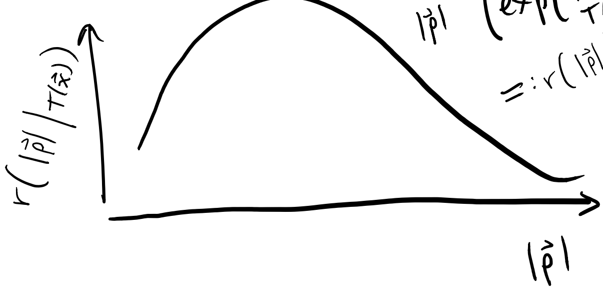
$$P\left(|\vec{p}| \in (r, r+dr)\right) \propto \int_{r \le \|\vec{p}\| \le r+dr} f(\vec{p})\, d\vec{p} =$$

$$= \iint_r^{r+dr} \left(\exp\left(\frac{|\vec{p}|}{T(\vec{x})}\right)-1\right)^{-1} r^{3-1}\, dr\, d\theta$$

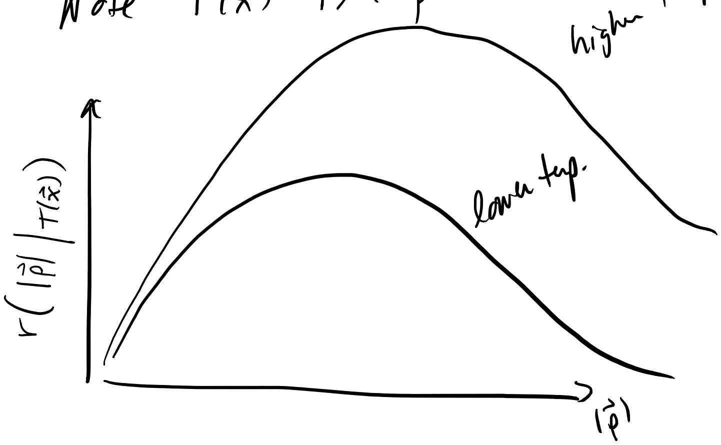
$$\propto \int_r^{r+dr} r^2 \left(\exp\left(\frac{r}{T(\vec{x})}\right)-1\right)^{-1} dr$$

∴ $|\vec{p}_1|, |\vec{p}_2|, \ldots$ are iid draws from

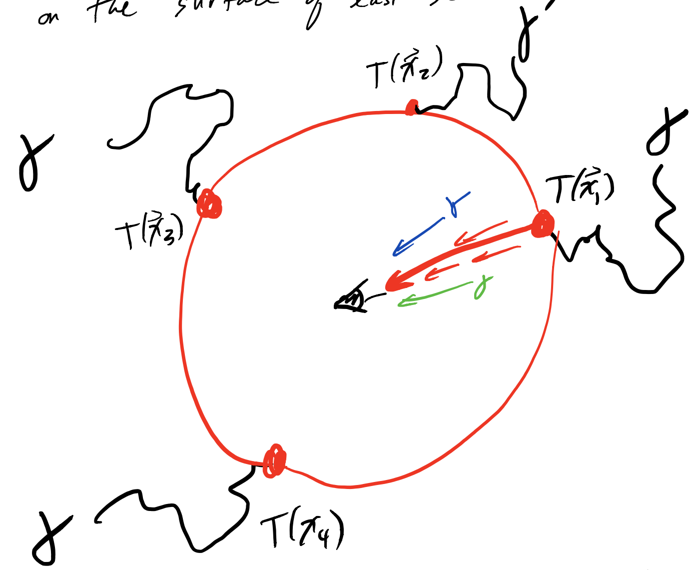
$$|\vec{p}|^2 \left(\exp\left(\frac{|\vec{p}|}{T(\vec{x})}\right)-1\right)^{-1} =: r\left(|\vec{p}| \,\middle|\, T(\vec{x})\right)$$



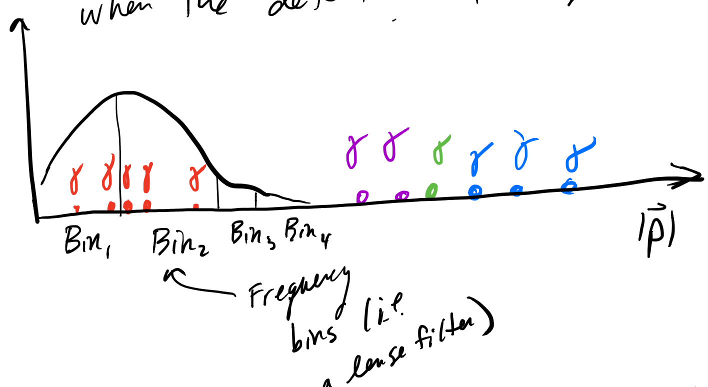
Note $T(\vec{x})$ is a parameter



- Also note that $|\vec{p}|$ shifts (left) if the bulb is moving away.
- We effectively measure $|\vec{p}|$ from a spherical trace of the photons on the surface of last scattering.



when the detector is pointing to $\vec{x}_1$



Let N denote the number of CMB photons γ entering the detector.

& the detector measures energy intensity in each Bin_i:

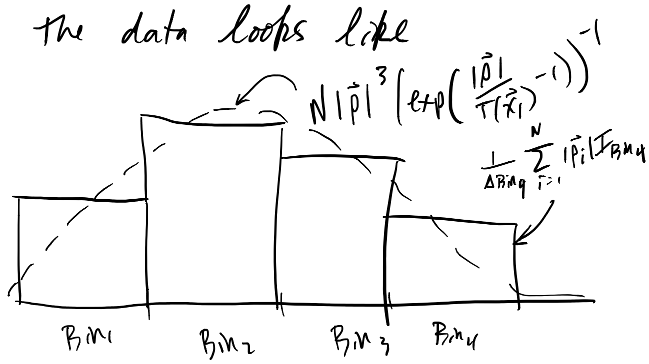
$$\sum_{i=1}^{N} |\vec{p}_i| I_{Bin_i} \approx N\, E\left(|\vec{p}|\, I_{Bin_i}(|\vec{p}|)\right)$$

$$= N \int_{Bin_i} |\vec{p}_i|\, r\left(|\vec{p}| \,\middle|\, T(\vec{x}_1)\right) d|\vec{p}_i|$$

$$\approx |\vec{p}_{Bin_i}|^3 \left(\exp\left(\frac{|\vec{p}_{Bin_i}|}{T(\vec{x}_1)}\right)-1\right)^{-1} (N \Delta Bin_i)$$

Not observed but is the same $\forall\, Bin_i$

$\therefore$ the data looks like



at each $\hat{n}_i = \frac{\vec{x}_i}{|\vec{x}_i|}$ = obs direction.

Now let $\hat{T}(\hat{n}_i)$ be an estimate (like MLE) of $T(\vec{x}_i)$ at each $\hat{n}_i$.

So there was some temperature map $\{T(\vec{x}): \vec{x} \in \mathbb{R}^3\}$ @ recombination. Our temperature estimates, as a function of obs direction $\hat{n}$

$$\{\hat{T}(\hat{n}) : \hat{n} \in S^2\}$$

is an approximation to a spherical trace of $T(\vec{x})$:

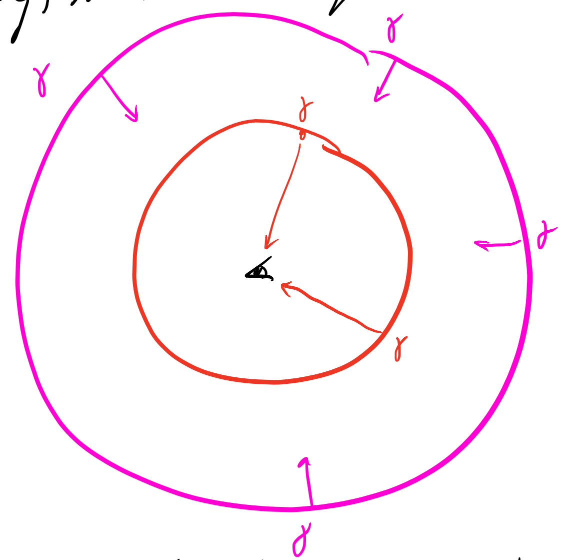
$$\{T(\vec{x}) : |\vec{x}| = r\}$$

<u>Quiz</u>: Are the CMB photons the most red shifted "i.e. distant" photons we will ever observe?

what happens in a million years ... will we look "past" the CMB?

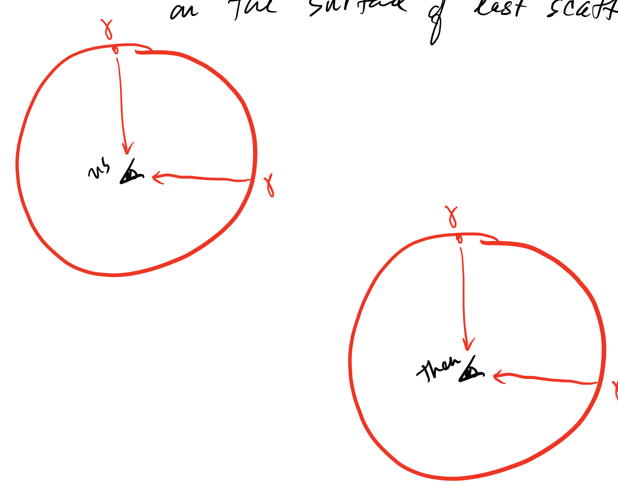
<u>Answer</u>:

- all the info behind the CMB is lost.
- in a Million years we will observe more "red" photons but these will still be from the surface of last scattering, but from a larger sphere



<u>Quiz</u>: What do aliens see when they look @ the CMB?

<u>Answer</u>: They see a different spherical trace of the $\{T(\vec{x}): \vec{x} \in \mathbb{R}^3\}$ on the surface of last scattering.



<u>Show pictures</u>