Lecture 11

Topics:

I) Lensing

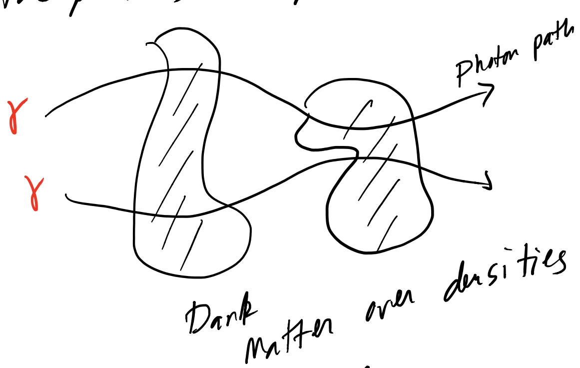
II) Polarization

III) the quadratic estimate

IV) The lensing bispectrum

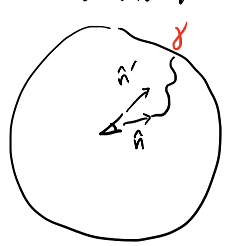
V) Bayesian lensing.

As a CMB photon travels towards us, it passes through/near clumps of dark matter. Since dark matter imposes a gravitation influence on the photon, its path is distorted



This distortion is called **Gravitational lensing**

The basic effect is that when we observe CMB photons (from a blackbody spectrum) in direction $\hat{n}$, they actually emitted from a distorted direction $\hat{n}'$



$\therefore$ on our observed temperature denoted $\tilde{T}(\hat{n})$ is actually a warped/lensed verion of the temp $T(\hat{n})$ on the surface of last sattering:

$$\tilde{T}(\hat{n}) = T(\hat{n}')$$

where $\hat{n}'$ varies with $\hat{n}$.

Flat sky

For the remainder of this lecture we focus on the flat sky approximation. i.e. the observed lensed temperature field (without noise) takes the form

$$\tilde{T}(x) = \underbrace{T(x')}_{\text{temp on the last scattering surface}}, \quad x, x' \in \mathbb{R}^2.$$

The lensing Potential

Under some simplified assumptions on the geometry of the photon path as well as on the nature of the dark matter the lensing of the CMB photons are characterized by a lensing (or gravitational) potential

$$\phi(x) = \text{a weighted line of sight integral over the 3-d gravitational potential.}$$

where $\nabla\phi(x)$ gives the photon displacement due to lensing &

$\Delta\phi(x) \simeq$ projected matter density.

3

$\therefore\ \tilde{T}(x) = T(x+\nabla\phi(x))$

Back of the envelope calculations (& data) establish that

$$\underbrace{\sqrt{E\,|\nabla\phi(x)|^2}}_{\text{typical size of the lensing displacement}} \approx 2.7 \text{ arcmins}$$

But that $\phi(x)$ is coherent (or correlated) over large spatial scales on the order of degrees.

Moreover, since the lensing power is strongest at the midpoint btwn us & the surface of last scattering the mass distribution is relatively smooth & diffuse.

Along with the Einstien principle one can therefore model $\phi(x)$ as a isotropic Gaussian random field with

$$E(\phi(x)) = 0$$

since only the fluctuations in $\phi(x)$ generate lensing.

with spectral density

$$E(\phi_k \phi_{k'}) = C_k^{\phi\phi}\,\delta_{k-k'}$$

Moreover we can assume

$$E(\phi_k T_{k'}) \equiv 0 \quad \forall k,k' \in \mathbb{R}^2$$

i.e. ϕ & T are essentially uncorrelated.

4

Is $\tilde{T}$ non-isotropic or non-Gaussian?

There are two ways to view the impact of lensing.

1) if one considers $\phi(x)$ fixed, i.e. conditioning on $\phi(x)$, then $\tilde{T}(x)$ is a non-stationary Gaussian random field since

$$\left(\tilde{T}(x_1),\ldots,\tilde{T}(x_n)\right)^T = \left(T(x_1+\nabla\phi(x_1)),\ldots,T(x_n+\nabla\phi(x_n))\right)^T \sim N(\underset{\approx 2.7K}{T_{CMB}}, \Sigma)$$

where $\Sigma_{ij} := C^T(|x_i - x_j + \nabla\phi(x_i) - \nabla\phi(x_j)|)$
$=$ non-stationary cor fun.

2) If, in contrast, one marginalizes over $\phi(x)$ then $\tilde{T}(x)$ becomes isotropic again but is now non-Gaussian. We will see this later by analyzing the Bispectrum of $\tilde{T}$ (which is zero for isotropic GRFs).

Show pictures of simulations of $\tilde{T}$, T & ϕ

The impact of lensing on $C^{\tilde T\tilde T}_{|k|}$ when marginalizing over $\phi(x)$

5

Modeling $\phi(x)$ as a isotropic GRF on can compute the flat sky spectral density of $\tilde T$, even though $\tilde T$ is non-Gaussian (so $C^{\tilde T\tilde T}_{|k|}$ will not completely characterize the distribution of $\tilde T$).

First take a Fourier decomposition of $T(x)$:

$$T(x) = \int \underline{dk}\, e^{ix\cdot k} T_k$$

this is my shorthand for $\frac{dk}{(2\pi)^{d/2}}$ where $d=2$ in this case.

$$\therefore\ \tilde T(x) = T(x+\nabla\phi(x))$$

$$= \int \underline{dk}\, e^{i(x+\nabla\phi(x))\cdot k} T_k$$

$$\therefore\ E(\tilde T(x)\tilde T(y)) = E\int \underline{dk}\,\underline{dk'}\, e^{i(x+\nabla\phi(x))\cdot k} e^{-i(y+\nabla\phi(y))k'} \times T_k T^*_{k'}$$

$$= \int \underline{dk}\,\underline{dk'}\, e^{ix\cdot k - iyk'} E\left(e^{i(\nabla\phi(x)\cdot k - \nabla\phi(y)\cdot k')} T_k T^*_{k'}\right)$$

Now

$$E\left(e^{i\nabla\phi(x)\cdot k - i\nabla\phi(y)\cdot k'} T_k T^*_{k'}\right)$$

$$= E\left[e^{i(\nabla\phi(x)k - \nabla\phi(y)\cdot k')}\right]\delta_{k-k'} C^{TT}_{|k|}$$

$$= \underbrace{E\left[e^{i(\nabla\phi(x)-\nabla\phi(y))k}\right]}\delta_{k-k'} C^{TT}_{|k|}$$

has the form $E(e^{iX\cdot k})$ for a gaussian vector $X\sim N(0,\Sigma)$ which, by characteristic functions, equals $\exp(-\frac{1}{2}k^T\Sigma k)$

6

The covariance matrix of $\nabla\phi(x)-\nabla\phi(y)$ is given by

$$E\left[(\nabla\phi(x)-\nabla\phi(y))(\nabla\phi(x)-\nabla\phi(y))^T\right]$$

$$= 2\left(\Sigma^{\nabla\phi}(0) - \Sigma^{\nabla\phi}(x-y)\right)$$

where $\Sigma^{\nabla\phi}(x-y) := E(\nabla\phi(x)\nabla\phi(y)^T)$.

$\therefore$ $E(\tilde T(x)\tilde T(y))$ is given by

$$\int \underline{dk}\,\underline{dk'}\, \exp\left(ix\cdot k - iyk + k^T\left[\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)\right]k\right)\delta_{k'-k} C^{TT}_{|k|}$$

$$= \int \underline{\underline{dk}}\, e^{i(x-y)k} e^{k^T\left[\Sigma^{\nabla\phi}(x-y)-\Sigma^{\nabla\phi}(0)\right]k} C^{TT}_{|k|}$$

$$= C^{\tilde T}(x-y)$$ so, indeed, $\tilde T$ stationary

In fact isotropy can be shown by analyzing how $\Sigma^{\nabla\phi}(R(x-y))$ behaves for rotations R.

Expanding $e^{k^T\left[\Sigma^{\nabla\phi}(x-y)-\Sigma^{\nabla\phi}(0)\right]k}$ to first order gives

$$E(\tilde T(x)\tilde T(0))$$

$$\approx \int \underline{\underline{dk}}\, e^{ix\cdot k}\left[1 + k^T\left[\Sigma^{\nabla\phi}(x-y) - \Sigma^{\nabla\phi}(0)\right]k\right] C^{TT}_{|k|}$$

$$= C^T(|x|) - \sigma_R^2\int \underline{\underline{dk}}\, e^{ix\cdot k} |k|^2 C^{TT}_{|k|}$$

$$+ \sum_{i,j=}^{2} \Sigma^{\nabla\phi}_{ij}(x)\int \underline{\underline{dk}}\, e^{ix\cdot k} k_i k_j C^{TT}_{|k|}$$

$$= C^T(|x|) - \sigma_R^2\int \underline{\underline{dk}}\, e^{ix\cdot k}|k|^2 C^{TT}_{|k|}$$

$$+ \frac{1}{2\pi}\sum_{i,j}^{2}\left[-\frac{\partial^2 C^\phi}{\partial x_i\partial x_j}(x)\right]\left[-\frac{\partial^2 C^T}{\partial x_i\partial x_j}(x)\right]$$

where $\sigma_R^2 = E\left(\frac{\partial\phi}{\partial x_i}(x)\right)^2 = \frac{1}{2}E|\nabla\phi(x)|^2 \sim 3\times10^{-7}$

so that the typical deflection size $\sqrt{E|\nabla\phi(x)|^2}$ is about 2.7 arcmins.

Notice that the third term is a sum of pixelwis products $A(x)B(x)$.

Recall how to take the fourier transform of such functions

$$\int d\underline{x}\; e^{-ix\cdot l} A(x)B(x) = \int d\underline{x}\, d\underline{k}\; e^{-ix\cdot l} A(x)\, e^{ix\cdot k} B_k$$

$$= \int d\underline{k}\; B_{-k} \int d\underline{x}\; e^{-ix(k+l)} A(x)$$

$$= \int d\underline{k}\; A_{k+l} B_{-k}.$$

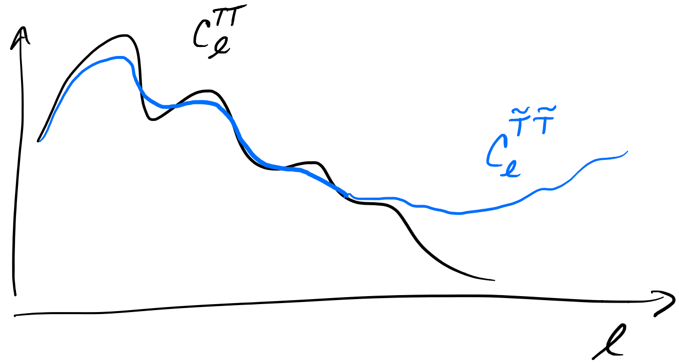
Now taking the fourier transform of $C^{\tilde T}(|x|) = E(\tilde T(x)\tilde T(0))$ gives $\frac{1}{(2\pi)^{d/2}} C^{\tilde T\tilde T}_{|k|}$ so that

$$C^{\tilde T\tilde T}_{|k|} = \left(1-\sigma_R^2 |k|^2\right) C^{TT}_{|k|} + \sum_{i,j=1}^{2} \int d\underline{k}'\, (k_i'+k_i)(k_j'+k_j)\, C^{\phi\phi}_{|k+k'|} \times (-k_i')(-k_j')\, C^{TT}_{|k'|}$$

$$= \left(1-\sigma_R^2|k|^2\right) C^{TT}_{|k|} + \int d\underline{k}' \left[(k+k')\cdot k'\right]^2 C^{\phi\phi}_{|k+k'|}\, C^{TT}_{|k'|}$$

$$= \left(1-\sigma_R^2|k|^2\right) C^{TT}_{|k|} + \int d\underline{k}' \left[(k-k')\cdot k'\right]^2 C^{\phi\phi}_{|k-k'|}\, C^{TT}_{|k'|}$$

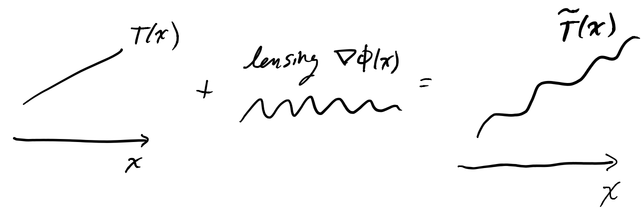
The last term is a convolution which smooths the peaks in $C^{TT}_{|k|}$ at low $|k|$ & adds power at high $|k|$



Intuitively the "peak smoothing" is due to $C_\ell^{\tilde T\tilde T}$ being thought of a a "average" of magnified or de-magnified C_ℓ^{TT} due to lensing.

The "added power" in $C_\ell^{\tilde T\tilde T}$ can be explained by the fact that at extremely small scales $T(x)$ is approximatly linear & $\nabla\phi(x)$ induces small ripples in this linear structure.

1-d analog at small scales



The quadratic estimate for $T(x)$

Consider the case where we are treating ϕ as fixed so that $\tilde{T}(x)$ is Gaussian but non-stationary.

Recall Bochers thm which states that any stationary RF $f(x)$ has uncorrelated fourier coefficients:

$$E(f_k f_{k'}^*) = C_k^{ff}\delta_{k-k'}$$

(assuming $E(f(x))=0$).

This condition is not only necessary but also sufficient.

$\therefore$ $\tilde{T}(x)$ must have correlated Fourier coefficients (due exclusively to ϕ).

To approximate this correlation (as a function of $\phi(x)$) Taylor expand $\tilde{T}(x)$ as follows

$$\tilde{T}(x) = T(x+\nabla\phi(x))$$

$$= \underbrace{T(x)}_{\text{isotropic}} + \underbrace{\nabla T(x)\cdot\nabla\phi(x)}_{\text{non-stationary}} + \mathcal{O}(\phi^2)$$

Therefore

$$\tilde{T}_k = T_k + \sum_{j=1}^{2}\int \underline{dk}'\ i(k'+k)_j T_{k'+k}\ i(-k')_j \phi_{-k'} + \mathcal{O}(\phi^2)$$

$$= T_k + \int \underline{dk}'\ (k'+k)\cdot k'\ T_{k'+k}\phi_{-k'} + \mathcal{O}(\phi^2)$$

Lets look at the covariance btwn $\tilde{T}_{k+\ell}$ & $\tilde{T}_k$:

$$E(\tilde{T}_{k+\ell}\tilde{T}_k^*) =$$

$$E(T_{k+\ell}T_k^*)$$

$$+ E\left(T_{k+\ell}\int \underline{dk}'\ (k'+k)\cdot k'\ T_{k'+k}^*\phi_{-k'}^*\right)$$

$$+ E\left(T_k^*\int \underline{dk}'\ (k'+k+\ell)\cdot k'\ T_{k'+k+\ell}\phi_{-k'}\right)$$

$$+ \mathcal{O}(\phi^2)$$

$$= \delta_\ell C_k^{TT}$$

$$+ \int \underline{dk}'\ (k'+k)\cdot k'\ \phi_{-k'}^*\ \underbrace{\delta_{k+\ell-k'-k}}_{k'=\ell}\ C_{k+\ell}^{TT}$$

$$+ \int \underline{dk}'\ (k'+k+\ell)\cdot k'\ \phi_{-k'}\ \underbrace{\delta_{-k+k'+k+\ell}}_{k'=-\ell}\ C_k^{TT}$$

$$+ \mathcal{O}(\phi^2)$$

$$= \delta_\ell C_k^{TT} + \frac{1}{2\pi}(k+\ell)\cdot\ell\ \phi_\ell C_{k+\ell}^{TT}$$

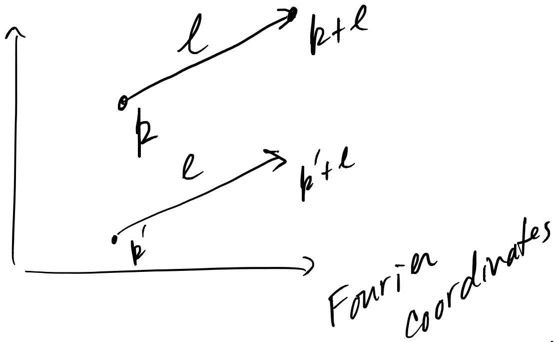
$$+ \frac{1}{2\pi}(k)(-\ell)\phi_\ell C_k^{TT}$$

$$+ \mathcal{O}(\phi^2)$$

$$= \underbrace{\delta_\ell C_k^{TT}}_{\text{This term is 0 when } \ell\neq 0} + \underbrace{\frac{\ell\cdot[(k+\ell)C_{k+\ell}^{TT} - kC_k^{TT}]}{2\pi}}_{\text{Call this } f_{k,\ell}\text{. known if } C_k^{TT} \text{ is known}}\phi_\ell + \mathcal{O}(\phi^2)$$

(11)

$\therefore \; E(\tilde T_{k+l}\tilde T_k^*) = f_{k,l}\,\phi_l + \mathcal{O}(\phi^2)$



A naive estimate of ϕ_l, at a fixed $l \in \mathbb{R}^2$ is then

$$\hat\phi_l = \frac{1}{N}\sum_{j=1}^{N} \frac{\tilde T_{k_j+l}\tilde T^*_{k_j}}{f_{k_j,l}}$$

when $\tilde T_k$ is observed at frequencies $k_1,\dots,k_N$ & $k_1+l,\dots,k_N+l$.

Note $\hat\phi_l$ is unbiased up to $\mathcal{O}(\phi^2)$.

$$E(\hat\phi_l) = \phi_l + \mathcal{O}(\phi^2)$$

Fixing ϕ & considering T_k as random

A much better estimate can be found by doing an approximate inverse weighted ave estimate ...

(12)

$$\hat\phi_l = \frac{1}{N}\sum_{j=1}^{N} \underbrace{w_{k_j,l}}_{\text{weights}} \frac{\tilde T_{k_j+l}\tilde T^*_{k_j}}{f_{k_j,l}}$$

s.t.

$$w_{k_j,l} \propto \frac{1}{var(\tilde T_{k_j+l}\tilde T^*_{k_j}/f_{k_j,l})} = \frac{|f_{k,l}|^2}{var(\tilde T_{k_j+l}\tilde T^*_{k_j})}$$

$$E(\hat\phi_l) = \phi_k + \mathcal{O}(\phi^2).$$

Working on the continuum we look for $w_{k,l}$ s.t.

$$\hat\phi_l = \int dk \; w_{k,l}\, \frac{\tilde T_{k+l}\tilde T^*_k}{f_{k,l}}.$$

To find $w_{k,l}$ we first need to compute

$var(\tilde T_{k+l}\tilde T^*_k)$ considering both $\tilde T$ & ϕ random!

$$= \underbrace{E(\tilde T_{k+l}\tilde T^*_k\tilde T^*_{k+l}\tilde T_k)}_{\text{approximate this by wicks thm}} - E(\tilde T_{k+l}\tilde T^*_k)E(\tilde T^*_{k+l}\tilde T_k)$$

These terms cancel

$$= E(\tilde T_{k+l}\tilde T^*_{k+l})E(\tilde T^*_k\tilde T_k) + E(\tilde T_{k+l}\tilde T_k)E(\tilde T^*_k\tilde T^*_{k+l})$$

$$= \delta_0\, C^{\tilde T\tilde T}_{k+l}\,\delta_0\, C^{\tilde T\tilde T}_k + \underbrace{\delta_{2k+l}\, C^{\tilde T\tilde T}_{k+l}\,\delta_{2k+l}\, C^{\tilde T\tilde T}}_{\text{only non-zero when } 2k=l \text{ so we ignore it}}$$

$$= \delta_0^2\, C^{\tilde T\tilde T}_{k+l}\, C^{\tilde T\tilde T}_k$$

Now replacing $w_{k,\ell}$ with

$$\frac{A_\ell\, 2\pi\, |f_{k,\ell}|^2}{\mathrm{var}(\tilde T_{k+\ell}\tilde T_k^*)} = \frac{A_\ell\, 2\pi\, |f_{k,\ell}|^2}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}$$

The yet to be determined normalizing constant

we have

$$\hat\phi_\ell = \int dk\; w_{k,\ell}\, \frac{\tilde T_{k+\ell}\tilde T_k^*}{f_{k,\ell}}$$

$$= A_\ell 2\pi \int dk\; \frac{|f_{k,\ell}|^2}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}\, \frac{\tilde T_{k+\ell}\tilde T_k^*}{f_{k,\ell}}$$

$$= A_\ell 2\pi \int dk\; \frac{f^*_{k,\ell}}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}\, \tilde T_{k+\ell}\tilde T_k^*$$

$$= A_\ell \int dk\; \frac{\ell\cdot[(k+\ell)C^{TT}_{k+\ell} - k\, C^{TT}_k]}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}\, \tilde T_{k+\ell}\tilde T_k^*$$

where A_ℓ is defined by setting

$$E(\hat\phi_\ell) = \phi_\ell + \mathcal{O}(\phi_\ell^2)$$

i.e.

$$\frac{E(\hat\phi_\ell)}{\phi_\ell} = 1 + \mathcal{O}(\phi_\ell)$$

$$A_\ell 2\pi \int dk\; \frac{|f_{k,\ell}|^2}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}\left[\frac{\phi_\ell + \mathcal{O}(\phi_\ell^2)}{\phi_\ell}\right]$$

$$\therefore\quad A_\ell := \left[2\pi \int dk\; \frac{|f_{k,\ell}|^2}{C^{\tilde T\tilde T}_{k+\ell}\, C^{\tilde T\tilde T}_k}\right]^{-1}$$

Note: when one has data with noise $d(x) = \tilde T(x) + n(x)$

Then

$$\hat\phi_\ell := A_\ell\, \ell\cdot \int dk\; \frac{(k+\ell)C^{TT}_{k+\ell} - k\, C^{TT}_k}{C^{dd}_{k+\ell}\, C^{dd}_k}\, d_{k+\ell}\, d_k^*$$

where

$$A_\ell := \left[2\pi \int dk\; \frac{|f_{k,\ell}|^2}{C^{dd}_{k+\ell}\, C^{dd}_k}\right]^{-1}$$

$$C^{dd}_k := C^{\tilde T\tilde T}_k + C^{nn}_k$$

A fast formula for $\hat{\phi}_\ell$

First Note that

$$\int \underline{dk}\, \frac{(k+\ell)C^{TT}_{k+\ell}}{C^{dd}_{k+\ell}\, C^{dd}_{k}}\, d_{k+\ell}\, d_k^{*}$$

$$= \int \underline{dk} \underbrace{\left[\frac{(k+\ell)C^{TT}_{k+\ell}\, d_{k+\ell}}{C^{dd}_{k+\ell}}\right]}\underbrace{\left[\frac{d_{-k}}{C^{dd}_{-k}}\right]}$$

$$:= -i\,[\,i(k+\ell)]\mathcal{L}_{k+\ell}\ \mathcal{H}_{-k}$$

$$= -i \int \underline{dx}\; e^{i\ell\cdot x}\, \nabla\mathcal{L}(x)\,\mathcal{H}(x)$$

And

$$\int \underline{dk}\, \frac{-k\, C^{TT}_{k}}{C^{dd}_{k+\ell}\, C^{dd}_{k}}\, d_{k+\ell}\, d_k^{*}$$

$$= -i \int \underline{dx}\; e^{i\ell\cdot x}\, \nabla\mathcal{L}(x)\,\mathcal{H}(x)$$

$$\therefore\ \hat{\phi}_\ell = -iA_\ell\, \ell\cdot \underbrace{\int \underline{dx}\; e^{i\ell\cdot x} \underbrace{\nabla\mathcal{L}(x)\,\mathcal{H}(x)}_{\text{pointwise pixel mult}}}_{\text{fourier transform}}$$

So the est $\hat{\phi}_\ell$ can be constructed as follows

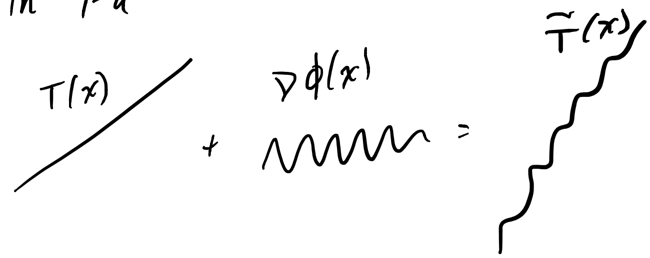
$$d(x) \xrightarrow[\text{mult.}]{\text{FFT \& pointwise}} \begin{cases} \mathcal{L}_k := \dfrac{C^{TT}_k}{C^{dd}_k}\, d_k \equiv \text{Filter } d \text{ to est } T \\[2ex] \mathcal{H}_k := \dfrac{d_k}{C^{dd}_k} \equiv \text{High pass filter} \end{cases}$$

$$\xrightarrow{\text{IFFT \& gradient}} \nabla\mathcal{L}(x)\,\mathcal{H}(x)$$

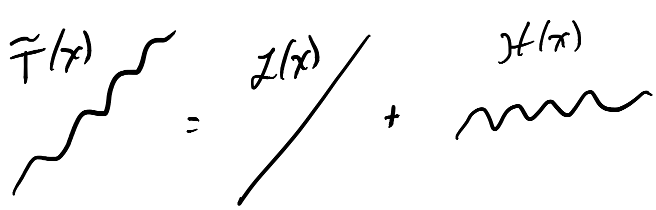
$$\xrightarrow[\text{mult.}]{\text{FFT \& pointwise}} -iA_\ell\,\ell\cdot\int \underline{dx}\; e^{ix\cdot\ell}\, \nabla\mathcal{L}(x)\,\mathcal{H}(x).$$

Interpretation of the fast formula

In 1-d



$\therefore$ it makes sense to look for a low-pass filter $\mathcal{L}(x)$ of the data & a high-pass filter $\mathcal{H}(x)$

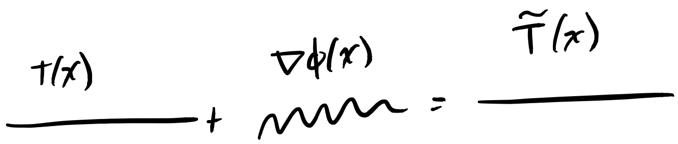


s.t. $\mathcal{L}(x)$ estimates $T(x)$ &

$\mathcal{H}(x)$ estimates $\nabla\phi(x)$.

However there is a problem...

Since the magnitude of $\mathcal{H}(x)$ is dependent on $\nabla T(x)$ as is illustrated in



so

$$\underline{\tilde T(x)} = \underbrace{\underline{\mathcal{L}(x)}}_{\text{this still approximates } T(x)} + \underbrace{\underline{\mathcal{H}(x) \simeq \nabla T(x)\nabla\phi(x)}}_{\text{but the magnitude of this is attinuated by } \nabla T(x).}$$

To solve this notice that

$$\nabla\mathcal{L}(x)\,\mathcal{H}(x) \simeq \nabla T(x)^2\,\nabla\phi(x)$$

so $\int dx\, e^{ix\cdot\ell}\,\nabla\mathcal{L}(x)\mathcal{H}(x)$ should approximate $\phi(x)$ up to a normalizing constant depending on C^{TT}_k & C^{dd}_k. This is exactly what $\mathcal{L}$ & $\mathcal{H}$ are in $\hat\phi_\ell$:

$$\mathcal{L}_k := \frac{C^{TT}_k}{C^{dd}_k}\, d_k = \text{Filter } d \text{ to est } T$$

$$\mathcal{H}_k := \frac{d_k}{C^{dd}_k} = \text{High pass filter}$$

The Frequentist properties of $\hat\phi_\ell$

One of the amazing things about $\hat\phi_\ell$ is the fact that one can get a good understanding of $C_\ell^{\hat\phi\hat\phi}$ marginalizing over both T, ϕ & n.

The variance of $\hat\phi_\ell$, conditional on ϕ_ℓ, is approximated by studying

$$\hat\phi^0_\ell := A_\ell\, \ell\cdot\int dk\, \frac{(k+\ell)C^{TT}_{k+\ell} - k\,C^{TT}_k}{C^{dd}_{k+\ell}\,C^{dd}_k}\, d^0_{k+\ell}\,\overline{d^0_k}$$

where $d^0_k = T_k + n_k$

so d^0_k has no lensing signal & $E(\hat\phi^0_\ell) \cong 0$

Now approximate the error distribution

$$\hat\phi_\ell - \phi_\ell \sim \hat\phi^0_\ell$$

Since

$$E\left(\hat\phi^0_\ell\,\overline{\hat\phi^0_{\ell'}}\right) = 2\pi^2 A_\ell A_{\ell'} \int dk\,dk'\, \frac{f^*_{k\ell}\,f_{k'\ell'}}{C^{dd}_{k+\ell}\,C^{dd}_k\,C^{dd}_{k'+\ell'}\,C^{dd}_{k'}} \times E\left(d^0_{k+\ell}\,\overline{d^0_k}\,\overline{d^0_{k'+\ell'}}\,d^0_{k'}\right)$$

where using Gaussian wik's thm one has

$$E\left(d^o_{k+\ell}\, \overline{d^o_k}\, \overline{d^o_{k'+\ell'}}\, d^o_{k'}\right)$$

$$= \cancel{\delta_\ell\, \delta_{\ell'}\, C^{\mathring{d}\mathring{d}}_{k+\ell}\, C^{\mathring{d}\mathring{d}}_{k'}} + \delta_{k+k'+\ell}\, \delta_{k+k'+\ell'}\, C^{\mathring{d}\mathring{d}}_{k+\ell}\, C^{\mathring{d}\mathring{d}}_{k'}$$

since we can assume either $\ell \neq 0$ or $\ell' \neq 0$

$$+ \delta_{k-k'+\ell-\ell'}\, \delta_{k-k'}\, C^{\mathring{d}\mathring{d}}_{k+\ell}\, C^{\mathring{d}\mathring{d}}_{k}$$

$$= 2\,\delta_{k-k'}\, \delta_{\ell-\ell'}\, C^{\mathring{d}\mathring{d}}_{k+\ell}\, C^{\mathring{d}\mathring{d}}_{k}$$

since $k+k'+\ell = k+k'+\ell' = 0 \Rightarrow k=k'$ & $\ell = \ell'$

& $k-k'+\ell-\ell' = k-k' = 0 \nearrow$

Plugging this back into $E(\hat\phi^o_\ell \overline{\hat\phi^o_{\ell'}})$ gives

$$E\left(\hat\phi^o_\ell \overline{\hat\phi^o_{\ell'}}\right) = 2\pi^2 A_\ell^2\, \delta_{\ell-\ell'}$$

$$\times \int \underline{\underline{dk}}\; \frac{|f_{k,\ell}|^2}{\left(C^{dd}_{k+\ell}\, C^{dd}_k\right)^2}\, 2\, C^{\mathring{d}\mathring{d}}_{k+\ell}\, C^{\mathring{d}\mathring{d}}_{k}$$

Using the approximation $C^{\mathring{d}\mathring{d}}_k \approx C^{dd}_k$

$$E\left(\hat\phi^o_\ell \overline{\hat\phi^o_{\ell'}}\right) \approx 2A_\ell^2\, \delta_{\ell-\ell'} \underbrace{\left[2\pi \int \underline{dk}\; \frac{|f_{k,\ell}|^2}{C^{dd}_{k+\ell}\, C^{dd}_k}\right]}_{A_\ell^{-1}}$$

$$= 2A_\ell\, \delta_{\ell-\ell'}$$

$\therefore$ Marginalizing over T, ϕ & n

$$E\left(\hat\phi_\ell\, \hat\phi^*_{\ell'}\right) \approx \delta_{\ell-\ell'} \Big(\underbrace{C^{\phi\phi}_\ell + 2A_\ell}_{\approx\, C^{\hat\phi\hat\phi}_\ell} \Big)$$

Estimating $C_\ell^{\phi\phi}$

Given data $d_k = \tilde{T}_k + n_k$

construct the quadratic estimate $\hat\phi_\ell$ and the associated bandpowers

$$\sigma^{\hat\phi}_\ell = \frac{1}{N_\ell} \sum_{k \in \Omega_\ell} \left[|\hat\phi_k|^2 / \delta_0 \right]$$

where Ω_ℓ indexes a set of frequencies in a collection of non-overlapping annuluses (with radius ℓ) & N_ℓ is the number of frequencies in each Ω_ℓ.

Now set $\hat C^{\phi\phi}_\ell$ to be

assuming rotationally invariant C^{nn}_k

$$\hat C^{\phi\phi}_\ell := \sigma^{\hat\phi}_\ell - 2A_\ell$$

So that

$$E\left(\hat C^{\phi\phi}_\ell\right) \cong C^{\phi\phi}_\ell$$

$$\mathrm{var}\left(\hat C^{\phi\phi}_\ell\right) \cong \frac{1}{N_\ell}\left(C^{\phi\phi}_\ell + 2A_\ell\right)^2.$$

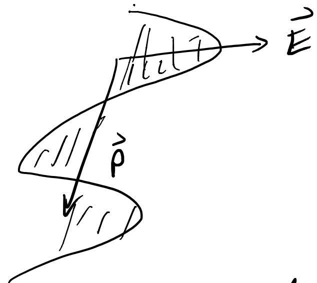
<u>Note</u>: There are bias correction terms $\mathcal{O}(C^{\phi\phi}_\ell) + \mathcal{O}((C^{\phi\phi}_\ell)^2) + \cdots$ the first two of which need to be accounted for when doing precision cosmology from $\hat C^{\phi\phi}_\ell$!

Polorization

21

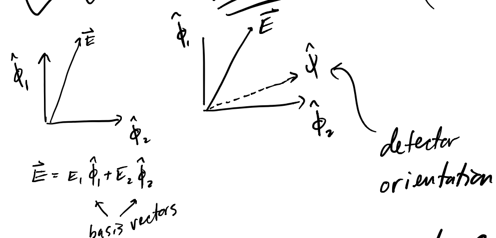
22

A photon is characterized by momentum $\vec{p} \in \mathbb{R}^3$ & an electromagnetic vector $\vec{E} \in \mathbb{R}^3$ describing the orientation of the propagating wave

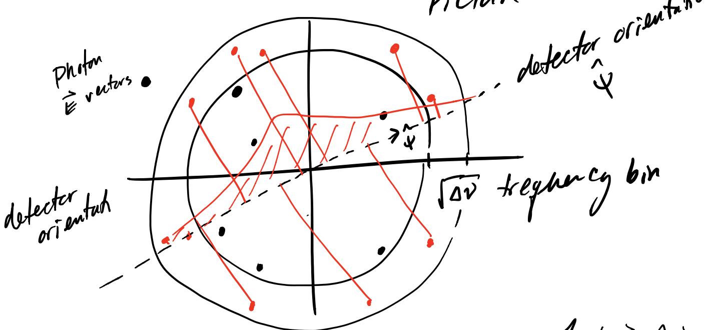


and $\vec{E}$ is equivalent to $-\vec{E}$. If one defines $\vec{E}$ to have magnitude equal to $|\vec{p}|^2$ then the response to a polarization detector

$$\xrightarrow{\text{response}} \left(\langle \vec{E}, \hat{\psi} \rangle\right)^2$$



Picture



The observation is the variance of $\langle \vec{E}, \hat{\psi} \rangle$ for $\vec{E}$ restricted to the frequency bin.

$\xrightarrow{\text{response}}$ $T_1 = E(E_1^2)$

$\xrightarrow{\text{response}}$ $T_2 = E(E_2^2)$

in units of intensity $E(|\vec{p}|) = E(|\vec{E}|^2)$

$\xrightarrow{\text{response}}$ $T_3 = E\left(\left\langle \vec{E}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\rangle^2\right) = E\left(\frac{(E_1+E_2)^2}{2}\right)$

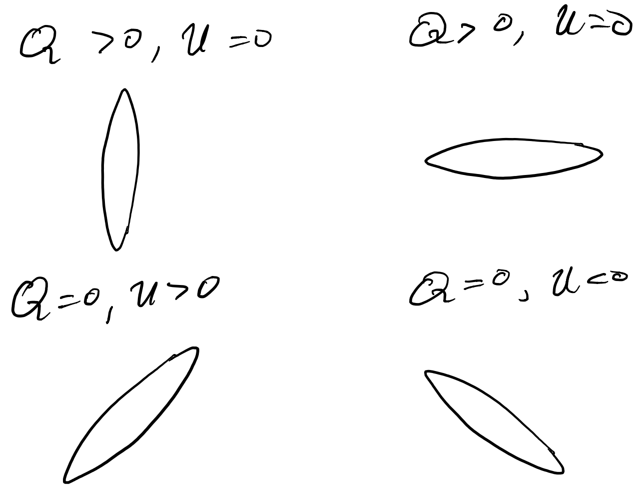
$\xrightarrow{\text{response}}$ $T_4 = E\left(\left\langle \vec{E}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\rangle^2\right) = E\left(\frac{(E_1-E_2)^2}{2}\right)$

$$\Sigma = \begin{pmatrix} I+Q & U \\ U & I-Q \end{pmatrix}$$

$\therefore$ $Q = (\Sigma_{11} - \Sigma_{22})/2$

$U = (2\,\Sigma_{12})/2$

measure the ellipticity of Σ as follows



This viewpoint of Q, U also makes clear how Q, U transform under rotations.

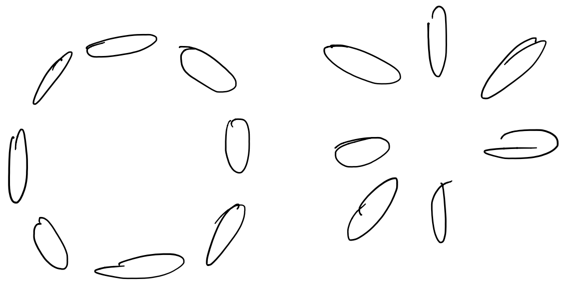
$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R_\theta} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\therefore\ \Sigma' = R_\theta \Sigma R_\theta^T$$

$$= \begin{pmatrix} I + Q\cos(2\theta) - U\sin(2\theta) & U\cos(2\theta) + Q\sin(2\theta) \\ U\cos(2\theta) + Q\sin(2\theta) & I - Q\cos(2\theta) + U\sin(2\theta) \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} Q\cos(2\theta) - U\sin(2\theta) \\ Q\sin(2\theta) + U\cos(2\theta) \end{pmatrix}$$

$E(\hat{n}), B(\hat{n})$ decomposition of $Q(\hat{n}), U(\hat{n})$



The covariance matrix of the distribution of $\vec{E}$ on frequency bin $\sqrt{\Delta\nu}$

$$\Sigma = \begin{pmatrix} E(E_1^2) & E(E_1 E_2) \\ E(E_1 E_2) & E(E_2^2) \end{pmatrix}$$