STA290 Presentation

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1. Show that if $X_1, \cdots, X_n$ are i.i.d $N(\mu, \sigma^2)$, then

$$\sqrt{n}(\bar{X} - \mu, \hat{\sigma^2} - \sigma^2) \to_D N(0, 0, \Sigma_0)$$

Where $\Sigma_0 = diag(\sigma^2, 2\sigma^4)$, $\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$.

<u>Sol.</u> Central Limit Theorem and Multivariate Delta Method

First Consider the limit distribution of $(\bar{X}, \frac{1}{n}\sum_{i=1}^n X_i^2)$, by CLT

$$\sqrt{n}(\bar{X} - \mu, \frac{1}{n}\sum_{i=1}^n X_i^2 - (\mu^2 + \sigma^2)) \to_D N(0, 0, \Sigma)$$

where Σ is the covariance of (X_i, X_i^2).

Actually $T = (X_i, X_i^2)$ is the sufficient statistics of normal distribution, so we could use properties of exponential family to obtain the covariance matrix of T.

Write the normal distribution as a canonical form of exponential family:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp\{\frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(\sigma^2)\} = \frac{1}{\sqrt{2\pi}} \exp(\eta_1 x + \eta_2 x^2 - A(\eta))$$

where $\eta_1 = \frac{\mu}{\sigma^2}$, $\eta_2 = -\frac{1}{2\sigma^2}$, $A(\eta) = -\frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\log(-2\eta_2)$.

$$\Sigma = \ddot{A}(\eta) = \begin{pmatrix} -\frac{1}{2\eta_2} & \frac{\eta_1}{2\eta_2^2} \\ \frac{\eta_1}{2\eta_2^2} & -\frac{\eta_1^2}{2\eta_2^3} + \frac{1}{2\eta_2^2} \end{pmatrix} = \begin{pmatrix} \sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & 4\sigma^2\mu^2 + 2\sigma^4 \end{pmatrix}$$

Last we apply multivariate delta method to T with
$(\bar{X}, \hat{\sigma}^2) = h(T) = (T_1, T_2 - T_1^2)$, we get
$\sqrt{n}(\bar{X} - \mu, \hat{\sigma}^2 - \sigma^2) \to_D N(0, 0, \Sigma_0)$, $\Sigma_0 = \nabla h \Sigma \nabla h^T = diag(\sigma^2, 2\sigma^4)$.

Again we see the independence of sample mean and sample covariance.