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Robust Model Predictive Control of PVTOL Aircraft ⋆

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Abstract: The paper presents a Tube Based Model Predictive Control (TB-MPC) strategy for robust stabilization of Planar Vertical Take-off and Landing (PVTOL) aircraft. PVTOL system is considered as a benchmark for investigating dynamics and control related issues for unmanned aerial vehicles (UAVs). Control problem of such a system is challenging due to its nonlinear dynamics and under actuated nature. Moreover, a major problem with these systems is the loss of controllability at certain points in the state space. To overcome these issues a unique approach has been used to approximate the nonlinear dynamics of the PVTOL system under additive disturbance with a set of linear systems to form polytopic linear differential inclusion (PLDI). Obtained approximation always satisfies the controllability condition. The proof of stability and feasibility for MPC has been derived. The effectiveness of the proposed control strategy is shown via simulations by stabilizing the aircraft, while rejecting the effects of disturbances.

Keywords: Model predictive control(MPC), Robust control, PVTOL, Disturbance rejection, LMI, unnamed aerial vehicle (UAV).

1. INTRODUCTION

PVTOL (Planar Vertical Take- off and landing) is a benchmark model for the aircrafts that evolve in the vertical plane with three degrees of freedom, see Hauser et al. (1992). The key advantage of VTOL systems is their capability to hover over an area, allowing for longer loiter time, and allow the craft to operate from a diverse array of airfields, using less space to get airborne. Its dynamics is nonlinear, under-actuated and non-minimum phase because of input coupling, see Chemori and Marchand (2008). VTOL ability and complex dynamics has attracted interest of researchers worldwide to develop controller for such system as it offers desirable flight properties in a restricted take-off/air-space and controlled hovering suitable for reconnaissance or rescue operations. There is also a particular interest for its offering of vertical propulsive landing, for the futuristic Reusable launch Vehicles (RLVs) that are under development by private space industry. To develop satisfactory controllers for, non-minimum phase, non-linear planar VTOL dynamics, different types of non-linear controller design approaches have been postulated, both for stabilization and trajectory tracking. In Hauser et al. (1992), a non-linear version of pole-zero cancellation is proposed. This feedback linearization for non-minimum phase system results in internally unstable dynamics, though in the context of input and output, the linearized system is stable. The global control strategies for control of PVTOL has been proposed in Chemori and Marchand (2008); Wood and Cazzolato (2007); Olfati-Saber (2002). In Chemori and Marchand (2008), a global discrete-time fast predictive control is proposed, where the complete problem was divided into two quadratic program - first problem focuses only on controlling roll angle of PVTOL and other on translation of PVTOL from initial point to final point. An Interconnection Damping Assignment passivity based controller (IDA-PBC) is for stabilization is proposed in Acosta et al. (2005). In Cardenas and Aguilar (2011), a sliding mode controller is designed. Most of the above mentioned strategies deal with stabilization of PVTOL at some desired point. There are many control strategies proposed in literature which deal with path following and trajectory tracking for e.g. Nielsen et al. (2008); Gruszka et al. (2011); Benvenuti et al. (1996) to mention a few.

Most of these non-linear control strategies do not take into account the constraints on the system explicitly in their design. For a PVTOL system, there are constraints on input thrust and also on the roll angle. Model Predictive Control (MPC) is an attractive solution to handle such systems with input and state constraints. MPC is used extensively in modern control applications since the availability of digital controllers and fast processors. MPC offers many advantages as the plant function can be controlled adaptively in the presence of disturbances by solving an open loop optimization problem online. The receding horizon implementation in MPC orders an inherently robust controller, see Mayne (2014). In order to curb the effect of large

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disturbances on system performance, many robust MPC techniques have been proposed. In particular, Min-Max approach in Magni et al. (2003) where the cost function is optimized for the worst case disturbance, and constraint tightening approach in Lee et al. (2002). Major drawback of these methods is high computational cost which makes them incompetent to fast processes. In this paper we present a tube based MPC (TB-MPC) technique that uses error invariant sets as tubes , constructed offline, whose center is the nominal trajectory tracked by nominal MPC without disturbances, refer Alvarado et al. (2007); Bohm and Allgöwer (2010); Mayne et al. (2011); Rakovic et al. (2012); Mayne (2014). TB-MPC uses nominal MPC along with an offline ancillary feedback controller based on error dynamics to keep the system inside the invariant tubes, thus the computation cost is reduced largely. The performance of this ancillary feedback controller depends on the type of error dynamics model used for its construction. A non-linear model captures the behavior of the system accurately but suffers from the drawback of high computational effort and complex mathematical tools for controller design. On the other hand, it is much easier to compute the offline feedback law using linear model of error dynamics but it is not valid for the entire state space/region of operation and hence not global. Thus the best tradeoff is to use PLDI (Polytopic Linear Differential Inclusion) model, which captures the system dynamics accurately as well as makes the control design simpler using linear control theoretic tools such as LMIs. For guaranteed closed loop stability and recursive feasibility of optimization in MPC, we use the standard Quasi-infinite horizon approach with terminal cost and terminal constrains as in Chen and Allgöwer (1998). Thus, TBMPC controller can work with fast processes along with noise handling capabilities. We shall examine the efficacy and robustness of the tube-based MPC as applied to PVTOL control problem of trajectory tracking.

Notations: $\mathbb{R}^n$ stands for the n-dimensional real space. For any vector $x \in \mathbb{R}^n$, $\|x\|_P$ denotes the P-weighted norm, defined by $\|x\|_P^2 = x^T P x$, and P is any positive-definite real symmetric matrix. The superscript 'T' stands for matrix transposition and $Diag([\cdots])$ indicates a block-diagonal matrix. The symbol $\mathbf{N}$ represents the set of all positive integers. Also, $x(t_i; t_j)$ represents the value of $x(t_i)$ produced at time t_j, where $t_i > t_j > 0$. If X_1 and X_2 are two sets then, Minkowski difference between the two sets can be represented as $X_1 \ominus X_2$. The convex hull of a set of points is given by $co[\cdots]$.

2. DYNAMICAL MODEL OF PVTOL AIRCRAFT

Considering the typical PVTOL aircraft in Fig 1,the thrust vector provided by the throttle and nozzle enables two-degrees-of freedom control in the pitch-yaw plane. The desired vertical position is easy to achieve by adjusting the thrust. However, For lateral manoeuvrability PVTOL a reaction control system (RCS) is employed to provide moment around the aircraft center of mass. This creates coupling between rolling moment of aircraft and the lateral force see Hauser et al. (1992); Wood and Cazzolato (2007); Zhang and Brandt (1999). Thus to achieve the desired lateral position the roll angle is controlled using rolling moment. The nominal system equation of PVTOL can be

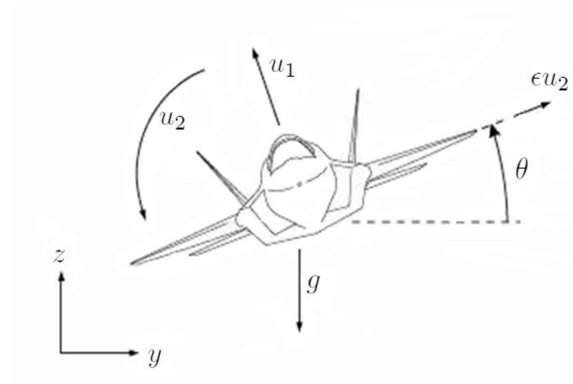

Fig. 1. PVTOL aircraft

found in Hauser et al. (1992). For robustness analysis we consider bounded additive disturbance. Thus the system equation can be written as follows.

$$\ddot{y} = -u_1 \sin\theta + \epsilon u_2 \cos\theta + \omega_1 \tag{1}$$

$$\ddot{z} = u_1 \cos\theta + \epsilon u_2 \sin\theta - g + \omega_2 \tag{2}$$

$$\ddot{\theta} = u_2 \tag{3}$$

Where, y and z represent the position of center of mass of PVTOL aircraft, while θ represents the roll angle of aircraft with respect to horizontal axis. ω_1 and ω_2 are the bounded disturbance forces acting in y and z direction. These terms can be used to simulate more practical scenarios like effect of wind and parametric uncertainties of the aircraft. The term g in the dynamics represents the acceleration due to gravity ($g = 9.81 m/s^2$) which can be normalized to 1. Here, u_1 and u_2 represent the normalized thrust required by aircraft for upward motion and rolling moment of the aircraft respectively. ϵ represents the coupling between lateral motion of aircraft and rolling moment ($0 < \epsilon < 1$). By defining, $x = \left[y\ \dot{y}\ z\ \dot{z}\ \theta\ \dot{\theta}\right]^T$ and $u = \left[u_1\ u_2\right]^T$ the above equations can be represented in more compact form as below,

$$\dot{x} = Ax(t) + B(t)u(t) + D(t) \tag{4}$$

Where,

$$A = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{bmatrix},\ D(t) = \begin{bmatrix} 0\\ \omega_1(t)\\ 0\\ -1+\omega_2(t)\\ 0\\ 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 & 0\\ -\sin(\theta(t)) & \varepsilon\cos(\theta(t))\\ 0 & 0\\ \cos(\theta(t)) & \varepsilon\sin(\theta(t))\\ 0 & 0\\ 0 & 1 \end{bmatrix}$$

Above system is uncontrollable at many points in state space. Thus we reformulate the dynamics before applying the proposed control strategy. To achieve controllability on complete state-space, a simple transformation of input $\bar{u}_1 = 1 + u_1$ is used, Similiar tranformation was used in Sun-Li et al. (2008). By using the transformation, system can be represented as,

$$\dot{x} = \bar{A}x(t) + \bar{B}(t)\bar{u}(t) + \bar{D}(t) \tag{5}$$

Where,

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \bar{D}(t) = \begin{bmatrix} 0 \\ \bar{\omega}_1(t) \\ 0 \\ \bar{\omega}_2(t) \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{B}(t) = \begin{bmatrix} 0 & 0 \\ -\sin(\theta(t)) & \varepsilon\cos(\theta(t)) \\ 0 & 0 \\ \cos(\theta(t)) & \varepsilon\sin(\theta(t)) \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$\bar{\omega}_1(t) = \omega_1(t) + \theta + \sin(\theta(t))$ and $\bar{\omega}_2(t) = -1 + \omega_2(t) + \cos(\theta(t))$. It can be noticed that even after the transformation disturbance remains bounded, as it depends on the 'sin' and 'cos' of roll angle θ. The system (5) is linear parameter varying system with the parameter θ. As it depends only on single parameter it can be easily reprensented as a polytopic linear differential inclusion (PLDI) with few vertices to define the polytope, see Boyd et al. (1994) for details. The nominal model without disturbance for (5) can be represented by,

$$\dot{\hat{x}} = \bar{A}\hat{x}(t) + \bar{B}(t)\hat{u}(t) \tag{6}$$

The error dynamics, $e(t) = x(t) - \hat{x}(t)$ between nominal and perturbed system is given by,

$$\dot{e}(t) = \bar{A}e(t) + \bar{B}(t)\bar{e}(t) + \bar{D}(t) \tag{7}$$

2.1 Nominal Model Predictive Control

In nominal Model Predictive Control we solve the following finite horizon optimal control problem at each sampling instance t_s.

Problem 1:

$$\hat{u} = \min_{\hat{u}} J(\hat{x}, \hat{u}) \tag{8}$$

subject to:

$$\dot{\hat{x}}(t) = \bar{A}\hat{x}(t) + \bar{B}(t)\hat{u}(t) \tag{9a}$$

$$\hat{x} \in \hat{X} \tag{9b}$$

$$\hat{u}(t) \in \hat{U} \tag{9c}$$

$$\hat{u}(t_k) - \hat{u}(t_{k-1}) \in \Delta \tag{9d}$$

$$\hat{x}_0 = \hat{x}(t_k) \tag{9e}$$

Where,

$$J(\hat{x}, \hat{u}) = \|\hat{x}(t_k + T, t_k)\|_P^2 + \int_{t_k}^{t_k+T} \|\hat{x}(t, t_k)\|_Q^2 + \|\hat{u}(t, t_k)\|_R^2 \, dt$$

T is the prediction horizon. R and Q are positive definite and positive semi-definite matrices. P is positive semi-definite matrices also known as terminal weight used to define terminal region for MPC. The value of P can be found by solving $A^T P + PA = -(Q + K^T RK)$. Here, K is static control feedback gain also known as terminal controller which can be calculated using many methods defined in literature Michalska and Mayne (1993); Mayne et al. (2000); Chen and Allgöwer (1998). Since, the MPC is designed for nominal system, the output will not violate system constraints in the absence of disturbance. However, the disturbance acts directly on the output, hence if system is operating near constraints boundaries it may force the system to violate the constraints. Let, X defines the constraint on state variable θ of pertubed system (5) i.e. $|\theta| \leq \frac{\pi}{2}$ and U is the constraint on input thrust $0 \leq u_1 \leq 4$. Then the tightened constraints on nominal system state and input variables are given by, $\hat{X} = X \ominus \Omega(e)$, $\hat{U} = U \ominus K_e\Omega(e)$. The method to calculate $\Omega(e)$ is explained in Section 3. The minkowski difference between two set can be easily found by using any geometric toolbox like Herceg et al. (2013). Lastly, the constraint (9d) is rate constraint on the thrust. It is used to obviate the abrupt change in input thrust, which may be out of the actuator's capability.

3. PROPOSED ROBUST CONTROL SCHEME

In this paper, our aim is to design a robust tube based MPC controller so as to stabilize the PVTOL at some predefined point in presence of a time varying bounded additive disturbance. The entire control strategy can be summarised as follows:

- S1: Solve **Problem 1** at every sample time t_k to get nominal control input $\hat{u}(t_k)$.
- S2: Compute an ancillary control law $\bar{u}_e(t_k) = K_e e(t_k)$ based on error dynamics (7) which aims at driving the error to origin.
- S3: The actual control inputs for perturbed system are then calculated by, $\bar{u}(t_k) = \hat{u}(t_k) + \bar{u}_e(t_k)$.

Step 1 in the algorithm uses nominal MPC scheme like Bohm and Allgöwer (2010); Chen and Allgöwer (1998). The process to find ancillary control law K_e in Step 2 is explained in *Lemma 1.* One of major advantage of using TB-MPC as mentioned in Mayne et al. (2011), it does not affect the computation time of MPC compared to nominal MPC, since the ancillary static feedback law is computed offline. Hence the proposed scheme is practically implementable.

3.1 Disturbance invariance of error system

The method to find invariant set is inline with Bohm and Allgöwer (2010). The error dynamics is given by (7), where, $\|\bar{\omega}\| = \left\| \begin{bmatrix} \bar{\omega}_1(t) \\ \bar{\omega}_2(t) \end{bmatrix} \right\| \leq \begin{bmatrix} \|\bar{\omega}_1\|_\infty \\ \|\bar{\omega}_2\|_\infty \end{bmatrix}$, Our objective is to keep this error in an invariant set so that actual trajectory does not diverge far away from nominal trajectory in presence of $\bar{\omega}_1(t)$ and $\bar{\omega}_2(t)$. To facilitate the calculation of invariant set the time varying linear system can be approximated by polytopic linear differential inclusion (PLDI), $\dot{e} = co[\bar{A} \quad \bar{B}_i \quad \bar{D}_{\bar{\omega}}]$, $i = 1, 2, 3 \cdots N$ represents the error dynamics on the ith vertex of the set formed by constraints (see APPENDIX) and $\bar{D}_{\bar{\omega}} = \frac{\partial \bar{D}}{\partial \bar{\omega}}$. The existence of PLDI can be guaranteed if $\dot{e}(t) = \bar{A}e(t) + \bar{B}_i u_e + \bar{D}_{\bar{\omega}}\bar{\omega} \in \bar{A}e(t) + \bar{B}(t)\bar{u}_e(t) + \bar{D}(t)$.

In the following lemma, the calculation of invariant set is explained. This lemma has been proposed in Bohm and Allgöwer (2010).

Lemma 1. If there exist a matrix $Y \in \mathbb{R}^{n \times n}$,a positive definite matrix $X \in \mathbb{R}^{n \times n}$, and scalar $\lambda, \mu > 0$ such that,

$$\begin{bmatrix} (\bar{A}X + \bar{B}_i Y)' + (\bar{A}X + \bar{B}_i Y) + \lambda X & \bar{D}_{\bar{\omega}_i} \\ * & -\mu I \end{bmatrix} \leq 0$$

$\forall i = 1, 2, ...N$. Where $*$ represents symmetric component. Then $V_e(t) = e(t)'Ee(t)$ forms control invariant set with control $\bar{u}_e = K_e e$, where $E = X^{-1}$ and $K_e = YX^{-1}$. Also, the invariant region is defined as $\Omega(e) = \{e|V_e(t) \leq \frac{\mu(\bar{\omega}_{max})^2}{\lambda}\}$, where $V_e(t) = e(t)'Ee(t)$

Proof: The LMI condition can be derived easily using s-procedure,

$$\frac{\mathrm{d}V_e(t)}{\mathrm{d}t} + \lambda V_e(t) - \mu\bar{\omega}'\bar{\omega} \leq 0 \tag{10}$$

from above we get,

$$\begin{bmatrix}(\bar{A}+\bar{B}_iK_e)'E + E(\bar{A}+\bar{B}_iK_e) + \lambda E & E\bar{D}_{\bar{\omega}_i} \\ * & -\mu I\end{bmatrix} \leq 0$$

by substituting, $K_e = YX^{-1}$ and $E = X^{-1}$, we get the LMI condition in Lemma 1. Now, multiplying (10) by $e^{\lambda t}$ on both the sides it can be rewritten as,

$$\frac{\mathrm{d}e^{\lambda t}V_e(t)}{\mathrm{d}t} \leq \mu e^{\lambda t}\bar{\omega}'\bar{\omega}$$

By integrating above equation from t_0 to t,

$$\begin{aligned} V_e(t) &\leq e^{-\lambda(t-t_0)}V_e(t_0) + \mu e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\tau}\bar{\omega}(\tau)'\bar{\omega}(\tau)d\tau \\ &\leq e^{-\lambda(t-t_0)}V_e(t_0) + \mu e^{-\lambda t}(\bar{\omega}_{max})^2\int_{t_0}^{t} e^{\lambda\tau}d\tau \\ &\leq \frac{\mu(\bar{\omega}_{max})^2}{\lambda} \end{aligned} \tag{11}$$

hence if $V_e(t)$ is the control invariant set for error dynamics (7), the control input $\bar{u}_e = K_e e$ can be obtained from the LMI condition derived in *Lemma* 1. Thus if the error dynamics is initialised once in the invariant set $V_e(t) \leq \frac{\mu(\bar{\omega}_{max})^2}{\lambda}$ then it will always be contained in the set for all future time t. The invariant region $V_e(t)$ depends on design parameters μ and λ. Special care must be taken while selecting the values of these design parameter as their selected values may tightened the constraints by large amount or may have negligible effect. If constraints are tightened by large amount then the MPC controller may have poor response and in worst case there may be no feasible solution. Thus a trial and error based reverse process is used, by which we can calculate the value of design parameters.

S1: Predetermine the amount by which the constraint on $\hat{u}(t)$ and $\hat{x}(t)$ be tightened.
S2: By fixing the value of either μ or λ, the LMI problem in *Lemma 1* can be solved
S3: Calculate the set $\Omega(e)$.
S4: Calculate tightened constraints in *Problem 1*. If it matches with the desired amount as decided in step S1 than stop else, repeat step S2 - S4 for new value of μ or λ.

Step S1 can be done by simulating the behavior of controller using nominal model and evaluate its performance by varying the boundaries of constraint by small amount. If the performance is within acceptable limits than we can proceed to next steps S1 - S4. Though, the above procedure may require many iteration, it is acceptable since this is done only one time at the start.

4. STABILITY AND FEASIBILITY ANALYSIS

The proposed strategy requires to solve an optimization problem recursively at each sampling time. One major problem associated with optimization problem is existence of solution is not guaranteed for all initial conditions. Hence feasibility analysis becomes an important part of controller design. Also, It is necessary to verify that the system is closed loop stable for proposed algorithm. The complete feasibility and stability analysis is explained in the following section.

4.1 Feasibility analysis

In this section, recursive feasibility is proved i.e. if for any initial state x_0 Problem 1 has a solution then it will have solutions for all future time.

Theorem 1. If Assumption 1-5 is satisfied and Problem 1 has solution for initial state x_0 at initial time t_k then Problem 1 is always feasible for future time.

Proof: Let, $\hat{u}(t)$ be the feasible input trajectory defined as,

$$\hat{u}(t) = \begin{cases}\hat{u}^*(t,t_k) & t \in [t_k+\delta, t_k+T] \\ K\hat{x}^*(t,t_k) & t \in [t_k+T, t_k+\delta+T]\end{cases} \tag{12}$$

the corresponding feasible state trajectory is given by,

$$\hat{x}(t) = \begin{cases}\hat{x}^*(t,t_k) & t \in [t_k+\delta, t_k+T] \\ \hat{x}(t,t_k) & t \in [t_k+T, t_k+\delta+T]\end{cases} \tag{13}$$

the state trajectory for time interval $[t_k+T, t_k+\delta+T]$ can be generated by by using system dynamics (6). As solution of Problem 1 is feasible for x_0, hence $\hat{u}^*(t,t_k) \in \hat{U}$, also $K\hat{x}^*(t,t_k) \in \hat{U}$ as it is a terminal controller designed to satisfy the constraints on input. Hence $\hat{u}(t) \in \hat{U}$. Similarly it can be showed that, $\hat{x}(t) \in \hat{X}$. Since $x(t_k+\delta) \in \hat{x}^*(t_k+\delta, t_k) \oplus \Omega(e)$, problem 1 has solution for time $t_k+\delta$. thus using the invariance property of $\Omega(e)$ recursive feasibility of problem 1 can be assured for all future time.

4.2 Stability analysis

For stability analysis let us consider a candidate lyapunov function, $V(\hat{x}) = \min_{\hat{u}^*(t)} J(\hat{x}(t), \hat{u}^*(t))$.

Theorem 2. If Assumption 1-5 and *Theorem* 1 is satisfied then for sufficiently small sampling time $\delta > 0$ system 1 is asymptotically stable.

Proof: If $x(0) \in \Omega(\varepsilon)$, from lemma 1 and lemma 4, system can be stabilized in error invariant region around equilibrium. If $x(0) \notin \Omega(\varepsilon)$ then to prove stability it is sufficient to proof, $\Delta J = J(\hat{x}(t,t_{k+1}), \hat{u}(t,t_{k+1})) - J(\hat{x}(t,t_k), \hat{u}(t,t_k)) \leq 0$.

$$\begin{aligned} \Delta J &= J(\hat{x}(t,t_{k+1}), \hat{u}(t,t_{k+1})) - J(\hat{x}(t,t_k), \hat{u}(t,t_k)) \\ &= \|\hat{x}(t_{k+1}+T, t_{k+1})\|_P^2 + \int_{t_{k+1}}^{t_{k+1}+T} \|\hat{x}(t,t_{k+1})\|_Q^2 \\ &\qquad + \|\hat{u}(t,t_{k+1})\|_R^2\,dt \\ &- \|\hat{x}(t_k+T, t_k)\|_P^2 + \int_{t_k}^{t_k+T} \|\hat{x}(t,t_k)\|_Q^2 + \|\hat{u}(t,t_k)\|_R^2\,dt \\ &= \|\hat{x}(t_{k+1}+T, t_{k+1})\|_P^2 + \int_{t_k+T}^{t_{k+1}+T} \|\hat{x}(t,t_{k+1})\|_Q^2 \\ &+ \|\hat{u}(t,t_{k+1})\|_R^2\,dt - \|\hat{x}(t_k+T, t_k)\|_P^2 - \int_{t_k}^{t_k+\delta} \|\hat{x}(t,t_k)\|_Q^2 \\ &\qquad + \|\hat{u}(t,t_k)\|_R^2\,dt \end{aligned}$$

by using the definition of terminal region the above equation can be rewritten as.

$$\Delta J = -\int_{t_k}^{t_k+\delta} \|\hat{x}(t,t_k)\|_Q^2 \, dt \tag{14}$$

since, $x(0) \notin \Omega(\varepsilon)$ in worst case,

$$\Delta J \leq -\delta\varepsilon \tag{15}$$

From above it can be seen that the nominal system will always decay with minimum given rate. Thus $V(\hat{x})$ is non increasing, and it can be shown that it will reach the stabilization point within finite time *i.e.* as $t \to \infty$, $\hat{x}(t) \to 0$. From Lemma 1 the trajectory of perturbed system is bounded by an invariant set $\Omega(e) = \{e | V_e(t) \leq \frac{\mu(\bar{\omega}_{max})^2}{\lambda}\}$ around the nominal trajectory. Thus trajectories of perturbed system decay alongwith nominal system and as $t \to \infty$, $x(t) \to \Omega$. Thus Ω acts as a attractive set to the nonlinear perturbed system (4).

5. SIMULATION

The proposed strategy was verified using a MATLAB simulation using (6) as a nominal model for MPC. The constraint on input thrust is $0 \leq u_1 \leq 4$ and $|\theta| \leq \frac{\pi}{2}$. The weight matrix Q and R are defined as diagonal matrix $Diag([3,1,3,1,3,1])$ and $Diag([1,1])$. Terminal matrix P and K can be found by solving Lypunov equation as proposed in section II-A. The error dynamic is PLDI which can be found out using method proposed in section III-A. As the system matrix A is constant, $\bar{B}$ is time varying and only depend on $\sin(\theta(t))$ and $\cos(\theta(t))$, above system can be replaced by PLDI formed by set of three pair of (A, B_i) where i = 1,2,3 defined in APPENDIX. The static feedback law required to make a set $\Omega(e)$ invariant set around origin can be found by solving LMI problem proposed in *Lemma 1* and $K = \begin{bmatrix} -0.067 & -0.165 & -0.856 & -3.208 & 0.242 & 0.352 \\ 5.220 & 12.765 & 0.046 & 0.198 & -14.810 & -6.196 \end{bmatrix}$. Simulation results are shown in Fig. 2-5, for simulation purpose the initial condition $x_0 = [0.707, 0, 0.707, 0, 0, 0]^T$ was selected and the target position was $x_F = [8, 0, 7, 0, 0, 0]^T$. The time varying bounded disturbance $\|\omega(t)\| \leq \left\|\begin{matrix} 0.1 \\ 0.1 \end{matrix}\right\|$ was considered. The effectiveness of the proposed controller can be seen in Fig. 2. It can be seen that even under time varying disturbance the PVTOL reaches its target position with very small error i.e 0.025 units in y-direction and 0.301 units in z-direction. The thrust input to the PVTOL is shown in Fig. 4, which satisfies the input constraints. Also, at time t=4 secs the roll angle reaches its maximum bound with high thrust, which helps the aircraft to have a lateral motion with decrease in altitude as seen in the Fig. 2. Thus the proposed strategy is able to stabilize the system under the effects of disturbance and input constraints.

6. CONCLUSION

In this paper, we have presented a Robust Model Predictive Control scheme for stabilisation of a PVTOL in presence of an additive bounded disturbance. The effects of these disturbances is diminished by an ancillary control law computed off-line based on error system. This task is simplified by expressing the original system in a modified form which facilitates the calculation of linear differential inclusion equivalent of a nonlinear system. The condition for recursive feasibility and stability of the overall closed-loop system have been analyzed. The effectiveness of proposed control strategy is proved via simulations.

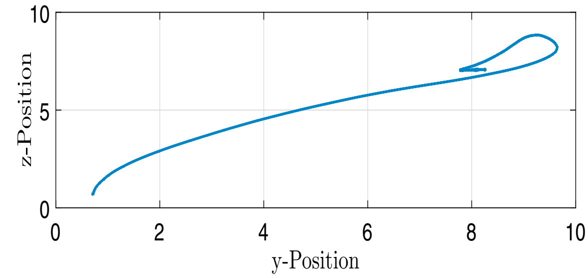

Fig. 2. movement of PVTOL in y-z plane

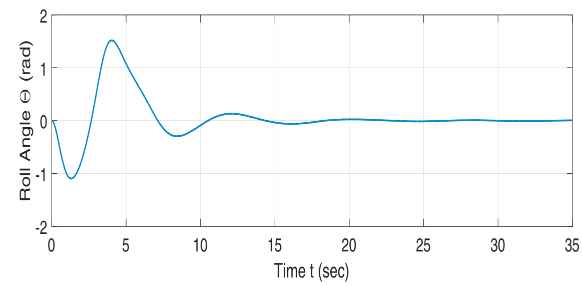

Fig. 3. Roll angle of PVTOL

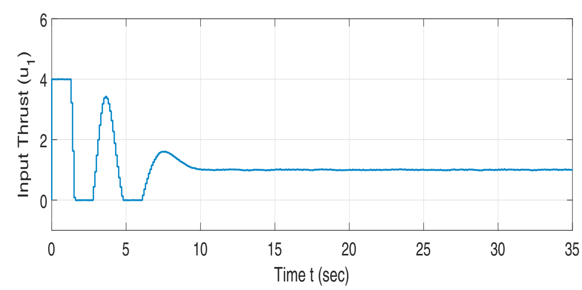

Fig. 4. Input Thrust

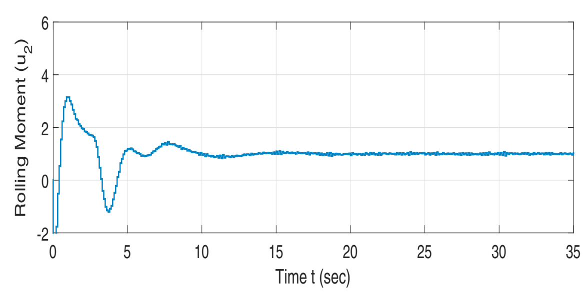

Fig. 5. Rolling moment

APPENDIX

The linear time varying system (5) can be represented by set of linear system $\dot{e} = co[A \quad B_i \quad D_{\bar{\omega}}]$, $i = 1,2,3$. The A matrix is constant hence it will be same on all vertice. The matrix B is parameter variant and it depends on θ. Since, θ varies between $\pm\pi/2$ we can consider three points on $\theta = \pi/2, -\pi/2$ and 0 as vertices. The time varying error dynamics (7) can be be replaced by set of lti systems $\dot{e} = co[A \;\; B_i \;\; D_{\bar{\omega}}]$, $i = 1, 2, 3$. Where B_i is matrix at the mentioned values of θ

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0.866 & 0.05 \\ 0 & 0 \\ 0.5 & -0.0866 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 0.866 & 0.05 \\ 0 & 0 \\ 0.5 & -0.0866 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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