The line is part of a circle

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November 26, 2022

1 Introduction

A mathematical object of interest X is sometime a subobject $X \rightarrowtail Y$ where Y has structure which makes reasoning about X simpler. For example, the real open interval $(0,1)$ can be embedded inside the circle $[0,1]/(0 \sim 1)$, which is a compact space although $(0,1)$ is not. Note also that the real numbers $\mathbb{R}$ can be embedded into the complex numbers $\mathbb{R} \rightarrowtail \mathbb{C}$ where $\mathbb{C}$ is an algebraically closed field and $\mathbb{R}$ is not. Today, we look at a categorical example of this phenomena, the *Yoneda embedding*.

2 Equivalences of categories and embeddings

Recall that a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is **faithful** if for all pairs of objects (X, Y) in $\mathscr{C}$ the function

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(FX,FY) \\ f &\longmapsto Ff \end{aligned}$$

is injective. Also, if this function is surjective then F is **full**.

Moreover, if for every object $D \in \mathscr{D}$ there exists an object $C \in \mathscr{C}$ such that $FC \cong D$ then F is **essentially surjective**.

This was introduced in Lecture 3 and we mentioned that it provides sufficient conditions for F to be part of the data of an *equivalence of categories*. At the time, we did not have the language of natural transformations, and so we did not give the definition, now though we can.

Definition 2.0.1. An **equivalence of categories** is a pair (F, G) of functors

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$
$$G : \mathscr{D} \longrightarrow \mathscr{C}$$

along with a pair of natural transformations $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_D$.

Exercise 2.0.2. Prove that (F, G) is an equivalence of categories if and only if F is a fully, faithful, essentially surjective functor.

Exercise 2.0.3 (Hard exercise)**.** Does your proof for the previous exercise hold in the first order theory of ZFC sets?

Thus, we can think of a fully, faithful functor (not necessarily essentially surjective) as an embedding of one category into another. The main result of today's lecture will exibit such an embedding.

3 An introduction to universal properties

In life, it is more important how an object *behaves* than it is what the object *is*. For instance, when pegging in a tent peg, I might use a rock as a hammer. Since the rock in that moment *behaved* like a hammer, does it really matter that what I had was a rock and strictly speaking *not* a hammer?

Mathematically, we can take the same approach.

Definition 3.0.1. Let X, Y be two sets. A **product** of X, Y consists of a set $X \times Y$ along with two functions $\pi_X : X \times Y \longrightarrow X, \pi_Y : X \times Y \longrightarrow Y$ which together satisfy the following property: if $f : U \longrightarrow X, g : U \longrightarrow Y$ are any two functions, then there exists a unique morphism $h : U \longrightarrow X \times Y$ which makes the following diagram commute.

$$\begin{tikzcd} X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\ & U \arrow[ul, "f"] \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd} \tag{1}$$

An example of a product is the cartesian product

$$(x, y) \in X \times Y \Leftrightarrow x \in X \text{ and } y \in Y \tag{2}$$

Proof. Let $f : U \longrightarrow X, g : U \longrightarrow Y$ be arbitrary. First we prove uniqueness. Say an appropriate $h : U \longrightarrow X \times Y$ exists. Then for any $u \in U$, the first entry of $h(u)$ is given by $\pi_X h(u) = f(u)$, and the second entry is given by $\pi_Y h(u) = g(u)$. This means $h(u) = (f(u), g(u))$ which we note is independent of h. We notice that this proves existence too. □

We notice that Definition 3.0.1 never defined what the set $X \times Y$ of a product *is*, but only defined a *property* of it.

This definition generalises to arbitrary categories immediately.

Definition 3.0.2. A **product** (if it exists) of two objects X, Y in a category $\mathscr{C}$ consists of an object $X \times Y$ along with a pair of morphisms $\pi_X : X \times Y \longrightarrow X, \pi_Y : X \times Y \longrightarrow Y$ which together satisfy the following propery: if $f : U \longrightarrow X, g : U \longrightarrow Y$ are any two functions, then there exists a unique morphism $h : U \longrightarrow X \times Y$ which makes the following diagram commute.

$$\begin{tikzcd} X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\ & U \arrow[ul, "f"] \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd} \tag{3}$$

Lemma 3.0.3. *If a product $(X \times Y, \pi_X, \pi_Y)$ exists, then it is unique up to unique isomorphism.*

Proof. Let $(X \hat{\times} Y, \rho_X, \rho_Y)$ be another product. Construct the following diagram, considering only the solid arrows for now.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[dd, dashed, bend left, "h"] & \\ X & & Y \\ & X \hat{\times} Y \arrow[ul, "\rho_X"] \arrow[ur, "\rho_Y"'] \arrow[uu, dashed, bend left, "j"] & \end{tikzcd} \tag{4}$$

The pairs of morphisms π_X, π_Y and ρ_X, ρ_Y each satisfy the other product's universal property. Thus we obtain two induced morphisms $h : X \times Y \longrightarrow X \hat{\times} Y, j : X \hat{\times} Y \longrightarrow X \times Y$ which makes the above diagram commute, considering all arrows now.

Figure 1: Yoneda Lemma core idea

$$
\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rrr, "\mathrm{Hom}(A,f)"] \arrow[ddd, "\Phi_A"'] & & & \mathrm{Hom}(A,X) \arrow[ddd, "\Phi_X"] \\
& \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & (Ff)u = \Phi_X(f) & \\
F(A) \arrow[rrr, "Ff"'] & & & F(X)
\end{tikzcd}
$$

The composition hj makes the following diagram commute

$$
\begin{tikzcd}
X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\
& X \times Y \arrow[ul, "f"] \arrow[u, dashed, "h"] \arrow[ur, "hj"'] &
\end{tikzcd}
\tag{5}
$$

and so does the identity morphism $\mathrm{id}_{X \times Y}$. By uniqueness of such morphsims, we have $hj = \mathrm{id}_{X \times Y}$. A similar argument shows that $jh = \mathrm{id}_{X \hat{\times} Y}$. □

4 The Yoneda Lemma

The following Lemma has been referred to as the only theorem in category theory.

Lemma 4.0.1. *Let $\mathscr{C}$ be a small category (that is, a category whose collection of objects is a set), and let $P : \mathscr{C} \longrightarrow \underline{\mathrm{Set}}$ be a functor. For any object $C \in \mathscr{C}$ there is a natural bijection*

$$
\begin{aligned}
\mathrm{Nat}(\mathrm{Hom}(C, \_), P) &\cong P(C) \\
\eta &\longmapsto \eta_C(\mathrm{id}_C)
\end{aligned}
$$

Proof. The core idea is the diagram shown in Figure 1. We notice that this proves injectivity and surjectivity. □

Exercise 4.0.2. Finish the proof of Lemma 4.0.1 by proving naturality. If you need a hint see [1].

Definition 4.0.3. A **contravariant functor** $F : \mathscr{C} \longrightarrow \mathscr{D}$ is an assignment of an object $FC \in \mathscr{D}$ to every object $C \in \mathscr{C}$ along with a function for each pair of obejcts (X, Y) in $\mathscr{C}$

$$Ff : \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(FY, FX) \tag{6}$$

(Note the change of order of X, Y), subject to the following conditions:

- For any pair of morphisms $f : X \longrightarrow Y, g : Y \longrightarrow Z$ in $\mathscr{C}$ we have $F(g \circ f) = F(f) \circ F(g)$,
- For any object $X \in \mathscr{C}$ we have $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

Exercise 4.0.4. Show that the data of a contravariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is equivalent to the data of a functor $F : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathscr{D}$.

Exercise 4.0.5. Show that there is a "contravariant" version of Yoneda's Lemma too. That is, prove the following.

Lemma 4.0.6. *Let $\mathscr{C}$ be a small category and $P : \mathscr{C} \longrightarrow \mathscr{D}$ a contravariant functor. For any object $C \in \mathscr{C}$ there is a natural bijection*

$$\mathrm{Nat}(\mathrm{Hom}(\_, C), P) \cong P(C)$$

In the special case where $P = \mathrm{Hom}(\_, D)$ for some object $D \in \mathscr{C}$, Yoneda's lemma implies the following natural isomorphism.

$$\mathrm{Nat}(\mathrm{Hom}(\_, C), \mathrm{Hom}(\_, D)) \cong \mathrm{Hom}(C, D) \tag{7}$$

That is, there is an embedding of categories:

$$\mathscr{C} \rightarrowtail \underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}} \tag{8}$$

Facts which we will not prove:

- $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ admits all products.
- $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ admits all coproducts.
- $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ admits all limits and colimits.

- $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ admits all exponential objects and a subobject classifier. In fact, $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ is a *topos*.

All of these holds whether $\mathscr{C}$ has any of this structure or *none*. In fact, even more can be said, we know what the excess in $\underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ is:

Proposition 4.0.7. *Every object $P \in \underline{\mathrm{Set}}^{\mathscr{C}^{\mathrm{op}}}$ is a colimit of elements in the image of $\mathscr{C}$ under the yoneda embedding.*

Suggestion: somebody makes a talk out of this.

References

[1] W. Troiani, *Course notes for the Séminaire étudiant de théorie des catégorie*, `https://williamtroiani.github.io/CategoryTheory/Lecture6.pdf`