HW 1 Math 126

UC Berkeley

June 26, 2025

1 Problem 1: Borthwick 2.2

In $\mathbb{R}^2$, we may use polar coordinates (r, θ) which are related to Cartesian coordinates by

$$x_1 = r\cos(\theta),\ x_2 = r\sin(\theta)$$

Part A) Use the chain rule to compute $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$.

Part B) Find an expression for Λ in terms of the (r, θ) coordinates. You should only have derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, and the radial derivative should agree with the radial Laplacian we computed in class.

Hint: Compute $\frac{\partial}{\partial x_1}$ in terms of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, and then compute $\frac{\partial^2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1}$.

2 Problem 2: Borthwick 3.1

Consider the conservation equation with a constant velocity $c > 0$: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ on the quadrant $t \geq 0, x \geq 0$. Suppose the initial and boundary conditions are

$$\begin{cases} u(0, x) = g(x) & x \geq 0 \\ u(t, 0) = h(t) & t \geq 0 \end{cases}$$

Suppose that $g(0) = h(0)$ and $g, h \in C^1([0, \infty))$. Find the formula for the solution $u(t, x)$ in terms of g and h.

3 Problem 3: Borthwick 3.2

Part A) A forcing term $f(t, x)$ is independent of the existing concentration (such as an intravenous injection in our bloodstream example). Assume $c \in R$, $f \in C^1(\mathbb{R}^2)$ and $g \in C^1(\mathbb{R})$. Solve the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = f,\ \ u(0, x) = g(x)$$

to find an explicit formula for $u(t, x)$ in terms of f and g.

Hint: Use method of characterstics, and your formula may have an integral in it!

Part B) A reaction term depends on the concentration u. The simplest case is a linear term γu where the coefficient is some function $\gamma(t, x)$ (this could represent oxygen absorption into the walls of the artery). Assume $c \in \mathbb{R}$, $\gamma \in C^1(\mathbb{R}^2)$ and $g \in C^1(\mathbb{R})$. Solve the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \gamma u,\ \ u(0, x) = g(x)$$

to find an explicit formula for $u(t, x)$ in terms of γ and g.

4 Problem 4: Burger's Equation

Suppose $a > b$ and u satisfies the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(0,x) = \begin{cases} a & x \leq 0 \\ a(1-x) + bx & 0 < x < 1 \\ b & x \geq 1 \end{cases}$$

Show that all characteristics originating from $x_0 \in [0,1]$ meet at the same point (creating a shock).

Hint: Compute the characteristics for all x_0 instead of just those for $x_0 \in [0,1]$. Draw some of these characteristics on an x,t-plane. Do you notice anything about the time $t = \frac{1}{a-b}$?