Extra Practice Problems

UCB

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1 Problem 1

Consider $(0,\pi)$ and the functions $\phi_k = \sqrt{\frac{2}{\pi}}\cos(kx)$, $\phi_0 = \sqrt{\frac{1}{\pi}}$. Show that ϕ_k gives an orthonormal basis for $L^2((0,\pi))$ as follows:

Part A) Show that the ϕ_k are orthonormal.

Part B) Let $f \in L^2((0,\pi))$ be such that $\int_0^\pi f\cos(kx)dx = 0$ for all k. Extend f to an even function on $(-\pi,\pi)$. Then, what is $\int_{-\pi}^{\pi} f(x)e^{ikx}dx$?
Part C) Recall that $\{e^{ikx}\}$ is a basis. What must f be? Conclude $\{\phi_k\}$ is a basis for $L^2((0,\pi))$.

2 Problem 2

Consider

$$I[w] = \int_0^\infty \int_{\mathbb{R}} \frac{1}{2}\left[(\frac{\partial u}{\partial t})^2 - (\frac{\partial u}{\partial x})^2\right] dxdt$$

Show that $u(t,x) \in H_0^1([0,\infty)\times\mathbb{R})$ is a stationary point of $I[w]$ iff. it is a weak solution to the wave equation

$$\begin{cases} (-\partial_t^2 + \partial_x^2)u = 0 \quad (0,\infty)\times\mathbb{R} \\ u(0,x) = 0 \\ u_t(0,x) = h(x) \end{cases}$$

3 Problem 3

Show that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for $x \neq 0$ as a distributional derivative as follows:
Part A)

$$\langle \ln|x|', \psi\rangle = \int_{-\infty}^{\infty} -\psi'(x)\ln|x|dx = \lim_{\epsilon\to 0} -\int_{|x|\geq\epsilon} \psi'(x)\ln|x|dx$$

Integrate by parts on each branch.
Part B) Show that $\lim_{\epsilon\to 0}[\psi(\epsilon)-\psi(-\epsilon)]\ln(\epsilon) = 0$ (Hint, try to use the definition of the derivative of ψ at 0 to help).
Part C) Conclude $\langle \ln|x|', \psi\rangle = \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{\psi(x)}{x}dx$ (Notice that this only works because the limit is taken symmetrically).