Math 54 Final Exam Review Sheet

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1 Systems and Matrices

1.) Start with a system

$$\begin{aligned} 3x + 5y + z &= 5 \\ 2x + 4y + 2z &= 7 \\ x + 2y + 3z &= 9 \end{aligned}$$

Write this system as a vector equation, then as a matrix equation. Solve this via Gaussian elimination. Where are the pivots (and what are pivots)?

2.) What does vector addition look like geometrically? Algebraically? What is the definition of span and what does $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \mathrm{span}\{e_1, e_2\}$ look like in $\mathbb{R}^3$?

3.) What are the requirements for Row Echelon Form and Reduced Row Echelon Form?

4.) When does a system have a solution? What critereon do we usually use to show that it fails to have a solution?

5.) What is the definition of linear independence? What are some equivalent versions?

6.) Are $\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 0 \\ 3 \end{bmatrix}\}$ linearly independent? If not, find a linearly independent subset of these vectors.

7.) Without using the properties of invertible matrices, show that a 2x2 matrix A has linearly independent columns iff. $Ax = b$ has a solution for all $b \in \mathbb{R}^2$.

8.) What is a linear transformation (what does linear mean)? When can we represent linear transformations by matrices? How do we compute that matrix?

9.) Let $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Find the matrix of T.

10.) What is the definition of one-to-one? What is the definition of onto? What criterea do we use for a matrix to determine whether it is one-to-one and onto (using pivots)?

11.) Is the matrix in problem 9 either one-to-one or onto? Come up with a couple examples of a one-to-one matrix (that are not all square).

12.) What is the general formula for a 2x2 matrix inverse?

13.) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 9 \\ 3 & 2 \end{bmatrix}$$

What is $(AB)^{-1} + C^T$?

14.) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

compute A^{-1} and use that to solve $Ax = e_1$.

15.) How does the algorithm to compute the inverse matrix work? If you don't remember, look back at elementary matrices.

16.) Do you remember the giant theorem about all the statements that are equivalent to a square matrix being invertible? Write down as many as you can remember and check against theorem 8 on page 121.

17.) If L is a $n \times n$ matrix and the equation $Lx = 0$ has the trivial solution, do the columns of L span $\mathbb{R}^n$.

18.) If the $n \times n$ matrices E and F have $EF = I$, is it true that E and F commute?

2 Bases, Dimension, Rank

1.) What is a subspace of $\mathbb{R}^n$. Give some examples. What is a basis of that subspace?

2.) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Find a basis for the column space of A and the null space of A.

3.) What does it mean to say that a subspace of $\mathbb{R}^n$ has dimension 2. Can you draw a 2-dimensional subspace in $\mathbb{R}^3$? What is a 0-dimensional subspace (and what is the basis for it– it isn't $\{0\}$)?

4.) What is the rank and what is our big theorem involving it? Use that theorem, and no comments about pivots or invertibility, to prove that a square matrix is onto iff. it is one-to-one.

5.) What is the rank of a 4x5 matrix with a 3-D null space?

3 Determinants and Cramer's Rule

1.) What is the 2x2 determinant formula? Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

2.) Write a full formula for the determinant of an arbitrary 3x3 matrix $A = (a_{i,j})$ (this is mainly to test you on cofactor expansion, so check your answer with another source and check that you have the right signs). You should write out the cofactor expansion across a row or column too.

3.) What do row operations do to a determinant? There are three row operations to check. Use these to compute the determinant of

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 0 & 5 & 6 \end{bmatrix}$$

by using row operations to turn this into an upper triangular matrix.

4.) Let A be a 2x2 matrix with odd entries only. Is the determinant of A odd, even, or can it be both? What if A is a 3x3 matrix?

5.) Now that we've had a bunch of computations, what do we usually use the determinant to tell us? What happens if the determinant is 0? What if it is nonzero?

6.) If we know the determinant of matrices A and B, what can we say about $\det(AB)$. Why is this true (hint: elementary matrices)? What do we know about $\det(A^T)$?

7.) This is looking a bit ahead, but use problem 6 to show that the determinant of a diagonalizable matrix is the product of its eigenvalues.

8.) Let's say we have a 100x100 matrix and we need to compute the 1-1 entry of the inverse ($(A^{-1})_{11}$). What (or whose) rule do we use to compute this without computing the whole inverse?

9.) For example, let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 3 & 10 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

Let $B = A^{-1}$. What is b_{44}?

10.) The Professor likes using the version of the formula that solves systems. Try solving the system from section 1 problem 1 using Cramer's rule.

11.) Geometrically, what does the determinant tell us? As a challenge, try to prove that $\det(AB) = \det(A)\det(B)$ when A and B are 2x2 matrices using geometry and a partitioned 4x4 matrix.

4 Abstract Vector Spaces and Subspaces

1.) What is your favorite vector space? What is your least favorite vector space?

2.) Use the axiomatic definition to check that the set of 2x2 matrices with real entries, $M_2(\mathbb{R})$, is a vector space. Recall the definition of symmetric ($S_2(\mathbb{R})$) and orthogonal (($O(2)$) matrices that we use in later sections. Use the subspace critereon to show that these are subspaces of the vector space of matrices.

3.) Show that the set of polynomials involving only even powers of x (notice 0 is even) is a subspace of the vector space of polynomials.

4.) What do the terms "kernel" and "image" mean? What do we usually call them when we talk about matrices?

5.) How do we define a basis in an abstract vector space? Consider the polynomials of degree at-most 3, P_3. Find a basis for P_3. What is the dimension of P_3?

6.) Can I find a basis for $\mathbb{R}^3$ containing the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

7.) Let $\text{span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^4$. What can I say about these 4 vectors? Is there a basis contained in the set $\{v_1, v_2, v_3, v_4\}$?

15/2.) We covered column spaces and null spaces above, but the Professor does them here. Just for extra practice, find a basis for the column space of the following matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & -1 \end{bmatrix}$$

-This concluded Midterm 1 Material. Practice with the midterm questions! We start below with coordinates

8.) Write the general form of a change-of-coordinates from $\mathcal{B} = \{b_1, ..., b_n\}$ to $\mathcal{C} = \{c_1, ..., c_n\}$. What is the product of two change of coordinate matrices? What is the inverse of a change-of-coordinates matrix?

9.)3. Suppose $\mathcal{B}$ and $\mathcal{C}$ are bases in $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}; \quad \mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Find the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

10.) In problem 5, you found a basis for P_3. The collection $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is also a basis. Write the polynomial $p(x) = 2x^2 + 3x$ as a vector in your coordinates and in these coordinates. Generate the change-of-coordinate matrix between these two bases.

11.) When can we represent linear transformations by matrices for arbitrary vector spaces? Try representing the derivative $\frac{d}{dx}$ acting on P_3 as a matrix with respect to your choice of coordinates.

5 Eigen-everything and Diagonalization

6 Orthogonality, Inner Products

7 Symmetric Matrices and Quadratic Forms

-This concluded the Midterm 2 Material. Practice with the midterm questions! We start below with ODEs.

8 Second-Order Constant Coefficient Differential Equations

9 Systems of Differential Equations

10 Fourier Series