Math 126 Midterm 1

UCB Summer 2025

July 9, 2025

Remember that the exam is closed-book. You will have 1 hour to complete the exam. There are 3 question. You may use the front and back of each page, and there will be plenty of space (no need for extra paper. If you need extra paper for any reason, please attach it via paperclips or staples). Please Write your full name and SID below. Good luck!

Name: ____

SID: ____

Formulas:

$$\int_U \nabla \cdot V \, dx = \int_{\partial U} \eta \cdot V \, dS$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + w = 0$$

$$\dot{x}(t) = v$$

$$\frac{Du}{Dt} + w = 0$$

Homogeneous Solution: $$u(t,x) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)d\tau$$

$$\mathcal{D}_{t,x} = \{(s,y) \in \mathbb{R}^2 \mid x - c(t-s) \leq y \leq x + c(t-s)\}$$

Inhomogeneous Wave equation, no initial data solution: $$u(t,x) = \frac{1}{2c}\int_{\mathcal{D}_{t,x}} f(s,y)dsdy$$

Poisson's Formula: $$u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{\mathbb{D}} \frac{g(x-ty)}{\sqrt{1-|y|^2}}dy\right) + \frac{t}{2\pi}\int_{\mathbb{D}} \frac{h(x-ty)}{\sqrt{1-|y|^2}}dy$$

1 Problem 1

The following questions may be answered with true or false. You do not need to explain your answer, but please make your answer clear and box it for each part.

Part A) The method of characteristics guarantees a solution to

$$\begin{cases} \partial_t u + u\partial_x u = 0 & \text{in } [0,\infty) \times \mathbb{R} \\ u(0,x) = \begin{cases} 1 & \leq 0 \\ 1-x & x \in (0,1) \\ 0 & x \geq 1 \end{cases} & \end{cases}$$

for all time $t \geq 0$ (Hint: $1 > 0$).

Part B) If you existed in 2 dimensions and someone clapped, you would hear a brief, sharp noise and followed by silence.

Part C) Let $U \subseteq \mathbb{R}^n$ be a bounded domain with piecewise-C^1 boundary. A solution $u \in C^2([0,\infty) \times \overline{U})$ to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 & \text{in } [0\infty) \times U \\ u(0,x) = g(x) & \text{in } U \\ \partial_t u(0,x) = h(x) & \text{in } U \\ u|_{\partial U} = 0 & \end{cases}$$

is unique for $f, g, h \in C^\infty$.

Part D) Eigenvalues of the Laplace operator in 1 dimension, i.e. λ solving the Helmholtz problem $-\frac{\partial^2}{\partial x^2}\phi = \lambda\phi$, correspond to frequencies we observe when a string fixed at both ends vibrates as governed by the Wave Equation.

Part E) If $U \subseteq \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary,

$$\int_U u \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \eta} dS + \int_U |\nabla u|^2 dx$$

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2 Problem 2

Let $u(t,x) \in C^1$ solve $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 1$ in the first quadrant $Q_1 = \{(t,x) \mid t \geq 0,\ x \geq 0\}$. Suppose

$$\begin{cases} u(0,x) = x & x \geq 0 \\ u(t,0) = t & t \geq 0 \end{cases}$$

Find an explicit formula for $u(t,x)$ using the method of characteristics.

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3 Problem 3

Assume that $u(t,x) \in C^2([0,\infty)_t \times \mathbb{R}_x)$ solves

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

and $v(t,x) \in C^2([0,\infty)_t \times \mathbb{R}_x)$ solves

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 1$$

Part A) Let $u(0,1) = 2$, $u(1,2) = 4)$ and $u(1,0) = 3$. What is $u(2,1)$?

Part B) Let $v(0,x) = u(0,x)$ and $\partial_t v(0,x) = \partial_t u(0,x)$. What is $v(2,1)$?

Part C) What is the range of influence of the point $(2,1)$?

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