HW 3: Heat Equation and Integration Theory

UCB

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1 Problem 1: Borthwick 6.3+6.4

Let $U \subset \mathbb{R}^n$ be a bounded domain with piecewise-C^1 boundary. Suppose that $u(t,x)$ is real-valued and satisfies

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

on $(0,\infty) \times U$.

First, define

$$\mathcal{U}(t) = \int_U u(t,x)dx$$

to be the total thermal energy at time $t \geq 0$.

Second, define

$$\eta(t) = \int_U u(t,x)^2 dx$$

for $t \geq 0$.

Part A) Assume u satisfies Neumann Boundary conditions $\frac{\partial u}{\partial \eta}|_{\partial U} = 0$ where η is the outward normal. Show that $\mathcal{U}$ is constant.

Part B)

Assume that $u(t,x)|_{\partial U} = 0$ for all $t \geq 0$. Show that η is decreasing as a function of time (this will use a similar trick to part A).

Part C)

Use Part B to show that a solution u to the heat equation satisfying boundary and initial conditions $u|_{t=0} = g$ and $u|_{x \in \partial U} = h$ for some continuous g and h is uniquely determined by this choice of g and h.

2 Problem 2: The Cantor Set

Recall that the symbol $\cap$ means that we take the points in both sets of interest: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. The $\cup$ symbol is like "or": $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

We produce a funny set contained in $[0,1]$ which is famous for its weird properties. First, we start with $I_0 = [0,1]$. Next, we "cut out" the middle-third to produce $I_1 = [0,1/3] \cup [2/3,1]$. We then do the same to each of these new intervals, so that $I_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Repeating this process gives levels $I_3, I_4, ...$ where each time we take the intervals and cut out their middle-third.

This produces the middle-third Cantor set: $C = \bigcap_{i=1}^{\infty} I_i$. This set is uncountable, nowhere-dense, and one of the most common fractals! It has "dimension" between 0 and 1 (it is precisely $\log_3(2)$).

Part A)

What is the measure of I_0? Recall that I_0 is a "rectangle".

Part B) One property of measures is that when A and B are disjoint sets (so that A and B have no points in common, such as the intervals $[0,1/3]$ and $[2/3,1]$), $m(A \cup B) = m(A) + m(B)$.

Using this property, what is $m(I_1)$? What about $m(I_2)$?

Can you come up with a formula for $m(I_k)$, the kth step in this process? *Hint:* I_1 is 2 intervals of length $1/3$, I_2 is 4 intervals of length $1/9$, I_3 is 8 intervals of length $1/27$...

Part C)

Another property of measures is that when $A \subset B$, $m(A) \leq m(B)$. This roughly amounts to the idea that "the part has less size than the whole". For example, $I_2 \subset I_1$, so as you calculated in part B, $m(I_2) \leq m(I_1)$.

Since $C \subset I_k$ for all k, what is $m(C)$?

3 Problem 3: Borthwick 7.2 + 7.3 + Extra

Part A) Consider the sequence of functions on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} ne^{-nx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

What is $\|f_n\|_1$? What is $\|f_n\|_2$? Show that $\lim_{n\to\infty} \|f_n\|_1 = 0$ but that $\lim_{n\to\infty} \|f_n\|_2 = 1$.

Part B) Similarly, consider

$$g_n(x) = n^{-1}\chi_{[0,n]}$$

Compute the L^1 and L^2 norm of g_n to show $\lim_{n\to\infty} \|g_n\|_2 = 0$ but $\lim_{n\to\infty} \|g_n\|_1 = 1$.

Part C)
What are the L^∞ norms of f_n and g_n. Do either approach 0 as $n \to \infty$?

4 Problem 4: Borthwick 7.6

Recall the solution $u(t,x)$ to the heat equation and the function $\eta(t)$ from problem 1 above.

Assume that $u|_{t=T} = 0 = u|_{x\in\partial U}$. The goal of this exercise is to show that the solution to the heat equation is uniquely determined by the boundary values at $t = T$, so that we will show $u = 0$ in the case given.

Part A) Use the Cauchy-Schwarz inequality on the L^2-inner product to show that

$$\eta'(t)^2 \leq 4\eta(t)\int_\Omega |\frac{\partial u}{\partial t}|^2 dx$$

Part B)

Recall that $u(t,x)$ is a smooth function.

Differentiating $\eta'(t) = \int_U 2u\Delta u dx$ with respect to time gives

$$\eta''(t) = \int_U 2(\frac{\partial u}{\partial t})\Delta(u) + 2u\Delta(\frac{\partial u}{\partial t})mdx$$

Use integration-by-parts and the heat equation to compute that

$$\eta''(t) = 4\int_U |\frac{\partial u}{\partial t}|^2 dx$$

This gives the inequality $\eta'(t)^2 \leq \eta(t)\eta''(t)$

Part C)

Suppose $\eta(0) > 0$. The above inequality shows that $(\log(\eta(t)))'' \geq 0$, which gives an inequality $\eta(t) \geq \eta(0)e^{-ct}$ for $c = -\eta'(0)/\eta(0) > 0$. You do not need to prove this and you may assume it as given (the proof is outlined in the exercise in Borthwick). Since $\eta(0) > 0$, η is strictly positive for all time.

Therefore, if $\eta(T) = 0$, we must have $\eta(0) = 0$. Use Problem 1 Part B to explain why this means $u = 0$ for all time (this answer should be one or two sentences at most. Hint: What must $\eta(t)$ be for all time? What is the definition of $\eta(t)$ with respect to u).