Extra Practice Problems

UCB

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1 Problem 1

Consider $(0, \pi)$ and the functions $\phi_k = \sqrt{\frac{2}{\pi}} \cos(kx)$. Show that ϕ_k gives an orthonormal basis for $L^2((0, \pi))$ as follows:

Part A) Show that the ϕ_k are orthonormal.

Part B) Let $f \in L^2((0, \pi))$ be such that $\int_0^\pi f \cos(kx)dx = 0$ for all k. Extend f to an even function on $(-\pi, \pi)$. Then, what is $\int_{-\pi}^{\pi} f(x)e^{ikx}dx$?

Part C) Recall that $\{e^{ikx}\}$ is a basis. What must f be? Conclude $\{\phi_k\}$ is a basis for $L^2((0, \pi))$.

2 Problem 2

Consider

$$I[w] = \int_0^\infty \int_{\mathbb{R}} \frac{1}{2} \left[(\frac{\partial u}{\partial t})^2 - (\frac{\partial u}{\partial x})^2 \right] dxdt$$

the variation Heat Lagrangian. Show that $u(t, x)$ is a minimizer of $I[w]$ iff. it is a weak solution to the heat equation

$$\begin{cases} (\partial_t - \partial_x^2)u = 0 & (0, \infty) \times \mathbb{R} \\ u(0, x) = 0 \end{cases}$$

Can you guess what functional we would construct to reduce the Wave equation $\begin{cases} (\partial_t^2 - \Delta)u = 0 \\ u(0, x) = 0 \\ \partial_t u(0, x) = 0 \end{cases}$?

3 Problem 3

Show that $\frac{d}{dx} \ln |x| = \frac{1}{x}$ for $x \neq 0$ as a distributional derivative as follows:

Part A)

$$\langle \ln |x|', \psi \rangle = \int_{-\infty}^{\infty} -\psi'(x) \ln |x| dx = \lim_{\epsilon \to 0} - \int_{|x| \geq \epsilon} \psi'(x) \ln |x| dx$$

Integrate by parts on each branch.

Part B) Show that $\lim_{\epsilon \to 0} [\psi(\epsilon) - \psi(-\epsilon)] \ln(\epsilon) = 0$ (Hint, try to use the definition of the derivative of ψ at 0 to help).

Part C) Conclude $\langle \ln |x|', \psi \rangle = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\psi(x)}{x} dx$ (Notice that this only works because the limit is taken symmetrically).