Math 54 Final Exam Review Sheet

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1 Introduction

Hello! This document is intended to help expose what you understand and do not understand from our Math 54 course, so that you can study effectively for the final. It is a mix of conceptual questions, computational questions, and some proof-style questions from the worksheets. It is not perfect nor comprehensive, but this document should cover most topics presented and give you an idea of an overall picture of the course.

The questions are based on the content from our text (Lay/Lay/Mcdonald, Linear Algebra and Differential Equations, Math 54, Second Custom Edition for UC Berkeley) and Professor Sharma's lectures and worksheets.

Please let me know if there are any typos and ask if anything is confusing.

2 Systems and Matrices

1.) Start with a system

$$\begin{aligned} 3x + 5y + z &= 5 \\ 2x + 4y + 2z &= 7 \\ x + 2y + 3z &= 9 \end{aligned}$$

Write this system as a vector equation, then as a matrix equation. Solve this via Gaussian elimination. Where are the pivots (and what are pivots)?

2.) What does vector addition look like geometrically? Algebraically? What is the definition of span and what does $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \text{span}\{e_1, e_2\}$ look like in $\mathbb{R}^3$?

3.) What are the requirements for Row Echelon Form and Reduced Row Echelon Form?

4.) When does a system have a solution? What critereon do we usually use to show that it fails to have a solution?

5.) What is the definition of linear independence? What are some equivalent versions?

6.) Are $\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 0 \\ 3 \end{bmatrix}\}$ linearly independent? If not, find a linearly independent subset of these vectors.

7.) Without using the properties of invertible matrices, show that a 2x2 matrix A has linearly independent columns iff. $Ax = b$ has a solution for all $b \in \mathbb{R}^2$.

8.) What is a linear transformation (what does linear mean)? When can we represent linear transformations by matrices? How do we compute that matrix?

9.) Let $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Find the matrix of T.

10.) What is the definition of one-to-one? What is the definition of onto? What criterea do we use for a matrix to determine whether it is one-to-one and onto (using pivots)?

11.) Is the matrix in problem 9 either one-to-one or onto? Come up with a couple examples of a one-to-one matrix (that are not all square).

12.) What is the general formula for a 2x2 matrix inverse?

13.) Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$
$$B = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 9 \\ 3 & 2 \end{bmatrix}$$

What is $(AB)^{-1} + C^T$?

14.) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

compute A^{-1} and use that to solve $Ax = e_1$.

15.) How does the algorithm to compute the inverse matrix work? If you don't remember, look back at elementary matrices.

16.) Do you remember the giant theorem about all the statements that are equivalent to a square matrix being invertible? Write down as many as you can remember and check against theorem 8 on page 121.

17.) If L is a $n \times n$ matrix and the equation $Lx = 0$ has the trivial solution, do the columns of L span $\mathbb{R}^n$.

18.) If the $n \times n$ matrices E and F have $EF = I$, is it true that E and F commute?

3 Bases, Dimension, Rank

1.) What is a subspace of $\mathbb{R}^n$. Give some examples. What is a basis of that subspace?

2.) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Find a basis for the column space of A and the null space of A.

3.) What does it mean to say that a subspace of $\mathbb{R}^n$ has dimension 2. Can you draw a 2-dimensional subspace in $\mathbb{R}^3$? What is a 0-dimensional subspace (and what is the basis for it– it isn't $\{0\}$)?

4.) What is the rank and what is our big theorem involving it? Use that theorem, and no comments about pivots or invertibility, to prove that a square matrix is onto iff. it is one-to-one.

5.) What is the rank of a 4x5 matrix with a 3-D null space?

4 Determinants and Cramer's Rule

1.) What is the 2x2 determinant formula? Compute the determinant of

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

2.) Write a full formula for the determinant of an arbitrary 3x3 matrix $A = (a_{i,j})$ (this is mainly to test you on cofactor expansion, so check your answer with another source and check that you have the right signs). You should write out the cofactor expansion across a row or column too.

3.) What do row operations do to a determinant? There are three row operations to check. Use these to compute the determinant of

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 0 & 5 & 6 \end{bmatrix}$$

by using row operations to turn this into an upper triangular matrix.

4.) Let A be a 2x2 matrix with odd entries only. Is the determinant of A odd, even, or can it be both? What if A is a 3x3 matrix?

5.) Now that we've had a bunch of computations, what do we usually use the determinant to tell us? What happens if the determinant is 0? What if it is nonzero?

6.) If we know the determinant of matrices A and B, what can we say about $\det(AB)$. Why is this true (hint: elementary matrices)? What do we know about $\det(A^T)$?

7.) This is looking a bit ahead, but use problem 6 to show that the determinant of a diagonalizable matrix is the product of its eigenvalues.

8.) Let's say we have a 100x100 matrix and we need to compute the 1-1 entry of the inverse $((A^{-1})_{11})$. What (or whose) rule do we use to compute this without computing the whole inverse?

9.) For example, let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 3 & 10 \\ 2 & 0 & 0 & -2 \end{bmatrix}$$

Let $B = A^{-1}$. What is b_{44}?

10.) The Professor likes using the version of the formula that solves systems. Try solving the system from section 1 problem 1 using Cramer's rule.

11.) Geometrically, what does the determinant tell us? As a challenge, try to prove that $\det(AB) = \det(A)\det(B)$ when A and B are 2x2 matrices using geometry and a partitioned 4x4 matrix.

12.) If A and B are 2x2 matrices and $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, prove that $\det(C) = \det(A)\det(B)$.

5 Abstract Vector Spaces and Subspaces

1.) What is your favorite vector space? What is your least favorite vector space?

2.) Use the axiomatic definition to check that the set of 2x2 matrices with real entries, $M_2(\mathbb{R})$, is a vector space. Recall the definition of symmetric ($S_2(\mathbb{R})$) and orthogonal (($O(2)$) matrices that we use in later sections. Use the subspace critereon to show that these are subspaces of the vector space of matrices.

3.) Show that the set of polynomials involving only even powers of x (notice 0 is even) is a subspace of the vector space of polynomials.

4.) What do the terms "kernel" and "image" mean? What do we usually call them when we talk about matrices?

5.) How do we define a basis in an abstract vector space? Consider the polynomials of degree at-most 3, P_3. Find a basis for P_3. What is the dimension of P_3?

6.) Can I find a basis for $\mathbb{R}^3$ containing the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

7.) Let $\text{span}\{v_1, v_2, v_3, v_4\} = \mathbb{R}^4$. What can I say about these 4 vectors? Is there a basis contained in the set $\{v_1, v_2, v_3, v_4\}$?

15/2.) We covered column spaces and null spaces above, but the Professor does them here. Just for extra practice, find a basis for the column space of the following matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & -1 \end{bmatrix}$$

-This concluded Midterm 1 Material. Practice with the midterm questions! We start below with coordinates

8.) Write the general form of a change-of-coordinates from $\mathcal{B} = \{b_1, ..., b_n\}$ to $\mathcal{C} = \{c_1, ..., c_n\}$. What is the product of two change of coordinate matrices? What is the inverse of a change-of-coordinates matrix?

9.)3. Suppose $\mathcal{B}$ and $\mathcal{C}$ are bases in $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}; \quad \mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Find the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

10.) In problem 5, you found a basis for P_3. The collection $\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ is also a basis. Write the polynomial $p(x) = 2x^2 + 3x$ as a vector in your coordinates and in these coordinates. Generate the change-of-coordinate matrix between these two bases.

11.) When can we represent linear transformations by matrices for arbitrary vector spaces? Try representing the derivative $\frac{d}{dx}$ acting on P_3 as a matrix with respect to your choice of coordinates.

12.) If A and B are matrices for which AB is defined, what can we say about rank(AB)?

6 Eigen-everything and Diagonalization

1.) What goals motivate us to try to force matrices to be diagonal? (Answer: ease of calculation and commuting matrices)

2.) Define an eigenvector, and an eigenvalue. These should be related via an important equation called the eigenvector equation. If I know what eigenvalue to use, how can I find all of the eigenvectors (pose it as the null space of some matrix)?

3.) How do I find the eigenvalues in the first place? Why does this work (it should connect to problem 2.)?

4.) Use your answer from problem 3 to find a matrix whose characteristic polynomial is $-\lambda^3 + 1$. (First, figure out what size matrix we need, then try to factor this polynomial and use those factors to construct an appropriate matrix).

4.) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

5.) When is a matrix diagonalizable? Define algebraic and geometric multiplicity of an eigenvalue, and connect a statement about these to the property you used for diagonalizability. Is the matrix of the previous problem diagonalizable? There is a very quick way to answer without computation.

6.) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

and use this to compute A^{100}.

7.) Does a matrix need to be invertible to be diagonalizable? Prove this or find a counterexample.

8.) Find a 4x4 matrix with no real eigenvalues (hint: find a 2x2 matrix with complex eigenvalues and use problem 12 from the determinants section).

9.) Define similar matrices. Give an example of $A \neq B$ so A and B are similar. If matrices have the same eigenvalues, are they similar? What if they have the same eigenvalues and are both diagonalizable?

10.) In the case of complex eigenvalues, we didn't diagonalize in this class and instead we used the polar decomposition. Compute such a decomposition for the matrix

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

(try to remember all the computational shortcuts).

11.) Recall that a matrix represents a linear transformation, so we can change the basis and have a new representation of a matrix. How does this relate to similarity? Let T be a linear transformation which, under the standard basis, has representation

$$\begin{bmatrix} 10 & 2 \\ -1 & 0 \end{bmatrix}$$

Compute the matrix for T under the basis $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ (and, as quick review, why is this a basis?).

7 Orthogonality, Inner Products

1.) What is the dot product and what does it do? What are some useful properties of the dot product? Let

$$u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

and compute $u \cdot v$, $v \cdot u$, $5u \cdot 5v$, $u \cdot v - 3u \cdot v$ (you should only need to compute one actual dot product to compute all of these).

2.) How do we define the length of a vector using the dot product? For u as in problem 1, what os $||u||$?

3.) How do we define orthogonal vectors? How do we compute the angle between two vectors? Directly compute the orthocomplement of $(1, 0, 2)$ in $\mathbb{R}^3$.

4.) Recall the formula $(\text{Row}(A))^\perp = \text{Nul}(A)$. Use this to compute a basis for the orthocomplement of $(1, 1, 1)$ in $\mathbb{R}^3$. What is the analogous statement for the column space?

5.) Is it true that $(W^\perp)^\perp = W$ in finite-dimensional vector spaces? Justify it. Is it also true that $\dim(W) + \dim(W^\perp) = n$ for $W \subset \mathbb{R}^n$ a subspace?

6.) Let $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Let $W = \text{span}\{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\}$. Compute $\text{proj}_W(u)$. What do we need to check before using the projection formula? Compute also the part of u orthogonal to W and the distance from u to W.

7.) What is an orthogonal matrix? Why is the name misleading? What properties do oethogonal matrices have with respect to the dot product? What values can the determinant of an orthogonal matrix take?

8.) If a matrix is orthogonal and upper triangular, what must it be?

9.) Say W is the subspace in problem 6. Write a matrix A such that $AX = \text{proj}_W(x)$ for every vector $x \in \mathbb{R}^3$.

10.) Let W now be an arbitrary subspace and A the matrix so $Ax = \text{proj}_W(x)$ for every x. Is A diagonalizable?

11.) Use Gram-Schmidt to compute an orthonormal basis for $W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 & 3/2 & 4 \end{bmatrix}\}$.

12.) As another application of Gram-Schmidt, define the QR-factorization and find the QR-factorization of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \\ 2 & 4 \end{bmatrix}$$

13.) How can we get close to solving $Ax = b$ when $b \notin \text{col}(A)$? What do we call this solution, and what equations do we use to compute it (Normal Equations)?

14.) Compute the least squares solution for the problem $Ax = b$ where $A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$.

15.) Compute the line of best for $y = mx + b$ through the points $(0, 1)$, $(1, 4)$, and $(2, 5)$.

16.) Suppose $y = x - 1$ is the line of best fit for the points $(x, y) = (0, 0), (2, 2), (4, 4), (a, b)$. Describe the set of all possible pairs (a, b).

17.) What axioms define an inner product on an arbitrary vector space?

18.) Consider the vector space $C([0, 1])$, the continuous functions on the interval $[0, 1]$. Consider $\langle f, g \rangle = \int_0^1 fgdx$. Show this is an inner product.

19.) Consider $C([0, 1])$ with the integral inner product as in the previous problem. Compute the projection of $\sin(x)$ onto the subspace P_2 of polynomials of degree at-most 2 (first you need to find an orthogonal basis for this subspace).

8 Symmetric Matrices and Quadratic Forms

-This concluded the Midterm 2 Material. Practice with the midterm questions! We start below with ODEs.

9 Second-Order Constant Coefficient Differential Equations

10 Systems of Differential Equations

11 Fourier Series