# Math 126 HW 2

UCB Summer 2025

July 8, 2025

## 1 Problem 1: Borthwick 4.1

Let $u(t,x) : [0,\infty)_t \times \mathbb{R} \to \mathbb{R}$ be a $C^2$ solution to the wave equation. Let $P$ be a parallelogram in the $(t,x)$-plane whose sides are characteristics. In particular, let it have vertices $(0,x)$, $(t/2, x-ct/2)$, $(t, x+ct/2)$, and $(t,x)$. Show that the value of $u$ at the vertex $(t,x)$ is determined by the value of $u$ at the other three vertices.

## 2 Problem 2: Borthwick 4.4

In example 4.8, we solved for $u(t,x)$ a solution to the 1D wave equation $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(t,x)$ with initial conditions $u(0,x) = 0 = \partial_t u(0,x)$.

Let the forcing term be $f(t,x) = \cos(\omega t)\sin(\omega_k x)$ for $\omega > 0$ and $\omega_k = \frac{k\pi}{l}$. Find the solution $u(t,x)$ with initial conditions $u(0,x) = 0 = \partial_t u(0,x)$. Include both case $\omega \neq \omega_k$ and $\omega = \omega_k$.

## 3 Problem 3: Borthwick 4.5

The telegraph equation is a variant of the wave equation that describes the propagation of electrical signals in the 1D cable:

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

where $u(t,x)$ is the line voltage, $c$ propagation speed, and $a, b > 0$ are constants such that $\frac{a^2}{4} = b$. Make the substituton $u(t,x) = e^{-at/2}w(t,x)$ to reduce to the ordinary wave equation for $w$:

$$e^{-at/2}(\partial_t^2 w - c^2\partial_x^2 w) = 0$$

Use the solution formula for $w$ in terms of initial conditions $u(0,x) = g$ and $\partial_t u(0,x) = h(x)$ for $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ to obtain a solution formula for $u$.

## 4 Problem 4: Borthwick 4.7

**Part A)** Let $u(t,x)$ be a $C^2$ solution to the free Schrödinger equation $\frac{\partial u}{\partial t} - i\Delta u = 0$ on the bounded $C^1$ domain $U$. Assume that $u$ us 0 on the boundary of $U$, i.e. $u|_{\partial U} = 0$. Show that $u$ conserves total probability:

$$\int_U |u(t,x)|^2 dx \left( = \int_U u\bar{u} dx \right) = \int_U |u(0,x)|^2 dx$$

*Hint:* What step did we take with the energy equation? Also, recall that i.) $\bar{\Delta u} = \Delta\bar{u}$ (complex conjugation commutes with the derivatives) and ii.) $\bar{a}\bar{b} = \overline{ab}$. What PDE does $\bar{u}$ satisfy?

**Part B)** Use the methods of our proof with energy to show that the solution $u$ from part A is uniquely determined by the initial condition $u(0,x)$.

# 5 Problem 5: Borthwick 4.8

In $\mathbb{R}^n$, consider the wave equation $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$. The *plane wave* solutions have the form $u(t,x) = e^{i(k \cdot x - wt)}$ for $w \in \mathbb{R}$ and $k \in \mathbb{R}^n$ constants.

**Part A)** Find the condition on $w = w(k)$ such that $u$ of the above form solves the wave equation (substitute $u$ of this form into the PDE and see what happens).

**Part B)** For a fixed $t, \theta \in \mathbb{R}$, consider $\{x \in \mathbb{R}^n \mid u(t,x) = e^{i\theta}\}$. Unravel the definition of $u$ to show that this is a set of planes perpendicular to $k$ (of the form $k \cdot x =$ constant. This explains the name *plane wave solution*.

**Part C)** Pick a plane out of the set above. As $t$ increases, what direction does the plane move? What speed does it move at? You only need to give an answer in words, but it will help to plot out an example plane and track what happens.