HW 5: Maximum Principles and Weak Solutions

UCB

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1 Problem 1: Borthwick 9.1

Suppose that $u, \phi \in C^2(U; \mathbb{R}) \cap C^0(\overline{U})$ on a bounded domain $U \subset \mathbb{R}^n$. Assume that u is subharmonic and ϕ is harmonic. Show that $u \leq \phi$ on U.

2 Problem 2: Borthwick 9.2

In complex analysis, Liouville's theorem says that a bounded holomorphic function is constant. As a harmonic function is the real part of a holomorphic function, we might expect the same to be true. This problem proves that. Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function such that $|u(x)| \leq M$ for all x.

Part A)
For $x_0 \in \mathbb{R}^n$, set $r_0 = |x_0|$. Show that

$$u(0) - u(x_0) = \frac{n}{A_n R^n} \left[\int_{B(0,R)} u dx - \int_{B(x_0,R)} u dx \right]$$

for any $R > 0$.

Part B) Assume that $R > r_0$. Show that

$$\text{vol}[B(0,R) \backslash B(x_0,R)] \leq \text{vol}[B(0,R) \backslash B(x_0/2, R - r_0/2)] = \frac{A_n}{n}[R^n - (R - \frac{r_0}{2})^n]$$

Hint: We are shifting to a circle contained in the intersection

Part C)
Apply these estimates and the fact that $|u| \leq M$ to estimate

$$|u(0) - u(x_0)| \leq 2M \left[\frac{R^n - (R - \frac{r_0}{2})^n}{R^n} \right]$$

Take the limit as $R \to \infty$ to show $u(x_0) = u(0)$.

3 Problem 3: Rankine-Hugoniot and Rarefaction

Consider the traffic equation

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0$$
$$u(0,x) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases}$$

In the case $a < b$, we saw the generation of shocks. Let us consider the case $a > b$. In particular, let $a = 1$ and $b = 0$. We recall that u is constant along characteristics for this equation. **Part A)** Plot the characteristics. Does the method of characteristics define a value for $u(0, 1)$?

Part B) Nevertheless, we may force a solution

$$u(t,x) = \begin{cases} 1 & x < \sigma(t) \\ 0 & \sigma(t) < x \end{cases}$$

assuming that u is a weak solution, use the Rankine-Hugoniot condition to find $\sigma(t)$.

Part C) We could try to "bridge the gap" between the two sides. Consider a solution

$$u(t,x) = \begin{cases} 1 & x < \sigma_1(t) \\ 1/2 & \sigma_1(t) < x < \sigma_2(t) \\ 0 & \sigma_2(t) < x \end{cases}$$

again, find σ_1 and σ_2.

Part D) Repeating this process "infinitely" gives

$$u(t,x) = \begin{cases} 1 & x < -t \\ \frac{1}{2} - \frac{x}{2t} & -t < x < t \\ 0 & t < x \end{cases}$$

Show by the Rankine-Hugoniot condition that this is a weak solution (don't forget to check that the middle definition is a classical solution).

4 Problem 4: Borthwick 10.1

On $\mathbb{R}$, consider the ODE $x\frac{du}{dx} = 1$.

Part A) Develop a weak formulation for this ODE in terms of pairing with a test function $\psi \in C_c^\infty(\mathbb{R})$.

Part B) Show that $u(x) = \log|x|$ is locally integrable and solves the equation in the weak sense.