9.1 Sum and product rules

Learn to:

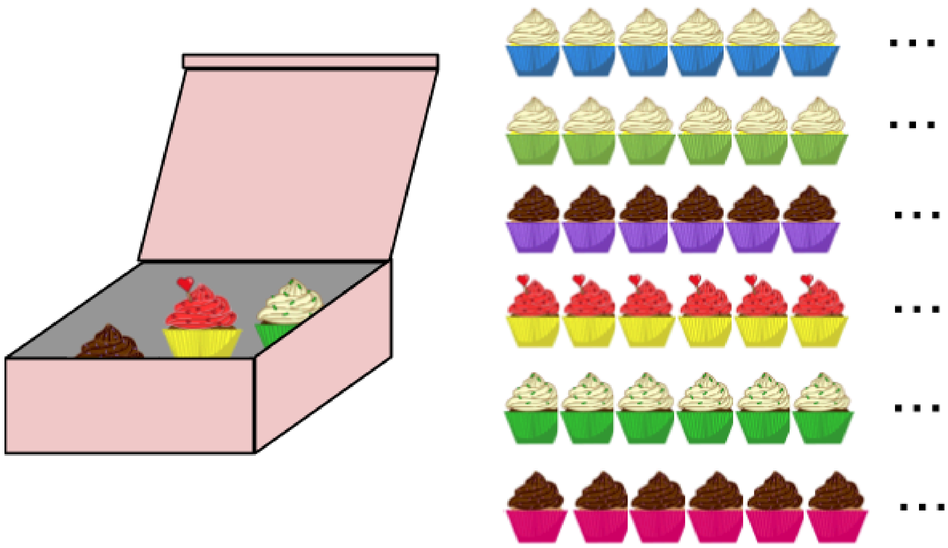
- Apply the product and sum rules to count the number of items in a set.
- Understand terminology and notation associated with strings.
- Apply counting techniques to find the number of strings that satisfy various conditions.

Introduction to counting

Counting, as simple as it may seem initially, is a central topic in discrete mathematics. Most children begin learning mathematics by learning to count: 1, then 2, and so forth. In discrete mathematics, the goal is to count the number of elements in (or the *cardinality* of) a finite set given a description of the set. Determining a set's cardinality often requires exploiting some mathematical structure of the set.

Figure 9.1.1: Counting the number of cupcake selections.

A bakery sells 6 different varieties of cupcakes (chocolate, vanilla, red velvet, etc.). How many ways can a box be filled with 24 cupcakes from the 6 varieties? The order in which the cupcakes are selected is unimportant; all that matters is the number of each variety in the box after they are chosen. The answer to the cupcake counting problem is $118{,}755$, which is too many to count by hand.



The answer to the cupcake counting question is $118{,}755$. This material covers a systematic technique to answer questions like the cupcake counting question. While counting cupcake selections may not seem like a compelling application, the same techniques can be used to count the number of ways 24 identical tasks can be assigned to a network of 6 processors, which in turn can be used to calculate the probability that a random assignment distributes the workload evenly among the 6 processors.

Counting is an important mathematical tool to analyze many problems that arise in computer science. Counting is useful, for example, to understand the amount of a particular resource used in a computer system or computation. Counting is also used to determine the number of valid passwords for a security system or addresses in a network to ensure that there are enough unique choices to meet the demand. Counting is also at the heart of discrete probability, which is central to many areas of science.

The product rule

The two most basic rules of counting are the sum rule and the product rule. These two rules applied in different combinations can be used to handle many counting problems. The ***product rule*** provides a way to count sequences. While sequences may not seem like a particularly common type of object to count, many sets can be expressed as sets of sequences.

Theorem 9.1.1: The product rule.

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$\left|A_1 \times A_2 \times \ldots \times A_n\right| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Consider a restaurant that has a breakfast special that includes a drink, a main course, and a side. The set of choices for each category are:

$$D = \{\text{coffee, orange juice}\}$$
$$M = \{\text{pancakes, eggs}\}$$
$$S = \{\text{bacon, sausage, hash browns}\}$$

Any particular breakfast selection can be described by a triplet indicating the choice of drink, main course, and side. For example, $(\text{coffee, pancakes, bacon})$ is one particular breakfast combination. The set of all possible choices is the same as the set of a triples where the first entry is a drink, the second entry is a main course, and the third entry is a side. The number of possible breakfast combinations is therefore:

$$|D \times M \times S| = |D| \cdot |M| \cdot |S| = 2 \cdot 2 \cdot 3 = 12$$

PARTICIPATION ACTIVITY 9.1.1: An example of the product rule: counting breakfast selections.

1 2 3 4 2x speed

Breakfast Special:
Drink choices: Coffee, OJ
Main course choices: pancakes, eggs
Side choices: bacon, sausage, hash browns

Breakfast selections: (Drink choice , Main course choice , Side choice)

Select a drink:
Select a main course:
Select a side choice:

(coffee	, pancakes	, bacon)
(coffee	, pancakes	, sausage)
(coffee	, pancakes	, hash browns)
(coffee	, eggs	, bacon)
(coffee	, eggs	, sausage)
(coffee	, eggs	, hash browns)
(OJ	, pancakes	, bacon)
(OJ	, pancakes	, sausage)
(OJ	, pancakes	, hash browns)
(OJ	, eggs	, bacon)
(OJ	, eggs	, sausage)
(OJ	, eggs	, hash browns)

Number of breakfast selections = $2 \times 2 \times 3 = 12$

For each choice of drink and main course, three choices for the side are available: bacon, sausage, or hash browns. There are $4 \cdot 3 = 12$ choices total.

Captions

1. A breakfast special includes a choice of drink, main course, and side.
2. Two choices for the drink are available: coffee or orange juice. There are two options so far for the breakfast special.

3. For each choice of drink, two choices for the main course are available: pancakes or eggs. There are $2 \cdot 2 = 4$ choices so far.
4. For each choice of drink and main course, three choices for the side are available: bacon, sausage, or hash browns. There are $4 \cdot 3 = 12$ choices total.

PARTICIPATION ACTIVITY

9.1.2: Applying the product rule: Counting burrito selections.

A burrito stand sells burritos with different choices of stuffing. The set of choices for each category are:

- $\text{Filling choices} = \{\text{chicken, beef, pork}\}$
- $\text{Bean choices} = \{\text{black, pinto}\}$
- $\text{Salsa choices} = \{\text{mild, medium, hot}\}$

1) If every burrito has a filling, beans, and salsa, then how many possible burrito combinations are there?

18

Check Show answer

Correct

18

A burrito selection is described by a triple of the form: $(\text{filling choice, bean choice, salsa choice})$. The number of selections is:

$$|\text{filling}| \cdot |\text{bean}| \cdot |\text{salsa}| = 3 \cdot 2 \cdot 3 = 18$$

2) Suppose that the customer can also select grilled veggies as a filling. Now how many selections are there?

24

Check Show answer

Correct

24

The set of choices for the filling is now $\{\text{chicken, beef, pork, veggies}\}$.
A burrito selection is described by a triple of the form: $(\text{filling choice, bean choice, salsa choice})$. The number of selections is:

$$|\text{filling}| \cdot |\text{bean}| \cdot |\text{salsa}| = 4 \cdot 2 \cdot 3 = 24$$

3) Suppose that the burrito stand introduces a choice between plain flour or whole wheat tortillas. The additional option to select veggies as a filling is still available as well. Now how many selections are there?

48

Check Show answer

Correct

48

A burrito selection is described by a 4-tuple of the form: $(\text{tortilla choice, filling choice, bean choice, salsa choice})$. The number of selections is:

$$|\text{tortilla}| \cdot |\text{filling}| \cdot |\text{bean}| \cdot |\text{salsa}| = 2 \cdot 4 \cdot 2 \cdot 3 = 48$$

Counting strings

If Σ is a set of characters (called an ***alphabet***), then Σ^n is the set of all strings of length n containing characters that come from the set Σ. For example, if $\Sigma = \{0, 1\}$, then Σ^6 is the set of all binary strings with 6 bits. The string 011101 is an example of an element in Σ^6. The strings $xxyzx$ and $zyyzy$ are examples of strings in the set $\{x, y, z\}^5$. The product rule can be applied directly to determine the number of strings of a given length over a finite alphabet:

$$|\Sigma^n| = |\underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{\text{n times}}| = \underbrace{|\Sigma| \cdot |\Sigma| \cdots |\Sigma|}_{\text{n times}} = |\Sigma|^n$$

For example, the number of binary strings of length n is 2^n since the size of the alphabet is 2 (that is, $\big|\{0, 1\}\big| = 2$).

The product rule can also be used to determine the number of strings in a set when one or more of the characters are restricted. Define S to be the set of binary strings of length 5 that start and end with 0. A string is in the set S if it has the form $0{*}{*}{*}0$, where each $*$ could be a 0 or a 1.

$$|S| = \big|\{0\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0\}\big| = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 8$$

PARTICIPATION ACTIVITY | 9.1.3: Using the product rule to count sets of strings.

1) How many six-bit binary strings are there?

Check Show answer

2) How many six-bit binary strings begin with "01"?

Check Show answer

3) How many strings of length 4 are there over the alphabet $\{a, b, c\}$?

Check Show answer

4) How many strings of length 4 are there over the alphabet $\{a, b, c\}$ that end with the character c?

Check Show answer

The sum rule

In the breakfast example, the product rule is applied because the customer selects a drink and a main course and a side. In contrast, the ***sum rule*** is applied when multiple choices are available, but only one selection is made. For example, suppose a customer just orders a drink. The customer selects a hot drink or a cold drink. The hot drink selections are $\{\text{coffee, hot cocoa, tea}\}$. The cold drink selections are $\{\text{milk, orange juice}\}$. The total number of choices is 5, namely 3 hot drink choices plus 2 cold drink choices. A formal statement of the sum rule is expressed in terms of sets below:

Theorem 9.1.2: The sum rule.

Consider n sets, $A_1, A_2, \ldots, A_n$. If the sets are pairwise disjoint (which means that $A_i \cap A_j = \emptyset$ for $i \neq j$), then:

$$\left|A_1 \cup A_2 \ldots \cup A_n\right| = \left|A_1\right| + \left|A_2\right| + \ldots \left|A_n\right|$$

In the example with the customer selecting a drink, $n = 2$ since two categories of drinks are available: hot drinks and cold drinks. Let C be the set of cold drinks and H the set of hot drinks. The fact that H and C are disjoint ($C \cap H = \emptyset$) means that no drink is categorized as both a hot drink and a cold drink. Applying the sum rule yields that the number of possible drinks is:

$$|C \cup H| = |C| + |H| = 2 + 3 = 5$$

Example 9.1.1: Product and sum rule in combination: Counting passwords.

Consider a system in which a password must be a string of length between 6 and 8. The characters can be any lowercase letter or digit.

Let L be the set of all lowercase letters and D be the set of digits. $|L| = 26$ and $|D| = 10$. The set of all allowed characters is $C = L \cup D$.. Since $D \cap L = \emptyset$, the sum rule can be applied to find the cardinality of C: $|C| = 26 + 10 = 36$.

Let A_j denote the strings of length j over the alphabet C. By the product rule, $|A_j| = 36^j$. Notice that for that two values j and k where $j \neq k$, A_j and A_k are disjoint because a string cannot have length j and length k at the same time. If the user must select a password of length 6 or 7 or 8, then the sum rule applies:

$$|A_6 \cup A_7 \cup A_8| = |A_6| + |A_7| + |A_8| = 36^6 + 36^7 + 36^8$$

In the next example, a customer purchasing a laptop can select three different sizes of screens and has a choice between two different processor speeds. For storage, the customer can select a hard disk drive or a solid state drive. The hard disk drive option comes in two sizes, and the solid state option has three different sizes. The manufacturer needs to know how many different configurations are possible. The number of choices is worked out in the animation below:

PARTICIPATION ACTIVITY 9.1.4: An example of the sum and product rule: Counting laptop selections.

Start 2x speed

Laptop customization choices:

Screen size $= \{14\text{in}, 15\text{in}, 17\text{in}\}$
Processor speed $= \{2.0\text{GHz}, 2.7\text{GHz}\}$
Storage $=$ **Solid State Drive (SSD) or Hard Disk Drive (HDD)**
SSD choices: $\{128\text{G}, 256\text{G}, 512\text{G}\}$
HDD choices: $\{256\text{G}, 512\text{G}\}$

Laptop Selection: (Screen size, Processor speed, Storage)

3 choices $\times$ 2 choices $\times$ SSD or HDD: SSD 3 choices, HDD 2 choices (Sum rule)

Number of choices:

$$3 \cdot 2 \cdot (3 + 2) = 30$$

Captions

PARTICIPATION ACTIVITY | 9.1.5: Applying the sum and product rule in combination: Counting sets of bit strings.

A bit string consists of 0s and 1s. For example, 0101 is a bit string with four bits.

1) How many six-bit strings begin and end with a 1, or start with 00?

Check Show answer

2) How many bit strings of length five or six start with a 1?

Check Show answer

CHALLENGE ACTIVITY | 9.1.1: Sum and product rules.

675114.5466804.qx3zqy7

Start

A deli sells sandwiches with different choices for bread, protein, and cheese. The categories are:

$$\text{Bread} = \{\text{kaiser roll, ciabatta, white, bagel}\}$$
$$\text{Protein} = \{\text{egg salad, roast beef, ham}\}$$
$$\text{Cheese} = \{\text{gouda, havarti}\}$$

If every sandwich has a bread, a protein, and a cheese, how many sandwich combinations are possible?

Use the math palette to type a number or a numeric expression.

1 2 3 4 5 6 7

1 2 3 4 5 6 7

Check Next

Additional exercises

EXERCISE 9.1.1: License plate combinations.

In a particular state, the license plates have 7 characters. Each character can be a capital letter or a digit except for 0. (The set of possible digits is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.) A person witnesses a crime and remembers some information about the license plate of the getaway car. The authorities need to figure out how many license plates need to be checked in each case. For each constraint given below, indicate the number of license plates that satisfy that constraint.

Note: The answer can contain multiplications and power. Calculating the number is not necessary.

(a) No constraints

(b) The license plate starts with a digit

(c) The first three characters are letters

(d) The first three characters are letters and the last four characters are numbers

EXERCISE 9.1.2: Counting passwords made up of letters, digits, and special characters.

Consider the following definitions for sets of characters:

- Digits$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Letters $= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
- Special characters $= \{*, \&, \$, \#\}$

Compute the number of passwords that satisfy the given constraints.

(a) Strings of length 6. Characters can be special characters, digits, or letters.

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

EXERCISE 9.1.3: Selecting lunch specials for the week.

(a) A Chinese restaurant offers $\mathbf{10}$ different lunch specials. Each weekday for one week, Fiona goes to the restaurant and selects a lunch special. How many different ways are there to select lunches for the week? Note that which lunch Fiona orders on which day matters, so the following two selections are considered different.
One possible selection:
- Mon: Kung pao chicken
- Tues: Beef with broccoli
- Wed: Kung pao chicken
- Thurs: Moo shu pork
- Fri: Beef with broccoli

A different selection:
- Mon: Beef with broccoli
- Tues: Kung pao chicken
- Wed: Kung pao chicken
- Thurs: Moo shu pork
- Fri: Beef with broccoli

(b) Now suppose that out of the $\mathbf{10}$ dishes that the restaurant offers, only $\mathbf{3}$ of them are vegetarian. If Fiona must select a vegetarian option on Friday, how many ways can the lunches be selected?

(c) Again, suppose that out of the $\mathbf{10}$ dishes that the restaurant offers, only $\mathbf{3}$ of them are vegetarian. If Fiona must go with a vegetarian option on both Monday and Friday, how many ways can the lunches be selected?

(d) Now suppose that Fiona can select a vegetarian or a nonvegetarian lunch on any day of the week. However, in addition to selecting a main course, Fiona must also select between water or tea to drink. How many ways can the lunches be selected?

EXERCISE 9.1.4: Dividing up a print job.

A $\mathbf{100}$-page document is being printed by four printers. Each page is printed exactly once.

(a) Suppose that there are no restrictions on how many pages a printer can print. How many ways are there for the $\mathbf{100}$ pages to be assigned to the four printers?
One possible combination: printer $\boldsymbol{A}$ prints out pages $\mathbf{2-50}$, printer $\boldsymbol{B}$ prints out pages $\mathbf{1}$ and $\mathbf{51-60}$; printer $\boldsymbol{C}$ prints out $\mathbf{61-80}$ and $\mathbf{86-90}$; printer $\boldsymbol{D}$ prints out pages $\mathbf{81-85}$ and $\mathbf{91-100}$.

(b) Suppose the first and the last page of the document must be printed in color, and only two printers can print in color. The two color printers can also print black-and-white. How many ways are there for the 100 pages to be assigned to the four printers?

(c) Suppose that all the pages are black-and-white, but each group of 25 consecutive pages ($1-25$, $26-50$, $51-75$, $76-100$) must be assigned to the same printer. Each printer can be assigned 0, 25, 50, 75, or 100 pages to print.
How many ways are there for the 100 pages to be assigned to the four printers?

Activity summary for assignment: 09 zyBooks Counting I

4 / 77 points

No due date

4 / 77 points submitted to canvas

Completion details