MA 3823 - Symmetric Groups Term Paper

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Isometries: Definition and Example

Definition: Isometry
An *isometry* of a metric space, M, is a function from M onto M that preserves distance.

Example: The function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(\vec{r}) = \vec{r} + \vec{a}, \quad \vec{a} \in \mathbb{R}^n \tag{1}$$

is simply a translation of points in $\mathbb{R}^n$ by the vector $\vec{a}$. Translations preserve distance in any space, so T is an *isometry*.

Isometries Acting on Geometric Objects

Theorem: An isometry maps:

- straight lines to straight lines
- segments to congruent segments
- triangles to congruent triangles
- angles to congruent angles

Proof: Let S be the isometry that maps A and B to A' and B' respectively. The distance between A and B is the same as the distance between A' and B', by the definition of an isometry. The approach here involves considering these distances and an arbitrary point on AB, call it C, and showing that C' must also be on the line $A'B'$.

$$\begin{aligned} AB &= AC + CB \\ \implies A'B' &= A'C' + C'B' \quad \text{(after applying } S\text{)} \end{aligned}$$

If C' is **not** on line $A'B'$, then we'll have that $A'B' < A'C' + C'B'$, via the triangle inequality. (Note that if 3 points do not all lie on the same line, they form the vertices of a triangle.)

However, we have shown that these segments are equal, so C' must lie on the line $A'B'$, the result of applying S to AB. This is true for any arbitrary point on $A'B'$, so $AB = A'B'$. ■

Definition: Symmetry Group

Let F be a set of points in M. The *symmetry group* of F in M is the set of all isometries of M that carry F onto itself, with the group operation of function composition.
A symmetry group is often referred to as the *full symmetry group* - such a group contains orientation-reversing isometries.

Definition: Proper Symmetry Group

The subset of a full symmetry group whose elements all preserve orientation is a subgroup, called a *proper symmetry group*.

Definition: Finite and Infinite Symmetry Groups

A symmetry group with a finite number of elements is said to be a *finite symmetry group*. A symmetry group with an infinite number of elements is said to be an *infinite symmetry group*.

Significance of the Metric Space

Conider the line segment. What is the order of symmetry group? Without specifying the metric space M, we cannot say. In $\mathbf{R}^1$ the symmetry is just Z_2, as given by the reflection around the midpoint of the of the line. If in $\mathbf{R}^2$, then we get D_2, as we now also have the perpendicular reflection and rotation by 180. Finally, if we look in $\mathbf{R^3}$, we have a symmetry group of infinite order, as we can choose an infinite number of planes which the segment exists on, end we can perform D_2 in any of these planes.

Proof of Finite Plane Symmetries

Theorem 0.1. *The only finite plane symmetry groups, up to isomorphism, are Z_n and D_n*

Proof. Let G be a finite plane symmetry group of some figure.

Observe G cannot contain a translation or glide reflection. Let S_t be the translation or glide operation. Notice that we can do $(S_t)^n$ as many times as we want, and so we can move arbitrarily far, tending to ∞ as n tends to ∞.

Two reflections give a translation if lines are parallel (impossible) or a rotation around the intersection point of lines of reflection. The intersection point of all reflections is at the same point:

Let f and f' be two distinct reflections in G. ff' preserves orientation, so it is a rotation. All geometries of a finite group of rotations must have common center, so the elements in G a common fixed point.

Let β be the smallest positive angle of rotation. Every rotation must be a power of R_β: Let $R_\sigma \in G$. By definition $\beta \leq \sigma$, and therefore is some integer t such that $t\beta \leq \sigma \leq (t+1)\beta$ Note that $R_{\sigma - t\beta} = R_\sigma \left(R_\beta\right)^{-t}$ is in G and $0 \leq \sigma - t\beta < \beta$, and since β is smallest positive angle, it must be 0 and therefore $R_\sigma = (R_\beta)^t$

If G has no reflections we proved it is isometric to Z_n, as R_β is a generator, with $n = 360/\beta$. If G has at least one reflection, f, then $f(R_\beta)^n$ are all also in G. Note, this is also all the reflections in G. To see this, note that if we take another reflection $g \in G$, then fg is a rotation and $fg = (R_\beta)^k$ for some k. Thus, $g = f^{-1}(R_\beta)^k = f(R_\beta)^k$. G is therefore the dihedral group D_n. □

Rotational Symmetries in $\mathbf{R}^3$

Similar to symmetries in a finite plane, rotational symmetries in $\mathbf{R}^3$ can only take on certain forms. These forms are Z_n, D_n, A_4, S_4, and A_5.

For the platonic solids, tetrahedron has A_4 symmetry, cube and octahedron have S_4 symmetry, and dodecahedron and icosahedron have A_5 symmetry. Note that the tetrahedron is unique in that it is the only solid of its symmetry group, a consequence of the fact that the tetrahedron is its own dual, as shown in Figure 2

Wallpaper Groups

***Wallpaper groups* are mathematical concepts that describe the symmetries of repeating patterns in 2D space. Encompasses 17 distinct types representing various combinations of translation, rotation, reflection, and glide reflection symmetries. In numerous *wallpaper groups*, translations are the sole transformations preserving the pattern's invariance.**

Figure 1: Aurelia insulinda, exhibiting Z_4 symmetry in nature

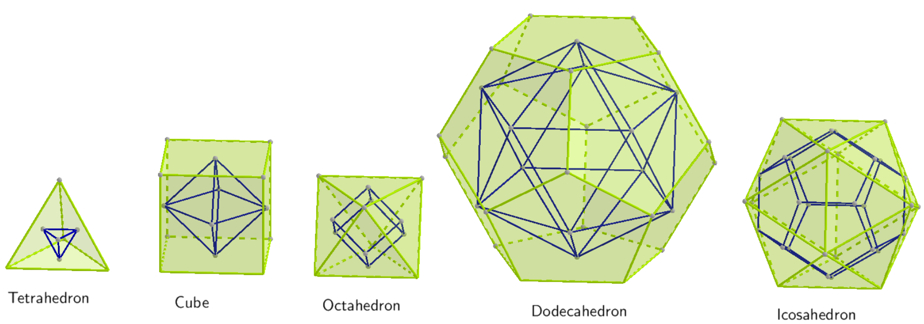

Figure 2: The platonic solids and their duals

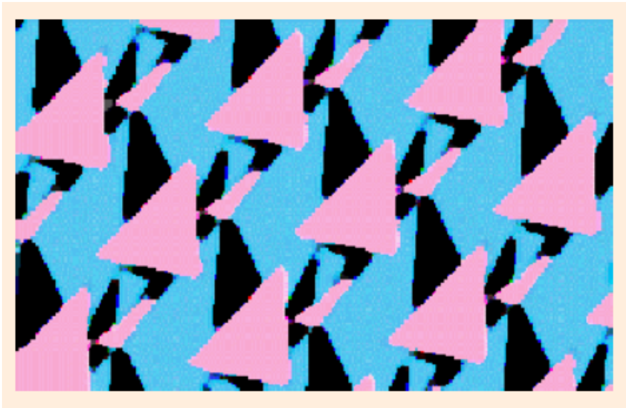

Figure 3: Symmetry group 1 (p1)

The first Symmetry group, p1, is the simplest symmetry group, comprising only translations, with no reflections, glide-reflections, or rotations. The two translation axes can be oriented at any angle. The lattice is parallelogrammatic, so a fundamental region for the symmetry group is the same as that for the translation group, namely, a parallelogram.

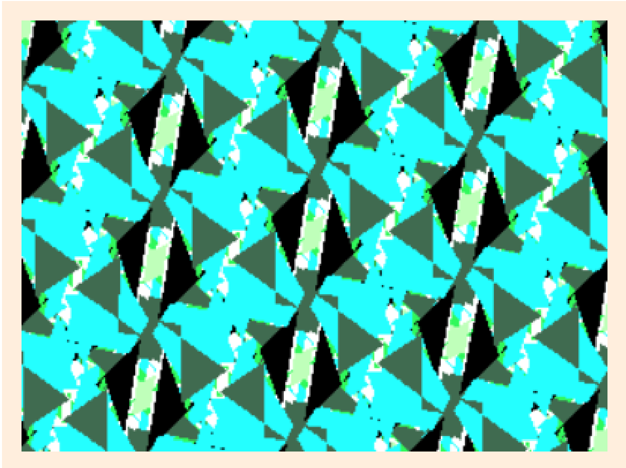

Figure 4: Symmetry group 2 (p2)

The second Symmetry group, p2, differs from the first group since it contains 180° rotations of order 2. There are translations but no reflections or glide reflections. The two translation axes have the flexibility to be inclined at various angles. The lattice maintains a parallelogrammatic structure. The symmetry group's fundamental region is half of a parallelogram, mirroring the fundamental region of the translation group.

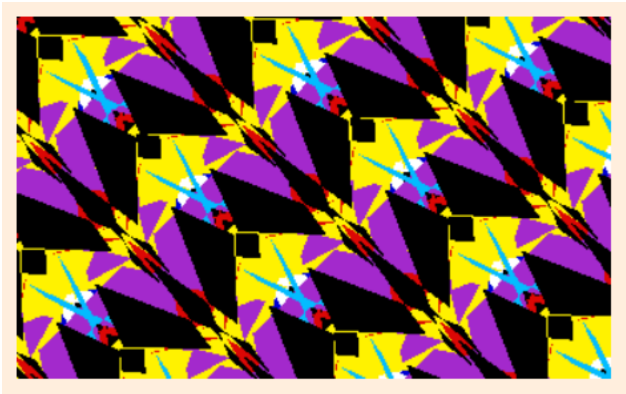

Figure 5: Symmetry group 3 (p3)

The third Symmetry group, p3, is the first group to consist of reflections and has no rotations or glide reflections. The axes of reflection are parallel to one axis of translation and perpendicular to the other axis of translation. A fundamental region is selected as a rectangle, and divided by an axis of reflection. Allowing one of the resulting half-rectangles to serve as a fundamental region for the symmetry group.

Neural Networks

Neural Networks are computational models inspired by the brain's structure, capable of learning and recognizing patterns from data by adjusting connections. Widely used in artificial intelligence, they excel at tasks like image recognition, speech processing, and complex data analysis.

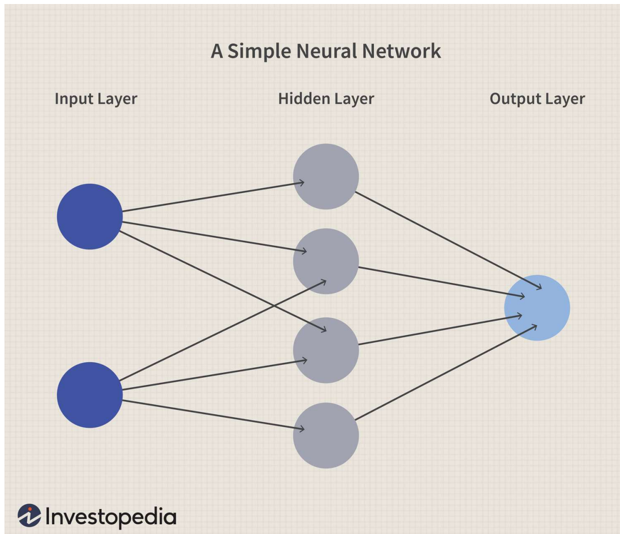

Figure 6: Simple Neural Network

The exploration of calculating symmetries within images plays a crucial role in the efficiency of *neural networks* training. Recognizing and leveraging symmetrical patterns contribute to a reduction in training data, enabling *neural networks* to generalize more effectively across diverse datasets.

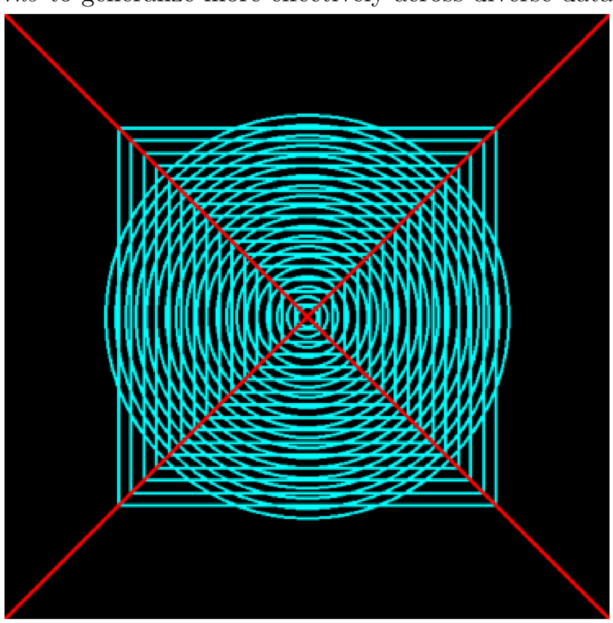

Figure 7: Symmetrical Pattern with Larger Shapes

The image is generated using the OpenCV Python library, consisting of circles, and squares to create a complex pattern and many of these can be used to train a *neural network* in computer vision applications. Using the software I was able to identify the symmetry points, indicated by red lines, similar to how a *neural network* would detect symmetry points.

Biomedical Research

Biomedical Research also has many applications for Symmetry Groups and Group Theory in general. *C. elegans* are microscopic organisms that are about 1 mm in length and commonly used in neuroscience research since they only have 302 neurons.

Figure 8: *C. elegans*
A *C. elegans* nematode. The history of C. elegans - sibelius natural products. Sibelius Natural Products. (n.d.). https://sibeliusnaturalproducts.com/the-history-of-c-elegans/

This is particularly useful when trying to figure out what mechanisms are causing certain diseases like Alzheimer's and to see what interventions improve the conditions of the nematodes. This is possible because of the discovery of Alzheimer's homologous genes such as $apl-1$ and $sel-12$.

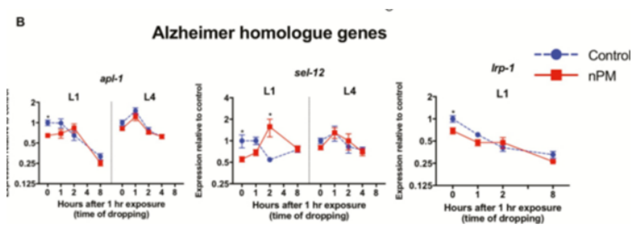

Figure 9: *Alzheimer's homologous genes*
Alzheimer's homologous genes – similar expression in humans and C. elegans. From WPI Systems Neuroscience Presentation.

Where Symmetry Groups can help is when it comes to mapping the connectome, or collection of all neurons and connections in a typical *C. elegans*.
A paper from 2019 successfully made a model that tries to find pseudosymmetry (where symmetry is described as two neurons sharing similar forward and backwards gap-junctions as an analogy to sharing a set of isomorphisms) to build circuits and simplify connectome data.

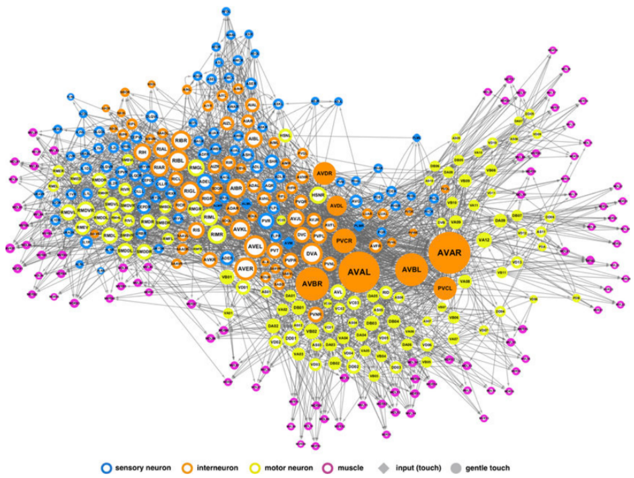

Figure 10: *The Connectome*
The Connectome. Yan, Gang Vértes, Petra Towlson, Emma Chew, Yee Lian Walker, Denise Schafer, William Barabási, Albert-László. (2017). Network control principles predict neuron function in the Caenorhabditis elegans connectome. Nature. 550. 10.1038/nature24056.

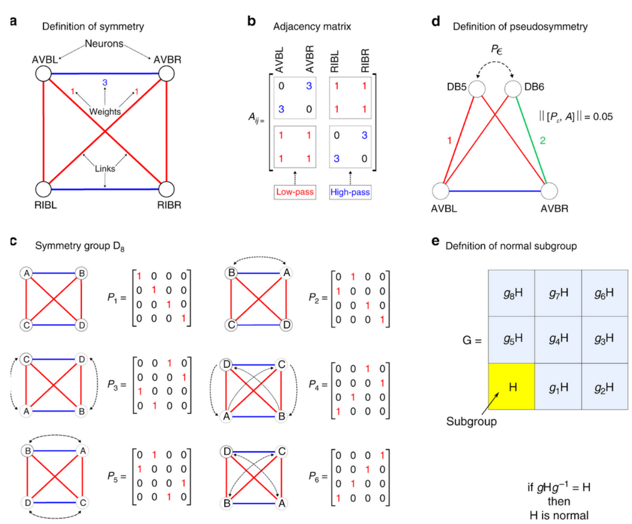

Figure 11:
General paradigm for simplifying connections. Morone, F., Makse, H.A. Symmetry group factorization reveals the structure-function relation in the neural connectome of Caenorhabditis elegans. Nat Commun 10, 4961 (2019). https://doi.org/10.1038/s41467-019-12675-8