An Introduction to Knot Polynomials

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1 Introduction

A knot is a closed curve in 3-space, they can be as simple as the unknot, shown in figure 1, or as complicated as this 10-crossing knot, shown in figure 2.

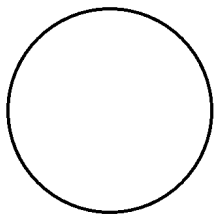


Figure 1: The Unknot

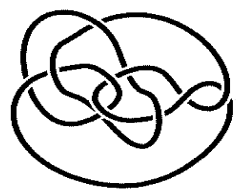


Figure 2: A 10-crossing knot Based on an image from The Knot Atlas, 10 119, `http://katlas.math.toronto.edu/wiki/10_119`

Knots may have an orientation, if a knot is oriented, this simply means that a direction is picked to go around the knot. Here is an example of an non-oriented and oriented trefoil.

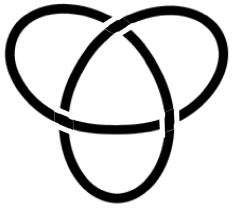


Figure 3: An Unoriented Trefoil Knot
Based on an image from Wikipedia, Trefoil knot, `https://en.wikipedia.org/wiki/Trefoil_knot`

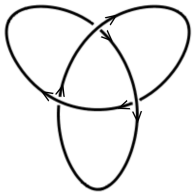


Figure 4: An Oriented Trefoil Knot
Based on an image from MathWorld, Trefoil Knot, `https://mathworld.wolfram.com/TrefoilKnot.html`

Notice the arrows on the right diagram, indicating which way to travel on the knot. However, if two knots have different orientations they are not necessarily different knots. Knots A and B are considered different if there is no way to deform knot A into knot B without cutting part of the knot. For example, consider the unknot and a trefoil knot shown below.

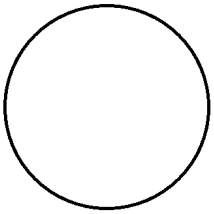


Figure 5: Knot A, an Unknot

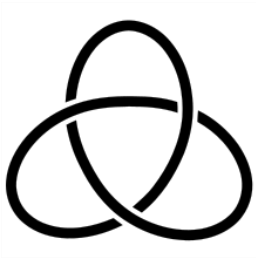


Figure 6: Knot B, a Trefoil knot
Based on an image from Wikipedia, Trefoil knot, `https://en.wikipedia.org/wiki/Trefoil_knot`

More specifically knot A and knot B are the same knot if and only if knot A can be transformed into knot B using a finite number of Reidemeister moves.

2 Reidemeister Moves

Reidemeister Moves are a way to translate between equivalent knot diagrams, that is diagrams representing the same knot. There are three

Reidemeister moves. The first is described as a twist or untwist in either direction, as shown in the diagram.

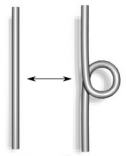


Figure 7: A type 1 Reidemeister Move
By YAMASHITA Makoto - Own work, CC BY-SA 3.0,
https://commons.wikimedia.org/w/index.php?curid=1178477

The second move is moving a strand over (or under) another.

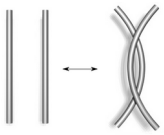


Figure 8: A type 2 Reidemeister Move
By YAMASHITA Makoto - Own work, CC BY-SA 3.0,
https://commons.wikimedia.org/w/index.php?curid=1178477

The third is moving a strand, completely above (or below) a crossing, over that crossing.

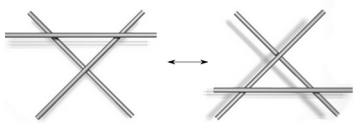


Figure 9: A type 3 Reidemeister Move
By YAMASHITA Makoto - Own work, CC BY-SA 3.0,
`https://commons.wikimedia.org/w/index.php?curid=1178477`

3 Invariants

Since a single knot can have an infinite number of diagrams it can be difficult properly label and classify knots. This is done using knot invariants, which is a quantity that will only change if the knot is different. The type of invariant we will be looking at is known as a knot polynomial, specifically the Bracket Polynomial, the Jones Polynomial, the Alexander Polynomial, and the HOMFLY polynomial. Some polynomials are not total invariants, for example the Alexander Polynomial cannot distinguish the right hand and left hand trefoil knots, shown below.

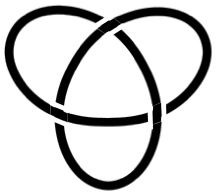


Figure 10: Left Hand Trefoil Knot

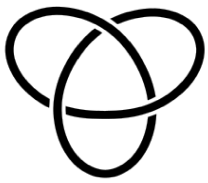


Figure 11: Right Hand Trefoil Knot

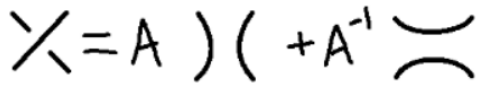
Based on an image from Wikipedia, Trefoil knot,
`https://en.wikipedia.org/wiki/Trefoil_knot`

4 Bracket Polynomial

The Kauffman Bracket Polynomial is an invariant of an unoriented knot diagram. This means different diagrams of the same knots can have different associated polynomials. The diagram may undergo smooth continuous

deformations without changing the polynomial. However, undergoing a Reidemeister move will alter the polynomial. Also note that the diagram need not be oriented, thus rotating a diagram will not affect the polynomial.

The bracket Polynomial is defined by three rules. The first states that a diagram with a positive crossing has the polynomial equivalent to the linear combination of A times the polynomial of the diagram with a vertical non crossing plus A inverse times the polynomial of the diagram with a horizontal non crossing, which is written as so:



The second rule states that the standard untwisted diagram of the unknot has an associated polynomial equal to one:

$$\langle \bigcirc \rangle = 1$$

The third rule states that the distant union of the standard unknot and some knot diagram L has a polynomial equal to:

$$\langle L \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$$

Note that the union of two knot diagrams is distant if and only if the diagrams have no shared crossings.

In order to find the Bracket Polynomial of a knot diagram we procedurally deconstruct the knot using our first rule, until we are left with an array of unknots. From our second and third rules we determine the value of each of the deconstructed diagrams then reconstruct the diagram and associated polynomial using rule 1. Several examples are illustrated below.

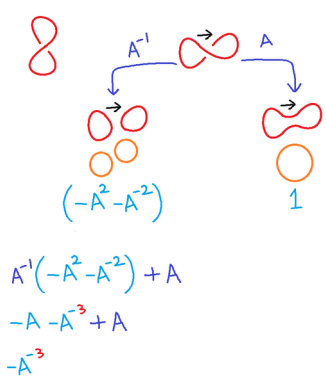


Figure 12:

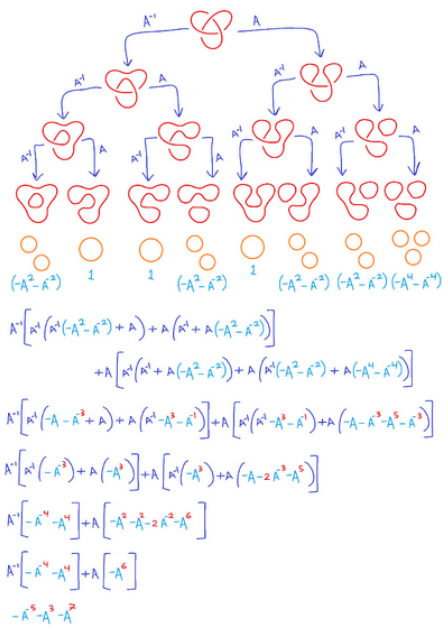


Figure 13:

5 Jones Polynomial

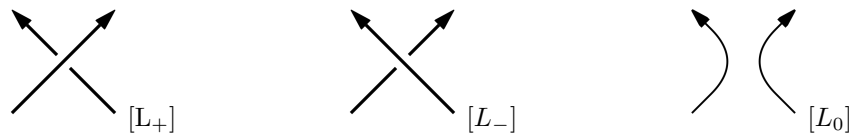
The Jones Polynomial was discovered in 1984 by Vaughn Jones. The Jones Polynomial is a knot invariant that can be considered a specific case of the HOMFLY polynomial. While it is true that if the Jones polynomial of two knots are different then they represent different knots, the Jones Polynomial isn't complete. This means that is possible for two distinct knots to share the same Jones Polynomial.

The Jones Polynomial is defined by the following skein relationship:

$$\mathrm{t}^{-1}V(L_+) - tV(L_-) + (t^{1/2} - t^{-1/2})V(L_0) = 0$$

Where V(O) = 1, and O is simply the unknot.

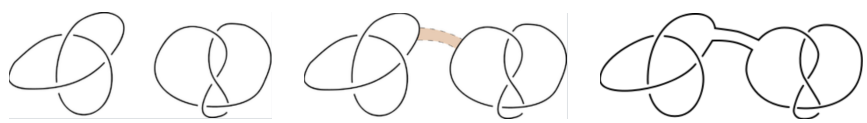
The coefficient's of the varying powers of t refer to the Jones Polynomial of some intersection along a knot which take three possible states:



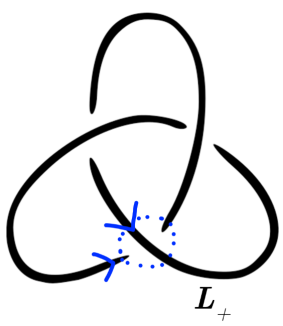
Images from Wikapedia Commons, https://bit.ly/3Lvw36r, https://bit.ly/3EWe9Ye
One interesting property of the Jones Polynomial is that it is multiplicative:

$$\mathrm{V}(\mathrm{K}_1 \# K_2) = V(K_1) * V(K_2)$$

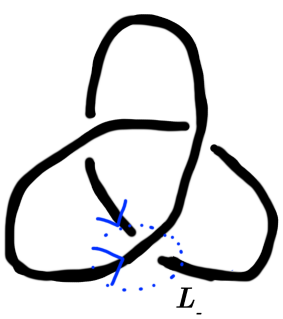
Where # denotes the connected sum operator which in the context of knots can be thought of as bridging two strands of a knot together as shown below:



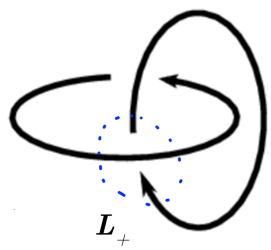
Images from Wikapedia, Connected Sum, https://bit.ly/38yF8gq
To give further insight into how the Jones Polynomial is calculated and applied to knots we will calculate the Jones Polynomial of a Right Trefoil Knot



(1) To start the calculation, we need to identify an intersection, find which type of the three crossings it is and then see what happens when we change the crossing to both of the other two types of crossings, giving us two cases to consider.

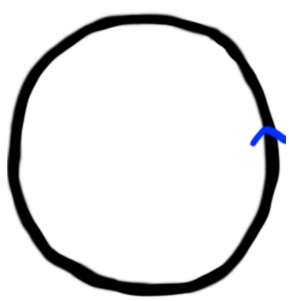


(2) Changing the original L_+ crossing into a L_- crossing we get this knot which is equivalent to the unknot, which is easy to verify if by replicating this knot diagram with a piece of string. Since we know $V(O) = 1$ by definition we only need to find the Jones Polynomial of what happens when we make the V_+ a V_0 before we use the skein relationship to calculate the Jones Polynomial of our trefoil.

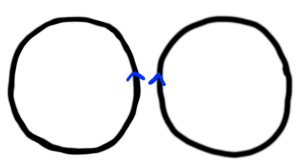


Research Gate, Thang Le, https://bit.ly/3vMbITQ

(3) Changing the original L_+ of the trefoil crossing into a L_0 crossing we get this knot, known as the hopf link. Since we don't know the Jones polynomial of this link, we must once again pick an intersection and change it to the two different types of crossings, meaning we take the L_+ crossing and make it into a L_- and L_0 crossing.



(4) Changing the original L_+ crossing from the hopf link into a L_0 crossing yields unknot which, again, we know the Jones polynomial of. This means we only have to consider what happens when we change the L_+ crossing of the hopf link into an L_- crossing now.



(5) Changing the original L_+ crossing from the hopf link into a L_- yields this "double unknot" which, while it contains zero intersections, does still have a L_0 crossing which we need to resolve, however making this L_0 crossing of the double unknot into either an L_+ or L_- crossing will result in an unknot.

This means we have finally taken the trefoil knot and traced out how we can change crossing to reduce it to unknots which we know the Jones Polynomial of. Now to calculate the value of the trefoil knot we work our way back up and apply the skein relationship.

Calculating the value of the L_0 crossing in the "double unknot" via the skein relationship:

$$\mathrm{t}^{-1}V(L_+) - tV(L_-) + (t^{1/2} + t^{-1/2})V(L_0) = 0$$

$$t^{-1}(1) - tV(1) - (t^{1/2} - t^{-1/2})\text{V("Double Unknot")} = 0$$

$$\text{V("Double Unknot")} = \frac{t-t^{-1}}{t^{1/2}-t^{-1/2}}$$

$$\text{V("Double Unknot")} = -(t^{-1/2} + t^{1/2})$$

Now that we have solved the Jones Polynomial of (5) and since the Jones Polynomial of (4) is the Jones Polynomial of the unknot which is known, we can solve for the Jones Polynomial of the hopf link

$$t^{-1}*\text{V(Hopf Link)}+t*(t^{-1/2} + t^{1/2}) - (t^{1/2} - t^{-1/2})(1) = 0$$

$$\text{V(Hopf Link)} = \frac{-t^{3/2}-t^{-1/2}}{t^{-1}}$$

$$\text{V(Hopf Link)} = -t^{5/2} - t^{1/2}$$

Now that we have found the Jones Polynomial of (3), the Hopf Link, and since we know the Joens Polynomial of (2), the unknot, we can finally calculate the Jones Polynomial of the Trefoil

$$t^{-1}\text{V(Trefoil)-tV(Unknot)}+(t^{1/2} + t^{-1/2})\text{V(Hopf Link)} = 0$$

$$t^{-1}\text{V(Trefoil)-t(1)}+(t^{1/2} + t^{-1/2}) * (-t^{5/2} - t^{1/2}) = 0$$

$$\text{V(Trefoil)} = t^2 + (t^{1/2} + t^{-1/2}) * (-t^{5/2} - t^{1/2})t$$

$$\text{V(Trefoil)} = t^2 + (t^2 + 1 - t^3 - t)t$$

$$\text{V(Trefoil)} = -t^4 + t^3 + t$$

After working through the trefoil example, we can see that the process of finding a Jones Polynomial of a knot involves decomposing the knot into unknots through changing crossings and then using the skein relationship to work your way back up to the knot in question.

6 Alexander Polynomial

The first step to calculating the Alexander Polynomial is to number the crossings and regions, a demonstration of this for the trefoil knot is given in figure 14.

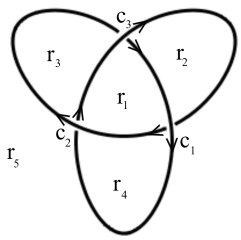


Figure 14:
Based on an image from MathWorld, Trefoil Knot,
`https://mathworld.wolfram.com/TrefoilKnot.html`

Notice that the diagram is oriented with arrows after each crossing. Notice also that there are exactly two more regions than crossings, which is always the case based on Euler's theorem on polyhedra. Now to find the Alexander Polynomial, create a Matrix, call it M_0, the rows of M_0 correspond to the crossings, and the columns of M_0 correspond to the regions. If the region is on the left of the under-crossing line before the crossing, the entry for that crossing-region pair in M_0 is a -t, if the region is on the right of the under-crossing line before the crossing, the crossing-region pair in M_0 is a 1. If the region is on the left of the under-crossing line after the crossing, the entry for that crossing-region pair in M_0 is a t, if the region is on the right of the under-crossing line after the crossing, the crossing-region pair in M_0 is a -1. If the region and crossing are not adjacent the entry is 0. This gives the following matrix for the knot in figure 14.

$$M_0 = \begin{bmatrix} -1 & 1 & 0 & t & -t \\ -1 & 0 & t & 1 & -t \\ -1 & t & 1 & 0 & -t \end{bmatrix}$$

Then by ignoring two consecutive columns, we create a square matrix of which we wish to find the determinant. We find the determinant of the matrix M_{12} to be

$$|M_{12}| = \begin{vmatrix} 0 & t & -t \\ t & 1 & -t \\ 1 & 0 & -t \end{vmatrix} = t^3 - t^2 + t = t^2(t^1 - 1 + t^{-1})$$

By ignoring another two consecutive columns, we get M_{45}, notice that its determinant differs from the $|M_{12}|$ by a factor of $\pm x^n$. The fact that this is true for any two consecutive columns was proven by Alexander in 1928.

$$|M_{45}| = \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & t \\ -1 & t & 1 \end{vmatrix} = t^2 - t + 1 = t(t^1 - 1 + t^{-1})$$

Now we have that the Alexander Polynomial for the trefoil knot is $t^1 - 1 + t^{-1}$. Next, we will show that the Alexander Polynomial is unchanged under type 1 and type 2 Reidemeister moves. Figure 15 shows the trefoil under a type 1 Reidemeister move, and $R1_0$ shows it's matrix.

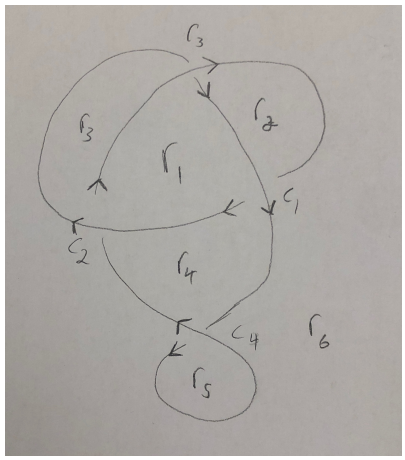


Figure 15: A Trefoil After type 1 Reidemeister move

$$R1_0 = \begin{bmatrix} -1 & 1 & 0 & t & 0 & -t \\ -1 & 0 & t & 1 & 0 & -t \\ -1 & t & 1 & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & t & -1-t \end{bmatrix}$$

$|R1_{12}|$ and $|R1_{45}|$ are then found to be:

$$|R1_{12}| = \begin{vmatrix} 0 & t & 0 & -t \\ t & 1 & 0 & -t \\ 1 & 0 & 0 & -t \\ 0 & 1 & t & -1-t \end{vmatrix} = -t^4 + t^3 - t^2 = -t^3(t^1 - 1 + t^{-1})$$

$$|R1_{45}| = \begin{vmatrix} -1 & 1 & 0 & t \\ -1 & 0 & t & 1 \\ -1 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = t^2 - t + 1 = t(t^1 - 1 + t^{-1})$$

This shows that the Alexander Polynomial is unchanged under type 1 Reidemeister moves. Figure 16 shows the trefoil under a type 2 Reidemeister move, and $R2_0$ shows it's matrix.

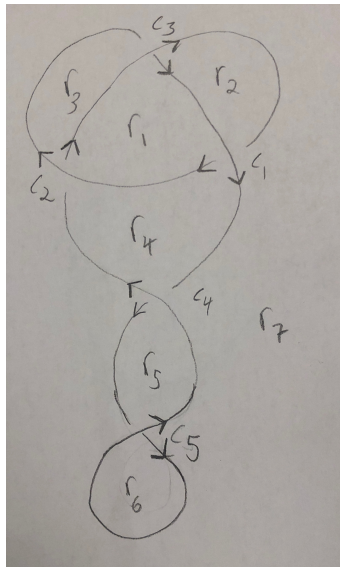


Figure 16: A Trefoil After type 2 Reidemeister move

$$R2_0 = \begin{bmatrix} -1 & 1 & 0 & t & 0 & 0 & -t \\ -1 & 0 & t & 1 & 0 & 0 & -t \\ -1 & t & 1 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & t & 0 & -1-t \\ 0 & 0 & 0 & 0 & -t & -1 & 1+t \end{bmatrix}$$

$|R1_{12}|$ and $|R1_{45}|$ are then found to be

$$|R1_{12}| = \begin{vmatrix} 0 & t & 0 & 0 & -t \\ t & 1 & 0 & 0 & -t \\ 1 & 0 & 0 & 0 & -t \\ 0 & 1 & t & 0 & -1-t \\ 0 & 0 & -t & -1 & 1+t \end{vmatrix} = -t^4 + t^3 - t^2 = -t^3(t^1 - 1 + t^{-1})$$

$$|R1_{45}| = \begin{vmatrix} -1 & 1 & 0 & t & 0 \\ -1 & 0 & t & 1 & 0 \\ -1 & t & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & -t \end{vmatrix} = t^2 - t + 1 = t(t^1 - 1 + t^{-1})$$

This shows that the Alexander Polynomial is unchanged under type 2 Reidemeister moves.

7 HOMFLY Polynomial

The HOMFLY Polynomial is a 2-variable oriented Knot Polynomial $H_K(z, a)$ influenced by the Jones Polynomial. The HOMFLY Polynomial was discovered about four months after the Jones polynomial was discovered, by Vaughn Jones. The discovery of the HOMFLY was a result of attempts to generalize the Alexander and Jones polynomial. The name HOMFLY is an acronym from all the names of its co-discoverers: Hoste, Ocneau, Millett, Freyd, Lickorish and Yetter. We denote the HOMFLY polynomial of a link K to be P_K.

$P_K(z, \alpha) = \alpha^{-w(K)} H_K(z, \alpha)$

Where w(k) denotes the writhe of a polynomial and is the total number of positive crossing minus the total number of negative crossings. Each crossing is assigned either positive or negative based on the orientation of the knot diagram. A crossing is positive when a counterclockwise rotation of the overcrossing line aligns it with the undercrossing line.

To Calculate the HOMFLY Polynomial, a skein relationship is used and the following rules are used:

1. $H_0 = 1$
2. $H_{K^+} - H_{K^-} = zH_{K^0}$
3. $H_{K^+} = \alpha H_K$ and $H_{K^-} = \alpha^{-1} H_K$

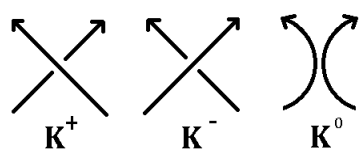


Figure 17: the Skein Relationship for the HOMFLY polynomial

Below is a table of some link diagrams and their HOMFLY polynomials.

K	H_K	P_K
	1	α
	α^{-1}	1
	$1 - z\alpha^{-1}$	$\alpha^2 - \alpha z$
	1	1
	$2z - z^2\alpha^{-1} - \alpha$	$-\alpha^4 + 2\alpha^3 z - \alpha^2 z^2$

Figure 18: HOMFLY polynomials of Small Links

Below is the computation of some link diagrams and their HOMFLY polynomials.

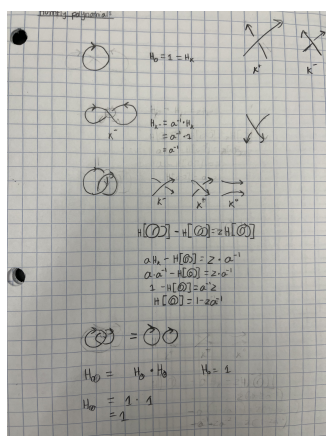


Figure 19: HOMFLY polynomials of Small Links

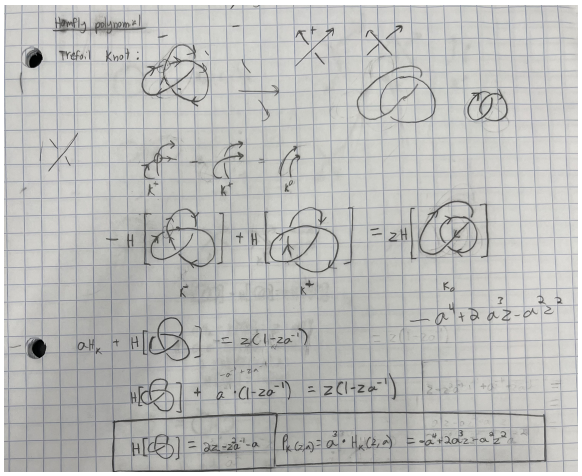


Figure 20: HOMFLY polynomial of a trefoil knot

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