Nets of a Cube

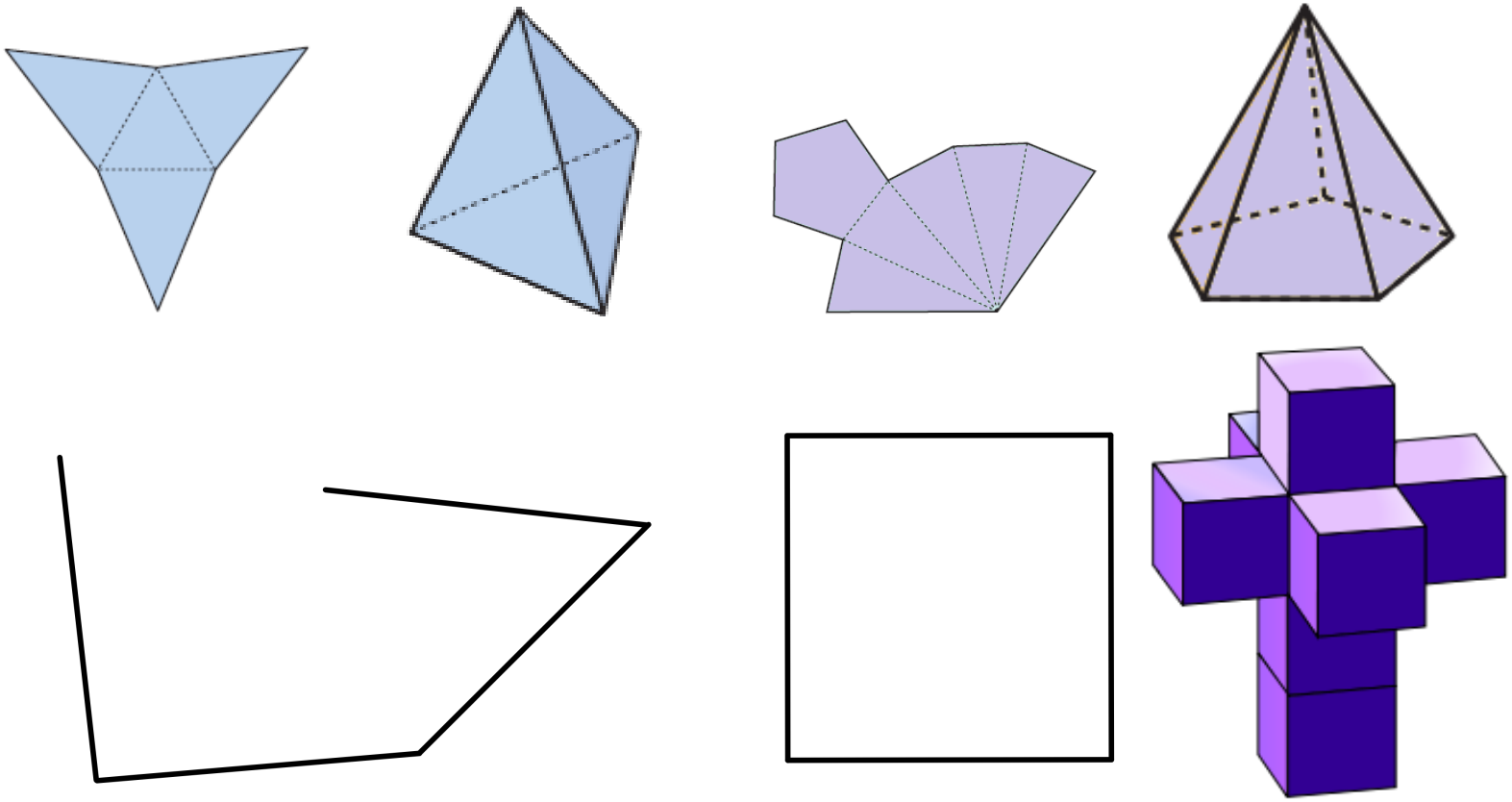
Ethan Hair

Oklahoma Baptist University

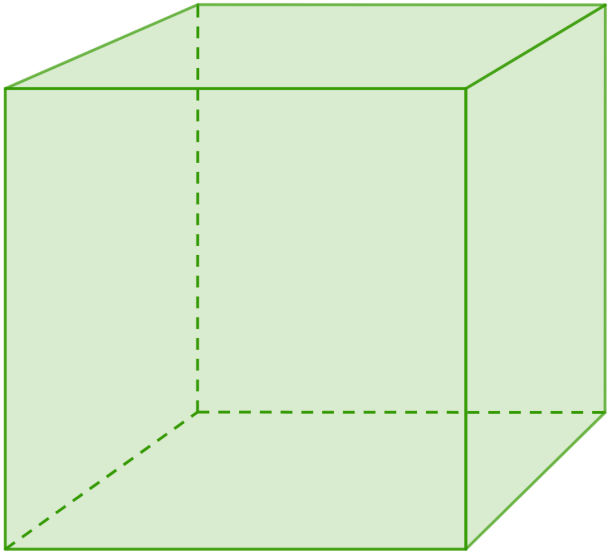
May 18, 2022

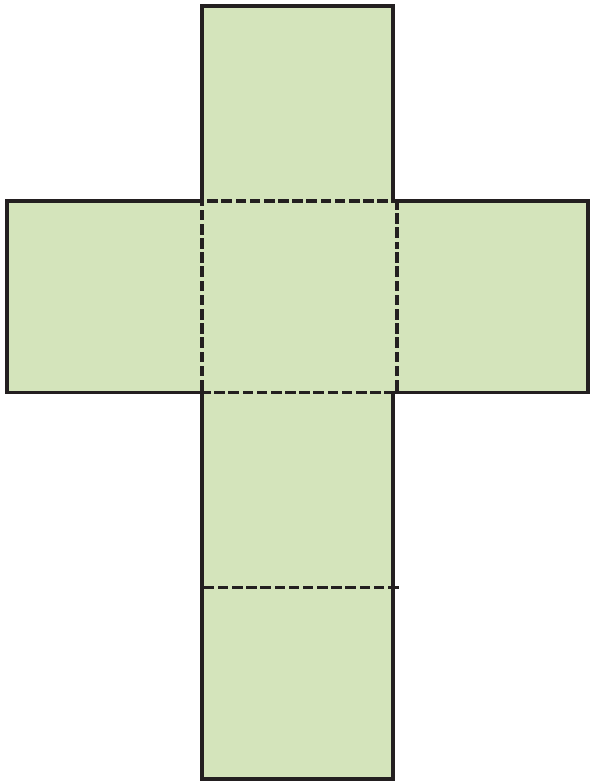
What is a net?

A net is an $n-1$ dimensional representation of an n dimensional shape that can be folded in the nth dimension and glued to become the shape. Some examples:

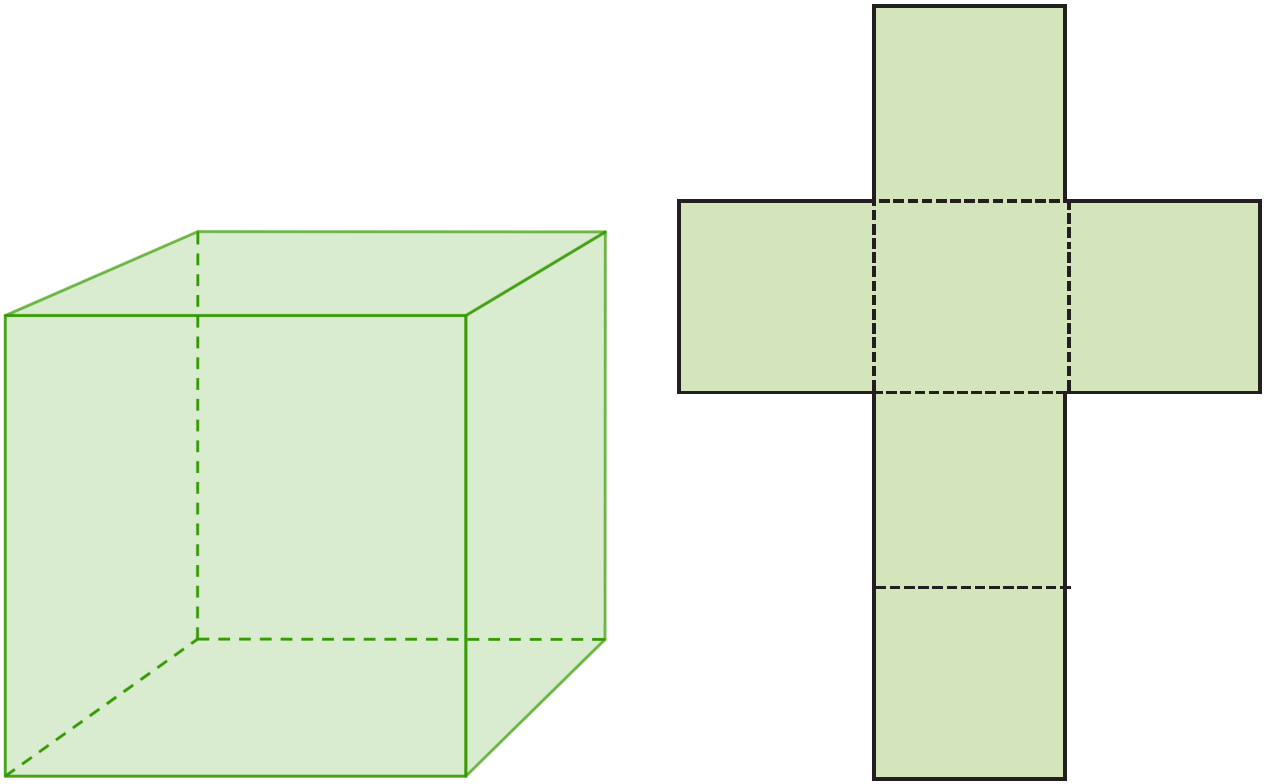


What about the Cube?





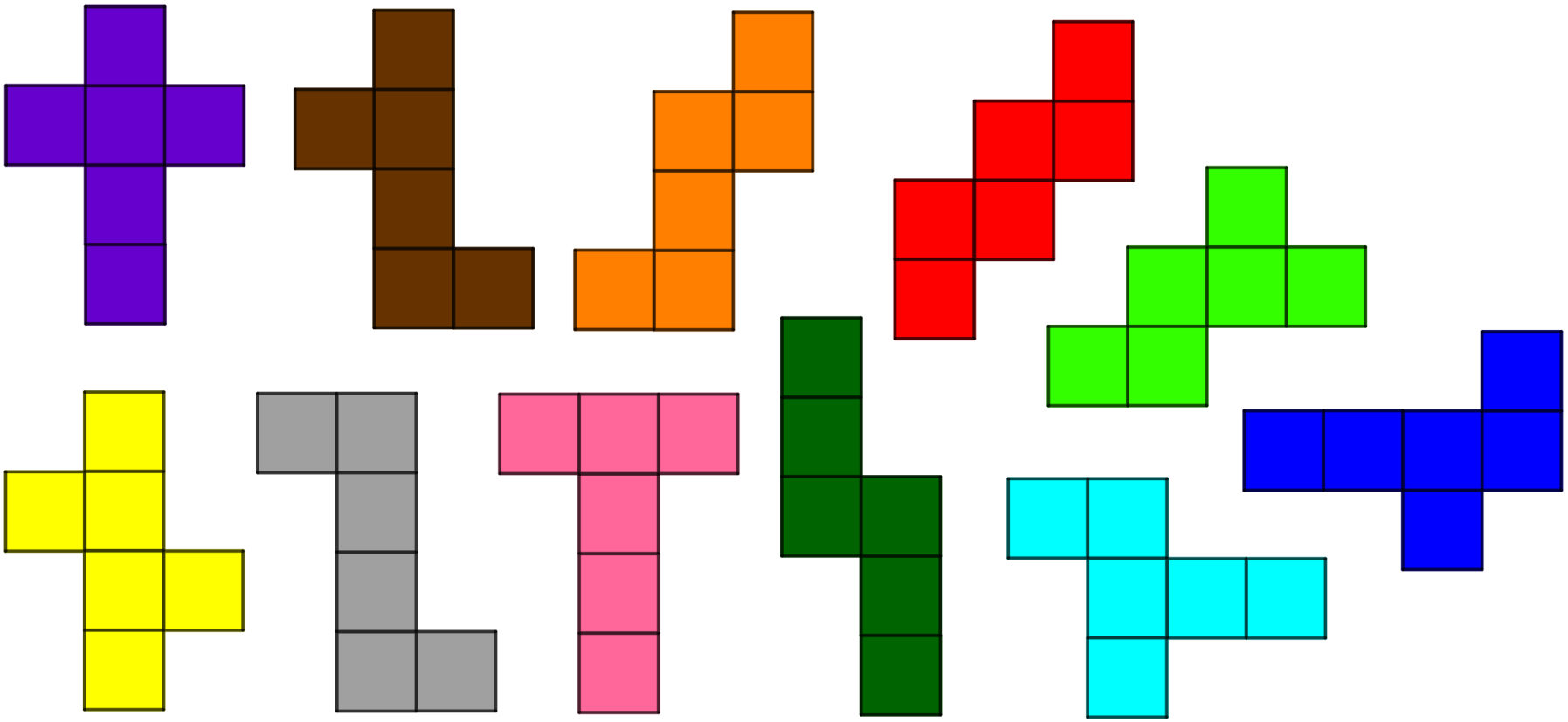
What about the Cube?



What if we were to move the squares around? Would it still be a net? And if so, how many nets are there?

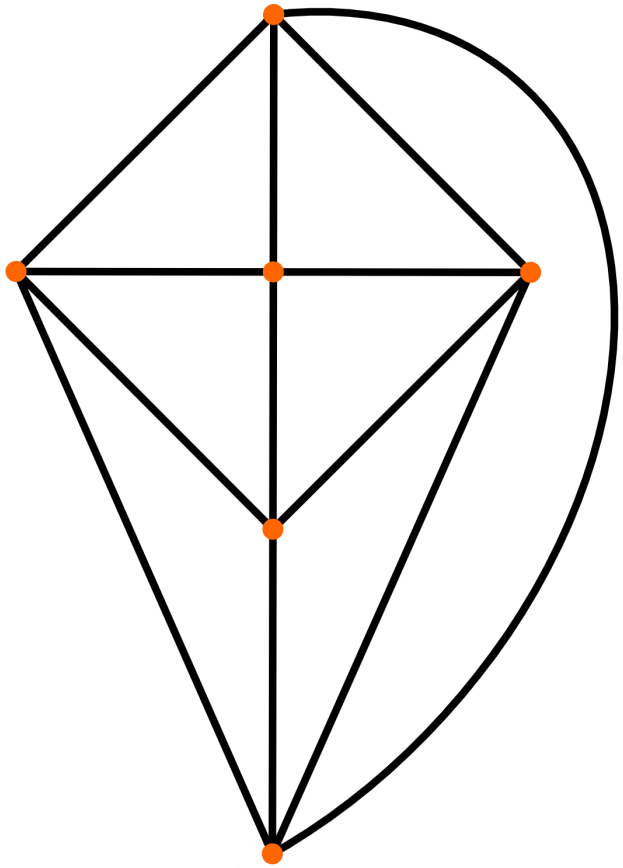
What about the Cube?

These are the 11 different nets of a cube



Why 11?

To discover how many nets of a shape there are we have to look at a graph that represents the shape. We will let the faces be vertices and if the faces are adjacent on the shape, there is an edge between the corresponding vertices.



Trees

We will now consider what are called the spanning trees of a graph.

Spanning Subgraph

A subgraph of a graph G is a graph having all of its vertices and edges in G. A spanning subgraph is a subgraph that contains all of the vertices of G.

Tree

A graph is called acyclic if it has no cycles. A tree is a connected acyclic graph.

Thus a spanning tree of a graph is a spanning subgraph that is a tree.

Trees

We will now consider what are called the spanning trees of a graph.

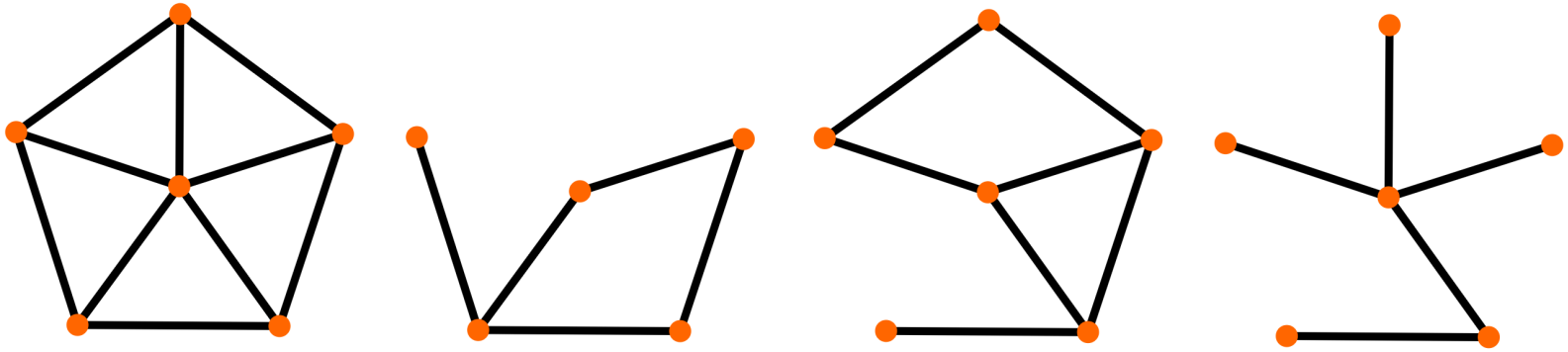
Spanning Subgraph

A subgraph of a graph G is a graph having all of its vertices and edges in G. A spanning subgraph is a subgraph that contains all of the vertices of G.

Tree

A graph is called acyclic if it has no cycles. A tree is a connected acyclic graph.

Thus a spanning tree of a graph is a spanning subgraph that is a tree.



Laplacian Matrix and Kirchhoff's Theorem

Laplacian Matrix

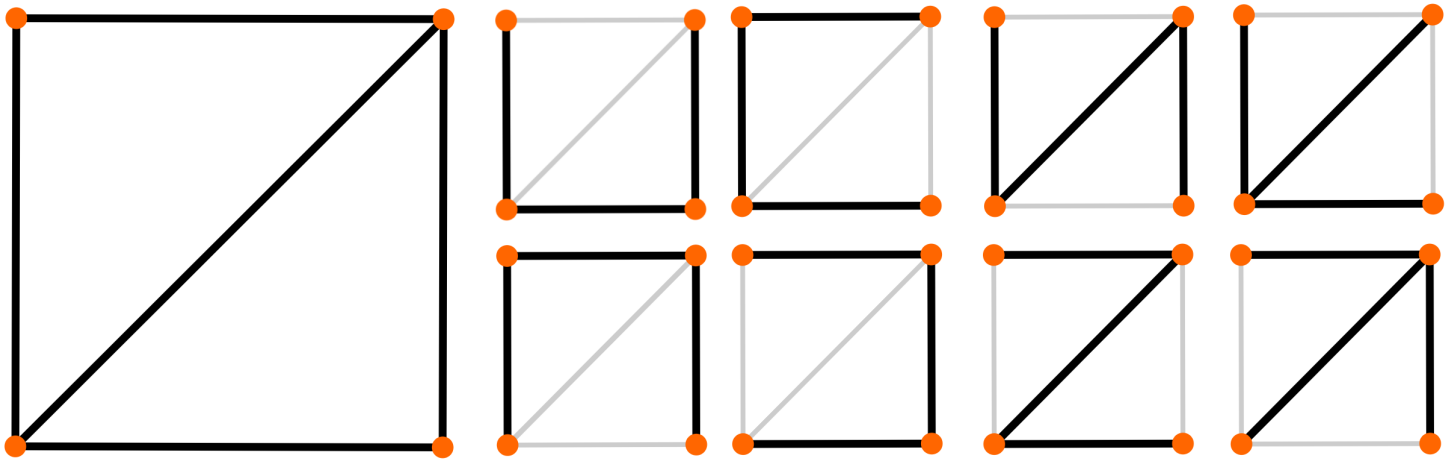
The Laplacian matrix of a graph is defined to be the square matrix $L = D - A$ where D is the diagonal matrix of the degrees of the vertices and A is the adjacency matrix where the ijth term is 1 if the vertices corresponding to i and j are adjacent and 0 if they are not.

Kirchhoff's Theorem (or Matrix-Tree Theorem)

All of the cofactors of the Laplacian matrix of a graph are equal and their common value is the number of spanning trees of the graph.

Laplacian Matrix and Kirchhoff's Theorem

Example:

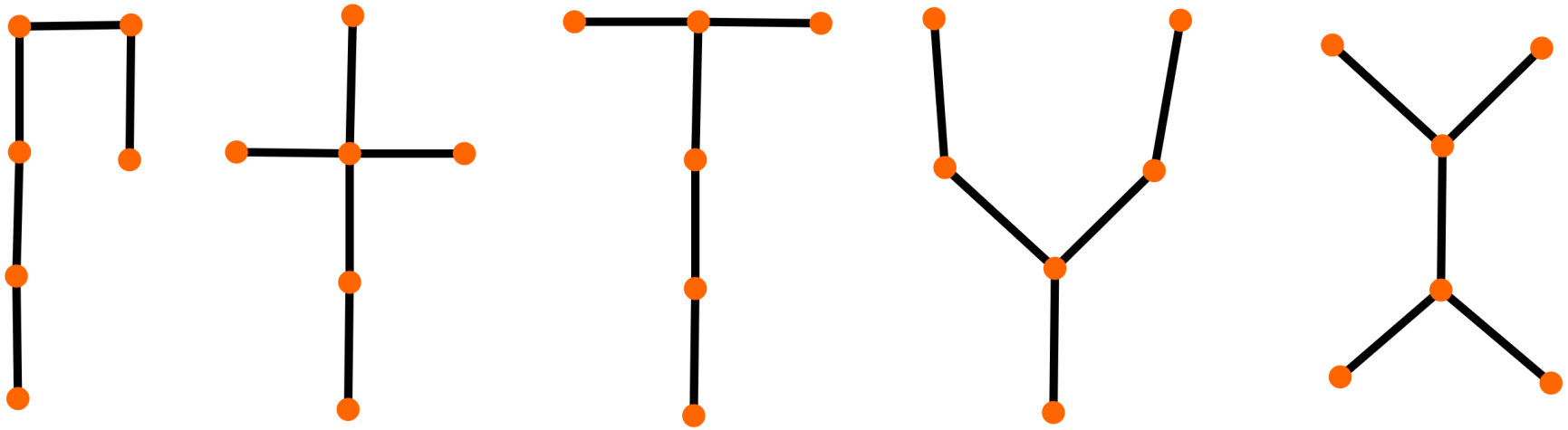


$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Cofactor of $L_{11} = \det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 8 \to 2$ different graphs

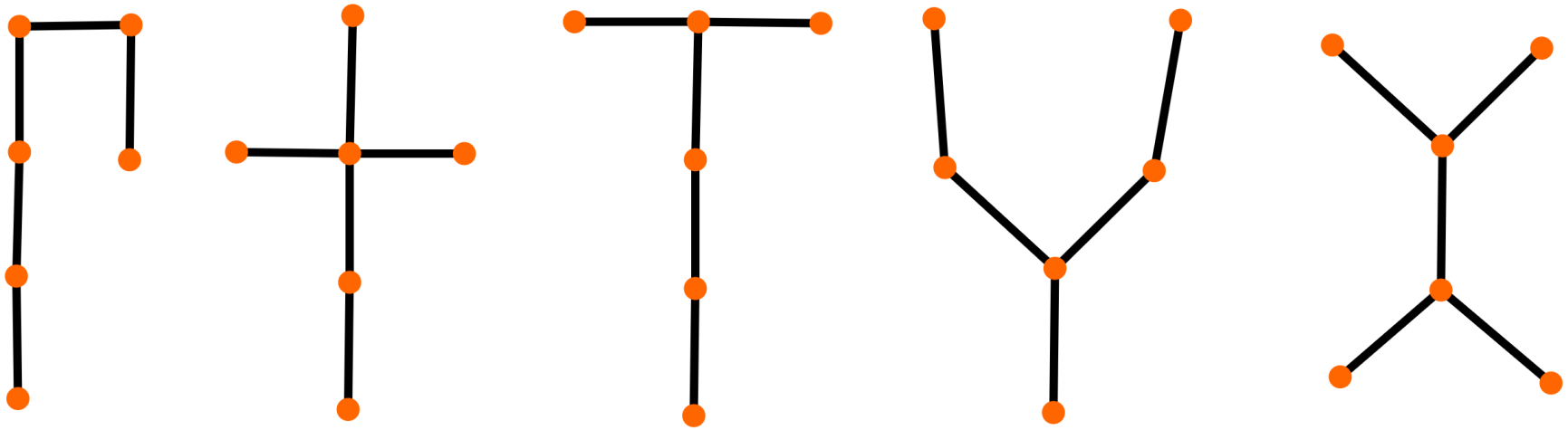
Spanning Trees of Cube Graph

Applying this to the graph that represents the cube we get that there are 384 spanning trees which can be reduced to the 5 different graphs below.



Spanning Trees of Cube Graph

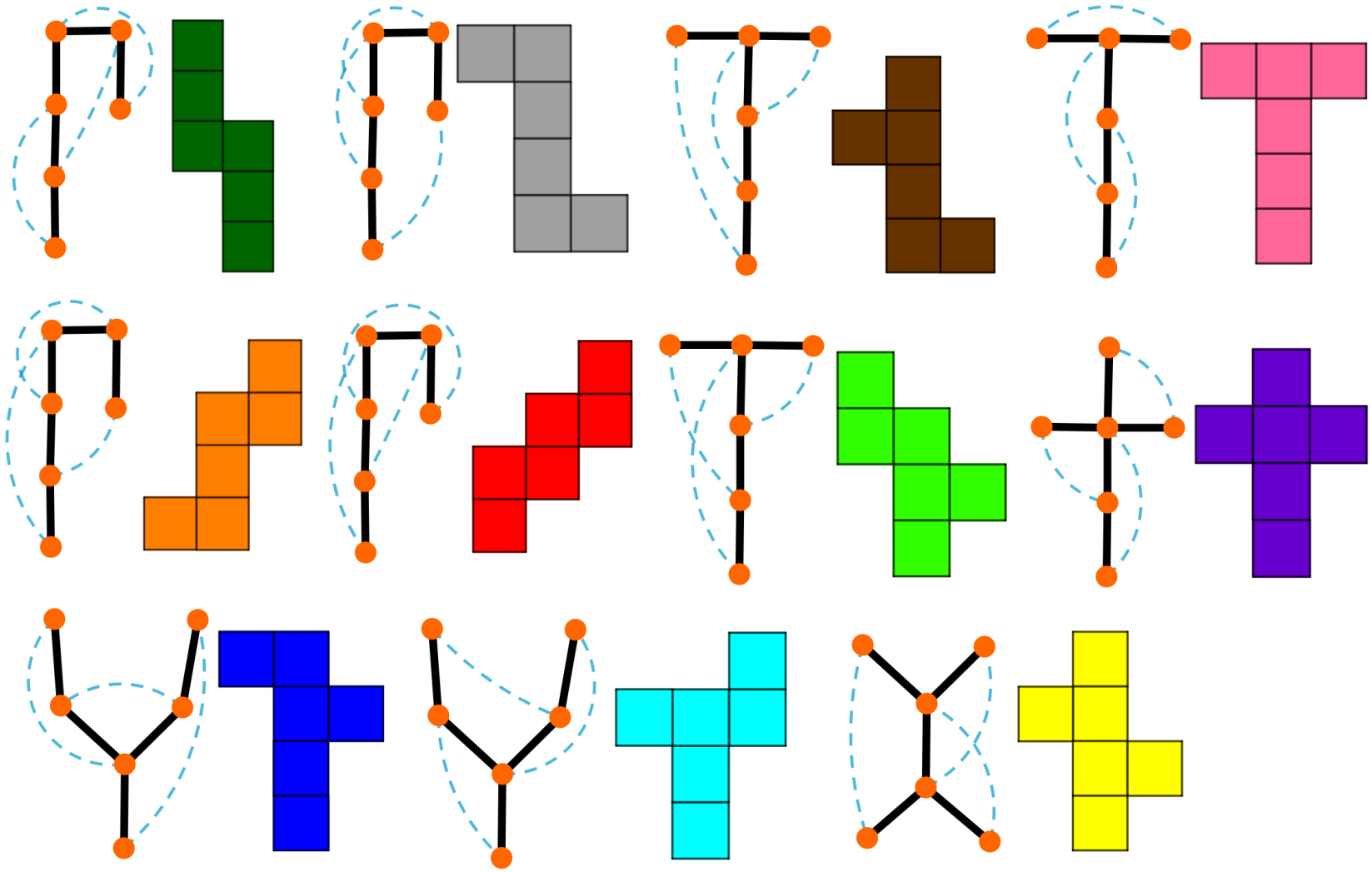
Applying this to the graph that represents the cube we get that there are 384 spanning trees which can be reduced to the 5 different graphs below.



These trees are not quite enough to get to the 11 nets of the cube. One more step is required. The vertices need to be paired up in such a way that there is at least one vertex is between them. These will be the opposite faces of the cube.

Spanning Trees of Cube Graph

There are 11 ways to pair up vertices on these graphs.



3D nets of the Hypercube

If we apply this method to the 4-D hypercube, there are 261 different nets of the hypercube.

3D nets of the Hypercube

If we apply this method to the 4-D hypercube, there are 261 different nets of the hypercube.

References I

[1] Frank Harary. *Graph Theory.* Addison-Wesley, 1972.

[2] Matt Parker (Stand-up Maths). How many 3D nets does the 4D hypercube have?. https://youtu.be/Yq3P-LhlcQo [Online: Accessed May 18, 2022].

[3] Laplacian Matrix. https://mathworld.wolfram.com/LaplacianMatrix.html. [Online: Accessed May 18, 2022].

[4] Kirchhoff's Theorem. https://de.zxc.wiki/wiki/Satz_von_Kirchhoff. [Online: Accessed May 18, 2022].

[5] Cofactor. https://byjus.com/maths/cofactor/. [Online: Accessed May 18, 2022].

References II

[6] All the Nets of a Cube. https://www.mechamath.com/geometry/nets-of-a-cube/. [Online: Accessed May 18, 2022].

[7] Nets of Solid Shapes. https://www.onlinemath4all.com/nets-of-solid-shapes.html. [Online: Accessed May 18, 2022].