

ECM1416 Forumlas

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1 Linear Algeobra

1.1 Vectors

1.1.1 Vector Norm

$$||v|| = \sqrt{\sum v_i^2} \quad (1)$$

1.1.2 Inner Product

$$u \cdot v = |u||v| \cos(\theta) \quad (2)$$

$$u \cdot v = u^T \cdot v = \sum_{i=1}^n u_i v_i \quad (3)$$

1.1.3 Rotating Vector

$$v' = \begin{bmatrix} v_x \cos(\theta) - v_y \sin(\theta) \\ v_x \sin(\theta) + v_y \cos(\theta) \end{bmatrix} \quad (4)$$

1.2 Matrices

1.2.1 Matrix Product ($A = (m \times n), B = (n \times p)$)

$$A \cdot B = \begin{bmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \cdots & \sum_{i=1}^n a_{1i}b_{ip} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{ni}b_{i1} & \cdots & \sum_{i=1}^n a_{ni}b_{ip} \end{bmatrix} \quad (5)$$

1.2.2 Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (6)$$

1.2.3 Scale Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (7)$$

1.2.4 Shear Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (8)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ \mu x + y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (9)$$

1.2.5 Transpose Properties

$$(A + B)^T = A^T + B^T \quad (10)$$

$$(AB)^T = A^T B^T \quad (11)$$

1.2.6 Identity Matrix

$$v = I_n v, \forall v \in \mathbb{R}^n \quad (12)$$

1.2.7 Determinant of a 2×2 Matrix

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc \quad (13)$$

1.2.8 Determinant of a $n \times n$ Matrix

$$\det(A) = \sum_{k=1}^n (-1)^{i+k} a_{ik} M_{ik} = \sum_{k=1}^n (-1)^{j+k} a_{kj} M_{kj} \quad (14)$$

1.2.9 Inverse of a 2×2 Matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (15)$$

1.2.10 Inverse of a $n \times n$ Matrix

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{21} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \quad (16)$$
$$C_{ij} = (-1)^{i+j} M_{ij}$$

1.2.11 Properties of an Inverse

$$(A^{-1})^{-1} = A \quad (17)$$

$$(A^T)^{-1} = (A^{-1})^T \quad (18)$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad (19)$$

$$(kA)^{-1} = k^{-1}A^{-1} \quad (20)$$

1.2.12 Solving a System of Linear Equations

$$Ax = b \quad (21)$$

$$x = A^{-1}b \quad (22)$$

2 Differentiation

2.1 Matrix Calculus

2.1.1 Gradient

$$\nabla f = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{bmatrix} \quad (23)$$

2.1.2 Hessian

$$H_f = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_mx_1} & f_{x_mx_2} & \cdots & f_{x_mx_n} \end{bmatrix} \quad (24)$$

2.1.3 Jacobian

$$D_f = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_n \end{bmatrix} = \begin{bmatrix} f_{1x_1} & f_{1x_2} & \cdots & f_{1x_n} \\ f_{2x_1} & f_{2x_2} & \cdots & f_{2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{nx_1} & f_{nx_2} & \cdots & f_{nx_n} \end{bmatrix} \quad (25)$$

2.2 Taylor Expantion

$$f(x) \approx \sum_{i=0}^n \frac{f^i(a)(x-a)^i}{i!} \quad (26)$$

2.3 Maclorian Expantion (a = 0)

2.3.1 $\sin(x)$

$$\sin(x) \approx \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad (27)$$

2.3.2 $\cos(x)$

$$\cos(x) \approx \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} \quad (28)$$

2.3.3 e^x

$$e^x \approx \sum_{k=0}^n \frac{x^k}{k!} \quad (29)$$

2.3.4 $\ln|1+x|$

$$\ln|1+x| \approx \sum_{k=0}^n (-1)^k \frac{x^{k+1}}{k+1} \quad (30)$$

2.3.5 $c(\frac{1}{1-x})$

$$c(\frac{1}{1-x}) \approx c \sum_{k=0}^n x^k \quad (31)$$

2.4 Solving ODEs

2.4.1 Intergrating Factor

$$\begin{aligned}\frac{dy}{dx} + P(x)y &= Q(x) \\ \text{I.F} &= e^{\int P(x)dx} \\ \frac{d}{dx}(e^{\int P(x)dx}y) &= Q(x)e^{\int P(x)dx} \\ y &= \frac{\int Q(x)e^{\int P(x)dx}dx}{e^{\int P(x)dx}}\end{aligned}\tag{32}$$

2.4.2 Seperation of Variables

$$\begin{aligned}\frac{dy}{dx} &= g(x)h(y) \\ \frac{dy}{dx} \frac{1}{h(y)} &= g(x) \\ \int \frac{1}{h(y)}dy &= \int g(x)dx\end{aligned}\tag{33}$$

2.4.3 Eulers Method

$$\begin{aligned}y_{n+1} &= y_n + hf(x_n, y_n) \\ x_{n+1} &= x_n + h\end{aligned}\tag{34}$$

3 Probability

3.1 Conditional Probability

3.1.1 Given That

$$P(A|B) = \frac{P(A \cap B)}{P(B)}\tag{35}$$

3.1.2 Independence

$$P(A \cap B) = P(A)P(B)\tag{36}$$

3.1.3 Law of Total Probability

$$P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n) = \sum_n P(B_n|A)P(A)\tag{37}$$

3.1.4 Bayes Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (38)$$

$$P(A|X) = \frac{P(X|A)P(A)}{P(X)} = \frac{P(X|A)P(A)}{P(X|A)P(A) + P(X|A^C)P(A^C)} \quad (39)$$

3.2 Random Variables

3.2.1 Discrete Random Variables

$$\begin{aligned} P(x_i) &= P(X = x_i) \\ P(X = x_i) &\geq 0, \forall i \\ \sum P(X = x_i) &= 1 \end{aligned} \quad (40)$$

3.2.2 Continuous Random Variables

$$\begin{aligned} P(a \leq X \leq b) &= \int_a^b f(x)dx \\ f(x) &\geq 0, \forall x \in X \\ \int_{\Omega} f(x)dx &= 1 \\ f(x) &= 0, x \notin X \end{aligned} \quad (41)$$

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= P(a \leq X < b) \\ &= P(a < X < b) \end{aligned} \quad (42)$$

3.3 Cumulative Distribution Function

3.3.1 General Case

$$F(x) = P(X \leq x) \quad (43)$$

3.3.2 X is Discrete

$$F(x) = P(X \leq x) = \sum_{\forall x_i \leq x} P(x_i) \quad (44)$$

3.3.3 X is Continuous

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt \quad (45)$$

$$P(a \leq X \leq b) = F(b) - F(a) \quad (46)$$

3.3.4 Properties of CDF

$$0 \leq F(x) \leq 1 \quad (47)$$

$$x < y \rightarrow F(x) \leq F(y) \quad (48)$$

$$\lim_{x \rightarrow \infty} F(x) = 1 \quad (49)$$

$$\lim_{x \rightarrow -\infty} F(x) = 0 \quad (50)$$

3.4 Mean, expectation

3.4.1 First Moment of X

$$\mu = E[x] = \begin{cases} \sum_{i=1}^n x_i P(X = x_i) & \text{Discrete R.V} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{Continus R.V} \end{cases} \quad (51)$$

3.4.2 nth Moment of X

$$E[x^n] = \begin{cases} \sum_{i=1}^N x_i^n P(X = x_i) & \text{Discrete R.V} \\ \int_{-\infty}^{\infty} x^n f(x) dx & \text{Continus R.V} \end{cases} \quad (52)$$

3.4.3 Properties of Expectation

$$x \geq 0 \iff E[x] \geq 0 \quad (53)$$

$$E[cx] = cE[x] \quad (54)$$

$$E[x + y] = E[x] + E[y] \quad (55)$$

3.4.4 Variance

$$\text{Var}(x) = \sigma^2 = E[x^2] - (E[x])^2 \quad (56)$$

3.4.5 Standard Deviation

$$\sigma = \sqrt{\text{Var}(x)} \quad (57)$$

$$\text{Var}(xC) = c^2 \text{Var}(x) \quad (58)$$

$$\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y) \quad \text{if } x, y \text{ are indepedent} \quad (59)$$

$$\text{Var}(x + y) = \text{Var}(x) + \text{Var}(y) + 2 \text{Cov}(x, y) \quad \text{if } x, y \text{ are dependant} \quad (60)$$

$$(61)$$

3.4.6 Covariance

$$\text{Cov}(x, y) = E[xy] - E[x]E[y] \quad (62)$$

$$\text{Cov}(x, x) = E[x^2] - (E[x])^2 = \text{Var}(x) \quad (63)$$

$$\text{Cov}(x, y) = 0 \quad x, y \text{ are indepedent} \quad (64)$$

4 Markov Chains