# ECM1416 Forumlas

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## Contents

# 1 Linear Algeobra

## 1.1 Vectors

### 1.1.1 Vector Norm

$$||v|| = \sqrt{\sum v_i^2} \tag{1}$$

### 1.1.2 Inner Product

$$u \cdot v = |u||v|\cos(\theta) \tag{2}$$

$$u \cdot v = u^T \cdot v = \sum_{i=1}^{n} u_i v_i \tag{3}$$

### 1.1.3 Rotating Vector

$$v' = \begin{bmatrix} v_x \cos(\theta) - v_y \sin(\theta) \\ v_x \sin(\theta) + v_y \cos(\theta) \end{bmatrix} \tag{4}$$

## 1.2 Matracies

### 1.2.1 Matrix Product $(A = (m \times n), B = (n \times p))$

$$A \cdot B = \begin{bmatrix} \Sigma_{i=1}^{n} a_{1i} b_{i1} & \cdots & \Sigma_{i=1}^{n} a_{1i} b_{ip} \\ \vdots & \ddots & \vdots \\ \Sigma_{i=1}^{n} a_{ni} b_{i1} & \cdots & \Sigma_{i=1}^{n} a_{ni} b_{ip} \end{bmatrix} \tag{5}$$

### 1.2.2 Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{6}$$

### 1.2.3 Scale Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda_1 x \\ \lambda_2 y \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

### 1.2.4 Shear Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ \mu x + y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9}$$

### 1.2.5 Transpose Properties

$$(A + B)^T = A^T + B^T \tag{10}$$

$$(AB)^T = A^T B^T \tag{11}$$

### 1.2.6 Identity Matrix

$$v = I_n v, \forall v \in \mathbb{R}^n \tag{12}$$

### 1.2.7 Determinant of a $2 \times 2$ Matrix

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc \tag{13}$$

### 1.2.8 Determinant of a $n \times n$ Matrix

$$\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} M_{ik} = \sum_{k=1}^{n} (-1)^{j+k} a_{kj} M_{kj} \tag{14}$$

### 1.2.9 Inverse of a $2 \times 2$ Matrix

$$A^-1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{15}$$

### 1.2.10 Inverse of a $n \times n$ Matrix

$$A^-1 = \frac{1}{\det(A)} C^T = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{21} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \tag{16}$$
$$C_{ij} = (-1)^{i+j} M_{ij}$$

### 1.2.11 Properties of an Inverse

$$(A^{-1})^{-1} = A \tag{17}$$
$$(A^T)^{-1} = (A^{-1})^T \tag{18}$$
$$(AB)^{-1} = B^{-1}A^{-1} \tag{19}$$
$$(kA)^{-1} = k^{-1}A^{-1} \tag{20}$$

### 1.2.12 Solving a System of Linear Equations

$$Ax = b \tag{21}$$
$$x = A^{-1}b \tag{22}$$

# 2 Differentation

## 2.1 Matrix Calculus

### 2.1.1 Gradient

$$\nabla f = \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{bmatrix} \tag{23}$$

### 2.1.2 Hessian

$$H_f = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_mx_1} & f_{x_mx_2} & \cdots & f_{x_mx_n} \end{bmatrix} \tag{24}$$

#### 2.1.3 Jacobian

$$D_f = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_n \end{bmatrix} = \begin{bmatrix} f_{1x_1} & f_{1x_2} & \cdots & f_{1x_n} \\ f_{2x_1} & f_{2x_2} & \cdots & f_{2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{nx_1} & f_{nx_2} & \cdots & f_{nx_n} \end{bmatrix} \tag{25}$$

### 2.2 Taylor Expantion

$$f(x) \approx \sum_{i=0}^{n} \frac{f^i(a)(x-a)^i}{i!} \tag{26}$$

### 2.3 Maclorian Expantion (a = 0)

#### 2.3.1 $\sin(x)$

$$\sin(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \tag{27}$$

#### 2.3.2 $\cos(x)$

$$\cos(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \tag{28}$$

#### 2.3.3 $e^x$

$$e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!} \tag{29}$$

#### 2.3.4 $\ln|1+x|$

$$\ln|1+x| \approx \sum_{k=0}^{n} (-1)^k \frac{x^{k+1}}{k+1} \tag{30}$$

#### 2.3.5 $c(\frac{1}{1-x})$

$$c(\frac{1}{1-x}) \approx c \sum_{k=0}^{n} x^k \tag{31}$$

## 2.4 Solving ODEs

### 2.4.1 Intergrating Factor

$$\begin{aligned} \frac{dy}{dx} + P(x)y &= Q(x) \\ \text{I.F} &= e^{\int P(x)dx} \\ \frac{d}{dx}(e^{\int P(x)dx}y) &= Q(x)e^{\int P(x)dx} \\ y &= \frac{\int Q(x)e^{\int P(x)dx}dx}{e^{\int P(x)dx}} \end{aligned} \tag{32}$$

### 2.4.2 Seperation of Variables

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \frac{dy}{dx}\frac{1}{h(y)} &= g(x) \\ \int \frac{1}{h(y)}dy &= \int g(x)dx \end{aligned} \tag{33}$$

### 2.4.3 Eulers Method

$$\begin{aligned} y_{n+1} &= y_n + hf(x_n, y+n) \\ x_{n+1} &= x_n + h \end{aligned} \tag{34}$$

# 3 Probability

## 3.1 Conditional Probability

### 3.1.1 Given That

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{35}$$

### 3.1.2 Independence

$$P(A \cap B) = P(A)P(B) \tag{36}$$

### 3.1.3 Law of Total Probability

$$P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n)P(B_n) = \sum_n P(B_n|A)P(A) \tag{37}$$

#### 3.1.4 Bayes Theorm

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{38}$$

$$P(A|X) = \frac{P(X|A)P(A)}{P(X)} = \frac{P(X|A)P(A)}{P(X|A)P(A) + P(X|A^C)P(A^C)} \tag{39}$$

## 3.2 Random Variables

#### 3.2.1 Descrete Random Variables

$$\begin{gathered} P(x_i) = P(X = x_i) \\ P(X = x_i) \geq 0, \forall i \\ \sum P(X = x_i) = 1 \end{gathered} \tag{40}$$

#### 3.2.2 Continus Random Variables

$$\begin{aligned} P(a \leq X \leq b) = \int_a^b f(x)dx \\ f(x) \geq 0, \forall x \in X \\ \int_\Omega f(x)dx = 1 \\ f(x) = 0, x \notin X \end{aligned} \tag{41}$$

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= P(a \leq X < b) \\ &= P(a < X < b) \end{aligned} \tag{42}$$

## 3.3 Cumulative Distrubution Function

#### 3.3.1 General Case

$$F(x) = P(X \leq x) \tag{43}$$

#### 3.3.2 X is Discrete

$$F(x) = P(X \leq x) = \sum_{\forall x_i \leq x} P(x_i) \tag{44}$$

#### 3.3.3 X is Continus

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt \tag{45}$$

$$P(a \leq X \leq b) = F(b) - F(a) \tag{46}$$

### 3.3.4 Properties of CDF

$$0 \leq F(x) \leq 1 \tag{47}$$
$$x < y \rightarrow F(x) \leq F(y) \tag{48}$$
$$\lim_{x\to\infty} F(x) = 1 \tag{49}$$
$$\lim_{x\to-\infty} F(x) = 0 \tag{50}$$

## 3.4 Mean, expectation

### 3.4.1 First Moment of X

$$\mu = E[x] = \begin{cases} \sum_{i=1}^{n} x_i P(X = x_i) & \text{Discrete R.V} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{Continus R.V} \end{cases} \tag{51}$$

### 3.4.2 $n$th Moment of X

$$E[x^n] = \begin{cases} \sum_{i=1}^{N} x_i^n P(X = x_i) & \text{Discrete R.V} \\ \int_{-\infty}^{\infty} x^n f(x) dx & \text{Continus R.V} \end{cases} \tag{52}$$

### 3.4.3 Properties of Expectation

$$x \geq 0 \iff E[x] \geq 0 \tag{53}$$
$$E[cx] = cE[x] \tag{54}$$
$$E[x+y] = E[x] + E[y] \tag{55}$$

### 3.4.4 Variance

$$\operatorname{Var}(x) = \sigma^2 = E[x^2] - (E[x])^2 \tag{56}$$

### 3.4.5 Standard Deviation

$$\sigma = \sqrt{\operatorname{Var}(x)} \tag{57}$$
$$\operatorname{Var}(xC) = c^2 \operatorname{Var}(x) \tag{58}$$
$$\operatorname{Var}(x+y) = \operatorname{Var}(x) + \operatorname{Var}(y) \qquad \text{if } x, y \text{ are indepedent} \tag{59}$$
$$\operatorname{Var}(x+y) = \operatorname{Var}(x) + \operatorname{Var}(y) + 2\operatorname{Cov}(x,y) \qquad \text{if } x, y \text{ are dependant} \tag{60}$$
$$\tag{61}$$

### 3.4.6 Covariance

$$\operatorname{Cov}(x,y) = E[xy] - E[x]E[y] \tag{62}$$
$$\operatorname{Cov}(x,x) = E[x^2] - (E[x])^2 = \operatorname{Var}(x) \tag{63}$$
$$\operatorname{Cov}(x,y) = 0 \qquad x, y \text{ are indepedent} \tag{64}$$

# 4 Markov Chains