Contents

Chapter 1

Definitions

1.1 Signals and Sequences

Definition 1. ***Kronecker Delta Function*** *The Kronecker delta function, $\delta[n]$, is a sequence defined as*

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 1 \end{cases}, n \in \mathbb{N} \tag{1.1}$$

Remark:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{1.2}$$

Definition 2. ***Periodic Sampling*** *A discrete-time representation $x[n]$ of a continuous signal $x_a(t)$ can be obtained from periodic sampling according to the relation*

$$x[n] = x_a(nT) \tag{1.3}$$

where T is the sampling period and $f_s = 1/T$ is the sampling frequency.

Definition 3. ***Linear Constant Coefficient Difference Equation*** *A linear constant coefficient difference equation (LCCDE) is an equation of the form*

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{1.4}$$

Notably, linear time-invariant systems can have difference equation representations.

1.2 Transforms

Definition 4. ***Continuous-Time Fourier Transform*** *If $x_a(t)$ is a function where $t \in \mathbb{R}$, then its continuous-time Fourier transform $X_a(\Omega)$ is*

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt. \tag{1.5}$$

and its inverse transformation is

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t}d\Omega. \tag{1.6}$$

Definition 5. ***Discrete-Time Fourier Transform*** *If $x[n]$ is a function where $n \in \mathbb{N}$, then its discrete-time Fourier transform $X(\omega)$ is*

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{1.7}$$

DTFT General Properties

1) $X(\omega)$ is a continuous function.

2) $X(\omega)$ is a periodic function.

3) Parseval's theorem holds that

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \tag{1.8}$$

Definition 6. ***Z-Transform*** *If $x[n]$ is a function where $n \in \mathbb{N}$, then its discrete-time Fourier transform $X(\omega)$ is*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{1.9}$$

If $X(z)$ exists when $|z| = 1$, $X(e^{j\omega})$ is equal to the DTFT of $x[n]$.
Remark: $Z\{x[n-k]\} = z^{-k}(X(z) - \sum_{i=-k}^{-1} x_i z^{-i})$.
Z-Transform General Properties

1)

$$\begin{aligned} ax_1[n] + bx_2[n] &\xleftrightarrow{Z} aX_1(z) + bX_2(z) \\ \text{ROC} &= R_{x1} \bigcap R_{x2} \end{aligned} \tag{1.10}$$

2)

$$\begin{aligned} x[n-n_0] &\xleftrightarrow{Z} z^{-n_0}X(z) \\ \text{ROC} &= R_x \end{aligned} \tag{1.11}$$

3)

$$\begin{aligned} nx[n] &\xleftrightarrow{Z} -z\frac{dx(z)}{dz} \\ \text{ROC} &= R_x \end{aligned} \tag{1.12}$$

4)

$$\begin{aligned} x^*[n] &\xleftrightarrow{Z} X^*(z^*) \\ \text{ROC} &= R_x \end{aligned} \tag{1.13}$$

5)

$$\begin{aligned} x_1[n] * x_2[n] &\xleftrightarrow{Z} X_1(z)X_2(z) \\ \text{ROC} &= R_{x1} \bigcap R_{x2} \end{aligned} \tag{1.14}$$

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=0}^{M} h[k]x[n-k] \\ &= \sum_{k=0}^{M} h[k]X(z)z^{-k} \\ &= X(z)H(z) \end{aligned} \tag{1.15}$$

6)

$$x[0] = \lim_{z\to\infty} X(z), x[n] = 0 \; \forall n < 0 \tag{1.16}$$

Definition 7. ***Region of Convergence*** *The region of convergence (ROC), or more generally the radius of convergence, is a circle which defines two sets of values, one of which converges under a transformation and one that diverges.*

ROC General Properties

1) The ROC is a ring or disk centered at the origin of a z-plane.

2) The ROC cannot contain any poles.

3) If $x[n]$ is always finite than the ROC is the entire z-plane with the exception of $z = 0$, or $z = \infty$.

4) If $x[n]$ is right sided, the ROC extends from the outermost pole.

5) If $x[n]$ is left sided, the ROC extends from the innermost pole to 0.

6) If $x[n]$ is two-sided, the ROC is a ring bounded by two poles.

7) The ROC is connected.

Chapter 2

The Sampling Theorem

2.1 Deriving the Sampling Theorem

Theorem 1. ***Sampling Theorem*** *The discrete-time Fourier transform of the sampled sequence $x[n]$ relates to the continuous-time Fourier transform of the signal $x_a(t)$ as*

$$X\left(\Omega T\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\frac{\omega}{T} + \frac{n2\pi}{T}\right). \tag{2.1}$$

Proof. By the sampling relation,

$$X\left(\omega\right) = \sum_{n=-\infty}^{\infty} x_a(nT)e^{-j\omega n}. \tag{2.2}$$

This can be expressed in term of the continuous-time Fourier transformation of $x_a(t)$,

$$\begin{aligned} X\left(\omega\right) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega nt}d\Omega e^{-j\omega n} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_a\left(\Omega\right)e^{-j\omega n}e^{j\Omega nt}d\Omega. \end{aligned} \tag{2.3}$$

Lemma 1. *Given $\omega \in [-\pi, \pi]$,*

$$\lim_{N\to\infty}\sum_{n=-N}^{N} e^{jn\omega} = 2\pi\delta\left(\omega\right). \tag{2.4}$$

Proof. Recall the geometric identity,

$$\begin{aligned} \lim_{N\to\infty}\sum_{n=-N}^{N} e^{jn\omega} &= \lim_{N\to\infty} e^{-j\omega N}\sum_{n=0}^{N} e^{jn\omega} \\ &= \lim_{N\to\infty} e^{-j\omega N}\frac{1-e^{j(2N+1)\omega}}{1-e^{j\omega}} \end{aligned} \tag{2.5}$$

As $\operatorname{sinc}(x) = e^{jx} - e^{-j}$,

$$\begin{aligned} \lim_{N\to\infty}\sum_{n=-N}^{N} e^{jn\omega} &= \lim_{N\to\infty}\left(2N+1\right)\frac{\operatorname{sinc}\left(\left(\frac{2N+1}{2}\right)\omega\right)}{\operatorname{sinc}\left(\frac{\omega}{2}\right)} \\ &= \alpha\delta(\omega) \end{aligned} \tag{2.6}$$

It is clear that the expression is proportional to the dirac delta function $\delta(\omega)$. Integrating the original expression over all values of ω will solve for α:

$$\int_{-\pi}^{\pi}\lim_{N\to\infty}\sum_{n=-N}^{N} e^{jn\omega}d\omega = 2\pi \tag{2.7}$$

Thus,

$$\lim_{N\to\infty}\sum_{n=-N}^{N} e^{jn\omega} = 2\pi\delta(\omega). \tag{2.8}$$

□

Accordingly, from Lemma 1,

$$X\left(\omega\right) = \int\limits_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_a\left(\Omega\right)\delta\left(\left(\Omega T-\omega\right)-n2\pi\right)d\Omega. \tag{2.9}$$

Simplify,

$$\begin{aligned} X\left(\omega\right) &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int\limits_{-\infty}^{\infty} X_a\left(\Omega\right)\delta\left(\Omega-\frac{\omega}{T}-\frac{n2\pi}{T}\right)d\Omega \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\frac{\omega}{T}+\frac{n2\pi}{T}\right). \end{aligned} \tag{2.10}$$

□

Chapter 3

Textbook Problems

3.1 Discrete Time Signals Problems

3.1.1 Impulse Response from LCCDE (2.4)

Problem

Consider the linear constant-coefficient difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1] \tag{3.1}$$

Determine $y[n]$ when $x[n] = \delta[n]$ and $y = 0, n < 0$.

Solution Using The Characteristic Polynomial

Assume a solution of the form,

$$y[n] = Aa^n u[n] \tag{3.2}$$

By substitution,

$$\left(Aa^n - \frac{3}{4}Aa^{n-1} + \frac{1}{8}Aa^{n-2}\right)u[n] = 2\delta[n-1] \tag{3.3}$$

Allow $n = 0$,

$$\begin{aligned} A - \frac{3}{4}Aa^{-1} + \frac{1}{8}Aa^{-2} &= 0 \\ 1 - \frac{3}{4}a^{-1} + \frac{1}{8}a^{-2} &= 0 \end{aligned} \tag{3.4}$$

By factorization, it is evident that the solutions for a are $\frac{1}{2}$, $\frac{1}{4}$. The homogeneous solution, then, becomes

$$y[n] = A_1\left(\frac{1}{2}\right)^n u[n] + A_2\left(\frac{1}{4}\right)^n u[n]. \tag{3.5}$$

Observe the system at two values,

$$\begin{matrix} A_1 + A_2 = 0 & n = 0 \\ \frac{1}{2}A_1 + \frac{1}{4}A_2 = 2 & n = 1 \end{matrix} \tag{3.6}$$

Therefore, the impulse response is,

$$\begin{aligned} y[n] &= 8\left(\frac{1}{2}\right)^n u[n] - 8\left(\frac{1}{4}\right)^n u[n] \\ &= 8\left(2^{-n} - 4^{-n}\right)u[n]. \end{aligned} \tag{3.7}$$

Solution Using the Z-Transform

Take the Z-transform of the LCCDE,

$$Z\left\{y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]\right\}=Z\left\{2\delta[n-1]\right\} \tag{3.8}$$

Considering the initial conditions,

$$\begin{aligned} Y(z)-\frac{3}{4}z^{-1}\left(Y(z)-y[-1]z\right)+\frac{1}{8}z^{-2}\left(Y(z)-y[-2]z^{2}-y[-1]z\right)=2 \\ Y(z)-\frac{3}{4}z^{-1}Y(z)+\frac{1}{8}z^{-2}Y(z)=2 \end{aligned} \tag{3.9}$$

Solve for $Y(z)$ and express it in expanded form,

$$Y(z)=\frac{2}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}=\frac{2}{\left(1-\frac{z^{-1}}{2}\right)\left(1-\frac{z^{-1}}{4}\right)}=\frac{8}{1-\frac{z^{-1}}{2}}-\frac{8}{1-\frac{z^{-1}}{4}} \tag{3.10}$$

Obtain $y[n]$ using the inverse Z-transform,

$$y[n]=Z^{-1}\left\{Y(z)\right\}=8\left(2^{-n}\right)u[n]-8\left(4^{-n}\right)u[n] \tag{3.11}$$

3.1.2 Step Response from LCCDE (2.5)

Problem

Consider a causal LTI system described by

$$y[n]-y[n-1]+6y[n-2]=2x[n-1]. \tag{3.12}$$

Determine the step response of the system.

Solution Using Discrete Convolution

First solve for the impulse response,

$$h[n]=2\left(3^{n}-2^{n}\right)u[n] \tag{3.13}$$

Recall that

$$y[n]=x[n]*h[n]=\sum_{m=-\infty}^{\infty}x[n-m]h[m]. \tag{3.14}$$

By substitution,

$$\begin{aligned} y[n]&=\sum_{m=-\infty}^{\infty}u[n-m]2\left(3^{m}-2^{m}\right)u[m] \\ &=\sum_{m=0}^{\infty}u[n-m]2\left(3^{m}-2^{m}\right) \\ &=\sum_{m=0}^{n}2\left(3^{m}-2^{m}\right) \end{aligned} \tag{3.15}$$

Note that this is the accumulation of $h[n]$ from 0 to n. Using the fact that

$$\sum_{m=0}^{n}r^{m}=\frac{1-r^{n}}{1-r}, \tag{3.16}$$

it is evident that

$$y[n]=\left(3\left(3^{n}\right)+6\left(2^{n}\right)+1\right)u[n]. \tag{3.17}$$

Solution Using the Z-Transform

Using the system rest condition, the initial conditions are set to zero. Accordingly, $Y(z)$ can be solved for as

$$\begin{aligned} Y(z) - z^{-1}5Y(z) + 6z^{-2}Y(z) &= 2z^{-1}Z\{u[n]\} \\ Y(z)\left(1 - 5z^{-1} + 6z^{-2}\right) &= 2z^{-1}\frac{1}{1-z^{-1}}. \end{aligned} \tag{3.18}$$

Hence,

$$Y(z) = \frac{2z^{-1}}{(1-3z^{-1})(1-2z^{-1})(1-z^{-1})}. \tag{3.19}$$

The inverse Z-transform can be solved for using partial fraction expansion,

$$Y(z) = \frac{3}{1-3z^{-1}} + \frac{6}{1-2z^{-1}} + \frac{1}{1-z^{-1}} \tag{3.20}$$

$$y[n] = Z^{-1}\{Y(z)\} = (3(3^n) + 6(2^n) + 1)u[n]. \tag{3.21}$$

3.1.3 Causal and Anti-Causal Solutions from LCCDE (2.16)

Problem

Consider the LCCDE

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] \tag{3.22}$$

Find the impulse response of the causal and anti-causal LTI systems characterized by the equation.

Solution

Using an assumed solution, the causal solution can be found as

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{4}\right)^n u[n]. \tag{3.23}$$

The anti-causal solution, then, is

$$h[n] = -2\left(\frac{1}{2}\right)^n u[-n-1] - \left(-\frac{1}{4}\right)^n u[-n-1]. \tag{3.24}$$

A system is said to be stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{3.25}$$

The causal absolute sum is

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=0}^{\infty} 2\left(\frac{1}{2}\right)^n + \left(-\frac{1}{4}\right)^n \\ &= \frac{24}{5} \end{aligned} \tag{3.26}$$

Being absolutely summable, the system is stable. On the other hand, for the anti-causal case,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=-\infty}^{-1} 2\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n \\ &= \infty \end{aligned} \tag{3.27}$$

Thus, the anti-causal system is unstable.

3.2 Sampling of Continuous Time Signals Problems

3.2.1 Determining Analog Frequency from Digital Frequency (4.2)

Problem

The sequence

$$x[n] = \cos\left(\frac{\pi}{4}n\right), -\infty < n < \infty \tag{3.28}$$

was obtained by sampling a continuous-time signal

$$x_c(t) = \cos(\Omega_0 t), -\infty < t < \infty \tag{3.29}$$

at a sampling rate of 1000 samples/s. Find the possible positive values of Ω_0 could have resulted in the sequence $x[n]$.

Solution

Recall that

$$x_c(nT) = x[n]. \tag{3.30}$$

Solve for when the two signals equal one another when $t = nT$,

$$\begin{aligned} \frac{\pi}{4}n + 2\pi nm &= \frac{\Omega_0 n}{1000} \\ \frac{\pi}{4} + 2\pi m &= \frac{\Omega_0}{1000} \\ \Rightarrow \Omega_0 &= 250\pi + 2000\pi m. \end{aligned} \tag{3.31}$$

Therefore, the first two positive analog frequencies are $\Omega_0 = 250\pi, 2250\pi$.

3.2.2 Determining Sampling Frequency from a Sampled Signal (4.4)

The continuous-time signal

$$x_c(t) = \sin(20\pi t) + \cos(40\pi t) \tag{3.32}$$

is sampled with a sampling period T to obtain the discrete-time signal

$$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right). \tag{3.33}$$

Find possible values of T.

Solution

Using the periodic sampling definition,

$$\begin{aligned} \frac{\pi n}{5} + 2\pi nm &= 20\pi nT \\ \Rightarrow T &= \frac{1}{100} + \frac{m}{10} \end{aligned} \tag{3.34}$$

Two values of T are $\frac{1}{100}, \frac{11}{100}$.

Chapter 4

ECE 465 Material

4.1 Homeworks

4.1.1 Homework 2

Problem

In a DSP application, it requires an A/D converter with a 1-volt full scale. It also requires that the RMS value of the quantization noise to be less than 0.001 volt.

1) What is the required resolution (how many bits) of this converter? What is the actual RMS value of the noise in this converter?

2) If one wants to record a 5-minute audio signal using that converter with a sampling rate of 44 kHz, how many bits of disk space is required?

Solution

Assuming a constant noise function, E, being the ideal quantizer. The RMS voltage can be expressed,

$$E_{rms} = \sqrt{\frac{1}{Q}\int_{-Q/2}^{Q/2} |E(e)|^2 de} = \sqrt{\frac{1}{Q}\int_{-Q/2}^{Q/2} e^2 de} = \frac{Q}{\sqrt{12}} \tag{4.1}$$

A linear relationship exists between E_{rms} and Q. Therefore, the specified noise requirement can be solved for

$$\begin{aligned} E_{rms} &= \frac{Q}{\sqrt{12}} \leqslant 0.001 \\ &\Rightarrow Q \leqslant 0.003464 \end{aligned} \tag{4.2}$$

The minimum number of bits can be solved for

$$\begin{aligned} B_{\min} &= \left\lceil \lg\left(\frac{R}{Q}\right) \right\rceil \\ &= \left\lceil \lg\left(\frac{1}{0.0035}\right) \right\rceil \\ &= 9 \end{aligned} \tag{4.3}$$

The ADC can be improved by using a more typical value of 10 bits.
Using 9 bits, the new RMS is

$$\begin{aligned} E_{rms} &= \frac{R}{2^B\sqrt{12}} \\ &= \frac{1}{2^9\sqrt{12}} \\ &\approx 0.00056 \end{aligned} \tag{4.4}$$

For the disk space per audio recording, the number of storage bits per signal is

$$\begin{aligned} N\left(B\right) &= \left(\frac{44k\text{ samples}}{\text{s}}\right)\left(\frac{60\text{ s}}{\text{min}}\right)\left(\frac{5\text{ min}}{\text{signal}}\right)\left(\frac{B\text{ bits}}{\text{sample}}\right) \\ &= 13.2\text{M}\cdot B \end{aligned} \tag{4.5}$$

Accordingly,

$$N(9) = 118.8\text{ Mb} = 118.8\text{ Mb} \tag{4.6}$$

This would consume 29k blocks on a hard disc with a minimum addressable unit of 4096 bits.

4.1.2 Homework 3

Problem

Determine the z-transform of the following sequences:

$$x[n] = (0.7)^n u[n] \tag{4.7}$$

$$x[n] = 4\delta[n] + (0.5)^n u[n-3] \tag{4.8}$$

Solution

The z-transform is defined as follow

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{4.9}$$

Substitute and reduce for the given $x[n]$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (0.7)^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} (0.7)^n z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{0.7}{z}\right)^n \end{aligned} \tag{4.10}$$

This reduces into a geometric series. Therefore the solution is

$$X(z) = \frac{z}{z-0.7}, |z| > 0.7 \tag{4.11}$$

Using similar methods,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(4\delta[n] + (0.5)^n u[n-3]\right) z^{-n} \\ &= 4\sum_{n=-\infty}^{\infty} \delta[n]z^{-n} + \sum_{n=3}^{\infty} (0.5)^n z^{-n} \\ &= 4 - \sum_{n=0}^{2} \left(\frac{0.5}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n \\ &= 4 - 1 - \frac{0.5}{z} - \frac{0.25}{z^2} + \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n \end{aligned} \tag{4.12}$$

Therefore, the solution is

$$X(z) = 3 - \frac{0.25}{z}\left(2 - \frac{1}{z}\right) + \frac{z}{z-0.5}, |z| > 0.5 \tag{4.13}$$

Problem

An LTI system has the following impulse response

$$h[n] = \cos\left(n\frac{\pi}{5}\right) u[n]. \tag{4.14}$$

Is the system stable? Why? Repeat the analysis for

$$h[n] = 0.9^n u[n]. \tag{4.15}$$

Solution

The absolute sum of $h[n]$ is

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |0.9^n u[n]| &= \sum_{n=0}^{\infty} 0.9^n \\ &= \frac{1}{0.1} \\ &= 10\end{aligned} \tag{4.16}$$

The system described by $h[n]$ is stable if and only if it is absolutely summable. That is,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{4.17}$$

The system is causal, so

$$\sum_{n=-\infty}^{\infty} \left|\cos\left(\frac{n\pi}{5}\right) u[n]\right| = \sum_{n=0}^{\infty} \left|\cos\left(\frac{n\pi}{5}\right)\right| \tag{4.18}$$

The terms in the sum repeat periodically, allowing the sum to be decomposed

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \left|\cos\left(\frac{n\pi}{5}\right) u[n]\right| = &\sum_{n=0}^{\infty} |\cos(0)| + \sum_{n=0}^{\infty} \left|\cos\left(\frac{\pi}{5}\right)\right| + \sum_{n=0}^{\infty} \left|\cos\left(\frac{2\pi}{5}\right)\right| + \\ &\sum_{n=0}^{\infty} \left|\cos\left(\frac{3\pi}{5}\right)\right| + \sum_{n=0}^{\infty} \left|\cos\left(\frac{4\pi}{5}\right)\right|\end{aligned} \tag{4.19}$$

One of the sums can be expressed as

$$\sum_{n=0}^{\infty} |\cos(0)| = \sum_{n=0}^{\infty} 1 \tag{4.20}$$

which diverges. Therefore, the original sum must also diverge. **The system is unstable**.
The sum converges. **The system is stable**.

4.2 Exams

4.2.1 Midterm

Problem

Consider the following signal

$$x(t) = \sin(10000\pi t) + \sin(20000\pi t) + \sin(60000\pi t) \tag{4.21}$$

The signal is pre-filtered by an anti-aliasing filter $H(f)$, and then sampled at the sampling rate of 40,000 Hz. The samples are then reconstructed by an ideal reconstruction filter to reproduce the analog signal. Find the output signal for the following cases:

1) $H(f) = 1$ i.e. no prefiltering

2) $H(f)$ is an ideal LPF with $f_c = 20,000$ Hz.

3) $H(f)$ is a filter with flat response up to 20,000 Hz, and attenuate at the rate of 60dB per Octave beyond 20,000 Hz.

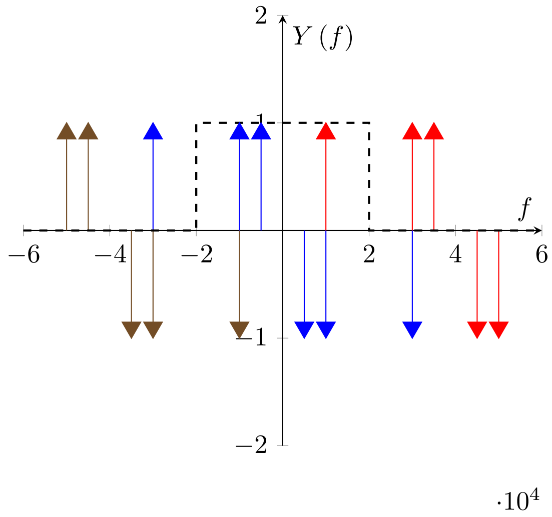

Figure 4.1: Spectrum of $y(t)$ produced from no prefiltering. Note that the aliases cancel out frequencies present in the Nyquist window.

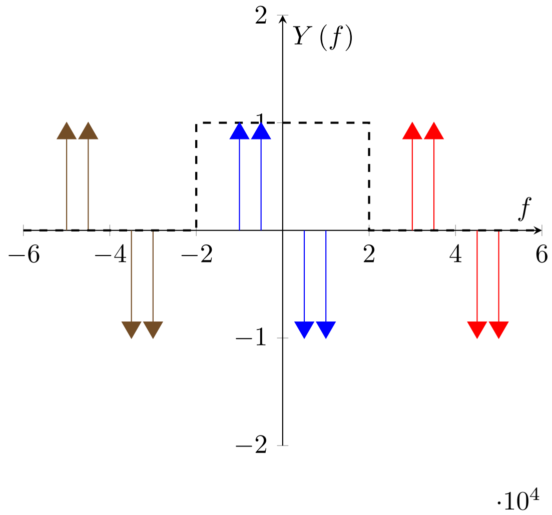

Figure 4.2: Spectrum of $y(t)$ produced from an ideal LPF with f_c = 20,000 Hz. Note that the 10,000 Hz tone is preserved.

Solution

Constructing the graph in figure 4.1, it is evident that

$$y(t) = \sin(10000\pi t)\,. \tag{4.22}$$

Referring to figure 4.2, for the ideal LPF,

$$y(t) = \sin(10000\pi t) + \sin(20000\pi t)\,. \tag{4.23}$$

For the case of the non-ideal LPF, calculate how many octaves away the 30 kHz tone is,

$$\lg\left(\frac{30}{20}\right) \approx 0.585 \text{ octaves} \tag{4.24}$$

The attenuation, then, is

$$60\lg\left(\frac{30}{20}\right) \approx 35.1 \text{ dB } \approx 0.0176 \tag{4.25}$$

Accordingly, for the non-ideal LPF,

$$y(t) = \sin(10000\pi t) + 0.982\sin(20000\pi t)\,. \tag{4.26}$$

Problem

An LTI system has the following impulse response

1) $h[n] = (-1)^n u[n]$. Is this system stable? Why?
2) Repeat with $h[n] = (0.9)^n \cos\left(\frac{n\pi}{5}\right) u[n]$

Solution

A system is said to be stable if its impulse response is absolutely stable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{4.27}$$

For 1),

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |(-1)^n u[n]| &= \sum_{n=0}^{\infty} |(-1)^n| \\ &= \infty\end{aligned} \tag{4.28}$$

For 2),

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \left|(0.9)^n \cos\left(\frac{n\pi}{5}\right) u[n]\right| &< \sum_{n=0}^{\infty} |(0.9)^n| \\ &= \frac{1}{1-0.9}\end{aligned} \tag{4.29}$$

Being less than 10, the system is stable.

Problem

An LTI system has a transfer function:

$$H(z) = \frac{3 + 0.1z^{-1}}{1 - 0.5z^{-1} - 0.66z^{-2}} \tag{4.30}$$

1) Find the causal impulse response $h[n]$.
2) Find the stable impulse response $h[n]$.

Solution

Expand $H(z)$ to facilitate solving for the inverse Z-transform,

$$H(z) = \frac{3 + 0.1z^{-1}}{(1 + 0.6z^{-1})(1 - 1.1z^{-1})} = \frac{2}{1 + 0.6z^{-1}} - \frac{3.8}{1 - 1.1z^{-1}}. \tag{4.31}$$

So, for the causal impulse response,

$$h[n] = Z^{-1}\left\{\frac{2}{1 + 0.6z^{-1}} - \frac{3.8}{1 - 1.1z^{-1}}\right\} = (2(0.6)^n - 3.8(1.1)^n)\, u[n]. \tag{4.32}$$

For the stable impulse response,

$$h[n] = Z^{-1}\left\{\frac{2}{1 + 0.6z^{-1}} - \frac{3.8}{1 - 1.1z^{-1}}\right\} = (2(0.6)^n u[n] + 3.8(1.1)^n u[-n-1]). \tag{4.33}$$