1 MAT301 Notes: Week 2 (Fall 2024)

1.1 Example. Let G be a group. Prove that

$$C_G(a) = \forall a \in G \iff G \text{ is Abelian}$$

Note that the centralizer $C_G(a)$ is defined by $C_G(a) = \{g \in G : ga = ag\}$

Proof. We begin by proving the $\Longleftarrow$ direction.

Let $a \in G$. Then, since G is Abelian, every element commutes with a. Therefore

$$C_G(a) = G$$

Conversely, let $x, y \in G$. Then we have

$$x \in C_G(x) = G = C_G(y) \implies xy = yx$$

$\square$

1.2 Example. Let G be a group. Prove that G is Abelian if and only if $(xy)^2 = x^2 y^2, x, y \in G$.

Proof. Suppose G is Abelian. Then

$$(xy)^2 = (xy)(xy) = (xx)(yy) = x^2 y^2$$

Next suppose $(xy)^2 = x^2 y^2$. Then

$$xxyy = x^2 y^2 = (xy)^2 = xyxy$$

Applying x^{-1} to the left and y^{-1} to the right obtains

$$x^{-1} xxyy y^{-1} = x^{-1} xyxy y^{-1} \implies xy = yx$$

$\square$