

# 1 Week 2 Homework - Ethan Hua

**1.1 Exercise.** (a) Prove that there exists an infinitely differentiable function  $\alpha : \mathbf{R} \rightarrow \mathbf{R}$  such that  $\alpha(t) = 0$  for all  $t \leq 0$ , and  $\alpha(t) > 0$  for all  $t > 0$ .

*Proof.* We define  $\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$

Trivially  $\alpha(t) = 0$  if  $t \leq 0$  and  $\alpha(t) > 0$  if  $t > 0$ . It remains to show that  $\alpha$  is infinitely differentiable.

Since 0 is infinitely differentiable, and  $e^{-\frac{1}{x}}$  is infinitely differentiable for  $x > 0$ , it suffices to show that derivatives of all orders of  $\alpha$  are continuous at  $t = 0$ .

We will continue by proving a lemma.

**Lemma.**  $\forall n \in \mathbb{N}$ , for  $t > 0$ ,  $\alpha^{(n)}(t) = Q(t)\alpha(t)$ , where  $Q(x)$  is a linear combination of nonpositive integer powers of  $t$ .

We will show this using induction.

*Proof.* When  $n = 1$ , fixing  $t > 0$ , we have

$$\alpha'(t) = \frac{1}{t^2} e^{-\frac{1}{t}} = \frac{1}{t^2} \alpha(t)$$

We let  $Q(t) = t^{-2}$  and we are done.

Now suppose that the lemma holds for all  $i \leq k$ , for some  $k \in \mathbb{N}$ .

Then for  $t > 0$ ,

$$\alpha^{(k)}(t) = P(t)\alpha(t), \text{ where } P(t) \text{ is a linear combination of nonpositive integer powers of } t$$

Taking the derivative of both sides with respect to  $t$ , we obtain

$$\alpha^{(k+1)}(t) = P'(t)\alpha(t) + P(t)\alpha'(t)$$

The lemma holds for  $i = 1$ , therefore for some  $R(t)$ ,

$$P'(t)\alpha(t) + P(t)\alpha'(t) = P'(t)\alpha(t) + P(t)R(t)\alpha(t) = (P'(t) + P(t)R(t))\alpha(t)$$

$P'(t) + P(t)R(t)$  is a linear combination of nonpositive integer powers of  $t$ , therefore letting  $Q(t) = P'(t) + P(t)R(t)$ , we get our desired conclusion.

By the principle of induction, the lemma holds true for all  $n \in \mathbb{N}$ .

□

We now continue in proving that derivatives of all orders of  $\alpha$  are continuous at 0. We show this by proving that

$$\lim_{t \rightarrow 0} \alpha^{(n)}(t) = 0, \text{ where } n \in \mathbb{N}$$

We will only worry about the right hand limit, as the left hand limit always evaluates to 0.

Let  $n \in \mathbb{N}$ . Since we only deal with positive  $t$ , by our lemma,

$$\alpha^{(n)}(t) = Q(t)\alpha(t)$$

Where  $Q(t)$  is a linear combination of nonpositive integer powers of  $t$ . So

$$\lim_{t \rightarrow 0^+} \alpha^{(n)}(t) = \lim_{t \rightarrow 0^+} Q(t)\alpha(t) = \lim_{t \rightarrow 0^+} \sum_{i=0}^k a_i t^{-i} \alpha(t)$$

Where  $a_i$  are real constants and  $k \in \mathbb{N}$ .

Consider an arbitrary term  $a_i t^{-i} \alpha(t) = a_i t^{-i} e^{\frac{-1}{t}}$ . We want to show that  $\lim_{t \rightarrow 0^+} a_i t^{-i} e^{\frac{-1}{t}}$  exists and is equal to 0.

First, we will perform the substitution  $x = \frac{1}{t}$ . Then the limit becomes

$$\lim_{x \rightarrow \infty} a_i x^i e^{-x} = 0$$

The proof for this fact is omitted, but applying L'Hopital's rule  $i$  times produces the same result. Thus,

$$\lim_{t \rightarrow 0^+} \alpha^{(n)}(t) = \lim_{t \rightarrow 0^+} \sum_{i=0}^k a_i t^{-i} \alpha(t) = \sum_{i=0}^k \lim_{t \rightarrow 0^+} a_i t^{-i} \alpha(t) = \sum_{i=0}^k 0 = 0$$

Thus  $\alpha$  is infinitely differentiable everywhere. □

- (b) Prove that there exists an infinitely differentiable function  $\beta : \mathbf{R} \rightarrow \mathbf{R}$  such that  $\beta(t) = 1$  for all  $t \geq 1$ , and  $\beta(t) = 0$  for all  $t \leq 0$ .

*Proof.* Define

$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}$$

If  $t \geq 1$ , we also have  $1-t \leq 0$ . Then

$$\beta(t) = \frac{\alpha(t)}{\alpha(t)} = 1$$

As well, if  $t \leq 0$ , we have

$$\beta(t) = 0$$

Since  $\alpha(t)$  is infinitely differentiable and  $\beta$  is composed of  $\alpha$ , it follows that  $\beta$  is also infinitely differentiable. □

- (c) Prove that there exists an infinitely differentiable function  $\varphi : \mathbf{R} \rightarrow \mathbf{R}$  such that  $\varphi(t) = 1$  for all  $t \in [2, 3]$ , and  $\varphi(t) = 0$  for  $t \in \mathbf{R} \setminus (1, 4)$ .

*Proof.* Define  $\phi(t) = \beta(t-1)\beta(4-t)$ . Since  $\beta$  is infinitely differentiable,  $\phi$  is as well. For  $t \in [2, 3]$ ,  $t-1 \geq 1$  and  $4-t \geq 1$ . Thus

$$\phi(t) = \beta(t-1)\beta(4-t) = 1$$

If  $t \in \mathbf{R} \setminus (1, 4)$ , then  $t-1 \leq 0$  or  $4-t \leq 0$ . In each case  $\beta(t-1) = 0$  or  $\beta(4-t) = 0$ , respectively, thus we have

$$\phi(t) = \beta(t-1)\beta(4-t) = 0$$

□

**1.2 Exercise.** Let  $S \subseteq \mathbb{R}^n$ . Consider the following three statements:

- $S$  is a bounded subset of  $(\mathbb{R}^n, \|\cdot\|_1)$ .
- $S$  is a bounded subset of  $(\mathbb{R}^n, \|\cdot\|_2)$ .
- $S$  is a bounded subset of  $(\mathbb{R}^n, \|\cdot\|_{\max})$ .

Among these statements, determine which implications are true and which are false. There are six implications to investigate. Supply proof or counterexample as appropriate. Include pictures.

**Claim.** Denote each statement as (1), (2), and (3), respectively. We claim that each statement implies all the other statements.

*Proof.* First of all, I am so sorry that you have to read this. Secondly, in this proof, we denote:

1. an open ball with respect to the 1-norm as  $B_1(p, \varepsilon)$
2. an open ball with respect to the Euclidean norm as  $B_2(p, \varepsilon)$
3. an open ball with respect to the max-norm as  $B_{\max}(p, \varepsilon)$

(1)  $\implies$  (2):

Suppose (1) is true. There exists a  $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$  and  $r > 0$  such that

$$S \subseteq B_1(p, r)$$

We will show that  $S \subseteq B_2(p, r)$ . Let  $x = (x_1, x_2, \dots, x_n) \in S$ . Since  $S \subseteq B_1(p, r)$ , by the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^n (x_i - p_i)^2} \leq \sum_{i=1}^n |x_i - p_i| = \|x - p\|_1 < r$$

Thus  $x \in B_2(p, r)$ .

(2)  $\implies$  (1):

We will show using induction on  $n$  that if for some open ball in  $\mathbb{R}^n$ ,  $S \subseteq B_2(p, r)$ , then  $S \subseteq B_1(p, \sqrt{n}r)$

Let  $n = 1$ . Suppose (2) is true. Fix  $x \in S \subseteq \mathbb{R}$ . Then by (2),

$$r > \|x - p\|_2 = \sqrt{(x - p)^2} = |x - p| = \|x - p\|_1 \implies S \subseteq B_1(p, \sqrt{1} \cdot r)$$

Thus the claim holds for  $n = 1$ . Now suppose the claim holds for  $n = k$ . Assume (2) is true. Let  $x \in S$ . By (2),

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{k+1} (x_i - p_i)^2} < r \implies \sum_{i=1}^{k+1} (x_i - p_i)^2 < r^2 \quad (*)$$

We want to show that  $x \in B_1(p, \sqrt{n}r)$ , or

$$\|x - p\|_1 = \sum_{i=0}^{k+1} |x_i - p_i| < \sqrt{n}r \iff \left( \sum_{i=0}^{k+1} |x_i - p_i| \right)^2 < nr^2$$

We have

$$\begin{aligned} \left( \sum_{i=0}^{k+1} |x_i - p_i| \right)^2 &= \left( \sum_{i=0}^k |x_i - p_i| + |x_{k+1} - p_{k+1}| \right)^2 \\ &= \left( \sum_{i=0}^k |x_i - p_i| \right)^2 + 2|x_{k+1} - p_{k+1}| \left( \sum_{i=0}^k |x_i - p_i| \right) + |x_{k+1} - p_{k+1}|^2 \end{aligned} \quad (**)$$

Manipulating the inequality from (\*), we obtain

$$\sum_{i=1}^k (x_i - p_i)^2 < r^2 - |x_{k+1} - p_{k+1}|^2$$

which implies that  $(x_1, x_2, \dots, x_k)$  is in the open ball  $B_2(p, r)$  on  $\mathbb{R}^k$ . From the induction hypothesis,

$$\begin{aligned} (**) &< k(r^2 - |x_{k+1} - p_{k+1}|^2) + 2|x_{k+1} - p_{k+1}| \left( \sum_{i=0}^k |x_i - p_i| \right) + r^2 - \sum_{i=1}^k (x_i - p_i)^2 \\ &= (k+1)r^2 - \left( k|x_{k+1} - p_{k+1}|^2 - 2|x_{k+1} - p_{k+1}| \left( \sum_{i=0}^k |x_i - p_i| \right) + \sum_{i=1}^k (x_i - p_i)^2 \right) \\ &= (k+1)r^2 - \sum_{i=1}^k \left( |x_{k+1} - p_{k+1}|^2 - 2|x_{k+1} - p_{k+1}| |x_i - p_i| + (x_i - p_i)^2 \right) \\ &= (k+1)r^2 - \sum_{i=1}^k (|x_{k+1} - p_{k+1}| - |x_i - p_i|)^2 \leq (k+1)r^2 \end{aligned}$$

In summary, we have that

$$\|x - p\|_1 < \sqrt{k+1}r$$

Thus  $x$  is in the open ball  $B_1(p, \sqrt{k+1}r)$ , which means that (1) is true. By the principle of induction, the claim holds for all  $n \in \mathbb{N}$ .

(1)  $\implies$  (3): Suppose (1). Then  $S$  is a subset of some open ball  $B_1(p, r)$ . We will show that  $S \subseteq B_{\max}(p, r)$ .

Let  $x \in S$ . Then by (1):

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n} \{|x_i - p_i|\} \leq \sum_{i=1}^n |x_i - p_i| = \|x - p\|_1 < r$$

Thus  $x \in B_{\max}(p, r)$ , so (3) is true.

(3)  $\implies$  (1):

Suppose that (3) holds true. Then for some open ball  $B_{\max}(p, r)$ . We will show that  $S \subseteq B_1(p, nr)$ .

Let  $x \in S$ . We have

$$\|x - p\|_1 = \sum_{i=1}^n |x_i - p_i| \leq n \max_{0 \leq i \leq n} |x_i - p_i| = n \|x - p\|_{\max} < nr$$

Thus  $x \in B_1(p, nr)$ , which implies that (1) is true.

(2)  $\implies$  (3):

Suppose (2) is true. Then  $S \subseteq B_2(p, r)$ , where  $B_2(p, r)$  is some open ball on the Euclidean norm. We want to show that  $S \subseteq B_{\max}(p, r)$ . Indeed, we have

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n} |x_i - p_i| \leq \sqrt{\sum_{i=1}^n |x_i - p_i|^2} < r$$

Thus  $S \subseteq B_{\max}(p, r)$ , so (3) is true.

(3)  $\implies$  (2):

Suppose that (3) holds true. Then for some open ball with the max-norm,  $S \subseteq B_{\max}(p, r)$ . We will show that  $S \subseteq B_2(p, nr)$

Let  $x \in S$ . Then be the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^n |a_i - p_i|^2} \leq \sum_{i=1}^n |a_i - p_i| \leq n \max_{1 \leq i \leq n} |x_i - p_i| < n \|x - p\|_{\max} < nr$$

Thus  $x \in B_2(p, nr)$ , so (2) holds.

Therefore, we have proven every implication to be true.

□

**1.3 Exercise.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed vector spaces. A linear mapping  $T : X \rightarrow Y$  is called **bounded** if there exists a constant  $M \geq 0$  such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Let  $B(X, Y)$  denote the set of these bounded linear operators. The **operator norm** on  $B(X, Y)$ , denoted by  $\|\cdot\|_{\text{op}}$ , is defined as follows:

$$\|T\|_{\text{op}} = \sup\{\|T(x)\|_Y : x \in X \text{ and } \|x\|_X \leq 1\}.$$

(a) Prove that  $B(X, Y)$  is a linear subspace of  $L(X, Y)$ .

*Proof.* The 0-transformation  $\|Z(x)\| = 0 \leq \|x\|_X, \forall x \in X$ , because of the definition of a norm. Thus  $0 \in B(X, Y)$ . Let  $T, U \in B(X, Y)$ ,  $c \in \mathbb{R}$ . Then for all  $x \in X$ , there exist  $M, N \geq 0$  such that

$$T(x) \leq M\|x\|_X \text{ and } U(x) \leq N\|x\|_X \implies T(x) + U(x) \leq (M + N)\|x\|_X$$

Which implies that  $T + U$  is a member of  $B(X, Y)$ . As well, from the first inequality,

$$T(x) \leq M\|x\|_X \implies cT(x) \leq cM\|x\|_X$$

Which implies that  $cT$  is a member of  $B(X, Y)$ .

Since  $0 \in B(X, Y)$  and  $B(X, Y)$  is closed under addition and scalar multiplication,  $B(X, Y)$  is a subspace of  $L(X, Y)$ . □

(b) Prove that  $\|\cdot\|_{\text{op}}$  is a norm on  $B(X, Y)$ .

*Proof.* To prove that the operator norm is a norm, we first verify that  $\|T\|_{\text{op}} = 0 \iff T = 0$ .

We denote the set of elements  $x$  in  $X$  such that  $\|x\|_X \leq 1$  as  $X'$ .

Fix  $T \in B(X, Y)$  and suppose that  $T = 0$ . Then for all  $x \in X'$ ,  $\|T(x)\|_Y = \|0\|_Y = 0$ . Thus  $\|T\|_{\text{op}} = 0$ .

Now suppose the converse, that  $\|T\|_{\text{op}} = 0$ . Then

$$\forall x \in X', \|T(x)\|_Y \leq \|T\|_{\text{op}} = 0.$$

But by the definition of the norm in  $Y$ ,

$$0 \leq \|T(x)\|_Y$$

It follows that  $T(x) = 0$ .

To show nonnegativity, we note that for  $x \in X'$ ,

$$\|T\|_{\text{op}} \geq \|T(x)\|_Y \geq 0$$

To show homogeneity, let  $T \in B(X, Y)$ ,  $c \in \mathbb{R}$ . Then

$$\|cT\|_{\text{op}} = \sup\{\|cT(x)\|_Y : x \in X'\} = \sup\{|c| \|T(x)\|_Y : x \in X'\} = |c| \sup\{\|T(x)\|_Y : x \in X'\} = |c| \|T\|_{\text{op}}$$

Now we show that the triangle inequality holds with respect to the operator norm.

Fix  $T, U \in B(X, Y)$ . Let  $x \in X'$ . By definition,

$$T(x) \leq \sup T(X') \text{ and } U(x) \leq \sup U(X')$$

Adding both together obtains

$$T(x) + U(x) \leq \sup T(X') + \sup U(X')$$

We see that  $\sup T(X') + \sup U(X')$  is an upper bound for  $T(x) + U(x)$ . By the definition of the least upper bound,

$$\sup\{T(X') + U(X')\} \leq \sup T(X') + \sup U(X') \implies \|T + U\|_{\text{op}} \leq \|T\|_{\text{op}} + \|U\|_{\text{op}}$$

Thus the operator norm is, indeed, a norm. □

- (c) Let  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the linear mapping given by  $T(x, y) = (x + y, x)$ . Find, with proof, the exact value of  $\|T\|_{\text{op}}$ . (Here,  $\mathbf{R}^2$  is equipped with the usual norm.)

**Claim.**  $\|T\|_{\text{op}} = \sqrt{\frac{3+\sqrt{5}}{2}}$

*Proof.* We will show that  $T(x, y)$  is bounded above by this value, and that equality is possible.

Let  $(x, y) \in \mathbb{R}^2$  such that  $\sqrt{x^2 + y^2} \leq 1 \implies y \leq \pm\sqrt{1 - x^2} \leq \sqrt{1 - x^2}$ . For such  $(x, y)$ ,

$$\|T(x, y)\|_2 = \|(x + y, x)\|_2 = \sqrt{(x + y)^2 + x^2} = \sqrt{2x^2 + 2xy + y^2}$$

By the monotonicity of the squareroot,

$$\sqrt{2x^2 + 2xy + y^2} \leq \sqrt{x^2 + 2x\sqrt{1 - x^2} + 1}$$

We attempt to maximize this function for  $x \in [0, 1]$  (The interval is the set of all  $x$  that satisfy the constraint  $x^2 + y^2 \leq 1$ ). Maximizing this function is synonymous to maximizing  $f(x) = x^2 + 2x\sqrt{1 - x^2} + 1$  on the interval  $[0, 1]$ . Taking the derivative, we get

$$f'(x) = 2x + 2\sqrt{1 - x^2} - \frac{2x^2}{\sqrt{1 - x^2}} = \frac{2x\sqrt{1 - x^2} + 2 - 4x^2}{\sqrt{1 - x^2}}$$

Now, we find every critical point. When  $f'$  is undefined,  $x = 1$ . Now, let  $f'(x) = 0$ ,  $x \neq 1$ . Then through a series of calculations I really don't want to type out,

$$\frac{2x\sqrt{1 - x^2} + 2 - 4x^2}{\sqrt{1 - x^2}} = 0 \implies 5x^4 - 5x^2 + 1 = 0 \implies x^2 = \frac{1}{2} \pm \frac{1}{2\sqrt{5}} \implies x = \sqrt{\frac{1}{2} \pm \frac{1}{2\sqrt{5}}}$$

We disregard the negative solution since we want  $x \in [0, 1]$ . Now we evaluate  $f$  at the endpoints, as well as at every point we found:

$$f(0) = 1$$

$$f(1) = 2$$

$$f\left(\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}\right) = \frac{3 + \sqrt{5}}{2}$$

$$f\left(\sqrt{\frac{1}{2} - \frac{1}{2\sqrt{5}}}\right) = \frac{3}{2} + \frac{3}{2\sqrt{5}}$$

It is not too hard to see that  $f$  achieves the maximum at  $x = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}$ . Then  $\sqrt{x^2 + 2x\sqrt{1 - x^2} + 1}$  also achieves a maximum at  $x = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}$ , which is  $\sqrt{\frac{3+\sqrt{5}}{2}}$ .

In summary, we have for all  $(x, y) \in \mathbb{R}^2$ ,

$$\|T(x, y)\|_2 \leq \sqrt{\frac{3 + \sqrt{5}}{2}}$$

Thus  $\sqrt{\frac{3+\sqrt{5}}{2}}$  is an upper bound for  $\|T(x, y)\|_2$ .

To show that  $\sqrt{\frac{3+\sqrt{5}}{2}}$  is the least upper bound, it suffices to show that  $\|T(x, y)\|_2$  can achieve that value. Indeed, if we let  $x = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)}$ ,  $y = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)}$  we see that

$$\|T(x, y)\| = \sqrt{\left(\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)} + \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)}\right)^2 + \left(\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)}\right)^2}$$

$$= \sqrt{2 \left( \frac{1}{2} + \frac{1}{2\sqrt{5}} \right) + 2 \sqrt{\left( \frac{1}{2} + \frac{1}{2\sqrt{5}} \right) \sqrt{\left( \frac{1}{2} - \frac{1}{2\sqrt{5}} \right) + \left( \frac{1}{2} - \frac{1}{2\sqrt{5}} \right)}}} = \sqrt{\frac{3 + \sqrt{5}}{2}}$$

Thus  $\|T\|_{op} = \sup\{\|T(x, y)\|_2 : \|(x, y)\|_2 \leq 1\} = \sqrt{\frac{3 + \sqrt{5}}{2}}$

□

(d) Find, with proof, an example of an unbounded linear operator.

*Proof.* Define  $\ell^0$  to be the set of all sequences that are eventually 0. Consider the metric spaces  $(\ell^0, \|\cdot\|_{\ell^\infty})$  and  $(C[0, 1], \|\cdot\|_{C^\infty})$ . Here, we denote  $\|\cdot\|_{\ell^\infty}$  as the sup norm on  $\ell^\infty$  and  $\|\cdot\|_{C^\infty}$  as the sup norm on  $C[0, 1]$ .

Let  $T : \ell^0 \rightarrow C[0, 1]$  be defined by

$$T((a_n)_n) = \sum_{i=0}^k a_i i^x, \text{ where } k \text{ is the last index where } a_k \neq 0$$

First, we will show that  $T$  is a linear transformation. Fix  $(a_n)_n, (b_n)_n \in \ell^0, c \in \mathbb{R}$ . Let  $k = \max\{k_a, k_b\}$ , where  $k_a, k_b$  are the last index where  $a_{k_a}$  and  $b_{k_b}$  are non-zero, respectively. Then

$$T(c(a_n) + (b_n)) = \sum_{i=0}^k (ca_i + b_i) i^x = c \sum_{i=0}^k a_i i^x + \sum_{i=0}^k b_i i^x = c \sum_{i=0}^{k_a} a_i i^x + \sum_{i=0}^{k_b} b_i i^x = cT((a_n)) + T((b_n))$$

This verifies that  $T$  is a linear transformation.

Now we show that  $T$  is unbounded. Fix  $M \geq 0$ . Let  $(a_n)_n \in \ell^0$  such that  $a_i = 1$  if  $i = M + 1$  and 0 otherwise. We have that

$$\|T((a_n))\|_{C^\infty} = \|(M + 1)^x\|_{C^\infty} = M + 1 > M = M \| (a_n) \|_{\ell^\infty}$$

Thus  $T$  is an unbounded linear operator.

□