3 Week 3 Homework

3.1 Exercise. Let $S \subseteq C[0,1]$. Consider the following two statements:

- S is an open subset of $(C[0,1], \|\cdot\|_1)$.
- S is an open subset of $(C[0,1], \|\cdot\|_\infty)$.

Determine if the first statement implies the second, and vice-versa. Supply proof or counterexample as appropriate.

Proof. We claim that the first statement implies the second, but not the converse.

Suppose that S is an open subset of $(C[0,1], \|\cdot\|_1)$. For any $g \in S$, there is an open ball with respect to the 1-norm centered around g with radius ε such that $B_1(g,\varepsilon) \subseteq S$. We proceed to show that $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon)$. Let $f \in B_\infty(g,\varepsilon)$. Then

$$\|f-g\|_1 = \int_0^1 |f-g| \leq \int_0^1 \sup\{|f-g|\} = \|f-g\|_\infty < \varepsilon$$

Thus $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon) \subseteq S$. Thus S is an open subset of $(C[0,1], \|\cdot\|_1)$.

Now we show that the converse is not necessarily true. Let $S = B_\infty(0,1)$. This is an open subset of $(C[0,1], \|\cdot\|_\infty)$. Consider $f(x) = 0 \in B_\infty(0,1)$. For every $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Let $g(x) = x^{n-1}$. Since $\int_0^1 g(x)dx = \frac{1}{n} < \varepsilon$, $g(x) \in B_1(0,\varepsilon)$. But $g(1) = 1$, which means that $g(x) \notin B_\infty(0,1)$. Thus f is not an interior point of $B_\infty(0,1)$ with respect to the 1-norm, which means that $B_\infty(0,1)$ is not an open subset of $(C[0,1], \|\cdot\|_1)$.

$\square$

3.2 Exercise. *Can linear subspaces be open and/or closed?*

(a) Let $C^\infty[0,1]$ denote the set of infinitely differentiably functions $f : [0,1] \to \mathbf{R}$. Prove that $C^\infty[0,1]$ is *not* a closed subset of $(C[0,1], \|\cdot\|_\infty)$.

Note: Do not use MAT159 sledgehammers to solve this problem. If you're not sure if your approach involves a sledgehammer, consult Ehsaan first.

(b) Let C be the set of **convergent** sequences of real numbers. Prove that C is a closed subset of $(\ell^\infty, \|\cdot\|_\infty)$.

(c) Let $(X, \|\cdot\|)$ be a normed vector space, and let M be a linear subspace of X. Prove that M is an open set if and only if $M = X$.

3.3 Exercise. *The Bolzano–Weierstrass Theorem.*

(a) Prove that every bounded sequence in $(\mathbf{R}^d, \|\cdot\|_2)$ has a convergent subsequence.

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ containing a sequence (x_n) which has no convergent subsequences.