Question 1. Define the predicates

$$P(n)\colon \text{ For any set } A, \text{ if } |A| = n \text{ then } |\mathcal{P}(A)| = 2^n$$

$$Q(A,n)\colon |A| = n \Longrightarrow |\mathcal{P}(A)| = 2^n$$

a) Prove $\forall n \in \mathbb{N}, P(n)$.

Proof. **Base Case.** To show $P(0)$, consider any set A such that $|A| = 0$. Then $A = \varnothing$ and its only subset is $\varnothing$. Thus $|\mathcal{P}(A)| = 1 = 2^0$, verifying that $P(0)$ is true.

Induction Step. Suppose $P(k)$ holds for some $k \in \mathbb{N}$. Now $P(k+1)$ will be proven to hold. Let A be a set such that $|A| = k+1$. $k+1$ is at least 1, so A possesses at least one element, which will be denoted as a.

Consider the set $A \setminus \{a\}$. Since $\big|A \setminus \{a\}\big| = k$, by the induction hypothesis,

$$\big|\mathcal{P}(A \setminus \{a\})\big| = 2^k$$

Notice that $\mathcal{P}(A \setminus \{a\})$ contains all the subsets of A that do not contain a. The remaining subsets must all contain a. The remaining subsets of A can be obtained by taking every individual element in $\mathcal{P}(A \setminus \{a\})$ and unioning it with $\{a\}$. Thus A contains twice as many subsets as $A \setminus \{a\}$. In mathematical terms,

$$|\mathcal{P}(A)| = 2 \cdot |\mathcal{P}(A \setminus \{a\})| = 2 \cdot 2^k = 2^{k+1}$$

It has been shown that $P(k+1)$ holds.

By the principle of simple induction, $\forall n \in \mathbb{N}, P(n)$.

$\square$

b) Prove that for every set A, $\forall n \in \mathbb{N}$, $Q(n)$. This method does not work. Here is the attempt at the proof:

Proof. Fix a set A. Proceed with using simple induction.

Base Case. Let $n = 0$. To show $Q(A,n)$ holds, suppose that $|A| = 0$. Then $A = \varnothing$. Thus $\mathcal{P}(A) = \{\varnothing\} \implies |\mathcal{P}(A)| = 1 = 2^0$.

Thus $Q(A, 0)$.

Induction Step. Suppose that $Q(A,k)$ holds for some $k \in \mathbb{N}$. To show $Q(A, k+1)$, suppose $|A| = k+1$. However, this is where the problem arises.

The induction hypothesis cannot be utilised since our assumption requires $|A| = k+1$, while the condition to use the induction hypothesis is $|A| = k$.

Thus the proof by induction cannot be continued.

$\square$

Question 2. Let $n, m \in \mathbb{N}$. Let A, B be arbitrary finite sets of size m and n respectively.

a) How many fuctions are there with domain A and co-domain B? It can be shown using simple induction on m that the answer to this question is n^m.

Proof. For $n, m \in \mathbb{N}$, define the predicate

$P(n, m)$: There are n^m functions with finite domain of size m and finite co-domain of size n

Fix $m \in \mathbb{N}$.

If $m = 0$,

Base Case. Let $m = 0$. There are no functions that can map to nothing, therefore the number of functions is $0 = 0^m$.

Induction Step. Suppose that $P(n, k)$ holds for every $n \in \mathbb{N}$, but only for some $k \in \mathbb{N}$. Let A, B be finite sets such that $|A| = k + 1$ and $|B| = n$. A contains at least one element a. Consider the set $A \setminus \{a\}$.

$\square$