4 Homework 4

Question 11. Let (X, d) be a metric space. A function $f : X \to X$ is called a **contraction mapping** if there exists a constant $M \in (0, 1)$ such that

$$d(f(x), f(y)) \leq Md(x, y) \quad \text{for all } x, y \in X.$$

(a) Suppose that (X, d) is a complete metric space, and that $f : X \to X$ is a contraction mapping. Prove that f has a unique fixed point; *i.e.* there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Let (X, d) be a complete metric space and f be a contraction mapping. In this proof, for $n \in \mathbb{N}$, we denote f^n to be a composition of f. First, we will prove a lemma:

Lemma. $\forall x \in X, k \in \mathbb{N}, d(x, f^n(x)) < C$, where C is a real constant.

To prove this, we will use an induction argument on k.

Let $k = 1$.

Now suppose that the claim holds true for $k = l$, where $l \in \mathbb{N}$. Then by the triangle inequality,

$$d(x, f^{l+1}(x)) \leq d(x, f^l(x)) + d(f^l(x), f^{l+1}(x))$$

□

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ and a contraction mapping $f : X \to X$ such that f does **not** have a fixed point.