1 MAT301 Notes: Week 2 (Fall 2024)

1.1 Example. Let G be a group. Prove that

$$C_G(a) = \forall a \in G \iff G \text{ is Abelian}$$

Note that the centralizer $C_G(a)$ is defined by $C_G(a) = \{g \in G : ga = ag\}$

Proof. We begin by proving the $\impliedby$ direction.

Let $a \in G$. Then, since G is Abelian, every element commutes with a. Therefore

$$C_G(a) = G$$

Conversely, let $x, y \in G$. Then we have

$$x \in C_G(x) = G = C_G(y) \implies xy = yx$$

□

1.2 Example. Let G be a group. Prove that G is Abelian if and only if $(xy)^2 = x^2y^2, x, y \in G$.

Proof. Suppose G is Abelian. Then

$$(xy)^2 = (xy)(xy) = (xx)(yy) = x^2y^2$$

Next suppose $(xy)^2 = x^2y^2$. Then

$$xxyy = x^2y^2 = (xy)^2 = xyxy$$

Applying x^{-1} to the left and y^{-1} to the right obtains

$$x^{-1}xxyyy^{-1} = x^{-1}xyxyy^{-1} \implies xy = yx$$

□

1.3 Exercise. Let G be a group. Let $a, b \in G$, where a has odd order. Prove that if $a = bab$, then $b = b^{-1}$.

Proof. Suppose that $a = bab$. Then if we apply ab to the right on both sides, we have

$$a(ab) = bab(ab) \implies a^2b = ba^2 \tag{1}$$

Since a is odd order, $a^{2n+1} = e \implies a = a^{2n+2}$, where $n \in \mathbb{N}$. Thus

$$a = bab = ba^{2n+2}b = b(a^2)^{n+1}b$$

We apply (1) $n+1$ times to obtain that

$$a = b(a^2)^{n+1}b = b^2(a^2)^{n+1} = b^2a^{2n+2} = b^2a \implies e = b^2 \implies b = b^{-1}$$

We are done.

□