4 Week 4

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Exercise 4.10. Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$ be a function. Prove that if f is continuous, then $f^{-1}(U)$ is an open subset of X if U is an open subset of Y.

Proof. Suppose that f is continuous. Let U be an open subset of Y. We will now show that $f^{-1}(U)$ is an open subset of X. Let $x_0 \in f^{-1}(U)$. It follows that $f(x_0) \in U$. We need to find an open ball centered around x_0 such that it is a subset of $f^{-1}(U)$.

Since U is open, there exists an $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq U$. By the continuity of f, there is a $\delta > 0$ such that $d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon$.

Take the open ball $B_X(x_0, \delta)$ in X. We want to show that $B_X(x_0, \delta) \subseteq f^{-1}(U)$.

Let $x \in B_X(x_0, \delta)$. Then $d_X(x_0, x) < \delta$, which implies that $d_Y(f(x_0), f(x)) < \varepsilon$ from continuity. It follows that $f(x) \in B_Y(f(x_0), \varepsilon) \subseteq U$. By definition, this means that $x \in f^{-1}(U)$. Thus $B_X(x_0, \delta) \subseteq f^{-1}(U)$, which means that $f^{-1}(U)$ is an open subset of X.

$\square$