1 Week 2 Homework - Ethan Hua

1.1 Exercise. (a) Prove that there exists an infinitely differentiable function $\alpha : \mathbf{R} \to \mathbf{R}$ such that $\alpha(t) = 0$ for all $t \leq 0$, and $\alpha(t) > 0$ for all $t > 0$.

Proof. We define $\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$

Trivially $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) > 0$ if $t > 0$. It remains to show that α is infinitely differentiable.

Since 0 is infinitely differentiable, and $e^{-\frac{1}{x}}$ is infinitely differentiable for $x > 0$, it suffices to show that derivatives of all orders of α are continuous at $t = 0$.

We will continue by proving a lemma.

Lemma. $\forall n \in \mathbb{N}$, for $t > 0$, $\alpha^{(n)}(t) = Q(t)\alpha(t)$, where $Q(x)$ is a linear combination of nonpositive integer powers of t.

We will show this using induction.

Proof. When $n = 1$, fixing $t > 0$, we have

$$\alpha'(t) = \frac{1}{t^2}e^{\frac{-1}{t}} = \frac{1}{t^2}\alpha(t)$$

We let $Q(t) = t^{-2}$ and we are done.

Now suppose that the lemma holds for all $i \leq k$, for some $k \in \mathbb{N}$.

Then for $t > 0$,

$$\alpha^{(k)}(t) = P(t)\alpha(t). \text{ where } P(t) \text{ is a linear combination of nonpositive integer powers of t}$$

Taking the derivative of both sides with respect to t, we obtain

$$\alpha^{(k+1)}(t) = P'(t)\alpha(t) + P(t)\alpha'(t)$$

The lemma holds for $i = 1$, therefore for some $R(t)$,

$$P'(t)\alpha(t) + P(t)\alpha'(t) = P'(t)\alpha(t) + P(t)R(t)\alpha(t) = (P'(t) + P(t)R(t))\alpha(t)$$

$P'(t) + P(t)R(t)$ is a linear conbination of nonpositive integer powers of t, therefore letting $Q(t) = P'(t) + P(t)R(t)$, we get our desired conclusion.

By the principle of induction, the lemma holds true for all $n \in \mathbb{N}$.

□

We now continue in proving that derivatives of all orders of α are continuous at 0. We show this by proving that

$$\lim_{t\to 0} \alpha^{(n)}(t) = 0, \text{ where } n \in \mathbb{N}$$

We will only worry about the right hand limit, as the left hand limit always evaluates to 0.

Let $n \in \mathbb{N}$. Since we only deal with positive t, by our lemma,

$$\alpha^{(n)}(t) = Q(t)\alpha(t)$$

Where $Q(t)$ is a linear combination of nonpositive integer powers of t. So

$$\lim_{t\to 0^+} \alpha^{(n)}(t) = \lim_{t\to 0^+} Q(t)\alpha(t) = \lim_{t\to 0^+} \sum_{i=0}^{k} a_i t^{-i}\alpha(t)$$

Where a_i are real constants and $k \in \mathbb{N}$.

Consider an arbitrary term $a_i t^{-i} \alpha(t) = a_i t^{-i} e^{\frac{-1}{t}}$. We want to show that $\lim_{t \to 0^+} a_i t^{-i} e^{\frac{-1}{t}}$ exists and is equal to 0.

First, we will perform the substitution $x = \frac{1}{t}$. Then the limit becomes

$$\lim_{x \to \infty} a_i x^i e^{-x} = 0$$

The proof for this fact is omitted, but applying L'Hopital's rule i times produces the same result. Thus,

$$\lim_{t \to 0^+} \alpha^{(n)}(t) = \lim_{t \to 0^+} \sum_{i=0}^{k} a_i t^{-i} \alpha(t) = \sum_{i=0}^{k} \lim_{t \to 0^+} a_i t^{-i} \alpha(t) = \sum_{i=0}^{k} 0 = 0$$

Thus α is infinitely differentiable everywhere.

$\square$

(b) Prove that there exists an infinitely differentiable function $\beta : \mathbf{R} \to \mathbf{R}$ such that $\beta(t) = 1$ for all $t \geq 1$, and $\beta(t) = 0$ for all $t \leq 0$.

Proof. Define

$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}$$

If $t \geq 1$, we also have $1 - t \leq 0$. Then

$$\beta(t) = \frac{\alpha(t)}{\alpha(t)} = 0$$

As well, if $t \leq 0$, we have

$$\beta(t) = 0$$

Since $\alpha(t)$ is infinitely differentiable and β is composed of α, it follows that β is also infinitely differentiable.

$\square$

(c) Prove that there exists an infinitely differentiable function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(t) = 1$ for all $t \in [2, 3]$, and $\varphi(t) = 0$ for $t \in \mathbf{R} \setminus (1, 4)$.

Proof. Define $\phi(t) = \beta(t-1)\beta(4-t)$. Since β is infinitely differentiable, ϕ is as well. For $t \in [2, 3]$, $t - 1 \geq 1$ and $4 - t \geq 1$. Thus

$$\phi(t) = \beta(t-1)\beta(4-t) = 1$$

If $t \in \mathbb{R} \setminus (1, 4)$, then $t - 1 \leq 0$ or $4 - t \leq 0$. In each case $\beta(t-1) = 0$ or $\beta(4-t) = 0$, respectively, thus we have

$$\phi(t) = \beta(t-1)\beta(4-t) = 0$$

$\square$

1.2 Exercise. Let $S \subseteq \mathbf{R}^n$. Consider the following three statements:

- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_1)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_2)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_{\max})$.

Among these statements, determine which implications are true and which are false. There are six implications to investigate. Supply proof or counterexample as appropriate. Include pictures.

Claim. Denote each statement as (1), (2), and (3), respectively. We claim that each statement implies all the other statements.

Proof. First of all, I am so sorry that you have to read this. Secondly, in this proof, we denote:

1. an open ball with respect to the 1-norm as $B_1(p, \varepsilon)$
2. an open ball with respect to the Euclidean norm as $B_2(p, \varepsilon)$
3. an open ball with respect to the max-norm as $B_{\max}(p, \varepsilon)$

(1) $\implies$ (2):

Suppose (1) is true. There exists a $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and $r > 0$ such that

$$S \subseteq B_1(p, r)$$

We will show that $S \subseteq B_2(p, r)$. Let $x = (x_1, x_2, \ldots, x_n) \in S$. Since $S \subseteq B_1(p, r)$, by the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - p_i)^2} \leq \sum_{i=0}^{n} |x_i - p_i| = \|x - p\|_1 < r$$

Thus $x \in B_2(p, r)$.

(2) $\implies$ (1):

We will show using induction on n that if for some open ball in $\mathbb{R}^n$, $S \subseteq B_2(p, r)$, then $S \subseteq B_1(p, \sqrt{n}r)$ Let $n = 1$. Suppose (2) is true. Fix $x \in S \subseteq \mathbb{R}$. Then by (2),

$$r > \|x - p\|_2 = \sqrt{(x - p)^2} = |x - p| = \|x - p\|_1 \implies S \subseteq B_1(p, \sqrt{1} \cdot r)$$

Thus the claim holds for $n = 1$. Now suppose the claim holds for $n = k$. Assume (2) is true. Let $x \in S$. By (2),

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{k+1} (x_i - p_i)^2} < r \implies \sum_{i=1}^{k+1} (x_i - p_i)^2 < r^2 \tag{*}$$

We want to show that $x \in B_1(p, \sqrt{n}r)$, or

$$\|x - p\|_1 = \sum_{i=0}^{k+1} |x_i - p_i| < \sqrt{n}r \iff \left(\sum_{i=0}^{k+1} |x_i - p_i| \right)^2 < nr^2$$

We have

$$\left(\sum_{i=0}^{k+1} |x_i - p_i| \right)^2 = \left(\sum_{i=0}^{k} |x_i - p_i| + |x_{k+1} - p_{k+1}| \right)^2$$

$$= \left(\sum_{i=0}^{k} |x_i - p_i| \right)^2 + 2 |x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i| \right) + |x_{k+1} - p_{k+1}|^2 \tag{**}$$

Manipulating the inequality from (*), we obtain

$$\sum_{i=1}^{k}(x_i - p_i)^2 < r^2 - |x_{k+1} - p_{k+1}|^2$$

which implies that $(x_1, x_2, \ldots, x_k)$ is in the open ball $B_2(p, r)$ on $\mathbb{R}^k$. From the indudction hypothesis,

$$\begin{aligned}
(**) &< k(r^2 - |x_{k+1} - p_{k+1}|^2) + 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + r^2 - \sum_{i=1}^{k}(x_i - p_i)^2 \\
&= (k+1)r^2 - \left(k\,|x_{k+1} - p_{k+1}|^2 - 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + \sum_{i=1}^{k}(x_i - p_i)^2\right) \\
&= (k+1)r^2 - \sum_{i=1}^{k}\left(|x_{k+1} - p_{k+1}|^2 - 2\,|x_{k+1} - p_{k+1}|\,|x_i - p_i| + (x_i - p_i)^2\right) \\
&= (k+1)r^2 - \sum_{i=1}^{k}\left(|x_{k+1} - p_{k+1}| - |x_i - p_i|\right)^2 \leq (k+1)r^2
\end{aligned}$$

In summary, we have that

$$\|x - p\|_1 < \sqrt{k+1}r$$

Thus x is in the open ball $B_1(p, \sqrt{k+1}r)$, which means that (1) is true. By the principle of induction, the claim holds for all $n \in \mathbb{N}$.

(1) $\implies$ (3): Suppose (1). Then S is a subset of some open ball $B_1(p, r)$. We will show that $S \subseteq B_{max}(p, r)$.

Let $x \in S$. Then by (1):

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n}\{|x_i - p_i|\} \leq \sum_{i=1}^{n} |x_i - p_i| = \|x - p\|_1 < r$$

Thus $x \in B_{\max}(p, r)$, so (3) is true.

(3) $\implies$ (1):

Suppose that (3) holds true. Then for some open ball $B_{\max}(p, r)$. We will show that $S \subseteq B_1(p, nr)$.

Let $x \in S$. We have

$$\|x - p\|_1 = \sum_{i=1}^{n} |x_i - p_i| \leq n \max_{0 \leq i \leq n} |x_i - p_i| = n\,\|x - p\|_{\max} < nr$$

Thus $x \in B_1(p, nr)$, which implies that (1) is true.

(2) $\implies$ (3):

Suppose (2) is true. Then $S \subseteq B_2(p, r)$, where $B_2(p, r)$ is some open ball on the Euclidean norm. We want to show that $S \subseteq B_{\max}(p, r)$. Indeed, we have

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n} |x_i - p_i| \leq \sqrt{\sum_{i=1}^{n} |x_i - p_i|^2} < r$$

Thus $S \subseteq B_{\max}(p, r)$, so (3) is true.

(3) $\implies$ (2):

Suppose that (3) holds true. Then for some open ball with the max-norm, $S \subseteq B_{\max}(p, r)$. We will show that $S \subseteq B_2(p, nr)$

Let $x \in S$. Then be the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{n} |a_i - p_i|} \leq \sum_{i=1}^{n} |a_i - p_i| \leq n \max_{1 \leq i \leq n} |x_i - p_i| < n\,\|x - p\|_{\max} < nr$$

Thus $x \in B_2(p, nr)$, so (2) holds.
Therefore, we have proven every implication to be true.

$\square$

1.3 Exercise. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. A linear mapping $T : X \to Y$ is called **bounded** if there exists a constant $M \geq 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Let $B(X,Y)$ denote the set of these bounded linear operators. The **operator norm** on $B(X,Y)$, denoted by $\|\cdot\|_{\text{op}}$, is defined as follows:

$$\|T\|_{\text{op}} = \sup\{\|T(x)\|_Y : x \in X \text{ and } \|x\|_X \leq 1\}.$$

(a) Prove that $B(X,Y)$ is a linear subspace of $L(X,Y)$.

Proof. The 0-transformation $\|Z(x)\| = 0 \leq \|x\|_X$, $\forall x \in X$, because of the definition of a norm. Thus $0 \in B(X,Y)$. Let $T, U \in B(X,Y)$, $c \in \mathbb{R}$. Then for all $x \in X$, there exist $M, N \geq 0$ such that

$$T(x) \leq M\|x\|_X \text{ and } U(x) \leq N\|x\|_X \implies T(x) + U(x) \leq (M+N)\|x\|_X$$

Which implies that $T+U$ is a member of $B(X,Y)$. As well, from the first inequality,

$$T(x) \leq M\|x\|_X \implies cT(x) \leq cM\|x\|_X$$

Which implies that cT is a member of $B(X,Y)$.

Since $0 \in B(X,Y)$and $B(X,Y)$ is closed under addition and scalar multiplication, $B(X,Y)$ is a subspace of $L(X,Y)$.

□

(b) Prove that $\|\cdot\|_{\text{op}}$ is a norm on $B(X,Y)$.

Proof. To prove that the operator norm is a norm, we first verify that $\|T\|_{op} = 0 \iff T = 0$.
We denote the set of elements x in X such that $\|x\|_X \leq 1$ as X'.
Fix $T \in B(X,Y)$ and suppose that $T = 0$. Then for all $x \in X'$, $\|T(x)\|_Y = \|0\|_Y = 0$. Thus $\|T\|_{op} = 0$.
Now suppose the converse, that $\|T\|_{op} = 0$. Then

$$\forall x \in X', \|T(x)\|_Y \leq \|T\|_{op} = 0.$$

But by the definition of the norm in Y,

$$0 \leq \|T(x)\|_Y$$

It follows that $T(x) = 0$.
To show nonnegativity, we note that for $x \in X'$,

$$\|T\|_{op} \geq \|T(x)\|_Y \geq 0$$

To show homogeity, let $T \in G(X,Y), c \in \mathbb{R}$. Then

$$\|cT\|_{op} = \sup\{\|cT(x)\|_Y : x \in X'\} = \sup\{|c|\,\|T(x)\|_Y : x \in X'\} = |c|\sup\{\|T(x)\|_Y : x \in X'\} = |c|\,\|T\|_{op}$$

Now we show that the triangle inequality holds with respect to the operator norm.
Fix $T, U \in B(X,Y)$. Let $x \in X'$. By definition,

$$T(x) \leq \sup T(X') \text{ and } U(x) \leq \sup U(X')$$

Adding both together obtains

$$T(x) + U(x) \leq \sup T(X') + \sup U(X')$$

We see that $\sup T(X') + \sup U(X')$ is an upper bound for $T(x) + U(x)$. By the definition of the least upper bound,

$$\sup\{T(X') + U(X')\} \leq \sup T(X') + \sup U(X') \implies \|T+U\|_{op} \leq \|T\|_{op} + \|U\|_{op}$$

Thus the operator norm is, indeed, a norm.

□

(c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping given by $T(x, y) = (x + y, x)$. Find, with proof, the exact value of $\|T\|_{\text{op}}$. (Here, $\mathbf{R}^2$ is equipped with the usual norm.)

Claim. $\|T\|_{op} = \sqrt{\frac{3+\sqrt{5}}{2}}$

Proof. We will show that $T(x, y)$ is bounded above by this value, and that equality is possible.

Let $(x, y) \in \mathbb{R}^2$ such that $\sqrt{x^2 + y^2} \leq 1 \implies y \leq \pm\sqrt{1 - x^2} \leq \sqrt{1 - x^2}$. For such (x, y),

$$\|T(x, y)\|_2 = \|(x + y, x)\|_2 = \sqrt{(x + y)^2 + x^2} = \sqrt{2x^2 + 2xy + y^2}$$

By the monotonicity of the squareroot,

$$\sqrt{2x^2 + 2xy + y^2} \leq \sqrt{x^2 + 2x\sqrt{1 - x^2} + 1}$$

We attempt to maximize this function for $x \in [0, 1]$ (The interval is the set of all x that satisfy the constraint $x^2 + y^2 \leq 1$). Maximizing this function is synonymous to maximizing $f(x) = x^2 + 2x\sqrt{1 - x^2} + 1$ on the interval $[0, 1]$. Taking the derivative, we get

$$f'(x) = 2x + 2\sqrt{1 - x^2} - \frac{2x^2}{\sqrt{1 - x^2}} = \frac{2x\sqrt{1 - x^2} + 2 - 4x^2}{\sqrt{1 - x^2}}$$

Now, we find every critical point. When f' is undefined, $x = 1$. Now, let $f'(x) = 0,\ x \neq 1$. Then through a series of calculations I really don't want to type out,

$$\frac{2x\sqrt{1 - x^2} + 2 - 4x^2}{\sqrt{1 - x^2}} = 0 \implies 5x^4 - 5x^2 + 1 = 0 \implies x^2 = \frac{1}{2} \pm \frac{1}{2\sqrt{5}} \implies x = \sqrt{\frac{1}{2} \pm \frac{1}{2\sqrt{5}}}$$

We disregard the negative solution since we want $x \in [0, 1]$. Now we evaluate f at the endpoints, as well as at every point we found:

$$f(0) = 1$$
$$f(1) = 2$$
$$f\left(\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}\right) = \frac{3 + \sqrt{5}}{2}$$
$$f\left(\sqrt{\frac{1}{2} - \frac{1}{2\sqrt{5}}}\right) = \frac{3}{2} + \frac{3}{2\sqrt{5}}$$

It is not too hard to see that f acheives the maximum at $x = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}$. Then $\sqrt{x^2 + 2x\sqrt{1 - x^2} + 1}$ also acheives a maximum at $x = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{5}}}$, which is $\sqrt{\frac{3+\sqrt{5}}{2}}$.

In summary, we have for all $(x, y) \in \mathbb{R}^2$,

$$\|T(x, y)\|_2 \leq \sqrt{\frac{3 + \sqrt{5}}{2}}$$

Thus $\sqrt{\frac{3+\sqrt{5}}{2}}$ is an upper bound for $\|T(x, y)\|_2$.

To show that $\sqrt{\frac{3+\sqrt{5}}{2}}$ is the least upper bound, it suffices to show that $\|T(x, y)\|_2$ can acheive that value. Indeed, if we let $x = \sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)}, y = \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)}$ we see that

$$\|T(x, y)\| = \sqrt{\left(\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)} + \sqrt{\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)}\right)^2 + \left(\sqrt{\left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)}\right)^2}$$

$$= \sqrt{2\left(\frac{1}{2}+\frac{1}{2\sqrt{5}}\right)+2\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{5}}\right)}\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{5}}\right)}+\left(\frac{1}{2}-\frac{1}{2\sqrt{5}}\right)} = \sqrt{\frac{3+\sqrt{5}}{2}}$$

Thus $\|T\|_{op} = \sup\{\|T(x,y)\|_2 : \|(x,y)\|_2 \le 1\} = \sqrt{\frac{3+\sqrt{5}}{2}}$

□

(d) Find, with proof, an example of an unbounded linear operator.

Proof. Define ℓ^0 to be the set of all sequences that are eventually 0. Consider the metric spaces $(\ell^0, \|\cdot\|_{\ell\infty})$ and $(C[0,1], \|\cdot\|_{C\infty})$. Here, we denote $\|\cdot\|_{\ell\infty}$ as the sup norm on ℓ^∞ and $\|\cdot\|_{C\infty}$ as the sup norm on $C[0,1]$..

Let $T : \ell^0 \to C[0,1]$ be defined by

$$T((a_n)_n) = \sum_{i=0}^{k} a_i i^x, \text{ where } k \text{ is the last index where } a_k \neq 0$$

First, we will show that T is a linear transformation. Fix $(a_n)_n, (b_n)_n \in \ell^0, c \in \mathbb{R}$. Let $k = \max\{k_a, k_b\}$, where k_a, k_b are the last index where a_{k_a} and b_{k_b} are non-zero, respectively. Then

$$T(c(a_n)+(b_n)) = \sum_{i=0}^{k}(ca_i+b_i)i^x = c\sum_{i=0}^{k} a_i i^x + \sum_{i=0}^{k} b_i i^x = c\sum_{i=0}^{k_a} a_i i^x + \sum_{i=0}^{k_b} b_i i^x = cT((a_n)) + T((b_n))$$

This verifies that T is a linear transformation.

Now we show that T is unbounded. Fix $M \geq 0$. Let $(a_n)_n \in \ell^0$ such that $a_i = 1$ if $i = M+1$ and 0 otherwise. We have that

$$\|T((a_n))\|_{C\infty} = \|(M+1)^x\|_{C\infty} = M+1 > M = M\,\|(a_n)\|_{\ell\infty}$$

Thus T is an unbounded linear operator.

□