5 Homework 5

13. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed vector spaces and let $T : X \to Y$ be a linear mapping.

(a) Prove that T is continuous if and only if T is a bounded linear mapping.

Proof. Suppose that T is continuous. Then at $x_0 = 0$, there exists a $\delta > 0$ such that

$$\|x\|_X < \delta \implies \|T(x)\|_2 \leq 1$$

We claim that our $M = \frac{2}{\delta}$.Let $x \in X$. Notice that $\left\| \dfrac{\delta \cdot x}{2\|x\|_X} \right\|_X <= \delta$. By the continuity of T we have that

$$\left\| T\left(\frac{\delta \cdot x}{2\|x\|_X}\right) \right\|_Y \leq 1 \implies \frac{\delta}{2 \cdot \|x\|_X} \|T(x)\|_Y \leq 1 \implies \|T(x)\|_Y \leq \frac{2}{\delta}\|x\|_X$$

Thus T is a bounded linear mapping.

Next, suppose that T is a bounded linear mapping. Then there is an $M > 0$ such that for all $x \in X$,

$$\|T(x)\|_Y \leq M\|x\|_X$$

We will show that T is continuous everywhere. Fix $a \in X$. Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{M}$. Let $x \in X$ and suppose that $\|x - a\|_X < \delta$. Then

$$\|T(x) - T(a)\|_Y = \|T(x - a)\|_Y \leq M\|x - a\|_X < \varepsilon$$

and we are done. $\square$

(b) Suppose that $(Y, \|\cdot\|_Y) = (\mathbf{R}^n, \|\cdot\|_2)$. Prove that T is continuous if and only if $\ker(T)$ is closed.

Proof. Suppose that T is continuous. Let a be a limit point for $\ker(T)$. We want to show that $T(a) = 0$. By definition, T being continuous implies that

$$\lim_{x \to a} T(x) = T(a)$$

or more formally, letting ε be arbitrary, there exists a δ such that

$$\|x - a\|_X < \delta \implies \|T(x) - T(a)\|_2$$

Since a is a limit point of $\ker(T)$, there exists $x \in \ker(T)$ such that

$$x \in B(a, \delta) \implies \|x - a\|_X \implies \|T(x) - T(a)\|_2 < \varepsilon \implies \|T(a)\|_2 < \varepsilon$$

By properties of norms $\|T(a)\|_2 \geq 0$, so we must have that $T(a) = 0 \implies a \in \ker(T)$. Thus $\ker(T)$ is closed.

Conversely, suppose that $\ker(T)$ is closed. Assume for contradiction that T is continuous. Then T is not a bounded operator.

For $n \in \mathbb{N}$, there exists an $x_n \in X$ so that $\|T(x_n)\|_2 \geq n\|x\|_X$.

Let $x_0 \notin \ker(T)$. Define a sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n = x_0 - \dfrac{\|T(x_0)\|_2}{\|T(x_n)\|_2} x_n$. Notice that

$$\|T(a_n)\| = \left\| T\left(x_0 - \frac{\|T(x_0)\|_2}{\|T(x_n)\|_2} x_n\right) \right\| = \left\| T(x_0) - \frac{\|T(x_0)\|_2}{\|T(x_n)\|_2} T(x_n) \right\| \leq \|T(x_0)\|_2 +$$

$\square$

17. *The magic number lemma.*

Let (X, d) be a metric space and let $\{U_i\}_{i \in I}$ be an open cover of X; this means that each U_i is an open subset of X, and that $X = \bigcup_{i \in I} U_i$. A **magic number** for $\{U_i\}_{i \in I}$ is a number $\delta > 0$ with the following property: if $A \subseteq X$ is a set with $\text{diam}(A) < \delta$, then $A \subseteq U_i$ for at least one index $i \in I$.

Suppose that (X, d) is a clustering metric space. Prove that every open cover has a magic number.

Proof. Suppose that (X, d) is a clustering metric space. Suppose for the sake of contradiction that there exists an open cover $\{U_i\}_{i \in I}$ that doesnt have a magic number.

For $n \in \mathbb{N}$, there is $A_n \subseteq X$ with $\text{diam} A_n < \frac{2}{n}$ so that $A_n \nsubseteq U_i$ for all indices $i \in I$. Since $\text{diam} A_n < \frac{2}{n}$, we can cover A_n with an open ball $B(a_n, \frac{1}{n})$, where a_n is some element in X.

Define a sequence $(a_n)_{n \in \mathbb{N}}$ in X such that a_n is equal to the one above.

By the clustering property of X, (a_n) has a convergent subsequence, which will be redefined as (a_n). Denote the limit of (a_n) as p.

p is an element of X, so it is contained in some U_i in the open cover. Since U_i is open, we can find $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U_i$. As well, since (a_n) converges to p we can find infinitely many entries of the sequence within the open ball $B(p, \frac{\varepsilon}{2})$. Thus we can find a large enough n such that $n > \frac{2}{\varepsilon}$, which gives $\frac{1}{n} < \frac{\varepsilon}{2}$, and still have that a_n is $\frac{\varepsilon}{2}$-close to p.

Now, consider the open ball $B(a_n, \frac{1}{n})$. We will show that $B(a_n, \frac{1}{n}) \subseteq B(p, \varepsilon)$.

Let $x \in B(a_n, \frac{1}{n})$. Then $d(x, a_n) < \frac{1}{n} < \frac{\varepsilon}{2}$, so we have

$$d(x, p) \leq d(x, a_n) + d(a_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies x \in B(p, \varepsilon)$$

which is what we wanted.

Recall that the set A_n is covered by $B(a_n, \frac{1}{n})$. Then we have

$$A_n \subseteq B(a_n, \frac{1}{n}) \subseteq B(p, \varepsilon) \subseteq U_i$$

contradicting the fact that $A_n \nsubseteq U_i$. Thus every open cover has a magic number.

$\square$