1 Week 2 Homework - Ethan Hua

1.1 Exercise. (a) Prove that there exists an infinitely differentiable function $\alpha : \mathbf{R} \to \mathbf{R}$ such that $\alpha(t) = 0$ for all $t \leq 0$, and $\alpha(t) > 0$ for all $t > 0$.

Proof. We define $\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$

Trivially $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) > 0$ if $t > 0$. It remains to show that α is infinitely differentiable.

Since 0 is infinitely differentiable, and $e^{-\frac{1}{x}}$ is infinitely differentiable for $x > 0$, it suffices to show that derivatives of all orders of α are continuous at $t = 0$.

We will prove using induction that

$$\forall n \in \mathbb{N} \cup \{0\},\ \lim_{t \to 0} \alpha^{(n)}(t) = 0$$

We will only worry about the right hand limit, as the left hand limit always evaluates to 0.

For $n = 0$,

$$\lim_{t \to 0^+} \alpha^{(0)}(t) = \lim_{t \to 0^+} \alpha(t) = \lim_{t \to 0^+} e^{-\frac{1}{x}} = 0$$

Thus the case for $n = 0$ holds.

Now suppose that the claim holds for $n = k$, for some $k \in \mathbb{N} \cup \{0\}$. It can be shown that when $t > 0$,

$$\alpha^{(k)}(t) = \sum_{i=0}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!} = \frac{(-1)^{k+1}(k+1)!}{t^{k+1}} + \sum_{i=1}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!}$$

□

(b) Prove that there exists an infinitely differentiable function $\beta : \mathbf{R} \to \mathbf{R}$ such that $\beta(t) = 1$ for all $t \geq 1$, and $\beta(t) = 0$ for all $t \leq 0$.

Hint: The shape you're looking for is $\dfrac{X}{X+Y}$.

(c) Prove that there exists an infinitely differentiable function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(t) = 1$ for all $t \in [2, 3]$, and $\varphi(t) = 0$ for $t \in \mathbf{R} \setminus (1, 4)$.

Hint: Your function $\beta(t)$ does half the job. Make a function $\gamma(t)$ that does the other half of the job. Then multiply them together.

1.2 Exercise. Let $S \subseteq \mathbf{R}^n$. Consider the following three statements:

- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_1)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_2)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_{\max})$.

Among these statements, determine which implications are true and which are false. There are six implications to investigate. Supply proof or counterexample as appropriate. Include pictures.

Denote each statement as (1), (2), and (3), respectively. We claim that each statement implies all the other statements.

Proof. In this proof, we denote:

1. an open ball with respect to the 1-norm as $B_1(p, \varepsilon)$
2. an open ball with respect to the Euclidean norm as $B_2(p, \varepsilon)$

3. an open ball with respect to the max-norm as $B_{\max}(p, \varepsilon)$

(1) $\implies$ (2):

Suppose (1) is true. There exists a $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ and $r > 0$ such that

$$S \subseteq B_1(p, r)$$

We will show that $S \subseteq B_2(p, r)$. Let $x = (x_1, x_2, \dots, x_n) \in S$. Since $S \subseteq B_1(p, r)$, by the triangle inequality,

$$\|x-p\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - p_i)^2} \leq \sum_{i=0}^{n} |x_i - p_i| = \|x-p\|_1 < r$$

Thus $x \in B_2(p, r)$.

(2) $\implies$ (1):

We will show using induction on n that if for some open ball in $\mathbb{R}^n$, $S \subseteq B_2(p, r)$, then $S \subseteq B_1(p, \sqrt{n}r)$
Let $n = 1$. Suppose (2) is true. Fix $x \in S \subseteq \mathbb{R}$. Then by (2),

$$r > \|x-p\|_2 = \sqrt{(x-p)^2} = |x-p| = \|x-p\|_1 \implies S \subseteq B_1(p, \sqrt{1} \cdot r)$$

Thus the claim holds for $n = 1$. Now suppose the claim holds for $n = k$. Assume (2) is true. Let $x \in S$. By (2),

$$\|x-p\|_2 = \sqrt{\sum_{i=1}^{k+1}(x_i - p_i)^2} < r \implies \sum_{i=1}^{k+1}(x_i - p_i)^2 < r^2 \tag{*}$$

We want to show that $x \in B_1(p, \sqrt{n}r)$, or

$$\|x-p\|_1 = \sum_{i=0}^{k+1} |x_i - p_i| < \sqrt{n}r \iff \left(\sum_{i=0}^{k+1} |x_i - p_i|\right)^2 < nr^2$$

We have

$$\left(\sum_{i=0}^{k+1} |x_i - p_i|\right)^2 = \left(\sum_{i=0}^{k} |x_i - p_i| + |x_{k+1} - p_{k+1}|\right)^2$$

$$= \left(\sum_{i=0}^{k} |x_i - p_i|\right)^2 + 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + |x_{k+1} - p_{k+1}|^2 \tag{**}$$

Manipulating the inequality from (*), we obtain

$$\sum_{i=1}^{k}(x_i - p_i)^2 < r^2 - |x_{k+1} - p_{k+1}|^2$$

which implies that $(x_1, x_2, \dots, x_k)$ is in the open ball $B_2(p, r)$ on $\mathbb{R}^k$. From the indudction hypothesis,

$$(**) < k(r^2 - |x_{k+1} - p_{k+1}|^2) + 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + |x_{k+1} - p_{k+1}|^2$$

$\square$

1.3 Exercise. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. A linear mapping $T : X \to Y$ is called **bounded** if there exists a constant $M \geq 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Let $B(X, Y)$ denote the set of these bounded linear operators. The **operator norm** on $B(X, Y)$, denoted by $\|\cdot\|_{\text{op}}$, is defined as follows:

$$\|T\|_{\text{op}} = \sup\{\|T(x)\|_Y : x \in X \text{ and } \|x\|_X \leq 1\}.$$

(a) Prove that $B(X, Y)$ is a linear subspace of $L(X, Y)$.

(In MAT257, the term "linear subspace" means what "subspace" meant in MAT240, *i.e.* "a nonempty subset of a vector space which is closed under addition and scalar multiplication.")

(b) Prove that $\| \cdot \|_{\mathrm{op}}$ is a norm on $B(X, Y)$.

(c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping given by $T(x, y) = (x + y, x)$. Find, with proof, the exact value of $\|T\|_{\mathrm{op}}$. (Here, $\mathbf{R}^2$ is equipped with the usual norm.)

(d) Find, with proof, an example of an unbounded linear operator.