

Question 1. Define the predicates

$P(n)$: For any set A , if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$

$Q(A, n)$: $|A| = n \implies |\mathcal{P}(A)| = 2^n$

a) Prove $\forall n \in \mathbb{N}, P(n)$.

Proof. Base Case. To show $P(0)$, consider any set A such that $|A| = 0$. Then $A = \emptyset$ and its only subset is \emptyset . Thus $|\mathcal{P}(A)| = 1 = 2^0$, verifying that $P(0)$ is true.

Induction Step. Suppose $P(k)$ holds for some $k \in \mathbb{N}$. Now $P(k+1)$ will be proven to hold. Let A be a set such that $|A| = k+1$. $k+1$ is at least 1, so A possesses at least one element, which will be denoted as a .

Consider the set $A \setminus \{a\}$. Since $|A \setminus \{a\}| = k$, by the induction hypothesis,

$$|\mathcal{P}(A \setminus \{a\})| = 2^k$$

Notice that $\mathcal{P}(A \setminus \{a\})$ contains all the subsets of A that do not contain a . The remaining subsets must all contain a . The remaining subsets of A can be obtained by taking every individual element in $\mathcal{P}(A \setminus \{a\})$ and unioning it with $\{a\}$. Thus A contains twice as many subsets as $A \setminus \{a\}$. In mathematical terms,

$$|\mathcal{P}(A)| = 2 \cdot |\mathcal{P}(A \setminus \{a\})| = 2 \cdot 2^k = 2^{k+1}$$

It has been shown that $P(k+1)$ holds.

By the principle of simple induction, $\forall n \in \mathbb{N}, P(n)$.

□

b) Prove that for every set A , $\forall n \in \mathbb{N}, Q(n)$. This method does not work. Here is the attempt at the proof:

Proof. Fix a set A . Proceed with using simple induction.

Base Case. Let $n = 0$. To show $Q(A, n)$ holds, suppose that $|A| = 0$. Then $A = \emptyset$. Thus $\mathcal{P}(A) = \{\emptyset\} \implies |\mathcal{P}(A)| = 1 = 2^0$.

Thus $Q(A, 0)$.

Induction Step. Suppose that $Q(A, k)$ holds for some $k \in \mathbb{N}$. To show $Q(A, k+1)$, suppose $|A| = k+1$. However, this is where the problem arises.

The induction hypothesis cannot be utilised since our assumption requires $|A| = k+1$, while the condition to use the induction hypothesis is $|A| = k$.

Thus the proof by induction cannot be continued.

□

Question 2. Let $n, m \in \mathbb{N}$. Let A, B be arbitrary finite sets of size m and n respectively.

- a) How many functions are there with domain A and co-domain B ? It can be shown using simple induction on m that the answer to this question is n^m .

Proof. For $n, m \in \mathbb{N}$, define the predicate

$P(m): \forall n \in \mathbb{N}$, there are n^m functions with finite domain of size m and finite co-domain of size n

Fix $m \in \mathbb{N}$.

Base Case. Let $m = 0$. There are no functions that can map to nothing, therefore the number of functions is $0^0 = 1$.

Induction Step. Suppose that $P(n, k)$ holds for every $n \in \mathbb{N}$, but only for some $k \in \mathbb{N}$. Let A, B be finite sets such that $|A| = k + 1$ and $|B| = n$. A contains at least one element a . Consider the set $A \setminus \{a\}$. By the induction hypothesis, there are n^k functions with domain $A \setminus \{a\}$ and co-domain B . For every such function f_k and some $b \in B$, define a new function

$$f_b(x) = \begin{cases} f_k(x), & \text{if } x \in A \setminus \{a\}; \\ b, & \text{if } x = a; \end{cases}$$

Every function that maps elements from A to B can be written in this form. Thus there are $n^k * n = n^{k+1}$ functions that map from A to B .

□

- b) Use part (a) to prove the original statement in Q1 directly without the use of induction.

Proof. Let A be a set such that $|A| = n$. Every subset A' of A can be defined as a function f that maps elements of A to $\{0, 1\}$:

For any $a \in A$, if $a \in A'$, $f(a) = 1$. Otherwise, $f(a) = 0$

From the previous part, there are 2^n different functions with domain A and co-domain $\{0, 1\}$, which also means that there are 2^n subsets of A , which implies that $\mathcal{P}(A) = 2^n$.

□