# MAT257 – Week 1 Homework

9/13 **1.** Let $A_1, A_2, A_3, \ldots$ be a sequence of countable sets. Prove that $\bigcup_{i\geq 1} A_i$ is countable.

*Proof.* Since $A_i$ is countable, denote $A_{ij}$ to be the $j$th element of set $A_i$.
Define $f : \bigcup_{i\geq 1} A_i \to \mathbb{N}$ as

$$f(A_{ij}) = 2^i 3^j$$

We will show that $f$ is injective. Let $A_{pq}, A_{rs} \in \bigcup_{i\geq 1} A_i$. Suppose that $f(A_{pq}) = f(A_{rs})$. Then

$$2^p 3^q = 2^r 3^s$$

Since every integer has a unique prime factorisation, it follows that $p = r$, $q = s$. Thus $A_{pq} = A_{rs}$.
Now, define the injection $g : \mathbb{N} \to \bigcup_{i\geq 1} A_i$ to be $g(n) = A_{1n}$.
Therefore by the Schröder–Bernstein theorem, $|\bigcup_{i\geq 1} A_i| = |\mathbb{N}|$. Thus the set is countable. □

9/13 **2.** Let $X, Y, Z$ be three vector spaces. Prove that $L^2(X, Y; Z)$ is isomorphic to $L(X, L(Y, Z))$.
(For two vector spaces $X, Y$, we use $L(X, Y)$ to denote the space of linear mappings from $X$ to $Y$. For three vector spaces $X, Y, Z$, and a a function $\beta : X \times Y \to Z$, we let $\beta(\,\cdot\,, y_0)$ denote the function $X \to Z$ given by $x \mapsto \beta(x, y_0)$, where $y_0 \in Y$ is some fixed vector; this is called the **$y_0$-slice** of $\beta$. The **$x_0$-slice** $\beta(x_0, \,\cdot\,)$ is defined similarly. We say that $\beta$ is **bilinear** if $\beta(\,\cdot\,, y_0)$ is linear for all $y_0 \in Y$, and $\beta(x_0, \,\cdot\,)$ is linear for all $x_0 \in X$. We use $L^2(X, Y; Z)$ the denote the space of *bilinear* mappings $\beta : X \times Y \to Z$. You should convince yourself that $L^2(X, Y; Z)$ is a vector space under pointwise operations: for two bilinear mappings $\alpha, \beta : X \times Y \to Z$ and a scalar $c \in \mathbf{R}$, we form the linear combination $c\alpha + \beta$ via $(x, y) \mapsto c\alpha(x, y) + \beta(x, y)$.)

*Proof.* For a bilinear map $T \in L^2(X, Y; Z)$, $(x, y) \in X \times Y$, Define $\phi : L^2(X, Y; Z) \to L(X, L(Y, Z))$ such that

$$\phi(T)(x, y) = T(x, \cdot)(y)$$

where $T(x, \cdot)$ is the $x$-slice of $T$. We claim that this transformation is an isomorphism.
First, let $\phi(T) = 0$. Then $\forall x \in X, T(x, \cdot)(y) = 0$, from which it follows that $T(x, y) = 0$, meaning $\phi$ is injective.
Next, fix $U \in L(X, L(Y, Z))$. Let $\beta$ be the basis for $X$. Let $T \in L^2(X, Y; Z)$ be the linear transformation such that $T(x, \cdot) = U(x)$, for all $x \in \beta$. We see that $\forall x, y \in X \times Y$,

$$\phi(T)(x, y) = T(x, \cdot)(y) = U(x)(y)$$

making $\phi$ surjective.
Thus $\phi$ is an isomorphism, and we get that $L^2(X, Y; Z) \cong L(X, L(Y, Z))$ as desired. □

9/13 **3.** Let $I = (a, b)$ and $J = (c, d)$ be two open intervals on the real line. Let $f : I \to J$ be an increasing function such that $f(I)$ is dense in $J$. Prove that $f$ is continuous.
(For two sets $D, S \subseteq \mathbf{R}$ we say that $D$ is **dense** in $S$ if $D \cap (s - \varepsilon, s + \varepsilon) \neq \varnothing$ for all $s \in S$ and all $\varepsilon > 0$.)

*Proof.* Fixing an $a \in I$, let $\varepsilon > 0$. We can assume without loss of generality that $\varepsilon$ is small enough that $(f(a) - \varepsilon, f(a) + \varepsilon) \subset J$. Since $f(I)$ is dense in $J$, we can always find an $y_1 \in (f(a) - \varepsilon, f(a))$ and $y_2 \in (f(a), f(a) + \varepsilon)$ such that $y_1, y_2 \in f(I)$. □