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Exercise 2.21. What can be said about $\overline{A \cup B}$?

Let (X, d) be a metric space, and let $A, B \in X$. We want to investigate whether the closure of a set is distributive over set union. That is,

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

We will now prove that this claim is true.

Proof. We will show set equality using double subset inclusion. Let $x \in \overline{A \cup B}$. By definition of a limit point, for every open ball of the form $B(x, \varepsilon)$, there exists a $y \in (A \cup B) \cap B(x, \varepsilon) = (A \cap B(x, \varepsilon)) \cup (B \cap B(x, \varepsilon))$. If $y \in A \cap B(x, \varepsilon)$, then $x \in \overline{A}$, or, likewise, if $y \in B \cap B(x, \varepsilon)$, then $x \in \overline{B}$. In other words, $x \in \overline{A} \cup \overline{B}$.

Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Conversely, let $x \in \overline{A} \cup \overline{B}$. We do the same process as before, but backwards. We have that

$$\forall \varepsilon > 0, \exists y \in X \text{ such that } y \in A \cap B(x, \varepsilon) \text{ or } y \in B \cap B(x, \varepsilon)$$

$$\implies y \in (A \cap B(x, \varepsilon)) \cup (B \cap B(x, \varepsilon)) \implies y \in (A \cup B) \cap B(x, \varepsilon) \implies x \in \overline{A \cup B}$$

Which shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

□

We will now see that distributivity also holds for a finite union of sets, as well.

Proposition. Let $A_i \in X, i \in \mathbb{N}$. Then $\forall n \in \mathbb{N}, \overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \overline{A_i}$.

Proof. We will show this by performing induction on n. When $n = 1$, obviously $\overline{\bigcup_{i=1}^1 A_i} = \overline{A_1} = \bigcup_{i=1}^1 \overline{A_i}$.

Now suppose this equality holds for $n = k$, for some natural k. Then from our previous claim, as well as our assumption,

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{\bigcup_{i=1}^{k} A_i \cup A_{k+1}} = \overline{\bigcup_{i=1}^{k} A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k+1} \overline{A_i}$$

Thus the equality holds for all $n \in \mathbb{N}$.

□