1 Week 2 Homework - Ethan Hua

1.1 Exercise. (a) Prove that there exists an infinitely differentiable function $\alpha : \mathbf{R} \to \mathbf{R}$ such that $\alpha(t) = 0$ for all $t \leq 0$, and $\alpha(t) > 0$ for all $t > 0$.

Proof. We define $\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$

Trivially $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) > 0$ if $t > 0$. It remains to show that α is infinitely differentiable.

Since 0 is infinitely differentiable, and $e^{-\frac{1}{x}}$ is infinitely differentiable for $x > 0$, it suffices to show that derivatives of all orders of α are continuous at $t = 0$.

We will prove using induction that

$$\forall n \in \mathbb{N} \cup \{0\},\ \lim_{t \to 0} \alpha^{(n)}(t) = 0$$

We will only worry about the right hand limit, as the left hand limit always evaluates to 0.

For $n = 0$,

$$\lim_{t \to 0^+} \alpha^{(0)}(t) = \lim_{t \to 0^+} \alpha(t) = \lim_{t \to 0^+} e^{-\frac{1}{x}} = 0$$

Thus the case for $n = 0$ holds.

Now suppose that the claim holds for $n = k$, for some $k \in \mathbb{N} \cup \{0\}$. It can be shown that when $t > 0$,

$$\alpha^{(k)}(t) = \sum_{i=0}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!} = \frac{(-1)^{k+1}(k+1)!}{t^{k+1}} + \sum_{i=1}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!}$$

□

(b) Prove that there exists an infinitely differentiable function $\beta : \mathbf{R} \to \mathbf{R}$ such that $\beta(t) = 1$ for all $t \geq 1$, and $\beta(t) = 0$ for all $t \leq 0$.

Hint: The shape you're looking for is $\dfrac{X}{X+Y}$.

(c) Prove that there exists an infinitely differentiable function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(t) = 1$ for all $t \in [2, 3]$, and $\varphi(t) = 0$ for $t \in \mathbf{R} \setminus (1, 4)$.

Hint: Your function $\beta(t)$ does half the job. Make a function $\gamma(t)$ that does the other half of the job. Then multiply them together.

1.2 Exercise. Let $S \subseteq \mathbf{R}^n$. Consider the following three statements:

- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_1)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_2)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_{\max})$.

Among these statements, determine which implications are true and which are false. There are six implications to investigate. Supply proof or counterexample as appropriate. Include pictures.

Denote each statement as (1), (2), and (3), respectively. We claim that each statement implies all the other statements.

Proof. In this proof, we denote:

1. an open ball with respect to the 1-norm as $B_1(p, \varepsilon)$
2. an open ball with respect to the Euclidean norm as $B_2(p, \varepsilon)$

3. an open ball with respect to the max-norm as $B_{\max}(p, \varepsilon)$

(1) $\implies$ (2):

Suppose (1) is true. There exists a $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ and $r > 0$ such that

$$S \subseteq B_1(p, r)$$

We will show that $S \subseteq B_2(p, r)$. Let $x = (x_1, x_2, \dots, x_n) \in S$. Since $S \subseteq B_1(p, r)$, by the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - p_i)^2} \leq \sum_{i=0}^{n} |x_i - p_i| = \|x - p\|_1 < r$$

Thus $x \in B_2(p, r)$.

(2) $\implies$ (1):

We will show using induction on n that if for some open ball in $\mathbb{R}^n$, $S \subseteq B_2(p, r)$, then $S \subseteq B_1(p, \sqrt{n}r)$

Let $n = 1$. Suppose (2) is true. Fix $x \in S \subseteq \mathbb{R}$. Then by (2),

$$r > \|x - p\|_2 = \sqrt{(x - p)^2} = |x - p| = \|x - p\|_1 \implies S \subseteq B_1(p, \sqrt{1} \cdot r)$$

Thus the claim holds for $n = 1$. Now suppose the claim holds for $n = k$. Assume (2) is true. Let $x \in S$. By (2),

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{k+1} (x_i - p_i)^2} < r \implies \sum_{i=1}^{k+1} (x_i - p_i)^2 < r^2 \tag{*}$$

We want to show that $x \in B_1(p, \sqrt{n}r)$, or

$$\|x - p\|_1 = \sum_{i=0}^{k+1} |x_i - p_i| < \sqrt{n}r \iff \left(\sum_{i=0}^{k+1} |x_i - p_i|\right)^2 < nr^2$$

We have

$$\left(\sum_{i=0}^{k+1} |x_i - p_i|\right)^2 = \left(\sum_{i=0}^{k} |x_i - p_i| + |x_{k+1} - p_{k+1}|\right)^2$$

$$= \left(\sum_{i=0}^{k} |x_i - p_i|\right)^2 + 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + |x_{k+1} - p_{k+1}|^2 \tag{**}$$

Manipulating the inequality from (*), we obtain

$$\sum_{i=1}^{k} (x_i - p_i)^2 < r^2 - |x_{k+1} - p_{k+1}|^2$$

which implies that $(x_1, x_2, \dots, x_k)$ is in the open ball $B_2(p, r)$ on $\mathbb{R}^k$. From the indudction hypothesis,

$$(**) < k(r^2 - |x_{k+1} - p_{k+1}|^2) + 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + r^2 - \sum_{i=1}^{k} (x_i - p_i)^2$$

$$= (k+1)r^2 - \left(k\,|x_{k+1} - p_{k+1}|^2 - 2\,|x_{k+1} - p_{k+1}| \left(\sum_{i=0}^{k} |x_i - p_i|\right) + \sum_{i=1}^{k} (x_i - p_i)^2\right)$$

$$= (k+1)r^2 - \sum_{i=1}^{k} \left(|x_{k+1} - p_{k+1}|^2 - 2\,|x_{k+1} - p_{k+1}|\,|x_i - p_i| + (x_i - p_i)^2\right)$$

$$= (k+1)r^2 - \sum_{i=1}^{k} (|x_{k+1} - p_{k+1}| - |x_i - p_i|)^2 \leq (k+1)r^2$$

In summary, we have that

$$\|x - p\|_1 < \sqrt{k+1}r$$

Thus x is in the open ball $B_1(p, \sqrt{k+1}r)$, which means that (1) is true. By the principle of induction, the claim holds for all $n \in \mathbb{N}$.

(1) $\implies$ (3): Suppose (1). Then S is a subset of some open ball $B_1(p, r)$. We will show that $S \subseteq B_{max}(p, r)$.

Let $x \in S$. Then by (1):

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n} \{|x_i - p_i|\} \leq \sum_{i=1}^{n} |x_i - p_i| = \|x - p\|_1 < r$$

Thus $x \in B_{\max}(p, r)$, so (3) is true.

(3) $\implies$ (1):

Suppose that (3) holds true. Then for some open ball $B_{\max}(p, r)$. We will show that $S \subseteq B_1(p, nr)$.

Let $x \in S$. We have

$$\|x - p\|_1 = \sum_{i=1}^{n} |x_i - p_i| \leq n \max_{0 \leq i \leq n} |x_i - p_i| = n \|x - p\|_{\max} < nr$$

Thus $x \in B_1(p, nr)$, which implies that (1) is true.

(2) $\implies$ (3):

Suppose (2) is true. Then $S \subseteq B_2(p, r)$, where $B_2(p, r)$ is some open ball on the Euclidean norm. We want to show that $S \subseteq B_{\max}(p, r)$. Indeed, we have

$$\|x - p\|_{\max} = \max_{1 \leq i \leq n} |x_i - p_i| \leq \sqrt{\sum_{i=1}^{n} |x_i - p_i|^2} < r$$

Thus $S \subseteq B_{\max}(p, r)$, so (3) is true.

(3) $\implies$ (2):

Suppose that (3) holds true. Then for some open ball with the max-norm, $S \subseteq B_{\max}(p, r)$. We will show that $S \subseteq B_2(p, nr)$

Let $x \in S$. Then be the triangle inequality,

$$\|x - p\|_2 = \sqrt{\sum_{i=1}^{n} |a_i - p_i|} \leq \sum_{i=1}^{n} |a_i - p_i| \leq n \max_{1 \leq i \leq n} |x_i - p_i| < n \|x - p\|_{\max} < nr$$

Thus $x \in B_2(p, nr)$, so (2) holds.

Therefore, we have proven every implication to be true.

□

1.3 Exercise. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. A linear mapping $T : X \to Y$ is called **bounded** if there exists a constant $M \geq 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Let $B(X, Y)$ denote the set of these bounded linear operators. The **operator norm** on $B(X, Y)$, denoted by $\|\cdot\|_{\text{op}}$, is defined as follows:

$$\|T\|_{\text{op}} = \sup\{\|T(x)\|_Y : x \in X \text{ and } \|x\|_X \leq 1\}.$$

(a) Prove that $B(X, Y)$ is a linear subspace of $L(X, Y)$.

Proof. The 0-transformation $\|Z(x)\| = 0 \leq \|x\|_X$, $\forall x \in X$, because of the definition of a norm. Thus $0 \in B(X,Y)$. Let $T, U \in B(X,Y)$, $c \in \mathbb{R}$. Then for all $x \in X$, there exist $M, N \geq 0$ such that

$$T(x) \leq M \|x\|_X \text{ and } U(x) \leq N \|x\|_X \implies T(x) + U(x) \leq (M+N) \|x\|_X$$

Which implies that $T + U$ is a member of $B(X,Y)$. As well, from the first inequality,

$$T(x) \leq M \|x\|_X \implies cT(x) \leq cM \|x\|_X$$

Which implies that cT is a member of $B(X,Y)$.

Since $0 \in B(X,Y)$and $B(X,Y)$ is closed under addition and scalar multiplication, $B(X,Y)$ is a subspace of $L(X,Y)$.

□

(b) Prove that $\|\cdot\|_{\mathrm{op}}$ is a norm on $B(X,Y)$.

Proof. To prove that the operator norm is a norm, we first verify that $\|T\|_{op} = 0 \iff T = 0$.

Fix $T \in B(X,Y)$ and suppose that $T = 0$. Then for all $x \in X$, $\|x\|_X \leq 1$, $\|T(x)\|_Y = \|0\|_Y = 0$. Thus $\|T\|_{op} = 0$.

Now suppose the converse, that $\|T\|_{op} = 0$. Then

$$\forall x \in X, \|x\|_X \leq 1, \|T(x)\|_Y \leq 0.$$

But by the definition of the norm in Y,

$$0 \leq \|T(x)\|_Y$$

It follows that $T(x) = 0$.

We denote the set of elements x in X such that $\|x\|_X \leq 1$ as X'.

To show nonnegativity, we note that for $x \in X'$,

$$\|T\|_{op} \geq \|T(x)\|_Y \geq 0$$

To show homogeity, let $T \in G(X,Y), c \in \mathbb{R}$. Then

$$\|cT\|_{op} = \sup\{\|cT(x)\|_Y : x \in X'\} = \sup\{c\|T(x)\|_Y : x \in X'\} = c\sup\{\|T(x)\|_Y : x \in X'\} = c\|T\|_{op}$$

Now we show that the triangle inequality holds with respect to the operator norm.

Fix $T, U \in B(X,Y)$. We denote the set of elements x in X such that $\|x\|_X \leq 1$ as X'. Let $x \in X'$. By definition,

$$T(x) \leq \sup T(X') \text{ and } U(x) \leq \sup U(X')$$

Adding both together obtains

$$T(x) + U(x) \leq \sup T(X') + \sup U(X')$$

We see that $\sup T(X') + \sup U(X')$ is an upper bound for $T(x) + U(x)$. By the definition of the least upper bound,

$$\sup\{T(X') + U(X')\} \leq \sup T(X') + \sup U(X') \implies \|T + U\|_{op} \leq \|T\|_{op} + \|U\|_{op}$$

Thus the operator norm is, indeed, a norm.

□

(c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping given by $T(x,y) = (x+y, x)$. Find, with proof, the exact value of $\|T\|_{\mathrm{op}}$. (Here, $\mathbf{R}^2$ is equipped with the usual norm.)

(d) Find, with proof, an example of an unbounded linear operator.

Proof. Let $X = Y = \mathbb{R}^n$, $\|\cdot\|_X = \|\cdot\|_2$ and $\|\cdot\|_Y = \|\cdot\|_{\max}$

Let $T(x) = x$. Fix $M \geq 0$. □