3 Week 3 Homework

3.1 Exercise. Let $S \subseteq C[0,1]$. Consider the following two statements:

- S is an open subset of $(C[0,1], \|\cdot\|_1)$.
- S is an open subset of $(C[0,1], \|\cdot\|_\infty)$.

Determine if the first statement implies the second, and vice-versa. Supply proof or counterexample as appropriate.

Proof. We claim that the first statement implies the second, but not the converse.

Suppose that S is an open subset of $(C[0,1], \|\cdot\|_1)$. For any $g \in S$, there is an open ball with respect to the 1-norm centered around g with radius ε such that $B_1(g,\varepsilon) \subseteq S$. We proceed to show that $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon)$. Let $f \in B_\infty(g,\varepsilon)$. Then

$$\|f-g\|_1 = \int_0^1 |f-g| \leq \int_0^1 \sup\{|f-g|\} = \|f-g\|_\infty < \varepsilon$$

Thus $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon) \subseteq S$. Thus S is an open subset of $(C[0,1], \|\cdot\|_1)$.

Now we show that the converse is not necessarily true. Let $S = B_\infty(0,1)$. This is an open subset of $(C[0,1], \|\cdot\|_\infty)$. Consider $f(x) = 0 \in B_\infty(0,1)$. For every $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Let $g(x) = x^{n-1}$. Since $\int_0^1 g(x)dx = \frac{1}{n} < \varepsilon$, $g(x) \in B_1(0,\varepsilon)$. But $g(1) = 1$, which means that $g(x) \notin B_\infty(0,1)$. Thus f is not an interior point of $B_\infty(0,1)$ with respect to the 1-norm, which means that $B_\infty(0,1)$ is not an open subset of $(C[0,1], \|\cdot\|_1)$.

□

3.2 Exercise. Let X be any set. The **diagonal** of $X \times X$ is the following set:

$$\Delta = \{(x,x) : x \in X\}.$$

Prove that if (X,d) is a metric space, then Δ is a closed subset of $X \times X$ (with respect to the product metric).

3.3 Exercise. *Can linear subspaces be open and/or closed?*

(a) Let $C^\infty[0,1]$ denote the set of infinitely differentiably functions $f : [0,1] \to \mathbf{R}$. Prove that $C^\infty[0,1]$ is *not* a closed subset of $(C[0,1], \|\cdot\|_\infty)$.

Proof. To show that this set is not closed, we just need to find a limit point that is not an element of the set. Let $f(x) = |x - \frac{1}{2}|$. Clearly $f \notin C^\infty[0,1]$. To prove f is a limit point of $C^\infty[0,1]$, first let $\varepsilon > 0$ and consider the open ball $B(f,\varepsilon)$. There exists an $n \in \mathbb{N}$ such that $n+1 > \frac{1}{\varepsilon} \implies asdfasfas$.

□

(b) Let C be the set of **convergent** sequences of real numbers. Prove that C is a closed subset of $(\ell^\infty, \|\cdot\|_\infty)$.

Proof. Let $(a_n)_{n\geq 1}$ be a limit point of C. Suppose for contradiction that (a_n) diverges. For any $\hat{\varepsilon} > 0$, there exists a convergent sequence $(b_n)_{n\geq 1} \in B((a_n), \hat{\varepsilon})$. Say that (b_n) converges to L. Then since (a_n) diverges,

$$\exists \tilde{\varepsilon} > 0 \text{ such that } \forall N \in \mathbb{N},\ \exists \tilde{n} \geq N \text{ such that } |a_{\tilde{n}} - L| \geq \tilde{\varepsilon} \geq \min\{\tilde{\varepsilon}, \hat{\varepsilon}\}$$

Let $\varepsilon = |\hat{\varepsilon} - \tilde{\varepsilon}|$. Since (b_n) converges, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |b_n - L| < \varepsilon$. If we take $\tilde{n}$ for this N, we have

$$\tilde{\varepsilon} \leq |a_{\tilde{n}} - L| = |a_{\tilde{n}} - b_{\tilde{n}} + b_{\tilde{n}} - L| \leq |a_{\tilde{n}} - b_{\tilde{n}}| + |b_{\tilde{n}} - L|$$

$$|a_{\tilde{n}} - b_{\tilde{n}}| + |b_{\tilde{n}} - L| < \hat{\varepsilon} + |\hat{\varepsilon} - \tilde{\varepsilon}|$$

If $\hat{\varepsilon} < \tilde{\varepsilon}$,

$$\hat{\varepsilon} + |\hat{\varepsilon} - \tilde{\varepsilon}| = \tilde{\varepsilon}$$

If $\hat{\varepsilon} \geq \tilde{\varepsilon}$,

$$\hat{\varepsilon} + |\hat{\varepsilon} - \tilde{\varepsilon}| = 2\hat{\varepsilon} - \tilde{\varepsilon} \leq \hat{\varepsilon}$$

regardless, this implies that

$$\tilde{\varepsilon} < \tilde{\varepsilon}$$

which is a contradiction. Thus (a_n) is convergent, which means that it is in C^∞. Since every limit point is in the set itself, C^∞ is closed.

□

(c) Let $(X, \|\cdot\|)$ be a normed vector space, and let M be a linear subspace of X. Prove that M is an open set if and only if $M = X$.

3.4 Exercise. *The Bolzano–Weierstrass Theorem.*

(a) Prove that every bounded sequence in $(\mathbf{R}^d, \|\cdot\|_2)$ has a convergent subsequence.

Proof. We take the Bolzano-Weierstrass Theorem in $\mathbb{R}$ for granted and use this to prove it for $\mathbb{R}^d$. We will do this using induction on d. When $d = 1$, it follows trivially from the theorem in $\mathbb{R}$.

Now suppose that the claim is true for some $d = k$, for some $k \in \mathbb{N}$. Let $(a_n)_{n\geq 1}$ be a bounded sequence in $(\mathbf{R}^k, \|\cdot\|_2)$. Define another sequence $(b_n)_{n\geq 1}$ in $\mathbb{R}^{k-1}$ such that b_i is the first $k-1$ components of a_i. From our assumption, b_i has a convergent subsequence $(b_{n_i})_{i\geq 1}$. Consider another sequence $(c_{n_i})_{i\geq 0}$ in $\mathbb{R}$, where c_{n_i} is equal to the last component of a_{n_i}. By the Theorem in $\mathbb{R}$, $(c_{n_i})_{i\geq 1}$ has a convergent subsequence $(c_{n_{m_i}})_{i\geq 1}$. Suppose that $(b_{n_{m_i}})_{i\geq 1}$ converges to $B = (B_1, B_2, \ldots, B_{k-1})$ and $(c_{n_{m_i}})_{i\geq 1}$ converges to C. We claim that $(a_{n_{m_i}})_{i\geq 1}$ is our desired subsequence, which converges to $(B_1, B_2, \ldots, B_{k-1}, C)$.

Let $\varepsilon > 0$. Since $(B_1, B_2, \ldots, B_{k-1})$ and $(c_{n_{m_i}})_{i\geq 1}$ converge,

$$\exists N_b, N_c > 0 \text{ such that } n_b > N_b \implies \|b_{n_b} - B\|_2 < \frac{\varepsilon}{2} \text{ and } n_c > N_c \implies |c_{n_c} - C| < \frac{\varepsilon}{2}$$

Let $N = \max\{N_b, N_c\}$. Let $n \in \mathbb{N}$, $n > N$. Then

$$\|a_n - (B_1, B_2, \ldots, B_{k-1}, C)\|_2 = \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2}$$

Using an inequality that I don't know the name of, we have

$$\sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2}$$

$$\leq \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2} + \sqrt{(a_k - C)^2} = \|b_n - B\|_2 + |c_n - C|_2$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $(a_{n_{m_i}})_{i\geq 1}$ converges, which means that $(a_n)_{n\geq 1}$ does indeed have a convergent subsequence. By the principle of induction, the Bolzano-Weierstrass Theorem holds in $(\mathbb{R}^d, \|\cdot\|_2)$ and we are done.

□

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ containing a (bounded?) sequence (x_n) which has no convergent subsequences.

Proof. Consider the normed vector space $(\ell^\infty, \|\cdot\|_\infty)$. Let $(\vec{x_n})_{n\geq 1}$ be a sequence in $(\ell^\infty, \|\cdot\|_\infty)$ defined by $\vec{x}_{ik} = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{otherwise.} \end{cases}$

Clearly $(\vec{x_n})_{n\geq 1}$ is bounded. Let $(\vec{x_{n_i}})_{i\geq 1}$ be a subsequence of $(\vec{x_n})_{n\geq 1}$. We will show that $(\vec{x_{n_i}})_{i\geq 1}$ diverges.

For any $(L_k)_{k\geq 1} \in \ell^\infty$ Let $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. If there are no values of $p \geq N$ so that $\|(\vec{x_p})_k - L_k\|_\infty < \varepsilon$, we are done. Now suppose the opposite, that there is a $p \geq N$ such that $\|(\vec{x_p})_k - L_k\|_\infty < \varepsilon$. Let $n = p + 1$. By the reverse triangle inequality,

$$\|(\vec{x_n})_k - L_k\|_\infty = \|(\vec{x_n})_k - (\vec{x_p})_k + (\vec{x_p})_k - L_k\|_\infty \geq |\|(\vec{x_n})_k - (\vec{x_p})_k\|_\infty - \|(\vec{x_p})_k - L_k\|_\infty|$$

$$= |1 - \|(\vec{x_p})_k - L_k\|_\infty| \geq 1 - \varepsilon = \frac{1}{2}$$

Thus $(\vec{x_{n_i}})_{i\geq 1}$ diverges.

$\square$