4 Homework 4

Question 11. Let (X, d) be a metric space. A function $f : X \to X$ is called a **contraction mapping** if there exists a constant $M \in (0, 1)$ such that

$$d(f(x), f(y)) \leq Md(x, y) \quad \text{for all } x, y \in X.$$

(a) Suppose that (X, d) is a complete metric space, and that $f : X \to X$ is a contraction mapping. Prove that f has a unique fixed point; *i.e.* there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Let (X, d) be a complete metric space and f be a contraction mapping. In this proof, for $n \in \mathbb{N}$, we denote f^n to be a composition of f n times.

Let $x \in X$. Define the sequence $(x_i)_{i\in\mathbb{N}}$ by $x_i = f^i(x)$. We show that this is a Cauchy sequence.

Let $\varepsilon > 0$. There exists an $N > 0$ such that $\dfrac{M^N \cdot d(x, f(x))}{1 - M} < \varepsilon$, since $\dfrac{d(x, f(x))}{1 - M}$ is constant. Then for all $m, n \geq N, m < n$,

$$d(x_m, x_n) = d(f^m(x), f^n(x)) \leq M^m d(x, f^{n-m}(x))$$

By the triangle inequality,

$$\begin{aligned}
M^m d(x, f^{n-m}(x)) &\leq M^m \left(d(x, f(x)) + d(f(x), f^2(x)) + d(f^2(x), f^3(x)) + \cdots + d(f^{n-m-1}(x), f^{n-m}(x))\right) \\
&\leq M^m \left(d(x, f(x)) + Md(x, f(x)) + M^2 d(x, f(x)) + \cdots + M^{n-m-1} d(x, f(x))\right) \\
&= M^m d(x, f(x))(1 + M + \cdots + M^{n-m-1}) = \frac{M^m(1 - M^{n-m})}{1 - M} d(x, f(x)) < \frac{M^m}{1 - M} d(x, f(x))
\end{aligned}$$

Since $0 < M < 1$ and $M \geq N$, $M^m < M^N$. Thus

$$\frac{M^m}{1 - M} d(x, f(x)) < \frac{M^N}{1 - M} d(x, f(x)) < \varepsilon$$

All in all, we have that

$$d(x_m, x_n) < \varepsilon$$

which shows that (x_i) is a Cauchy sequence. By the completeness of X, (x_n) converges to some value which will be denoted as x_0. In fact, x_0 is the fixed point we want. To prove this, we will show that if (x_n) converges to x_0, then it also converges to $f(x_0)$.

Suppose $(x_n) \to x_0$. Letting $\varepsilon > 0$, there exists an $N' > 0$ such that $d(f^n(x), x_0) < \varepsilon$ for all $n \geq N'$. Let $N = N' + 1$. For all $n \geq N$,

$$d(f^n(x), f(x_0)) \leq Md(f^{n-1}(x), x_0)$$

$n - 1 \geq N'$ so

$$Md(f^{n-1}(x), x_0) < \varepsilon$$

We have shown that (x_0) converges to both x_0 and $f(x_0)$. By the uniqueness of limits, we have $x_0 = f(x_0)$.

□

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ and a contraction mapping $f : X \to X$ such that f does **not** have a fixed point.

Proof. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ be defined by $f(x) = \frac{x}{2}$, where $\mathbb{R} \setminus \{0\}$ is endowed with the usual metric in $\mathbb{R}$. First we verify that f is a contraction mapping. Let $M = \frac{2}{3}$ and $x, y \in \mathbb{R} \setminus \{0\}$. Then indeed,

$$|f(x) - f(y)| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| \leq \frac{2}{3}|x - y| = M|x - y|$$

Suppose for contradiction that f has a fixed point $x_0 \in \mathbb{R} \setminus \{0\}$. Then

$$x_0 = f(x_0) \implies x_0 = \frac{x_0}{2} \implies \frac{x_0}{2} = 0 \implies x_0 = 0$$

But $0 \notin \mathbb{R} \setminus \{0\}$. There is a contradiction. Thus f has no fixed point.

$\square$

Question 12. *The Intermediate Value Theorem.*

(a) A subset $I \subseteq \mathbf{R}$ is called an **interval** if $a, b \in I$ implies $[a, b] \subseteq I$.

Let $I \subseteq \mathbf{R}$. Prove that I is connected (with respect to the usual metric on $\mathbf{R}$) if and only if I is an interval.

Proof. We prove the equivalent statement I is disconnected if and only if I is not an interval.

Suppose I is disconnected. Then there exist disjoint, open, non-empty sets $A, B \subseteq I$ such that $A \cup B = I$. Take any $a \in A$ and $b \in B$. We can assume without loss of generality that $a < b$.

Let $S = [a, b] \cap A$. Notice that for all elements $x \in S$, $x \leq b$. Thus b is an upper bound for S. Since $\mathbb{R}$ possesses the least upper bound property, this set has a supremum $m \in \mathbb{R}$. By the definition of least upper bound, we see that $a \leq m \leq b$, so $m \in [a, b]$. Now, we consider two cases.

If $m \notin A$, it suffices to show that $m \notin B$. Since m is the supremem of S, we can always find an element in S that is greater than $m - \varepsilon$, so there is always an element in A that is within the ε ball surrounding m. Thus m is a limit point of A, which means that it is impossible for m to be an element of B, which is an open set. Thus we have that $m \notin A \cup B = I$ as desired.

If $m \in A$, then since A is an open subset of I, m is an interior point of A. But we also know that m is greater than or equal to any other element in A. It must be that for some $\varepsilon > 0$, the interval $(m, m + \varepsilon)$ is disjoint from I. Otherwise, it contradicts the fact that A is open in I. Then take any value x from the set $(m, \varepsilon) \cap [a, b]$ and notice that $x \notin I$ but $x \in [a, b]$.

In both cases, we have shown that I is not an interval.

Conversely, suppose that I is not an interval. Then for $p < q \in I$, there is a $c \in [p, q]$ such that $c \notin I$. Define subsets A and B in I as $A = \{x \in I \colon x < c\}$ and $B = \{x \in I \colon x > c\}$. A and B are non-empty because $p \in A$ and $q \in B$. The sets are also disjoint by construction.

To show that A is open, first take any $a \in A$. For this value of a, take $\varepsilon = c - a > 0$. For all $x \in B_I(a, \varepsilon)$, note that $x \in I$. As well, if $x < a$, immediately we have $x < a < c \implies x \in A$. If $x > a$, since x is within the open ball surrounding a, $x - a = |x - a| < c - a \implies x < c \implies x \in A$. Thus every element of A is an interior point, so A is open.

We can use a very similar argument for the set B, so the proof is omitted. We conclude that B is open as well. Therefore I is disconnected.

□

(b) Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$ be a continuous function. Prove that if C is a connected subset of X, then $f(C)$ is a connected subset of Y.

Proof. We examine the contrapositive: if $f(C)$ is a disconnnected subset of Y, then C is a disconnected subset of X.

Suppose that $f(C)$ is a disconnected subset of Y. Then there exist non-empty, disjoint, open subsets $A, B \subseteq f(C)$ such that $A \cup B = f(C)$. Consider the sets $f^{-1}(A), f^{-1}(B) \subseteq C$. Notice that if $x \in f^{-1}(A)$, $f(x) \in A$. Since A, B are disjoint, $f(x) \notin B$, so $x \notin f^{-1}(B)$. It follows that $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint.

Now, we will show that $f^{-1}(A)$ is non-empty and open.

We know that A is non-empty, so there exists an element $y \in A$. By the definition of image, $y = f(x)$ for some $x \in C$. it follows that $x \in f^{-1}(A)$. Thus $f^{-1}(A)$ is non-empty.

To show that $f^{-1}(A)$ is open, we use the topological definition of continuity of f to conclude that since A is open, $f^{-1}(A)$ is open as well.

We can apply the same argument for $f^{-1}(B)$, so B is non-empty and open as well.

Thus $f^{-1}(A), f^{-1}(B)$ are non-empty, disjoint, and open. It remains to show that $f^{-1}(A) \cup f^{-1}(B) = C$. Immediately, we know that $f^{-1}(A) \subseteq C$ and $f^{-1}(B) \subseteq C$, so $f^{-1}(A) \cup f^{-1}(B) \subseteq C$.

For the other direction, let $x \in C$. We know that $f(x) \in f(C)$, which means that either $f(x) \in A$ or $f(x) \in B$. It follows that $x \in f^{-1}(A)$ or $x \in f^{-1}(B) \implies x \in f^{-1}(A) \cup f^{-1}(B)$. Thus $f^{-1}(A) \cup f^{-1}(B) = C$.

We conclude that C is a disconnected subset of X, which shows that the contrapositive of the original statement is true, completing the proof.

$\square$

(c) Recall the **Intermediate Value Theorem** from single-variable calculus. *"Let $I \subseteq \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$ be a continuous function. Suppose that $a, b \in f(I)$ are two numbers such that $a < b$, and suppose that $a < y_0 < b$. Then there exists $x_0 \in I$ such that $f(x_0) = y_0$."*

Prove that this theorem immediately follows from (a) and (b). Thus, (b) is a generalization of the Intermediate Value Theorem.

Proof. Let $a, b, y_0 \in f(I)$ and suppose that $a < y_0 < b$. Since I is an interval, by part (a) I is connected. As well, by continuity of f, it follows from the result of (b) that $f(I)$ is a connected subset of $\mathbb{R}$. By (a) once again, $f(I)$ is an interval. Thus $[a, b] \subseteq f(I)$. Since $a < y_0 < b$, $y_0 \in [a, b] \subseteq f(I)$ which implies that $y_0 = f(x_0)$ for some $x_0 \in I$, which is what we were looking for. $\square$