4 Homework 4

Question 11. Let (X, d) be a metric space. A function $f : X \to X$ is called a **contraction mapping** if there exists a constant $M \in (0, 1)$ such that

$$d(f(x), f(y)) \leq Md(x, y) \quad \text{for all } x, y \in X.$$

(a) Suppose that (X, d) is a complete metric space, and that $f : X \to X$ is a contraction mapping. Prove that f has a unique fixed point; *i.e.* there exists a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Proof. Let (X, d) be a complete metric space and f be a contraction mapping. In this proof, for $n \in \mathbb{N}$, we denote f^n to be a composition of f. First, we will prove a lemma:

Lemma. $\forall x \in X, k \in \mathbb{N}, d(x, f^n(x)) < C$, where C is a real constant.

To prove this, we will use an induction argument on k.

Let $k = 1$.

Now suppose that the claim holds true for $k = l$, where $l \in \mathbb{N}$. Then by the triangle inequality,

$$d(x, f^{l+1}(x)) \leq d(x, f^l(x)) + d(f^l(x), f^{l+1}(x))$$

□

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ and a contraction mapping $f : X \to X$ such that f does **not** have a fixed point.

Question 12. *The Intermediate Value Theorem.*

(a) A subset $I \subseteq \mathbf{R}$ is called an **interval** if $a, b \in I$ implies $[a, b] \subseteq I$.

Let $I \subseteq \mathbf{R}$. Prove that I is connected (with respect to the usual metric on $\mathbf{R}$) if and only if I is an interval.

Proof. We prove the equivalent statement I is disconnected if and only if I is not an interval.

Suppose I is disconnected. Then there exist disjoint, open, non-empty sets $A, B \subseteq I$ such that $A \cup B = I$. Take any $a \in A$ and $b \in B$. We can assume without loss of generality that $a < b$ and consider the interval $[a, b]$.

Conversely, suppose that I is not an interval. Then for $p < q \in I$, there is a $c \in [p, q]$ such that $c \notin I$. Define subsets A and B in I as $A = \{x \in I \colon x < c\}$ and $B = \{x \in I \colon x > c\}$. A and B are non-empty because $p \in A$ and $q \in B$. The sets are also disjoint by construction. To show that A is open, take any $a \in A$.

For this value of a, take $\varepsilon = c - a > 0$. For all $x \in B_I(a, \varepsilon)$, note that $x \in I$. if $x < a$, immediately we have $x < a < c \implies x \in A$. If $x > a$, since x is within the open ball surrounding a, $x - a = |x - a| < c - a \implies x < c \implies x \in A$. Thus every element of A is an interior point, so A is open.

□

(b) Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f : X \to Y$ be a continuous function. Prove that if C is a connected subset of X, then $f(C)$ is a connected subset of Y.

(c) Recall the **Intermediate Value Theorem** from single-variable calculus. *"Let $I \subseteq$ **R** be an open interval and $f : I \to$ **R** be a continuous function. Suppose that $a, b \in f(I)$ are two numbers such that $a < b$, and suppose that $a < y_0 < b$. Then there exists $x_0 \in I$ such that $f(x_0) = y_0$."*

Prove that this theorem immediately follows from (a) and (b). Thus, (b) is a generalization of the Intermediate Value Theorem.