3 Week 3 Homework

3.1 Exercise. Let $S \subseteq C[0,1]$. Consider the following two statements:

- S is an open subset of $(C[0,1], \|\cdot\|_1)$.
- S is an open subset of $(C[0,1], \|\cdot\|_\infty)$.

Determine if the first statement implies the second, and vice-versa. Supply proof or counterexample as appropriate.

Proof. We claim that the first statement implies the second, but not the converse.

Suppose that S is an open subset of $(C[0,1], \|\cdot\|_1)$. For any $g \in S$, there is an open ball with respect to the 1-norm centered around g with radius ε such that $B_1(g,\varepsilon) \subseteq S$. We proceed to show that $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon)$. Let $f \in B_\infty(g,\varepsilon)$. Then $\|f-g\|_\infty < \varepsilon$. Thus we have

$$\|f-g\|_1 = \int_0^1 |f-g| \leq \int_0^1 \sup\{|f-g|\} = \|f-g\|_\infty < \varepsilon$$

Thus $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon) \subseteq S$. Thus S is an open subset of $(C[0,1], \|\cdot\|_1)$.

Now we show that the converse is not necessarily true. Let $S = B_\infty(0,1)$. This is an open subset of $(C[0,1], \|\cdot\|_\infty)$. Consider $f(x) = 0 \in B_\infty(0,1)$. For every $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Let $g(x) = x^{n-1}$. Since $\int_0^1 g(x)dx = \frac{1}{n} < \varepsilon$, $g(x) \in B_1(0,\varepsilon)$. But $g(1) = 1$, which means that $g(x) \notin B_\infty(0,1)$. Thus f is not an interior point of $B_\infty(0,1)$ with respect to the 1-norm, which means that $B_\infty(0,1)$ is not an open subset of $(C[0,1], \|\cdot\|_1)$.

$\square$

3.2 Exercise. Let X be any set. The **diagonal** of $X \times X$ is the following set:

$$\Delta = \{(x, x) : x \in X\}.$$

Prove that if (X, d) is a metric space, then Δ is a closed subset of $X \times X$ (with respect to the product metric).

Proof. We define d as the metric in X and d_X to be the product metric of $X \times X$. Suppose that (X, d) is a metric space. We want to show that every limit point of Δ is a member of Δ.

Let $(a, b) \in X \times X$ be a limit point of Δ and suppose for contradiction that (a, b) is not an element of X. This implies that $d(a, b) > 0$. Consider the open ball $B((a, b), d(a, b))$. For all $(x, x) \in X$, by the triangle inequality,

$$d_X((a, b), (x, x)) = d(a, x) + d(b, x) \geq d(a, b)$$

This means that no element lies within this open ball centered around (a, b), contradicting the fact that (a, b) is a limit point of X. Thus $(a, b) \in X$.

$\square$

3.3 Exercise. *Can linear subspaces be open and/or closed?*

(a) Let $C^\infty[0,1]$ denote the set of infinitely differentiably functions $f : [0,1] \to \mathbf{R}$. Prove that $C^\infty[0,1]$ is *not* a closed subset of $(C[0,1], \|\cdot\|_\infty)$.

Proof. To show that this set is not closed, we just need to find a limit point that is not an element of the set. Let $f(x) = |x - \frac{1}{2}|$. Clearly $f \notin C^\infty[0,1]$. To prove f is a limit point of $C^\infty[0,1]$, first let $\varepsilon > 0$ and consider the open ball $B(f, \varepsilon)$. There exists an $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon^2} \implies \frac{1}{\sqrt{n}} < \varepsilon$.

Let $g_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}}$. Since $x^2 + \frac{1}{n}$ is always positive and the squareroot function is infinitely differentiable for all positive real numbers, g_n is infinitely differentiable for all $x \in [0,1]$. Our goal is to find an upper bound for $|f - g_n|$. For convenience, we perform the change of variables $t = x - \frac{1}{2}$. Define a new function in $C[-\frac{1}{2}, \frac{1}{2}]$ as $h(t) = f(t) - g_n(t)$. We find the global maximum and minimum of h. Taking the derivative with respect to t, we obtain

$$h'(t) = \frac{|t|}{t} - \frac{t}{\sqrt{t^2 + \frac{1}{n}}}$$

We see that h has a critical point at $t = 0$ since $h'(0)$ is undefined. Note that $\pm t \leq |t| \leq \sqrt{t^2 + \frac{1}{n}} \implies \sqrt{t^2 + \frac{1}{n}} \pm t > 0$. If $t > 0$,

$$h'(t) = 1 - \frac{t}{\sqrt{t^2 + \frac{1}{n}}} = \frac{\sqrt{t^2 + \frac{1}{n}} - t}{\sqrt{t^2 + \frac{1}{n}}} > 0$$

If $t < 0$,

$$h'(t) = -1 - \frac{t}{\sqrt{t^2 + \frac{1}{n}}} = \frac{-\sqrt{t^2 + \frac{1}{n}} - t}{\sqrt{t^2 + \frac{1}{n}}} < 0$$

Therefore h acheives a local minimum at $t = 0$. We evaluate h at its critical points and endpoints:

$$h\left(-\frac{1}{2}\right) = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{n}}$$

$$h(0) = -\frac{1}{\sqrt{n}}$$

$$h\left(\frac{1}{2}\right) = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{n}}$$

By the monotonicity of the squareroot, $-\frac{1}{\sqrt{n}} < \sqrt{\frac{1}{4} + \frac{1}{n}} < \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{n}} < 0$. Thus $-\frac{1}{\sqrt{n}}$ is the global minimum of h and $\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{n}}$ is the global maximum of h. Note that $|h(0)| > |h(\frac{1}{2})|$. Thus

$$\|f - g_n\|_\infty = \sup_{0 \leq x \leq 1} \{|f(x) - g_n(x)|\} = \sup_{-\frac{1}{2} \leq t \leq \frac{1}{2}} \{|h(t)\} = \frac{1}{\sqrt{n}} < \varepsilon$$

Thus f is a limit point of C^∞, but is not a member of the set. Thus C^∞ is not closed.

□

(b) Let C be the set of **convergent** sequences of real numbers. Prove that C is a closed subset of $(\ell^\infty, \|\cdot\|_\infty)$.

Proof. Let $(a_n)_{n\geq 1}$ be a limit point of C. We want to show to (a_n) is convergent. We will do this by showing that (a_n) is a Cauchy sequence.

Let $\varepsilon > 0$. Using the fact that (a_n) is a limit point of C, there exists a convergent sequence $b_n \in B((a_n), \frac{\varepsilon}{3}) \implies \forall k \in \mathbb{N}, |a_k - b_k| < \|(b_n) - (a_n)\|_\infty < \frac{\varepsilon}{3}$. Since (b_n) converges, there is a $N' \in \mathbb{N}$ such that $m, n \geq N \implies |b_m - b_n| < \frac{\varepsilon}{3}$.

Let $N = N'$. Then for all $m, n \geq N$, we have

$$|a_m - a_n| = |(a_m - b_m) + (b_m - b_n) + (b_n - a_n)| \leq |(a_m - b_m)| + |b_m - b_n| + |b_n - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus (a_n) is a Cauchy sequence, and therefore converges, which means that $(a_n) \in C$. Thus C is closed.

□

(c) Let $(X, \|\cdot\|)$ be a normed vector space, and let M be a linear subspace of X. Prove that M is an open set if and only if $M = X$.

Proof. First, suppose that $M = X$. Then for every $x \in X$, any open ball centered around x is obviously a subset of X. Thus M is open.

Next, suppose that M is an open. To show that $X = M$, it suffices to show that $X \subseteq M$, since we already have $M \subseteq X$ from M being a subspace of X.

Since M is a subspace, we know that $0 \in M$. Thus there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq M$.

Let β be the basis for X. For every $\vec{x} \in \beta$, $\left\| \frac{\varepsilon \vec{x}}{2\|\vec{x}\|} \right\| = \frac{\varepsilon \|\vec{x}\|}{2\|\vec{x}\|} = \frac{\varepsilon}{2} < \varepsilon$, which means that $\frac{\varepsilon \vec{x}}{2\|\vec{x}\|} \in M$. Since M is a subspace, it follows that $\beta \subseteq M \implies X = \operatorname{span} \beta \subseteq M$. Thus $M = X$ and we are done.

□

3.4 Exercise. *The Bolzano–Weierstrass Theorem.*

(a) Prove that every bounded sequence in $(\mathbf{R}^d, \|\cdot\|_2)$ has a convergent subsequence.

Proof. We take the Bolzano-Weierstrass Theorem in $\mathbb{R}$ for granted and use this to prove it for $\mathbb{R}^d$. We will do this using induction on d. When $d = 1$, it follows trivially from the theorem in $\mathbb{R}$.

Now suppose that the claim is true for some $d = k - 1$, for some $k \in \mathbb{N}$. Let $(a_n)_{n\geq 1}$ be a bounded sequence in $(\mathbf{R}^k, \|\cdot\|_2)$. Define another sequence $(b_n)_{n\geq 1}$ in $\mathbb{R}^{k-1}$ such that b_i is the first $k-1$ components of a_i. From our assumption, b_i has a convergent subsequence $(b_{n_i})_{i\geq 1}$. Consider another sequence $(c_{n_i})_{i\geq 0}$ in $\mathbb{R}$, where c_{n_i} is equal to the last component of a_{n_i}. By the Theorem in $\mathbb{R}$, $(c_{n_i})_{i\geq 1}$ has a convergent subsequence $(c_{n_{m_i}})_{i\geq 1}$. Since $(b_{n_i})_{i\geq 1}$ converges, we know that $(b_{n_{m_i}})_{i\geq 1}$ converges to the same value as well. Suppose that $(b_{n_{m_i}})_{i\geq 1}$ converges to $B = (B_1, B_2, \dots, B_{k-1})$ and $(c_{n_{m_i}})_{i\geq 1}$ converges to C. We claim that $(a_{n_{m_i}})_{i\geq 1}$ is our desired subsequence, which converges to $(B_1, B_2, \dots, B_{k-1}, C)$.

Let $\varepsilon > 0$. Since $(B_1, B_2, \dots, B_{k-1})$ and $(c_{n_{m_i}})_{i\geq 1}$ converge,

$$\exists N_b, N_c > 0 \text{ such that } n_b > N_b \implies \|b_{n_b} - B\|_2 < \frac{\varepsilon}{2} \text{ and } n_c > N_c \implies |c_{n_c} - C| < \frac{\varepsilon}{2}$$

Let $N = \max\{N_b, N_c\}$. Let $n \in \mathbb{N}$, $n > N$. Then

$$\|a_n - (B_1, B_2, \dots, B_{k-1}, C)\|_2 = \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2}$$

Using an inequality that I don't know the name of, we have

$$\sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2}$$
$$\leq \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \cdots + (a_{k-1} - B_{k-1})^2} + \sqrt{(a_k - C)^2} = \|b_n - B\|_2 + |c_n - C|_2$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $(a_{n_{m_i}})_{i\geq 1}$ converges, which means that $(a_n)_{n\geq 1}$ does indeed have a convergent subsequence. By the principle of induction, the Bolzano-Weierstrass Theorem holds in $(\mathbb{R}^d, \|\cdot\|_2)$ and we are done.

□

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ containing a (bounded?) sequence (x_n) which has no convergent subsequences.

Proof. Consider the normed vector space $(\ell^\infty, \|\cdot\|_\infty)$. Let $(\vec{x_n})_{n\geq 1}$ be a sequence in $(\ell^\infty, \|\cdot\|_\infty)$ defined by $\vec{x_i}_k = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{otherwise.} \end{cases}$

Clearly $(\vec{x_n})_{n\geq 1}$ is bounded. Let $(\vec{x_{n_i}})_{i\geq 1}$ be a subsequence of $(\vec{x_n})_{n\geq 1}$. We will show that $(\vec{x_{n_i}})_{i\geq 1}$ diverges.

For any $(L_k)_{k\geq 1} \in \ell^\infty$ Let $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. If there are no values of $p \geq N$ so that $\|(\vec{x_p})_k - L_k\|_\infty < \varepsilon$, we are done. Now suppose the opposite, that there is a $p \geq N$ such that $\|(\vec{x_p})_k - L_k\|_\infty < \varepsilon$. Let $n = p + 1$. By the reverse triangle inequality,

$$\|(\vec{x_n})_k - L_k\|_\infty = \|(\vec{x_n})_k - (\vec{x_p})_k + (\vec{x_p})_k - L_k\|_\infty \geq \big|\|(\vec{x_n})_k - (\vec{x_p})_k\|_\infty - \|(\vec{x_p})_k - L_k\|_\infty\big|$$
$$= |1 - \|(\vec{x_p})_k - L_k\|_\infty| \geq 1 - \varepsilon = \frac{1}{2}$$

Thus $(\vec{x_{n_i}})_{i\geq 1}$ diverges.

□