

3 Week 3 Homework

3.1 Exercise. Let $S \subseteq C[0, 1]$. Consider the following two statements:

- S is an open subset of $(C[0, 1], \|\cdot\|_1)$.
- S is an open subset of $(C[0, 1], \|\cdot\|_\infty)$.

Determine if the first statement implies the second, and vice-versa. Supply proof or counterexample as appropriate.

Proof. We claim that the first statement implies the second, but not the converse.

Suppose that S is an open subset of $(C[0, 1], \|\cdot\|_1)$. For any $g \in S$, there is an open ball with respect to the 1-norm centered around g with radius ε such that $B_1(g, \varepsilon) \subseteq S$. We proceed to show that $B_\infty(g, \varepsilon) \subseteq B_1(g, \varepsilon)$. Let $f \in B_\infty(g, \varepsilon)$. Then $\|f - g\|_\infty < \varepsilon$. Thus we have

$$\|f - g\|_1 = \int_0^1 |f - g| \leq \int_0^1 \sup\{|f - g|\} = \|f - g\|_\infty < \varepsilon$$

Thus $B_\infty(g, \varepsilon) \subseteq B_1(g, \varepsilon) \subseteq S$. Thus S is an open subset of $(C[0, 1], \|\cdot\|_1)$.

Now we show that the converse is not necessarily true. Let $S = B_\infty(0, 1)$. This is an open subset of $(C[0, 1], \|\cdot\|_\infty)$. Consider $f(x) = 0 \in B_\infty(0, 1)$. For every $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Let $g(x) = x^{n-1}$. Since $\int_0^1 g(x)dx = \frac{1}{n} < \varepsilon$, $g(x) \in B_1(0, \varepsilon)$. But $g(1) = 1$, which means that $g(x) \notin B_\infty(0, 1)$. Thus f is not an interior point of $B_\infty(0, 1)$ with respect to the 1-norm, which means that $B_\infty(0, 1)$ is not an open subset of $(C[0, 1], \|\cdot\|_1)$. □

3.2 Exercise. Let X be any set. The **diagonal** of $X \times X$ is the following set:

$$\Delta = \{(x, x) : x \in X\}.$$

Prove that if (X, d) is a metric space, then Δ is a closed subset of $X \times X$ (with respect to the product metric).

Proof. Suppose that (X, d) is a metric space. We want to show that every limit point of Δ is a member of Δ .

Let $(a, b) \in X \times X$ be a limit point of Δ . □

3.3 Exercise. Can linear subspaces be open and/or closed?

- (a) Let $C^\infty[0, 1]$ denote the set of infinitely differentiable functions $f : [0, 1] \rightarrow \mathbf{R}$. Prove that $C^\infty[0, 1]$ is not a closed subset of $(C[0, 1], \|\cdot\|_\infty)$.

Proof. To show that this set is not closed, we just need to find a limit point that is not an element of the set. Let $f(x) = |x - \frac{1}{2}|$. Clearly $f \notin C^\infty[0, 1]$. To prove f is a limit point of $C^\infty[0, 1]$, first let $\varepsilon > 0$ and consider the open ball $B(f, \varepsilon)$. There exists an $n \in \mathbb{N}$ such that $n + 1 > \frac{1}{\varepsilon} \implies asdfasfas$. □

- (b) Let C be the set of **convergent** sequences of real numbers. Prove that C is a closed subset of $(\ell^\infty, \|\cdot\|_\infty)$.

Proof. Let $(a_n)_{n \geq 1}$ be a limit point of C . We want to show (a_n) is convergent. We will do this by showing that (a_n) is a Cauchy sequence.

Let $\varepsilon > 0$. Using the fact that (a_n) is a limit point of C , there exists a convergent sequence $b_n \in B((a_n), \frac{\varepsilon}{3}) \implies \forall k \in \mathbb{N}, |a_k - b_k| < \|(b_n) - (a_n)\|_\infty < \frac{\varepsilon}{3}$. Since (b_n) converges, there is a $N' \in \mathbb{N}$ such that $m, n \geq N \implies |b_m - b_n| < \frac{\varepsilon}{3}$.

Let $N = N'$. Then for all $m, n \geq N$, we have

$$|a_m - a_n| = |(a_m - b_m) + (b_m - b_n) + (b_n - a_n)| \leq |a_m - b_m| + |b_m - b_n| + |b_n - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Thus (a_n) is a Cauchy sequence, and therefore converges, which means that $(a_n) \in C$. Thus C is closed. □

- (c) Let $(X, \|\cdot\|)$ be a normed vector space, and let M be a **linear subspace** of X . Prove that M is an open set if and only if $M = X$.

Proof. First, suppose that $M = X$. Then for every $x \in X$, any open ball in X is obviously a subset of X . Thus M is open.

Next, suppose that M is an open subset of X . To show that $X = M$, it suffices to show that $X \subseteq M$, since we already have $M \subseteq X$.

Since M is a subspace, we know that $0 \in M$. Thus there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subseteq M$. Let β be the basis for X .

For every $\vec{x} \in \beta$, $\left\| \frac{\varepsilon \vec{x}}{2\|\vec{x}\|} \right\| = \frac{\varepsilon \|\vec{x}\|}{2\|\vec{x}\|} = \frac{\varepsilon}{2} < \varepsilon$, which means that $\frac{\varepsilon \vec{x}}{2\|\vec{x}\|} \in M$. Since M is a subspace, it follows that $\beta \subseteq M \implies X = \text{span } \beta \subseteq M$. Thus $M = X$ and we are done. □

3.4 Exercise. The Bolzano–Weierstrass Theorem.

- (a) Prove that every bounded sequence in $(\mathbf{R}^d, \|\cdot\|_2)$ has a convergent subsequence.

Proof. We take the Bolzano–Weierstrass Theorem in \mathbb{R} for granted and use this to prove it for \mathbb{R}^d . We will do this using induction on d . When $d = 1$, it follows trivially from the theorem in \mathbb{R} .

Now suppose that the claim is true for some $d = k$, for some $k \in \mathbb{N}$. Let $(a_n)_{n \geq 1}$ be a bounded sequence in $(\mathbf{R}^k, \|\cdot\|_2)$. Define another sequence $(b_n)_{n \geq 1}$ in \mathbb{R}^{k-1} such that b_i is the first $k - 1$ components of a_i . From our assumption, b_i has a convergent subsequence $(b_{n_i})_{i \geq 1}$. Consider another sequence $(c_{n_i})_{i \geq 1}$ in \mathbb{R} , where c_{n_i} is equal to the last component of a_{n_i} . By the Theorem in \mathbb{R} , $(c_{n_i})_{i \geq 1}$ has a convergent subsequence $(c_{n_{m_i}})_{i \geq 1}$. Suppose that $(b_{n_{m_i}})_{i \geq 1}$ converges to $B = (B_1, B_2, \dots, B_{k-1})$ and $(c_{n_{m_i}})_{i \geq 1}$ converges to C . We claim that $(a_{n_{m_i}})_{i \geq 1}$ is our desired subsequence, which converges to $(B_1, B_2, \dots, B_{k-1}, C)$.

Let $\varepsilon > 0$. Since $(B_1, B_2, \dots, B_{k-1})$ and $(c_{n_{m_i}})_{i \geq 1}$ converge,

$$\exists N_b, N_c > 0 \text{ such that } n_b > N_b \implies \|b_{n_b} - B\|_2 < \frac{\varepsilon}{2} \text{ and } n_c > N_c \implies |c_{n_c} - C| < \frac{\varepsilon}{2}$$

Let $N = \max\{N_b, N_c\}$. Let $n \in \mathbb{N}$, $n > N$. Then

$$\|a_n - (B_1, B_2, \dots, B_{k-1}, C)\|_2 = \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \dots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2}$$

Using an inequality that I don't know the name of, we have

$$\begin{aligned} & \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \dots + (a_{k-1} - B_{k-1})^2 + (a_k - C)^2} \\ & \leq \sqrt{(a_1 - B_1)^2 + (a_2 - B_2)^2 + \dots + (a_{k-1} - B_{k-1})^2} + \sqrt{(a_k - C)^2} = \|b_n - B\|_2 + |c_n - C|_2 \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus $(a_{n_{m_i}})_{i \geq 1}$ converges, which means that $(a_n)_{n \geq 1}$ does indeed have a convergent subsequence. By the principle of induction, the Bolzano–Weierstrass Theorem holds in $(\mathbb{R}^d, \|\cdot\|_2)$ and we are done. □

- (b) Give an example of a normed vector space $(X, \|\cdot\|)$ containing a (bounded?) sequence (x_n) which has no convergent subsequences.

Proof. Consider the normed vector space $(\ell^\infty, \|\cdot\|_\infty)$. Let $(\vec{x}_n)_{n \geq 1}$ be a sequence in $(\ell^\infty, \|\cdot\|_\infty)$ defined by $\vec{x}_{ik} = \begin{cases} 1, & \text{if } k = i; \\ 0, & \text{otherwise.} \end{cases}$

Clearly $(\vec{x}_n)_{n \geq 1}$ is bounded. Let $(\vec{x}_{n_i})_{i \geq 1}$ be a subsequence of $(\vec{x}_n)_{n \geq 1}$. We will show that $(\vec{x}_{n_i})_{i \geq 1}$ diverges.

For any $(L_k)_{k \geq 1} \in \ell^\infty$ Let $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$. If there are no values of $p \geq N$ so that $\|(\vec{x}_p)_k - L_k\|_\infty < \varepsilon$, we are done. Now suppose the opposite, that there is a $p \geq N$ such that $\|(\vec{x}_p)_k - L_k\|_\infty < \varepsilon$. Let $n = p + 1$. By the reverse triangle inequality,

$$\begin{aligned} \|(\vec{x}_n)_k - L_k\|_\infty &= \|(\vec{x}_n)_k - (\vec{x}_p)_k + (\vec{x}_p)_k - L_k\|_\infty \geq \|(\vec{x}_n)_k - (\vec{x}_p)_k\|_\infty - \|(\vec{x}_p)_k - L_k\|_\infty \\ &= |1 - \|(\vec{x}_p)_k - L_k\|_\infty| \geq 1 - \varepsilon = \frac{1}{2} \end{aligned}$$

Thus $(\vec{x}_{n_i})_{i \geq 1}$ diverges.

□