3 Week 3 Homework

3.1 Exercise. Let $S \subseteq C[0,1]$. Consider the following two statements:

- S is an open subset of $(C[0,1], \|\cdot\|_1)$.
- S is an open subset of $(C[0,1], \|\cdot\|_\infty)$.

Determine if the first statement implies the second, and vice-versa. Supply proof or counterexample as appropriate.

Proof. We claim that the first statement implies the second, but not the converse.

Suppose that S is an open subset of $(C[0,1], \|\cdot\|_1)$. For any $g \in S$, there is an open ball with respect to the 1-norm centered around g with radius ε such that $B_1(g,\varepsilon) \subseteq S$. We proceed to show that $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon)$. Let $f \in B_\infty(g,\varepsilon)$. Then

$$\|f-g\|_1 = \int_0^1 |f-g| \leq \int_0^1 \sup\{|f-g|\} = \|f-g\|_\infty < \varepsilon$$

Thus $B_\infty(g,\varepsilon) \subseteq B_1(g,\varepsilon) \subseteq S$. Thus S is an open subset of $(C[0,1], \|\cdot\|_1)$.

Now we show that the converse is not necessarily true. Let $S = B_\infty(0,1)$. This is an open subset of $(C[0,1], \|\cdot\|_\infty)$. Consider $f(x) = 0 \in B_\infty(0,1)$. For every $\varepsilon > 0$, we can always find $n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$. Let $g(x) = x^{n-1}$. Since $\int_0^1 g(x)dx = \frac{1}{n} < \varepsilon$, $g(x) \in B_1(0,\varepsilon)$. But $g(1) = 1$, which means that $g(x) \notin B_\infty(0,1)$. Thus f is not an interior point of $B_\infty(0,1)$ with respect to the 1-norm, which means that $B_\infty(0,1)$ is not an open subset of $(C[0,1], \|\cdot\|_1)$.

□

3.2 Exercise. *Can linear subspaces be open and/or closed?*

(a) Let $C^\infty[0,1]$ denote the set of infinitely differentiably functions $f:[0,1] \to \mathbf{R}$. Prove that $C^\infty[0,1]$ is *not* a closed subset of $(C[0,1], \|\cdot\|_\infty)$.

Proof. To show that this set is not closed, we just need to find a limit point that is not an element of the set. Let $f(x) = |x - \frac{1}{2}|$. Clearly $f \notin C^\infty[0,1]$. To prove f is a limit point of $C^\infty[0,1]$, first let $\varepsilon > 0$ and consider the open ball $B(f,\varepsilon)$. There exists an $n \in \mathbb{N}$ such that $n+1 > \frac{1}{\varepsilon} \implies asdfasfas$. Let

$$g_n(x) = \int_{\frac{1}{2}}^{x} \frac{e^{-\frac{n}{t}} - e^{-\frac{n}{1-t}}}{e^{-\frac{n}{t}} + e^{-\frac{n}{1-t}}} dt,\ x \in [0,1]$$

Since the integrand is constructed using infinitely differentiable functions, g_n is infinitely differentiable as well, thus $g_n \in C^\infty[0,1]$. Notice that $g_n(1) = \int_{\frac{1}{2}}^{1} \frac{e^{-\frac{n}{t}} - e^{-\frac{n}{1-t}}}{e^{-\frac{n}{t}} + e^{-\frac{n}{1-t}}} dt$. If we perform the substitution $u = 1-t$, we see that

$$g_n(1) = \int_{\frac{1}{2}}^{1} \frac{e^{-\frac{n}{t}} - e^{-\frac{n}{1-t}}}{e^{-\frac{n}{t}} + e^{-\frac{n}{1-t}}} dt = -\int_{\frac{1}{2}}^{0} \frac{e^{-\frac{n}{1-u}} - e^{-\frac{n}{u}}}{e^{-\frac{n}{1-u}} + e^{-\frac{n}{u}}} du = g_n(0)$$

Thus we can conclude that

$$g_n(1) = \frac{1}{2}(g_n(1) + g_n(0)) = \frac{1}{2}\left(\int_{\frac{1}{2}}^{1} \frac{e^{-\frac{n}{t}} - e^{-\frac{n}{1-t}}}{e^{-\frac{n}{t}} + e^{-\frac{n}{1-t}}} dt + \int_{\frac{1}{2}}^{0} \frac{e^{-\frac{n}{t}} - e^{-\frac{n}{1-t}}}{e^{-\frac{n}{1-t}} + e^{-\frac{n}{t}}} dt\right) = \frac{1}{2}()$$

which we can solve by using the substitution

We compute $\|f - g_n\|_\infty = \sup\{|f(x) - g_n(x)| : x \in [0,1]\}$.

Let $h(x) = f(x) - g_n(x)$. We try to maximize $|h(x)|$. Taking its derivative with respect to x, we get

$$h'(x) = f'(x) - g_n'(x) = \frac{x - \frac{1}{2}}{|x - \frac{1}{2}|} - \frac{e^{-\frac{n}{x}} - e^{-\frac{n}{1-x}}}{e^{-\frac{n}{x}} + e^{-\frac{n}{1-x}}}$$

There is a critical point at $x = \frac{1}{2}$ since $h'(\frac{1}{2})$ is undefined. Otherwise, if $x > \frac{1}{2}$,

$$h'(x) = 1 - \frac{e^{-\frac{n}{x}} - e^{-\frac{n}{1-x}}}{e^{-\frac{n}{x}} + e^{-\frac{n}{1-x}}} = \frac{2e^{-\frac{n}{1-x}}}{e^{-\frac{n}{x}} + e^{-\frac{n}{1-x}}} > 0$$

If $x < 0$,

$$h'(x) = -1 - \frac{e^{-\frac{n}{x}} - e^{-\frac{n}{1-x}}}{e^{-\frac{n}{x}} + e^{-\frac{n}{1-x}}} = \frac{-2e^{-\frac{n}{x}}}{e^{-\frac{n}{x}} + e^{-\frac{n}{1-x}}} < 0$$

Checking all critical points and endpoints, we see that

$$h(\frac{1}{2}) = 0h(0) =$$

$\square$

(b) Let C be the set of **convergent** sequences of real numbers. Prove that C is a closed subset of $(\ell^\infty, \|\cdot\|_\infty)$.

(c) Let $(X, \|\cdot\|)$ be a normed vector space, and let M be a linear subspace of X. Prove that M is an open set if and only if $M = X$.

3.3 Exercise. *The Bolzano–Weierstrass Theorem.*

(a) Prove that every bounded sequence in $(\mathbf{R}^d, \|\cdot\|_2)$ has a convergent subsequence.

Proof. We take the Bolzano-Weierstrass Theorem in $\mathbb{R}$ for granted and use this to prove it for $\mathbb{R}^d$. We will do this using induction on d. When $d = 1$, it follows trivially from the theorem in $\mathbb{R}$.

Now suppose that the claim is true for some $d = k$, for some $k \in \mathbb{N}$. Let $(a_n)_{n\geq 1}$ be a bounded sequence in $(\mathbf{R}^k, \|\cdot\|_2)$. Define another sequence $(b_n)_{n\geq 1}$ in $\mathbb{R}^{k-1}$ such that b_i is the first $k-1$ components of a_i. From our assumption, b_i has a convergent subsequence $(b_{n_i})_{i\geq 1}$. Consider the sequence $(a_{n_i})_{i\geq 0}$. $\square$

(b) Give an example of a normed vector space $(X, \|\cdot\|)$ containing a sequence (x_n) which has no convergent subsequences.