MAT257 – Week 1 Homework

9/13 **1.** Let $A_1, A_2, A_3, \ldots$ be a sequence of countable sets. Prove that $\bigcup_{i\geq 1} A_i$ is countable.
My previous submission did not take into account the fact that the sets could have been not disjoint. This resubmission should resolve that issue.
Changelog:

1. Fixed the argument to address the non-disjoint case.

Proof. Notice that

$$\bigcup_{i\geq 1} A_i = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cap A_2)) \cup \ldots$$

We define a new collection of sets $\{B_i\}$ as follows:

$$B_1 = A_1$$

$$B_k = A_k \setminus \left(\bigcap_{i=1}^{k-1} A_i\right), \, k > 1$$

Since A_i is countable, denote A_{ij} to be the jth element of set A_i. Noting that $\bigcup_{i\geq 1} A_i = \bigcup_{i\geq 1} B_i$, Define $f : \bigcup_{i\geq 1} B_i \to \mathbb{N}$ as

$$f(A_{ij}) = 2^i 3^j$$

We will show that f is injective. Let $A_{pq}, A_{rs} \in \bigcup_{i\geq 1} B_i$. Suppose that $f(A_{pq}) = f(A_{rs})$. Then

$$2^p 3^q = 2^r 3^s$$

Since every integer has a unique prime factorisation, it follows that $p = r$, $q = s$. Thus $A_{pq} = A_{rs}$.

Now, define the injection $g : \mathbb{N} \to \bigcup_{i\geq 1} A_i$ to be $g(n) = A_{1n}$.
Therefore by the Schröder–Bernstein theorem, $|\bigcup_{i\geq 1} A_i| = |\mathbb{N}|$. Thus the set is countable.

$\square$

9/13 **2.** Let X, Y, Z be three vector spaces. Prove that $L^2(X, Y; Z)$ is isomorphic to $L(X, L(Y, Z))$.

(For two vector spaces X, Y, we use $L(X, Y)$ to denote the space of linear mappings from X to Y. For three vector spaces X, Y, Z, and a a function $\beta : X \times Y \to Z$, we let $\beta(\,\cdot\,, y_0)$ denote the function $X \to Z$ given by $x \mapsto \beta(x, y_0)$, where $y_0 \in Y$ is some fixed vector; this is called the y_0**-slice** of β. The x_0**-slice** $\beta(x_0, \,\cdot\,)$ is defined similarly.

Proof. For a bilinear map $T \in L^2(X, Y; Z)$, $(x, y) \in X \times Y$, Define $\phi : L^2(X, Y; Z) \to L(X, L(Y, Z))$ such that

$$\phi(T)(x, y) = T(x, \cdot)(y)$$

where $T(x, \cdot)$ is the x-slice of T. We claim that this transformation is an isomorphism.

First, let $\phi(T) = 0$. Then $\forall x \in X, T(x, \cdot)(y) = 0$, from which it follows that $T(x, y) = 0$, meaning ϕ is injective.

Next, fix $U \in L(X, L(Y, Z))$. Let β be the basis for X. Let $T \in L^2(X, Y; Z)$ be the linear transformation such that $T(x, \cdot) = U(x)$, for all $x \in \beta$. We see that $\forall x, y \in X \times Y$,

$$\phi(T)(x, y) = T(x, \cdot)(y) = U(x)(y)$$

making ϕ surjective.

Thus ϕ is an isomorphism, and we get that $L^2(X, Y; Z) \cong L(X, L(Y, Z))$ as desired. □

9/13 **3.** Let $I = (a, b)$ and $J = (c, d)$ be two open intervals on the real line. Let $f : I \to J$ be an increasing function such that $f(I)$ is dense in J. Prove that f is continuous.

(For two sets $D, S \subseteq \mathbf{R}$ we say that D is **dense** in S if $D \cap (s - \varepsilon, s + \varepsilon) \neq \varnothing$ for all $s \in S$ and all $\varepsilon > 0$.)

Proof. Fixing an $a \in I$, let $\varepsilon > 0$. We can assume without loss of generality that ε is small enough that $(f(a) - \varepsilon, f(a) + \varepsilon) \subseteq J$. Since $f(I)$ is dense in J, we can always find an $y_1 \in (f(a) - \varepsilon, f(a))$ and $y_2 \in (f(a), f(a) + \varepsilon)$ such that $y_1, y_2 \in f(I)$, which means $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in I$.

Take $\delta = \min\{|a - x_1|, |a - x_2|\}$. Let $x \in I$. Suppose that $|x - a| \leq \delta$. If $x = a$, clearly $|f(x) - f(a)| < \varepsilon$. Consider when $x < a$.

We see that due to the choice of δ, we have that $x_1 < x < a$. Using the fact that f is increasing, we obtain

$$f(a) - \varepsilon < y_1 = f(x_1) < f(x) < f(a) \implies -\varepsilon < f(x) - f(a) < 0 \implies f(a) - f(x) = |f(x) - f(a)| < \varepsilon$$

The argument for the case when $x < a$ is almost the exact same, except for the use of x_2 and y_2 instead of x_1 and y_1, as well as the inequalities being swapped.

With this, we can conclude that f is continuous.

□