1 Week 2 Homework - Ethan Hua

1.1 Exercise. (a) Prove that there exists an infinitely differentiable function $\alpha : \mathbf{R} \to \mathbf{R}$ such that $\alpha(t) = 0$ for all $t \leq 0$, and $\alpha(t) > 0$ for all $t > 0$.

Proof. We define $\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\frac{1}{t}}, & \text{if } t > 0. \end{cases}$

Trivially $\alpha(t) = 0$ if $t \leq 0$ and $\alpha(t) > 0$ if $t > 0$. It remains to show that α is infinitely differentiable.

Since 0 is infinitely differentiable, and $e^{-\frac{1}{x}}$ is infinitely differentiable for $x > 0$, it suffices to show that derivatives of all orders of α are continuous at $t = 0$.

We will prove using induction that

$$\forall n \in \mathbb{N} \cup \{0\},\ \lim_{t\to 0} \alpha^{(n)}(t) = 0$$

We will only worry about the right hand limit, as the left hand limit always evaluates to 0.

For $n = 0$,

$$\lim_{t\to 0^+} \alpha^{(0)}(t) = \lim_{t\to 0^+} \alpha(t) = \lim_{t\to 0^+} e^{-\frac{1}{x}} = 0$$

Thus the case for $n = 0$ holds.

Now suppose that the claim holds for $n = k$, for some $k \in \mathbb{N} \cup \{0\}$. It can be shown that when $t > 0$,

$$\alpha^{(k)}(t) = \sum_{i=0}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!} = \frac{(-1)^{k+1}(k+1)!}{t^{k+1}} + \sum_{i=1}^{\infty} \frac{(-1)^{i+k+1}(i+k+1)!}{t^{i+k+1} i!(i+1)!}$$

$\square$

(b) Prove that there exists an infinitely differentiable function $\beta : \mathbf{R} \to \mathbf{R}$ such that $\beta(t) = 1$ for all $t \geq 1$, and $\beta(t) = 0$ for all $t \leq 0$.

Hint: The shape you're looking for is $\dfrac{X}{X+Y}$.

(c) Prove that there exists an infinitely differentiable function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(t) = 1$ for all $t \in [2, 3]$, and $\varphi(t) = 0$ for $t \in \mathbf{R} \setminus (1, 4)$.

Hint: Your function $\beta(t)$ does half the job. Make a function $\gamma(t)$ that does the other half of the job. Then multiply them together.

1.2 Exercise. Let $S \subseteq \mathbf{R}^n$. Consider the following three statements:

- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_1)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_2)$.
- S is a bounded subset of $(\mathbf{R}^n, \|\cdot\|_{\max})$.

Among these statements, determine which implications are true and which are false. There are six implications to investigate. Supply proof or counterexample as appropriate. Include pictures.

1.3 Exercise. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces. A linear mapping $T : X \to Y$ is called **bounded** if there exists a constant $M \geq 0$ such that

$$\|T(x)\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Let $B(X, Y)$ denote the set of these bounded linear operators. The **operator norm** on $B(X, Y)$, denoted by $\|\cdot\|_{\text{op}}$, is defined as follows:

$$\|T\|_{\text{op}} = \sup\{\|T(x)\|_Y : x \in X \text{ and } \|x\|_X \leq 1\}.$$

(a) Prove that $B(X, Y)$ is a linear subspace of $L(X, Y)$.

(In MAT257, the term "linear subspace" means what "subspace" meant in MAT240, *i.e.* "a nonempty subset of a vector space which is closed under addition and scalar multiplication.")

(b) Prove that $\|\cdot\|_{\mathrm{op}}$ is a norm on $B(X, Y)$.

(c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping given by $T(x, y) = (x + y, x)$. Find, with proof, the exact value of $\|T\|_{\mathrm{op}}$. (Here, $\mathbf{R}^2$ is equipped with the usual norm.)

(d) Find, with proof, an example of an unbounded linear operator.