## MAT257 - Week 1 Homework - Ethan Hua

9/13 1. Let  $A_1, A_2, A_3, \ldots$  be a sequence of countable sets. Prove that  $\bigcup_{i>1} A_i$  is countable.

*Proof.* Since  $A_i$  is countable, denote  $A_{ij}$  to be the jth element of set  $A_i$ .

Define  $f: \bigcup_{i\geq 1} A_i \to \mathbb{N}$  as

$$f(A_{ij}) = 2^i 3^j$$

We will show that f is injective. Let  $A_{pq}, A_{rs} \in \bigcup_{i>1} A_i$ . Suppose that  $f(A_{pq}) = f(A_{rs})$ . Then

$$2^p 3^q = 2^r 3^s$$

Since every integer has a unique prime factorisation, it follows that p = r, q = s. Thus  $A_{pq} = A_{rs}$ .

Now, define the injection  $g: \mathbb{N} \to \bigcup_{i>1} A_i$  to be  $g(n) = A_{1n}$ .

Therefore by the Schröder-Bernstein theorem,  $|\bigcup_{i>1} A_i| = |\mathbb{N}|$ . Thus the set is countable.

9/13 2. Let X, Y, Z be three vector spaces. Prove that  $L^2(X, Y; Z)$  is isomorphic to L(X, L(Y, Z)).

(For two vector spaces X,Y, we use L(X,Y) to denote the space of linear mappings from X to Y. For three vector spaces X,Y,Z, and a a function  $\beta: X\times Y\to Z$ , we let  $\beta(\cdot,y_0)$  denote the function  $X\to Z$  given by  $x\mapsto \beta(x,y_0)$ , where  $y_0\in Y$  is some fixed vector; this is called the  $y_0$ -slice of  $\beta$ . The  $x_0$ -slice  $\beta(x_0,\cdot)$  is defined similarly.

*Proof.* For a bilinear map  $T \in L^2(X,Y;Z)$ ,  $(x,y) \in X \times Y$ , Define  $\phi: L^2(X,Y;Z) \to L(X,L(Y,Z))$  such that

$$\phi(T)(x,y) = T(x,\cdot)(y)$$

where  $T(x,\cdot)$  is the x-slice of T. We claim that this transformation is an isomorphism.

First, let  $\phi(T) = 0$ . Then  $\forall x \in X, T(x, \cdot)(y) = 0$ , from which it follows that T(x, y) = 0, meaning  $\phi$  is injective.

Next, fix  $U \in L(X, L(Y, Z))$ . Let  $\beta$  be the basis for X. Let  $T \in L^2(X, Y; Z)$  be the linear transformation such that  $T(x, \cdot) = U(x)$ , for all  $x \in \beta$ . We see that  $\forall x, y \in X \times Y$ ,

$$\phi(T)(x,y) = T(x,\cdot)(y) = U(x)(y)$$

making  $\phi$  surjective.

Thus  $\phi$  is an isomorphism, and we get that  $L^2(X,Y;Z) \cong L(X,L(Y,Z))$  as desired.

9/13 3. Let I = (a, b) and J = (c, d) be two open intervals on the real line. Let  $f: I \to J$  be an increasing function such that f(I) is dense in J. Prove that f is continuous.

(For two sets  $D, S \subseteq \mathbf{R}$  we say that D is **dense** in S if  $D \cap (s - \varepsilon, s + \varepsilon) \neq \emptyset$  for all  $s \in S$  and all  $\varepsilon > 0$ .)

*Proof.* Fixing an  $a \in I$ , let  $\varepsilon > 0$ . We can assume without loss of generality that  $\varepsilon$  is small enough that  $(f(a) - \varepsilon, f(a) + \varepsilon) \subseteq J$ . Since f(I) is dense in J, we can always find an  $y_1 \in (f(a) - \varepsilon, f(a))$  and  $y_2 \in (f(a), f(a) + \varepsilon)$  such that  $y_1, y_2 \in f(I)$ , which means  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$  for some  $x_1, x_2 \in I$ .

Take  $\delta = \min\{|a - x_1|, |a - x_2|\}$ . Let  $x \in I$ . Suppose that  $|x - a| \le \delta$ . If x = a, clearly  $|f(x) - f(a)| < \varepsilon$ . Consider when x < a.

We see that due to the choice of  $\delta$ , we have that  $x_1 < x < a$ . Using the fact that f is increasing, we obtain

$$f(a) - \varepsilon < y_1 = f(x_1) < f(x) < f(a) \implies -\varepsilon < f(x) - f(a) < 0 \implies f(a) - f(x) = |f(x) - f(a)| < \varepsilon$$

The argument for the case when x < a is almost the exact same, except for the use of  $x_2$  and  $y_2$  instead of  $x_1$  and  $y_1$ , as well as the inequalities being swapped.

With this, we can conclude that f is continuous.