Question 1. Define the predicates

$$P(n)\colon \text{ For any set } A \text{, if } |A| = n \text{ then } |\mathcal{P}(A)| = 2^n$$

$$Q(A, n)\colon |A| = n \implies |\mathcal{P}(A)| = 2^n$$

a) Prove $\forall n \in \mathbb{N}, P(n)$.

Proof. **Base Case.** To show $P(0)$, consider any set A such that $|A| = 0$. Then $A = \varnothing$ and its only subset is $\varnothing$. Thus $|\mathcal{P}(A)| = 1 = 2^0$, verifying that $P(0)$ is true.

Induction Step. Suppose $P(k)$ holds for some $k \in \mathbb{N}$. Now $P(k+1)$ will be proven to hold. Let A be a set such that $|A| = k+1$. $k+1$ is at least 1, so A possesses at least one element, which will be denoted as a.

Consider the set $A \setminus \{a\}$. Since $\big|A \setminus \{a\}\big| = k$, by the induction hypothesis,

$$\big|\mathcal{P}(A \setminus \{a\})\big| = 2^k$$

Notice that $\mathcal{P}(A \setminus \{a\})$ contains all the subsets of A that do not contain a. The remaining subsets must all contain a. The remaining subsets of A can be obtained by taking every individual element in $\mathcal{P}(A \setminus \{a\})$ and unioning it with $\{a\}$. Thus A contains twice as many subsets as $A \setminus \{a\}$. In mathematical terms,

$$|\mathcal{P}(A)| = 2 \cdot |\mathcal{P}(A \setminus \{a\})| = 2 \cdot 2^k = 2^{k+1}$$

It has been shown that $P(k+1)$ holds.

By the principle of simple induction, $\forall n \in \mathbb{N}, P(n)$.

$\square$

b) Prove that for every set A, $\forall n \in \mathbb{N}$, $Q(n)$.