1 Seperablility Stuff

1.1 Theorem. *Let (X, d) be a seperable metric space. Then $A \subseteq X$ is seperable with respect to the metric d.*

Proof. Since X is seperable, there exists a countable dense set $D \subseteq X$.

For any $a_i \in A$ and any $k \in \mathbb{N}$, there exists an element $d_{ik} \in D$ such that $d_{ik} \in B(a_i, \frac{1}{k})$.

Consider the set

$$D' = \bigcup_{i \in I} \left(\bigcup_{k \in \mathbb{N}} d_{ik} \right)$$

Notice that $D' \subseteq D$ so it is at most countable.

For all $a \in A$, $\varepsilon > 0$, consider the open ball $B(a, \varepsilon)$. By the Archimedian property, there exists $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon} \implies \frac{1}{k} < \varepsilon$. It follows that $B(a, \frac{1}{k}) \subseteq B(a, \varepsilon)$, and we know we can find a $d_{ik} \in D'$ within this open ball. Thus D' is dense in A.

Since D' is dense and at most countable, A is seperable. □

1.2 Theorem. *Let X, Y be metric spaces let and $f \colon C \to X$ be a continuous function. If $A \subseteq X$ is seperable, then $f(A)$ is seperable.*

Proof. Define d_X and d_Y to be metrics on X and Y respectively. Suppose that A is seperable. We will show that $f(A)$ is seperable by equivalently showing that it is pre-totally bounded.

Let $\varepsilon > 0$. By the continuity of f, for every $a \in A$, there exists $\delta > 0$ such that for all $x \in A$, $d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon$. Keep this value of δ.

Since A is seperable, by Theorem y.x (replace this with the thm number idk what it is), A is pre-totally bounded. By definition, A is covered by a countable subcover $\{B_X(a_i, \delta)\}_{i \in \mathbb{N}}$.

Consider the countable collection $\{B_Y(f(a_i), \varepsilon)\}_{i \in \mathbb{N}}$. We will show that this collection covers $f(A)$. Let $y \in f(A)$. Then $y = f(x)$ for some $x \in A \subseteq \{B_X(a_i, \delta)\}_{i \in \mathbb{N}}$. This implies that x is an some open ball $B_X(a_k, \delta) \implies d_X(x, a_k) < \delta$. By the continuity of f, this implies that $d_Y(y, f(a_k)) = d_Y(f(x), f(a_k)) < \varepsilon \implies y \in B_Y(f(a_k), \varepsilon) \subseteq \{B_Y(f(a_i), \varepsilon)\}_{i \in \mathbb{N}}$.

We see that $f(A)$ is pre-totally bounded, which implies by thm 29384qwurjhq that $f(A)$ is seperable. □