

Question 1.

Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Use row and column operations on A to obtain a matrix B of the form in Theorem 53. Use that work to find invertible matrices P, Q so that $B = PAQ$.

Proof. We perform the following row and column operations:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \rightarrow r_3 - 2r_1]{r_2 \rightarrow r_2 - r_1} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{r_3 \rightarrow r_3 - r_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[c_4 \rightarrow c_4 - c_1 + c_2]{c_3 \rightarrow c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Define this matrix we obtained as B . We will perform the same row and column operations above on I_3 and I_4 , respectively in order to define P and Q . We have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \rightarrow r_3 - 2r_1]{r_2 \rightarrow r_2 - r_1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 \rightarrow r_3 - r_2} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4 \rightarrow c_4 - c_1 + c_2]{c_3 \rightarrow c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We see that

$$PAQ = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= B
\end{aligned}$$

as required. □

Question 2.

Let $A = \begin{pmatrix} 1 & -2 & -4 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

- (a) Verify that A is invertible, by row-reducing the augmented matrix $(A|I_3)$.
- (b) Use (a) to find A^{-1} .
- (c) Express A as a product of elementary matrices.

Proof.

(a): We see that

$$\begin{aligned}
(A|I_3) &= \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_2 \rightarrow r_2 - r_1, r_3 \rightarrow r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array} \right) \\
&\xrightarrow{r_1 \rightarrow r_1 + r_3, r_2 \rightarrow r_2 - r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array} \right) \xrightarrow{r_3 \rightarrow r_3 - 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array} \right) \\
&\xrightarrow{r_1 \rightarrow r_1 + \frac{1}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array} \right) \xrightarrow{r_3 \rightarrow \frac{1}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array} \right)
\end{aligned}$$

Since A can be row reduced into the identity matrix, A is invertible.

(b):

By our row reductions above, we know that $A^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & -1 \\ -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$.

(c):

To express A as a product of elementary matrices, we can apply the opposite row operations to the identity matrix in reverse order. That is,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

□

Question 3.

Find the explicit formula for the linear transformation $T : \mathbb{Q}^4 \rightarrow \mathbb{Q}^3$ which satisfies:

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Proof. Notice that

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is a basis for \mathbb{Q}^4 . We attempt to find the general form for a vector $(x, y, z, w) \in \mathbb{Q}^4$ in terms of these vectors. By inspection, we see that

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = (x - 2y + z) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (y - z) \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (z - w) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus

$$\begin{aligned} T \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} &= (x - 2y + z) T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (y - z) T \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (z - w) T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &= (x - 2y + z) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (y - z) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (z - w) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x - 2y + z \\ 2x - 3y + z \\ 3x - 5y + 3z - w \end{pmatrix} \end{aligned}$$

□

Question 4.

Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathcal{M}_{2 \times 2}(\mathbb{F})$. Consider the linear map $T : \mathcal{M}_{2 \times 2}(\mathbb{F}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{F})$ given by $T(A) = A^T$. Set $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$.

Question 5.

Let $T : \mathcal{M}_{n \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{n \times n}(\mathbb{F})$ be the linear map given by $T(A) = A + A^T$.

- (a) Find $N(T)$ and $\dim N(T)$.

We claim that $N(T)$ is the set of all skew symmetric matrices with zeroes on the diagonal, which has dimension $\frac{1}{2}n(n-1)$.

Set $T(A) = A + A^T = 0$. We have that $A_{ij} + A_{ji} = 0$ for each $0 < i, j \leq n$. In particular, we have that $A_{ii} = 0$ if $i = j$ and $A_{ij} = -A_{ji}$ otherwise. But this describes exactly all skew symmetric matrices with zeroes on the diagonal. The basis for this set is

$$\beta = \{E_{ij} - E_{ji} : 0 < i < j \leq n\}$$

and there are $\frac{1}{2}n(n-1)$ vectors in this set, so $\dim N(T) = \frac{1}{2}n(n-1)$.

- (b) What is $\text{im}(T)$?

We claim that $\text{im}(T)$ is the set of all symmetric matrices S_n . We see that

$$(A + A^t)_{ij} = A_{ij} + A_{ij}^t = A_{ij} + A_{ji} = A_{ji} + A_{ji}^t = (A + A^t)_{ji}$$

so $\text{im}(T) \subseteq S_n$. To show set equality, suppose that B is a symmetric matrix. Let $A = \frac{1}{2}B$ then

$$T(A) = \frac{1}{2}T(B) = \frac{1}{2}(B + B^t) = B$$

Thus $\text{im}(T) = S_n$ and has basis

$$\gamma = \{E_{ij} : 0 < i \leq j \leq n\}.$$

and is dimension $\frac{1}{2}n(n+1)$.

- (c) Is $\mathcal{M}_{n \times n}(\mathbb{F}) = \text{im}(T) \oplus N(T)$?

Yes.

To show this, notice that $\beta \cap \gamma = \emptyset$, so $\text{im}(T) \oplus N(T)$ has basis $\alpha = \beta \cup \gamma$. But notice that $|\alpha| = \frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$, which is the dimension of $\mathcal{M}_n(\mathbb{F})$. Therefore α is actually a basis for $\mathcal{M}_n(\mathbb{F})$ and thus $\mathcal{M}_n(\mathbb{F}) = \text{im}(T) \oplus N(T)$.

Question 6.

Let V, W be vector spaces over a field \mathbb{F} and $T : V \rightarrow W$ a linear map. Prove that T is injective if and only if $N(T) = \{\mathbf{0}_V\}$. (Make no assumption here about $\dim V, \dim W$.)

Proof. Suppose that T is injective. Let $T(x) = 0$, for some $x \in V$. Recall that $T(0) = 0$ for any linear map. Therefore by injectivity $x = 0$, so $N(T) = \{0\}$.

Conversely, suppose that $N(T) = \{0\}$. Let $x, y \in V$ such that $T(x) = T(y)$. By linearity, we have that $T(x - y) = 0$, but this implies that $x - y = 0$, so $x = y$ and T is injective.

□

Question 7.

Let V, W be vector spaces over a field \mathbb{F} , and $T : V \rightarrow W$ a linear map. Find a condition on T which is equivalent to " $T(S)$ spans W for any spanning set $S \subseteq V$ of V ".
(Hint: Write down the definition of $T(S)$ is spanning to get started.)

Proof. We claim that this statement is equivalent to saying that T is surjective. Suppose that for any set $S \subseteq V$ that spans V , $T(S)$ spans W . We prove that T is surjective. Let $w \in W$. We can write w as a linear combination of some number of vectors in $T(S)$. That is, for some $k \in \mathbb{N}$ and $s_i \in S$, $c_i \in \mathbb{F}$, $i \in \{1, \dots, k\}$,

$$w = \sum_{i=1}^k c_i T(s_i) = T \left(\sum_{i=1}^k c_i s_i \right)$$

so T is surjective.

Conversely, suppose that T is surjective. Let S be a spanning set of V . We will show that $T(S)$ spans W . Let $w \in W$. By surjectivity, there exists $v \in V$ so that $T(v) = w$. We can rewrite

$$v = \sum_{i=1}^k c_i s_i$$

for some number of vectors $s_i \in S$ and $c_i \in \mathbb{F}$. Then

$$T \left(\sum_{i=1}^k c_i s_i \right) = w \implies \sum_{i=1}^k c_i T(s_i) = w$$

Notice that $T(s_i) \in T(S)$, from which it follows that $T(S)$ spans W , and the proof is complete. □

Question 8.

Let $P \in \mathcal{M}_{n \times n}(\mathbb{F})$. Prove the following three conditions are equivalent.

- (a) P is invertible.
- (b) There exists bases β, γ of \mathbb{F}^n so that $P = [I_{\mathbb{F}^n}]_{\beta}^{\gamma}$.
- (c) For any n -dimensional vector space V over \mathbb{F} , there exists bases β, γ of V so that $P = [I_V]_{\beta}^{\gamma}$.

Proof. Suppose (a). We prove (b) and (c) at the same time.

Question 9.

Consider the relation \equiv on $\mathcal{M}_{m \times n}(\mathbb{F})$ defined by $A \equiv B$ if $A \rightarrow B$ using a combination of row and/or column operations.

- (a) Prove that \equiv is an equivalence relation on $\mathcal{M}_{m \times n}(\mathbb{F})$.
- (b) Find a condition on A, B which is equivalent to $A \equiv B$. (Hint: Theorem 53.)
- (c) Classify the equivalence classes for this relation, and prove that there are exactly $1 + \min\{n, m\}$ such classes.

Proof.

(a):

We show reflexivity, symmetry, and transitivity in that order.

Reflexivity: Since $IA = A$, and I is considered a row operation, $A \equiv A$.

Symmetry: Suppose that $A \equiv B$ then for some invertible matrices P, Q we have that $PAQ = B$. But at the same time this means that $P^{-1}BQ^{-1} = A$ so $B \equiv A$.

Transitivity: Suppose that $A \equiv B$ and $B \equiv C$. Then for invertible matrices P, Q, R, S , $PAQ = B$ and $RBS = C$, so $(RP)A(QS) = R(PAQ)S = RBS = C$. Since RP, QS are also invertible, we have that $A \equiv C$.

(b):

We claim that an equivalent condition is $\text{rank}A = \text{rank}B$. Suppose that $A \equiv B$. Then $PAQ = B$ for some invertible matrices P, Q , but it is known that rank is preserved by multiplication with invertible matrices, so $\text{rank}A = \text{rank}PAQ = \text{rank}B$.

Conversely, suppose that $r := \text{rank}A = \text{rank}B$. By Theorem 53, there exist row/column operations so that

$$A, B \rightarrow \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right).$$

We denote this matrix by J_r . that is, for invertible matrices P, Q, R, S , $PAQ = I' = RBS$. It follows that $R^{-1}PAQS^{-1} = B$, so $A \equiv B$ as desired.

(c):

We can classify the equivalence classes by matrix rank. That is, each equivalence class is of the form

$$[J_r] = \{A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \text{rank}A = r\}.$$

The possible ranks of $m \times n$ matrices range from 0 to $\min\{n, m\}$, so there are $\min\{n, m\} + 1$ different values of r . We will verify that these equivalence classes are exhaustive and disjoint. Every $m \times n$ matrix must have a rank, so it belongs to at least one of the classes, but at the same time, a matrix can possibly only have one rank, so it necessarily belongs to exactly one equivalence class.

□

Question 10.

Let V, W be finite dimensional vector spaces over \mathbb{F} , and $T : V \rightarrow W$ a linear map with $\text{rank} T = 2$. Set $n = \dim V$, $m = \dim W$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}^n$ be two non-parallel vectors. Prove there exists bases β, γ of V, W respectively, so that $[T]_{\beta}^{\gamma} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$. (Hint: use problems 7,8.)

Proof. By the Dimension Theorem, $\text{null}(T) = n - 2$. Let $\alpha = \{v_1, \dots, v_{n-2}\}$ be a basis for $N(T)$ and extend this basis into a basis $\{a, b, v_1, \dots, v_{n-2}\}$ which we set as γ .

□

Question 11.

Let $T : V \rightarrow V$ be linear. We say that a subspace $W \subseteq V$ is “ T -invariant” if $T(W) \subseteq W$. For example, if $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is counter-clockwise rotation around the z -axis by angle θ , then $P_{xy} = \{(x, y, 0) \in \mathbb{R}^3\}$ is T -invariant, as is L_z (the z -axis).

- Verify the claims made above, by showing that P_{xy} and L_z are T -invariant.
- Show that $\mathbb{R}^3 = P_{xy} \oplus L_z$ by finding a basis $\beta = \beta_1 \cup \beta_2$ for \mathbb{R}^3 so that β_1 is a basis for P_{xy} and β_2 is a basis for L_z .
- Using your basis β from (b), find $[T]_{\beta}$.

Question 12.

Let V be a finite dimensional vector space over \mathbb{F} , $T \in \mathcal{L}(V)$, and $W_1 \subseteq V$ a T -invariant subspace with basis β_1 . Set $k = \dim W_1$.

We will generalize what we saw in #11c.

- Extend β_1 to a basis β of V . Show that $[T]_{\beta} = \begin{pmatrix} A & C \\ O_{n-k,k} & B \end{pmatrix}$, where A is $k \times k$, B is $(n - k) \times (n - k)$, and C is $k \times (n - k)$.
- Suppose that W_2 is a subspace so that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$, where β_2 is any basis for W_2 .

Prove that if W_2 is T -invariant, then $[T]_{\beta} = \begin{pmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{pmatrix}$ is block diagonal.

- Is the converse of (b) true or false? Justify your answer.

Question 13.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

- (a) Let $\beta = \{e_1, \dots, e_n\}$ be the standard basis for \mathbb{F}^n , and $\gamma = \{v_1, \dots, v_n\}$ a basis for \mathbb{F}^n . Then there exists a sequence of row operations that takes β to γ . (That is, v_i is obtained from e_i using the same row operations for all i .)
- (b) Let V be a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ a linear map. If β, γ are bases for V so that $[T]_{\beta}^{\gamma} = I_n$, then $T = I_V$.
- (c) Let V be a finite dimensional vector space over \mathbb{F} and $S, T : V \rightarrow V$ linear maps. If $\text{rank } T = \text{rank } S$, then there exist bases $\beta, \beta', \gamma, \gamma'$ for V so that $[S]_{\beta}^{\gamma} = [T]_{\beta'}^{\gamma'}$.
- (d) Let $A, B \in \mathcal{M}_{n \times n}(\mathbb{F})$. If $A^2 \sim B^2$, then $A \sim B$.