Exercise 13.14

Solvers: Ethan
Recall the statement of Theorem 13.6 and the definition of a simple mapping:
Theorem 13.6. Let $U, V \subseteq \mathbf{R}^n$ be two open sets and let $\Phi : U \to V$ be a diffeomorphism. Let $E \subseteq \mathbf{R}^n$ be a measurable set. Then $\Phi(E)$ is also measurable, and its measure is

$$\mu(\Phi(E)) = \int_E |\det J\Phi|.$$

Definition 13.12. Let $\Phi : U \to \mathbf{R}^n$ be a function defined on some set $U \subseteq \mathbf{R}^n$, and let $\varphi_1, \ldots, \varphi_n$ be the component functions of Φ:

$$\Phi(p) = (\varphi_1(p), \ldots, \varphi_n(p)) \quad \text{for all } p \in U.$$

If the ith component function is just x_i — *i.e.* $\varphi_i(x_1, \ldots, x_n) = x_i$ for some i — then we say that Φ is x_i**-simple**. We say that Φ is **simple** if it is x_i-simple for all but one i.
Exercise 13.14. Show that Theorem 13.6 holds if Φ is simple.

Proof. Let $U, V \subseteq \mathbb{R}^n$ be open, and let $\Phi : U \to V$ be a simple diffeomorphism. Without loss of generality, let the only non-simple component of Φ be φ_n, and let $E = \prod_{i=1}^n [a_i, b_i] \subseteq U$ be a closed box. Since Φ is a diffeomorphism, this implies that $\Phi(E)$ is measurable, and as well, because Φ is simple,

$$\Phi(E) = \left\{ (x_1, ..., x_{n-1}, \varphi(x)) \in V : x = (x_1, ..., x_n) \in \prod_{i=1}^n [a_i, b_i] \right\}$$

Let $E' = \prod_{i=1}^{n-1} [a_i, b_i]$, and define $\Psi : E' \to \mathbb{R}$ by

$$\Psi(x') = \int_{\varphi(x', [a_n, b_n])} 1 \; dx_n.$$

Φ is a diffeomorphism, so the restriction $\varphi(x', \cdot)$ is also a diffeomorphism. It follows that the image $\varphi(x', [a_n, b_n])$ is an interval. Now, we want to apply single-variable change of variables with $\varphi(x', u) = x_n$. Notice that

$$dx_n = \frac{\partial \varphi}{\partial u}(x', u) du$$

and

$$\det J\Phi = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{\partial \varphi}{\partial u}(x', u) \end{vmatrix} = \frac{\partial \varphi}{\partial u}(x', u)$$

so it follows that

$$\Psi(x') = \int_{\varphi(x', [a_n, b_n])} 1 \; dx_n = \int_{a_n}^{b_n} |\det J\Phi|$$

Finally, using Fubini's Theorem, we get that

$$\mu(\Phi(E)) = \int_{\Phi(E)} 1 = \int_{E'} \Psi = \int_{E'} \int_{a_n}^{b_n} |\det J\Phi| = \int_E |\det J\Phi|$$

as needed. $\square$