Question 1.

Use row operations on the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix}$ to obtain an upper triangular matrix, then use Theorem 59 to find $\det A$. (You will get no credit for using a row/column expansion.)

We have

$$\begin{aligned}
\det A &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 4 & -10 & -4 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -12 & -6 \end{pmatrix} \\
&= -6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix} \\
&= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{pmatrix} \\
&= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \\
&= 6(1)(2)(2)\left(\frac{1}{2}\right) \\
&= 12
\end{aligned}$$

Question 2.

Let $T = T_A : \mathbb{Q}^5 \to \mathbb{Q}^5$ where $A = \begin{pmatrix} 1 & 0 & 1 & -2 & 0 \\ 3 & 0 & 1 & 0 & -2 \\ 2 & 0 & 0 & 2 & -2 \\ 2 & 0 & 0 & 1 & -2 \\ 2 & 0 & 1 & -2 & -1 \end{pmatrix}$.

(a) Find C_T and the eigenvalues of T.

We have

$$\begin{aligned} C_T(\lambda) &= \det(\lambda I - T) \\ &= \det \begin{pmatrix} \lambda-1 & 0 & -1 & 2 & 0 \\ -3 & \lambda & -1 & 0 & 2 \\ -2 & 0 & \lambda & -2 & 2 \\ -2 & 0 & 0 & \lambda-1 & 2 \\ -2 & 0 & -1 & 2 & \lambda+1 \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} \lambda-1 & -1 & 2 & 0 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda-1 & 2 \\ -2 & -1 & 2 & \lambda+1 \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} \lambda+1 & 0 & 0 & -\lambda-1 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda-1 & 2 \\ -2 & -1 & 2 & \lambda+1 \end{pmatrix} \\ &= -\lambda \left((\lambda+1) \det \begin{pmatrix} \lambda & -2 & 2 \\ 0 & \lambda-1 & 2 \\ -1 & 2 & \lambda+1 \end{pmatrix} + (\lambda+1) \det \begin{pmatrix} -2 & \lambda & -2 \\ -2 & 0 & \lambda-1 \\ -2 & -1 & 2 \end{pmatrix} \right) \\ &= \lambda(\lambda+1)\left(-(\lambda-1)(\lambda(\lambda+1)+2) + 2(2\lambda-2) + -2(2\lambda-2) + (\lambda-1)(2+2\lambda)\right) \\ &= \lambda(\lambda+1)\left((\lambda-1)(2-\lambda-\lambda^2-2+2\lambda\right) \\ &= \lambda(\lambda+1)(\lambda-1)(\lambda-\lambda^2) \\ &= -\lambda^2(\lambda-1)^2(\lambda+1) \end{aligned}$$

The eigenvalues are the roots of C_T, which are $\lambda = 0, 1, -1$.

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

For $\lambda = 0$, we solve the equation $Ax = 0$ via row reduction:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & 1 & -2 & 0 \\ 2 & 0 & 1 & -2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We get that $x_1 = 0, x_4 = 0, x_3 + x_5 = 0$. We parametrize $x_2 = t, x_3 = s$ and get

$$x = \begin{pmatrix} 0 \\ t \\ s \\ 0 \\ -s \end{pmatrix} = te_2 + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Thus a basis for $E_0(T)$ is $\left\{ e_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

For $\lambda = 1$, to solve $(A - I)x = 0$, we get the system

$$\begin{aligned} \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 3 & -1 & 1 & 0 & -2 & 0 \\ 2 & 0 & -1 & 2 & -2 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 1 & -2 & -2 & 0 \end{array}\right) &\to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & -1 & 2 & -2 & 0 & 0 \\ 2 & 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 \end{array}\right) \\ &\to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ &\to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{aligned}$$

We parametrize $x_4 = t, x_5 = s$ to get

$$x = \begin{pmatrix} s \\ 2t + s \\ 2t \\ t \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Thus a basis for $E_1(T)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

For $\lambda = -1$, we solve $(A + I)x = 0$:

$$\left(\begin{array}{ccccc|c} 2 & 0 & 1 & -2 & 0 & 0 \\ 3 & 1 & 1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 2 & -2 & 0 \\ 2 & 0 & 0 & 2 & -2 & 0 \\ 2 & 0 & 1 & -2 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 2 & 0 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 2 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & -2 & 1 & -4 & 4 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_5 = t$. The general solution is

$$x = t \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So a basis for $E_{-1}(T)$ is $\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(c) Determine if T is diagonalizable, and if so, find a basis β so that $[T]_\beta$ is diagonal.

Since the dimension of each eigenspace matches the algebraic multiplicity of each corresponding eigenvalue, T is diagonalizable and the basis β that makes $[T]_\beta$ diagonal is exactly the basis consisting of the basis vectors of each eigenspace. In particular,

$$\beta = \left\{ e_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Question 3.

1. Read the proof of Theorem 58 from the additional file in the Week 10 Readings on the course page.
2. Prove Part 1 of Theorem 59 using a strategy similar to the proof of Theorem 58. (You cannot use other parts of Theorem 59 in this proof.)

Question 4.

Assume that Parts 1 and 2 of Theorem 59 have been proved. You cannot use Parts 4 through 7 of Theorem 59 in the following problem.

1. Prove Part 3 using induction on n. (Check $n = 1, 2$ by hand, then in the inductive step assume $n + 1 \geq 3$.)
2. Prove Part 4 using row-swapping matrices and properties of determinants.

Question 5.

Prove that if $U \in M_{n\times n}(F)$ is upper triangular, then $\det U = \prod_{i=1}^{n} U_{ii}$.

Question 6.

Let V be a vector space over F, and $T : V \to V$ a linear map. If $W \subseteq V$ is a T-invariant subspace, then the restriction map $T_W : W \to W$ is defined as follows.

1. Prove that $A = [T_W]_{\beta_W}$ using the relationship $[T]_\beta = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.
2. Prove that $\det M = (\det A)(\det C)$.

Question 7.

Deduce from Question 6 that if W is a T-invariant subspace, then C_{T_W} divides C_T.

Question 8.

Let V be a finite-dimensional vector space over a field F, and $W_1, W_2 \subseteq V$ such that $V = W_1 \oplus W_2$. Define the projection maps $P_i : V \to V$ by $P(x) = x_i$ where $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$.

1. Prove that P_i is linear.
2. Prove that $P_i^2 = P_i$.
3. Prove that each W_j is P_i-invariant.
4. Determine if P_i is diagonalizable and justify your answer.

Question 9.

Define the direct sum for more than two subspaces. Let $W_1, \ldots, W_k \subseteq V$ be subspaces such that $V = W_1 \oplus \cdots \oplus W_k$.

1. Prove that every basis β for V gives a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_n$ where $\dim W_i = 1$.
2. Prove the converse: If $V = W_1 \oplus \cdots \oplus W_n$ with $\dim W_i = 1$, then choosing non-zero $w_i \in W_i$ forms a basis for V.
3. Let $T : V \to V$ be linear. Show that $[T]_\beta$ is block diagonal.

Question 10.

Let $W_1, \ldots, W_k \subseteq V$ with bases $\beta_1, \ldots, \beta_k$. Prove that $V = W_1 \oplus \cdots \oplus W_k$ if and only if $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V.

Question 11.

Determine whether the following statements are true or false. Justify your answers.

1. If $V = W_1 \oplus W_2$ and T_{W_1}, T_{W_2} are diagonalizable, then T is diagonalizable.
2. If $W_i \cap W_j = \{0\}$ for $i \neq j$ and $V = W_1 + W_2 + W_3$, then $V = W_1 \oplus W_2 \oplus W_3$.
3. If $\dim V = 7$, $\dim N(T) = 3$, and $\operatorname{rank}(T - I) = 4$, then T is diagonalizable.