Exercise 9.10

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Writeup: Ethan

Suppose that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix, and let Q_D be the corresponding quadratic form. *Q_D achieves a minimum at the origin if and only if $\lambda_i \geq 0$ for all $i \in \{1, ..., n\}$.*

Proof. First note that $Q_D(x) = x^T D x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} \lambda_i x_i^2$.

Suppose that every $\lambda_i \geq 0$. It follows that $\lambda_i x_i^2 \geq 0$. Thus

$$Q_D(x) = \sum_{i=1}^{n} \lambda_i x_i^2 \geq 0$$

Since $Q_D(0) = 0$, Q_D achieves a minimum at the origin.

Conversely, suppose that $\lambda_j < 0$, for some $j \in \{1, ..., n\}$. $Q_D(0) = 0$, but $Q_D(e_j) = \lambda_j < 0$, so the origin is not a minimum.

$\square$