Question 1.

Find the error term for the derivative approximation:

$$f''(x_0) \approx \frac{2f(x_0-h) - 3f(x_0) + f(x_0+2h)}{3h^2}.$$

We write the polynomial expansion for each term on the right:

$$f(x_0-h) = f(x_0) - f'(x_0)h + f''(x_0)h^2 - \frac{f'''(\xi_1)}{6}h^3$$

$$f(x_0) = f(x_0)$$

$$f(x_0+2h) = f(x_0) + 2f'(x_0)h + 4f''(x_0)h^2 + \frac{4f'''(\xi_2)}{3}h^3$$

Then

$$2f(x_0-h) - 3f(x_0) + f(x_0+2h) = 6f''(x_0)h^2 - \frac{2f'''(\xi_1)}{6}h^3 + \frac{4f'''(\xi_2)}{3}h^3$$

$$\frac{2f(x_0-h) - 3f(x_0) + f(x_0+2h)}{3h^2} = 2f''(x_0)h^2 - \frac{1}{9}f'''(\xi_1)h + \frac{4}{9}f'''(x_0)h$$

so the error term is

$$f''(x_0) - \left[2f''(x_0)h^2 + \frac{1}{9}f'''(\xi_1)h - \frac{4}{9}f'''(x_0)h\right] = -f''(x_0)h^2 + \frac{1}{9}f'''(\xi_1)h - \frac{4}{9}f'''(x_0)h$$

Question 2.

Find the error term for the quadrature method, and state its degree of precision.

$$\int_{x_0}^{x_0+2h} f(x)\,dx \approx \frac{h}{2}\left[3f\left(x_0 + \frac{4}{3}h\right) + f(x_0)\right].$$

We expand the left hand side:

$$\int_{x_0}^{x_0+2h} f(x)\,dx$$
$$= \int_{x_0}^{x_0+2h} f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \frac{f^{(4)}(\xi_1)}{24}(x-x_0)^4\,dx$$
$$= 2f(x_0)h + 2f'(x_0)h^2 + \frac{4}{3}f''(x_0)h^3 + \frac{2}{3}f'''(x_0)h^4 + \frac{4}{15}f^{(4)}(\xi_1)h^5$$

Now we expand each term on the right hand side:

$$f\left(x_0 + \frac{4}{3}h\right) = f(x_0) + \frac{4}{3}f'(x_0)h + \frac{8}{9}f''(x_0) + \frac{32}{81}f'''(x_0)h^3 + \frac{32}{243}f^{(4)}(\xi_2)h^4.$$

Thus

$$\frac{h}{2}\left[3f\left(x_0+\frac{4}{3}h\right)+f(x_0)\right]=2f(x_0)h+2f'(x_0)h^2+\frac{4}{3}f''(x_0)+\frac{16}{27}f'''(x_0)h^4+\frac{16}{81}f^{(4)}(\xi_2)h^5.$$

and the error term is

$$\int_{x_0}^{x_0+2h}f(x)\;dx-\frac{h}{2}\left[3f\left(x_0+\frac{4}{3}h\right)+f(x_0)\right]=\frac{2}{27}f'''(x_0)h^4+\frac{4}{15}f^{(4)}(\xi_1)h^5-\frac{16}{81}f^{(4)}(\xi_2)h^5.$$

and its degree of precision is 2.

Question 3.

Consider the integral $\int_1^7 \cos(x^2)\;dx$

(a) Use the composite Simpson's rule to approximate the value of this integral using $n=3$ intervals.

We split the interval $[1,7]$ into the three intervals $[1,3]$, $[3,5]$, $[3,7]$ and approximate the integral on each interval. Using Simpson's rule,

$$\begin{aligned}\int_1^7 \cos(x^2)\;dx &= \int_1^3 \cos(x^2)\;dx+\int_3^5 \cos(x^2)\;dx+\int_5^7 \cos(x^2)\;dx\\ &\approx \frac{(\cos 1^2+4\cos 2^2+\cos 3^2)+(\cos 3^2+4\cos 4^2+\cos 5^2)}{3}\\ &\quad+\frac{\cos 5^2+4\cos 6^2+\cos 7^2}{3}\\ &\approx -1.985\end{aligned}$$

(b) Determine the number of intervals n needed to guarantee an error of at most 10^{-4}.

Using composite Simpson's rule, the absolute error for using n intervals is $\frac{6^5}{180n^4}|f^{(4)}(\xi)|$, for some $\xi\in(1,7)$. If we calculate the fourth derivative of f, we get

$$f^{(4)}(x)=48x^2\sin(x^2)+(16x^4-12)\cos(x^2)$$

so since $\xi\in(1,7)$,

$$|f^{(4)}(\xi)|\le 48\cdot 7^2+16\cdot 7^4-12=40756$$

Thus the error bound is $\frac{6^5\cdot 40756}{180n^4}$. We solve for n such that $\frac{6^5\cdot 40756}{180n^4}<10^{-4}$ to get

$$n^4>\frac{6^5\cdot 10^4\cdot 40756}{180}\implies n>\sqrt[4]{\frac{6^5\cdot 10^4\cdot 40756}{180}}\approx 364.3$$

Thus we need around 365 intervals to guarantee an error better than 10^{-4}.

Question 4.

Consider the IVP:

$$2\dot{y} + y = t^4 + 1, \;\; y(1) = 2.$$

Apply the second degree Taylor method with $h = 0.5$ to this ODE to approximate $y(2)$. Show the details in each step.

We let $f(t, y) = \dfrac{t^4 - y + 1}{2}$ and rearrange the ODE to see that

$$\dot{y} = f(t, y).$$

Set $t_0 = 1, y_0 = y(1) = 2$. Then

$$y_1 = y_0 + \frac{1}{2} f\left(t_0 + \frac{1}{2}, y_0\right) = 2 + \frac{1}{2} f\left(\frac{3}{2}, 2\right) = 2 + \frac{1}{2} \cdot \frac{65}{32} = \frac{129}{64}.$$

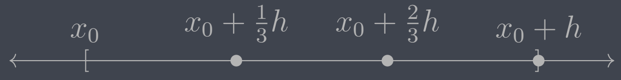
$$\begin{aligned} y_2 = y_1 + \frac{1}{2} f(t_1, y_1) = \frac{129}{64} + \frac{1}{2} f\left(2, \frac{129}{64}\right) &= \frac{129}{64} + \frac{64 \cdot 2^4 - 129 + 64}{128} \\ &= \frac{129}{64} + \frac{959}{128} = \frac{1217}{128} \end{aligned}$$

Thus

$$y(2) \approx \frac{1217}{128}$$

Question 5.

Derive an ODE solver based on the stencil and corresponding integration formula.



Formula: $\frac{h}{4}\left(3f\left(x_0 + \frac{1}{3}h\right) + f(x_0 + h)\right) + O(h^4)$

Suppose we have the IVT:

$$\dot{y} = f(t, y), \;\; y(t_0) = y_0$$

Let $h = t_{i+1} - t_i$. Define

$$\begin{aligned} k_1 &= f(t_i, y_i) \\ k_2 &= f\left(t_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right) \\ k_3 &= f(t_i + h, y_i + hk_2). \end{aligned}$$

Our update step is

$$\begin{aligned} y_{i+1} &= y_i + \int_{t_i}^{t_{i+1}} f(t, y)\ dt \\ &= y_i + \frac{h}{4}(3k_2 + k_3). \end{aligned}$$