Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that O_n is a smooth manifold, and find its dimension.

Proof. First, we note that $O_n(\mathbb{R})$ is the zero set of the function $f : M_n(\mathbb{R}) \to S^n$ defined by

$$f(A) = A^t A - I_n$$

where S^n is the set of symmetric $n \times n$ matrices. We will show that $Jf(X)(h) = X^t h + h^T X$. Indeed,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(X+h) - f(X) - X^t h - h^t X}{\|h\|} &= \lim_{h \to 0} \frac{(X+h)^t(X+h) - X^t X - X^t h - h^t X}{\|h\|} \\
&= \lim_{h \to 0} \frac{h^t h}{\|h\|} \\
&= 0
\end{aligned}$$

Next, we want to show that $\operatorname{rank} Jf(X) = \frac{1}{2}n(n+1)$ for all $X \in O_n(\mathbb{R})$. It suffices to show that $Jf(X)$ is surjective to S^n.
Let $Y \in S^n$. Let $h = \frac{1}{2}XY$. We see that

$$\begin{aligned}
Jf(X)(h) = X^t \left(\frac{1}{2}XY\right) + \left(\frac{1}{2}XY\right)^t X &= \frac{1}{2}\left(X^t XY + Y^t X^t X\right) \\
&= \frac{1}{2}(Y + Y^t) && (X \text{ is orthogonal}) \\
&= Y && (Y \text{ is symmetric})
\end{aligned}$$

Thus $R(Jf(X)) = S^n$ so $\operatorname{rank} Jf(X) = \dim S^n = \frac{1}{2}n(n+1)$.

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