

Question 42.

Let $M \subseteq \mathbf{R}^N$ be a smooth n -manifold (with or without boundary!).

- (a) Show that if $n < N$, then M is a *Lebesgue null set*.
- (b) Show that if $n = N$ and M is closed and its boundary is nonempty, then ∂M coincides with the usual topological boundary (as defined on Handout #2).
- (c) Show that if M is compact and its boundary is nonempty, then M is Jordan measurable.

Proof.

(a):

We begin by proving a number of lemmas:

Lemma 1: An open cover of any subset $M \subseteq \mathbf{R}^n$ has a countable subcover.

We know that \mathbf{R}^n is separable, so M is also separable. Let C be a countable dense subset of M . Let \mathcal{U} be an open cover for M . We construct the countable subcover \hat{U} as follows. For each $q \in C$ and $k \in \mathbb{Q}$, consider the open ball $B(q, k)$. If there exists a $U_{qk} \in \mathcal{U}$ such that $B(q, k) \subseteq U_{qk}$, include it in \hat{U} . Notice that \hat{U} is at most countable. We claim that it is also an open cover.

Let $x \in M$. Then it is contained in some open set $U \in \mathcal{U}$. As well, we can find an open ball such that $B(x, \delta) \subseteq U$. Since C is dense, we can find $q \in C$ such that $q \in B(x, \frac{\delta}{4})$. Let $k \in \mathbb{Q}$ such that $\frac{\delta}{4} < k < \frac{\delta}{2}$. Then $x \in B(q, k) \subseteq B(x, \delta)$, because for all $y \in B(q, k)$,

$$\|x - y\| \leq \|x - q\| + \|q - y\| < \frac{\delta}{4} + \frac{\delta}{2} < \delta$$

It follows that $B(q, k) \subseteq U$, so it is guaranteed that some U_{qk} from our construction exists. Thus $x \in U_{qk} \in \hat{U}$ so \hat{U} is indeed an open cover and we are done.

Lemma 2: A countable union of sets with Jordan measure 0 is a Lebesgue null set.

Let $E = \bigcup_{i \geq 1} E_i$, where $\mu(E_i) = 0$. Let $\varepsilon > 0$. For each E_i , we can find a finite union of boxes B_i such that $B_i \supseteq E_i$ and $\text{vol}(B_i) < \frac{\varepsilon}{2^i}$. We see that $\bigcup_{i \geq 1} B_i$ is a countable union of boxes, $E \subseteq \bigcup_{i \geq 1} B_i$, and

$$\sum_{i=1}^{\infty} \text{vol}(B_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2(1 - \frac{1}{2})} = \varepsilon$$

as desired.

Now, we prove the problem at hand. Let $M \subseteq \mathbf{R}^N$ be a smooth n -manifold with $n < N$. Let $\{(U_i, \varphi)\}_{i \in I}$ be an atlas for M . By Lemma 1, we can assume without loss of generality that the atlas is countable. We can also assume that each U_i is bounded, for if not, we can take a countable union of open balls that cover the unbounded U_i , and restrict the embedding to each ball.

For each chart (U_i, φ_i) , assume that the domain \hat{U} is measurable.

(b):

Let p be a point in the topological boundary of M . We will show that $p \in \partial M$. Suppose for contradiction that p is in M° , meaning it is contained in a chart (U, φ) that is diffeomorphic

Question 34.

Let $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a C^1 mapping.

- (a) Suppose that $n > m = 1$. Show that Φ cannot be injective.
- (b) Suppose that $n < m$. Show that if $K \subseteq \mathbf{R}^n$ is a compact set, then $\Phi(K)$ is a Jordan measurable set, and has Jordan measure zero.

Proof.

(a):

Suppose for contradiction that $n > m = 1$ and Φ is a C^1 injective function. Since Φ cannot be a constant function, by the results of Big List #26, there is a $p \in \mathbf{R}^n$ so that $\nabla \Phi(p) \neq 0$. In particular, we will say that $\frac{\partial \Phi}{\partial x_j} \neq 0$. Define $\alpha : \mathbf{R}^n \rightarrow \mathbf{R}$ by $\alpha(x) = \Phi(x) - \Phi(p)$. Injectivity is translation-invariant, so α is injective. Notice that $\alpha(p) = 0$. We can apply the implicit function theorem to obtain an open set $W \subseteq \mathbf{R}^{n-1}$ that contains $p' = (p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n)$ and a C^1 function $\Psi : W \rightarrow \mathbf{R}$ such that for all $x = (x_1, \dots, x_{n-1}) \in W$,

$$\alpha(x_1, \dots, x_{j-1}, \Psi(x), x_j, \dots, x_{n-1}) = 0$$

Then, since W is open and contains p' , we can find another distinct point $q \in W$. We have

$$\alpha(p_1, \dots, p_{j-1}, \Psi(p'), p_{j+1}, \dots, p_n) = 0 = \alpha(q_1, \dots, q_{j-1}, \Psi(q), q_j, \dots, q_{n-1})$$

which contradicts the fact that α is injective.

(b):

Since K is compact, and thus bounded, we can enclose it in a closed box $B = [-L, L]^n$, for some positive L . It suffices to show that $\Phi(B)$ has measure 0, as we can apply the monotonicity of measure to conclude that $\Phi(K)$ has measure 0.

First, let $\hat{\Phi} : \mathbf{R}^m \rightarrow \mathbf{R}^m$ be defined by $\hat{\Phi}(x) = \Phi(\pi_{\mathbf{R}^n}(x))$. That is, $\hat{\Phi}$ first projects vectors in \mathbf{R}^m onto \mathbf{R}^n and then composes it with Φ . Let $\hat{B} = B \times \{0\}^{m-n}$. Then note that $\hat{\Phi}(\hat{B}) = \Phi(B)$. Trivially, \hat{B} has measure 0. We now show that $\Phi(B)$ also has measure 0.

Let $\varepsilon > 0$. Since $\hat{\Phi}$ is C^1 , its derivative is continuous. By the extreme value theorem, each component derivative attains a maximum on \hat{B} . Let α a positive number greater than all the maximums. Since \hat{B} has measure 0, we can find a finite number of cubes B_1, \dots, B_k with side length d such that

$$\hat{B} \subseteq \bigcup_{i=1}^k B_i \text{ and } \sum_{i=1}^k \text{vol}(B_i) < \frac{\varepsilon}{m^m \alpha^m}$$

Consider some cube $B_i = \prod_{j=1}^m [a_{ij}, a_{ij} + d]$. For each component $\hat{\Phi}_j : \mathbf{R}^m \rightarrow \mathbf{R}$, it must be true that $\hat{\Phi}_j$ attains a maximum $\hat{\Phi}_j(M_{ij})$ and minimum $\hat{\Phi}_j(m_{ij})$, for some $M_{ij}, m_{ij} \in B_i$. Define $g : [0, 1] \rightarrow \mathbf{R}$ by

$$g(t) = \hat{\Phi}_j(tM_{ij} + (1-t)m_{ij})$$

By the Mean Value Theorem, there exists some $c \in (0, 1)$ such that

$$g(1) - g(0) = g'(c) \implies \hat{\Phi}_j(M) - \hat{\Phi}_j(m) = \hat{\Phi}'_j(cM_{ij} + (1-c)m_{ij})(M_{ij} - m_{ij})$$

