Question 35.

Perturbing the roots of a polynomial.
Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a **monic** polynomial with **no repeated real roots**. This means that $a_n = 1$, and that all real roots of f have multiplicity 1.

(a) Let r be a root of $f(x)$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that: if $g(x) = \sum_{i=0}^{n} b_i x^i$ is a monic polynomial with coefficients b_i satisfying $|a_i - b_i| < \delta$, then $g(x)$ has at least one root in the interval $(r - \varepsilon, r + \varepsilon)$.

This shows that slight perturbations of the coefficients results in slight perturbations of the roots.

(b) Suppose that f has fewer than n real roots. Prove that number of real roots of f does not change under small perturbation of the coefficients.

Proof.
(a):
First, we prove the following lemma:
Lemma. Let $r \in \mathbb{R}$ be a root of a polynomial p. Then r is a repeated root if and only if $p'(r) = 0$.
Suppose that p has a repeated root. Then we can factor p as $(x - r)^k q(x)$, for some $k > 1$ and $q \in \mathbb{P}(\mathbb{R})$. We can take the derivative of this and get that

$$p'(x) = k(x - r)^{k-1} q(x) + (x - r)^k q'(x)$$

$$\implies p'(r) = 0$$

Conversely, suppose that $p'(r) = 0$, for some $r \in \mathbb{R}$. We can write

$$p'(x) = (x - r) \sum_{i=0}^{m} c_i x^i, \text{ for constants } c_0, \ldots, c_m$$

We can integrate both sides to get that

$$p(x) = \int (x - r) \sum_{i=0}^{m} c_i x_i \, dx = \int x \sum_{i=0}^{m} c_i x_i - r \sum_{i=0}^{m} c_i x_i \, dx$$

$$= \sum_{i=0}^{m} \frac{c_i}{i+2} x^{i+2} - r \sum_{i=0}^{m} \frac{c_i}{i+1} x^{i+1} + C$$

In order for r to be a root of p, we must have that

$$p(r) = \sum_{i=0}^{m} \left(\frac{c_i}{i+2} r^{i+2} - \frac{c_i}{i+1} r^{i+2} \right) + C = 0$$

$$\implies C = \sum_{i=0}^{m} \left(\frac{c_i}{i+1} r^{i+2} - \frac{c_i}{i+2} r^{i+2} \right)$$

Therefore (im sorry)

$$
\begin{aligned}
p(x) &= \sum_{i=0}^{m}\left(\frac{c_i}{i+2}x^{i+2} - \frac{c_i}{i+2}r^{i+2} + \frac{c_i}{i+1}r^{i+2} - r\frac{c_i}{i+1}x^{i+1}\right) \\
&= \sum_{i=0}^{m}\left(\left(\frac{c_i}{i+2}\right)(x^{i+2} - r^{i+2}) - \left(\frac{c_i r}{i+1}\right)(x^{i+1} - r^{i+1})\right) \\
&= \sum_{i=0}^{m}\left(\left(\frac{c_i}{i+2}\right)(x-r)\sum_{j=0}^{i+1} x^j r^{i-j+1} - \left(\frac{c_i}{i+1}\right)(x-r)\sum_{j=0}^{i} x^j r^{i-j+1}\right) \\
&= (x-r)\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\left((i+1)\left(x^{i+1} + \sum_{j=0}^{i} x^j r^{i-j+1}\right) - (i+2)\sum_{j=0}^{i} x^j r^{i-j+1}\right)\right) \\
&= (x-r)\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\left((i+1)x^{i+1} - \sum_{j=0}^{i} x^j r^{i-j+1}\right)\right) \\
&= (x-r)\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\sum_{j=0}^{i}(x^{i+1} - x^j r^{i-j+1})\right) \\
&= (x-r)\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\sum_{j=0}^{i} x^j(x^{i-j+1} - r^{i-j+1})\right) \\
&= (x-r)\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\sum_{j=0}^{i} x^j(x-r)\sum_{k=0}^{i-j} x^k r^{i-j-k+1}\right) \\
&= (x-r)^2\sum_{i=0}^{m}\left(\frac{c_i}{(i+1)(i+2)}\sum_{j=0}^{i}\sum_{k=0}^{i-j} x^{k+j} r^{i-j-k+1}\right)
\end{aligned}
$$

which implies that r is a repetaed root of p.
Define the C^1 function $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}$ by

$$\Phi(y_0, y_1, ..., y_{n+1}, x) = \sum_{i=0}^{n} y_i x^i.$$

Let $a = (a_0, ..., a_n) \in \mathbb{R}^{n+1}$. We have that $\Phi(a, r) = 0$ and as well, by our lemma, $\dfrac{\partial \Phi}{\partial x}(a, r) = f'(r) \neq 0$. Applying the Implicit Function Theorem, we get that for some open set $W \subseteq \mathbb{R}^{n+1}$ and C^1 function $\Psi : W \to \mathbb{R}$ such that $\Psi(a) = r$,

$$\Phi(b, \Psi(b)) = 0, \text{ for all } b \in W.$$

Now, let $\varepsilon > 0$. by the continuity of Ψ at a, there exists a $\delta > 0$ so that for all $b \in W$ with

$\|b - a\| < (n+1)\delta$, $|\Psi(b) - r| < \varepsilon$, so $\Psi(b) \in (r - \varepsilon, r + \varepsilon)$. Consider the polynomial

$$g(x) = \sum_{i=0}^{n} b_i x^i = \Phi(b, x)$$

Where $|a_i - b_i| < \delta$. It follows that

$$(n+1)\delta > \sum_{i=0}^{n} |a_i - b_i| > \|a - b\|$$

which implies that $\Psi(b) \in (r - \varepsilon, r + \varepsilon)$. But notice that $\Psi(b)$ is actually a root of $g(x)$, as desired.

(b):
From part (a), we know that the number of roots can never decrease under small perturbations. Thus, it suffices to show that it will never increase, either.
First, we will show that there exists $\xi > 0$ so that for any polynomial g with coefficients b_i and $\|a_i - b_i\| < \xi$, g has no repeated roots.
Recall the δ we got from part (a). We claim that if we shrink it down to $\eta = \min\{\frac{\delta}{2}, \xi\}$, the number of roots does not change. Let g be a polynomial with coefficients b_i such that $|a_i - b_i| < m$

$\square$