

Question 27.

Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be a continuously differentiable function.

- (a) Show that the partial function  $\mathbf{R} \rightarrow \mathbf{R}$ ,  $t \mapsto f(x, t)$  is integrable (over any bounded interval in  $\mathbf{R}$ ).
- (b) By (a), we can define a function  $\varphi : \mathbf{R} \rightarrow \mathbf{R}$  by

$$\varphi(x) = \int_a^b f(x, t) \, dt.$$

Show that  $\varphi$  is differentiable, and that its (classical) derivative is given by

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t) \, dt.$$

This formula is known as **differentiation under the integral sign**, or **Feynmann's trick**.

- (c) Use Feynmann's trick to solve the single-variable integral:

$$\int_0^\infty e^{-t^2} \, dt$$

*Proof.* (a):

Since  $f$  is continuous, it follows immediately that its partial function is continuous, which implies that it is integrable.

(b):

Let  $x_0 \in \mathbf{R}$ . We will show that as  $h \rightarrow 0$ ,

$$\frac{1}{h} \left( \int_a^b f(x_0 + h, t) dt - \int_a^b f(x_0, t) dt - h \int_a^b \frac{\partial f}{\partial x}(x_0, t) dt \right) \rightarrow 0,$$

which is equivalent to saying

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t) \, dt.$$

Let  $\varepsilon > 0$ . By the partial differentiability of  $f$ , we obtain a  $\delta$  so that

$$\left| f(x_0 + h, t) - f(x_0, t) - h \frac{\partial f}{\partial x}(x_0, t) \right| < \frac{|h|\varepsilon}{b-a}$$

for all  $0 < |h| < \delta$ .

Fix  $h \in \mathbf{R}$  so that  $0 < |h| < \delta$ . By the linearity of the integral,

$$\left| \frac{1}{h} \left( \int_a^b f(x_0 + h, t) dt - \int_a^b f(x_0, t) dt - h \int_a^b \frac{\partial f}{\partial x}(x_0, t) dt \right) \right|$$





Question 28.

Let  $U$  be an open set in a normed vector space  $X$  and let  $f : U \rightarrow Y$  be a **twice continuously differentiable** function, meaning that the second derivative  $f'' : U \rightarrow B(X, B(X, Y))$  exists and is continuous on  $U$ . We also say that  $f$  is a  **$C^2$ -function**.

- (a) Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be given by  $f(x, y) = x^2 - xy + y^2$ . Find, with proof, an explicit formula for the linear mapping  $f''(2, -1)$ . Also, write down the matrix that represents this linear mapping with respect to a suitable “standard” basis.
- (b) Now we investigate the case  $X = \mathbf{R}^n$  and  $Y = \mathbf{R}$ , and let  $f : U \rightarrow \mathbf{R}$  be some function defined on an open set  $U \subseteq \mathbf{R}^n$ . We use the notation  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  to refer to the  $(i, j)$ th **second partial derivative** of  $f$ : this is the  $i$ th partial derivative of the  $j$ th partial derivative  $\frac{\partial f}{\partial x_j}$ .
- (i) Show that  $f$  is twice continuously differentiable if and only if all second partial derivatives exist and are continuous.
- (ii) Let  $f$  be twice continuously differentiable. Let  $v \in \mathbf{R}^n$  and let  $D_v f : U \rightarrow \mathbf{R}$  be the directional derivative of  $f$  along  $v$ . Show that  $D_v f$  is continuously differentiable.
- (iii) Let  $f$  be twice continuously differentiable and let  $v \in \mathbf{R}^n$ . By (ii), we know that  $D_v f$  is  $C^1$ , hence differentiable in every direction  $w \in \mathbf{R}^n$ . Show that the directional derivatives commute:

$$D_v(D_w f) = D_w(D_v f) \quad \text{for all } v, w \in \mathbf{R}^n.$$

- (iv) Deduce **Clairaut’s Theorem**: that the second partial derivatives commute.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

*Proof.* (a):

We claim that  $f''(2, -1) : \mathbf{R}^2 \rightarrow B(\mathbf{R}^2, \mathbf{R})$  is a bounded linear map given by

$$f''(2, -1)(p, q)(x, y) = (2p - q)x - (p - 2q)y$$

Let  $(p, q), (r, s) \in \mathbf{R}^2$  and  $c \in \mathbf{R}$ . Then for all  $(x, y) \in \mathbf{R}^2$ ,

$$\begin{aligned} f''(2, -1)(cp + r, cq + s)(x, y) &= [2(cp + r) - (cq + s)]x - [(cp + r) - 2(cq + s)]y \\ &= c[(2p - q)x - (p - 2q)y] + [(2r - s)x - (r - 2s)y] = cf''(2, -1)(p, q)(x, y) + f''(2, -1)(r, s)(x, y) \end{aligned}$$

so  $f''(2, -1)$  is linear.

Recall that  $f'(p, q)(x, y) = (2p - q)x - (p - 2q)y$ . In particular, for  $(p, q) = (2, -1)$ ,  $f'(2, -1)(x, y) = 5x - 4y$ .

Fix  $(x, y) \in \mathbf{R}^2$ . Then

$$\lim_{h \rightarrow 0} \frac{f'(2 + h_1, -1 + h_2)(x, y) - f'(2, -1)(x, y) - L_p(h)}{\|h\|}$$





Question 29.

Let  $U \subseteq \mathbf{R}^2$  be an open set, and let  $f : U \rightarrow \mathbf{R}$  be some differentiable function. This data defines an **explicit surface** in  $\mathbf{R}^3$ , also known as the **graph** of  $f$ :

$$S = S(f) = \{(x, y, z) \in \mathbf{R}^3 : (x, y) \in U, z = f(x, y)\}.$$

Colloquially, we often say “Let  $z = f(x, y)$  be a surface in  $\mathbf{R}^3$ .”

- (a) Find the equation of the tangent plane to the surface  $z = x + xy^2 - y^3$  at the point  $p = (2, 1, 3)$ , as follows.
- (i) First, fix  $x = 2$  and set  $y = 1 + t$ ; write  $z$  as a function of  $t$ , and find  $z'(0)$ . This is the “slope in the  $x$  direction.”
  - (ii) Similarly, find the “slope in the  $y$  direction.”
  - (iii) Now you have two slopes in orthogonal directions. This gives you two vectors which span a plane. Shift this plane so that it becomes tangent at  $p$ . Write the equation of this plane in the form  $Ax + By + Cz = D$ , where  $A, B, C, D \in \mathbf{R}$ .
- (b) Find — with proof! — the equations of all planes in  $\mathbf{R}^3$  which (i) are tangent to the surface  $z = x + xy^2 - y^3$ ; (ii) are parallel to the vector  $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ ; and (iii) pass through the point  $p = (-1, -2, 3)$ .
- (c) Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be a continuously differentiable function, and define a new function  $g : \mathbf{R}^2 \rightarrow \mathbf{R}$  by

$$g(x, y) = f(f(xy, x), f(y, xy)).$$

You are given the following information about  $f$  and  $g$ :

- $f(3, 1) = 5$  and  $\nabla f(3, 1) = (1, 2)$
- $f(3, 3) = 2$  and  $\nabla f(3, 3) = (0, -1)$
- $g(1, 3) = 6$  and  $\nabla g(1, 3) = (3, 4)$

Find  $\nabla f(5, 2)$ .

*Proof.* (a)(i):

Fix  $x = 2$  and set  $y = 1 + t$ . Then

$$\begin{aligned} z(t) &= 2 + 2(1+t)^2 - (1+t)^3 \implies z'(t) = 4(1+t) - 3(1+t)^2 \\ &\implies z'(0) = 1 \end{aligned}$$

(a)(ii):

Fix  $y = 1$  and set  $x = 2 + t$ . Then

$$z(t) = (2+t) + (2+t) - 1 = 2t + 3 \implies z'(0) = 2$$

(a)(iii):

