Question 1.

Suppose $\widetilde{p}$ must approximate $p$ with relative error at most $10^{-3}$. Find the largest interval in which $\widetilde{p}$ must lie if $p = 900$.

*Proof.* Since we want the relative error to be at most $10^{-3}$, we set

$$\frac{|\widetilde{p} - p|}{|p|} \leq 10^{-3}$$

Substitute $p = 900$ to get

$$\frac{|\widetilde{p} - 900|}{900} \leq 10^{-3} \implies |\widetilde{p} - 900| \leq \frac{9}{10} \implies 900 - \frac{9}{10} \leq \widetilde{p} \leq 900 + \frac{9}{10}.$$

Thus $\widetilde{p}$ lies within the interval $\left[900 - \frac{9}{10}, 900 + \frac{9}{10}\right]$.

$\square$

Question 2.

Compute the absolute error and relative error of the following approximation of $e$:

$$\sum_{n=0}^{5} \frac{1}{n!}$$

Using Octave, we find that the absolute error is

$$\left| e - \sum_{n=0}^{5} \frac{1}{n!} \right| = \left| e - \left( 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \right) \right| = \left| e - \frac{326}{120} \right| \approx \texttt{1.615161792378306e-03}$$

and the relative error is

$$\left| \frac{e - \sum_{n=0}^{5} \frac{1}{n!}}{e} \right| = \left| \frac{e - \frac{326}{120}}{e} \right| \approx \texttt{5.941848175815963e-04}$$

Question 3.

Find the second Taylor polynomial, $P_2(x)$, $f(x) = e^x \cos(x)$ about $x_0 = 0$.

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound on the error $|f(0.5) - P_2(0.5)|$ using the remainder term and compare it to the actual error.

First, we find $P_2(x)$. We calculate that

(b) Find a bound on the error $|f(x) - P_2(x)|$ good on the interval $[0, 1]$.

(c) Approximate $\int_0^1 f(x)\, dx$ by calculating $\int_0^1 P_2(x)\, dx$ instead.

(d) Find an upper bound for the error in (c) using $\int_0^1 |R(x)|\, dx$ and compare the bound to the actual error.

Question 4.

Find a theoretical upper bound, as a function of $x$, for the absolute error in using $T_4(x)$ to approximate $f(x) = \frac{10}{x} + \sin(10x)$; $x_0 = \pi$.

Question 5.

Let $(p_n) = \left\langle \frac{3n^5-5n}{1-n^5} \right\rangle \to -3$ Find the (fastest) rate of convergence of the form $\mathcal{O}\left(\frac{1}{n^p}\right)$ or $\mathcal{O}\left(\frac{1}{a^n}\right)$ for each. If this is not possible, suggest a reasonable rate of convergence.

Question 6.

(a) Suppose you are trying to find the root of $f(x) = x - e^{-x}$ using the bisection method. Find an integer $a$ such that the interval $[a, a+2]$ is an appropriate one in which to start the search.

(b) Use the bisection method to find the root in your interval in (a), accurate to $10^{-4}$. Provide the Octave code you used to produce your result.