

Question 1.

Suppose \tilde{p} must approximate p with relative error at most 10^{-3} . Find the largest interval in which \tilde{p} must lie if $p = 900$.

Proof. Since we want the relative error to be at most 10^{-3} , we set

$$\frac{|\tilde{p} - p|}{|p|} \leq 10^{-3}$$

Substitute $p = 900$ to get

$$\frac{|\tilde{p} - 900|}{900} \leq 10^{-3} \implies |\tilde{p} - 900| \leq \frac{9}{10} \implies 900 - \frac{9}{10} \leq \tilde{p} \leq 900 + \frac{9}{10}.$$

Thus \tilde{p} lies within the interval $\left[900 - \frac{9}{10}, 900 + \frac{9}{10}\right]$.

□

Question 2.

Compute the absolute error and relative error of the following approximation of e :

$$\sum_{n=0}^5 \frac{1}{n!}$$

Using Octave, we find that the absolute error is

$$\left| e - \sum_{n=0}^5 \frac{1}{n!} \right| = \left| e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \right) \right| = \left| e - \frac{326}{120} \right| \approx 1.615161792378306\text{e-}03$$

and the relative error is

$$\left| \frac{e - \sum_{n=0}^5 \frac{1}{n!}}{e} \right| = \left| \frac{e - \frac{326}{120}}{e} \right| \approx 5.941848175815963\text{e-}04$$

Question 3.

Find the second Taylor polynomial, $P_2(x)$, $f(x) = e^x \cos(x)$ about $x_0 = 0$.

- (a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound on the error $|f(0.5) - P_2(0.5)|$ using the remainder term and compare it to the actual error.

First, we find $P_2(x)$. We calculate that

$$\begin{aligned} f(x_0) &= 1 \\ f'(x_0) &= e^{x_0}(\cos(x_0) - \sin(x_0)) = 1 \\ f''(x_0) &= -2e^{x_0} \sin(x_0) = 0 \\ f^{(3)}(x) &= -2e^x(\sin(x) - \cos(x)) \end{aligned}$$

Thus $P_2(x) = 1 + x$, so $f(0.5) \approx P_2(0.5) = 1.5$. The error term is

$$|R_2(0.5)| = \left| \frac{f^{(3)}(\xi)}{3!} (0.5)^3 \right| = \left| \frac{e^\xi(\sin(\xi) - \cos(\xi))}{24} \right|, \text{ for } \xi \in (0, 0.5).$$

Since $e^\xi < e^{0.5}$, $\sin(\xi), \cos(\xi) \leq 1$, we get that

$$|R_2(0.5)| < \frac{e^{0.5}}{12} \approx 0.1374.$$

and the actual absolute error is

$$|e^{0.5} \cos(0.5) - 1.5| \approx 0.053111$$

- (b) Find a bound on the error $|f(x) - P_2(x)|$ good on the interval $[0, 1]$.

Similar to the previous part, the error term is

$$|R_2(x)| = \left| \frac{e^\xi(\sin(\xi) - \cos(\xi))}{3} x^3 \right|, \text{ for } \xi \in (0, x)$$

We know that $e^\xi < e^x$, so

$$|R_2(x)| < \frac{2}{3} x^3 e^x.$$

- (c) Approximate $\int_0^1 f(x) dx$ by calculating $\int_0^1 P_2(x) dx$ instead.

We have that

$$\int_0^1 f(x) dx \approx \int_0^1 P_2(x) dx = \int_0^1 1 + x dx = x + \frac{x^2}{2} \Big|_0^1 = 1.5$$

- (d) Find an upper bound for the error in (c) using $\int_0^1 |R(x)| dx$ and compare the bound to the actual error.

From part (b), the error term $|R_2(x)|$ is bounded above by $\frac{2}{3} x^3 e^x$. Thus the error for the computation is

$$\int_0^1 |R_2(x)| dx < \int_0^1 \frac{2}{3} x^3 e^x dx = \frac{2}{3} (x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x) \Big|_0^1$$

Question 4.

Find a theoretical upper bound, as a function of x , for the absolute error in using $T_4(x)$ to approximate $f(x) = \frac{10}{x} + \sin(10x)$; $x_0 = \pi$.

We can find the series expansion for $\sin(10x)$ quite easily by noticing that

$$\sin(10x) = \sin(10x - 10\pi) = \sin(10(x - \pi)) = \sum_{i=0}^{\infty}$$

Question 5.

Let $(p_n) = \left\langle \frac{3n^5 - 5n}{1 - n^5} \right\rangle \rightarrow -3$ Find the (fastest) rate of convergence of the form $\mathcal{O}\left(\frac{1}{n^p}\right)$ or $\mathcal{O}\left(\frac{1}{a^n}\right)$ for each. If this is not possible, suggest a reasonable rate of convergence.

We claim that (p_n) converges to $p = -3$ with rate of convergence $\mathcal{O}\left(\frac{1}{n^4}\right)$. To prove this, first let $n_0 = 1$ and $\lambda = 7$. We see that for all $n > n_0$,

$$\begin{aligned} |p_n - p| &= \left| \frac{3n^5 - 5n}{1 - n^5} + 3 \right| = \left| \frac{3 - 5n}{(1 - n)(1 + n + n^2 + n^3 + n^4)} \right| \\ &= \left| \frac{5 - 5n - 2}{(1 - n)(1 + n + n^2 + n^3 + n^4)} \right| = \left| \frac{5}{1 + n + n^2 + n^3 + n^4} + \frac{2}{(n - 1)(1 + n + n^2 + n^3 + n^4)} \right| \end{aligned}$$

We can remove the absolute values because the term inside is positive. Since $n - 1 > 1$, we get

$$\frac{5}{1 + n + n^2 + n^3 + n^4} + \frac{2}{(n - 1)(1 + n + n^2 + n^3 + n^4)} \leq \frac{5}{n^4} + \frac{2}{n^4} = \frac{7}{n^4}$$

which is what we wanted.

To show that this is indeed the fastest rate of convergence, we can show that the sequence $\left(\frac{1}{n^4}\right)$ converges to 0 with rate of convergence $\mathcal{O}\left(\frac{1}{n^4}\right)$. To do this, let $n_0 = 1$, $\lambda = 1$. Let $n > n_0$. From the work we did in the previous part, we know that

$$|p_n - p| = \frac{5}{1 + n + n^2 + n^3 + n^4} + \frac{2}{(n - 1)(1 + n + n^2 + n^3 + n^4)}$$

Since $\frac{2}{(n - 1)(1 + n + n^2 + n^3 + n^4)} > 0$ and $1 < n < n^2 < n^3 < n^4$, we get

$$|p_n - p| > \frac{5}{1 + n + n^2 + n^3 + n^4} > \frac{5}{n^4 + n^4 + n^4 + n^4 + n^4} = \frac{1}{n^4}$$

as needed.

Question 6.

- (a) Suppose you are trying to find the root of $f(x) = x - e^{-x}$ using the bisection method. Find an integer a such that the interval $[a, a+2]$ is an appropriate one in which to start the search.

Let $a = 0$. We see that $f(0) = -1 < 0$, $f(2) = 2 - \frac{1}{e^2} > 0$, so the interval $[0, 2]$ satisfies the conditions to use the bisection method.

- (b) Use the bisection method to find the root in your interval in (a), accurate to 10^{-4} . Provide the Octave code you used to produce your result.

Using the method, the root was found at $x \approx 0.5672$.

Below is the Octave code used, along with the command used in command line:

```
1 function [m,M,i]=bisection(a,b,f,tol)
2 N=ceil((log(b-a)-log(tol))/log(2));
3 L=f(a);
4 for i=1:N
5 m=(a+b)/2;
6 M=f(m);
7 if (M==0)
8 return;
9 end%if
10 if (L*M<0)
11 b=m;
12 else
13 a=m;
14 L=M;
15 end%if
16 end%for
17 i=N;
18 end%function
```

```
>> [m,M,i] = bisection(0, 2, @(x) x-e^(-x), 0.0001)
```