Question 1.

Count the number of five-card hands that can be formed from a standard deck such that:

- the hand contains more red cards than black cards, **and**
- there are no repeated ranks in the hand.

Solution.
We will handle the first condition by separating the problem into 3 cases: the hands with 3, 4, and 5 red cards respectively. We will also consider the second condition in every case.

First, we pick our 5 distinct ranks, which there are $\binom{13}{5}$ different ways to do so. After picking the 5 ranks, we want to assign each colored card a rank. For 3, 4, and 5 red cards, there are $\binom{5}{3}, \binom{5}{4}, \binom{5}{5}$ ways respectively to assign each red card to a different rank. For each rank, there are 2 ways to choose a red card and 2 ways to choose a black card. Since we are choosing a total of 5 cards, no matter if we are choosing 3, 4, or 5 red cards, the number of ways turns out to be 2^5. We can add all the cases together to get that the answer is

$$\binom{13}{5} \cdot \binom{5}{3} \cdot 2^5 + \binom{13}{5} \cdot \binom{5}{4} \cdot 2^5 + \binom{13}{5} \cdot \binom{5}{5} \cdot 2^5$$

$$= \binom{13}{5} \cdot 2^5 \cdot (10 + 5 + 1) = \binom{13}{5} \cdot 2^9$$

Question 2.

In the expansion of $(x^3 + (2y - 3z)^9)^{202}$, determine the coefficients of the following terms:

(a) z^{1818}

(b) $x^{303}y^{404}z^{505}$

Solution.
We can solve this problem by applying the binomial theorem twice. We see that

$$\left(x^3 + (2y - 3z)^9\right)^{202} = \sum_{k=0}^{202} \binom{202}{k} x^{3(202-k)}(2y - 3z)^{9k}$$

$$= \sum_{k=0}^{202} \binom{202}{k} x^{3(202-k)} \sum_{i=0}^{9k} \binom{9k}{i} (2y)^{9k-i}(-3z)^i$$

For part (a), we are interested in the term with $k = 202$, $i = 9k = 1818$, as that is the only way we can get z^{1818}. The coefficient for this is $\binom{202}{202}\binom{1818}{1818}(-3)^{1818} = 3^{1818}$.

For part (b), to obtain the term with $x^{303}y^{404}z^{505}$, we set $k = 101$, $i = 505$, and see that the coefficient is $\binom{202}{101}\binom{909}{505}2^{909-505}(-3)^{505}$.