Question 23.

Let S^2 denote the unit sphere in \mathbb{R}^3 . Let N = (0,0,1) denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to \mathbb{R}^2 . To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbb{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in \mathbb{R}^3 .

- (a) Given P = (x, y, z), find an explicit formula for $\Phi(P)$ in terms of x, y, z.
- (b) Deduce that Φ is continuous.
- (c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.
- (d) Deduce that Φ is a homeomorphism.

Proof. (a):

Let P = (x, y, z). First, we find the equation of the line that passes P and N. Consider the equation of the line L(t) = (tx, ty, (z-1)t+1). Notice that L(0) = N and L(1) = P, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when (z-1)t+1=0. Solving for t gives $t=\frac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$

Thus

$$\Phi(P) = \frac{1}{1-z} (x, y).$$

(b):

(c)

Let $p = (s, t) \in \mathbb{R}^2$. Our goal is to find $(x, y, z) \in S^2 \setminus \{N\}$ such that $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s, t)$. Immediately, we obtain the following system of equations:

$$\frac{x}{1-z} = s,$$

$$\frac{y}{1-z} = t,$$

$$x^{2} + y^{2} + z^{2} = 1$$

We also have the restriction $z \neq 1$ because $(x, y, z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^{2}(1-z)^{2} + t^{2}(1-z)^{2} + z^{2} = 1 \implies (s^{2} + t^{2} + 1)z^{2} - 2(s^{2} + t^{2})z + s^{2} + t^{2} - 1 = 0$$

We can replace the term $t^2 + s^2$ with $||p||_2^2$, and the equation becomes

$$(\|p\|_2^2 + 1)z^2 - 2\|p\|_2^2z + \|p\|_2^2 - 1 = 0$$

Using the quadratic formula:

$$z = \frac{2(\|p\|_2^2) \pm \sqrt{4(\|p\|_2^2)^2 - 4(\|p\|_2^2 + 1)(\|p\|_2^2 - 1)}}{2(\|p\|_2^2 + 1)}$$

$$\implies z = \frac{\|p\|_2^2 \pm \sqrt{\|p\|_2^4 - (\|p\|_2^4 - 1)}}{\|p\|_2^2 + 1}$$

$$z = \frac{\|p\|_2^2 \pm 1}{\|p\|_2^2 + 1}$$

Notice that we cannot use the positive solution, for then

$$z = \frac{\|p\|_2^2 + 1}{\|p\|_2^2 + 1} = 1$$

Thus it must be true that

$$z = \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}$$

$$x = s(1 - z) = \frac{2s}{\|p\|_2^2 + 1}$$

$$y = \frac{2t}{\|p\|_2^2 + 1}$$

It can be verified that these values of x, y, z result in $\Phi(x, y, z) = (s, t)$. In fact, using this, we obtain that the formula for Φ^{-1} is

$$\Phi^{-1}(s,t) = \left(\frac{2s}{\|p\|_2^2 + 1}, \frac{2t}{\|p\|_2^2 + 1}, \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}\right)$$

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Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

- (i) X is finite-dimensional.
- (ii) The unit ball $\overline{B}(\vec{0},1)$ is compact
- (iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that \overline{U} is compact.

Proof. It will be proven that (i) \implies (ii) \iff (iii) \implies (i). (i) \implies (ii):

Suppose that X has finite dimension n. Consider the linear isomorphism $\Phi: X \to \mathbb{R}^n$ by mapping elements of an ordered basis for X to the basis $\beta = \{e_1, ..., e_n\}$ for \mathbb{R}^n . This is a homeomorphism,

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