

Question 1.

Find all solutions to the following complex equations.

1. $(1 + i)\bar{z} = i(2 + 8i)$
2. $z^3 = -8i$
3. $e^{\bar{z}} = -2 + 2i$

Proof.

1. $(1 + i)\bar{z} = i(2 + 8i)$.

Suppose that z is of the form $z = a + bi$, for $a, b \in \mathbb{R}$. Then the equation becomes

$$(1 + i)(a - bi) = i(2 + 8i) \implies a + b + (a - b)i = -8 + 2i.$$

Equating coefficients, we get

$$a + b = -8 \text{ and } a - b = 2.$$

Solving the system of equations gives us $a = -3$ and $b = -5$, so $z = -3 - 5i$.

2. $z^3 = -8i$.

Suppose that z is of the form $z = re^{i\theta}$, for $r, \theta \in \mathbb{R}$. Then the equation becomes

$$r^3 e^{3i\theta} = -8i \implies r^3 e^{3i\theta} = 8e^{-i(\frac{\pi}{2} + 2n\pi)}, \text{ for } n \in \mathbb{Z}$$

Equating the coefficient and exponent gives us

$$r^3 = 8 \text{ and } 3\theta = \frac{\pi}{2} + 2n\pi \implies r = 2, \theta = \frac{\pi}{6} + \frac{2n\pi}{3}.$$

Therefore

$$z = 2e^{i(\frac{\pi}{6} + \frac{2n\pi}{3})} = 2 \cos\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right) + 2i \sin\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right).$$

We can convert this into the standard form by considering cases when $n = 0, 1, 2$, as any other value will give us a value of z that is already accounted for. Therefore

$$z = \sqrt{3} + i, -\sqrt{3} + i, -2i$$

3. $e^{\bar{z}} = -2 + 2i$.

Let $z = a + bi$, for $a, b \in \mathbb{R}$. Converting the right hand side of the equation into polar form, we get

$$e^a e^{bi} = 2\sqrt{2}e^{i(\frac{3\pi}{4} + 2n\pi)}, \text{ where } n \in \mathbb{Z}.$$

We can equate real and complex parts to get that

$$e^a = 2\sqrt{2} \text{ and } b = \frac{3\pi}{4} + 2n\pi$$

so

$$z = \frac{3}{2} \ln(2) + i \left(\frac{3\pi}{2} + 2n\pi \right).$$

□

Question 2.

Find all solutions to the following equations in \mathbb{Z}_9 , or show that they have no solution.

(a) $[4]x + [3] = [1]$

(b) $[6]x + [3] = [5]$

(c) $x^2 = [0]$.

Proof. (a) $[4]x + [3] = [1]$

Adding $[6]$ to both sides of the equation yields

$$[4]x = [7].$$

Multiplying both sides by $[7]$, we get

$$[28]x = [49]$$

$$\implies x = [4].$$

(b) $[6]x + [3] = [5]$

This equation has no solution. To show this, we first simplify the equation to $[6]x = [2]$ by adding $[6]$ to both sides. We can substitute $x = [0], \dots, [8]$ into the left hand side and see that it does not equal the right hand side:

$$[6][1] = [6], [6][2] = [3], [6][3] = [0], [6][4] = [6], [6][5] = [3], [6][6] = [0],$$

$$[6][7] = [6], [6][8] = [3],$$

As shown, the left hand side can never equal $[5]$, so the equation has no solution.

(c) $x^2 = [0]$

We can solve this by substituting every element in \mathbb{Z}_9 into the left hand side. We see that

$$[0]^2 = [0], [1]^2 = [1], [2]^2 = [4], [3]^2 = [0], [4]^2 = [7],$$

$$[5]^2 = [7], [6]^2 = [0], [7]^2 = [4], [8]^2 = [1].$$

Thus the solutions to this equation are $x = [0], [3], [6]$.

□

Question 3.

Let $\mathbb{Z}_3[i] = \{a + bi \mid a, b \in \mathbb{Z}_3\}$, where we define operations $+$, \cdot by:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Set $1 = [1] + [0]i$ and $0 = [0] + [0]i$.

- Using only the definition of the operations above, and the fact that \mathbb{Z}_3 is a field, show that $\mathbb{Z}_3[i]$ satisfies Axioms 1-4, as well as the existence of additive inverses.
- Compute the multiplication table for $\mathbb{Z}_3[i]$ to verify that multiplicative inverses exist, and hence conclude that $\mathbb{Z}_3[i]$ is a field.
- What is the characteristic of $\mathbb{Z}_3[i]$? (See question #6 for the definition of characteristic of a field.)

Proof.

(a):

Let $a, b, c, d, p, q \in \mathbb{Z}_3$, so $z = a + bi$, $w = c + di$, and $x = p + qi$ are elements of $\mathbb{Z}_3[i]$.

To show closure under addition and multiplication, we use the closure of \mathbb{Z}_3 to see that $a + c \in \mathbb{Z}_3$ and $b + d \in \mathbb{Z}_3$. It follows that $z + w = (a + c) + (b + d)i \in \mathbb{Z}_3[i]$.

As well, we also have that $ac - bd, ad + bc \in \mathbb{Z}_3$, so $zw = (ac - bd) + (ad + bc)i \in \mathbb{Z}_3[i]$.

To show the commutativity of addition and multiplication, we note that $a + c = c + a$ and $b + d = d + b$, so

$$z + w = (a + c) + (b + d)i = (c + a) + (d + b)i = w + z$$

Likewise, since $ac = ca$, $bd = db$, $ad = da$, and $bc = cb$,

$$zw = (ac - bd) + (ad + bc)i = (ca - db) + (da + cb)i = wz$$

To show associativity, we again use the field properties of \mathbb{Z}_3 to see that

$$\begin{aligned} (z + w) + x &= ((a + c) + (b + d)i) + p + qi \\ &= ((a + c) + p) + ((b + d) + q)i \\ &= (a + (c + p)) + (b + (d + q))i && \text{(associativity of } \mathbb{Z}_3) \\ &= a + bi + (c + p) + (d + q)i \\ &= z + (w + x) \end{aligned}$$

Question 4.

We introduce a new definition in this question:

Definition: Let \mathbb{F} be a field. We say a subset $\mathbb{K} \subseteq \mathbb{F}$ is a **subfield** of \mathbb{F} if \mathbb{K} is also a field, using the same operations as \mathbb{F} .

For example: \mathbb{Q} is a subfield of \mathbb{R} . \mathbb{R} is a subfield of \mathbb{C} . \mathbb{Z}_3 is not a subfield of \mathbb{Q} , since \mathbb{Z}_3 is not a subset of \mathbb{Q} .

- (a) Let $\mathbb{K} \subseteq \mathbb{F}$ be a subfield. Let $0_{\mathbb{F}}, 1_{\mathbb{F}}$ denote the additive and multiplicative identities in \mathbb{F} . Similarly, we denote by $0_{\mathbb{K}}, 1_{\mathbb{K}}$ the identities in \mathbb{K} . Prove that $0_{\mathbb{F}} = 0_{\mathbb{K}}$ and $1_{\mathbb{F}} = 1_{\mathbb{K}}$. (Hint: Prove that in a field, the only solution to the equation $x^2 = x$ are $x = 0, x = 1$.)
- (b) Let $\mathbb{K} \subseteq \mathbb{F}$ be a subfield. Prove that for all $x \in \mathbb{K}$, we have $-x \in \mathbb{K}$, and that for all $x \in \mathbb{K} \setminus \{0\}$ we have $x^{-1} \in \mathbb{K}$. (Here $-x$ is the additive inverse of x **treated as an element of \mathbb{F}** and x^{-1} is the multiplicative inverse of x **treated as an element of \mathbb{F}** .)
- (c) Prove that a subset $\mathbb{K} \subseteq \mathbb{F}$ is a subfield if and only if the following conditions are met:
 - (i) $0, 1 \in \mathbb{K}$.
 - (ii) For all $x, y \in \mathbb{K}$, we have $x + y, x \cdot y \in \mathbb{K}$.
 - (iii) For all $x \in \mathbb{K}$, we have $-x \in \mathbb{K}$.
 - (iv) For all $x \in \mathbb{K} \setminus \{0\}$, we have $x^{-1} \in \mathbb{K}$.

(Hints: For the \implies direction: this is “part c” for a reason. For the \impliedby direction, you only need one or two short sentences to argue why addition and multiplication in \mathbb{K} satisfy Axioms 1-3. Axioms 4 and 5 should also have fairly short proofs. If you find yourself with a very long argument, you should rethink your argument.)

Proof.

(a):

Fix $x \in \mathbb{K}$. Then because $x \in \mathbb{F}$,

$$\begin{aligned} 0_{\mathbb{F}} + x = x = 0_{\mathbb{K}} + x & \quad \text{(existence of additive identity in } \mathbb{F} \text{ and } \mathbb{K}) \\ \implies 0_{\mathbb{F}} = 0_{\mathbb{K}} & \quad \text{(by cancellation)} \end{aligned}$$

Similarly for multiplication,

$$1_{\mathbb{F}} \cdot x = x = 1_{\mathbb{K}} \cdot x \implies 1_{\mathbb{F}} = 1_{\mathbb{K}}$$

(b):

Let $x \in \mathbb{K}$. Since \mathbb{K} is a field, x has an additive inverse $-x_{\mathbb{K}}$. Note that $-x_{\mathbb{K}} \in \mathbb{F}$ as well, so $-x_{\mathbb{K}}$ is an inverse for x in \mathbb{F} . By the uniqueness of additive inverses in \mathbb{F} , we have that $-x_{\mathbb{K}} = -x$.

Similarly, x has a multiplicative inverse $x_{\mathbb{K}}^{-1}$ in \mathbb{K} , which is also an inverse of x with respect to \mathbb{F} . It follows by uniqueness of inverses that $x_{\mathbb{K}}^{-1} = x^{-1}$.

(c):

Question 5.

Let $\mathbb{Q}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Q}\}$. Prove that if \mathbb{K} is a subfield of \mathbb{C} and $\sqrt{-2} \in \mathbb{K}$, then $\mathbb{Q}[\sqrt{-2}] \subseteq \mathbb{K}$.

Proof. Suppose that \mathbb{K} is a subfield of \mathbb{C} and $\sqrt{-2} \in \mathbb{K}$. Fix $z \in \mathbb{Q}[\sqrt{-2}]$. Then $z = a + b\sqrt{-2}$, for some $a, b \in \mathbb{Q}$. First, we will show that for all $c \in \mathbb{Q}$, $c \in \mathbb{K}$.

Letting $c \in \mathbb{Q}$, we can write $c = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. By the existence of the additive identity, we have that $1 \in \mathbb{K}$, and we can repeatedly use the closure of addition to see that

$$\underbrace{1 + \dots + 1}_{q \text{ times}} = q \in \mathbb{K} \text{ and } \underbrace{1 + \dots + 1}_{p \text{ times}} = p \in \mathbb{K}.$$

By the existence of inverses in \mathbb{K} , we know that $\frac{1}{q} \in \mathbb{K}$, and by closure under multiplication, we have that

$$p \cdot \frac{1}{q} = c \in \mathbb{K}$$

as needed.

This implies that $a, b \in \mathbb{K}$ as well. Since $\sqrt{-2} \in \mathbb{K}$, we use closure again to conclude that $b\sqrt{-2} \in \mathbb{K}$, and therefore $z = a + b\sqrt{-2} \in \mathbb{K}$, so $\mathbb{Q}[\sqrt{-2}] \subseteq \mathbb{K}$, proving the statement. \square

Question 6.

In this exercise we introduce a new definition:

Definition: Let \mathbb{F} be a field. The smallest non-negative integer n so that $\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = 0$

is called the characteristic of \mathbb{F} . If no such n exists, then we say \mathbb{F} has characteristic 0.

We denote this non-negative integer by $\text{char}(\mathbb{F})$.

For example: \mathbb{Z}_3 has characteristic 3 because $1 + 1 + 1 = 0$ in \mathbb{Z}_3 , but $1 + 1 \neq 0$ in \mathbb{Z}_3 . So $n = 3$ is the smallest integer so that $\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = 0$ in \mathbb{Z}_3 .

However, \mathbb{Q} has characteristic 0, because for any n we have $\underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n \neq 0$ in \mathbb{Q} .

(a) Prove that $\text{char}(\mathbb{Z}_p) = p$.

(b) Prove that $\text{char}(\mathbb{F})$ must either be prime or 0. (Hint: For the case that $\text{char}(\mathbb{F})$ is non-zero, use contradiction.)

Proof.

(a):

This result is quite fast, as we can add $[1]$ to itself p times to check:

$$\underbrace{[1] + [1] + \dots + [1]}_{p \text{ times}} = [p] = [0]$$

(b):

Assume seeking contradiction that \mathbb{F} is a field and $\text{char}(\mathbb{F})$ is non-zero and non-prime. We disregard the case where $\text{char}(\mathbb{F}) = 1$, because that means that $1 = 0$, which is impossible. It follows that $\text{char}(\mathbb{F})$ can be written as a product of two integers $a \cdot b$, where $1 < a, b < \text{char}(\mathbb{F})$. By definition, we see that

$$\underbrace{1 + 1 + \dots + 1}_{a \cdot b \text{ times}} = 0$$

Group the 1's into groups of a like so:

$$\underbrace{\underbrace{(1 + \dots + 1)}_{a \text{ times}} + \dots + \underbrace{(1 + \dots + 1)}_{a \text{ times}}}_{b \text{ times}} = 0$$

Denote each term as x_a . We can repeatedly use the axiom of distributivity to see that

$$x_a \cdot \underbrace{(1 + \dots + 1)}_{b \text{ times}} = \underbrace{x_a + \dots + x_a}_{b \text{ times}} = 0$$

Let $x_b = \underbrace{(1 + \dots + 1)}_{b \text{ times}}$, so

$$x_a \cdot x_b = 0$$

This means that we must have either $x_a = 0$ or $x_b = 0$. Regardless, we have found a value $p < \text{char}(\mathbb{F})$ such that repeated addition of 1 up to p times results in 0, which is a contradiction.

□

Question 7.

In this question we introduce a new definition:

Definition: Let $f, g \in \mathbb{P}(\mathbb{F})$. We say that a polynomial $d \in \mathbb{P}(\mathbb{F})$ is a **greatest common divisor** of f and g if:

- d is a divisor of both f and g , and;
- for any other divisor d' of f and g , we have $\deg d \geq \deg d'$.

(a) Prove that if d is a common divisor of f and g , then for all $a \in \mathbb{F} \setminus \{0\}$, the polynomial ad is also a common divisor for f and g . Explain why this shows that there is no "unique" greatest common divisor for f and g like there is for integers.

(b) Prove that if d_1, d_2 are both greatest common divisors for f and g , then $d_1 = ad_2$ for some non-zero field element a .

(c) Prove that we can compute a greatest common divisor for f and g like we do for integers: repeatedly apply long division until the remainder is 0, then the last non-zero remainder is a greatest common divisor for f and g .

(d) Deduce from (c) that if d is a greatest common division for f and g , then we can write $d = pf + qg$ for some polynomials p, q .

Proof.

(a):

Suppose that d is a common divisor of f and g . By definition,

$$f = dp \text{ and } g = dq, \text{ for some } p, q \in \mathbb{P}(\mathbb{F}).$$

Let $a \in \mathbb{F} \setminus \{0\}$. We know that a^{-1} exists because \mathbb{F} is a field. It follows that

$$\begin{aligned} f &= dp \\ &= 1 \cdot dp && \text{(additive identity)} \\ &= (a \cdot a^{-1})dp \\ &= a(a^{-1}d)p && \text{(associativity)} \\ &= a(da^{-1})p && \text{(commutativity)} \\ &= (ad)(a^{-1}p) && \text{(associativity)} \end{aligned}$$

Likewise for g ,

$$\begin{aligned} g &= dq \\ &= 1 \cdot dq && \text{(additive identity)} \\ &= (a \cdot a^{-1})dq \\ &= a(a^{-1}d)q && \text{(associativity)} \\ &= a(da^{-1})q && \text{(commutativity)} \\ &= (ad)(a^{-1}q) && \text{(associativity)} \end{aligned}$$

The equations above imply that ad divides both f and g , so ad is a common divisor.

Question 8.

Apply the procedures in Question 7 to compute a greatest common divisor for the polynomials $f(x) = x^4 + x^2 + 1$, $g(x) = x^4 + 2x^3 + x^2 + 1 \in \mathbb{P}(\mathbb{Q})$, and express this divisor as a combination of f and g .

(In particular, you should not try to factor f , g to find the greatest common divisor, and doing so will not receive any credit.)

Question 9.

Let $p \in \mathbb{P}(\mathbb{C})$ be a polynomial with real coefficients. Prove that if a is a root of p , then \bar{a} is a root of p . (Hint: Write down an equation that means " a is a root of p ". Conjugate this equation.)

Question 10.

Using Question 9 and the Fundamental Theorem of Algebra, prove that the only irreducible polynomials over \mathbb{R} are linear and quadratics with no real roots. Use this to deduce our Theorem from class (Week 2) about the factorization of real polynomials.