Question 23.

Let S^2 denote the unit sphere in $\mathbf{R}^3$. Let $N = (0,0,1)$ denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to $\mathbf{R}^2$. To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbf{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in $\mathbf{R}^3$.

(a) Given $P = (x, y, z)$, find an explicit formula for $\Phi(P)$ in terms of x, y, z.

(b) Deduce that Φ is continuous.

(c) Prove that Φ is a bijection; in fact, given $p = (s,t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.

(d) Deduce that Φ is a homeomorphism.

Proof. (a):
Let $P = (x, y, z)$. First, we find the equation of the line that passes P and N. Consider the equation of the line $L(t) = (tx, ty, (z-1)t + 1)$. Notice that $L(0) = N$ and $L(1) = P$, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when $(z-1)t + 1 = 0$. Solving for t gives $t = \dfrac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z}\left(x, y\right).$$

(b):
(c):
Let $p = (s,t) \in \mathbb{R}^2$. Our goal is to find $(x,y,z) \in S^2 \setminus \{N\}$ such that $\Phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s,t)$. Immediately, we obtain the following system of equations:

$$\frac{x}{1-z} = s,$$

$$\frac{y}{1-z} = t,$$

$$x^2 + y^2 + z^2 = 1$$

We also have the restriction $z \neq 1$ because $(x,y,z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^2(1-z)^2 + t^2(1-z)^2 + z^2 = 1 \implies (s^2+t^2+1)z^2 - 2(s^2+t^2)z + s^2 + t^2 - 1 = 0$$

We can replace the term $t^2 + s^2$ with $\|p\|_2^2$, and the equation becomes

$$(\|p\|_2^2 + 1)z^2 - 2\|p\|_2^2 z + \|p\|_2^2 - 1 = 0$$

Using the quadratic formula:

$$z = \frac{2(\|p\|_2^2) \pm \sqrt{4(\|p\|_2^2)^2 - 4(\|p\|_2^2 + 1)(\|p\|_2^2 - 1)}}{2(\|p\|_2^2 + 1)}$$

$$\implies z = \frac{\|p\|_2^2 \pm \sqrt{\|p\|_2^4 - (\|p\|_2^4 - 1)}}{\|p\|_2^2 + 1}$$

$$z = \frac{\|p\|_2^2 \pm 1}{\|p\|_2^2 + 1}$$

Notice that we cannot use the positive solution, for then

$$z = \frac{\|p\|_2^2 + 1}{\|p\|_2^2 + 1} = 1$$

Thus it must be true that

$$z = \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}$$

$$x = s(1 - z) = \frac{2s}{\|p\|_2^2 + 1}$$

$$y = \frac{2t}{\|p\|_2^2 + 1}$$

It can be verified that these values of x, y, z result in $\Phi(x, y, z) = (s, t)$. In fact, using this, we obtain that the formula for Φ^{-1} is

$$\Phi^{-1}(s, t) = \left(\frac{2s}{\|p\|_2^2 + 1}, \frac{2t}{\|p\|_2^2 + 1}, \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1} \right)$$

$\square$

Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

(i) X is finite-dimensional.

(ii) The unit ball $\overline{B}(\vec{0}, 1)$ is compact.

(iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that $\overline{U}$ is compact.

Proof. It will be proven that (i) $\implies$ (ii) $\iff$ (iii) $\implies$ (i).
(i) $\implies$ (ii):
Suppose that X has finite dimension n. Consider the linear isomorphism $\Phi : X \to \mathbb{R}^n$ by mapping elements of an ordered basis for X to the basis $\beta = \{e_1, ..., e_n\}$ for $\mathbb{R}^n$. This is a homeomorphism,
(ii) $\implies$ (iii):
Suppose that the unit ball $\overline{B}(\vec{0}, 1)$ is compact. Let $p \in X$. We claim that $U = B(p, 1)$. Consider $\overline{U} = \overline{B}(p, 1)$. There is a homeomorphism Φ from $\overline{B}(0, 1)$ to $\overline{B}(p, 1)$ defined by $\Phi(x) = p + x$. Compactness is preserved by the homeomorphism, so it follows that $\overline{B}(p, 1)$ is compact. Thus X is locally compact.
(iii) $\implies$ (i):

□