Question 1.

Suppose $\widetilde{p}$ must approximate p with relative error at most 10^{-3}. Find the largest interval in which $\widetilde{p}$ must lie if $p = 900$.

Proof. Since we want the relative error to be at most 10^{-3}, we set

$$\frac{|\widetilde{p} - p|}{|p|} \leq 10^{-3}$$

Substitute $p = 900$ to get

$$\frac{|\widetilde{p} - 900|}{900} \leq 10^{-3} \implies |\widetilde{p} - 900| \leq \frac{9}{10} \implies 900 - \frac{9}{10} \leq \widetilde{p} \leq 900 + \frac{9}{10}.$$

Thus $\widetilde{p}$ lies within the interval $\left[900 - \frac{9}{10}, 900 + \frac{9}{10}\right]$.

□

Question 2.

Compute the absolute error and relative error of the following approximation of e:

$$\sum_{n=0}^{5} \frac{1}{n!}$$

Using Octave, we find that the absolute error is

$$\left| e - \sum_{n=0}^{5} \frac{1}{n!} \right| = \left| e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}\right) \right| = \left| e - \frac{326}{120} \right| \approx \texttt{1.615161792378306e-03}$$

and the relative error is

$$\left| \frac{e - \sum_{n=0}^{5} \frac{1}{n!}}{e} \right| = \left| \frac{e - \frac{326}{120}}{e} \right| \approx \texttt{5.941848175815963e-04}$$

Question 3.

Find the second Taylor polynomial, $P_2(x)$, $f(x) = e^x \cos(x)$ about $x_0 = 0$.

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound on the error $|f(0.5) - P_2(0.5)|$ using the remainder term and compare it to the actual error.

First, we find $P_2(x)$. We calculate that

$$\begin{aligned} f(x_0) &= 1 \\ f'(x_0) &= e^{x_0}(\cos(x_0) - \sin(x_0)) = 1 \\ f''(x_0) &= -2e^{x_0}\sin(x_0) = 0 \\ f^{(3)}(x) &= -2e^x(\sin(x) - \cos(x)) \end{aligned}$$

Thus $P_2(x) = 1 + x$, so $f(0.5) \approx P_2(0.5) = 1.5$. The error term is

$$|R_2(0.5)| = \left|\frac{f^{(3)}(\xi)}{3!}(0.5)^3\right| = \left|\frac{e^\xi(\sin(\xi) - \cos(\xi))}{24}\right|, \text{ for } \xi \in (0, 0.5).$$

Since $e^\xi < e^{0.5}$, $\sin(\xi), \cos(\xi) \leq 1$, we get that

$$|R_2(0.5)| < \frac{e^{0.5}}{12} \approx 0.1374.$$

and the actual absolute error is

$$\left|e^{0.5}\cos(0.5) - 1.5\right| \approx 0.053111$$

(b) Find a bound on the error $|f(x) - P_2(x)|$ good on the interval $[0, 1]$.

Similar to the previous part, the error term is

$$|R_2(x)| = \left|\frac{e^\xi(\sin(\xi) - \cos(\xi))}{3}x^3\right|, \text{ for } \xi \in (0, x)$$

We know that $e^\xi < e^x$, so

$$|R_2(x)| < \frac{2}{3}x^3e^x.$$

(c) Approximate $\int_0^1 f(x)\,dx$ by calculating $\int_0^1 P_2(x)\,dx$ instead.

We have that

$$\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx = \int_0^1 1 + x\,dx = x + \frac{x^2}{2}\bigg|_0^1 = 1.5$$

(d) Find an upper bound for the error in (c) using $\int_0^1 |R(x)|\,dx$ and compare the bound to the actual error.

From part (b), the error term $|R_2(x)$ is bounded above by $\frac{2}{3}x^3e^x$. Thus the error for the computation is

$$\int_0^1 |R_2(x)|\,dx < \int_0^1 \frac{2}{3}x^3e^x\,dx = \frac{2}{3}\left(x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right)\bigg|_0^1$$

$$= \frac{2}{3}(-2e + 6) \approx 0.3756.$$

and the actual error is

$$\left| \int_0^1 f(x)\ dx - 1.5 \right| = \left| \int_0^1 e^x \cos(x)\ dx - 1.5 \right| = \left| \frac{1}{2} e^x (\cos(x) + \sin(x)) \Big|_0^1 - 1.5 \right|$$

$$= \left| \frac{e(\cos(1) + \sin(1))}{2} - \frac{1}{2} - 1.5 \right| \approx 0.1220.$$

Question 4.

Find a theoretical upper bound, as a function of x, for the absolute error in using $T_4(x)$ to approximate $f(x) = \frac{10}{x} + \sin(10x)$; $x_0 = \pi$.

Question 5.

Let $(p_n) = \left\langle \frac{3n^5 - 5n}{1 - n^5} \right\rangle \to -3$ Find the (fastest) rate of convergence of the form $\mathcal{O}\left(\frac{1}{n^p}\right)$ or $\mathcal{O}\left(\frac{1}{a^n}\right)$ for each. If this is not possible, suggest a reasonable rate of convergence.

Question 6.

(a) Suppose you are trying to find the root of $f(x) = x - e^{-x}$ using the bisection method. Find an integer a such that the interval $[a, a+2]$ is an appropriate one in which to start the search.

(b) Use the bisection method to find the root in your interval in (a), accurate to 10^{-4}. Provide the Octave code you used to produce your result.