Question 32.

Let M be a subset of $\mathbf{R}^n$, let $p_0 \in M$ be a point, and let $\vec{v} \in \mathbf{R}^n$ be a vector. We say that $\vec{v}$ is a **tangent vector** to M at p_0 if there exists $\delta > 0$ and a C^1 function $\alpha : (-\delta, \delta) \to M$ such that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. In other words, $\vec{v}$ is the velocity vector of a curve through M.

(a) Suppose now that M is the *zero set* of some C^1 function $f : U \to \mathbf{R}$, where U is an open set in $\mathbf{R}^n$: thus

$$M = \{p \in U : f(p) = 0\}.$$

Suppose that $p_0 \in M$ is a point such that $\nabla f(p_0) \neq \vec{0}$, and let $\vec{v} \in \mathbf{R}^n$ be a vector. Show that $\vec{v}$ is a tangent vector to M at p_0 if and only if $\nabla f(p_0) \cdot \vec{v} = 0$.

(b) Let E be the ellipsoid in $\mathbf{R}^3$ defined by the following equation:

$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$

Find the equation of the tangent plane to M at the point $p_0 = (1, 2, 3)$.

Hint: Define an appropriate function f, then find two vectors which are orthogonal to $\nabla f(p_0)$. By (a), these two vectors span the tangent plane. I recommend using graphing software to confirm your result.

Proof. (a):
Suppose that $\vec{v}$ is a tangent vector to M at p_0. Then there exists a function $\alpha : (-\delta, \delta) \to M$ so that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. Define $g : (-\delta, \delta) \to \mathbb{R}$ by $g(t) = f(\alpha(t))$. For all $t \in (-\delta, \delta)$, $\alpha(t) \in M$, so $g(t) = 0$. It follows that

$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

Substituting $t = 0$ yields

$$f(p_0) \cdot \vec{v} = 0$$

as needed.

Conversely, suppose that $\nabla f(p_0) \cdot \vec{v} = 0$.

□

Question 33.

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a **monic** polynomial with **no repeated real roots**. This means that $a_n = 1$ and that all real roots of f have multiplicity 1.

(a) Let r be a root of $f(x)$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that: if $g(x) = \sum_{i=0}^{n} b_i x^i$ is a monic polynomial with coefficients b_i satisfying $|a_i - b_i| < \delta$, then $g(x)$ has at least one root in the interval $(r - \varepsilon, r + \varepsilon)$.

This shows that slight perturbations of the coefficients results in slight perturbations of the roots.

Hint: You will need the Implicit Function Theorem.

(b) Suppose that f has fewer than n real roots. Prove that number of real roots of f does not change under small perturbation of the coefficients.