Exercise 10.9

Solvers: Ethan
Writeup: Ethan
Let $U \subseteq \mathbb{R}^n$ be open, and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping. Suppose that there is a point $p_0 \in U$ such that $\Phi'(p_0) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

Define the function $\Psi = \Phi^{-1}$. From the previous exercises, we know that $\Psi'(q) = \Phi'(\Psi(q))^{-1}$. It remains to show that Ψ' is continuous.
Exercise 10.9 Ψ' is continuous.

Proof. Consider the inversion map $T : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$. That is, for an invertible linear mapping A in $\mathbb{R}^n$, $T(A) = A^{-1}$. It will be shown that T is continuous.
Fix $A \in \mathrm{GL}\mathbb{R}^n$ and let $\varepsilon > 0$. Let $\delta =$. For $h \in \mathrm{GL}(\mathbb{R}^n)$ so that $\|h\|_{\mathrm{op}} < \delta$, we have that

$$\|T(A+h) - T(A)\|_{\mathrm{op}} = \|(A+h)^{-1} \circ (A+h) \circ ((A+h)^{-1} - A^{-1})\|_{\mathrm{op}}$$

$$= \|(A+h)^{-1} \circ ((A+h) \circ (A+h)^{-1} - (A+h) \circ A^{-1})\|_{\mathrm{op}} = \|(A+h)^{-1} \circ (-h \circ A^{-1})\|_{\mathrm{op}}$$

$$\leq \|(A+h)^{-1}\|_{\mathrm{op}} \|h\|_{\mathrm{op}} \|A^{-1}\|_{\mathrm{op}}$$

□