Question 1.

(Based on 2.2, #24) Let $g(x) = \left(\frac{1}{2}\right)^x + \left(\frac{1}{5}\right)^x - 10^{-5}$.

a. Show that if g has a zero at p, then the function $f(x) = x + cg(x)$ has a fixed point at p.

   Suppose that g has a zero at p. Then $g(p) = 0$. It follows immediately that $f(p) = p + cg(p) = p$, so f has a fixed point at p.

b. Find a value of c for which fixed point iteration of $f(x)$ will successfully converge for any starting value, p_0, in the interval $[16, 17]$. (*Note: You don't need to include the graphs.)

   To guarantee convergence, we will find c such that $|f'(x)| < 1$ for all $x \in [16, 17]$. First, we rule out $c = 0$, as despite $f(x) = x$ converging to a fixed point everywhere, it is unable to tell us about the roots of g. Now, we compute that

$$f'(x) = 1 + c\left(\left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right) + \left(\frac{1}{5}\right)^x \cdot \ln\left(\frac{1}{5}\right)\right) = 1 - c\left(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5\right)$$

   We note that if $c < 0$, then $-c\left(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5\right) > 0$, so $f'(x) > 1$, which is not what we want. If $c > 0$, f' is an increasing function. Since $16 \le x \le 17$ we get that

$$1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) \le f'(x) \le 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5)$$

   We solve for c in the following inequality:

$$\begin{aligned} 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) < 1 &\implies c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) > 0 \\ &\implies c > 0 \end{aligned}$$

   We also want the lower bound of $f'(x)$ to be -1:

$$\begin{aligned} 1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) > -1 &\implies c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) < 2 \\ &\implies c < \frac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5} \end{aligned}$$

   Thus any value of c between 0 and $\dfrac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}$ will work, so we can just pick $c = \dfrac{1}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}$.

c. Use the function from part (b) with the value of c you have determined to find a root of $g(x)$ accurate to within 10^{-4}. State the value you used for p_0 and show the last three iterations. How many iterations did it take?

   We will use fixed point iteration on $f(x) = x + cg(x)$ with $p_0 = 16.5$. Below is the Octave code, input, and output:

```
function [m] = fixedpoint(f,x,N,tol)
  for j=1:N
    m = f(x);
```

```
    disp(["Value at iteration number " num2str(j) ": " num2str(m)]);
    if abs(m - x) <= tol
    disp(["Fixed point within given tolerance found in " num2str(j) " iterations."])
    return;
    else
    x = m;
    end%if
  end%for
  disp("Method failed. Max iterations exceeded.")
end%function

>> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5))*(1/2^x + 1/5^x - 10^(-5)), 16.5, 1000, 10^(-4))
Value at iteration number 1: 16.5747
Value at iteration number 2: 16.5979
Value at iteration number 3: 16.6056
Value at iteration number 4: 16.6083
Value at iteration number 5: 16.6092
Value at iteration number 6: 16.6095
Value at iteration number 7: 16.6096
Value at iteration number 8: 16.6096
Fixed point within given tolerance found in 8 iterations.
m = 16.610
```

We found a fixed point for f around $x = 16.610$, which implies that g has a root around that point as well.

d. Now repeat part (c) and find a root of g accurate to within 10^{-7}, using potentially other values for c as necessary. Explain your process and how you picked an appropriate c and x_0.

We continue using fixed point iteration, keeping the value of c the same. We know that our fixed point is close to $x = 16.610$, so that will be where we start the next fixed point iteration. Below is the Octave commands used and the output:

```
>> format long
>> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5))*(1/2^x + 1/5^x - 10^(-5)), 16.610, 1000, 10^(-7))
Value at iteration number 1: 16.6098
Value at iteration number 2: 16.6097
Value at iteration number 3: 16.6097
Value at iteration number 4: 16.6096
Value at iteration number 5: 16.6096
Value at iteration number 6: 16.6096
Value at iteration number 7: 16.6096
```

```
Value at iteration number 8: 16.6096
Value at iteration number 9: 16.6096
Fixed point within given tolerance found in 9 iterations.
m = 16.60964085351603
```

Question 2.

(2.3, #9) The function $g(x) = \sqrt[3]{5-3x}$ satisfies the hypotheses of Proposition 5 over the interval $[1, 1.3]$.
Find a bound on the number of iterations required to find the fixed point to within 10^{-5} accuracy starting with initial value x_0 of your choice.
Let $\hat{x}$ be the fixed point of g within the interval $[1, 1.3]$. To find the M value in proposition 5, we take the derivative of g:

$$g'(x) = \frac{-1}{(5-3x)^{\frac{2}{3}}}, \text{ where } x \in [1, 1.3].$$

Since $1 \leq x \leq 1.3$, we have

$$1.1 \leq 5 - 3x \leq 2 \implies \frac{1}{5-3x} \leq \frac{1}{1.1} < 1$$

Since the function $t^{\frac{2}{3}}$ is increasing for positive t, we have that

$$\frac{1}{(5-3x)^{\frac{2}{3}}} \leq \frac{1}{1.1^{\frac{2}{3}}} < 1$$

Thus $|g'(x)| \leq \frac{1}{1.1^{\frac{2}{3}}} < 1$, so $M = \frac{1}{1.1^{\frac{2}{3}}} \approx 0.9384$ is our desired value.
Let $x_0 = 1$. Notice that $1 \leq \hat{x} \leq 1.3$. By proposition 5, we have that

$$|x_{165} - \hat{x}| \leq M^{165}|1 - \hat{x}| = M^{165}(\hat{x} - 1) \leq M^{165}(1.3 - 1) = 0.3 \cdot M^{165} \approx 8.393 \cdot 10^{-6} < 10^{-5}$$

Thus the upper bound on the number of iterations is 165.

Question 3.

Consider the function $g(x) = \ln(\sin x + 1.5)$.
Find an initial value x_0 (to four decimal places) so that Newton's method fails at the second iteration. That is, Newton's method finds x_1 but cannot find x_2.

$$\frac{\pi}{2} = x_0 - \frac{\ln(\sin x_0 + 1.5)}{\frac{\cos x_0}{\sin x_0 + 1.5}}$$

Question 4.

Let $g(x) = \cos x - e^{-x/2} + 1.0005$, which has one negative root in $[-1, 0]$. Using $x_0 = -1$ and $x_1 = 0$, determine x_2 and x_3 when using:

a. the bracketed Newton's method, and

b. the bracketed secant method.

Show the results of your computation in a table and explain your steps.
Let l denote the left endpoint of the current interval and r denote the right endpoint of the current interval. To begin, we set $l = x_0$ and $r = x_1$.

k	l	r	$g(l)$	$g(r)$	$g'(x_k)$	Candidate	x_{k+1}	$f(x_{k+1})$
1	-1	0	-0.1079	1.0005	0.5	-2.001	-0.5	0.5941
2	-1	-0.5	-0.1079	0.5941	1.121	-1.030	-0.75	0.2772

Question 5.

Let $g(x) = \cos x - e^{-x/2} + 1.0005$.
Using any of the root-finding methods discussed in Chapter 2, find all of its positive roots to within 10^{-4}. Explain how you know you've found all of them.

Question 6.

(3.2, #11, 12) A Lagrange interpolating polynomial is constructed for the function $f(x) = (\sqrt{2})^x$ using $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

a. If this polynomial is used to approximate $f(1.5)$, find a bound on the error in this approximation.

   Let P_3 be the interpolating polynomial of least degree that passes through the points $(0,1), (1,\sqrt{2}), (2,2)$ and $(3, 2\sqrt{2})$. We know that there exists $\xi \in (0,3)$ such that the absolute error

$$|f(1.5) - P(1.5)| = \left|\frac{f^{(4)}(\xi)}{4!}1.5(1.5-1)(1.5-2)(1.5-3)\right| = \frac{5625}{24000}\left|\left(\frac{\ln 2}{2}\right)^4 (\sqrt{2})^\xi\right|$$

   f is increasing, so it follows that

$$|f(1.5) - P(1.5)| \le \frac{5625 \ln 2}{24000 \cdot 2^4}(\sqrt{2})^3 = \frac{5625 \ln 2}{24000 \cdot 2^3}\sqrt{2} \approx 0.2871$$

b. Find the Lagrange interpolating polynomial, and use it to approximate $f(1.5)$. Then calculate the actual error in approximation.

First, we will find p_1, p_2, p_3, and p_4:

$$p_1(x) = f(0) \cdot \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3)$$

$$p_2(x) = f(1) \cdot \frac{x(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \sqrt{2}x(x-2)(x-3)$$

$$p_3(x) = f(2) \cdot \frac{x(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -x(x-1)(x-3)$$

$$p_4(x) = f(3) \cdot \frac{x(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{\sqrt{2}}{3}x(x-1)(x-2)$$

Thus the Lagrange interpolating polynomial is

$$P_3(x) = p_1(x) + p_2(x) + p_3(x) + p_4(x)$$

$$= -\frac{1}{6}(x-1)(x-2)(x-3) + \sqrt{2}x(x-2)(x-3) - x(x-1)(x-3) + \frac{\sqrt{2}}{3}x(x-1)(x-2)$$