Exercise 8.12 (c) (iii)

Solvers: Alan, Ethan
Writeup: Ethan

Let $U \subseteq \mathbb{R}$ be an open set and let $f : U \to \mathbb{R}$ be a scalar function. Then if all partial derivatives exist and are continuous on U, then f is continuously differentiable and $f'(p)$ is given by $f'(p)(v) = D_v f(p)$.

Proof. Let $L_p = f'(p)$. We will show that $\dfrac{|f(p+h) - f(p) - L_p(h)|}{\|h\|} \to 0$.

Let $\varepsilon > 0$. Utilising part (i), we define a sequence of points $p_0, ..., p_n \in X$ by

$$p_0 = p \text{ and } p_i = p_{i-1} + h_i e_i.$$

We know that for $\|h\|$ smaller than some positive γ, $p_i \in U$.

Let $\delta =$. Let $h \in U$ so that $\|h\| < \delta$. Notice that $p + h = p_n$ and $p = p_0$. We have that

$$\frac{|f(p+h) - f(p) - L_p(h)}{\|h\|} = \frac{|f(p_n) - f(p_0) - L_p(h)|}{\|h\|}$$

We can expand the numerator by adding and subtracting every term of p_i and substituting

$$L_p(h) = D_h f = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(p),$$

which yields

$$\frac{\left|\sum_{i=1}^{n} \left(f(p_i) - f(p_{i-1}) - h_i \frac{\partial f}{\partial x_i}(p) \right)\right|}{\|h\|} \leq \sum_{i=1}^{n} \frac{\left| f(p_i) - f(p_{i-1}) - h_i \frac{\partial f}{\partial x_i}(p) \right|}{\|h\|}.$$

Now, by part (ii), we can rewrite $f(p_i) - f(p_{i-1})$ as $h_i \dfrac{\partial f}{\partial x_i}(q_i)$, for some $q_i \in U$, so the expression becomes

$$\sum_{i=1}^{n} \frac{\left| h_i \frac{\partial f}{\partial x_i}(q_i) - h_i \frac{\partial f}{\partial x_i}(p) \right|}{\|h\|} = \frac{1}{\|h\|} \sum_{i=1}^{n} |h_i| \left| \frac{\partial f}{\partial x_i}(q_i - p) \right|$$

$\square$