Question 37.

The following set is called the n**-simplex**:

$$\Delta_n := \{\vec{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \cdots + x_n \leq 1\}.$$

You can assume, without proof, that Δ_n is Jordan measurable.
Find, with proof, an explicit formula for $\mu(\Delta_n)$ in terms of n.

Proof. First, we show that Δ_n is the same as the set

$$S = \left\{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, ..., 0 \leq x_n \leq 1 - \sum_{i=1}^{n-1} x_i\right\}$$

Let $x \in \Delta_n$. We want to show that $0 \leq x_i \leq 1 - \sum_{j=1}^{i-1} x_j$. We get that $x_i \geq 0$ immediately. As well, since $\sum_{j=1}^{n} x_j \leq 1$ and every component is non-negative, we have that

$$x_i \leq 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^{n} x_j \leq 1 - \sum_{j=1}^{i-1} x_j$$

which shows that $\Delta_n \subseteq S$.
Now, let $x \in S$. We know every x_i is non-negative and additionally

$$x_n \leq 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^{n} x_i \leq 1$$

so $S \subseteq \Delta_n$.
Now, we proceed to find $\mu(S) = \mu(\Delta_n)$. Using Fubini's Theorem, we get

$$\mu(S) = \int_S 1 = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 \, dx_n \cdots dx_2 \, dx_1$$

Let $I : \mathbb{N} \times [0, 1] \to \mathbb{R}$ be defined recursively as follows:

$$\begin{aligned} I(1, \alpha) &= \int_0^{1-\alpha} 1 \, dt \\ I(k, \alpha) &= \int_0^{1-\alpha} I(k-1, \alpha + t) \, dt, & \text{for } > 1. \end{aligned}$$

Notice that if we continue applying the definition, we get that

$$I(n, 0) = \mu(S)$$

Now, we will prove using induction on n that for all $\alpha \in [0, 1]$, $I(n, \alpha) = \dfrac{1}{n!}(1-\alpha)^n$.
Let $n = 1$. Then

$$I(1, \alpha) = \int_0^{1-\alpha} 1 \, dt = 1 - \alpha$$

Now, suppose that $I(k,\alpha) = \frac{1}{k!}(1-\alpha)^k$ holds for all $\alpha \in [0,1]$ and some $k \in \mathbb{N}$. We want to show that the same holds for $k+1$ as well. For an arbitrary α, we get

$$\begin{aligned} I(k+1,\alpha) &= \int_0^{1-\alpha} I(k,\alpha+t)\ dt \\ &= \int_0^{1-\alpha} \frac{1}{k!}(1-\alpha-t)^k\ dt \\ &= -\frac{1}{(k+1)!}(1-\alpha-t)^{k+1}\Big|_0^{1-\alpha} \\ &= \frac{1}{(k+1)!}(1-\alpha)^{k+1} \end{aligned}$$

as desired. Thus we get that

$$\mu(S) = I(n,0) = \frac{1}{n!}$$

$\square$