Question 1.

Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Use row and column operations on A to obtain a matrix B of the form in Theorem 53. Use that work to find invertible matrices P, Q so that $B = PAQ$.

Proof. We perform the following row and column operations:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Define this matrix we obtained as B. We will perform the same row and column operations above on I_3 and I_4, respectively in order to define P and Q. We have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We see that

$$PAQ = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= B$$

as required.

□

Question 2.

Let $A = \begin{pmatrix} 1 & -2 & -4 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

1. Verify that A is invertible, by row-reducing the augmented matrix $(A|I_3)$.
2. Use (a) to find A^{-1}.
3. Express A as a product of elementary matrices.

Question 3.

Find the explicit formula for the linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^3$ which satisfies:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question 4.

Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathcal{M}_{2\times 2}(\mathbb{F})$. Consider the linear map $T : \mathcal{M}_{2\times 2}(\mathbb{F}) \to \mathcal{M}_{2\times 2}(\mathbb{F})$ given by $T(A) = A^T$. Set $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$.

1. Find P - the change of coordinate matrix from γ to β coordinates.
2. Find P^{-1} - the change of coordinate matrix from β to γ coordinates.
3. Find $A = [T]_\beta$.
4. Find $B = [T]_\gamma$.
5. Confirm that $A = PBP^{-1}$ using (a)-(d).

Question 5.

Let $T : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathcal{M}_{n\times n}(\mathbb{F})$ be the linear map given by $T(A) = A + A^T$.

1. Find $N(T)$ and $\dim N(T)$.
2. What is $\mathrm{im}(T)$?
3. Is $\mathcal{M}_{n\times n}(\mathbb{F}) = \mathrm{im}(T) \oplus N(T)$?

Question 6.

Let V, W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$ a linear map. Prove that T is injective if and only if $N(T) = \{\mathbf{0}_V\}$. (Make no assumption here about $\dim V, \dim W$.)

Proof. Suppose that T is injective. Let $T(x) = 0$, for some $x \in V$. Recall that $T(0) = 0$ for any linear map. Therefore by injectivity $x = 0$, so $N(T) = \{0\}$.
Conversely, suppose that $N(T) = \{0\}$. Let $x, y \in V$ such that $T(x) = T(y)$. By linearity, we have that $T(x - y) = 0$, but this implies that $x - y = 0$, so $x = y$ and T is injective.

$\square$

Question 7.

Let V, W be vector spaces over a field $\mathbb{F}$, and $T : V \to W$ a linear map. Find a condition on T which is equivalent to "$T(S)$ spans W for any spanning set $S \subseteq V$ of V".
(Hint: Write down the definition of $T(S)$ is spanning to get started.)

Question 8.

Let $P \in \mathcal{M}_{n\times n}(\mathbb{F})$. Prove the following three conditions are equivalent.

1. P is invertible.
2. There exists bases β, γ of $\mathbb{F}^n$ so that $P = [\mathrm{I}_{\mathbb{F}^n}]_\beta^\gamma$.
3. For any n-dimensional vector space V over $\mathbb{F}$, there exists bases β, γ of V so that $P = [\mathrm{I}_V]_\beta^\gamma$.

Question 9.

Consider the relation $\equiv$ on $\mathcal{M}_{m\times n}(\mathbb{F})$ defined by $A \equiv B$ if $A \to B$ using a combination of row and/or column operations.

1. Prove that $\equiv$ is an equivalence relation on $\mathcal{M}_{m\times n}(\mathbb{F})$.
2. Find a condition on A, B which is equivalent to $A \equiv B$. (Hint: Theorem 53.)
3. Classify the equivalence classes for this relation, and prove that there are exactly $\min\{n, m\}$ such classes.

Question 10.

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, and $T : V \to W$ a linear map. Set $n = \dim V$, $m = \dim W$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}^n$ be two non-parallel vectors. Prove there exists bases β, γ of V, W respectively, so that $[T]_\beta^\gamma = (\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{0}\ \cdots\ \mathbf{0})$. (Hint: use problems 7,8.)

Question 11.

Let $T : V \to V$ be linear. We say that a subspace $W \subseteq V$ is "T-invariant" if $T(W) \subseteq W$. For example, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is counter-clockwise rotation around the z-axis by angle θ, then $P_{xy} = \{(x, y, 0) \in \mathbb{R}^3\}$ is T-invariant, as is L_z (the z-axis).

1. Verify the claims made above, by showing that P_{xy} and L_z are T-invariant.
2. Show that $\mathbb{R}^3 = P_{xy} \oplus L_z$ by finding a basis $\beta = \beta_1 \cup \beta_2$ for $\mathbb{R}^3$ so that β_1 is a basis for P_{xy} and β_2 is a basis for L_z.
3. Using your basis β from (b), find $[T]_\beta$.

Question 12.

Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$, and $W_1 \subseteq V$ a T-invariant subspace with basis β_1. Set $k = \dim W_1$.
We will generalize what we saw in #11c.

1. Extend β_1 to a basis β of V. Show that $[T]_\beta = \begin{pmatrix} A & C \\ O_{n-k,k} & B \end{pmatrix}$, where A is $k \times k$, B is $(n-k) \times (n-k)$, and C is $k \times (n-k)$.
2. Suppose that W_2 is a subspace so that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$, where β_2 is any basis for W_2.

   Prove that if W_2 is T-invariant, then $[T]_\beta = \begin{pmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{pmatrix}$ is block diagonal.
3. Is the converse of (b) true or false? Justify your answer.

Question 13.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

1. Let $\beta = \{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{F}^n$, and $\gamma = \{v_1, \dots, v_n\}$ a basis for $\mathbb{F}^n$. Then there exists a sequence of row operations that takes β to γ. (That is, v_i is obtained from e_i using the same row operations for all i.)
2. Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. If β, γ are bases for V so that $[T]_\beta^\gamma = I_n$, then $T = I_V$.
3. Let V be a finite dimensional vector space over $\mathbb{F}$ and $S, T : V \to V$ linear maps. If $\operatorname{rank} T = \operatorname{rank} S$, then there exist bases $\beta, \beta', \gamma, \gamma'$ for V so that $[S]_\beta^\gamma = [T]_\beta^\gamma$.
4. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$. If $A^2 \sim B^2$, then $A \sim B$.