Question 35.

Perturbing the roots of a polynomial.
Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a **monic** polynomial with **no repeated real roots**. This means that $a_n = 1$, and that all real roots of f have multiplicity 1.

(a) Let r be a root of $f(x)$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that: if $g(x) = \sum_{i=0}^{n} b_i x^i$ is a monic polynomial with coefficients b_i satisfying $|a_i - b_i| < \delta$, then $g(x)$ has at least one root in the interval $(r - \varepsilon, r + \varepsilon)$.

This shows that slight perturbations of the coefficients results in slight perturbations of the roots.

(b) Suppose that f has fewer than n real roots. Prove that number of real roots of f does not change under small perturbation of the coefficients.

Proof.
(a):
First, we prove the following lemma:
Lemma. Let $r \in \mathbb{R}$ be a root of a polynomial p. Then r is a repeated root if and only if $p'(r) = 0$.
Suppose that p has a repeated root. Then we can factor p as $(x-r)^k q(x)$, for some $k > 1$ and $q \in \mathbb{P}(\mathbb{R})$. We can take the derivative of this and get that

$$p'(x) = k(x-r)^{k-1} q(x) + (x-r)^k q'(x)$$

$$\implies p'(r) = 0$$

Conversely, suppose that $p'(r) = 0$, for some $r \in \mathbb{R}$. We can write

$$p'(x) = (x-r) \sum_{i=0}^{m} c_i x^i, \text{ for constants } c_0, ..., c_m$$

We can integrate both sides to get that

$$p(x) = \int (x-r) \sum_{i=0}^{m} c_i x_i \, dx = \int x \sum_{i=0}^{m} c_i x_i - r \sum_{i=0}^{m} c_i x_i \, dx$$

$$= \sum_{i=0}^{m} \frac{c_i}{i+2} x^{i+2} - r \sum_{i=0}^{m} \frac{c_i}{i+1} x^{i+1} + C$$

In order for r to be a root of p, we must have that

$$p(r) = \sum_{i=0}^{m} \left(\frac{c_i}{i+2} r^{i+2} - \frac{c_i}{i+1} r^{i+2} \right) + C = 0$$

$$\implies C = \sum_{i=0}^{m} \left(\frac{c_i}{i+1} r^{i+2} - \frac{c_i}{i+2} r^{i+2} \right)$$

Therefore

$$\begin{aligned}
p(x) &= \sum_{i=0}^{m} \left(\frac{c_i}{i+2} x^{i+2} - \frac{c_i}{i+2} r^{i+2} + \frac{c_i}{i+1} r^{i+2} - r \frac{c_i}{i+1} x^{i+1} \right) \\
&= \sum_{i=0}^{m} \left(\left(\frac{c_i}{i+2} \right) (x^{i+2} - r^{i+2}) - \left(\frac{c_i r}{i+1} \right) (x^{i+1} - r^{i+1}) \right) \\
&= \sum_{i=0}^{m} \left(\left(\frac{c_i}{i+2} \right) (x-r) \sum_{j=0}^{i+1} x^j r^{i-j+1} - \left(\frac{c_i}{i+1} \right) (x-r) \sum_{j=0}^{i} x^j r^{i-j} \right) \\
&= (x-r) \sum_{i=0}^{m} \left(\frac{c_i}{(i+1)(i+2)} \left((i+1) \left(x^{i+1} + \sum_{j=0}^{i} x^j r^{i-j+1} \right) - (i+2) \sum_{j=0}^{i} x^j r^{i-j} \right) \right)
\end{aligned}$$

Define the C^1 function $\Phi : \mathbb{R}^{n+2} \to \mathbb{R}$ by

$$\Phi(y_0, y_1, ..., y_{n+1}, x) = \sum_{i=0}^{n} y_i x^i.$$

Let $a = (a_0, ..., a_n) \in \mathbb{R}^{n+1}$. We have that $\Phi(a, r) = 0$ and as well, by our lemma, $\frac{\partial \Phi}{\partial x}(a, r) = f'(r) \neq 0$. Applying the Implicit Function Theorem, we get that for some open set $W \subseteq \mathbb{R}^{n+1}$ and C^1 function $\Psi : W \to \mathbb{R}$ such that $\Psi(a) = r$,

$$\Phi(b, \Psi(b)) = 0, \text{ for all } b \in W.$$

Now, let $\varepsilon > 0$. by the continuity of Ψ at a, there exists a $\delta > 0$ so that for all $b \in W$ with $\|b - a\| < \delta$, $|\Psi(b) - r| < \varepsilon$, so $\Psi(b) \in (r - \varepsilon, r + \varepsilon)$. For any b that satisfies the condition above, consider the polynomial

$$g(x) = \sum_{i=0}^{n} b_i x^i = \Phi(b, x)$$

$\square$