Exercise 8.12 (c) (iii)

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Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a scalar function. Then if all partial derivatives exist and are continuous on U, then f is continuously differentiable and $f'(p)$ is given by $f'(p)(v) = D_v f(p)$.

Recall the statement in (ii): There exists a point $q_k \in U$ such that

$$\frac{f(p_k) - f(p_{k-1})}{h_k} = \frac{\partial f}{\partial x_k}(q_k)$$

We make the additional distinction that q_k is of the form $p_{k-1} + \gamma e_k$, where $|\gamma| < |h_k|$.

Proof. Define $\|\cdot\|$ on U to be the 1-norm.

Let $L_p = f'(p)$. We will show that $\dfrac{|f(p+h) - f(p) - L_p(h)|}{\|h\|} \to 0$.

Let $\varepsilon > 0$. Utilising part (i), we define a sequence of points $p_0, ..., p_n \in X$ by

$$p_0 = p \text{ and } p_i = p_{i-1} + h_i e_i.$$

We know that for $\|h\|$ smaller than some positive δ_1, $p_i \in U$. By the uniform continuity of $\dfrac{\partial f}{\partial x_i}$ there is also a δ_2 such that for all $a, b \in U$ such that $\|a - b\| < \delta_2$, $\left|\dfrac{\partial f}{\partial x_i}(a-b)\right| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Let $h \in U$ so that $\|h\| < \delta$. Notice that $p + h = p_n$ and $p = p_0$. We have that

$$\frac{|f(p+h) - f(p) - L_p(h)}{\|h\|} = \frac{|f(p_n) - f(p_0) - L_p(h)|}{\|h\|}$$

We can expand the numerator by adding and subtracting every term p_i and substituting

$$L_p(h) = D_h f(p) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(p),$$

which yields

$$\frac{\left|\sum_{i=1}^{n}\left(f(p_i) - f(p_{i-1}) - h_i \frac{\partial f}{\partial x_i}(p)\right)\right|}{\|h\|} \le \sum_{i=1}^{n} \frac{\left|f(p_i) - f(p_{i-1}) - h_i \frac{\partial f}{\partial x_i}(p)\right|}{\|h\|}.$$

Now, by part (ii), we can rewrite $f(p_i) - f(p_{i-1})$ as $h_i \dfrac{\partial f}{\partial x_i}(q_i)$, for some $q_i \in U$, so the expression becomes

$$\sum_{i=1}^{n} \frac{\left|h_i \frac{\partial f}{\partial x_i}(q_i) - h_i \frac{\partial f}{\partial x_i}(p)\right|}{\|h\|} = \frac{1}{\|h\|}\sum_{i=1}^{n} |h_i| \left|\frac{\partial f}{\partial x_i}(q_i - p)\right|$$

Note that p_i can also be written as $p + \sum_{j=1}^{i} h_j e_j$. Thus we can say that $q_i = p_{i-1} + \gamma e_i = p + \gamma e_i + \sum_{j=1}^{i-1} h_j e_j$. We get that

$$\frac{1}{\|h\|}\sum_{i=1}^{n} |h_i| \left|\frac{\partial f}{\partial x_i}(q_i - p)\right| = \frac{1}{\|h\|}\sum_{i=1}^{n} |h_i| \left|\frac{\partial f}{\partial x_i}\left(\gamma e_i + \sum_{j=1}^{i-1} h_j e_j\right)\right|$$

We see that the norm of the argument inside the partial derivative is

$$\left\|\gamma e_i + \sum_{j=1}^{i-1} h_j e_j\right\| \le |\gamma| + \sum_{j=1}^{i-1} |h_j| < \sum_{j=1}^{i} |h_j| \le \sum_{j=1}^{n} |h_j| = \|h\| < \delta_2,$$

so by the continuity of $\dfrac{\partial f}{\partial x_i}$,

$$\frac{1}{\|h\|}\sum_{i=1}^{n}|h_i|\left|\frac{\partial f}{\partial x_i}\left(\gamma e_i+\sum_{j=1}^{i-1}h_je_j\right)\right|<\frac{1}{\|h\|}\sum_{i=1}^{n}|h_i|\cdot\varepsilon=\frac{\|h\|}{\|h\|}\cdot\varepsilon=\varepsilon$$

and the proof is complete.

$\square$