Question 27.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuously differentiable function.

(a) Show that the partial function $\mathbf{R} \to \mathbf{R}$, $t \mapsto f(x,t)$ is integrable (over any bounded interval in $\mathbf{R}$).

(b) By (a), we can define a function $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(x) = \int_a^b f(x,t)\ dt.$$

Show that $\varphi$ is differentiable, and that its (classical) derivative is given by

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0,t)\ dt.$$

This formula is known as **differentiation under the integral sign**, or **Feynmann's trick**.

(c) Use Feynmann's trick to solve the single-variable integral:

$$\int_0^\infty e^{-t^2}\ dt$$

*Proof.* (a):
Since $f$ is continuous, it follows immediately that its partial function is continuous, which implies that it is integrable.
(b):
Let $x_0 \in \mathbb{R}$. We will show that as $h \to 0$,

$$\frac{1}{h}\left(\int_a^b f(x_0+h,t)dt - \int_a^b f(x_0,t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0,t)dt\right) \to 0,$$

which is equivalent to saying

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0,t)\ dt.$$

Let $\varepsilon > 0$. By the partial differentiability of $f$, we obtain a $\delta$ so that

$$\left|f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t)\right| < \frac{|h|\varepsilon}{b-a}$$

for all $0 < |h| < \delta$.
Fix $h \in \mathbb{R}$ so that $0 < |h| < \delta$. By the linearity of the integral,

$$\left|\frac{1}{h}\left(\int_a^b f(x_0+h,t)dt - \int_a^b f(x_0,t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0,t)dt\right)\right|$$

$$= \left| \frac{1}{h} \int_a^b (f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t))dt \right|$$
$$\leq \frac{1}{|h|} \int_a^b \left| f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t) \right| dt$$
$$< \frac{1}{|h|} \int_a^b \frac{|h|\varepsilon}{b-a} dt = \varepsilon$$

as desired. Thus we have found an expression for $\varphi'(x_0)$.
(c):
Let $I = \int_0^\infty e^{-t^2}\, dt$. We will first show that this integral converges. We make the observation that the integrand is continuous and positive. Splitting the integral in two, we see that

$$\int_0^\infty e^{-t^2}\, dt = \int_0^1 e^{-t^2}\, dt + \int_1^\infty e^{-t^2}\, dt$$

The first integral has finite bounds so it is convergent. For $t \geq 1$, $-t^2 \leq -t$. Since $\exp(t)$ is an increasing function,

$$e^{-t^2} \leq e^{-t} \implies \int_1^\infty e^{-t}\, dt \leq \int_1^\infty e^{-t}\, dt = 1$$

Therefore $I$ converges by the basic comparison test.
Define $\varphi : [0,\infty) \to \mathbb{R}$ by

$$\varphi(x) = \int_0^\infty \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt$$

Notice that $\int_0^\infty \frac{e^{-x^2(t^2+1)}}{t^2+1}\, dt \leq \int_0^\infty e^{-x^2(t^2+1)}\, dt$, so $\varphi$ always converges for similar reasons.
From part (b), we see that

$$\varphi'(x) = -2x \int_0^\infty e^{-x^2(t^2+1)} dt = -2xe^{-x^2} \int_0^\infty e^{-x^2t^2} dt$$

We perform the substitution $u = xt$. Changing $u$ back to $t$ gives us

$$\varphi'(x) = -2e^{-x^2} \int_0^\infty e^{-t^2} dt$$

Thus we have that

$$\varphi'(x) = -2e^{-x^2} I$$

We can take the definite integral of both sides with respect to $x$:

$$\int_0^\infty \varphi'(x) dx = \int_0^\infty -2e^{-x^2} I dx \implies \lim_{n\to\infty} \varphi(n) - \varphi(0) = -2I^2$$

Now we will analyse $\lim_{n\to\infty} \varphi(n)$ and $\varphi(0)$ separately.

First, we will show that $\lim_{n\to\infty} \varphi(n) = 0$. Let $\varepsilon > 0$. For $n > N$, where $N$ is a fixed number, we have that $e^{-n^2(t^2+1)} < \dfrac{2\varepsilon}{\pi}$, so

$$|\varphi(n)| = \left| \int_0^\infty \frac{e^{-n^2(t^2+1)}}{t^2+1}\, dt \right| \leq \int_0^\infty \left| \frac{e^{-n^2(t^2+1)}}{t^2+1}\, dt \right| < \frac{2}{\pi} \int_0^\infty \frac{\varepsilon}{t^2+1}\, dt$$

$$\implies \frac{2\varepsilon}{\pi} \arctan(t) \Big|_0^\infty = \varepsilon.$$

Thus we can conclude that $\lim_{n\to\infty} \varphi(n) = 0$.
Now, notice that

$$\varphi(0) = \int_0^\infty \frac{1}{t^2+1}\, dt = \arctan(t) \Big|_0^\infty = \frac{\pi}{2}$$

Applying this to our original equation,

$$0 - \frac{\pi}{2} = -2I^2 \implies 2I^2 = \frac{\pi}{2} \implies I^2 = \frac{\pi}{4}$$

Finally, taking the squareroot of both sides gives us the result:

$$\int_0^\infty e^{-t^2}\, dt = I = \frac{\sqrt{\pi}}{2}$$

$\square$

Question 28.

Let $U$ be an open set in a normed vector space $X$ and let $f : U \to Y$ be a **twice continuously differentiable** function, meaning that the second derivative $f'' : U \to B(X, B(X, Y))$ exists and is continuous on $U$. We also say that $f$ is a $C^2$**-function**.

(a) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = x^2 - xy + y^2$. Find, with proof, an explicit formula for the linear mapping $f''(2, -1)$. Also, write down the matrix that represents this linear mapping with respect to a suitable "standard" basis.

(b) Now we investigate the case $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, and let $f : U \to \mathbf{R}$ be some function defined on an open set $U \subseteq \mathbf{R}^n$. We use the notation $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ to refer to the $(i, j)$**th second partial derivative** of $f$: this is the $i$th partial derivative of the $j$th partial derivative $\dfrac{\partial f}{\partial x_j}$.

(i) Show that $f$ is twice continuously differentiable if and only if all second partial derivatives exist and are continuous.

(ii) Let $f$ be twice continuously differentiable Let $v \in \mathbf{R}^n$ and let $D_v f : U \to \mathbf{R}$ be the directional derivative of $f$ along $v$. Show that $D_v f$ is continuously differentiable.

(iii) Let $f$ be twice continuously differentiable and let $v \in \mathbf{R}^n$. By (ii), we know that $D_v f$ is $C^1$, hence differentiable in every direction $\in \mathbf{R}^n$. Show that the directional derivatives commute:

$$D_v(D_w f) = D_w(D_v f) \quad \text{for all } v, w \in \mathbf{R}^n.$$

(iv) Deduce **Clairaut's Theorem**: that the second partial derivatives commute.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

*Proof.* (a):
We claim that $f''(2, -1) : \mathbb{R}^2 \to B(\mathbb{R}^2, \mathbb{R})$ is a bounded linear map given by

$$f''(2, -1)(p, q)(x, y) = (2p - q)x - (p - 2q)y$$

Let $(p, q), (r, s) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then for all $(x, y) \in \mathbb{R}^2$,

$$f''(2, -1)(cp + r, cq + s)(x, y) = [2(cp + r) - (cq + s)]x - [(cp + r) - 2(cq + s)]y$$

$$= c[(2p - q)x - (p - 2q)y] + [(2r - s)x - (r - 2s)y] = cf''(2, -1)(p, q)(x, y) + f''(2, -1)(r, s)$$

so $f''(2, -1)$ is linear.
Recall that $f'(p, q)(x, y) = (2p - q)x - (p - 2q)y$. In particular, for $(p, q) = (2, -1)$, $f'(2, -1)(x, y) = 5x - 4y$.
Fix $(x, y) \in \mathbb{R}^2$. Then

$$\lim_{h \to 0} \frac{f'(2 + h_1, -1 + h_2)(x, y) - f'(2, -1)(x, y) - L_p(h)}{\|h\|}$$

$$= \lim_{h \to 0} \frac{[(5 + 2h_1 - h_2)x - (4 + h_1 - 2h_2)y] - [5x - 4y] - L_p(h)}{\|h\|}$$

$$= \lim_{h \to 0} \frac{(2h_1 - h_2)x - (h_1 - 2h_2)y - L_p(h)}{\|h\|} = 0$$

which verifies that $f''(2, -1)(p, q)(x, y) = (2p - q)x - (p - 2q)y$.
(b):
Let $f : U \to \mathbb{R}$ be a function, where $U$ is open in $\mathbb{R}^n$.
(b)(i):
Suppose that $f$ is twice continuously differentiable. This implies that all its partial derivatives exist.
Consider an arbitrary partial derivative $\dfrac{\partial f}{\partial x_j}$ for $i \in \{1, ..., n\}$. It will be shown that their directional derivatives all exist and are continuous.
Let $v \in \mathbb{R}^n$. We see that

$$\lim_{t \to 0} \frac{1}{t} \left( \frac{\partial f}{\partial x_j}(p + tv) - \frac{\partial f}{\partial x_j}(p) \right)$$

Recall that $f'(q)(v) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(q)$. Now, notice that

$$\frac{\partial f}{\partial x_j}(p + tv) = f'(p + tv)(x_j) \text{ and } \frac{\partial f}{\partial x_j}(p) = f'(p)(x_j)$$

We make this substitution in the limit expression, so it becomes

$$\lim_{t \to 0} \frac{1}{t}(f'(p + tv)(x_j) - f'(p)(x_j)) = D_v f'(p)(x_j)$$

by definition. Thus $D_v \left( \frac{\partial f}{\partial x_j}(p) \right) = D_v f'(p)(x_j)$. This is continuous because every directional derivative of $f'$ is continuous by our assumption. Taking $v = x_i$, we conclude that all the second partial derivatives of $f$ exist and are continuous.
Conversely, suppose that all second partial derivatives of $f$ exist and are continuous. Since all first partial derivatives will exist and are continuous, then $f$ is continuously differentiable, meaning that $f'$ exists.
Let $i \in \{1, ..., n\}$. We claim that $\dfrac{\partial f'}{\partial x_i}$ exists and is continuous, where $x_i$ represents the $i$th standard ordered basis vector of $\mathbb{R}^n$.
Fix $p \in \mathbb{R}^n$. Recall that $f'(p)(\vec{v}) = \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}(p)$ from the differentiability theorem. For $\vec{v} \in \mathbb{R}^n$, we have that

$$\lim_{h \to 0} \frac{f'(p + hx_i)(\vec{v}) - f'(p)(\vec{v})}{h} = \lim_{h \to 0} \sum_{j=1}^n \frac{v_i}{h} \left( \frac{\partial f}{\partial x_j}(p + hx_i) - \frac{\partial f}{\partial x_j}(p) \right)$$

Since all second partial derivatives exist, the limit evaluates to

$$\sum_{j=1}^n v_i \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

By definition, this is the $i$th partial derivative of $f'$.
Notice that it is also continuous, as it is a finite linear combination of second partial derivatives, which are continuous by assumption.
Since we have that all partial derivatives of $f'$ exist and are continuous, by the differentiability theorem, $f'$ is continuously differentiable, which implies that $f$ is twice continuously differentiable.
Therefore we can conclude that twice continuously differentiable is equivalent to having all second partial derivatives exist and continuous.

(b)(ii):
Suppose that $f$ is twice continuously differentiabe. Consider the directional derivative $D_v f$. We will show that for all $p \in U$, $D_v f(p) = f'(p)(v)$.
We can assume that $p+v \in U$ by making $\|v\|$ small enough. Define a function $\alpha : [0,1] \to U$ by $\alpha(t) = p + tv$. The derivative is given by $\alpha'(t) = v$. Now, construct a new function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = (f \circ \alpha)(t)$. By the chain rule, we see that $g'(t) = f'(\alpha(t))(v)$. Thus we get that

$$D_v f(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0) = f'(\alpha(0))(v) = f'(p)(v),$$

Now, we can compute the derivative of $D_v f$ directly and obtain that $(D_v f)'(p) = (f')'(p)(v)$. But we know by that $f$ is twice continuously differentiable, so $(D_v f)'$ is continuous.
Thus $D_v f$ is continuously differentiable.

(b)(iii):
We will evaluate $D_v(D_w f)$ from the definition and show that it is equal to $D_w(D_v f)$.
Fix $p \in U$. There exists $\delta > 0$ so that $p + \delta v \in U$. For any such $h$, construct a function $g : [0, \delta] \to \mathbb{R}$ defined as

$$g(t) = D_w f(p + tv)$$

$g$ is differentiable on its domain because it is a composition of differentiable functions. By the chain rule, we see that

$$g'(t) = (D_w f)'(p + tv)(v)$$

Using the fact that for all continuously differentiable functions, $D_v h(q) = h'(q)(v)$, we get

$$g'(t) = D_v(D_w f)(p + tv)$$

Now we compute $D_v(D_w f)$:

$$\lim_{h \to 0} \frac{D_w f(p + hv) - D_w f(p)}{h}$$

For $|h| < \delta$, the limit becomes

$$D_w(D_v f)(p) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0) = D_v(D_w f)(p)$$

Therefore the directional derivatives commute.

(b)(iv):
From the previous part, simply take $v = x_i$ and $w = x_j$ and the result follows immediately.

$\square$

Question 29.

Let $U \subseteq \mathbf{R}^2$ be an open set, and let $f : U \to \mathbf{R}$ be some differentiable function. This data defines an **explicit surface** in $\mathbf{R}^3$, also known as the **graph** of $f$:

$$S = S(f) = \{(x, y, z) \in \mathbf{R}^3 : (x, y) \in U, z = f(x, y)\}.$$

Colloquially, we often say "Let $z = f(x, y)$ be a surface in $\mathbf{R}^3$."

(a) Find the equation of the tangent plane to the surface $z = x + xy^2 - y^3$ at the point $p = (2, 1, 3)$, as follows.

(i) First, fix $x = 2$ and set $y = 1 + t$; write $z$ as a function of $t$, and find $z'(0)$. This is the "slope in the $x$ direction."

(ii) Similarly, find the "slope in the $y$ direction."

(iii) Now you have two slopes in orthogonal directions. This gives you two vectors which span a plane. Shift this plane so that it becomes tangent at $p$. Write the equation of this plane in the form $Ax + By + Cz = D$, where $A, B, C, D \in \mathbf{R}$.

(b) Find — with proof! — the equations of all planes in $\mathbf{R}^3$ which (i) are tangent to the surface $z = x + xy^2 - y^3$; (ii) are parallel to the vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$; and (iii) pass through the point $p = (-1, -2, 3)$.

(c) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuously differentiable function, and define a new function $g : \mathbf{R}^2 \to \mathbf{R}$ by

$$g(x, y) = f(f(xy, x), f(y, xy)).$$

You are given the following information about $f$ and $g$:

- $f(3, 1) = 5$ and $\nabla f(3, 1) = (1, 2)$
- $f(3, 3) = 2$ and $\nabla f(3, 3) = (0, -1)$
- $g(1, 3) = 6$ and $\nabla g(1, 3) = (3, 4)$

Find $\nabla f(5, 2)$.

*Proof.* (a)(i):
Fix $x = 2$ and set $y = 1 + t$. Then

$$z_y(t) = 2 + 2(1 + t)^2 - (1 + t)^3 \implies z'(t) = 4(1 + t) - 3(1 + t)^2$$
$$\implies z_y'(0) = 1$$

(a)(ii):
Fix $y = 1$ and set $x = 2 + t$. Then

$$z_x(t) = (2 + t) + (2 + t) - 1 = 2t + 3 \implies z_x'(0) = 2$$

(a)(iii):

From the previous two parts gave us two vectors $(1, 0, 2)$ and $(0, 1, 1)$. This spans the plane given by the equation $2x + y - z = 0$. Translate this plane by the vector $(2, 1, 3)$ which yields the equation

$$2(x-2) + (y-1) - (z-3) = 0 \implies 2x + y - z = 2$$

(b):
We claim that there are only two planes that satisfy all three conditions:

$$5x - 14y - z = 20 \text{ and } 5x - 11y - 4z = 5$$

Let $(p, q) \in \mathbb{R}^n$ and suppose that the plane tangent to the surface $z$ at this point has the properties above. We find the equation of this tangent plane like the previous part.
Fix $x = p$ and set $y = q + t$. Then

$$z_y(t) = p + p(q+t)^2 - (q+t)^3 \implies z_y'(t) = 2p(q+t) - 3(q+t)^2$$

$$\implies z_y'(0) = 2pq - 3q^2$$

Next, fix $y = q$ and set $x = p + t$. Then

$$z_x(t) = (x+t) + (x+t)q^2 - q^3 \implies z_x'(t) = q^2 + 1$$

This gives two vectors $(0, 1, 2pq - 3q^2)$ and $(1, 0, q^2 + 1)$ that span a plane given by the equation $(q^2+1)x + (2pq - 3q^2)y - z = 0$. Translating this plane by $(p, q, p + pq^2 - q^3)$ gives another plane with the equation

$$(q^2+1)(x-p) + (2pq - 3q^2)(y-q) - (z + q^3 - pq^2 - p) = 0$$

$$\implies (q^2+1)x + (2pq - 3q^2)y - z = p(q^2+1) + q(2pq - 3q^2) + (q^3 - pq^2 - p)$$

$$\implies (q^2+1)x + (2pq - 3q^2)y - z = 2pq^2 - 2q^3$$

Now, take the initial vectors $(0, 1, 2pq - 3q^2)$ and $(1, 0, q^2 + 1)$ and compute their cross product to find the normal vector $\vec{n}$:

$$\vec{n} = (0, 1, 2pq - 3q^2) \times (1, 0, q^2 + 1) = (q^2 + 1, 2pq - 3q^2, -1)$$

This vector is normal to the plane. We only want planes that are parallel to $\vec{v}$, which means that this normal vector should be normal to $\vec{v}$ as well. We calculate the dot product of this normal vector and $\vec{v}$:

$$\vec{n} \cdot \vec{v} = 3(q^2+1) + (2pq - 3q^2) - 1$$

Set this equal to 0. Simplifying, we have that

$$3(q^2+1) + (2pq - 3q^2) - 1 = 0 \implies 2pq = -2$$

$$\implies pq = -1$$

We see that $p, q \neq 0$ and get the relationship $p = -\dfrac{1}{q}$. Substitute this into our plane equation and we get that

$$(q^2+1)x + (2pq - 3q^2)y - z = 2pq^2 - 2q^3$$

$$\implies (q^2+1)x + (-2-3q^2)y - z = -2q - 2q^3$$
$$\implies (q^2+1)x - (3q^2+2)y - z = -2q(q^2+1)$$

Finally, recall that our plane should pass through the point $p = (-1, -2, 3)$. Substituting $(x, y, z) = (-1, -2, 3)$, we can solve for $q$ and obtain our answer:

$$-(q^2+1) + 2(3q^2+2) - 3 = -2q(q^2+1)$$
$$\implies 2q^3 + 5q^2 + 2q = 0$$

We disregard the case where $q = 0$ and end up with the quadratic equation

$$2q^2 + 5q^2 + 2 = 0 \implies q = \frac{-5 \pm \sqrt{25-16}}{4} \implies q_1 = -2, q_2 = -\frac{1}{2}$$

Substituting these values into our plane equation, we see that there are two planes that satisfy the above properties:

$$5x - 14y - z = 20 \text{ and } 5x - 11y - 4z = 5$$

(c):
In general,

$$\nabla g(p,q) = \left(\frac{\partial g}{\partial x}(p,q), \frac{\partial g}{\partial y}(p,q)\right)$$

For $a(x,y) = xy$, $b(x,y) = x$, and $c(x,y) = y$, we make the quick note that

$$\begin{bmatrix} a'(x,y) \end{bmatrix} = \begin{bmatrix} y & x \end{bmatrix}, \ \begin{bmatrix} b'(x,y) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} c'(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Differentiating $g$ at $(1,3)$, by multiple applications of the chain rule, we get

$$\begin{bmatrix} g'(1,3) \end{bmatrix} = \begin{bmatrix} f'(f(3,1), f(3,3)) \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} f'(3,1) \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} a'(1,3) \end{bmatrix} \\ \begin{bmatrix} b'(1,3) \end{bmatrix} \end{bmatrix} \\ \\ \begin{bmatrix} f'(3,3) \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} c'(1,3) \end{bmatrix} \\ \begin{bmatrix} a'(1,3) \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} f'(5,2) \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} f'(5,2) \end{bmatrix} \times \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$$

Suppose that $\begin{bmatrix} f'(5,2) \end{bmatrix} = \begin{bmatrix} r & s \end{bmatrix}$, for some $r, s \in \mathbb{R}$. Then

$$\begin{bmatrix} g'(1,3) \end{bmatrix} = \begin{bmatrix} f'(5,2) \end{bmatrix} \times \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} r & s \end{bmatrix} \times \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 5r - 3s & r - s \end{bmatrix}$$

$$\implies 5r - 3s = 3 \text{ and } r - s = 4$$

Solving the system of equations gives

$$r = -\frac{9}{2} \text{ and } s = -\frac{17}{2}$$

Therefore $\nabla f(5, 2) = \left(-\frac{9}{2}, -\frac{17}{2}\right)$.

$\square$