Question 1.

Let V be a vector space over the field $\mathbb{F}$, and S a (non-empty) set. Let $\mathcal{F}(S, V) = \{f : S \to V\}$ be the set of V-valued functions.
We define addition and scaling on $\mathcal{F}(S, V)$ pointwise:

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

We will verify some of the vector space axioms required to prove that $\mathcal{F}(S, V)$ is a vector space over $\mathbb{F}$.

(a) Why do these operations make sense?

(b) Prove (using only the definitions above, and the fact that V is a vector space) that $c(f + g) = cf + cg$ for all $f, g \in \mathcal{F}(S, V)$ and $c \in \mathbb{F}$.

(c) Prove that for all $f \in \mathcal{F}(S, V)$ there exists $g \in \mathcal{F}(S, V)$, so that $f + g = 0$. (Here $0 : S \to V$ is the constant function defined by $0(s) = 0_V$ for $s \in S$.)

Proof.
(a):
To show that these operations are well defined, notice that $f(s), g(s) \in V$, so by the axiom of closure on V we have that

$$(f + g)(s) = f(s) + g(s) \in V \text{ and } (cf)(s) = cf(s) \in V$$

(b):
We will do this by showing that for all $s \in S$, we have $c(f(s) + g(s)) = cf(s) + cg(s)$.
Fix $s \in S$. It follows that $f(s), g(s) \in V$, so by the axiom of distributivity in V, we have that

$$c(f(s) + g(s)) = cf(s) + cg(s)$$

(c):
Let $f \in \mathcal{F}(S, V)$. Choose $g = (-1 \cdot f)$. Then for all $s \in S$,

$$f(s) + g(s) = f(s) + (-f(s)) = 0$$

as needed.

$\square$

Question 2.

Let $W = \left\{ (x,y,z,w) \in \mathbb{Q}^4 \;\middle|\; \begin{array}{l} x + 5w = y + 5z \\ y = 4w - 3z \\ x + y + z = 3w \end{array} \right\}$.

Do not use Q3 to solve this problem. This problem is a "warm up" for Q3.

(a) Rearrange the equations defining W to show that W is the set of solutions to a homogeneous system of equations.

(b) Solve the system using row-reduction and express the general solution as a linear combination of the "basic solutions".

(c) Show that $W = \operatorname{span} S$, for some set $S \subseteq \mathbb{Q}^4$.

(d) Deduce that W is a subspace of $\mathbb{Q}^4$.

Proof.

(a):

Rearranging, the equations become

$$\begin{cases} x - y - 5z + 5w = 0 \\ \quad\;\; y + 3z - 4w = 0 \\ x + y + \;\; z - 3w = 0 \end{cases}$$

(b):

The augmented matrix associated with this system of equations is

$$\left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right)$$

Row reducing this, we get

$$\left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right) \xrightarrow{r_3 \to r_3 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 2 & 6 & -8 & 0 \end{array}\right) \xrightarrow[r_3 \to r_3 - 2r_2]{r_1 \to r_1 + r_2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We parameterize z and w to obtain that

$$\begin{aligned} x &= 2s - t \\ y &= -3s + 4t \\ z &= s \\ w &= t \end{aligned}$$

so the general solution of this system of equations is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

(c):

Let $S = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$. Let $\vec{v} \in \operatorname{span} S$. It follows that

$$\vec{v} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

for some $s, t \in \mathbb{Q}$. But notice that this is actually a solution to the system in W, so $\vec{v} \in W$. Now let $\vec{w} \in W$, so $\vec{w}$ solves the system of equations in W, but this means that we can write $\vec{w}$ as a linear combination of the vectors in S, so $\vec{w} \in \operatorname{span} S$, so $W = \operatorname{span} S$.

(d):

By the previous part, W is actually a spanning set, and we know that all spanning sets are subspaces, so we conclude that W is a subspace of $\mathbb{Q}^4$.

$\square$

Question 3.

We now generalize Q2. Consider a linear system with m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

We saw in Week 3 that any solution $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ can be expressed as $x = \sum_{i=1}^{k} t_i x_i$, where $t_i \in \mathbb{F}$ are the parameters, and $x_i \in \mathbb{F}^n$ are the "basic solutions".
Let W be the set of solutions to this system.

(a) Prove that $W = \operatorname{span} S$ for some set S, and hence that W is a subspace of $\mathbb{F}^n$.

(b) Prove that the set $\{x_1, x_2, \ldots, x_k\}$ is linearly independent.

(Hint: Think about the variables which correspond to the choice of parameters. There is exactly one vector for each such parameter. Use the corresponding entry to show that if $t_1x_1 + t_2x_2 + \cdots + t_kx_k = 0$ then $t_i = 0$ for each i.)

(c) Find a basis for W.

Proof.
(a):
Let $S = \{x_1, ..., x_k\}$. We note that any linear conbination of the vectors in S form a solution to the system, but not only that, any solution to the system can be written as a linear combination of these vectors, so $\text{span}S = W$.

(b):
Suppose for contradiction that $\sum_{i=1}^{k} t_i x_i = 0$ and not all coefficients $t_i = 0$, let's say that $t_j \neq 0$. Then t_j^{-1} exists and we can rearrange to get that

$$x_j = -t_j^{-1} \sum_{i=1, i\neq j}^{k} t_i x_i.$$

Thus for any general solution x to the homogenous equation,

$$x = \sum_{i=1}^{k} t_i x_i = \sum_{i=1, i\neq j}^{k} t_i(1 - t_j^{-1})x_i.$$

Notice that this is a linear combination of $k-1$ vectors, which implies that the general solution has $k-1$ parameters, which is a contradiction, as we assumed that it had k parameters.
Thus $\{x_1, ..., x_k\}$ is linearly independent.

(c):
From part (a) and part (b), the set $S = \{x_1, ..., x_k\}$ spans W and is linearly independent, which by definition forms a basis of W.

$\square$

Question 4.

Is the set $S = \{e_1 + 2e_2 - 3e_3, e_1 + e_2 - e_3, e_2 - e_3\} \subseteq \mathbb{Q}^3$ a basis for $\mathbb{Q}^3$? Justify your answer.

Proof. We claim that S is indeed a basis for $\mathbb{Q}^3$. For convenience, denote the vectors in S by v_1, v_2, v_3 respectively. Notice that

$$e_1 = v_2 - v_3,\ e_2 = -v_1 + v_2 + 2v_3,\ e_3 = -v_1 + v_2 + v_3.$$

Thus, for any $x \in \mathbb{Q}^3$, since $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{Q}^3$, for some $a, b, c \in \mathbb{Q}$, we have that

$$x = ae_1 + be_2 + ce_3 = a(v_2 - v_3) + b(-v_1 + v_2 + 2v_3) + c(-v_1 + v_2 + v_3)$$

$$\implies x = (-b - c)v_1 + (a + b + c)v_2 + (-a + c)v_3$$

which shows that S spans $\mathbb{Q}^3$.
Now, for constants $p, q, r \in \mathbb{Q}$, suppose that

$$0 = pv_1 + qv_2 + rv_3$$

Substituting back our values, we get that

$$\begin{aligned} 0 &= p(e_1 + 2e_2 - 3e_3) + q(e_1 + e_2 - e_3) + r(e_2 - e_3) \\ &= (p + q)e_1 + (2p + q + r)e_2 + (-3p - q - r)e_3 \end{aligned}$$

By the linear independence of the standard vectors, we have that

$$\begin{aligned} p + q \qquad &= 0 \\ 2p + q + r &= 0 \\ -3p - q - r &= 0 \end{aligned}$$

We can solve for p, q, r to get that $p = q = r = 0$.
Thus we can conclude that S is a basis for $\mathbb{Q}^3$.

$\square$

Question 5.

Let V be a finite dimensional vector space over a field $\mathbb{F}$.

(a) Prove that if $W \subseteq V$ is a subspace with basis β_W, then there exists a linearly independent set α so that $\beta = \beta_W \cup \alpha$ is a basis for V. (We say that β "extends" β_W. So you are proving that "every basis of a subspace W can be extended to a basis of V".)

(b) Prove that for any linearly independent set I and spanning set S, we have $|I| \leq \dim V \leq |S|$.

Proof.
(a):
Let γ be a basis for V. Since β_W is linearly independent, we apply the Replacement Theorem to get that there exists a subset $\alpha \subseteq \gamma$ such that $\beta_W \cup \alpha$ is a basis for V, and we are done.

(b):
Let I be a linearly independent set. Then $W = \operatorname{span}(I)$ is a subspace of V with basis I. By part (a), we can extend I to a basis β of V, where $\beta = I \cup \alpha$. Since $|\alpha| \geq 0$, we have that

$$\dim V = |\beta| = |I| + |\alpha| \geq |I|$$

Now, let S be a spanning set of V. If $\dim V = 0$, then $V = \{0\}$ and its basis is $\beta = \varnothing$. S must be either $\varnothing$ or $\{0\}$, so $\dim V \leq |S|$.
If $\dim V > 0$, it contains a non-zero vector, so S also contains a non-zero vector. Pick $s_0 \in S$, and note that $S_0 = \{s_0\}$ is linearly independent.
If there are no elements w_i in S such that $\{s_1, w_i\}$ is linearly independent, that is, $s = c_i w_i$ for some $c_i \in \mathbb{F}$, then for all $v \in V$, because S is a spanning set, for m vectors in S we have that

$$v = \sum_{i=1}^{m} a_i w_i = \sum_{i=1}^{m} a_i \cdot c_i s_0$$

which implies that $\operatorname{span}(\{s_0\}) = V$, and the result that we want follows immediately after.
If not, we can find another non-zero vector s_1 so that $S_1 = \{s_0, s_1\}$ is linearly independent. We repeat this process until we have a linearly independent set $S_n = \{s_0, ..., s_n\}$, where $n = \dim V$. We claim that S_n is a basis for V, and it suffices to show that S_n spans V. First, if $S_n = S$, then our result is immediate. Otherwise, let $s_j \in S$ so that $s_j \notin S_n$. Consider the set $S_n \cup \{s_j\}$, whose number of elements is greater than $\dim V$. By the first half of this proof, we know that no linearly independent set can have a size larger than $\dim V$, so it must be true that $S_n \cup \{s_j\}$ is linearly dependent. In particular, for constants $a_j, a_{ij} \in \mathbb{F}$ not all zero,

$$0 = a_j s + \sum_{i=1}^{n} a_{ij} s_i.$$

Notice that $a_j \neq 0$, for if not, then we get that

$$0 = \sum_{i=1}^{n} a_{ij} s_i$$

which implies that every $a_{ij} = 0$: a contradiction. Thus we can rearrange for s to obtain

$$s = a_j^{-1} \sum_{i=1}^{n} a_{ij} s_i$$

Now, let $v \in V$. Since S spans V, we have that for m vectors $s_j \in S$ and non-zero $c_j \in \mathbb{F}$,

$$v = \sum_{j=1}^{m} c_j s_j = \sum_{j=1}^{m} \left(c_j \cdot a_j^{-1} \sum_{i=1}^{n} a_{ij} s_i \right)$$

which shows that v is a linear combination of the vectors in S_n, so S_n is a linearly independent spanning set for V. Therefore we can conclude that S_n is a basis for V and we are done.

$\square$

Question 6.

Consider a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{F})$. Given $p \in \{1,\dots,n\}$ we can split M into "blocks":

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A is $k\times k$, B is $k\times(n-k)$, C is $(n-k)\times k$ and D is $(n-k)\times(n-k)$.
For example, if $n=5$ and $k=2$, then such a block matrix would be of the form

$$M = \left(\begin{array}{cc|ccc} 1 & 2 & 3 & 2 & 3 \\ -5 & 3 & 3 & 1 & 1 \\ \hline 1 & 2 & 0 & -1 & 1 \\ 3 & 1 & 3 & -1 & 7 \\ 1 & 0 & -1 & 3 & 5 \end{array}\right)$$

where $A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 7 \\ -1 & 3 & 5 \end{pmatrix}$.

Prove that if $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $N = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right)$, then

$$\alpha M + N = \left(\begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array}\right)$$

Proof. Let $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), N = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) \in \mathcal{M}_n(\mathbb{F})$, let $k \in \mathbb{N}$ Suppose that $A, A' \in \mathcal{M}_k(\mathbb{F})$, so $B, B' \in \mathcal{M}_{k\times(n-k)}$, $C, C' \in \mathcal{M}_{(n-k)\times k}(\mathbb{F})$, and $D, D' \in \mathcal{M}_{(n-k)\times(n-k)}(\mathbb{F})$. Fix $\alpha \in \mathbb{F}$ and define

$$P = \left(\begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array}\right)$$

Now, let $i, j \in \mathbb{N}$ so that $1 \le i, j \le n$. We will show that $(\alpha M + N)_{ij} = P_{ij}$ by taking cases on i and j.

Case 1: $1 \le i, j \le k$

In this case, $(\alpha M + N)_{ij} = \alpha M_{ij} + N_{ij} = \alpha A_{ij} + A'_{ij} = (\alpha A + A')_{ij} = P_{ij}$.

Case 2: $1 \le i \le k,\ k+1 \le j \le n$

We have that

$$(\alpha M + N)_{ij} = \alpha M_{ij} + N_{ij} = \alpha B_{ij} + B'_{ij} = (\alpha B + B')_{ij} = P_{ij}$$

Case 3: $k+1 \le i \le n,\ 1 \le j \le k$

Similarly,

$$(\alpha M + N)_{ij} = \alpha M_{ij} + N_{ij} = \alpha C_{ij} + C'_{ij} = (\alpha C + C')_{ij} = P_{ij}$$

Case 4: $k+1 \le i \le n,\ k+1 \le j \le n$

Finally, we have

$$(\alpha M + N)_{ij} = \alpha M_{ij} + N_{ij} = \alpha D_{ij} + D'_{ij} = (\alpha D + D')_{ij} = P_{ij}$$

Thus we have shown that $(\alpha M + N)_{ij} = P_{ij}$ for all $1 \le i, j \le n$ and we are done.

$\square$

Question 7.

Let $W = \left\{ A \in \mathcal{M}_{2n\times 2n}(\mathbb{F}) \mid A = \left(\begin{array}{c|c} \frac{X-X^T}{O_n} & \frac{O_n}{X+X^T} \end{array}\right) \text{ with } X \in \mathcal{M}_{n\times n}(\mathbb{F}) \right\}$.
(Assume char$(\mathbb{F}) \neq 2$.)

(a) Let $n = 2$. Find a basis for W.

(b) Now generalize to arbitrary n. Find a basis for W, and use it to compute $\dim W$.

Proof.
(a):
Let $n = 2$. We claim that a basis for W is given by

$$\beta = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

For convenience, we label these matrices in order as f_1, f_2, f_3, and f_4. It is pretty clear that β is linear independent. To show that β is spanning, let $A \in W$. Then

$$A = \left(\begin{array}{c|c} X - X^T & 0_n \\ \hline 0_n & X + X^T \end{array}\right), \text{ for some } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{n\times n}(\mathbb{F}).$$

We substitute a, b, c, d to get

$$A = \begin{pmatrix} 0 & b-c & 0 & 0 \\ c-b & 0 & 0 & 0 \\ 0 & 0 & 2a & b+c \\ 0 & 0 & b+c & 2d \end{pmatrix} = 2af_1 + bf_2 + cf_3 + 2df_4$$

so we can conclude that W is spanned by β.

(b):
Let $n \in \mathbb{N}$. We define e_{ij} to be the $n \times n$ matrix with all its entries equal to 0 except the entry at the ith row and jth column. We claim that

$$\beta = \left\{ \left(\begin{array}{c|c} e_{ij} - e_{ij}^T & 0_n \\ \hline 0_n & e_{ij} - e_{ij}^T \end{array}\right), 1 \leq i, j \leq n \right\}$$

is a basis for W.
Similarly to the previous part, define each matrix associated with e_{ij} by f_{ij}. Let $A \in W$, so

$$A = \left(\begin{array}{c|c} X - X^T & 0_n \\ \hline 0_n & X + X^T \end{array}\right), \text{ for an } n \times n \text{ matrix } X.$$

We can write X in terms of the basis for $\mathcal{M}_{n\times n}(\mathbb{F})$ to obtain that

$$X = \sum_{i,j=1}^{n} c_{ij} e_{ij}.$$

By the linearity of matrix transposition, it follows that

$$\begin{aligned}
A &= \left(\begin{array}{c|c} \sum_{i,j=1}^{n} c_{ij}e_{ij} - \left(\sum_{i,j=1}^{n} c_{ij}e_{ij}\right)^T & 0_n \\ \hline 0_n & \sum_{i,j=1}^{n} c_{ij}e_{ij} + \left(\sum_{i,j=1}^{n} c_{ij}e_{ij}\right)^T \end{array}\right) \\
&= \left(\begin{array}{c|c} \sum_{i,j=1}^{n} c_{ij}e_{ij} - \sum_{i,j=1}^{n} c_{ij}e_{ij}^T & 0_n \\ \hline 0_n & \sum_{i,j=1}^{n} c_{ij}e_{ij} + \sum_{i,j=1}^{n} c_{ij}e_{ij}^T \end{array}\right) \\
&= \sum_{i,j=1}^{n} c_{ij} \left(\begin{array}{c|c} e_{ij} - e_{ij}^T & 0_n \\ \hline 0_n & e_{ij} + e_{ij}^T \end{array}\right) \\
&= \sum_{i,j=1}^{n} c_{ij} f_{ij}
\end{aligned}$$

so W is spanned by β.
To show independence, for constants $c_{ij} \in \mathbb{F}$, let

$$\sum_{i,j=1}^{n} c_{ij} f_{ij} = 0$$

We get that

$$\sum_{i,j=1}^{n} c_{ij} \left(\begin{array}{c|c} e_{ij} - e_{ij}^T & 0_n \\ \hline 0_n & e_{ij} + e_{ij}^T \end{array}\right) = 0$$

In order for this to be true, we must have that

$$\sum_{i,j=1}^{n} c_{ij}(e_{ij} - e_{ij}^T) = 0$$

and

$$\sum_{i,j=1}^{n} c_{ij}(e_{ij} + e_{ij}^T) = 0$$

We add both equations together to see that

$$2 \sum_{i,j=1}^{n} c_{ij}e_{ij} = 0.$$

Since the e_{ij}'s form a basis on $\mathcal{M}_{n\times n}(\mathbb{F})$, it follows that $c_{ij} = 0$ for all i, j, and we are done.

$\square$

Question 8.

(a) Prove that if $W_1, W_2 \subseteq V$ are subspaces, then $W_1 + W_2$ is a subspace.

(b) Let $W_1 = \{(x, y, x+y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$. Find two subspaces W_2, W_3 so that:

- $W_1 + W_2 = \mathbb{F}^3$ but $\mathbb{F}^3 \neq W_1 \oplus W_2$.
- $W_1 \oplus W_3 = \mathbb{F}^3$.

(c) Find another subspace $U \subseteq \mathbb{F}^3$ so that $W_1 \oplus U = \mathbb{F}^3$.

Proof.
(a):
Let W_1, W_2 be subspaces of V. We verify that $W_1 + W_2$ is also a subspace of V.
First, note that $0 \in W_1, W_2$, so $0 + 0 = 0 \in W_1 + W_2$. Next, let $c \in \mathbb{F}$, $\vec{v}, \vec{w} \in W_1 + W_2$. Then $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{w} = \vec{w}_1 + \vec{w}_2$, for some $\vec{v}_1, \vec{w}_1 \in W_1$ and $\vec{v}_2, \vec{w}_2 \in W_2$. Since W_1, W_2 are subspaces, it is true that

$$c\vec{v}_1 + \vec{w}_1 \in W_1 \text{ and } c\vec{v}_2 + \vec{w}_2 \in W_2$$

which implies that

$$\vec{v} + \vec{w} = (c\vec{v}_1 + \vec{w}_1) + (c\vec{v}_2 + \vec{w}_2) \in W_1 + W_2,$$

verifying that $W_1 + W_2$ is indeed a subspace.

(b):
Let $W_2 = \mathbb{F}^3$, $W_3 = \text{span}\{e_3\}$. We start by showing that $\mathbb{F}^3 = W_1 + W_2$. The backward direction is instant so we will only show that $\mathbb{F}^3 \subseteq W_1 + W_2$.
Let $x \in \mathbb{F}^3$. We know that x is the same as $0 + x$, and $0 \in W_1$ and $x \in W_2$, so $x \in W_1 + W_2$. However, $W_1 \subseteq W_2$, so $W_1 \cap W_2 = W_1 \neq \{0\}$, so $\mathbb{F}^3$ is not a direct sum of W_1 and W_2.
Now, we will show that $W_1 \oplus W_3 = \mathbb{F}^3$. Again, the fact that $W_1 + W_3 \subseteq \mathbb{F}^3$ is obvious. To show that $\mathbb{F}^3 \subseteq W_1 + W_3$, let $(x, y, z) \in \mathbb{F}^3$. Notice that

$$(x, y, z) = (x, y, x+y) + (0, 0, z - x - y)$$

and

$$(x, y, x+y) \in W_1 \text{ and } (0, 0, z - x - y) \in W_3,$$

so $(x, y, z) \in W_1 + W_3$.
Now let $(a, b, c) \in W_1 \cap W_3$. Then we have that $a = b = 0$, but this implies that $c = a + b = 0$, so $(a, b, c) = 0$. Thus $W_1 \cap W_3 = \{0\}$ and we can conclude that $\mathbb{F}^3 = W_1 \oplus W_2$.

(c):
Let $U = \text{span}\{(0, 1, 2)\}$. It is clear that $U \neq W_3$, so U is in fact another subspace. We follow the same structure as before in part (b).
Let $(x, y, z) \in \mathbb{F}^3$. We rewrite

$$(x, y, z) = (x, x + 2y - z, 2x + 2y - z) + (0, -x - y + z, 2(-x - y + z))$$

and note that $(x, x + 2y - z, 2x + 2y - z) \in W_1$ and $(0, -x - y + z, 2(-x - y + z)) \in W_2$, so $(x, y, z) \in W_1 + W_2$.

Now, let $(a, b, c) \in W_1 \cap W_2$. Since $(a, b, c) \in W_2$, it must be true that $a = 0$ and $2b = c$. At the same time, since $(a, b, c) \in W_1$ we have that $a + b = c$, which implies that $b = c$, which can only be true if $b = c = 0$, so $(a, b, c) = 0$, thus showing that $W_1 \oplus W_2 = \mathbb{F}^3$.

$\square$

Question 9.

Let V be a finite dimensional vector space over $\mathbb{F}$, and $W_1, W_2 \subseteq V$ subspaces with mutually disjoint bases β_1, β_2 respectively. Prove that $V = W_1 \oplus W_2$ if and only if $\beta = \beta_1 \cup \beta_2$ is a basis for V.

Proof. Let $m = |\beta_1|$, $k = |\beta_2|$.
Suppose that $V = W_1 \oplus W_2$. We will show that $\beta = \beta_1 \cup \beta_2$ is a basis for V.
Let $x \in V$. By our assumption, $x = w_1 + w_2$, for some $w_1 \in W_1$ and $w_2 \in W_2$. These vectors can in turn be written as

$$w_1 = \sum_{i=1}^{m} a_i v_i \text{ and } w_2 = \sum_{i=1}^{k} b_i w_i$$

where $v_i \in \beta_1$ and $w_i \in \beta_2$. Thus x can be written as a linear combination of vectors in β:

$$x = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{k} b_i w_i$$

so β spans V.
To show that β is linearly independent, suppose that

$$\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{k} b_i w_i = 0$$

We put the vectors of each subspace on each side to get

$$\sum_{i=1}^{m} a_i v_i = -\sum_{i=1}^{k} b_i w_i$$

By the closure property of subspaces, $\sum_{i=1}^{m} a_i v_i \in W_1$ and $\sum_{i=1}^{k} b_i w_i \in W_2$, but since they are equal, it must be true that $\sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i \in W_1 \cap W_2 = \{0\}$, so $\sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i = 0$. Since β_1, β_2 are linearly independent, it must be true that $a_i = 0$ and $b_i = 0$, which was what we wanted to show. Therefore β is indeed a basis for V.
Conversely, suppose that β is a basis for V. We want to show that $V = W_1 \oplus W_2$. It is obvious that $W_1 + W_2 \subseteq V$, so it suffices to prove that $V \subseteq W_1 \oplus W_2$ and $W_1 \cap W_2 = \{0\}$.
Let $x \in V$. Then since β is a basis, we have that

$$x = \sum_{j=1}^{m} a_i v_i + \sum_{j=1}^{k} b_i w_i, \text{ for } a_i, b_i \in \mathbb{F}, v_i \in \beta_1, \text{ and } w_i \in \beta_2.$$

By closure, we have that $\sum_{j=1}^{m} a_i v_i \in W_1$ and $\sum_{j=1}^{k} b_i w_i \in W_2$, so we see that $x \in W_1 + W_2$. Thus $V = W_1 + W_2$.
To show that $W_1 \cap W_2 = \{0\}$, it suffices to show that if $x \in W_1 \cap W_2$, then it must be true that $x = 0$. Indeed, if $x \in W_1 \cap W_2$, we can write it as a two linear combinations of vectors in either β_1 or β_2:

$$x = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i$$

$$\implies \sum_{i=1}^{m} a_i v_i - \sum_{i=1}^{k} b_i w_i = 0$$

By the linear independence of β, we have that $a_i = b_i = 0$, for all i, which means that $x = 0$, as desired, and the proof is complete.

$\square$

Question 10.

Let $J = \left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right)$ and $\mathbb{F} = \mathbb{C}$.

(a) Verify that $J^2 = -I_4$.

(b) Find all $X \in \mathcal{M}_{4\times 4}(\mathbb{F})$ so that $XJ = JX$.

(c) Show that $\mathfrak{sp}_4 = \{X \in \mathcal{M}_{4\times 4}(\mathbb{F}) | XJ = JX\}$ is a subspace of $\mathcal{M}_{4\times 4}(\mathbb{F})$.

(d) Find dim $\mathfrak{sp}_4$ by finding a basis for $\mathfrak{sp}_4$.

Proof.
(a):
Indeed, we have that

$$J^2 = \left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right)\left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right) = \left(\begin{array}{c|c} O^2 + -I_2^2 & O(-I_2) + -I_2 O \\ \hline OI_2 + I_2 O & -I_2^2 + O^2 \end{array}\right) = \left(\begin{array}{c|c} -I_2 & O \\ \hline O & -I_2 \end{array}\right) = -I_4$$

(b):
For $A, B, C, C \in \mathcal{M}_2(\mathbb{F})$, let $X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathcal{M}_4(\mathbb{F})$ and suppose that $XJ = JX$. Then we have that

$$\begin{aligned} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right) &= \left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \\ \implies \left(\begin{array}{c|c} AO + BI_2 & A(-I_2) + BO \\ \hline CO + DI_2 & C(-I_2) \end{array}\right) &= \left(\begin{array}{c|c} OA + -I_2 C & OB - I_2 D \\ \hline I_2 A + OC & I_2 B + OD \end{array}\right) \\ \implies \left(\begin{array}{c|c} B & -A \\ \hline D & -C \end{array}\right) &= \left(\begin{array}{c|c} -C & -D \\ \hline A & B \end{array}\right) \\ \implies B = -C &\text{ and } A = -D \end{aligned}$$

Therefore all X that satisfy this equation are of the form

$$X = \left(\begin{array}{c|c} P & -Q \\ \hline -Q & P \end{array}\right), \text{ where } P, Q \in M_2(\mathbb{F}).$$

(c):
Notice that $X \in \mathfrak{sp}_4$ if and only if X can be decomposed into a block matrix such that

$$X = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right)$$

for some matrices $P, Q \in M_2(\mathbb{F})$.
We know that $0 \in \mathfrak{sp}_4$ because as a 4×4 matrix, $0 = \left(\begin{array}{c|c} O & O \\ \hline O & O \end{array}\right)$, which is clearly satisfies the condition outlined above.

Now, let $c \in \mathbb{F}$ and $X, Y \in \mathfrak{sp}_4$. Then for some 2×2 matrices P, Q, R, S,

$$X = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right) \text{ and } Y = \left(\begin{array}{c|c} R & S \\ \hline -S & -R \end{array}\right)$$

It follows that

$$cX + Y = c\left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right) + \left(\begin{array}{c|c} R & S \\ \hline -S & -R \end{array}\right) = \left(\begin{array}{c|c} cP + R & cQ + S \\ \hline -cQ - S & -cP - R \end{array}\right)$$

so $cX + Y \in \mathfrak{sp}_4$. Thus we can conclude that $\mathfrak{sp}_4$ is indeed a subspace.

(d):

For $1 \leq i, j \leq 2$, let e_{ij} be the (i, j)th standard basis vector for $M_2(\mathbb{F})$. We claim that a basis for $\mathfrak{sp}_4$ is given by

$$\beta = \left\{ X \in \mathcal{M}_4(\mathbb{F}) : X = \left(\begin{array}{c|c} e_{ij} & O \\ \hline O & -e_{ij} \end{array}\right) \text{ or } X = \left(\begin{array}{c|c} O & e_{ij} \\ \hline -e_{ij} & O \end{array}\right) \right\}.$$

Let $A \in \mathfrak{sp}_4$. Then for matrices $P, Q \in \mathcal{M}_2(\mathbb{R})$,

$$A = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right)$$

We can rewrite P, Q as linear combinations of basis vectors:

$$A = \left(\begin{array}{c|c} \sum_{i,j=1}^{2} p_{ij} e_{ij} & \sum_{i,j=1}^{2} q_{ij} e_{ij} \\ \hline -\sum_{i,j=1}^{2} p_{ij} e_{ij} & -\sum_{i,j=1}^{2} q_{ij} e_{ij} \end{array}\right) = \sum_{i,j=1}^{2} p_{ij} \left(\begin{array}{c|c} e_{ij} & O \\ \hline O & -e_{ij} \end{array}\right) + \sum_{i,j=1}^{2} q_{ij} \left(\begin{array}{c|c} O & e_{ij} \\ \hline -e_{ij} & O \end{array}\right)$$

Thus β spans $\mathfrak{sp}_4$.

Now, suppose that

$$= \sum_{i,j=1}^{2} p_{ij} \left(\begin{array}{c|c} e_{ij} & O \\ \hline O & -e_{ij} \end{array}\right) + \sum_{i,j=1}^{2} q_{ij} \left(\begin{array}{c|c} O & e_{ij} \\ \hline -e_{ij} & O \end{array}\right) = 0$$

Then it must be true that

$$\sum_{i,j=1}^{2} p_{ij} e_{ij} = \sum_{i,j=1}^{2} q_{ij} e_{ij} = 0$$

Since the standard basis vectors are linearly independent, it follows that all $p_{ij} = q_{ij} = 0$, thus proving that β is a basis for $\mathfrak{sp}_4$.

For every of the four standard basis vector e_{ij} of $\mathcal{M}_2(\mathbb{F})$, there are two vectors in β that correspond to it, so $\dim V = |\beta| = 2 \cdot 4 = 8$.

□

Question 11.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

(a) Let V be a finite dimensional vector space over $\mathbb{F}$. If $I \subseteq V$ is a linearly independent set so that for any $x \in V \setminus I$, the set $I \cup \{x\}$ is linearly dependent, then I is a basis for V.

(b) Let V be a finite dimensional vector space over $\mathbb{F}$. If $S \subseteq V$ is a spanning set so that $|S| = \dim V$, then S is a basis for V.

(c) Let V be a finite dimensional vector space over $\mathbb{F}$. If $W \subseteq V$ a subspace, then there exists a unique subspace $U \subseteq V$ so that $V = W \oplus U$.

Proof.
(a):
Let V be a finite dimensional vector space over $\mathbb{F}$. Suppose $I \subseteq V$ is linearly independent and that adding any vector in $V \setminus I$ will result in the set no longer being linearly independent, and note that the same also applies when choosing a vector that is in I. Then for any $x \in V$, we have that for some vectors $v_1, ..., v_n \in I$,

$$cx + \sum_{i=1}^{n} c_i v_i = 0$$

for $c, c_i \in \mathbb{F}$ not all zero. We make the important note that it is necessary for $c \neq 0$, because if not, then

$$\sum_{i=1}^{n} c_i v_i = 0$$

which implies that all coefficients are zero by independence, which contradicts our claim that not all coefficients were zero. It follows that c has an inverse c^{-1} and

$$x = \sum_{i=1}^{n} -c^{-1} c_i v_i.$$

Since every $x \in V$ is a linear combination of vectors in I, it follows that I is indeed a basis for V.

(b):
Suppose for contradiction that S is not a basis for V. Then S is not linearly independent, that is, for some $s \in S$, $c_i \in \mathbb{F}$, $s_i \in S \setminus \{s\}$,

$$s = \sum_{i=1}^{n} c_i s_i$$

This means that $S \backslash \{s\}$ is also a spanning set. But $|S \backslash \{s\}| < \dim V$, which is a contradiction, as no spanning set can have a size less than $\dim V$. Thus S is a basis for V.

(c):
This statement is false. Recall the result of Question 8: If we have the subspace $W_1 \subseteq \mathbb{F}^3$ defined by $W_1 = \{(x, y, x+y) \in \mathbb{F} | x, y \in \mathbb{F}\}$, then $W_3 = \text{span}\{e_3\}$ and $U = \text{span}\{(0, 1, 2)\}$ are two distinct subspaces such that $\mathbb{F}^3 = W_1 \oplus W_3 = W_1 \oplus U$.

$\square$