

Question 22.

For a normed vector space $(X, \|\cdot\|)$, let $X^* = B(X, \mathbf{R})$ denote the set of bounded linear mappings from X to \mathbf{R} . Here X^* is equipped with the operator norm $\|\cdot\|_{\text{op}}$, and the normed vector space $(X^*, \|\cdot\|_{\text{op}})$ is called the **topological dual** of X .

Let c_0 denote the set of sequences converging to zero. Prove that $c_0^* \equiv \ell^1$.

Proof. Equip ℓ^1 with the 1-norm and c_0 with the sup-norm. Define the function $f : \ell^1 \rightarrow c_0^*$ by $f(\vec{x})(\vec{a}) = \sum_{i=1}^{\infty} x_i \cdot a_i$. We claim that this is a bijective isometry.

Let $A = \{|f(x)(a) - f(y)(a)| : a \in c_0, \|a\|_{\infty} \leq 1\}$.

To show that this function is an isometry, let $x, y \in \ell^1$. It will be shown that $\|x - y\|_1 = \sup A$.

Let $a \in c_0$ such that $\|a\|_{\infty} \leq 1$. Then

$$|f(x)(a) - f(y)(a)| = \left| \sum_{i=1}^{\infty} (x_i \cdot a_i - y_i \cdot a_i) \right| = \left| \sum_{i=1}^{\infty} a_i (x_i - y_i) \right| \leq \sum_{i=1}^{\infty} |a_i| |x_i - y_i| \leq \sum_{i=1}^{\infty} |x_i - y_i|$$

$$\|x - y\|_1.$$

This means that $\|x - y\|_1$ is an upper bound for A .

□

Question 23.

Let S^2 denote the unit sphere in \mathbf{R}^3 . Let $N = (0, 0, 1)$ denote the “north pole”. In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to \mathbf{R}^2 . To do this, we define a function $\Phi : S^2 \setminus \{N\} \rightarrow \mathbf{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N , and let $\Phi(P)$ denote the point where this line intersects the xy -plane in \mathbf{R}^3 .

- (a) Given $P = (x, y, z)$, find an explicit formula for $\Phi(P)$ in terms of x, y, z .
- (b) Deduce that Φ is continuous.
- (c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.
- (d) Deduce that Φ is a homeomorphism.

Proof. (a):

Let $P = (x, y, z)$. First, we find the equation of the line that passes P and N . Consider the equation of the line $L(t) = (tx, ty, (z-1)t + 1)$. Notice that $L(0) = N$ and $L(1) = P$, so L satisfies what we were looking for. Now we find the point where L intersects with the xy -plane. This happens exactly when $(z-1)t + 1 = 0$. Solving for t gives $t = \frac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z} (x, y).$$

(b):

(c):

Let $p = (s, t) \in \mathbf{R}^2$. Our goal is to find $(x, y, z) \in S^2 \setminus \{N\}$ such that $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s, t)$. Immediately, we obtain the following system of equations:

$$\begin{aligned} \frac{x}{1-z} &= s, \\ \frac{y}{1-z} &= t, \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

We also have the restriction $z \neq 1$ because $(x, y, z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^2(1-z)^2 + t^2(1-z)^2 + z^2 = 1 \implies (s^2 + t^2 + 1)z^2 - 2(s^2 + t^2)z + s^2 + t^2 - 1 = 0$$

Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

- (i) X is finite-dimensional.
- (ii) The unit ball $\overline{B}(\vec{0}, 1)$ is compact.
- (iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that \overline{U} is compact.

Proof. It will be proven that (i) \implies (ii) \iff (iii) \implies (i).

(i) \implies (ii):

Suppose that X has finite dimension n . Then there is a continuous linear isomorphism Φ between X and \mathbb{R}^n . Since $\overline{B}(0, 1)$ is closed and bounded, $\Phi(\overline{B}(0, 1))$ is also closed and bounded in \mathbb{R}^n , so the set is compact. Since homeomorphisms preserve compactness, we can conclude that the closed unit ball in X is compact.

(ii) \implies (iii):

Suppose that the unit ball $\overline{B}(\vec{0}, 1)$ is compact. Let $p \in X$. We claim that $U = B(p, 1)$. Consider $\overline{U} = \overline{B}(p, 1)$. There is an isometry Φ from $\overline{B}(0, 1)$ to $\overline{B}(p, 1)$ defined by $\Phi(x) = p + x$. Since the closed unit ball is compact, it follows that $\overline{B}(p, 1)$ is compact. Thus X is locally compact.

(iii) \implies (ii):

Suppose that X is locally compact. Then $\vec{0}$ is contained in an open set U such that \overline{U} is compact. Since U is open, $B(0, \varepsilon) \subseteq U$ for some ε -ball centered around 0. It follows that $\overline{B}(0, \varepsilon)$ is compact, as it is a closed subset of \overline{U} . Since there is a homeomorphism from this closed ball to $\overline{B}(0, 1)$, the closed unit ball is also compact, as desired.

(iii) \implies (i):

Suppose that X is locally compact. We know previously that this implies that the closed unit ball is compact. Firstly, a quick lemma will be proven.

Lemma 1. $\forall x \in X, r > 0, B(x, r) = \{x\} + B(0, r)$.

Let $y \in B(x, r)$. Notice that $y - x \in B(0, r)$. Thus $y = x + (y - x)$ so $y \in \{x\} + B(0, r)$.

Then, let $y \in \{x\} + B(0, r)$, so $y = x + s$, for some $s \in B(0, r)$. Since $\|y - x\| = \|s\| < r$, it follows that $y \in B(x, r)$ and we are done.

Moving on, we construct a finite set of vectors in the following way:

Since $\overline{B}(0, 1)$ is totally bounded, we can find a finite set of vectors β such that $\bigcup_{x \in \beta} B(x, \frac{1}{2})$.

Lemma 2. $B(0, 1) \subseteq \text{span}(\beta) + 2^{-n}B(0, 1), \forall n \in \mathbb{N}$.

To prove this, let $n \in \mathbb{N}$ and $y \in B(0, 1)$. It follows that $y \in B(x_1, \frac{1}{2})$ for some $x_1 \in \beta$. Using Lemma 1,

$$y \in \{x_1\} + B\left(0, \frac{1}{2}\right) = \{x_1\} + \frac{1}{2}B(0, 1)$$

We can repeatedly use this argument to obtain that

$$y \in \left\{x_1 + \frac{1}{2}x_2 + \frac{1}{2^2}x_3 + \cdots + \frac{1}{2^{n-1}}x_n\right\} + \frac{1}{2^n}B(0, 1)$$

