Exercise 10.9

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Let $U \subseteq \mathbb{R}^n$ be open, and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping. Suppose that there is a point $p_0 \in U$ such that $\Phi'(p_0) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

Define the function $\Psi = \Phi^{-1}$. From the previous exercises, we know that $\Psi'(q) = \Phi'(\Psi(q))^{-1}$. It remains to show that Ψ' is continuous.

Exercise 10.9 Ψ' is continuous.

Proof. Consider the inversion map $T : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$. That is, for an invertible linear mapping A in $\mathbb{R}^n$, $T(A) = A^{-1}$. It will be shown that T is continuous.

Fix $A \in \mathrm{GL}(\mathbb{R}^n)$ and let $\varepsilon > 0$. Since $\mathrm{GL}(\mathbb{R}^n)$ is open, for some δ_0, $B(A, \delta_0) \subseteq \mathrm{GL}(\mathbb{R}^n)$. Let $\delta = \min\{\delta_0, \|A^{-1}\|_{\mathrm{op}}^{-2}\}$. Let $h \in \mathrm{GL}(\mathbb{R}^n)$ so that $\|h\|_{\mathrm{op}} < \delta$. Fix $x \in \mathbb{R}^n$ such that $\|x\| = 1$. Since $(A+h)^{-1}$ is surjective, we have $x = (A+h)(y)$, for some $y \in \mathbb{R}^n$. We see that

$$\|(A+h)^{-1}(x) - A^{-1}(x)\| = \|y - (y + A^{-1}(h(y)))\| = \|A^{-1} \circ h(y)\| \leq \|A^{-1}\|_{\mathrm{op}}\|h\|_{\mathrm{op}}\|y\|$$

Since A is a bounded linear operator,

$$\|y\| = \|A^{-1}(A(y))\| \leq \|A^{-1}\|_{\mathrm{op}}\|A(y)\| = \|A^{-1}\|_{\mathrm{op}}\|x - h(y)\| \leq \|A^{-1}\|_{\mathrm{op}}\left(\|x\| + \|h(y)\right)$$

$$\|T(A+h) - T(A)\|_{\mathrm{op}} = \|(A+h)^{-1} \circ (A+h) \circ ((A+h)^{-1} - A^{-1})\|_{\mathrm{op}}$$
$$= \|(A+h)^{-1} \circ ((A+h) \circ (A+h)^{-1} - (A+h) \circ A^{-1})\|_{\mathrm{op}} = \|(A+h)^{-1} \circ (-h \circ A^{-1})\|_{\mathrm{op}}$$
$$\leq \|(A+h)^{-1}\|_{\mathrm{op}}\|h\|_{\mathrm{op}}\|A^{-1}\|_{\mathrm{op}}$$

$\square$