Question 36.

In Handout #7, we defined differentiability for functions on open sets. Now we give a definition that works over arbitrary sets. For this problem, you will need to read Piazza Post @274 and use Theorem 1.1.

Let $A \subseteq \mathbf{R}^n$ be an arbitrary set, let $f : A \to \mathbf{R}$ be a function, and let $p \in A$ be a point. We say that f is **differentiable** at p if there exists an open neighborhood U of p and a function $\widehat{f} : U \to \mathbf{R}$ such that $\widehat{f}$ is differentiable at p (in the sense of Handout #7) and $\widehat{f}|_{U \cap A} = f|_{U \cap A}$.

(a) Prove that f is differentiable at every point of A if and only if f extends to a differentiable function defined on an open set containing A.

(b) Suppose further that A is closed. Prove that f is differentiable at every point of A if and only if f extends to a differentiable function on $\mathbf{R}^n$.

Proof.

(a):

Suppose that f extends to a differentiable function $\hat{f}$ on an open set $U \supseteq A$. That is, $\hat{f}|_A = f$. Let $x \in A$. Since U is open, we can find an open ball such that $B(x, \varepsilon) \subseteq U$. Immediately, we get that the function $\hat{f}|_{B(x,\varepsilon)}$ is the desired extension of f at x, as $\hat{f}$ is differentiable at x and $\hat{f}|_{B(x,\varepsilon)\cap A} = f_{B(x,\varepsilon)\cap A}$.

Conversely, suppose that f is differentiable at every point of A. For all $p_a \in A$, there exists an open neighborhood U_a of p_a and a function $f_a : U_a \to \mathbb{R}$ that is differentiable at p_a and $f_a|_{U_a \cap A} = f|U \cap A$. Notice that $U = \{U_a\}_{a\in I}$ forms an open cover of A. Thus we can find a partition of unity $\{\varphi_a : \mathbb{R}^n \to [0,1]\}_{a\in I}$. In particular,

- $\varphi_a \in C^\infty$,
- $\operatorname{supp}(\varphi_a) \cap U \subseteq U_a$,
- $\{\operatorname{supp}(\varphi_a)\}$ is locally finite,
- $\sum\limits_{a\in I} \varphi_a = 1$.

We claim that $\hat{f} = \sum_{a\in I} \varphi_a \cdot f_a$ is the function that extends f to $\{U_a\}_{a\in I}$. First, note that this sum is well defined because the partition of unity is locally finite, so the sum is finite at every point, and as well, any point outside the domain of f_a implies that $\varphi_a = 0$, so $\varphi_a \cdot f_a = 0$.

Moreover, for all $x \in U$, since $\{\varphi_a\}_{a\in I}$ is locally finite, for some open neighborhood V of x, for some finite sequence of $a_n \in I$,

$$\hat{f}(y) = \sum_{n=1}^{N} \varphi_{a_n}(y) \cdot f_{a_n}(y), \text{ where } y \in V.$$

Since each term is differentiable, $\hat{f}$ is differentiable at x. Thus $\hat{f}$ is differentiable on the entirety of U. Additionally, for each a_n, $x \in U_{a_n}$, so by our initial assumption,

$$\sum_{n=1}^{N} \varphi_{a_n}(x) \cdot f_{a_n}(x) = f(x) \cdot \sum_{n=1}^{N} \varphi_{a_n}(x) = f(x)$$

which shows that $\hat{f}|_A = f$ as needed.

(b):

$\square$

Question 37.

The following set is called the n-**simplex**:

$$\Delta_n := \{\vec{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \cdots + x_n \leq 1\}.$$

You can assume, without proof, that Δ_n is Jordan measurable.
Find, with proof, an explicit formula for $\mu(\Delta_n)$ in terms of n.

Proof. First, we show that Δ_n is the same as the set

$$S = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, ..., 0 \leq x_n \leq 1 - \sum_{i=1}^{n-1} x_i \right\}$$

Let $x \in \Delta_n$. We want to show that $0 \leq x_i \leq 1 - \sum_{j=1}^{i-1} x_j$. We get that $x_i \geq 0$ immediately. As well, since $\sum_{j=1}^{n} x_j \leq 1$ and every component is non-negative, we have that

$$x_i \leq 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^{n} x_j \leq 1 - \sum_{j=1}^{i-1} x_j$$

which shows that $\Delta_n \subseteq S$.
Now, let $x \in S$. We know every x_i is non-negative and additionally

$$x_n \leq 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^{n} x_i \leq 1$$

so $S \subseteq \Delta_n$.
Now, we proceed to find $\mu(S) = \mu(\Delta_n)$. Using Fubini's Theorem, we get

$$\mu(S) = \int_S 1 = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 \, dx_n \cdots dx_2 \, dx_1$$

Let $I : \mathbb{N} \times [0,1] \to \mathbb{R}$ be defined recursively as follows:

$$\begin{aligned} I(1, \alpha) &= \int_0^{1-\alpha} 1 \, dt \\ I(k, \alpha) &= \int_0^{1-\alpha} I(k-1, \alpha + t) \, dt, && \text{for } > 1. \end{aligned}$$

Notice that if we continue applying the definition, we get that

$$I(n, 0) = \mu(S)$$

Now, we will prove using induction on n that for all $\alpha \in [0,1]$, $I(n, \alpha) = \dfrac{1}{n!}(1-\alpha)^n$.
Let $n = 1$. Then

$$I(1, \alpha) = \int_0^{1-\alpha} 1 \, dt = 1 - \alpha$$

Now, suppose that $I(k, \alpha) = \frac{1}{k!}(1-\alpha)^k$ holds for all $\alpha \in [0, 1]$ and some $k \in \mathbb{N}$. We want to show that the same holds for $k + 1$ as well. For an arbitrary α, we get

$$\begin{aligned} I(k+1, \alpha) &= \int_0^{1-\alpha} I(k, \alpha + t)\ dt \\ &= \int_0^{1-\alpha} \frac{1}{k!}(1 - \alpha - t)^k\ dt \\ &= -\frac{1}{(k+1)!}(1 - \alpha - t)^{k+1}\Big|_0^{1-\alpha} \\ &= \frac{1}{(k+1)!}(1-\alpha)^{k+1} \end{aligned}$$

as desired. Thus we get that

$$\mu(S) = I(n, 0) = \frac{1}{n!}$$

$\square$