Question 23.

Let S^2 denote the unit sphere in $\mathbf{R}^3$. Let $N = (0, 0, 1)$ denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to $\mathbf{R}^2$. To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbf{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in $\mathbf{R}^3$.

(a) Given $P = (x, y, z)$, find an explicit formula for $\Phi(P)$ in terms of x, y, z.

(b) Deduce that Φ is continuous.

(c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.

(d) Deduce that Φ is a homeomorphism.

Proof. (a):
Let $P = (x, y, z)$. First, we find the equation of the line that passes P and N. Consider the equation of the line $L(t) = (tx, ty, (z-1)t + 1)$. Notice that $L(0) = N$ and $L(1) = P$, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when $(z-1)t + 1 = 0$. Solving for t gives $t = \frac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z}\,(x, y)\,.$$

(b):
(c):
Let $(s, t) \in \mathbb{R}^2$.

□

Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

(i) X is finite-dimensional.

(ii) The unit ball $\overline{B}(\vec{0}, 1)$ is compact.

(iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that $\overline{U}$ is compact.

Proof. idk $\square$