Question 43.

Transverse intersction. Let M, N be two smooth surfaces in $\mathbf{R}^3$. We say that M and N **intersect transversally** if $T_pM \neq T_pN$ for all $p \in M \cap N$.

(a) Prove that if M, N intersect transversally, then $M \cap N$ is a smooth curve in $\mathbf{R}^3$.

(b) Show by example that the conclusion of (a) fails without the assumption of transverse intersection.

Proof.
(a):
Suppose that M and N intersect transversally. We will show that $M \cap N$ is a smooth 1-manifold in $\mathbb{R}^3$. Let $p \in M \cap N$. Then there is some relatively open neighborhood U of M and V of N that is the zero set of some smooth functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, that is, $U = Z(f)$ and $V = Z(g)$. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\Phi = (f, g)$. We claim that there exists a chart containing p by showing that $U \cap V$ is the zero set of Φ, and $J\Phi(q)$ is surjective for all $q \in U \cap V$. We first start by verifying that $U \cap V$ is relatively open to $M \cap N$. This is quick, as we know that U and V are relatively open to M and N respectively, for each point in $U \cap V$, we can choose two open balls with radii r_1, r_2 that stays within M and N respectively. We then take the lesser of the radii as our radius.
Moving on, we see that Φ is smooth, and $Z(\Phi) = U \cap V$. It remains to show that $J\Phi(q)$ has rank 2 for all $q \in U \cap V$. The Jacobian of Φ is a 2×3 matrix given by

$$J\Phi(q) = \begin{pmatrix} \nabla f(q) \\ \nabla g(q) \end{pmatrix}.$$

It necessarily has rank at most 2 and at least 1 (because $\nabla f, \nabla g \neq 0$). Suppose for contradiction that $\operatorname{rank} J\Phi(q) = 1$. Then $\nabla f(q) = c\nabla g(q)$ for some non-zero constant c. We know that the tangent space T_qM is given by the set of tagged vectors orthogonal to $\nabla f(q)$. Likewise, T_qN consists of tagged vectors orthogonal to $\nabla g(q)$. But notice that for $v \in T_qM$, we have $\nabla f(q) \cdot v = 0$ but also $c\nabla f(q) \cdot v = \nabla g(q) \cdot v = 0$. If we additionally apply the same argument to $u \in T_qN$, we can see that $T_qM = T_qN$ which is a contradiction. Thus $J\Phi(q)$ must have rank 2. From here, it follows that $\Phi^{-1}(\{0\}) = U \cap V$ is a smooth manifold with dimension 1, so we can conclude that $M \cap N$ is a smooth curve.

(b):
Let M be the xy-plane, and let N be the graph of $f(x, y) = x^2 + y^2$. Then $M \cap N$ is simply the origin, which is not a smooth curve.

$\square$

Question 44.

Suppose that M is a smooth manifold, and let $\mathcal{A}$ be an open cover of M by pairwise consistently oriented charts. Let $\mathcal{A}^+$ be the collection of all charts on M which are positively oriented with $\mathcal{A}$; likewise, let $\mathcal{A}^-$ be the collection of all charts on M which are negatively oriented with $\mathcal{A}$.
Now suppose that $\mathcal{B}$ is some other open cover of M by charts, such that any two (overlapping) charts in $\mathcal{B}$ are consistently oriented. Prove that if M is connected, then either $\mathcal{B}$ is completely contained in $\mathcal{A}^+$, or else it is completely contained in $\mathcal{A}^-$.

Proof. First, we will prove a form of transitivity for manifold charts.
Lemma. From a manifold M, take two charts (U, φ), (V, ψ) that are consistently oriented with each other. Then (U, φ) is positively oriented with $\mathcal{A}$ if (V, ψ) is positively oriented with $\mathcal{A}$.
Suppose that (U, φ) is positively oriented with $\mathcal{A}$. Consider the chart $(U \cap V, \varphi|_{U\cap V})$ that is consistently oriented with some chart (W, γ) in $\mathcal{A}$. We will show that (V, ψ) is consistently oriented with (W, γ). Notice that

$$\psi \circ \gamma^{-1} = (\psi \circ \varphi^{-1}) \circ (\varphi \circ \gamma^{-1})$$

so

$$J(\psi \circ \gamma^{-1}) = J(\psi \circ \varphi^{-1}) \cdot J(\varphi \circ \gamma^{-1}) \text{ and } \det J(\psi \circ \gamma^{-1}) > 0$$

as needed.
Next, we prove the main result. Let $\mathcal{B}$ be an open cover of M by charts. Pick a chart (U, φ) in $\mathcal{B}$, and say for convenience that it is contained in $\mathcal{A}^+$. If the chart picked was in $\mathcal{A}^-$ the argument is analogous. We claim that all other charts are also contained in $\mathcal{A}^+$. Let $(V, \psi) \in \mathcal{B}$. Take any two points $p \in U$ and $q \in V$. Since M is connected, we can find a continuous function $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Consider $C = \gamma[0, 1]$. Since it is compact, we can find a finite number of charts $\{(U_n, \varphi_n)\}_{n \le N}$ that cover C, such that the open cover includes (U, φ) and (V, ψ). As well, we can always find a sequence of m charts $(U_{n_k})_{1\le k\le m}$ such that

$$U \cap U_{n_1} \neq \varnothing, U_{n_1} \cap U_{n_2} \neq \varnothing, \ldots, U_{n_m} \cap V \neq \varnothing.$$

Suppose if not. Then consider $\mathcal{C}$, the collection of charts in the finite subcover that can be reached from U using a chain of charts. Let $\overline{\mathcal{C}}$ be the complement; the colleciton of charts that cannot be reached from V. Then the charts of $\mathcal{C}$ and $\mathcal{D}$ are pairwise disjoint from one another, and $V \in \overline{\mathcal{C}}$. We have that $\mathcal{C} \cup \overline{\mathcal{C}}$ cover $\gamma[0, 1]$, but $\bigcup_{C_i \in \mathcal{C}} C_i$ is disjoint from $\bigcup_{D_i \in \overline{\mathcal{C}}} D_i$, which contradicts the fact that $\gamma[0, 1]$ is connected, so such a sequence of charts does indeed exist. Since (U, φ) and (U_{n_1}, φ_{n_1}) are consistently oriented with each other and (U, φ) is positively oriented with $\mathcal{A}$, by the lemma, (U_{n_1}, φ_{n_1}) is positively oriented with $\mathcal{A}$. We can apply this argument inductively to conclude that V is also positively oriented with $\mathcal{A}$ and thus $V \in \mathcal{A}^+$.
Therefore, if a single chart in $\mathcal{B}$ is contained in $\mathcal{A}^+$, then in fact every chart in $\mathcal{B}$ is contained in $\mathcal{A}^+$. The same applies for the case where there is a chart that is contained in $\mathcal{A}^-$.

□