Question 42.

Let $M \subseteq \mathbf{R}^N$ be a smooth $n$-manifold (with or without boundary!).

(a) Show that if $n < N$, then $M$ is a *Lebesgue null set.*

(b) Show that if $n = N$ and $M$ is closed and its boundary is nonempty, then $\partial M$ coincides with the usual topological boundary (as defined on Handout #2).

(c) Show that if $M$ is compact and its boundary is nonempty, then $M$ is Jordan measurable.

*Proof.*
(a):
We begin by proving a lemma:
**Lemma:** A countable union of sets with Jordan measure 0 is a Lebesgue null set.
Let $E = \bigcup_{i \geq 0} E_i$, where $\mu(E_i) = 0$.

(b):
Let $p$ be a point in the topological boundary of $M$. We will show that $p \in \partial M$. Suppose for contradiction that $p$ is in $M^\circ$, meaning it is contained in a chart $(U, \varphi)$ that is diffeomorphic to an open set $\hat{U}$ in $\mathbb{R}^N$. Note that $\varphi$ is a diffeomorphism with domain $\hat{U}$ and codomain $U$. Then it must be true that $\varphi(\hat{U}) = U$ is an open subset of $\mathbb{R}^N$. But this is a contradiction, as that would imply that the boundary point $p$ is in the interior of $M$. Therefore $p \in \partial M$.
Next, let $p \in \partial M$ and again suppose for contradiction that $p$ is not in the topological boundary of $M$. Then it must be true that $p$ is in the topological interior of $M$. Recall that $p \in \partial M$ implies that it is contained in a chart $(U, \varphi)$ that is diffeomorphic to $\overline{\mathbb{H}^n}$ and $p \in \mathrm{bd}(\mathbb{H}^n)$. Since $p$ is in the topological interior of $M$, we can find an open ball $B(p, r) \subseteq M$ which is also open in $\mathbb{R}^N$. Then $\varphi^{-1}(B(p, r))$ should also be open in $\mathbb{R}^N$. But this implies that for small enough $\delta$, $\varphi(p) - (0, ..., \delta) \in \overline{\mathbb{H}^N}$, which cannot happen.

(c):
First, we prove that a compact Lebesgue null set $E$ has Jordan measure 0. It suffices to show that the upper measure $\mu^*(E) = 0$.
Let $\varepsilon > 0$. By definition, we can find a countable union of boxes $B = \bigcup_{i=1}^{\infty} B_i$ such that $E \subseteq B$ and $\mathrm{vol}(B) < \varepsilon$. But since $E$ is compact, it can be covered by finitely many boxes $B_{n_i}$, $0 < i \leq N$. Thus

$$\mathrm{vol}\left(\bigcup_{i=1}^{N} B_{n_i}\right) = \sum_{i=1}^{N} \mathrm{vol}(B_{n_i}) \leq \mathrm{vol}(B) < \varepsilon.$$

Since $\varepsilon$ was chosen arbitrarily, we can conclude that $\mu^*(E) = 0$, and $E$ has Jordan measure 0.
Now, suppose that $M$ is compact and its boundary is nonempty. If $\dim M < N$, $M$ is a Lebesgue null set and has Jordan measure 0, and therefore measurable. Otherwise, if $\dim M = N$, since the boundary of $M$ is non-empty, the topological boundary of $M$ is actually a smooth manifold of dimension $(N-1)$, and therefore a Lebesgue null set. As well, the topological boundary of $M$ is compact, so it is Jordan measure 0, which implies that $M$ is Jordan measurable.