

Question 35.

*Perturbing the roots of a polynomial.*

Let  $f(x) = \sum_{i=0}^n a_i x^i$  be a **monic** polynomial with **no repeated real roots**. This means that  $a_n = 1$ , and that all real roots of  $f$  have multiplicity 1.

- (a) Let  $r$  be a root of  $f(x)$ . Prove that for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that: if  $g(x) = \sum_{i=0}^n b_i x^i$  is a monic polynomial with coefficients  $b_i$  satisfying  $|a_i - b_i| < \delta$ , then  $g(x)$  has at least one root in the interval  $(r - \varepsilon, r + \varepsilon)$ .

This shows that slight perturbations of the coefficients results in slight perturbations of the roots.

- (b) Suppose that  $f$  has fewer than  $n$  real roots. Prove that number of real roots of  $f$  does not change under small perturbation of the coefficients.

*Proof.*

(a):

First, we prove the following lemma:

**Lemma.** Let  $r \in \mathbb{R}$  be a root of a polynomial  $p$ . Then  $r$  is a repeated root if and only if  $p'(r) = 0$ .

Suppose that  $p$  has a repeated root. Then we can factor  $p$  as  $(x - r)^k q(x)$ , for some  $k > 1$  and  $q \in \mathbb{P}(\mathbb{R})$ . We can take the derivative of this and get that

$$\begin{aligned} p'(x) &= k(x - r)^{k-1} q(x) + (x - r)^k q'(x) \\ &\implies p'(r) = 0 \end{aligned}$$

Conversely, suppose that  $p'(r) = 0$ , for some  $r \in \mathbb{R}$ . We can write

$$p'(x) = (x - r) \sum_{i=0}^m c_i x^i, \text{ for constants } c_0, \dots, c_m$$

We can integrate both sides to get that

$$\begin{aligned} p(x) &= \int (x - r) \sum_{i=0}^m c_i x^i dx = \int x \sum_{i=0}^m c_i x^i - r \sum_{i=0}^m c_i x^i dx \\ &= \sum_{i=0}^m \frac{c_i}{i+2} x^{i+2} - r \sum_{i=0}^m \frac{c_i}{i+1} x^{i+1} + C \end{aligned}$$

In order for  $r$  to be a root of  $p$ , we must have that

$$\begin{aligned} p(r) &= \sum_{i=0}^m \left( \frac{c_i}{i+2} r^{i+2} - \frac{c_i}{i+1} r^{i+2} \right) + C = 0 \\ &\implies C = \sum_{i=0}^m \left( \frac{c_i}{i+1} r^{i+2} - \frac{c_i}{i+2} r^{i+2} \right) \end{aligned}$$



