

Exercise 10.9

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Writeup: Ethan

Let $U \subseteq \mathbb{R}^n$ be open, and let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 mapping. Suppose that there is a point $p_0 \in U$ such that $\Phi'(p_0) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism.

Define the function $\Psi = \Phi^{-1}$. From the previous exercises, we know that $\Psi'(q) = \Phi'(\Psi(q))^{-1}$. It remains to show that Ψ' is continuous.

Exercise 10.9 Ψ' is continuous.

Proof. Consider the inversion map $T : \text{GL}(\mathbb{R}^n) \rightarrow \text{GL}(\mathbb{R}^n)$. That is, for an invertible linear mapping A in \mathbb{R}^n , $T(A) = A^{-1}$. It will be shown that T is continuous.

Fix $A \in \text{GL}(\mathbb{R}^n)$ and let $\varepsilon > 0$. Since $\text{GL}(\mathbb{R}^n)$ is open, for some δ_0 , $B(A, \delta_0) \subseteq \text{GL}(\mathbb{R}^n)$. Let $\delta = \min\{\delta_0, \|A\|^{-2}\}$. Let $h \in \text{GL}(\mathbb{R}^n)$ so that $\|h\|_{\text{op}} < \delta$. Fix $x \in \mathbb{R}^n$ such that $\|x\| = 1$. Since $(A + h)^{-1}$ is surjective, we have $x = (A + h)^{-1}(y)$, for some $y \in \mathbb{R}^n$. We see that

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$$\begin{aligned} \|T(A + h) - T(A)\|_{\text{op}} &= \|(A + h)^{-1} \circ (A + h) \circ ((A + h)^{-1} - A^{-1})\|_{\text{op}} \\ &= \|(A + h)^{-1} \circ ((A + h) \circ (A + h)^{-1} - (A + h) \circ A^{-1})\|_{\text{op}} = \|(A + h)^{-1} \circ (-h \circ A^{-1})\|_{\text{op}} \\ &\leq \|(A + h)^{-1}\|_{\text{op}} \|h\|_{\text{op}} \|A^{-1}\|_{\text{op}} \end{aligned}$$

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