Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that $O_n$ is a smooth manifold, and find its dimension.

*Proof.* First, we will prove the Regular Level-Set Theorem:

Let $X, Y$ be normed vector spaces with dimensions $n, m$ and ordered bases $\alpha, \beta$ respectively, where $n > m$. Let $F : X \to Y$ be a smooth function. Define $M = F^{-1}(0_Y)$. If $F'(p)$ is surjective for all $p \in M$, then $M$ is a smooth manifold of dimension $n - m$. That is, if $\phi_\alpha : X \to \mathbb{R}^n$ is the coordinate isomorphism corresponding to $\alpha$, then $M$ is a smooth $k$-manifold if $\phi_\alpha(M)$ is a smooth $k$-manifold in the usual sense.

This problem reduces to trying to prove that $N := \phi_\alpha(M)$ is a smooth manifold of dimension $n - m$. Notice that $N = \phi_\alpha(F^{-1}(0_Y)) = \phi_\alpha(F^{-1}(\phi_\beta(0_{\mathbb{R}^m})))$. Since $\phi_\alpha, \phi_\beta$ are isomorphisms, we have that $N = \phi_\alpha \circ F^{-1} \circ \phi_\beta(0_{\mathbb{R}^m})$. Let $\hat{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a function defined by $\hat{F} = \phi_\alpha \circ F^{-1} \circ \phi_\beta$, and notice that $N$ is the zero set of $\hat{F}$. For any $p \in N$, notice that $\hat{F}'(p)$ is surjective, because

$$\hat{F}'(p) = (\phi_\beta \circ F \circ \phi_\alpha^{-1})'(p) = \phi_\beta \circ F'(\phi_\alpha^{-1}) \circ \phi_\alpha^{-1}$$

is just $F'(\phi_\alpha^{-1})$–a surjective map–composed with linear isomorphisms. This implies that $R(J\hat{F}(p)) = \mathbb{R}^m$ and $\operatorname{rank} J\hat{F}(p) = m$. We can write

$$J\hat{F}(p) = \begin{pmatrix} A \mid B \end{pmatrix}$$

where $A$ is a $m \times n - m$ matrix and $B$ is a $m \times m$ matrix, and assume without loss of generality that $B$ is invertible, for if not, $J\hat{F}(p)$ is still guaranteed to contain an invertible $m \times m$ submatrix, and we can perform column swaps to move the matrix to the right, which does not change the conclusion of our statement.

Recall that since $N$ is the zero set of $\hat{F}$, $\hat{F}(p) = 0$. Thus, we write $p = (a, b)$ for $a \in \mathbb{R}^{n-m}$, $b \in \mathbb{R}^m$ and apply the Implicit Function Theorem and obtain an open set $\hat{U} \subseteq \mathbb{R}^{n-m}$ containing $a$ and a $C^\infty$ function $\Phi : \hat{U} \to \mathbb{R}^m$ so that

$$\hat{F}(x, \Phi(x)) = 0$$

for all $x \in \hat{U}$. We claim that $\varphi : \hat{U} \to \Phi(\hat{U})$ defined by

$$\varphi(x) = (x, \Phi(x))$$

is our desired smooth regular embedding. It is fairly clear that $\varphi$ is smooth. Additionally, $J\varphi(x) = \left(\dfrac{I_{n-m}}{J\Phi}\right)$ is a $(n - m) \times n$ matrix and has at least $n - m$ linearly independent rows, so $\varphi$ is regular. Finally, if we let $\varphi(x) = \varphi(y)$, we have that $(x, \Phi(x)) = (y, \Phi(y))$, from which we get $x = y$, so $\varphi$ is injective, and therefore bijective to its image. In addition, we can explicitly find $\varphi^{-1}(y) = \pi_{\mathbb{R}^{n-m}}(y)$, which is continuous because it is a linear map. Therefore

$\varphi$ is a homeomorphism onto its image, so $N$ is a smooth $(n-m)$-manifold. It follows that $\phi_\alpha^{-1}(N) = M$ is also a smooth manifold of dimension $n-m$ so we are done.

Now, we may proceed to prove that $O_n(\mathbb{R})$ is a smooth manifold of dimension $\frac{1}{2}n(n-1)$. We note that $O_n(\mathbb{R})$ is the zero set of the function $f : M_n(\mathbb{R}) \to S^n$ defined by

$$f(A) = A^t A - I_n$$

where $S^n$ is the set of symmetric $n \times n$ matrices. Notice that $f$ is smooth as it is constructed by smooth functions. Additionally, we show that $f'(X)(h) = X^t h + h^T X$. Indeed,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(X+h) - f(X) - X^t h - h^t X}{\|h\|} &= \lim_{h\to 0} \frac{(X+h)^t(X+h) - X^t X - X^t h - h^t X}{\|h\|} \\
&= \lim_{h\to 0} \frac{h^t h}{\|h\|} \\
&= 0
\end{aligned}$$

Next, we want to show that $f'(X)$ is surjective on $S^n$ for all $X \in O_n(\mathbb{R})$.
Let $Y \in S^n$. Let $h = \frac{1}{2}XY$. We see that

$$\begin{aligned}
f'(X)(h) = X^t \left(\frac{1}{2}XY\right) + \left(\frac{1}{2}XY\right)^t X &= \frac{1}{2}\left(X^t XY + Y^t X^t X\right) \\
&= \frac{1}{2}(Y + Y^t) && (X \text{ is orthogonal}) \\
&= Y && (Y \text{ is symmetric})
\end{aligned}$$

Now, we have the hypotheses needed to apply the Regular Level-Set Theorem, and conclude that $O_n(\mathbb{R})$ is a smooth manifold of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

$\square$

Question 41.

Let $0 < a < b$. In the $xz$-plane, draw a circle of radius $a$ centered at the point $(b, 0, 0)$; rotate this circle about the $z$-axis. The resulting subset of $\mathbf{R}^3$ is called a **torus**, denoted by $\mathbf{T} = \mathbf{T}_{a,b}$.

(a) Find a smooth function $f : U \to \mathbf{R}$, defined on some open set $U \subseteq \mathbf{R}^3$, so that $\mathbf{T}$ is equal to the zero set of $f$.

(b) Show that $\mathbf{T}$ is a smooth manifold.

(c) Find the surface area of $\mathbf{T}$, in terms of $a$ and $b$.

*Proof.*
(a):
Notice that in cylindrical coordinates, the torus can be defined by

$$T = \{(r, \theta, z) : (r - b)^2 + z^2 = a^2\}.$$

If we map the polar part of the set back to cartesian coordinates, we see that $T$ is actually the zero set of the function

$$f(x, y, z) = (\sqrt{x^2 + y^2} - b)^2 + z^2 - a^2$$

This function is smooth everywhere except for when $x = 0 = y$, so we let $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$.

(b):
Notice that for all $p = (x, y, z) \in T$, $f'(p)$ is rank 1, because $f'(p)$ is a $1 \times 3$ matrix, so its rank is at most 1, but it cannot be rank 0 because

$$\frac{\partial f}{\partial x}(x, y, z) = 2\frac{\sqrt{x^2 + y^2} - b}{\sqrt{x^2 + y^2}} = 2 - \frac{2b}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = 2z$$

so being rank 0 implies that

$$f'(p) = \left(2 - \frac{2b}{\sqrt{x^2 + y^2}}, 2 - \frac{2b}{\sqrt{x^2 + y^2}}, 2z\right) = (0, 0, 0) \implies \sqrt{x^2 + y^2} = b \text{ and } z = 0$$

but if this is the case, $f(p) = -a^2 \neq 0$, so $p \notin T$, which is a contradiction.
Thus $Jf(p)$ is always rank 1, so according to the Regular Level-Set Theorem, $T$ is a smooth manifold of dimension $3 - 1 = 2$.

(c):
We split $T$ into 4 quadrants of equal volume, but we will only do the computation for one quadrant. Let $Q$ be the set

$$Q = \{(x, y, z) \in T : x > 0, z > 0\}.$$

We parametrize $Q$ with the function $\varphi : (b-a, b+a) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ defined by

$$\varphi(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{a^2 - (r-b)^2}).$$

To see that $\varphi$ is a smooth regular embedding, we see that every component is smooth on $Q$ (the third component is smooth when $r \neq b \pm a$, but $r \in (b-a, b+a)$). As well, the first two components are the polar coordinate transform, so $J\varphi(p)$ is rank 2 for all $p \in (b-a, b+a) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Finally, notice that $\varphi^{-1}(x, y, z) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)$ is continuous, so $\varphi$ is a homeomorphism, thus confirming that $\varphi$ is a smooth regular embedding. Now, we find an expression for $V(J\varphi(r, \theta))$:

$$\begin{aligned}
V(J\varphi(r,\theta)) &= \sqrt{\det((J\varphi(r,\theta))^t J\varphi(r,\theta))} \\
&= \sqrt{\det\left(\begin{pmatrix} \cos\theta & \sin\theta & -\frac{r-b}{\sqrt{a^2-(r-b)^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -\frac{r-b}{\sqrt{a^2-(r-b)^2}} & 0 \end{pmatrix}\right)} \\
&= \sqrt{\det\begin{pmatrix} \frac{a^2}{a^2-(r-b)^2} & 0 \\ 0 & r^2 \end{pmatrix}} = \sqrt{\frac{a^2r^2}{a^2-(r-b)^2}} \\
&= \frac{ar}{\sqrt{a^2-(r-b)^2}}
\end{aligned}$$

Now we can calculate the volume of $Q$:

$$\begin{aligned}
\mathrm{vol}(Q) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{b-a}^{b+a} V(J\varphi(r,\theta))\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{b-a}^{b+a} \frac{ar}{\sqrt{a^2-(r-b)^2}}\,dr\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-a}^{a} \frac{a(r+b)}{\sqrt{a^2-r^2}}\,dr\,d\theta && \text{(substitution } r \mapsto r-b\text{)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-a}^{a} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-a}^{a} \frac{ab}{\sqrt{a^2-r^2}}\,dr\,d\theta \\
&= 0 + ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arcsin\left(\frac{r}{a}\right)\Big|_{-a}^{a}\,d\theta && \text{(first integrand is odd)} \\
&= ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi\,d\theta \\
&= \pi^2 ab
\end{aligned}$$

Finally, we multiply the surface area of the quadrant by 4 to obtain that the total surface area is $4\pi^2 ab$.

$\square$