Question 27.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuously differentiable function.

(a) Show that the partial function $\mathbf{R} \to \mathbf{R}$, $t \mapsto f(x,t)$ is integrable (over any bounded interval in $\mathbf{R}$).

(b) By (a), we can define a function $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(x) = \int_a^b f(x,t)\ dt.$$

Show that φ is differentiable, and that its (classical) derivative is given by

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0,t)\ dt.$$

This formula is known as **differentiation under the integral sign**, or **Feynmann's trick**.

(c) Use Feynmann's trick to solve the single-variable integral:

$$\int_0^\infty e^{-t^2}\ dt$$

Proof. (a):
Since f is continuous, it follows immediately that its partial function is continuous, which implies that it is integrable.
(b):
Let $x_0 \in \mathbb{R}$. We will show that as $h \to 0$,

$$\frac{1}{h}\left(\int_a^b f(x_0+h,t)dt - \int_a^b f(x_0,t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0,t)dt\right) \to 0,$$

which is equivalent to saying

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0,t)\ dt.$$

Let $\varepsilon > 0$. By the partial differentiability of f, we obtain a δ so that

$$\left|f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t)\right| < \frac{|h|\varepsilon}{b-a}$$

for all $0 < |h| < \delta$.
Fix $h \in \mathbb{R}$ so that $0 < |h| < \delta$. By the linearity of the integral,

$$\left|\frac{1}{h}\left(\int_a^b f(x_0+h,t)dt - \int_a^b f(x_0,t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0,t)dt\right)\right|$$

$$= \left| \frac{1}{h} \int_a^b (f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t))dt \right|$$

$$\leq \frac{1}{|h|} \int_a^b \left| f(x_0+h,t) - f(x_0,t) - h\frac{\partial f}{\partial x}(x_0,t) \right| dt$$

$$< \frac{1}{|h|} \int_a^b \frac{|h|\varepsilon}{b-a} dt < \varepsilon$$

as desired. Thus we have found an expression for $\varphi'(x_0)$.
(c):
Let $I = \int_0^\infty e^{-t^2}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \int_0^\infty \frac{e^{-x^2(t^2+1)}}{t^2+1} dt$$

From part (b), we see that

$$\varphi'(x) = -2x \int_0^\infty e^{-x^2(t^2+1)} dt = -2xe^{-x^2} \int_0^\infty e^{-x^2t^2} dt$$

We perform the substitution $u = xt$. Changing u back to t gives us

$$\varphi'(x) = -2e^{-x^2} \int_0^\infty e^{-t^2} dt$$

Thus we have that

$$\varphi'(x) = -2e^{-x^2} I$$

We can take the definite integral of both sides with respect to x:

$$\int_0^\infty \varphi'(x)dx = \int_0^\infty -2e^{-x^2} I dx \implies \lim_{n\to\infty} \varphi(n) - \varphi(0) = -2I^2$$

Now we will analyse $\lim_{n\to\infty} \varphi(n)$ and $\varphi(0)$ separately.
First, we will show that $\lim_{n\to\infty} \varphi(n) = 0$. Let $\varepsilon > 0$. For $n > N$, where N is a fixed number, we have that $e^{-n^2(t^2+1)} < \frac{2\varepsilon}{\pi}$, so

$$|\varphi(n)| = \left| \int_0^\infty \frac{e^{-n^2(t^2+1)}}{t^2+1} \, dt \right| \leq \int_0^\infty \left| \frac{e^{-n^2(t^2+1)}}{t^2+1} \, dt \right| < \frac{2}{\pi} \int_0^\infty \frac{\varepsilon}{t^2+1} \, dt$$

$$\implies \frac{2\varepsilon}{\pi} \arctan(t) \Big|_0^\infty = \varepsilon.$$

Thus we can conclude that $\lim_{n\to\infty} \varphi(n) = 0$.
Now, notice that

$$\varphi(0) = \int_0^\infty \frac{1}{t^2+1} \, dt = \arctan(t) \Big|_0^\infty = \frac{\pi}{2}$$

Applying this to our original equation,

$$0 - \frac{\pi}{2} = -2I^2 \implies 2I^2 = \frac{\pi}{2} \implies I^2 = \frac{\pi}{4}$$

Finally, taking the squareroot of both sides gives us the result:

$$\int_0^\infty e^{-t^2}\, dt = I = \frac{\sqrt{\pi}}{2}$$

$\square$

Question 28.

Let U be an open set in a normed vector space X and let $f : U \to Y$ be a **twice continuously differentiable** function, meaning that the second derivative $f'' : U \to B(X, B(X, Y))$ exists and is continuous on U. We also say that f is a C^2**-function**.

(a) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = x^2 - xy + y^2$. Find, with proof, an explicit formula for the linear mapping $f''(2, -1)$. Also, write down the matrix that represents this linear mapping with respect to a suitable "standard" basis.

(b) Now we investigate the case $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, and let $f : U \to \mathbf{R}$ be some function defined on an open set $U \subseteq \mathbf{R}^n$. We use the notation $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ to refer to the (i, j)**th second partial derivative** of f: this is the ith partial derivative of the jth partial derivative $\dfrac{\partial f}{\partial x_j}$.

(i) Show that f is twice continuously differentiable if and only if all second partial derivatives exist and are continuous.

(ii) Let f be twice continuously differentiable Let $v \in \mathbf{R}^n$ and let $D_v f : U \to \mathbf{R}$ be the directional derivative of f along v. Show that $D_v f$ is continuously differentiable.

(iii) Let f be twice continuously differentiable and let $v \in \mathbf{R}^n$. By (ii), we know that $D_v f$ is C^1, hence differentiable in every direction $\in \mathbf{R}^n$. Show that the directional derivatives commute:

$$D_v(D_w f) = D_w(D_v f) \quad \text{for all } v, w \in \mathbf{R}^n.$$

(iv) Deduce **Clairaut's Theorem**: that the second partial derivatives commute.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

Proof. (a):
We claim that $f''(2, -1) : \mathbb{R}^2 \to B(\mathbb{R}^2, \mathbb{R})$ is a bounded linear map which is represented by the matrix

$$[f''] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(b)(i):
(b)(ii):
(b)(iii):
(b)(iv):

□