

Question 27.

Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a continuously differentiable function.

- (a) Show that the partial function $\mathbf{R} \rightarrow \mathbf{R}$, $t \mapsto f(x, t)$ is integrable (over any bounded interval in \mathbf{R}).
- (b) By (a), we can define a function $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ by

$$\varphi(x) = \int_a^b f(x, t) \, dt.$$

Show that φ is differentiable, and that its (classical) derivative is given by

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t) \, dt.$$

This formula is known as **differentiation under the integral sign**, or **Feynmann's trick**.

- (c) Use Feynmann's trick to solve the single-variable integral:

$$\int_0^\infty e^{-t^2} \, dt$$

Proof. (a):

Since f is continuous, it follows immediately that its partial function is continuous, which implies that it is integrable.

(b):

Let $x_0 \in \mathbf{R}$. We will show that as $h \rightarrow 0$,

$$\frac{1}{h} \left(\int_a^b f(x_0 + h, t) dt - \int_a^b f(x_0, t) dt - h \int_a^b \frac{\partial f}{\partial x}(x_0, t) dt \right) \rightarrow 0,$$

which is equivalent to saying

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t) \, dt.$$

Let $\varepsilon > 0$. By the partial differentiability of f , we obtain a δ so that

$$\left| f(x_0 + h, t) - f(x_0, t) - h \frac{\partial f}{\partial x}(x_0, t) \right| < \frac{|h|\varepsilon}{b-a}$$

for all $0 < |h| < \delta$.

Fix $h \in \mathbf{R}$ so that $0 < |h| < \delta$. By the linearity of the integral,

$$\left| \frac{1}{h} \left(\int_a^b f(x_0 + h, t) dt - \int_a^b f(x_0, t) dt - h \int_a^b \frac{\partial f}{\partial x}(x_0, t) dt \right) \right|$$

Question 28.

Let U be an open set in a normed vector space X and let $f : U \rightarrow Y$ be a **twice continuously differentiable** function, meaning that the second derivative $f'' : U \rightarrow B(X, B(X, Y))$ exists and is continuous on U . We also say that f is a **C^2 -function**.

- (a) Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be given by $f(x, y) = x^2 - xy + y^2$. Find, with proof, an explicit formula for the linear mapping $f''(2, -1)$. Also, write down the matrix that represents this linear mapping with respect to a suitable “standard” basis.
- (b) Now we investigate the case $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, and let $f : U \rightarrow \mathbf{R}$ be some function defined on an open set $U \subseteq \mathbf{R}^n$. We use the notation $\frac{\partial^2 f}{\partial x_i \partial x_j}$ to refer to the (i, j) th **second partial derivative** of f : this is the i th partial derivative of the j th partial derivative $\frac{\partial f}{\partial x_j}$.
- (i) Show that f is twice continuously differentiable if and only if all second partial derivatives exist and are continuous.
- (ii) Let f be twice continuously differentiable. Let $v \in \mathbf{R}^n$ and let $D_v f : U \rightarrow \mathbf{R}$ be the directional derivative of f along v . Show that $D_v f$ is continuously differentiable.
- (iii) Let f be twice continuously differentiable and let $v \in \mathbf{R}^n$. By (ii), we know that $D_v f$ is C^1 , hence differentiable in every direction $w \in \mathbf{R}^n$. Show that the directional derivatives commute:

$$D_v(D_w f) = D_w(D_v f) \quad \text{for all } v, w \in \mathbf{R}^n.$$

- (iv) Deduce **Clairaut’s Theorem**: that the second partial derivatives commute.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

Proof. (a):

We claim that $f''(2, -1) : \mathbf{R}^2 \rightarrow B(\mathbf{R}^2, \mathbf{R})$ is a bounded linear map given by

$$f''(2, -1)(p, q)(x, y) = (2p - q)x - (p - 2q)y$$

Let $(p, q), (r, s) \in \mathbf{R}^2$ and $c \in \mathbf{R}$. Then for all $(x, y) \in \mathbf{R}^2$,

$$\begin{aligned} f''(2, -1)(cp + r, cq + s)(x, y) &= [2(cp + r) - (cq + s)]x - [(cp + r) - 2(cq + s)]y \\ &= c[(2p - q)x - (p - 2q)y] + [(2r - s)x - (r - 2s)y] = cf''(2, -1)(p, q)(x, y) + f''(2, -1)(r, s) \end{aligned}$$

so $f''(2, -1)$ is linear.

Recall that $f'(p, q)(x, y) = (2p - q)x - (p - 2q)y$. In particular, for $(p, q) = (2, -1)$, $f'(2, -1)(x, y) = 5x - 4y$.

Fix $(x, y) \in \mathbf{R}^2$. Then

$$\lim_{h \rightarrow 0} \frac{f'(2 + h_1, -1 + h_2)(x, y) - f'(2, -1)(x, y) - L_p(h)}{\|h\|}$$

Question 29.

Let $U \subseteq \mathbf{R}^2$ be an open set, and let $f : U \rightarrow \mathbf{R}$ be some differentiable function. This data defines an **explicit surface** in \mathbf{R}^3 , also known as the **graph** of f :

$$S = S(f) = \{(x, y, z) \in \mathbf{R}^3 : (x, y) \in U, z = f(x, y)\}.$$

Colloquially, we often say “Let $z = f(x, y)$ be a surface in \mathbf{R}^3 .”

- (a) Find the equation of the tangent plane to the surface $z = x + xy^2 - y^3$ at the point $p = (2, 1, 3)$, as follows.
- (i) First, fix $x = 2$ and set $y = 1 + t$; write z as a function of t , and find $z'(0)$. This is the “slope in the x direction.”
 - (ii) Similarly, find the “slope in the y direction.”
 - (iii) Now you have two slopes in orthogonal directions. This gives you two vectors which span a plane. Shift this plane so that it becomes tangent at p . Write the equation of this plane in the form $Ax + By + Cz = D$, where $A, B, C, D \in \mathbf{R}$.
- (b) Find — with proof! — the equations of all planes in \mathbf{R}^3 which (i) are tangent to the surface $z = x + xy^2 - y^3$; (ii) are parallel to the vector $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$; and (iii) pass through the point $p = (-1, -2, 3)$.
- (c) Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a continuously differentiable function, and define a new function $g : \mathbf{R}^2 \rightarrow \mathbf{R}$ by

$$g(x, y) = f(f(xy, x), f(y, xy)).$$

You are given the following information about f and g :

- $f(3, 1) = 5$ and $\nabla f(3, 1) = (1, 2)$
- $f(3, 3) = 2$ and $\nabla f(3, 3) = (0, -1)$
- $g(1, 3) = 6$ and $\nabla g(1, 3) = (3, 4)$

Find $\nabla f(5, 2)$.

Proof. (a)(i):

Fix $x = 2$ and set $y = 1 + t$. Then

$$\begin{aligned} z(t) &= 2 + 2(1+t)^2 - (1+t)^3 \implies z'(t) = 4(1+t) - 3(1+t)^2 \\ &\implies z'(0) = 1 \end{aligned}$$

(a)(ii):

Fix $y = 1$ and set $x = 2 + t$. Then

$$z(t) = (2+t) + (2+t) - 1 = 2t + 3 \implies z'(0) = 2$$

(a)(iii):

