Question 38.

Basel Problem. Here you will use multivariable calculus to establish the following famous equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

To do it, you will evaluate the (improper) double integral $\int_U \frac{1}{1-xy}$ in two ways. Let $f : (0,1)^2 \to \mathbf{R}$ be the function given by $f(x,y) = \frac{1}{1-xy}$, and let K_N denote the closed box $\left[\frac{1}{N}, 1-\frac{1}{N}\right]^2$.

(a) Evaluate $\int_{K_N} f$ using Fubini's theorem.

(b) Evaluate $\int_{K_N} f$ using the Change of Variables formula twice: first using the linear diffeomorphism $(x,y) = (u+v, u-v)$, then using the polar coordinates transform.

(c) Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof.
(a):
By Fubini's:

$$\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{\frac{1}{N}}^{1-\frac{1}{N}} \frac{1}{1-xy}\, dy\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x} \ln(1-xy)\Big|_{\frac{1}{N}}^{1-\frac{1}{N}}\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x} \left(\ln\left(1-\left(1-\frac{1}{N}\right)x\right) - \ln\left(1-\frac{1}{N}x\right)\right)\, dx
\end{aligned}$$

Notice that $-1 < -\left(1-\frac{1}{N}\right), -\frac{1}{N} < 1$, so we can use the power series expansion of $\ln(1+t)$

to get that

$$\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x}\left(\sum_{n=1}^{\infty} -\frac{\left(1-\frac{1}{N}\right)^n x^n}{n} + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{N}\right)^n x^n}{n}\right)\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\left(\left(1-\frac{1}{N}\right)^n - \left(\frac{1}{N}\right)^n\right)\, dx \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n^2}\left(\left(1-\frac{1}{N}\right)^n - \left(\frac{1}{N}\right)^n\right)\Bigg|_{\frac{1}{N}}^{1-\frac{1}{N}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\left(1-\frac{1}{N}\right)^n - \left(\frac{1}{N}\right)^n\right)^2
\end{aligned}$$

(b):
Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by

$$\Phi(x,y) = (x+y, x-y).$$

Notice that $\det J\Phi = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$ and $\Phi^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$.
Define

$$E = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{N} \le x \le 1 - \frac{1}{N}, \left|x - \frac{1}{2}\right| - \frac{1}{2} + \frac{1}{2N} \le y \le \frac{1}{2} - \frac{1}{2N} - \left|x - \frac{1}{2}\right|\right\}.$$

We claim that $\Phi(E) = K_N$.
Let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$. Recall that our constraints for x and y are $\frac{1}{N} \le x, y \le 1 - \frac{1}{N}$. It follows that

$$\frac{1}{N} \le u \le 1 - \frac{1}{N}$$
$$\frac{1}{N} \le u + v \le 1 - \frac{1}{N} \implies \frac{1}{N} - u \le v \le 1 - \frac{1}{N} - u$$
$$\frac{1}{N} \le u - v \le 1 - \frac{1}{N} \implies u - 1 + \frac{1}{N} \le v \le u - \frac{1}{N}$$

If $\frac{1}{N} \le u < \frac{1}{2}$, then $u - 1 + \frac{1}{N} < \frac{1}{N} - u$, so the inequality $\frac{1}{N} - u \le v \le u - \frac{1}{N}$ suffices. On the other hand, if $\frac{1}{2} \le u \le 1 - \frac{1}{N}$, then the inequality $u - 1 + \frac{1}{N} \le v \le 1 - \frac{1}{N} - u$ suffices. Therefore, we have that

$$\begin{aligned}
\int_{K_N} f &= \int_E f \circ \Phi \cdot |\det J\Phi| \\
&= 2\left(\int_{\frac{1}{N}}^{\frac{1}{2}} \int_{\frac{1}{N}-u}^{u-\frac{1}{N}} \frac{1}{1-(u+v)(u-v)}\, dv\, du + \int_{\frac{1}{2}}^{1-\frac{1}{N}} \int_{u-1+\frac{1}{N}}^{1-\frac{1}{N}-u} \frac{1}{1-(u+v)(u-v)}\, dv\, du\right) \\
&= 2\left(\int_{\frac{1}{N}}^{\frac{1}{2}} \int_{\frac{1}{N}-u}^{u-\frac{1}{N}} \frac{1}{1-u^2+v^2}\, dv\, du + \int_{\frac{1}{2}}^{1-\frac{1}{N}} \int_{u-1+\frac{1}{N}}^{1-\frac{1}{N}-u} \frac{1}{1-u^2+v^2}\, dv\, du\right)
\end{aligned}$$

By symmetry along $y=0$, the integral above is equal to

$$4\left(\int_{\frac{1}{N}}^{\frac{1}{2}}\int_0^{u-\frac{1}{N}}\frac{1}{1-u^2+v^2}\,dv\,du+\int_{\frac{1}{2}}^{1-\frac{1}{N}}\int_0^{1-\frac{1}{N}-u}\frac{1}{1-u^2+v^2}\,dv\,du\right)$$

We do both integrals side-by-side:

$$=4\left(\int_{\frac{1}{N}}^{\frac{1}{2}}\frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{u-\frac{1}{N}}{\sqrt{1-u^2}}\right)\,du+\int_{\frac{1}{2}}^{1-\frac{1}{N}}\frac{1}{\sqrt{1-u^2}}\arctan\left(\frac{1-\frac{1}{N}-u}{\sqrt{1-u^2}}\right)\,du\right)$$

For the first integral, let $u=\sin\theta$. For the second integral, let $u=\cos\theta$. We have

$$=4\left(\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}}\arctan\left(\frac{\sin\theta-\frac{1}{N}}{|\cos\theta|}\right)\,d\theta-\int_{\frac{\pi}{3}}^{\arccos\left(1-\frac{1}{N}\right)}\arctan\left(\frac{1-\frac{1}{N}-\cos\theta}{|\sin\theta|}\right)\,d\theta\right)$$

We remove the absolute values and simplify to get that the integral is equal to

$$4\left(\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}}\arctan\left(\tan\theta-\frac{1}{N}\sec\theta\right)\,d\theta-\int_{\frac{\pi}{3}}^{\arccos\left(1-\frac{1}{N}\right)}\arctan\left(\tan\left(\frac{\theta}{2}\right)-\frac{1}{N}\csc\theta\right)\,d\theta\right)$$

It is quite difficult to find a closed form for these integrals due to the pesky $\frac{1}{N}$, but we will deal with it in part (c).

(c):

Recall from part (a) that $\int_{K_N}f=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left(1-\frac{1}{N}\right)^n-\left(\frac{1}{N}\right)^n\right)^2$. It is not hard to see that as $N\to\infty$, $\int_{K_N}f\to\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$. Now, we will show that

$$4\left(\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}}\arctan\left(\tan\theta-\frac{1}{N}\sec\theta\right)\,d\theta-\int_{\frac{\pi}{3}}^{\arccos\left(1-\frac{1}{N}\right)}\arctan\left(\tan\left(\frac{\theta}{2}\right)-\frac{1}{N}\csc\theta\right)\,d\theta\right)$$

converges to

$$\frac{\pi^2}{6}.$$

First, we restrict $N>2$ to ensure that $\arcsin\left(\frac{1}{N}\right)<\frac{\pi}{6}$ and $\arccos\left(1-\frac{1}{N}\right)<\frac{\pi}{3}$. Since $\sec t$ is bounded by $\frac{2}{\sqrt{3}}$ on $\left[\arcsin\left(\frac{1}{N}\right),\frac{\pi}{6}\right]$, by the monotonicity of $\arctan t$ we have that

$$\arctan\left(\tan\theta-\frac{2}{\sqrt{3}N}\right)\leq\arctan\left(\tan\theta-\frac{1}{N}\sec\theta\right)\leq\arctan\left(\tan\theta+\frac{2}{\sqrt{3}N}\right)$$

We note that $\arctan t$ is uniformly continuous on a closed interval, so for sufficiently large N, we have that for any $\theta\in\left[\arcsin\left(\frac{1}{N}\right),\frac{\pi}{6}\right]$,

$$\arctan(\tan\theta)-\frac{6\varepsilon}{\pi}<\arctan\left(\tan\theta-\frac{2}{\sqrt{3}N}\right)$$

and

$$\arctan(\tan\theta) + \frac{6\varepsilon}{\pi} > \arctan\left(\tan\theta + \frac{2}{\sqrt{3}N}\right)$$

for some $\varepsilon > 0$ that can be made arbitrarily small. By the monotonicity of the integral,

$$\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan(\tan\theta) - \frac{6\varepsilon}{\pi}\ d\theta < \int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan\left(\tan\theta - \frac{1}{N}\sec\theta\right)\ d\theta$$

and

$$\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan\left(\tan\theta - \frac{1}{N}\sec\theta\right)\ d\theta < \int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan(\tan\theta) + \frac{6\varepsilon}{\pi}\ d\theta$$

It is easy to evaluate the two integrals that bound the original one. We get that

$$\begin{aligned}
\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan(\tan\theta) - \frac{6\varepsilon}{\pi}\ d\theta &= \int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \theta - \frac{6\varepsilon}{\pi}\ d\theta \\
&= \frac{1}{2}\left(\frac{\pi^2}{36} - \arcsin^2\left(\frac{1}{N}\right)\right) - \frac{6\varepsilon}{\pi}\left(\frac{\pi}{6} - \arcsin\left(\frac{1}{N}\right)\right) \\
\int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan(\tan\theta) + \frac{6\varepsilon}{\pi}\ d\theta &= \int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \theta + \frac{6\varepsilon}{\pi}\ d\theta \\
&= \frac{1}{2}\left(\frac{\pi^2}{36} - \arcsin^2\left(\frac{1}{N}\right)\right) + \frac{6\varepsilon}{\pi}\left(\frac{\pi}{6} - \arcsin\left(\frac{1}{N}\right)\right)
\end{aligned}$$

We apply Squeeze Theorem to see that

$$\lim_{N\to\infty} \int_{\arcsin\left(\frac{1}{N}\right)}^{\frac{\pi}{6}} \arctan\left(\tan\theta - \frac{1}{N}\sec\theta\right)\ d\theta = \frac{\pi^2}{72}.$$

By a similar argument, we can show that

$$\begin{aligned}
-\lim_{N\to\infty} \int_{\frac{\pi}{3}}^{\arccos\left(1-\frac{1}{N}\right)} \arctan\left(\tan\left(\frac{\theta}{2}\right) - \frac{1}{N}\csc\theta\right)\ d\theta &= \int_0^{\frac{\pi}{3}} \frac{\theta}{2}\ d\theta \\
&= \frac{\pi^2}{36}
\end{aligned}$$

Thus our original integral converges to

$$4\left(\frac{\pi^2}{72} + \frac{\pi^2}{36}\right) = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

Therefore we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{N\to\infty} \int_{K_N} f = \frac{\pi^2}{6}$$

and we are done.

$\square$