

Question 23.

Let  $S^2$  denote the unit sphere in  $\mathbf{R}^3$ . Let  $N = (0, 0, 1)$  denote the “north pole”. In this problem, you will show that  $S^2 \setminus \{N\}$  is homeomorphic to  $\mathbf{R}^2$ . To do this, we define a function  $\Phi : S^2 \setminus \{N\} \rightarrow \mathbf{R}^2$  known as the **stereographic projection**: given a point  $P$  in  $S^2 \setminus \{N\}$ , draw a line between  $P$  and  $N$ , and let  $\Phi(P)$  denote the point where this line intersects the  $xy$ -plane in  $\mathbf{R}^3$ .

- (a) Given  $P = (x, y, z)$ , find an explicit formula for  $\Phi(P)$  in terms of  $x, y, z$ .
- (b) Deduce that  $\Phi$  is continuous.
- (c) Prove that  $\Phi$  is a bijection; in fact, given  $p = (s, t) \in \mathbf{R}^2$ , find an explicit formula for  $\Phi^{-1}(p)$ .
- (d) Deduce that  $\Phi$  is a homeomorphism.

*Proof.* (a):

Let  $P = (x, y, z)$ . First, we find the equation of the line that passes  $P$  and  $N$ . Consider the equation of the line  $L(t) = (tx, ty, (z-1)t + 1)$ . Notice that  $L(0) = N$  and  $L(1) = P$ , so  $L$  satisfies what we were looking for. Now we find the point where  $L$  intersects with the  $xy$ -plane. This happens exactly when  $(z-1)t + 1 = 0$ . Solving for  $t$  gives  $t = \frac{1}{1-z}$ . This value is always defined as  $z \neq 1$ . As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z} (x, y).$$

(b):

(c):

Let  $p = (s, t) \in \mathbf{R}^2$ . Our goal is to find  $(x, y, z) \in S^2 \setminus \{N\}$  such that  $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s, t)$ . Immediately, we obtain the following system of equations:

$$\begin{aligned} \frac{x}{1-z} &= s, \\ \frac{y}{1-z} &= t, \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

We also have the restriction  $z \neq 1$  because  $(x, y, z) \neq N$ . Isolating for  $x$  and  $y$  yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^2(1-z)^2 + t^2(1-z)^2 + z^2 = 1 \implies (s^2 + t^2 + 1)z^2 - 2(s^2 + t^2)z + s^2 + t^2 - 1 = 0$$



Question 24.

Let  $X$  be a normed vector space. Prove that the following statements are equivalent.

- (i)  $X$  is finite-dimensional.
- (ii) The unit ball  $\overline{B}(\vec{0}, 1)$  is compact.
- (iii)  $X$  is **locally compact**: each point  $p \in X$  is contained in some open set  $U$  such that  $\overline{U}$  is compact.

*Proof.* It will be proven that (i)  $\implies$  (ii)  $\iff$  (iii)  $\implies$  (i).

(i)  $\implies$  (ii):

Suppose that  $X$  has finite dimension  $n$ . Consider the linear isomorphism  $\Phi : X \rightarrow \mathbb{R}^n$  by mapping elements of an ordered basis for  $X$  to the basis  $\beta = \{e_1, \dots, e_n\}$  for  $\mathbb{R}^n$ . This is a homeomorphism,

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