Exercise 15.14

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Let M be a (smooth) n-manifold. Show that each point $p \in M$ has a relatively open neighborhood $U \subseteq M$ such that U homeomorphic to $\mathbb{R}^n$. In fact, show that there is a smooth regular embedding $\varphi : \mathbb{R}^n \to U$ such that $\varphi(\vec{0}) = p$.

Proof. Let $p \in M$. By our assumption, we can find a smooth regular embedding $\hat{\varphi} : \hat{V} \to V$, where $\hat{V} \subseteq \mathbb{R}^n$ is open and V is a relatively open subset of M containing p. Furthermore, there exists $q \in \hat{V}$ such that $\hat{\varphi}(q) = p$ and an open ball $B_{\max}(q, r) \subseteq \hat{V}$. Note that we make use of an open ball with respect to the max-norm, for reasons that will become clear soon. Let $U = \hat{\varphi}(B_{\max}(q, r))$. We are guaranteed that U is open because $\hat{\varphi}$ is a homeomorphism. Now, we define a smooth homeomorphism between $B_{\max}(q, r)$ and $\mathbb{R}^n$. Let $\Phi : \mathbb{R}^n \to B_{\max}(q, r)$ be defined by

$$\Phi(\vec{x}) = \left(\frac{2}{\pi r} \arctan(x_1) + q_1, ..., \frac{2}{\pi r} \arctan(x_n) + q_n \right)$$

Notice that $B_{\max}(q, r)$ is a cartesian product of intervals $\prod_{i=1}^{n} [q_i - r, q_i + r]$. The function $\Phi_i(x) = \frac{2}{\pi r} \arctan(x) + q_i$ is a well known bijection between $\mathbb{R}$ and $[q_i - r, q_i + r]$, so it is clear that Φ is a bijection with a continuous inverse. Moreover, each component is smooth, so Φ is a smooth homeomorphism. We claim that our desired function $\varphi : \mathbb{R}^n \to U$ is given by

$$\varphi(x) = \hat{\varphi}(\Phi(x)).$$

φ is a composition of smooth functions, and therefore smooth. $J\Phi$ is a diagonal matrix with non-zero diagonals, so it is rank n. It follows that the Jacobian $J\varphi(x) = J\hat{\varphi}(\Phi(x)) \cdot J\Phi(x)$ is rank n for all $x \in \mathbb{R}^n$. Finally, φ is a homeomorphism since it is a composition of homeomorphisms (so $U \simeq \mathbb{R}^n$, as needed). Thus we can conclude that φ is a smooth regular embedding. As well,

$$\varphi(0) = \hat{\varphi}(\Phi(0)) = \hat{\varphi}(q) = p.$$

and the proof is done.

$\square$

This exercise shows that we can assume WLOG that the domain of every smooth regular embedding is $\mathbb{R}^n$.