Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that O_n is a smooth manifold, and find its dimension.

Proof. First, we will prove the Regular Level-Set Theorem:

Let X, Y be normed vector spaces with dimensions n, m and ordered bases α, β respectively, where $n > m$. Let $F : X \to Y$ be a smooth function. Define $M = F^{-1}(0_Y)$. If $F'(p)$ is surjective for all $p \in M$, then M is a smooth manifold of dimension $n - m$. That is, if $\phi_\alpha : X \to \mathbb{R}^n$ is the coordinate isomorphism corresponding to α, then M is a smooth k-manifold if $\phi_\alpha(M)$ is a smooth k-manifold in the usual sense.

This problem reduces to trying to prove that $N := \phi_\alpha(M)$ is a smooth manifold of dimension $n - m$. Notice that $N = \phi_\alpha(F^{-1}(0_Y)) = \phi_\alpha(F^{-1}(\phi_\beta(0_{\mathbb{R}^m})))$. Since ϕ_α, ϕ_β are isomorphisms, we have that $N = \phi_\alpha \circ F^{-1} \circ \phi_\beta(0_{\mathbb{R}^m})$. Let $\hat{F} = \phi_\alpha \circ F^{-1} \circ \phi_\beta$, and notice that N is the zero set of $\hat{F}$.

We note that $O_n(\mathbb{R})$ is the zero set of the function $f : M_n(\mathbb{R}) \to S^n$ defined by

$$f(A) = A^t A - I_n$$

where S^n is the set of symmetric $n \times n$ matrices. Notice that f is smooth as it is constructed by smooth functions. Additionally, we show that $Jf(X)(h) = X^t h + h^T X$. Indeed,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(X+h) - f(X) - X^t h - h^t X}{\|h\|} &= \lim_{h \to 0} \frac{(X+h)^t(X+h) - X^t X - X^t h - h^t X}{\|h\|} \\
&= \lim_{h \to 0} \frac{h^t h}{\|h\|} \\
&= 0
\end{aligned}$$

Next, we want to show that $\operatorname{rank} Jf(X) = \frac{1}{2}n(n+1)$ for all $X \in O_n(\mathbb{R})$. It suffices to show that $Jf(X)$ is surjective to S^n.

Let $Y \in S^n$. Let $h = \frac{1}{2}XY$. We see that

$$\begin{aligned}
Jf(X)(h) = X^t\left(\frac{1}{2}XY\right) + \left(\frac{1}{2}XY\right)^t X &= \frac{1}{2}\left(X^t XY + Y^t X^t X\right) \\
&= \frac{1}{2}(Y + Y^t) && (X \text{ is orthogonal}) \\
&= Y && (Y \text{ is symmetric})
\end{aligned}$$

Thus $R(Jf(X)) = S^n$ so $\operatorname{rank} Jf(X) = \dim S^n = \frac{1}{2}n(n+1)$.

We now prove that $O_n(\mathbb{R})$ is a smooth manifold of dimension $\frac{1}{2}n(n+1)$. Let $p \in O_n(\mathbb{R})$. Then $f(p) = 0$ and $Jf(p)$ has the maximal rank of $\frac{1}{2}n(n+1)$. We write

$$Jf(p) = \left(A \mid B\right)$$

where A is a $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n-1)$ matrix and B is a $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$ matrix and assume without loss of generality that B is an invertible submatrix of $Jf(p)$. We can do this because we can swap the components of f, and therefore columns of $Jf(p)$ without affecting the conclusion of the statement (because manifolds are invariant under diffeomorphims). Thus, we write $p=(a,b)$ for $a\in\mathbb{R}^{\frac{1}{2}n(n-1)}$, $b\in\mathbb{R}^{\frac{1}{2}n(n+1)}$ and apply the Implicit Function Theorem and obtain an open set $\hat{U}\subseteq\mathbb{R}^{\frac{1}{2}n(n-1)}$ containing a and a C^∞ function $\Phi:\hat{U}\to\mathbb{R}^{\frac{1}{2}n(n+1)}$ so that

$$f(x,\Phi(x))=0$$

for all $x\in\hat{U}$. We claim that $\varphi:\hat{U}\to\Phi(\hat{U})$ defined by

$$\varphi(x)=(x,\Phi(x))$$

is our desired smooth regular embedding. It is fairly clear that φ is smooth. Additionally, $J\varphi(x)=\begin{pmatrix}I_{\frac{1}{2}n(n-1)}\\ \hline J\Phi\end{pmatrix}$ is a $n^2\times\frac{1}{2}n(n-1)$ matrix and has at least $\frac{1}{2}n(n-1)$ linearly independent rows, so φ is regular. Finally, if we let $\varphi(x)=\varphi(y)$, we have that $(x,\Phi(x))=(y,\Phi(y))$, from which we get $x=y$, so φ is injective, and therefore bijective to its image. In addition, we can explicitly find $\varphi^{-1}(y)=\pi_{\mathbb{R}^{\frac{1}{2}n(n-1)}}(y)$, which is continuous because it is a linear map. Therefore φ is a homeomorphism onto its image, and we are done.

□

Question 41.

Let $0 < a < b$. In the xz-plane, draw a circle of radius a centered at the point $(b, 0, 0)$; rotate this circle about the z-axis. The resulting subset of $\mathbf{R}^3$ is called a **torus**, denoted by $\mathbf{T} = \mathbf{T}_{a,b}$.

(a) Find a smooth function $f : U \to \mathbf{R}$, defined on some open set $U \subseteq \mathbf{R}^3$, so that $\mathbf{T}$ is equal to the zero set of f.

(b) Show that $\mathbf{T}$ is a smooth manifold.

(c) Find the surface area of $\mathbf{T}$, in terms of a and b.

Proof.
(a):
Notice that in cylindrical coordinates, the torus can be defined by

$$T = \{(r, \theta, z) : (r - b)^2 + z^2 = a^2\}.$$

If we map the polar part of the set back to cartesian coordinates, we see that T is actually the zero set of the function

$$f(x, y, z) = (\sqrt{x^2 + y^2} - b)^2 + z^2 - a^2$$

This function is smooth everywhere except for when $x = 0 = y$, so we let $U = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$.

(b):
Since T is the zero set of a smooth function, T is automatically a smooth manifold by a result in the tutorial which was (kind of) proved in the previous question above. However, we will show that T is a 2-manifold for fun. This is still pretty quick, as $Jf(p)$ is a 1×3 matrix, so its maximum rank is 1, but its rank cannot be 0 because

$$\frac{\partial f}{\partial x}(x, y, z) = 2\frac{\sqrt{x^2 + y^2} - b}{\sqrt{x^2 + y^2}} = 2 - \frac{2b}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = 2z$$

Thus $Jf(p)$ is always rank 1, so according to the tutorial result, T is of dimension $3 - 1 = 2$. $\square$