Question 22.

For a normed vector space $(X, \|\cdot\|)$, let $X^* = B(X, \mathbf{R})$ denote the set of bounded linear mappings from X to $\mathbf{R}$. Here X^* is equipped with the operator norm $\|\cdot\|_{\text{op}}$, and the normed vector space $(X^*, \|\cdot\|_{\text{op}})$ is called the **topological dual** of X.
Let c_0 denote the set of sequences converging to zero. Prove that $c_0^* \equiv \ell^1$.

Proof. Equip ℓ^1 with the 1-norm and c_0 with the sup-norm. Define the function $f : \ell^1 \to c_0^*$ by $f(\vec{x})(\vec{a}) = \sum_{i=1}^{\infty} x_i \cdot a_i$. This sum is convergent because for sufficiently large $N > 0$, $|a_n| < 1$ for $n > N$, so

$$\sum_{i=1}^{\infty} x_i \cdot a_i = \sum_{i=1}^{N} x_i \cdot a_i + \sum_{i=N+1}^{\infty} x_i \cdot a_i < \sum_{i=1}^{N} x_i \cdot a_i + \sum_{i=N+1}^{\infty} x_i < \infty$$

We claim that this is a bijective isometry.
First, it can be quickly verified that this function is linear. Letting $a_n \in c_0$, for $x, y \in \ell^1$, $k > 0$,

$$f(kx + y)(a_n) = \sum_{i=1}^{\infty} a_i \cdot (kx_i + y_i) = k \sum_{i=1}^{\infty} a_i \cdot x_i + \sum_{i=1}^{\infty} a_i \cdot y_i = kf(x)(a_n) + f(y)(a_n).$$

To show that this function is an isometry, let $x, y \in \ell^1$.
Let $A = \{|f(x)(a) - f(y)(a)| : a \in c_0, \|a\|_\infty \le 1\}$. It will be shown that $\|x - y\|_1 = \sup A$.
Let $a \in c_0$ such that $\|a\|_\infty \le 1$. Then

$$|f(x)(a) - f(y)(a)| = \left|\sum_{i=1}^{\infty}(x_i \cdot a_i - y_i \cdot a_i)\right| = \left|\sum_{i=1}^{\infty} a_i(x_i - y_i)\right| \le \sum_{i=1}^{\infty} |a_i||x_i - y_i| \le \sum_{i=1}^{\infty} |x_i - y_i|$$
$$= \|x - y\|_1.$$

This means that $\|x - y\|_1$ is an upper bound for A. To prove that this is the least upper bound, let $\varepsilon > 0$. There is an $N > 0$ such that $\sum_{N+1}^{\infty} |x_i - y_i| < \varepsilon$. Define the sequence (a_n) in c_0 as the sequence of 1's until and including $n = N$ and 0 afterwards. Then

$$|f(x)(a) - f(y)(a)| = \left|\sum_{i=1}^{\infty} a_i(x_i - y_i)\right| = \left|\sum_{i=1}^{\infty} (1 - 1 + a_i)(x_i - y_i)\right|$$
$$\ge \left|\sum_{i=1}^{\infty}(x_i - y_i)\right| - \left|\sum_{i=1}^{\infty} (1 - a_i)(x_i - y_i)\right|$$
$$= \left|\sum_{i=1}^{\infty}(x_i - y_i)\right| - \sum_{i=N+1}^{\infty} |x_i - y_i| > \|x - y\|_1 - \varepsilon$$

Thus we have that

$$\|f(x) - f(y)\|_{\text{op}} = \sup A = \|x - y\|_1$$

so f is an isometry.

To prove that f is bijective, it suffices to show that f is surjective, since we already have injectivity from the isometry.
Let $T \in c_0^*$. Construct a sequence (x_n) in ℓ^1 defined by $x_n = T(e_n)$, where (e_n) is the sequence with the only non-zero term being $e_n = 1$.
First, we need to verify that $(x_n) \in \ell^1$. To do this, we bound the absolute sum by a finite number. We have that

$$\sum_{i=1}^{\infty} |x_i| = \sum_{i=1}^{\infty} |T(e_i)|$$

For all non-zero terms, we can rewrite it as $|T(e_i)| \cdot \dfrac{T(e_i)}{T(e_i)} = T\left(\dfrac{|T(e_i)|}{T(e_i)} e_i\right)$. Define (b_n) to be a sequence defined by

$$b_n = \begin{cases} 0, & \text{if } T(e_i) = 0; \\ \dfrac{|T(e_i)|}{T(e_i)} e_i, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i=1}^{\infty} |T(e_i)| = T(b_n)$$

Notice that $|b_n| \leq 1$ for all $n \in \mathbb{N}$. Thus $\|b_n\|_\infty = 1$, so

$$\sum_{i=1}^{\infty} |x_i| = T(b_n) \leq \|T\|_{\text{op}}$$

By the monotone bounded convergence, $\sum_{i=1}^{\infty} |x_i|$ converges. Therefore $(x_n) \in \ell^1$.
Now, we want to show that $f(x_n) = T$. For $(a_n) \in c_0$, we have that

$$f(x_n)(a_n) = \sum_{i=1}^{\infty} x_i \cdot a_i = \sum_{i=1}^{\infty} T(e_i) \cdot a_i = T\left(\sum_{i=1}^{\infty} a_i(e_i)\right) = T((a_n))$$

so f is surjective.
Thus f is a bijective isometry, so $c_0^* \equiv \ell^1$.

$\square$

Question 23.

Let S^2 denote the unit sphere in $\mathbf{R}^3$. Let $N = (0, 0, 1)$ denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to $\mathbf{R}^2$. To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbf{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in $\mathbf{R}^3$.

(a) Given $P = (x, y, z)$, find an explicit formula for $\Phi(P)$ in terms of x, y, z.

(b) Deduce that Φ is continuous.

(c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.

(d) Deduce that Φ is a homeomorphism.

Proof. We will work off the assumption that both metric spaces are equipped with the max-norm. We can do this because all norms on $\mathbb{R}^n$ are equivalent.

(a):

Let $P = (x, y, z)$. First, we find the equation of the line that passes P and N. Consider the equation of the line $L(t) = (tx, ty, (z-1)t + 1)$. Notice that $L(0) = N$ and $L(1) = P$, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when $(z-1)t + 1 = 0$. Solving for t gives $t = \dfrac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z}(x, y).$$

(b):

To show that Φ is continuous, fix $(a, b, c) \in \mathbb{R}^3$. Let $\varepsilon > 0$. Then let $\delta = \min\{\frac{1-c}{2}, \frac{(1-c)\varepsilon}{4}, \frac{(1-c)^2\varepsilon}{4|a|}\}$. For $(x, y, z) \in \mathbb{R}^3$ such that $\|(x, y, z) - (a, b, c)\|_{\max} < \delta$, notice that

$$z - c < |z - c| \leq \|(x, y, z) - (a, b, c)\|_{\max} < \frac{1-c}{2}$$

Manipulating this, we obtain

$$z < \frac{1+c}{2} \implies 1 - z > \frac{1-c}{2} \implies \frac{1}{1-z} < \frac{2}{1-c}.$$

So

$$\begin{aligned}
\|\Phi(x, y, z) - \Phi(a, b, c)\|_{\max} &= \left\| \left(\frac{x}{1-z}, \frac{y}{1-z}\right) - \left(\frac{a}{1-c}, \frac{b}{1-c}\right) \right\|_{\max} \\
&= \frac{\|(x(1-c) - a(1-z), y(1-c) - b(1-z))\|_{\max}}{(1-z)(1-c)}
\end{aligned}$$

$$< \frac{2\|(x(1-c)-a(1-z), y(1-c)-b(1-z))\|_{\max}}{(1-c)^2}$$

Without loss of generality, suppose that $|x(1-c)-a(1-z)| \geq |y(1-c)-b(1-z)|$. Then

$$\frac{2\|(x(1-c)-a(1-z), y(1-c)-b(1-z))\|_{\max}}{(1-c)^2} = \frac{2|x(1-c)-a(1-z)|}{(1-c)^2}$$

$$= \frac{2|(x-a)(1-c)+a(z-c)|}{(1-c)^2} \leq \frac{2(|x-a||1-c|+|a||z-c|)}{(1-c)^2}$$

(c):
Let $p = (s,t) \in \mathbb{R}^2$. Our goal is to find $(x,y,z) \in S^2 \backslash \{N\}$ such that $\Phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s,t)$. Immediately, we obtain the following system of equations:

$$\frac{x}{1-z} = s,$$

$$\frac{y}{1-z} = t,$$

$$x^2 + y^2 + z^2 = 1$$

We also have the restriction $z \neq 1$ because $(x,y,z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^2(1-z)^2 + t^2(1-z)^2 + z^2 = 1 \implies (s^2+t^2+1)z^2 - 2(s^2+t^2)z + s^2 + t^2 - 1 = 0$$

We can replace the term $t^2 + s^2$ with $\|p\|_2^2$, and the equation becomes

$$(\|p\|_2^2 + 1)z^2 - 2\|p\|_2^2 z + \|p\|_2^2 - 1 = 0$$

Using the quadratic formula:

$$z = \frac{2(\|p\|_2^2) \pm \sqrt{4(\|p\|_2^2)^2 - 4(\|p\|_2^2+1)(\|p\|_2^2-1)}}{2(\|p\|_2^2+1)}$$

$$\implies z = \frac{\|p\|_2^2 \pm \sqrt{\|p\|_2^4 - (\|p\|_2^4 - 1)}}{\|p\|_2^2 + 1}$$

$$z = \frac{\|p\|_2^2 \pm 1}{\|p\|_2^2 + 1}$$

Notice that we cannot use the positive solution, for then

$$z = \frac{\|p\|_2^2 + 1}{\|p\|_2^2 + 1} = 1$$

Thus it must be true that

$$z = \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}$$

$$x = s(1 - z) = \frac{2s}{\|p\|_2^2 + 1}$$

$$y = \frac{2t}{\|p\|_2^2 + 1}$$

It can be verified that these values of x, y, z result in $\Phi(x, y, z) = (s, t)$. In fact, using this, we obtain that the formula for Φ^{-1} is

$$\Phi^{-1}(s, t) = \left(\frac{2s}{\|p\|_2^2 + 1}, \frac{2t}{\|p\|_2^2 + 1}, \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1} \right)$$

$\square$

Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

(i) X is finite-dimensional.

(ii) The unit ball $\overline{B}(\vec{0}, 1)$ is compact.

(iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that $\overline{U}$ is compact.

Proof. It will be proven that (i) $\implies$ (ii) $\iff$ (iii) $\implies$ (i).
(i) $\implies$ (ii):
Suppose that X has finite dimension n. Then there is a continuous linear isomorphism Φ between X and $\mathbb{R}^n$. Since $\overline{B}(0,1)$ is closed and bounded, $\Phi(\overline{B}(0,1))$ is also closed and bounded in $\mathbb{R}^n$, so the set is compact. Since homeomorphisms preserve compactness, we can conclude that the closed unit ball in X is compact.
(ii) $\implies$ (iii):
Suppose that the unit ball $\overline{B}(\vec{0}, 1)$ is compact. Let $p \in X$. We claim that $U = B(p, 1)$. Consider $\overline{U} = \overline{B}(p, 1)$. There is an isometry Φ from $\overline{B}(0, 1)$ to $\overline{B}(p, 1)$ defined by $\Phi(x) = p + x$. Since the closed unit ball is compact, it follows that $\overline{B}(p, 1)$ is compact. Thus X is locally compact.
(iii) $\implies$ (ii):
Suppose that X is locally compact. Then $\vec{0}$ is contained in an open set U such that $\overline{U}$ is compact. Since U is open, $B(0, \varepsilon) \subseteq U$ for some ε-ball centered around 0. It follows that $\overline{B}(0, \varepsilon)$ is compact, as it is a closed subset of $\overline{U}$. Since there is a homeomorphism from this closed ball to $\overline{B}(0, 1)$, the closed unit ball is also compact, as desired.
(iii) $\implies$ (i):
Suppose that X is locally compact. We know previously that this implies that the closed unit ball is compact. Firstly, a quick lemma will be proven.
Lemma 1. $\forall x \in X$, $r > 0$, $B(x, r) = \{x\} + B(0, r)$.
Let $y \in B(x, r)$. Notice that $y - x \in B(0, r)$. Thus $y = x + (y - x)$ so $y \in \{x\} + B(0, r)$. Then, let $y \in \{x\} + B(0, r)$, so $y = x + s$, for some $s \in B(0, r)$. Since $\|y - x\| = \|s\| < r$, it follows that $y \in B(x, r)$ and we are done.
Moving on, we construct a finite set of vectors in the following way:
Since $\overline{B}(0, 1)$ is totally bounded, we can find a finite set of vectors β such that $\bigcup_{x \in \beta} B\left(x, \frac{1}{2}\right)$.
Lemma 2. $B(0, 1) \subseteq \operatorname{span}(\beta) + 2^{-n} B(0, 1)$, $\forall n \in \mathbb{N}$.
To prove this, let $n \in \mathbb{N}$ and $y \in B(0, 1)$. It follows that $y \in B(x_1, \frac{1}{2})$ for some $x_1 \in \beta$. Using Lemma 1,

$$y \in \{x_1\} + B\left(0, \frac{1}{2}\right) = \{x_1\} + \frac{1}{2} B(0, 1)$$

We can repeatedly use this argument to obtain that

$$y \in \left\{x_1 + \frac{1}{2} x_2 + \frac{1}{2^2} x_3 + \cdots + \frac{1}{2^{n-1}} x_n\right\} + \frac{1}{2^n} B(0, 1)$$

Note that x_i are not necessarily distinct. This simplifies down to

$$y \in \operatorname{span}(\beta) + \frac{1}{2^n}B(0,1) \implies B(0,1) \subseteq \operatorname{span}(\beta) + \frac{1}{2^n}B(0,1)$$

which is what we wanted.
Finally, we claim that β is the basis for X.
Let $v \in X$. Since $\dfrac{v}{2\|v\|} \in B(0,1)$, by Lemma 2,

$$\frac{v}{2\|v\|} \in \operatorname{span}(\beta) + \frac{1}{2^n}B(0,1), \text{ for all } n \in \mathbb{N}$$

Then we construct a sequence $\{v_n\}_{n\geq 1}$ in $\operatorname{span}(\beta)$ where $v_n \in \operatorname{span}(\beta)$ such that $y = v_n + p$, for some $p \in \frac{1}{2^n}B(0,1)$. Notice that

$$\|y - v_n\| = \|p\| < \frac{1}{2^n}$$

which can become arbitrarily small. This implies that y is a limit point for $\operatorname{span}(\beta)$. However, this set is closed because it is finite dimensional, so we also have that $y \in \operatorname{span}(\beta)$. Therefore $X \subseteq \operatorname{span}(\beta)$, and since X is spanned by a finite set, it is finite dimensional.
These implications are sufficient for proving that (i), (ii), (iii) are equivalent.

$\square$