Question 27.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuously differentiable function.

(a) Show that the partial function $\mathbf{R} \to \mathbf{R}$, $t \mapsto f(x, t)$ is integrable (over any bounded interval in $\mathbf{R}$).

(b) By (a), we can define a function $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(x) = \int_a^b f(x, t)\ dt.$$

Show that φ is differentiable, and that its (classical) derivative is given by

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t)\ dt.$$

This formula is known as **differentiation under the integral sign**, or **Feynmann's trick**.

(c) Use Feynmann's trick to solve the single-variable integral:

$$\int_0^\infty e^{-t^2}\ dt$$

Proof. (a):
Since f is continuous, it follows immediately that its partial function is continuous, which implies that it is integrable.
(b):
Let $x_0 \in \mathbb{R}$. We will show that as $h \to 0$,

$$\frac{1}{h}\left(\int_a^b f(x_0 + h, t)dt - \int_a^b f(x_0, t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0, t)dt\right) \to 0,$$

which is equivalent to saying

$$\frac{d\varphi}{dx}(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t)\ dt.$$

Let $\varepsilon > 0$. By the partial differentiability of f, we obtain a δ so that

$$\left| f(x_0 + h, t) - f(x_0, t) - h\frac{\partial f}{\partial x}(x_0, t)\right| < \frac{|h|\varepsilon}{b - a}$$

for all $0 < |h| < \delta$.
Fix $h \in \mathbb{R}$ so that $0 < |h| < \delta$. By the linearity of the integral,

$$\left|\frac{1}{h}\left(\int_a^b f(x_0 + h, t)dt - \int_a^b f(x_0, t)dt - h\int_a^b \frac{\partial f}{\partial x}(x_0, t)dt\right)\right|$$

$$= \left| \frac{1}{h} \int_a^b (f(x_0 + h, t) - f(x_0, t) - h \frac{\partial f}{\partial x}(x_0, t)) dt \right|$$

$$\leq \frac{1}{|h|} \int_a^b \left| f(x_0 + h, t) - f(x_0, t) - h \frac{\partial f}{\partial x}(x_0, t) \right| dt$$

$$< \frac{1}{|h|} \int_a^b \frac{|h|\varepsilon}{b-a} dt = \varepsilon$$

as desired. Thus we have found an expression for $\varphi'(x_0)$.
(c):
Let $I = \int_0^\infty e^{-t^2}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \int_0^\infty \frac{e^{-x^2(t^2+1)}}{t^2+1} dt$$

From part (b), we see that

$$\varphi'(x) = -2x \int_0^\infty e^{-x^2(t^2+1)} dt = -2xe^{-x^2} \int_0^\infty e^{-x^2 t^2} dt$$

We perform the substitution $u = xt$. Changing u back to t gives us

$$\varphi'(x) = -2e^{-x^2} \int_0^\infty e^{-t^2} dt$$

Thus we have that

$$\varphi'(x) = -2e^{-x^2} I$$

We can take the definite integral of both sides with respect to x:

$$\int_0^\infty \varphi'(x) dx = \int_0^\infty -2e^{-x^2} I dx \implies \lim_{n\to\infty} \varphi(n) - \varphi(0) = -2I^2$$

Now we will analyse $\lim_{n\to\infty} \varphi(n)$ and $\varphi(0)$ separately.
First, we will show that $\lim_{n\to\infty} \varphi(n) = 0$. Let $\varepsilon > 0$. For $n > N$, where N is a fixed number, we have that $e^{-n^2(t^2+1)} < \frac{2\varepsilon}{\pi}$, so

$$|\varphi(n)| = \left| \int_0^\infty \frac{e^{-n^2(t^2+1)}}{t^2+1} \, dt \right| \leq \int_0^\infty \left| \frac{e^{-n^2(t^2+1)}}{t^2+1} \, dt \right| < \frac{2}{\pi} \int_0^\infty \frac{\varepsilon}{t^2+1} \, dt$$

$$\implies \frac{2\varepsilon}{\pi} \arctan(t) \Big|_0^\infty = \varepsilon.$$

Thus we can conclude that $\lim_{n\to\infty} \varphi(n) = 0$.
Now, notice that

$$\varphi(0) = \int_0^\infty \frac{1}{t^2+1} \, dt = \arctan(t) \, \Big|_0^\infty = \frac{\pi}{2}$$

Applying this to our original equation,

$$0 - \frac{\pi}{2} = -2I^2 \implies 2I^2 = \frac{\pi}{2} \implies I^2 = \frac{\pi}{4}$$

Finally, taking the squareroot of both sides gives us the result:

$$\int_0^\infty e^{-t^2}\,dt = I = \frac{\sqrt{\pi}}{2}$$

$\square$

Question 28.

Let U be an open set in a normed vector space X and let $f : U \to Y$ be a **twice continuously differentiable** function, meaning that the second derivative $f'' : U \to B(X, B(X, Y))$ exists and is continuous on U. We also say that f is a C^2**-function**.

(a) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = x^2 - xy + y^2$. Find, with proof, an explicit formula for the linear mapping $f''(2, -1)$. Also, write down the matrix that represents this linear mapping with respect to a suitable "standard" basis.

(b) Now we investigate the case $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, and let $f : U \to \mathbf{R}$ be some function defined on an open set $U \subseteq \mathbf{R}^n$. We use the notation $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ to refer to the (i, j)**th second partial derivative** of f: this is the ith partial derivative of the jth partial derivative $\dfrac{\partial f}{\partial x_j}$.

   (i) Show that f is twice continuously differentiable if and only if all second partial derivatives exist and are continuous.

   (ii) Let f be twice continuously differentiable Let $v \in \mathbf{R}^n$ and let $D_v f : U \to \mathbf{R}$ be the directional derivative of f along v. Show that $D_v f$ is continuously differentiable.

   (iii) Let f be twice continuously differentiable and let $v \in \mathbf{R}^n$. By (ii), we know that $D_v f$ is C^1, hence differentiable in every direction $\in \mathbf{R}^n$. Show that the directional derivatives commute:

$$D_v(D_w f) = D_w(D_v f) \quad \text{for all } v, w \in \mathbf{R}^n.$$

   (iv) Deduce **Clairaut's Theorem**: that the second partial derivatives commute.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

Proof. (a):
We claim that $f''(2, -1) : \mathbb{R}^2 \to B(\mathbb{R}^2, \mathbb{R})$ is a bounded linear map given by

$$f''(2, -1)(p, q)(x, y) = (2p - q)x - (p - 2q)y$$

Let $(p, q), (r, s) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then for all $(x, y) \in \mathbb{R}^2$,

$$f''(2, -1)(cp + r, cq + s)(x, y) = [2(cp + r) - (cq + s)]x - [(cp + r) - 2(cq + s)]y$$

$$= c[(2p - q)x - (p - 2q)y] + [(2r - s)x - (r - 2s)y] = cf''(2, -1)(p, q)(x, y) + f''(2, -1)(r, s)$$

so $f''(2, -1)$ is linear.
Recall that $f'(p, q)(x, y) = (2p - q)x - (p - 2q)y$. In particular, for $(p, q) = (2, -1)$, $f'(2, -1)(x, y) = 5x - 4y$.
Fix $(x, y) \in \mathbb{R}^2$. Then

$$\lim_{h \to 0} \frac{f'(2 + h_1, -1 + h_2)(x, y) - f'(2, -1)(x, y) - L_p(h)}{\|h\|}$$

$$= \lim_{h \to 0} \frac{[(5+2h_1-h_2)x-(4+h_1-2h_2)y]-[5x-4y]-L_p(h)}{\|h\|}$$

$$= \lim_{h \to 0} \frac{(2h_1-h_2)x-(h_1-2h_2)y-L_p(h)}{\|h\|} = 0$$

which verifies that $f''(2,-1)(p,q)(x,y) = (2p-q)x-(p-2q)y$.
(b):
Let $f : U \to \mathbb{R}$ be a function, where U is open in $\mathbb{R}^n$.
(b)(i):
Suppose that f is twice continuously differentiable. This implies that all its partial derivatives exist.
Consider an arbitrary partial derivative $\dfrac{\partial f}{\partial x_i}$ for $i \in \{1, ..., n\}$. It will be shown that each are continuously totally differentiable.
Conversely, suppose that all second partial derivatives of f exist and are continuous. Since all first partial derivatives will exist and are continuous, then f is continuously differentiable, meaning that f' exists.
Let $i \in \{1, ..., n\}$. We claim that $\dfrac{\partial f'}{\partial x_i}$ exists and is continuous, where x_i represents the ith standard ordered basis vector of $\mathbb{R}^n$.
Fix $p \in \mathbb{R}^n$. Recall that $f'(p)(\vec{v}) = \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}(p)$ from the differentiability theorem. For $\vec{v} \in \mathbb{R}^n$, we have that

$$\lim_{h \to 0} \frac{f'(p+hx_i)(\vec{v}) - f'(p)(\vec{v})}{h} = \lim_{h \to 0} \sum_{j=1}^n \frac{v_i}{h} \left(\frac{\partial f}{\partial x_j}(p+hx_i) - \frac{\partial f}{\partial x_j}(p) \right)$$

Since all second partial derivatives exist, the limit evaluates to

$$\sum_{j=1}^n v_i \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

By definition, this is the ith partial derivative of f'.
Notice that it is also continuous, as it is a finite linear combination of second partial derivatives, which are continuous by assumption.
Since we have that all partial derivatives of f' exist and are continuous, by the differentiability theorem, f' is continuously differentiable, which implies that f is twice continuously differentiable.
Therefore we can conclude that twice continuously differentiable is equivalent to having all second partial derivatives exist and continuous.

(b)(ii):
Suppose that f is twice continuously differentiabe. Consider the directional derivative $D_v f$. We will show that for all $p \in U$, $D_v f(p) = f'(p)(v)$.
We can assume that $p+v \in U$ by making $\|v\|$ small enough. Define a function $\alpha : [0,1] \to U$ by $\alpha(t) = p + tv$. The derivative is given by $\alpha'(t) = v$. Now, construct a new function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = (f \circ \alpha)(t)$. By the chain rule, we see that $g'(t) = f'(\alpha(t))(v)$.

Now, recall that

$$D_v(p) = \lim_{h\to 0} \frac{f(p+hv) - f(p)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) = f'(\alpha(0))(v) = f'(p)(v),$$

but we know by that f is twice continuously differentiable, so $f'(p)(v) = D_v f$ is continuously differentiable.

(b)(iii):
(b)(iv):

$\square$