Question 1.

Use row operations on the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix}$ to obtain an upper triangular matrix, then use Theorem 59 to find $\det A$. (You will get no credit for using a row/column expansion.)

We have

$$\begin{aligned}
\det A &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 4 & -10 & -4 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -12 & -6 \end{pmatrix} \\
&= -6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix} \\
&= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{pmatrix} \\
&= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \\
&= 6(1)(2)(2)\left(\frac{1}{2}\right) \\
&= 12
\end{aligned}$$

Question 2.

Let $T = T_A : \mathbb{Q}^5 \to \mathbb{Q}^5$ where $A = \begin{pmatrix} 1 & 0 & 1 & -2 & 0 \\ 3 & 0 & 1 & 0 & -2 \\ 2 & 0 & 0 & 2 & -2 \\ 2 & 0 & 0 & 1 & -2 \\ 2 & 0 & 1 & -2 & -1 \end{pmatrix}$.

(a) Find C_T and the eigenvalues of T.

We have

$$\begin{aligned}
C_T(\lambda) &= \det(\lambda I - T) \\
&= \det \begin{pmatrix} \lambda - 1 & 0 & -1 & 2 & 0 \\ -3 & \lambda & -1 & 0 & 2 \\ -2 & 0 & \lambda & -2 & 2 \\ -2 & 0 & 0 & \lambda - 1 & 2 \\ -2 & 0 & -1 & 2 & \lambda + 1 \end{pmatrix} \\
&= -\lambda \det \begin{pmatrix} \lambda - 1 & -1 & 2 & 0 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda - 1 & 2 \\ -2 & -1 & 2 & \lambda + 1 \end{pmatrix} \\
&= -\lambda \det \begin{pmatrix} \lambda + 1 & 0 & 0 & -\lambda - 1 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda - 1 & 2 \\ -2 & -1 & 2 & \lambda + 1 \end{pmatrix} \\
&= -\lambda \left((\lambda + 1) \det \begin{pmatrix} \lambda & -2 & 2 \\ 0 & \lambda - 1 & 2 \\ -1 & 2 & \lambda + 1 \end{pmatrix} + (\lambda + 1) \det \begin{pmatrix} -2 & \lambda & -2 \\ -2 & 0 & \lambda - 1 \\ -2 & -1 & 2 \end{pmatrix} \right) \\
&= \lambda(\lambda + 1) \left(-(\lambda - 1)(\lambda(\lambda + 1) + 2) + 2(2\lambda - 2) + -2(2\lambda - 2) + (\lambda - 1)(2 + 2\lambda) \right) \\
&= \lambda(\lambda + 1) \left((\lambda - 1)(2 - \lambda - \lambda^2 - 2 + 2\lambda \right) \\
&= \lambda(\lambda + 1)(\lambda - 1)(\lambda - \lambda^2) \\
&= -\lambda^2(\lambda - 1)^2(\lambda + 1)
\end{aligned}$$

The eigenvalues are the roots of C_T, which are $\lambda = 0, 1, -1$.

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

For $\lambda = 0$, we solve the equation $Ax = 0$ via row reduction:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & 1 & -2 & 0 \\ 2 & 0 & 1 & -2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We get that $x_1 = 0, x_4 = 0, x_3 + x_5 = 0$. We parametrize $x_2 = t, x_3 = s$ and get

$$x = \begin{pmatrix} 0 \\ t \\ s \\ 0 \\ -s \end{pmatrix} = te_2 + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Thus a basis for $E_0(T)$ is $\left\{ e_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

For $\lambda = 1$, to solve $(A - I)x = 0$, we get the system

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 3 & -1 & 1 & 0 & -2 & 0 \\ 2 & 0 & -1 & 2 & -2 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 1 & -2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & -1 & 2 & -2 & 0 & 0 \\ 2 & 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We parametrize $x_4 = t, x_5 = s$ to get

$$x = \begin{pmatrix} s \\ 2t + s \\ 2t \\ t \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Thus a basis for $E_1(T)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$.

For $\lambda = -1$, we solve $(A+I)x = 0$:

$$
\begin{aligned}
\left(\begin{array}{ccccc|c} 2 & 0 & 1 & -2 & 0 & 0 \\ 3 & 1 & 1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 2 & -2 & 0 \\ 2 & 0 & 0 & 2 & -2 & 0 \\ 2 & 0 & 1 & -2 & 0 & 0 \end{array}\right)
&\to \left(\begin{array}{ccccc|c} 2 & 0 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 2 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & -2 & 1 & -4 & 4 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & -2 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}
$$

Let $x_5 = t$. The general solution is

$$x = t \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So a basis for $E_{-1}(T)$ is $\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(c) Determine if T is diagonalizable, and if so, find a basis β so that $[T]_\beta$ is diagonal.

Since the dimension of each eigenspace matches the algebraic multiplicity of each corresponding eigenvalue, T is diagonalizable and the basis β that makes $[T]_\beta$ diagonal is exactly the basis consisting of the basis vectors of each eigenspace. In particular,

$$\beta = \left\{ e_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Question 3.

(a) Read the proof of Theorem 58 from the additional file in the Week 10 Readings on the course page.

(b) Prove Part 1 of Theorem 59 using a strategy similar to the proof of Theorem 58. (You cannot use other parts of Theorem 59 in this proof.)

Let $A \in M_n(\mathbb{F})$ with $n \geq 2$. If A has a row of 0's, then $\det A = 0$.

Proof. Write $A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$, where r_j represents the rows of A. Suppose that $r_i = \vec{0}$. If $i = 1$, the result is immediate by cofactor expansion. Otherwise, if $i > 1$, we do induction on n. Let $n = 2$. The only possibility is $i = 2$, so denote

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

From here, it is easy to see that $\det A = 0$.
Now, suppose that this is true for some n. We will show that it is true for $n+1$. Define $\tilde{r}_{j,k}$ to be the row obtained by deleting the kth entry of r_j. Using cofactor expansion along the first row, we have

$$\det A = \sum_{k=1}^{n+1} A_{1k} \det \tilde{A}_{1k}$$

Observe that $\tilde{A}_{1k}$ are $n \times n$ matrices, and since the ith row was 0 in the original matrix (and $i > 1$), the $i-1$th row in $\tilde{A}_{1k}$ is 0, so by the induction hypothesis $\det \tilde{A}_{1k} = 0$, thus $\det A = 0$ and we are done.

$\square$

Question 4.

Assume that Parts 1 and 2 of Theorem 59 have been proved. You cannot use Parts 4 through 7 of Theorem 59 in the following problem.

(a) Prove Part 3 using induction on n. (Check $n = 1, 2$ by hand, then in the inductive step assume $n + 1 \geq 3$.)

(b) Prove Part 4 using row-swapping matrices and properties of determinants.

Proof.
(a):
We prove Part 3 using induction on n.
Let $n = 1$. The statement is vacuously true, as A cannot have 2 identical rows.
Let $n = 2$. Then it must be true that

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \text{ for } a, b \in \mathbb{F}.$$

Then $\det A = ab - ab = 0$ as expected.
Now, suppose that this statement is true for some $n \in \mathbb{N}$, where $n > 1$. We will show it also holds for $n+1$. Let r_i, r_j be the identical rows. Since $n + 1 > 2$, we are guaranteed to have one other row r_k that is not r_i or r_j. We perform a row k expansion of $\det A$ and see that

$$\det A = \sum_{l=1}^{n+1} A_{kl} \det \tilde{A}_{kl}$$

Notice that $\tilde{A}_{kl}$ is a $n \times n$ matrix, and contain both r_i and r_j with the lth entry deleted. But these rows are still identical because the same entry got deleted. By the induction hypothesis,

$$\det A = \sum_{l=1}^{n+1} A_{kl} 0 = 0$$

which was what we wanted.

(b):
Suppose that B is obtained from A by swapping row i and row j. Denote these rows as r_i, r_j respectively. Using linearity in one row of the determinant, and the previous result we proved,

$$0 = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i + r_j \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix} = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix}$$

$$\implies 0 = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix}$$

$$\implies 0 = \det A + 0 + \det B + 0$$

$$\implies \det B = -\det A$$

as needed.

□

Question 5.

Prove that if $U \in M_{n\times n}(F)$ is upper triangular, then $\det U = \prod_{i=1}^{n} U_{ii}$.

Proof. We proceed using induction on n. If $n = 1$, the result is immediate. Suppose that the statement holds for some $n \in \mathbb{N}$. We will show the same is the case for $n + 1$.
Let $U \in M_{n+1}(\mathbb{F})$ be upper triangular. We have

$$\det U = \sum_{j=1}^{n+1} U_{1j} \det \tilde{U}_{1j}.$$

For $j \neq 1$, notice that the entries of the first column of $\tilde{U}_{1j}$ are 0, as $(\tilde{U}_{1j})_{i1} = U_{(i+1)1} = 0$. Thus

$$\det \tilde{U}_{1j} = \det \tilde{U}_{1j}^t = 0$$

as the transpose has a row of 0's. The cofacter expansion of $\det U$ reduces to

$$\det U = U_{11} \det \tilde{U}_{11}$$

but $\tilde{U}_{11} \in M_n(\mathbb{F})$ is upper triangular, so

$$\det U = U_{11} \prod_{i=1}^{n} \tilde{U}_{ii} = U_{11} \prod_{i=2}^{n+1} U_{ii} = \prod_{i=1}^{n+1} U_{ii}.$$

which completes the proof.

□

Question 6.

Let V be a vector space over F, and $T : V \to V$ a linear map. If $W \subseteq V$ is a T-invariant subspace, then we can restrict T to W, to obtain a map $T_W : W \to W$. We call T_W the restriction map.

(a) Let β_W be a basis for W. In HW#3 we proved that if $\beta = \beta_W \beta_1$ is an extension of β_W to a basis for V, then $[T]_\beta = \left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right)$. Prove that $A = [T_W]_{\beta_W}$.

(b) Let $M = \left(\begin{array}{c|c} A & B \\ \hline O & C \end{array}\right)$. Prove that $\det M = \det A \det C$.

Proof.
(a):
Let $n = \dim V$, $k = \dim W$. Denote $\beta = \{w_1, ..., w_n\}$. Then the jth column of $\left(\begin{array}{c} A \\ \hline O \end{array}\right)$ is $[T(w_j)]_\beta$, so

$$T(w_j) = \sum_{i=1}^{k} A_{ij} w_i.$$

But since $w_j \in W$, we have

$$T_W(w_j) = T(w_j) = \sum_{i=1}^{k} A_{ij} w_i.$$

which implies that $[T_W(w_j)]_{\beta_W}$ is exactly the jth column of A, from which we can conclude that $[T_W]_{\beta_W} = A$.
(b):
We begin by proving 2 lemmas:
Lemma 1: Let $k < n$. For matrices $B \in M_{k\times(n-k)}(\mathbb{F})$, $C \in M_{(n-k)\times(n-k)}(\mathbb{F})$, Let $M = \left(\begin{array}{c|c} I_k & B \\ \hline O & C \end{array}\right) \in M_n(\mathbb{F})$. Then $\det M = \det C$.
Proceed using induction on k. If $k = 1$, then

$$\det M = \det C + \sum_{j=2}^{n} M_{1j} \det \tilde{M}_{1j}.$$

For $j > 1$, $\tilde{M}_{1j}$ has a column full of 0's, so $\det \tilde{M}_{1j} = 0$ and the result follows.
Now suppose that the lemma is true for some $k \in \mathbb{N}$. Using a similar argument,

$$\det M = \sum_{j=1}^{n} M_{1j} \det \tilde{M}_{1j} = \det \tilde{M}_{11} + \sum_{j=k+2}^{n} M_{1j} \det \tilde{M}_{1j} = \det \tilde{M}_{11}$$

Notice that $\tilde{M}_{11}$ satisfies our assumption in the induction hypthoesis, so $\det M = \det \tilde{M}_{11} = \det C$.

Lemma 2: $\det \left(\begin{array}{c|c} A & O \\ \hline O & I \end{array} \right) = \det A$.
The proof is analogous to the proof of Lemma 1, so we omit it. We now proceed with the main result.
First, consider the case where A is not invertible. Then its columns are linearly dependent. But this means that M also has linearly dependent columns, so $\det M = 0 = \det A \det C$.
Otherwise, if A is invertible, define $N = \left(\begin{array}{c|c} A^{-1} & O \\ \hline O & I \end{array} \right)$. Then $MN = \left(\begin{array}{c|c} I & B \\ \hline O & C \end{array} \right)$ and we get

$$\det C = \det MN = \det M \det N = \det M \det A^{-1} = \frac{\det M}{\det A}$$
$$\implies \det M = \det A \det C$$

and we are done.

$\square$

Question 7.

Deduce from Question 6 that if W is a T-invariant subspace, then C_{T_W} divides C_T.

Question 8.

Let V be a finite-dimensional vector space over a field F, and $W_1, W_2 \subseteq V$ such that $V = W_1 \oplus W_2$. Define the projection maps $P_i : V \to V$ by $P(x) = x_i$ where $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$.

1. Prove that P_i is linear.
2. Prove that $P_i^2 = P_i$.
3. Prove that each W_j is P_i-invariant.
4. Determine if P_i is diagonalizable and justify your answer.

Question 9.

Define the direct sum for more than two subspaces. Let $W_1, \dots, W_k \subseteq V$ be subspaces such that $V = W_1 \oplus \cdots \oplus W_k$.

1. Prove that every basis β for V gives a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_n$ where $\dim W_i = 1$.
2. Prove the converse: If $V = W_1 \oplus \cdots \oplus W_n$ with $\dim W_i = 1$, then choosing non-zero $w_i \in W_i$ forms a basis for V.
3. Let $T : V \to V$ be linear. Show that $[T]_\beta$ is block diagonal.

Question 10.

Let $W_1, \dots, W_k \subseteq V$ with bases $\beta_1, \dots, \beta_k$. Prove that $V = W_1 \oplus \cdots \oplus W_k$ if and only if $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V.

Question 11.

Determine whether the following statements are true or false. Justify your answers.

1. If $V = W_1 \oplus W_2$ and T_{W_1}, T_{W_2} are diagonalizable, then T is diagonalizable.
2. If $W_i \cap W_j = \{0\}$ for $i \neq j$ and $V = W_1 + W_2 + W_3$, then $V = W_1 \oplus W_2 \oplus W_3$.
3. If $\dim V = 7$, $\dim N(T) = 3$, and $\operatorname{rank}(T - I) = 4$, then T is diagonalizable.