6 Week 6? i think

Solvers: Sanchit, Ethan
Writeup: Ethan

Exercise 6.33. Prove that $[0,1]^2$ is homeomorphic to the closed unit ball $\overline{B}(0,1)$ in $\mathbf{R}^2$.

Proof. Now, we continue to the main result. It is pretty easy to see that the closed box $[0,1]^2$ is homeomorphic to $[-1,1]^2$. We define the function $h : [-1,1]^2 \to \overline{B}(0,1)$ by

$$h(x,y) = \begin{cases} \dfrac{\|(x,y)\|_{\max}}{\|(x,y)\|_2}(x,y), & \text{if } (x,y) \neq (0,0); \\ (0,0), & \text{if } (x,y) = (0,0). \end{cases}$$

We can verify that this function is indeed well defined because for $(x,y) \in [-1,1]^2$,

$$\|h(x,y)\|_2 \leq \left\| \frac{\|(x,y)\|_{\max}}{\|(x,y)\|_2}(x,y) \right\|_2 = \frac{\|(x,y)\|_{\max}}{\|(x,y)\|_2} \|(x,y)\|_2 = \|(x,y)\|_{\max} \leq 1$$

which implies that $h(x,y) \in \overline{B}(0,1)$.

We show that this is a homeomorphism by first showing continuity, and then showing that the inverse is continuous.

Fact! Norms are continuous. Therefore h is continuous when $(x,y) \neq 0$. It remains to show continuity at $(0,0)$.

Let $\varepsilon > 0$. By the strong equivalence of norms on $\mathbb{R}^2$, there is an $M > 0$ such that $\dfrac{\|(x,y)\|_{\max}}{\|(x,y)\|_2} \leq M$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = \frac{\varepsilon}{M}$. Let $(x,y) \in \mathbb{R}^2$ such that $\|(x,y)\| < \delta$. Then

$$\|h(x,y)\| = \left\| \frac{\|(x,y)\|_{\max}}{\|(x,y)\|_2}(x,y) \right\| = \frac{\|(x,y)\|_{\max}}{\|(x,y)\|_2} \|(x,y)\| \leq M \|(x,y)\| < \varepsilon$$

Thus h is continuous everywhere. We can explicitly define the inverse h^{-1} as

$$h^{-1}(x,y) = \begin{cases} \dfrac{\|(x,y)\|_2}{\|(x,y)\|_{\max}}(x,y), & \text{if } (x,y) \neq 0; \\ (0,0), & \text{if } (x,y) = (0,0). \end{cases}$$

By a similar argument, we can prove that h^{-1} is continuous. Therefore h is indeed a homeomorphism.

Since $[0,1]^2 \cong [-1,1]^2 \cong \overline{B}(0,1)$, $[0,1]^2 \cong \overline{B}(0,1)$ by transitivity.

$\square$