Question 32.

Let M be a subset of $\mathbf{R}^n$, let $p_0 \in M$ be a point, and let $\vec{v} \in \mathbf{R}^n$ be a vector. We say that $\vec{v}$ is a **tangent vector** to M at p_0 if there exists $\delta > 0$ and a C^1 function $\alpha : (-\delta, \delta) \to M$ such that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. In other words, $\vec{v}$ is the velocity vector of a curve through M.

(a) Suppose now that M is the *zero set* of some C^1 function $f : U \to \mathbf{R}$, where U is an open set in $\mathbf{R}^n$: thus

$$M = \{p \in U : f(p) = 0\}.$$

Suppose that $p_0 \in M$ is a point such that $\nabla f(p_0) \neq \vec{0}$, and let $\vec{v} \in \mathbf{R}^n$ be a vector. Show that $\vec{v}$ is a tangent vector to M at p_0 if and only if $\nabla f(p_0) \cdot \vec{v} = 0$.

(b) Let E be the ellipsoid in $\mathbf{R}^3$ defined by the following equation:

$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$

Find the equation of the tangent plane to M at the point $p_0 = (1, 2, 3)$.

Hint: Define an appropriate function f, then find two vectors which are orthogonal to $\nabla f(p_0)$. By (a), these two vectors span the tangent plane. I recommend using graphing software to confirm your result.

Proof. (a):
Suppose that $\vec{v}$ is a tangent vector to M at p_0. Then there exists a function $\alpha : (-\delta, \delta) \to M$ so that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. Define $g : (-\delta, \delta) \to \mathbb{R}$ by $g(t) = f(\alpha(t))$. For all $t \in (-\delta, \delta)$, $\alpha(t) \in M$, so $g(t) = 0$. It follows that

$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

Substituting $t = 0$ yields

$$\nabla f(p_0) \cdot \vec{v} = 0$$

as needed.

Conversely, suppose that $\nabla f(p_0) \cdot \vec{v} = 0$. Since $\nabla f(p_0) \neq 0$, $\dfrac{\partial f}{\partial x_i}(p_0) \neq 0$ for some $i \in \{0, ..., n\}$. Define the C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ as the function that swaps the ith and nth coordinate. That is,

$$g(x_1, ..., x_n) = f(x_1, ..., x_{i-1}, x_n, x_{i+1}, ..., x_i)$$

Notice that $\dfrac{\partial g}{\partial x_n}(p_0) = \dfrac{\partial f}{\partial x_i}(p_0) \neq 0$. Let $x' = \pi_{\mathbb{R}^{n-1}}(x)$. Applying the Implicit Function Theorem with $k = 1$, there exists an open set W and a function $\psi : W \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$g(x', \psi(x')) = 0$$

Since W is open, there exists $\delta > 0$ so that for all $\|t\| < \delta$,

(b):

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$f(x, y, z) = x^2 + yz + y^2 - xy - xz + z^2 - 3$$

The gradient of f is

$$\nabla f(x, y, z) = (2x - y - z, z + 2y - x, y - x + 2z)$$

and its zero set is exactly defined by the set of solutions of the equation

$$M$$

Substituting p_0 into this gradient, we have

$$\nabla f(1, 2, 3) = (-3, 6, 7)$$

Let $\vec{v}_1 = (2, 1, 0)$, $\vec{v}_2 = (-7, 14, -15)$. Notice that $\nabla f(1, 2, 3) \cdot \vec{v}_1 = \nabla f(1, 2, 3) \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_2 = 0$. By part (a), $\vec{v}_1$ and $\vec{v}_2$ are tangent vectors to

$\square$

Question 33.

(a) Let $g : U \to \mathbf{R}$ be a C^1 function defined on an open set $U \subseteq \mathbf{R}^n$, and let M be its zero set:

$$M = \{p \in U : g(p) = 0\}.$$

Suppose that we have a C^1 function $f : U \to \mathbf{R}$, defined on an open set $U \subseteq \mathbf{R}^n$ which contains M, and we wish to find the maximum of f on M. Assume that M is compact, and that f achieves its maximum on M at some point $p_0 \in M$. Prove that there exists a real number $\lambda \in \mathbf{R}$ such that

$$\nabla f(p_0) = \lambda \nabla g(p_0).$$

This number λ is known as the **Lagrange multiplier**.

(b) Use Lagrange multipliers to solve the following optimization problem: *Find the point(s) on the ellipsoid $x^2 + yz + y^2 - xy - xz + z^2 = 3$ which are* ***closest*** *and* ***furthest*** *from the origin.*