1 Seperablility Stuff

1.1 Theorem. *Let (X, d) be a separable metric space. Then $A \subseteq X$ is separable with respect to the metric d.*

Proof. Since X is separable, it is pre-totally bounded. We will show that a metric subspace A is separable by showing that it is pre-totally bounded.

Let $\varepsilon > 0$. Since X is pre-totally bounded, there exists a finite collection of open balls $\{B(x_i, \frac{\varepsilon}{2})\}_{i \leq n}$ that covers X. We define the desired collection of open balls in A with the following method:

If $B(x_i, \frac{\varepsilon}{2})$ contains some element $a_i \in A$, we add $B_A(a_i, \varepsilon)$ to the collection.

The number of open balls in this collection is at most the number of open balls in the original collection in X, so $\{B_A(a_i, \varepsilon)\}$ is finite.

It remains to show that the collection covers A. Let $a \in A$. Then $a \in B(x_i, \frac{\varepsilon}{2})$ for some i. It follows that $a \in B(x_i, \frac{\varepsilon}{2}) \cap A \subseteq B_A(a_i, \varepsilon)$. Thus A is covered by this collection, so A is pre-totally bounded.

Thus we can conclude that A is separable.

$\square$

1.2 Theorem. *Let X, Y be metric spaces let and $f : C \to X$ be a continuous function. If $A \subseteq X$ is seperable, then $f(A)$ is seperable.*

Proof. Define d_X and d_Y to be metrics on X and Y respectively. Suppose that A is seperable. We will show that $f(A)$ is seperable by equivalently showing that it is pre-totally bounded.

Let $\varepsilon > 0$. By the continuity of f, for every $a \in A$, there exists $\delta > 0$ such that for all $x \in A$, $d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon$. Keep this value of δ.

Since A is seperable, by Theorem y.x (replace this with the thm number idk what it is), A is pre-totally bounded. By definition, A is covered by a countable subcover $\{B_X(a_i, \delta)\}_{i \in \mathbb{N}}$.

Consider the countable collection $\{B_Y(f(a_i), \varepsilon)\}_{i \in \mathbb{N}}$. We will show that this collection covers $f(A)$. Let $y \in f(A)$. Then $y = f(x)$ for some $x \in A \subseteq \{B_X(a_i, \delta)\}_{i \in \mathbb{N}}$. This implies that x is an some open ball $B_X(a_k, \delta) \implies d_X(x, a_k) < \delta$. By the continuity of f, this implies that $d_Y(y, f(a_k)) = d_Y(f(x), f(a_k)) < \varepsilon \implies y \in B_Y(f(a_k), \varepsilon) \subseteq \{B_Y(f(a_i), \varepsilon)\}_{i \in \mathbb{N}}$.

We see that $f(A)$ is pre-totally bounded, which implies by thm 29384qwurjhq that $f(A)$ is seperable.

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