

### Question 1.

Let  $V$  be a vector space over the field  $\mathbb{F}$ , and  $S$  a (non-empty) set. Let  $\mathcal{F}(S, V) = \{f : S \rightarrow V\}$  be the set of  $V$ -valued functions.

We define addition and scaling on  $\mathcal{F}(S, V)$  pointwise:

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

We will verify some of the vector space axioms required to prove that  $\mathcal{F}(S, V)$  is a vector space over  $\mathbb{F}$ .

- (a) Why do these operations make sense?
- (b) Prove (using only the definitions above, and the fact that  $V$  is a vector space) that  $c(f + g) = cf + cg$  for all  $f, g \in \mathcal{F}(S, V)$  and  $c \in \mathbb{F}$ .
- (c) Prove that for all  $f \in \mathcal{F}(S, V)$  there exists  $g \in \mathcal{F}(S, V)$ , so that  $f + g = 0$ . (Here  $0 : S \rightarrow V$  is the constant function defined by  $0(s) = 0_V$  for  $s \in S$ .)

*Proof.*

(a):

These operations make sense because they allow us to use the properties of the sets underlying  $\mathcal{F}(S, V)$ .

(b):

We will do this by showing that for all  $s \in S$ , we have  $c(f(s) + g(s)) = cf(s) + cg(s)$ .

Fix  $s \in S$ . It follows that  $f(s), g(s) \in V$ , so by the axiom of distributivity in  $V$ , we have that

$$c(f(s) + g(s)) = cf(s) + cg(s)$$

(c):

Let  $f \in \mathcal{F}(S, V)$ . Choose  $g = (-1 \cdot f)$ . Then for all  $s \in S$ ,

$$f(s) + g(s) = f(s) + (-f(s)) = 0$$

as needed.

□

Question 2.

$$\text{Let } W = \left\{ (x, y, z, w) \in \mathbb{Q}^4 \left| \begin{array}{l} x + 5w = y + 5z \\ y = 4w - 3z \\ x + y + z = 3w \end{array} \right. \right\}.$$

Do not use Q3 to solve this problem. This problem is a “warm up” for Q3.

- (a) Rearrange the equations defining  $W$  to show that  $W$  is the set of solutions to a homogeneous system of equations.
- (b) Solve the system using row-reduction and express the general solution as a linear combination of the “basic solutions”.
- (c) Show that  $W = \text{span } S$ , for some set  $S \subseteq \mathbb{Q}^4$ .
- (d) Deduce that  $W$  is a subspace of  $\mathbb{Q}^4$ .

*Proof.*

(a):

Rearranging, the equations become

$$\begin{cases} x - y - 5z + 5w = 0 \\ y + 3z - 4w = 0 \\ x + y + z - 3w = 0 \end{cases}$$

(b):

The augmented matrix associated with this system of equations is

$$\left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array} \right)$$

Row reducing this, we get

$$\left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array} \right) \xrightarrow{r_3 \rightarrow r_3 - r_1} \left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 2 & 6 & -8 & 0 \end{array} \right) \xrightarrow[r_3 \rightarrow r_3 - 2r_2]{r_1 \rightarrow r_1 + r_2} \left( \begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

We parameterize  $z$  and  $w$  to obtain that

$$\begin{aligned} x &= 2s - t \\ y &= -3s + 4t \\ z &= s \\ w &= t \end{aligned}$$

so the general solution of this system of equations is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$



### Question 3.

We now generalize Q2. Consider a linear system with  $m$  equations and  $n$  unknowns:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.\end{aligned}$$

We saw in Week 3 that any solution  $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$  can be expressed as  $x = \sum_{i=1}^k t_i x_i$ , where  $t_i \in \mathbb{F}$  are the parameters, and  $x_i \in \mathbb{F}^n$  are the “basic solutions”. Let  $W$  be the set of solutions to this system.

- (a) Prove that  $W = \text{span } S$  for some set  $S$ , and hence that  $W$  is a subspace of  $\mathbb{F}^n$ .
- (b) Prove that the set  $\{x_1, x_2, \dots, x_k\}$  is linearly independent.  
(Hint: Think about the variables which correspond to the choice of parameters. There is exactly one vector for each such parameter. Use the corresponding entry to show that if  $t_1x_1 + t_2x_2 + \cdots + t_kx_k = 0$  then  $t_i = 0$  for each  $i$ .)
- (c) Find a basis for  $W$ .

*Proof.*

(a):

Let  $S = \{x_1, \dots, x_k\}$ . We note that any linear combination of the vectors in  $S$  form a solution to the system, but not only that, any solution to the system can be written as a linear combination of these vectors, so  $\text{span } S = W$ .

(b):

Suppose that  $\sum_{i=1}^k t_i x_i = 0$ .

(c):

From part (a) and part (b), the set  $S = \{x_1, \dots, x_k\}$  spans  $W$  and is linearly independent, which by definition forms a basis of  $W$ .

□

Question 4.

Is the set  $S = \{e_1 + 2e_2 - 3e_3, e_1 + e_2 - e_3, e_2 - e_3\} \subseteq \mathbb{Q}^3$  a basis for  $\mathbb{Q}^3$ ? Justify your answer.

*Proof.* We claim that  $S$  is indeed a basis for  $\mathbb{Q}^3$ . For convenience, denote the vectors in  $S$  by  $v_1, v_2, v_3$  respectively. Notice that

$$e_1 = v_2 - v_3, \quad e_2 = -v_1 + v_2 + 2v_3, \quad e_3 = -v_1 + v_2 + v_3.$$

Thus, for any  $x \in \mathbb{Q}^3$ , since  $\{e_1, e_2, e_3\}$  is a basis for  $\mathbb{Q}^3$ , for some  $a, b, c \in \mathbb{Q}$ , we have that

$$x = ae_1 + be_2 + ce_3 = a(v_2 - v_3) + b(-v_1 + v_2 + 2v_3) + c(-v_1 + v_2 + v_3)$$

$$\implies x = (-b - c)v_1 + (a + b + c)v_2 + (-a + c)v_3$$

which shows that  $S$  spans  $\mathbb{Q}^3$ .

Now, for constants  $p, q, r \in \mathbb{Q}$ , suppose that

$$0 = pv_1 + qv_2 + rv_3$$

Substituting back our values, we get that

$$\begin{aligned} 0 &= p(e_1 + 2e_2 - 3e_3) + q(e_1 + e_2 - e_3) + r(e_2 - e_3) \\ &= (p + q)e_1 + (2p + q + r)e_2 + (-3p - q - r)e_3 \end{aligned}$$

By the linear independence of the standard vectors, we have that

$$\begin{aligned} p + q &= 0 \\ 2p + q + r &= 0 \\ -3p - q - r &= 0 \end{aligned}$$

We can solve for  $p, q, r$  to get that  $p = q = r = 0$ .

Thus we can conclude that  $S$  is a basis for  $\mathbb{Q}^3$ .

□

Question 5.

Let  $V$  be a finite dimensional vector space over a field  $\mathbb{F}$ .

1. Prove that if  $W \subseteq V$  is a subspace with basis  $\beta_W$ , then there exists a linearly independent set  $\alpha$  so that  $\beta = \beta_W \cup \alpha$  is a basis for  $V$ . (We say that  $\beta$  “extends”  $\beta_W$ . So you are proving that “every basis of a subspace  $W$  can be extended to a basis of  $V$ ”.)
2. Prove that for any linearly independent set  $I$  and spanning set  $S$ , we have  $|I| \leq \dim V \leq |S|$ .

Question 6.

Consider a matrix  $M \in \mathcal{M}_{n \times n}(\mathbb{F})$ . Given  $p \in \{1, \dots, n\}$  we can split  $M$  into “blocks”:

$$M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

where  $A$  is  $k \times k$ ,  $B$  is  $k \times (n - k)$ ,  $C$  is  $(n - k) \times k$  and  $D$  is  $(n - k) \times (n - k)$ . For example, if  $n = 5$  and  $k = 2$ , then such a block matrix would be of the form

$$M = \left( \begin{array}{cc|ccc} 1 & 2 & 3 & 2 & 3 \\ -5 & 3 & 3 & 1 & 1 \\ \hline 1 & 2 & 0 & -1 & 1 \\ 3 & 1 & 3 & -1 & 7 \\ 1 & 0 & -1 & 3 & 5 \end{array} \right)$$

where  $A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $D = \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 7 \\ -1 & 3 & 5 \end{pmatrix}$ .

Prove that if  $M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$  and  $N = \left( \begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right)$ , then

$$\alpha M + N = \left( \begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array} \right)$$

Question 7.

Let  $W = \left\{ A \in \mathcal{M}_{2n \times 2n}(\mathbb{F}) \mid A = \left( \frac{X-X^t}{O_n} \middle| \frac{O_n}{X+X^t} \right) \text{ with } X \in \mathcal{M}_{n \times n}(\mathbb{F}) \right\}$ .  
(Assume  $\text{char}(\mathbb{F}) \neq 2$ .)

1. Let  $n = 2$ . Find a basis for  $W$ .
2. Now generalize to arbitrary  $n$ . Find a basis for  $W$ , and use it to compute  $\dim W$ .



Question 8.

- (a) Prove that if  $W_1, W_2 \subseteq V$  are subspaces, then  $W_1 + W_2$  is a subspace.
- (b) Let  $W_1 = \{(x, y, x + y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$ . Find two subspaces  $W_2, W_3$  so that:
- $W_1 + W_2 = \mathbb{F}^3$  but  $\mathbb{F}^3 \neq W_1 \oplus W_2$ .
  - $W_1 \oplus W_3 = \mathbb{F}^3$ .
- (c) Find another subspace  $U \subseteq \mathbb{F}^3$  so that  $W_1 \oplus U = \mathbb{F}^3$ .

*Proof.*

(a):

Let  $W_1, W_2$  be subspaces of  $V$ . We verify that  $W_1 + W_2$  is also a subspace of  $V$ .

First, note that  $0 \in W_1, W_2$ , so  $0 + 0 = 0 \in W_1 + W_2$ . Next, let  $c \in \mathbb{F}$ ,  $\vec{v}, \vec{w} \in W_1 + W_2$ . Then  $\vec{v} = \vec{v}_1 + \vec{v}_2$  and  $\vec{w} = \vec{w}_1 + \vec{w}_2$ , for some  $\vec{v}_1, \vec{w}_1 \in W_1$  and  $\vec{v}_2, \vec{w}_2 \in W_2$ . Since  $W_1, W_2$  are subspaces, it is true that

$$c\vec{v}_1 + \vec{w}_1 \in W_1 \text{ and } c\vec{v}_2 + \vec{w}_2 \in W_2$$

which implies that

$$\vec{v} + \vec{w} = (c\vec{v}_1 + \vec{w}_1) + (c\vec{v}_2 + \vec{w}_2) \in W_1 + W_2,$$

verifying that  $W_1 + W_2$  is indeed a subspace.

(b):

Let  $W_2 = \mathbb{F}^3$ ,  $W_3 = \text{span}\{e_3\}$ .

□

Question 9.

Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ , and  $W_1, W_2 \subseteq V$  subspaces with bases  $\beta_1, \beta_2$  respectively. Prove that  $V = W_1 \oplus W_2$  if and only if  $\beta = \beta_1 \cup \beta_2$  is a basis for  $V$ .

*Proof.* Let  $n = \dim V$ ,  $m = |\beta_1|$ ,  $k = |\beta_2|$ .

Suppose that  $V = W_1 \oplus W_2$ . We will show that  $\beta = \beta_1 \cup \beta_2$  is a basis for  $V$ .

Let  $x \in V$ . By our assumption,  $x = w_1 + w_2$ , for some  $w_1 \in W_1$  and  $w_2 \in W_2$ . These vectors can in turn be written as

$$w_1 = \sum_{i=1}^m a_i v_i \text{ and } w_2 = \sum_{i=1}^k b_i w_i$$

where  $v_i \in \beta_1$  and  $w_i \in \beta_2$ . Thus  $x$  can be written as a linear combination of vectors in  $\beta$ :

$$x = \sum_{i=1}^m a_i v_i + \sum_{i=1}^k b_i w_i$$

so  $\beta$  spans  $V$ .

To show that  $\beta$  is linearly independent, suppose that

$$\sum_{i=1}^m a_i v_i + \sum_{i=1}^k b_i w_i = 0$$

We put the vectors of each subspace on each side to get

$$\sum_{i=1}^m a_i v_i = - \sum_{i=1}^k b_i w_i$$

By the closure property of subspaces,  $\sum_{i=1}^m a_i v_i \in W_1$  and  $\sum_{i=1}^k b_i w_i \in W_2$ , but since they are equal, it must be true that  $\sum_{i=1}^m a_i v_i = \sum_{i=1}^k b_i w_i \in W_1 \cap W_2 = \{0\}$ , so  $\sum_{i=1}^m a_i v_i = \sum_{i=1}^k b_i w_i = 0$ . Since  $\beta_1, \beta_2$  are linearly independent, it must be true that  $a_i = 0$  and  $b_i = 0$ , which was what we wanted to show. Therefore  $\beta$  is indeed a basis for  $V$ .

Conversely, suppose that  $\beta$  is a basis for  $V$ .

□

Question 10.

Let  $J = \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right)$  and  $\mathbb{F} = \mathbb{C}$ .

1. Verify that  $J^2 = -I_4$ .
2. Find all  $X \in \mathcal{M}_{4 \times 4}(\mathbb{F})$  so that  $XJ = JX$ .
3. Show that  $sp_4 = \{X \in \mathcal{M}_{4 \times 4}(\mathbb{F}) | XJ = JX\}$  is a subspace of  $\mathcal{M}_{4 \times 4}(\mathbb{F})$ .
4. Find  $\dim sp_4$  by finding a basis for  $sp_4$ .

Question 11.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

- (a) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $I \subseteq V$  is a linearly independent set so that for any  $x \in V \setminus I$ , the set  $I \cup \{x\}$  is linearly dependent, then  $I$  is a basis for  $V$ .
- (b) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $S \subseteq V$  is a spanning set so that  $|S| = \dim V$ , then  $S$  is a basis for  $V$ .
- (c) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $W \subseteq V$  a subspace, then there exists a unique subspace  $U \subseteq V$  so that  $V = W \oplus U$ .