

### Question 1.

Let  $V$  be a vector space over the field  $\mathbb{F}$ , and  $S$  a (non-empty) set. Let  $\mathcal{F}(S, V) = \{f : S \rightarrow V\}$  be the set of  $V$ -valued functions.

We define addition and scaling on  $\mathcal{F}(S, V)$  pointwise:

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

We will verify some of the vector space axioms required to prove that  $\mathcal{F}(S, V)$  is a vector space over  $\mathbb{F}$ .

- (a) Why do these operations make sense?
- (b) Prove (using only the definitions above, and the fact that  $V$  is a vector space) that  $c(f + g) = cf + cg$  for all  $f, g \in \mathcal{F}(S, V)$  and  $c \in \mathbb{F}$ .
- (c) Prove that for all  $f \in \mathcal{F}(S, V)$  there exists  $g \in \mathcal{F}(S, V)$ , so that  $f + g = 0$ . (Here  $0 : S \rightarrow V$  is the constant function defined by  $0(s) = 0_V$  for  $s \in S$ .)

*Proof.*

(a):

To show that these operations are well defined, notice that  $f(s), g(s) \in V$ , so by the axiom of closure on  $V$  we have that

$$(f + g)(s) = f(s) + g(s) \in V \text{ and } (cf)(s) = cf(s) \in V$$

(b):

We will do this by showing that for all  $s \in S$ , we have  $c(f(s) + g(s)) = cf(s) + cg(s)$ .

Fix  $s \in S$ . It follows that  $f(s), g(s) \in V$ , so by the axiom of distributivity in  $V$ , we have that

$$c(f(s) + g(s)) = cf(s) + cg(s)$$

(c):

Let  $f \in \mathcal{F}(S, V)$ . Choose  $g = (-1 \cdot f)$ . Then for all  $s \in S$ ,

$$f(s) + g(s) = f(s) + (-f(s)) = 0$$

as needed.

□

Question 2.

$$\text{Let } W = \left\{ (x, y, z, w) \in \mathbb{Q}^4 \left| \begin{array}{l} x + 5w = y + 5z \\ y = 4w - 3z \\ x + y + z = 3w \end{array} \right. \right\}.$$

Do not use Q3 to solve this problem. This problem is a “warm up” for Q3.

- (a) Rearrange the equations defining  $W$  to show that  $W$  is the set of solutions to a homogeneous system of equations.
- (b) Solve the system using row-reduction and express the general solution as a linear combination of the “basic solutions”.
- (c) Show that  $W = \text{span } S$ , for some set  $S \subseteq \mathbb{Q}^4$ .
- (d) Deduce that  $W$  is a subspace of  $\mathbb{Q}^4$ .

*Proof.*

(a):

Rearranging, the equations become

$$\begin{cases} x - y - 5z + 5w = 0 \\ y + 3z - 4w = 0 \\ x + y + z - 3w = 0 \end{cases}$$

(b):

The augmented matrix associated with this system of equations is

$$\left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array} \right)$$

Row reducing this, we get

$$\left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array} \right) \xrightarrow{r_3 \rightarrow r_3 - r_1} \left( \begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 2 & 6 & -8 & 0 \end{array} \right) \xrightarrow[r_3 \rightarrow r_3 - 2r_2]{r_1 \rightarrow r_1 + r_2} \left( \begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

We parameterize  $z$  and  $w$  to obtain that

$$\begin{aligned} x &= 2s - t \\ y &= -3s + 4t \\ z &= s \\ w &= t \end{aligned}$$

so the general solution of this system of equations is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$



### Question 3.

We now generalize Q2. Consider a linear system with  $m$  equations and  $n$  unknowns:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0.\end{aligned}$$

We saw in Week 3 that any solution  $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$  can be expressed as  $x = \sum_{i=1}^k t_i x_i$ , where  $t_i \in \mathbb{F}$  are the parameters, and  $x_i \in \mathbb{F}^n$  are the “basic solutions”. Let  $W$  be the set of solutions to this system.

- (a) Prove that  $W = \text{span } S$  for some set  $S$ , and hence that  $W$  is a subspace of  $\mathbb{F}^n$ .
- (b) Prove that the set  $\{x_1, x_2, \dots, x_k\}$  is linearly independent.  
(Hint: Think about the variables which correspond to the choice of parameters. There is exactly one vector for each such parameter. Use the corresponding entry to show that if  $t_1x_1 + t_2x_2 + \cdots + t_kx_k = 0$  then  $t_i = 0$  for each  $i$ .)
- (c) Find a basis for  $W$ .

*Proof.*

(a):

Let  $S = \{x_1, \dots, x_k\}$ . We note that any linear combination of the vectors in  $S$  form a solution to the system, but not only that, any solution to the system can be written as a linear combination of these vectors, so  $\text{span } S = W$ .

(b):

Suppose that  $\sum_{i=1}^k t_i x_i = 0$ .

(c):

From part (a) and part (b), the set  $S = \{x_1, \dots, x_k\}$  spans  $W$  and is linearly independent, which by definition forms a basis of  $W$ .

□

Question 4.

Is the set  $S = \{e_1 + 2e_2 - 3e_3, e_1 + e_2 - e_3, e_2 - e_3\} \subseteq \mathbb{Q}^3$  a basis for  $\mathbb{Q}^3$ ? Justify your answer.

*Proof.* We claim that  $S$  is indeed a basis for  $\mathbb{Q}^3$ . For convenience, denote the vectors in  $S$  by  $v_1, v_2, v_3$  respectively. Notice that

$$e_1 = v_2 - v_3, \quad e_2 = -v_1 + v_2 + 2v_3, \quad e_3 = -v_1 + v_2 + v_3.$$

Thus, for any  $x \in \mathbb{Q}^3$ , since  $\{e_1, e_2, e_3\}$  is a basis for  $\mathbb{Q}^3$ , for some  $a, b, c \in \mathbb{Q}$ , we have that

$$x = ae_1 + be_2 + ce_3 = a(v_2 - v_3) + b(-v_1 + v_2 + 2v_3) + c(-v_1 + v_2 + v_3)$$

$$\implies x = (-b - c)v_1 + (a + b + c)v_2 + (-a + c)v_3$$

which shows that  $S$  spans  $\mathbb{Q}^3$ .

Now, for constants  $p, q, r \in \mathbb{Q}$ , suppose that

$$0 = pv_1 + qv_2 + rv_3$$

Substituting back our values, we get that

$$\begin{aligned} 0 &= p(e_1 + 2e_2 - 3e_3) + q(e_1 + e_2 - e_3) + r(e_2 - e_3) \\ &= (p + q)e_1 + (2p + q + r)e_2 + (-3p - q - r)e_3 \end{aligned}$$

By the linear independence of the standard vectors, we have that

$$\begin{aligned} p + q &= 0 \\ 2p + q + r &= 0 \\ -3p - q - r &= 0 \end{aligned}$$

We can solve for  $p, q, r$  to get that  $p = q = r = 0$ .

Thus we can conclude that  $S$  is a basis for  $\mathbb{Q}^3$ .

□

Question 5.

Let  $V$  be a finite dimensional vector space over a field  $\mathbb{F}$ .

- (a) Prove that if  $W \subseteq V$  is a subspace with basis  $\beta_W$ , then there exists a linearly independent set  $\alpha$  so that  $\beta = \beta_W \cup \alpha$  is a basis for  $V$ . (We say that  $\beta$  “extends”  $\beta_W$ . So you are proving that “every basis of a subspace  $W$  can be extended to a basis of  $V$ ”.)
- (b) Prove that for any linearly independent set  $I$  and spanning set  $S$ , we have  $|I| \leq \dim V \leq |S|$ .

*Proof.*

(a):

Let  $\gamma$  be a basis for  $V$ . Since  $\beta_W$  is linearly independent, we apply the Replacement Theorem to get that there exists a subset  $\alpha \subseteq \gamma$  such that  $\beta_W \cup \alpha$  is a basis for  $V$ , and we are done.

(b):

Let  $I$  be a linearly independent set. Then  $W = \text{span}(I)$  is a subspace of  $V$  with basis  $I$ . By part (a), we can extend  $I$  to a basis  $\beta$  of  $V$ , where  $\beta = I \cup \alpha$ . Since  $|\alpha| \geq 0$ , we have that

$$\dim V = |\beta| = |I| + |\alpha| \geq |I|$$

Now, let  $S$  be a spanning set of  $V$ . If  $\dim V = 0$ , then  $V = \{0\}$  and its basis is  $\beta = \emptyset$ .  $S$  must be either  $\emptyset$  or  $\{0\}$ , so  $\dim V \leq |S|$ .

If  $\dim V > 0$ , it contains a non-zero vector, so  $S$  also contains a non-zero vector. Pick  $s_0 \in S$ , and note that  $S_0 = \{s_0\}$  is linearly independent.

If there are no elements  $w_i$  in  $S$  such that  $\{s_1, w_i\}$  is linearly independent, that is,  $s = c_i w_i$  for some  $c_i \in \mathbb{F}$ , then for all  $v \in V$ , because  $S$  is a spanning set, for  $m$  vectors in  $S$  we have that

$$v = \sum_{i=1}^m a_i w_i = \sum_{i=1}^m a_i \cdot c_i s_0$$

which implies that  $\text{span}(\{s_0\}) = V$ , and the result that we want follows immediately after.

If not, we can find another non-zero vector  $s_1$  so that  $S_1 = \{s_0, s_1\}$  is linearly independent.

We repeat this process until we have a linearly independent set  $S_n = \{s_0, \dots, s_n\}$ , where  $n = \dim V$ . We claim that  $S_n$  is a basis for  $V$ , and it suffices to show that  $S_n$  spans  $V$ . First, if  $S_n = S$ , then our result is immediate. Otherwise, let  $s_j \in S$  so that  $s_j \notin S_n$ . Consider the set  $S_n \cup \{s_j\}$ , whose number of elements is greater than  $\dim V$ . By the first half of this proof, we know that no linearly independent set can have a size larger than  $\dim V$ , so it must be true that  $S_n \cup \{s_j\}$  is linearly dependent. In particular, for constants  $a_j, a_{ij} \in \mathbb{F}$  not all zero,

$$0 = a_j s + \sum_{i=1}^n a_{ij} s_i.$$

Notice that  $a_j \neq 0$ , for if not, then we get that

$$0 = \sum_{i=1}^n a_{ij} s_i$$



Question 6.

Consider a matrix  $M \in \mathcal{M}_{n \times n}(\mathbb{F})$ . Given  $p \in \{1, \dots, n\}$  we can split  $M$  into “blocks”:

$$M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

where  $A$  is  $k \times k$ ,  $B$  is  $k \times (n - k)$ ,  $C$  is  $(n - k) \times k$  and  $D$  is  $(n - k) \times (n - k)$ . For example, if  $n = 5$  and  $k = 2$ , then such a block matrix would be of the form

$$M = \left( \begin{array}{cc|ccc} 1 & 2 & 3 & 2 & 3 \\ -5 & 3 & 3 & 1 & 1 \\ \hline 1 & 2 & 0 & -1 & 1 \\ 3 & 1 & 3 & -1 & 7 \\ 1 & 0 & -1 & 3 & 5 \end{array} \right)$$

where  $A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $D = \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 7 \\ -1 & 3 & 5 \end{pmatrix}$ .

Prove that if  $M = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$  and  $N = \left( \begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right)$ , then

$$\alpha M + N = \left( \begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array} \right)$$

*Proof.*

□



Question 7.

Let  $W = \left\{ A \in \mathcal{M}_{2n \times 2n}(\mathbb{F}) \mid A = \left( \begin{array}{c|c} \frac{X - X^T}{0_n} & \frac{0_n}{X + X^T} \end{array} \right) \text{ with } X \in \mathcal{M}_{n \times n}(\mathbb{F}) \right\}$ .

(Assume  $\text{char}(\mathbb{F}) \neq 2$ .)

(a) Let  $n = 2$ . Find a basis for  $W$ .

(b) Now generalize to arbitrary  $n$ . Find a basis for  $W$ , and use it to compute  $\dim W$ .

*Proof.*

(a):

Let  $n = 2$ . We claim that a basis for  $W$  is given by

$$\beta = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

For convenience, we label these matrices in order as  $f_1, f_2, f_3$ , and  $f_4$ . It is pretty clear that  $\beta$  is linear independent. To show that  $\beta$  is spanning, let  $A \in W$ . Then

$$A = \left( \begin{array}{c|c} \frac{X - X^T}{0_n} & \frac{0_n}{X + X^T} \end{array} \right), \text{ for some } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{n \times n}(\mathbb{F}).$$

We substitute  $a, b, c, d$  to get

$$A = \begin{pmatrix} 0 & b - c & 0 & 0 \\ c - b & 0 & 0 & 0 \\ 0 & 0 & 2a & b + c \\ 0 & 0 & b + c & 2d \end{pmatrix} = 2af_1 + bf_2 + cf_3 + 2df_4$$

so we can conclude that  $W$  is spanned by  $\beta$ .

(b):

Let  $n \in \mathbb{N}$ . We define  $e_{ij}$  to be the  $n \times n$  matrix with all its entries equal to 0 except the entry at the  $i$ th row and  $j$ th column. We claim that

$$\beta = \left\{ \left( \begin{array}{c|c} \frac{e_{ij} - e_{ij}^T}{0_n} & \frac{0_n}{e_{ij} - e_{ij}^T} \end{array} \right), 1 \leq i, j \leq n \right\}$$

is a basis for  $W$ .

Similarly to the previous part, define each matrix associated with  $e_{ij}$  by  $f_{ij}$ . Let  $A \in W$ , so

$$A = \left( \begin{array}{c|c} \frac{X - X^T}{0_n} & \frac{0_n}{X + X^T} \end{array} \right), \text{ for an } n \times n \text{ matrix } X.$$

We can write  $X$  in terms of the basis for  $\mathcal{M}_{n \times n}(\mathbb{F})$  to obtain that

$$X = \sum_{i,j=1}^n c_{ij} e_{ij}.$$



Question 8.

- (a) Prove that if  $W_1, W_2 \subseteq V$  are subspaces, then  $W_1 + W_2$  is a subspace.
- (b) Let  $W_1 = \{(x, y, x + y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$ . Find two subspaces  $W_2, W_3$  so that:
- $W_1 + W_2 = \mathbb{F}^3$  but  $\mathbb{F}^3 \neq W_1 \oplus W_2$ .
  - $W_1 \oplus W_3 = \mathbb{F}^3$ .
- (c) Find another subspace  $U \subseteq \mathbb{F}^3$  so that  $W_1 \oplus U = \mathbb{F}^3$ .

*Proof.*

(a):

Let  $W_1, W_2$  be subspaces of  $V$ . We verify that  $W_1 + W_2$  is also a subspace of  $V$ . First, note that  $0 \in W_1, W_2$ , so  $0 + 0 = 0 \in W_1 + W_2$ . Next, let  $c \in \mathbb{F}$ ,  $\vec{v}, \vec{w} \in W_1 + W_2$ . Then  $\vec{v} = \vec{v}_1 + \vec{v}_2$  and  $\vec{w} = \vec{w}_1 + \vec{w}_2$ , for some  $\vec{v}_1, \vec{w}_1 \in W_1$  and  $\vec{v}_2, \vec{w}_2 \in W_2$ . Since  $W_1, W_2$  are subspaces, it is true that

$$c\vec{v}_1 + \vec{w}_1 \in W_1 \text{ and } c\vec{v}_2 + \vec{w}_2 \in W_2$$

which implies that

$$\vec{v} + \vec{w} = (c\vec{v}_1 + \vec{w}_1) + (c\vec{v}_2 + \vec{w}_2) \in W_1 + W_2,$$

verifying that  $W_1 + W_2$  is indeed a subspace.

(b):

Let  $W_2 = \mathbb{F}^3$ ,  $W_3 = \text{span}\{e_3\}$ . We start by showing that  $\mathbb{F}^3 = W_1 + W_2$ . The backward direction is instant so we will only show that  $\mathbb{F}^3 \subseteq W_1 + W_2$ .

Let  $x \in \mathbb{F}^3$ . We know that  $x$  is the same as  $0 + x$ , and  $0 \in W_1$  and  $x \in W_2$ , so  $x \in W_1 + W_2$ . However,  $W_1 \subseteq W_2$ , so  $W_1 \cap W_2 = W_1 \neq \{0\}$ , so  $\mathbb{F}^3$  is not a direct sum of  $W_1$  and  $W_2$ .

Now, we will show that  $W_1 \oplus W_3 = \mathbb{F}^3$ . Again, the fact that  $W_1 + W_3 \subseteq \mathbb{F}^3$  is obvious. To show that  $\mathbb{F}^3 \subseteq W_1 + W_3$ , let  $(x, y, z) \in \mathbb{F}^3$ . Notice that

$$(x, y, z) = (x, y, x + y) + (0, 0, z - x - y)$$

and

$$(x, y, x + y) \in W_1 \text{ and } (0, 0, z - x - y) \in W_3,$$

so  $(x, y, z) \in W_1 + W_3$ .

Now let  $(a, b, c) \in W_1 \cap W_3$ . Then we have that  $a = b = 0$ , but this implies that  $c = a + b = 0$ , so  $(a, b, c) = 0$ . Thus  $W_1 \cap W_3 = \{0\}$  and we can conclude that  $\mathbb{F}^3 = W_1 \oplus W_3$ .

(c):

Let  $U = \text{span}\{(0, 1, 2)\}$ . We follow the same structure as before in part (b).

Let  $(x, y, z) \in \mathbb{F}^3$ . We rewrite

$$(x, y, z) = (x, x + 2y - z, 2x + 2y - z) + (0, -x - y + z, 2(-x - y + z))$$

and note that  $(x, x + 2y - z, 2x + 2y - z) \in W_1$  and  $(0, -x - y + z, 2(-x - y + z)) \in W_2$ , so  $(x, y, z) \in W_1 + W_2$ .



Question 9.

Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ , and  $W_1, W_2 \subseteq V$  subspaces with mutually disjoint bases  $\beta_1, \beta_2$  respectively. Prove that  $V = W_1 \oplus W_2$  if and only if  $\beta = \beta_1 \cup \beta_2$  is a basis for  $V$ .

*Proof.* Let  $m = |\beta_1|$ ,  $k = |\beta_2|$ .

Suppose that  $V = W_1 \oplus W_2$ . We will show that  $\beta = \beta_1 \cup \beta_2$  is a basis for  $V$ .

Let  $x \in V$ . By our assumption,  $x = w_1 + w_2$ , for some  $w_1 \in W_1$  and  $w_2 \in W_2$ . These vectors can in turn be written as

$$w_1 = \sum_{i=1}^m a_i v_i \text{ and } w_2 = \sum_{i=1}^k b_i w_i$$

where  $v_i \in \beta_1$  and  $w_i \in \beta_2$ . Thus  $x$  can be written as a linear combination of vectors in  $\beta$ :

$$x = \sum_{i=1}^m a_i v_i + \sum_{i=1}^k b_i w_i$$

so  $\beta$  spans  $V$ .

To show that  $\beta$  is linearly independent, suppose that

$$\sum_{i=1}^m a_i v_i + \sum_{i=1}^k b_i w_i = 0$$

We put the vectors of each subspace on each side to get

$$\sum_{i=1}^m a_i v_i = - \sum_{i=1}^k b_i w_i$$

By the closure property of subspaces,  $\sum_{i=1}^m a_i v_i \in W_1$  and  $\sum_{i=1}^k b_i w_i \in W_2$ , but since they are equal, it must be true that  $\sum_{i=1}^m a_i v_i = \sum_{i=1}^k b_i w_i \in W_1 \cap W_2 = \{0\}$ , so  $\sum_{i=1}^m a_i v_i = \sum_{i=1}^k b_i w_i = 0$ . Since  $\beta_1, \beta_2$  are linearly independent, it must be true that  $a_i = 0$  and  $b_i = 0$ , which was what we wanted to show. Therefore  $\beta$  is indeed a basis for  $V$ .

Conversely, suppose that  $\beta$  is a basis for  $V$ . We want to show that  $V = W_1 \oplus W_2$ . It is obvious that  $W_1 + W_2 \subseteq V$ , so it suffices to prove that  $V \subseteq W_1 \oplus W_2$  and  $W_1 \cap W_2 = \{0\}$ .

Let  $x \in V$ . Then since  $\beta$  is a basis, we have that

$$x = \sum_{j=1}^m a_j v_j + \sum_{j=1}^k b_j w_j, \text{ for } a_j, b_j \in \mathbb{F}, v_j \in \beta_1, \text{ and } w_j \in \beta_2.$$

By closure, we have that  $\sum_{j=1}^m a_j v_j \in W_1$  and  $\sum_{j=1}^k b_j w_j \in W_2$ , so we see that  $x \in W_1 + W_2$ . Thus  $V = W_1 + W_2$ .

To show that  $W_1 \cap W_2 = \{0\}$ , it suffices to show that if  $x \in W_1 \cap W_2$ , then it must be true that  $x = 0$ . Indeed, if  $x \in W_1 \cap W_2$ , we can write it as a two linear combinations of vectors in either  $\beta_1$  or  $\beta_2$ :

$$x = \sum_{i=1}^m a_i v_i = \sum_{i=1}^k b_i w_i$$



Question 10.

Let  $J = \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right)$  and  $\mathbb{F} = \mathbb{C}$ .

- (a) Verify that  $J^2 = -I_4$ .
- (b) Find all  $X \in \mathcal{M}_{4 \times 4}(\mathbb{F})$  so that  $XJ = JX$ .
- (c) Show that  $\mathfrak{sp}_4 = \{X \in \mathcal{M}_{4 \times 4}(\mathbb{F}) \mid XJ = JX\}$  is a subspace of  $\mathcal{M}_{4 \times 4}(\mathbb{F})$ .
- (d) Find  $\dim \mathfrak{sp}_4$  by finding a basis for  $\mathfrak{sp}_4$ .

*Proof.*

(a):

Indeed, we have that

$$J^2 = \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right) \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right) = \left( \begin{array}{c|c} O^2 + -I_2^2 & O(-I_2) + -I_2O \\ \hline OI_2 + I_2O & -I_2^2 + O^2 \end{array} \right) = \left( \begin{array}{c|c} -I_2 & 0 \\ \hline 0 & -I_2 \end{array} \right) = -I_4$$

(b):

For  $A, B, C, D \in \mathcal{M}_2(\mathbb{F})$ , let  $X = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \in \mathcal{M}_4(\mathbb{F})$  and suppose that  $XJ = JX$ . Then we have that

$$\begin{aligned} & \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right) = \left( \begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array} \right) \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \\ \implies & \left( \begin{array}{c|c} AO + BI_2 & A(-I_2) + BO \\ \hline CO + DI_2 & C(-I_2) \end{array} \right) = \left( \begin{array}{c|c} OA + -I_2C & OB - I_2D \\ \hline I_2A + OC & I_2B + OD \end{array} \right) \\ \implies & \left( \begin{array}{c|c} B & -A \\ \hline D & -C \end{array} \right) = \left( \begin{array}{c|c} -C & -D \\ \hline A & B \end{array} \right) \\ \implies & B = -C \text{ and } A = -D \end{aligned}$$

Therefore all  $X$  that satisfy this equation are of the form

$$X = \left( \begin{array}{c|c} P & -Q \\ \hline -Q & P \end{array} \right), \text{ where } P, Q \in M_2(\mathbb{F}).$$

(c):

Notice that  $X \in \mathfrak{sp}_4$  if and only if  $X$  can be decomposed into a block matrix such that

$$X = \left( \begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array} \right)$$

for some matrices  $P, Q \in M_2(\mathbb{F})$ .

We know that  $0 \in \mathfrak{sp}_4$  because as a  $4 \times 4$  matrix,  $0 = \left( \begin{array}{c|c} O & O \\ \hline O & O \end{array} \right)$ , which clearly satisfies the condition outlined above.





Question 11.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

- (a) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $I \subseteq V$  is a linearly independent set so that for any  $x \in V \setminus I$ , the set  $I \cup \{x\}$  is linearly dependent, then  $I$  is a basis for  $V$ .
- (b) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $S \subseteq V$  is a spanning set so that  $|S| = \dim V$ , then  $S$  is a basis for  $V$ .
- (c) Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . If  $W \subseteq V$  a subspace, then there exists a unique subspace  $U \subseteq V$  so that  $V = W \oplus U$ .

*Proof.*

(a):

Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . Suppose  $I \subseteq V$  is linearly independent and that adding any vector in  $V \setminus I$  will result in the set no longer being linearly independent, and note that the same also applies when choosing a vector that is in  $I$ . Then for any  $x \in V$ , we have that for some vectors  $v_1, \dots, v_n \in I$ ,

$$cx + \sum_{i=1}^n c_i v_i = 0$$

for  $c, c_i \in \mathbb{F}$  not all zero. We make the important note that it is necessary for  $c \neq 0$ , because if not, then

$$\sum_{i=1}^n c_i v_i = 0$$

which implies that all coefficients are zero by independence, which contradicts our claim that not all coefficients were zero. It follows that  $c$  has an inverse  $c^{-1}$  and

$$x = \sum_{i=1}^n -c^{-1} c_i v_i.$$

Since every  $x \in V$  is a linear combination of vectors in  $I$ , it follows that  $I$  is indeed a basis for  $V$ .

(b):

Suppose for contradiction that  $S$  is not a basis for  $V$ . Then  $S$  is not linearly independent, that is, for some  $s \in S$ ,  $c_i \in \mathbb{F}$ ,  $s_i \in S \setminus \{s\}$ ,

$$s = \sum_{i=1}^n c_i s_i$$

This means that  $S \setminus \{s\}$  is also a spanning set. But  $|S \setminus \{s\}| < \dim V$ , which is a contradiction by Question 5. Thus  $S$  is a basis for  $V$ .

