# Question 32

Let M be a subset of  $\mathbb{R}^n$ , let  $p_0 \in M$  be a point, and let  $\vec{v} \in \mathbb{R}^n$  be a vector. We say that  $\vec{v}$  is a **tangent vector** to M at  $p_0$  if there exists  $\delta > 0$  and a  $C^1$  function  $\alpha : (-\delta, \delta) \to M$  such that  $\alpha(0) = p_0$  and  $\alpha'(0) = \vec{v}$ . In other words,  $\vec{v}$  is the velocity vector of a curve through M.

(a) Suppose now that M is the zero set of some  $C^1$  function  $f: U \to \mathbf{R}$ , where U is an open set in  $\mathbf{R}^n$ : thus

$$M = \{ p \in U : f(p) = 0 \}.$$

Suppose that  $p_0 \in M$  is a point such that  $\nabla f(p_0) \neq \vec{0}$ , and let  $\vec{v} \in \mathbf{R}^n$  be a vector Show that  $\vec{v}$  is a tangent vector to M at  $p_0$  if and only if  $\nabla f(p_0) \cdot \vec{v} = 0$ .

(b) Let E be the ellipsoid in  $\mathbb{R}^3$  defined by the following equation:

$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$

Find the equation of the tangent plane to M at the point  $p_0 = (1, 2, 0)$ .

Hint: Define an appropriate function f, then find two vectors which are orthogonal to  $\nabla f(p_0)$ . By (a), these two vectors span the tangent plane. I recommend using graphing software to confirm your result.

# Proof. (a):

Suppose that  $\vec{v}$  is a tangent vector to M at  $p_0$ . Then there exists a function  $\alpha: (-\delta, \delta) \to M$  so that  $\alpha(0) = p_0$  and  $\alpha'(0) = \vec{v}$ . Define  $g: (-\delta, \delta) \to \mathbb{R}$  by  $g(t) = f(\alpha(t))$ . For all  $t \in (-\delta, \delta)$ ,  $\alpha(t) \in M$ , so g(t) = 0. It follows that

$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

Substituting t = 0 yields

$$\nabla f(p_0) \cdot \vec{v} = 0$$

as needed.

Conversely, suppose that  $\nabla f(p_0) \cdot \vec{v} = 0$ . Since  $\nabla f(p_0) \neq 0$ ,  $\frac{\partial f}{\partial x_i}(p_0) \neq 0$  for some  $i \in \{0,...,n\}$ . Define the  $C^1$  function  $g: \mathbb{R}^n \to \mathbb{R}$  as the function that swaps the *i*th and *n*th coordinate. That is,

$$g(x_1,...,x_n) = f(x_1,...,x_{i-1},x_n,x_{i+1},...,x_i)$$

Let p' be the vector in  $\mathbb{R}^{n-1}$  whose components are the same as  $p_0$  except that its *i*th component is  $p_n$ . In particular,

$$p' = (p_1, ..., p_{i-1}, p_n, p_{i+1}, ..., p_{n-1}).$$

Notice that  $\frac{\partial g}{\partial x_n}(p', p_i) = \frac{\partial f}{\partial x_i}(p_0) \neq 0$ . Applying the Implicit Function Theorem with k = 1, there exists an open set  $W \subseteq \mathbb{R}^{n-1}$  that contains p' and a continuously differentiable function  $\psi: W \to \mathbb{R}$  such that for all  $x' \in W$ ,  $\psi(p') = p_i$  and

$$g(x', \psi(x')) = f(x_1, ..., x_{i-1}, \psi(x'), x_{i+1}, ..., x_{n-1}) = 0$$

Since W is open, there exists  $\delta > 0$  so that for all  $||t|| < \delta$ ,  $p' + t\pi_{\mathbb{R}^{n-1}}(\vec{v}) \in W$ . Let  $\hat{v}$  be  $\vec{v}$  with swapped *i*th and *n*th components. Let  $\alpha : (-\delta, \delta) \to M$  be defined by

$$\alpha(t) = (p_1 + tv_1, ..., p_{i-1} + tv_{i-1}, \psi(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1)), p_{i+1} + tv_{i+1}, ..., p_n + tv_n)$$

We can see that  $\alpha(0) = p_0$ . Next, we want to show that  $\alpha'(0) = \vec{v}$ . We first find the total derivative of  $\psi(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1))$ . For  $j \in \{1, ..., n-1\}$ , we know that the jth partial derivative is given by

$$\frac{\frac{\partial g}{\partial x_j}(p'+tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}),-1))\cdot v_i\cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p'+tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}),-1))\cdot (-v_i)} = -\frac{\frac{\partial g}{\partial x_j}(p'+tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}),-1))\cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p'+tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}),-1))}$$

substituting t=0 into this expression gives us

$$-\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p')} = -\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)}$$

For  $j \neq i$ 

$$-\frac{\frac{\partial g}{\partial x_j}(p')\cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)} = -\frac{\frac{\partial f}{\partial x_j}(p_0)\cdot v_j}{\frac{\partial f}{\partial x_i}(p_0)}.$$

For j = i,

$$-\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)} = -\frac{\frac{\partial f}{\partial x_n}(p_0) \cdot v_n}{\frac{\partial f}{\partial x_i}(p_0)}$$

Thus the total derivative is given by

$$-\frac{1}{\frac{\partial f}{\partial x_i}(p_0)} \sum_{j \in \{1, \dots, i-1, i+1, \dots, n\}} \frac{\partial f}{\partial x_j}(p_0) \cdot v_j$$

Recall that  $\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p_0) \cdot v_j = \nabla f(p_0) \cdot \vec{v} = 0$ . Using this, our expression for the total derivative at t = 0 becomes

$$-\frac{1}{\frac{\partial f}{\partial x_i}(p_0)} \cdot \left(-\frac{\partial f}{\partial x_i}(p_0) \cdot v_i\right) = v_i$$

Therefore, we conclude that

$$\alpha'(0) = (v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n) = \vec{v},$$

Which verifies that  $\vec{v}$  is indeed a tangent vector to M at  $p_0$ , finishing the proof.

(b):

Let  $f: \mathbb{R}^3 \to \mathbb{R}$  be a function defined by

$$f(x, y, z) = x^{2} + yz + y^{2} - xy - xz + z^{2} - 3$$

Its zero set is exactly M and its gradient is

$$\nabla f(x, y, z) = (2x - y - z, z + 2y - x, y - x + 2z).$$

Substituting  $p_0$ , we get  $\nabla f(p_0) = (0, 3, 1) \neq 0$ . Let  $\vec{v}_1 = (1, 0, 0), \vec{v}_2 = (0, 1, -3)$ . Notice that

$$\nabla f(p_0) \cdot \vec{v}_1 = \nabla f(p_0) \cdot \vec{v}_2 = 0$$

By the results of part (a),  $\vec{v}_1$  and  $\vec{v}_2$  are tangent vectors to M at  $p_0$ . The equation of the plane spanned by  $\vec{v}_1$  and  $\vec{v}_2$  is given by

$$3y + z = 0$$

We shift this equation to  $p_0$ , giving us

$$3y + z = 6$$

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# Question 33

(a) Let  $g:U\to \mathbf{R}$  be a  $C^1$  function defined on an open set  $U\subseteq \mathbf{R}^n$ , and let M be its zero set:

$$M = \{ p \in U : g(p) = 0 \}.$$

Suppose that we have a  $C^1$  function  $f: U \to \mathbf{R}$ , defined on an open set  $U \subseteq \mathbf{R}^n$  which contains M, and we wish to find the maximum of f on M. Assume that M is compact, and that f achieves its maximum on M at some point  $p_0 \in M$ . Prove that there exists a real number  $\lambda \in \mathbf{R}$  such that

$$\nabla f(p_0) = \lambda \nabla g(p_0).$$

This number  $\lambda$  is known as the **Lagrange multiplier**.

(b) Use Lagrange multipliers to solve the following optimization problem: Find the point(s) on the ellipsoid  $x^2 + yz + y^2 - xy - xz + z^2 = 3$  which are **closest** and **furthest** from the origin.

Proof.

(a):

Let  $h: U \times \mathbb{R} \to \mathbb{R}$  be the  $C^1$  function defined by

$$h(x,y) = yg(x) + y - f(x)$$

Notice that  $h(p_0, f(p_0)) = f(p_0) \cdot g(p_0) + f(p_0) - f(p_0) = 0$  and  $\frac{\partial h}{\partial y}(p_0, f(p_0)) = g(p_0) + 1 = 1 \neq 0$ . By the implicit function theorem, there exists an open set  $W \subseteq \mathbb{R}^n$  and  $C^1$  function  $\psi : W \to \mathbb{R}$  such that for all  $x \in W$ ,

$$h(x, \psi(x)) = \psi(x)g(x) + \psi(x) - f(x) = 0$$

Taking the derivative of both sides at  $p_0$  with respect to x, we see that

$$g(p_0)\nabla\psi(p_0) + \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

$$\implies \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

Also note that  $\frac{\partial \psi}{\partial x_i}(p_0) = \frac{\frac{\partial h}{\partial x_i}(p_0, f(p_0))}{\frac{\partial h}{\partial y}(p_0, f(p_0))} = \frac{\partial g}{\partial x_i}(p_0) - \frac{\partial f}{\partial x_i}(p_0)$ . Thus our equation becomes

$$(\psi(p_0) + 1)\nabla g(p_0) - 2\nabla f(p_0) = 0 \implies \nabla f(p_0) = \frac{\psi(p_0) + 1}{2}g(p_0)$$

Therefore the value  $\lambda = \frac{\psi(p_0) + 1}{2}$  is the one that we needed.

(b):

Define

$$g(x, y, z) = x^{2} + yz + y^{2} - xy - xz + z^{2} - 3.$$

Its zero set M is defined by the ellipsoid

$$x^2 + yz + y^2 - xy - xz + z^2 - 3 = 0.$$

We will attempt to minimize and maximize  $f(x, y, z) = x^2 + y^2 + z^2$  on M. We know that if f achieves a maximum or minimum, by part (a), there exists  $\lambda \in \mathbb{R}$  so that

$$\nabla f(p_0) = \lambda \nabla g(p_0) \implies (2x, 2y, 2z) = \lambda (2x - y - z, z + 2y - x, y - x + 2z)$$

We simply evaluate f at these points and check which ones are maximums, minimums, or neither. From the above equality, we end up with the system of equations

$$2(1 - \lambda)x + y + z = 0$$

$$x + 2(1 - \lambda)y - z = 0$$

$$x - y + 2(1 - \lambda)z = 0$$

Consider if  $\lambda = \frac{3}{2}$ . The equations collect and become

$$x = y + z$$

From the ellipsoid, we have

$$x^{2} + yz + y^{2} - xy - xz + z^{2} = 3 \implies f(x, y, z) = x(y + z) - yz + 3$$

but at the same time, we have

$$x^{2} + yz + y^{2} - xy - xz + z^{2} = 3 \implies x^{2} + yz + y^{2} - (y+z)y - (y+z)z + z^{2} = 3$$

Expand the brackets and simplify to get

$$x^2 - yz = 3 \implies x^2 - yz + 3 = 6.$$

Thus

$$f(x, y, z) = 6$$
, when  $\lambda = \frac{3}{2}$ 

Otherwise, we take the first and second equation in the original system and add them together to obtain

$$(2(1 - \lambda) + 1)x + (2(1 - \lambda) + 1)y = 0$$

$$\implies (2(1 - \lambda) + 1)(x + y) = 0$$
(1)

Likewise, we can subtract the third equation from the second to get

$$(2(1-\lambda)+1)y - (2(1-\lambda)+1)z = 0$$

$$(2(1-\lambda)+1)(y-z) = 0$$
(2)

Since  $(2(1-\lambda)+1) \neq 0$ , from equations (1) and (2), we get that z=y=-x. We substitute every variable into the original equation for x and see that

$$x^{2} + x^{2} + x^{2} + x^{2} + x^{2} + x^{2} + x^{2} = 3 \implies x = \frac{1}{\sqrt{2}}, \ y = z = -\frac{1}{\sqrt{2}}$$

# Question 34.

Let  $\Phi: \mathbf{R}^n \to \mathbf{R}^m$  be a  $C^1$  mapping.

- (a) Suppose that n > m = 1. Show that  $\Phi$  cannot be injective.
- (b) Suppose that n < m. Show that if  $K \subseteq \mathbf{R}^n$  is a compact set, then  $\Phi(K)$  is a Jordan measurable set, and has Jordan measure zero.

# Proof.

(a):

Suppose for contradiction that n > m = 1 and  $\Phi$  is an injective function.

(b):