Question 1.

Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Use row and column operations on A to obtain a matrix B of the form in Theorem 53. Use that work to find invertible matrices P, Q so that $B = PAQ$.

Proof. We perform the following row and column operations:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Define this matrix we obtained as B. We will perform the same row and column operations above on I_3 and I_4, respectively in order to define P and Q. We have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We see that

$$PAQ = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= B$$

as required.

$\square$

Question 2.

Let $A = \begin{pmatrix} 1 & -2 & -4 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

(a) Verify that A is invertible, by row-reducing the augmented matrix $(A|I_3)$.

(b) Use (a) to find A^{-1}.

(c) Express A as a product of elementary matrices.

Proof.
(a): We see that

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 \to r_2 - r_1, r_3 \to r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 + r_3, r_2 \to r_2 - r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 + \frac{1}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array}\right) \xrightarrow{r_3 \to \frac{1}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)$$

Since A can be row reduced into the identity matrix, A is invertible.

(b):

By our row reductions above, we know that $A^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & -1 \\ -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$.

(c):

To express A is a product of elementary matrices, we can apply the opposite row operations to the identity matrix in reverse order. That is,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\square$

Question 3.

Find the explicit formula for the linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^3$ which satisfies:

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad T\begin{pmatrix}2\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Proof. Notice that

$$\beta = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\}$$

is a basis for $\mathbb{Q}^4$. We attempt to find the general form for a vector $(x, y, z, w) \in \mathbb{Q}^4$ in terms of these vectors. By inspection, we see that

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = (x - 2y + z)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (y - z)\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + (z - w)\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + w\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Thus

$$\begin{aligned} T\begin{pmatrix}x\\y\\z\\w\end{pmatrix} &= (x - 2y + z)T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (y - z)T\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + (z - w)T\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + wT\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \\ &= (x - 2y + z)\begin{pmatrix}1\\2\\3\end{pmatrix} + (y - z)\begin{pmatrix}0\\1\\1\end{pmatrix} + (z - w)\begin{pmatrix}0\\0\\1\end{pmatrix} + w\begin{pmatrix}0\\0\\0\end{pmatrix} \\ &= \begin{pmatrix} x - 2y + z \\ 2x - 3y + z \\ 3x - 5y + 3z - w \end{pmatrix} \end{aligned}$$

$\square$

Question 4.

Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathcal{M}_{2\times 2}(\mathbb{F})$. Consider the linear map $T : \mathcal{M}_{2\times 2}(\mathbb{F}) \to \mathcal{M}_{2\times 2}(\mathbb{F})$ given by $T(A) = A^T$. Set $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$.

(a) Find P - the change of coordinate matrix from γ to β coordinates.

We have

$$\begin{aligned} P &= \begin{pmatrix} [E_{11}]_\beta & [E_{22}]_\beta & [E_{12}+E_{21}]_\beta & [E_{12}-E_{21}]_\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

(b) Find P^{-1} - the change of coordinate matrix from β to γ coordinates.

Similarly,

$$\begin{aligned} P^{-1} &= \begin{pmatrix} [E_{11}]_\gamma & [E_{12}]_\gamma & [E_{21}]_\gamma & [E_{22}]_\gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{aligned}$$

(c) Find $A = [T]_\beta$.

We see that

$$\begin{aligned} A &= \begin{pmatrix} [T(E_{11})]_\beta & [T(E_{12})]_\beta & [T(E_{21})]_\beta & [T(E_{22})]_\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(d) Find $B = [T]_\gamma$.

Once again,

$$\begin{aligned} B &= \begin{pmatrix} [T(E_{11})]_\gamma & [T(E_{22})]_\gamma & [T(E_{12}+E_{21})]_\gamma & [T(E_{12}-E_{21})]_\gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

(e) Confirm that $A = PBP^{-1}$ using (a)-(d).

$$PBP^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= A
\end{aligned}
$$

as expected.

Question 5.

Let $T : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathcal{M}_{n\times n}(\mathbb{F})$ be the linear map given by $T(A) = A + A^T$.

(a) Find $N(T)$ and $\dim N(T)$.

We claim that $N(T)$ is the set of all skew symmetric matrices with zeroes on the diagonal, which has dimension $\frac{1}{2}n(n-1)$.

Set $T(A) = A + A^T = 0$. We have that $A_{ij} + A_{ji} = 0$ for each $0 < i, j \leq n$. In particular, we have that $A_{ij} = 0$ if $i = j$ and $A_{ij} = -A_{ji}$ otherwise. But this describes exactly all skew symmetric matrices with zeroes on the diagonal. The basis for this set is

$$\beta = \{E_{ij} - E_{ji} : 0 < i < j \leq n\}$$

and there are $\frac{1}{2}n(n-1)$ vectors in this set, so $\dim N(T) = \frac{1}{2}n(n-1)$.

(b) What is $\operatorname{im}(T)$?

We claim that $\operatorname{im}(T)$ is the set of all symmetric matrices S_n. We see that

$$(A + A^t)_{ij} = A_{ij} + A^t_{ij} = A_{ij} + A_{ji} = A_{ji} + A^t_{ji} = (A + A^t)_{ji}$$

so $\operatorname{im}(T) \subseteq S_n$. To show set equality, suppose that B is a symmetric matrix. Let $A = \frac{1}{2}B$ then

$$T(A) = \frac{1}{2}T(B) = \frac{1}{2}(B + B^t) = B$$

Thus $\operatorname{im}(T) = S_n$ and has basis

$$\gamma = \{E_{ij} : 0 < i \leq j \leq n\}.$$

and is dimension $\frac{1}{2}n(n+1)$.

(c) Is $\mathcal{M}_{n\times n}(\mathbb{F}) = \operatorname{im}(T) \oplus N(T)$?

Yes.

To show this, notice that $\beta \cap \gamma = \varnothing$, so $\operatorname{im}(T) \oplus N(T)$ has basis $\alpha = \beta \cup \gamma$. But notice that $|\alpha| = \frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$, which is the dimension of $\mathcal{M}_n(\mathbb{F})$. Therefore α is actually a basis for $\mathcal{M}_n(\mathbb{F})$ and thus $\mathcal{M}_n(\mathbb{F}) = \operatorname{im}(T) \oplus N(T)$.

Question 6.

Let V, W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$ a linear map. Prove that T is injective if and only if $N(T) = \{\mathbf{0}_V\}$. (Make no assumption here about $\dim V, \dim W$.)

Proof. Suppose that T is injective. Let $T(x) = 0$, for some $x \in V$. Recall that $T(0) = 0$ for any linear map. Therefore by injectivity $x = 0$, so $N(T) = \{0\}$.
Conversely, suppose that $N(T) = \{0\}$. Let $x, y \in V$ such that $T(x) = T(y)$. By linearity, we have that $T(x - y) = 0$, but this implies that $x - y = 0$, so $x = y$ and T is injective.

$\square$

Question 7.

Let V, W be vector spaces over a field $\mathbb{F}$, and $T : V \to W$ a linear map. Find a condition on T which is equivalent to "$T(S)$ spans W for any spanning set $S \subseteq V$ of V".
(Hint: Write down the definition of $T(S)$ is spanning to get started.)

Proof. We claim that this statement is equivalent to saying that T is surjective.
Suppose that for any set $S \subseteq V$ that spans V, $T(S)$ spans W. We prove that T is surjective. Let $w \in W$. We can write w as a linear combination of some number of vectors in $T(S)$. That is, for some $k \in \mathbb{N}$ and $s_i \in S$, $c_i \in \mathbb{F}$, $i \in \{1, ..., k\}$,

$$w = \sum_{i=1}^{k} c_i T(s_i) = T\left(\sum_{i=1}^{k} c_i s_i\right)$$

so T is surjective.
Conversely, suppose that T is surjective. Let S be a spanning set of V. We will show that $T(S)$ spans W. Let $w \in W$. By surjectivity, there exists $v \in V$ so that $T(v) = w$. We can rewrite

$$v = \sum_{i=1}^{k} c_i s_i$$

for some number of vectors $s_i \in S$ and $c_i \in \mathbb{F}$. Then

$$T\left(\sum_{i=1}^{k} c_i s_i\right) = w \implies \sum_{i=1}^{k} c_i T(s_i) = w$$

Notice that $T(s_i) \in T(S)$, from which it follows that $T(S)$ spans W, and the proof is complete.

□

Question 8.

Let $P \in \mathcal{M}_{n\times n}(\mathbb{F})$. Prove the following three conditions are equivalent.

(a) P is invertible.

(b) There exists bases β, γ of $\mathbb{F}^n$ so that $P = [\mathrm{I}_{\mathbb{F}^n}]_\beta^\gamma$.

(c) For any n-dimensional vector space V over $\mathbb{F}$, there exists bases β, γ of V so that $P = [\mathrm{I}_V]_\beta^\gamma$.

Proof. Suppose (a). We prove (b) and (c) at the same time.

Let $\beta = \{v_1, ..., v_n\}, \beta' = \{v'_1, ..., v'_n\}$ be bases for $\mathbb{F}^n$ and V respectively. For $i = 1, ..., n$, let

$$u_i = \sum_{j=1}^{n} P_{ji} v_j \text{ and } u'_i = \sum_{j=1}^{n} P_{ji} v'_j.$$

Define $\gamma = \{u_1, ..., u_n\}, \gamma' = \{u'_1, ..., u'_n\}$. We claim that they are bases for $\mathbb{F}^n$ and V. We will only show that is the case for γ, because the argument is the same for γ'.
It suffices to show that γ is linearly independent, as it is a set of n vectors, from which it will follow that γ is a basis for $\mathbb{F}^n$. For constants $c_i \in \mathbb{F}$, let

$$\sum_{i=1}^{n} c_i \sum_{j=1}^{n} P_{ji} v_j = 0$$

By the linear independence of β, for each j,

$$\sum_{i=1}^{n} c_i P_{ji} = 0$$

But this is the same as saying

$$P \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

Since P^{-1} exists, we perform left multiplication by P^{-1} to see that

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$$

which means that $c_i = 0$ for all i, thus showing that γ is linearly independent and indeed a basis for $\mathbb{F}^n$, and by the same argument, γ' is also a basis for V.
Finally, notice that for each u_i, $[u_i]_\beta$ is equal to the ith row of P which confirms that $P = [I_{\mathbb{F}^n}]_\gamma^\beta$. The same applies for u'_i so $P = [I_V]_{\gamma'}^{\beta'}$.
Consider the linear transformation $T_P : \mathbb{F}^n \to \mathbb{F}^n$. Let β be an ordered basis for $\mathbb{F}^n$. We will show that $\gamma = T_P(\beta)$ is also an ordered basis for $\mathbb{F}^n$. Since P is invertible, T_P has an inverse $(T_P)^{-1} = T_{P^{-1}}$, so T_P is surjective and $\text{span}(T_P(\beta)) = \mathbb{F}^n$. Since $|T_P(\beta)| = n$, $T_P(\beta)$ is indeed an ordered basis. Thus we can conclude that P is a change of basis matrix from β to γ.

Suppose (c). We prove (a).
For some bases β, γ, $P = [I_V]_\beta^\gamma$. We claim that $P^{-1} = [I_V]_\gamma^\beta$. Indeed,

$$PP^{-1} = [I_V]_\beta^\gamma [I_V]_\gamma^\beta = [I_V]_\gamma = I_n$$

This covers all the equivalences and we are done.

$\square$

Question 9.

Consider the relation $\equiv$ on $\mathcal{M}_{m\times n}(\mathbb{F})$ defined by $A \equiv B$ if $A \to B$ using a combination of row and/or column operations.

(a) Prove that $\equiv$ is an equivalence relation on $\mathcal{M}_{m\times n}(\mathbb{F})$.

(b) Find a condition on A, B which is equivalent to $A \equiv B$. (Hint: Theorem 53.)

(c) Classify the equivalence classes for this relation, and prove that there are exactly $1 + \min\{n, m\}$ such classes.

Proof.
(a):
We show reflexivity, symmetry, and transitivity in that order.
Reflexivity: Since $IA = A$, and I is considered a row operation, $A \equiv A$.
Symmetry: Suppose that $A \equiv B$ then for some invertible matrices P, Q we have that $PAQ = B$. But at the same time this means that $P^{-1}BQ^{-1} = A$ so $B \equiv A$.
Transitivity: Suppose that $A \equiv B$ and $B \equiv C$. Then for invertible matrices P, Q, R, S, $PAQ = B$ and $RBS = C$, so $(RP)A(QS) = R(PAQ)S = RBS = C$. Since RP, QS are also invertible, we have that $A \equiv C$.

(b):
We claim that an equivalent condition is $\text{rank}A = \text{rank}B$. Suppose that $A \equiv B$. Then $PAQ = B$ for some invertible matrices P, Q, but it is known that rank is preserved by multiplication with invertible matrices, so $\text{rank}A = \text{rank}PAQ = \text{rank}B$.
Conversely, suppose that $r := \text{rank}A = \text{rank}B$. By Theorem 53, there exist row/column operations so that

$$A, B \to \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

We denote this matrix by J_r. that is, for invertible matrices P, Q, R, S, $PAQ = I' = RBS$. It follows that $R^{-1}PAQS^{-1} = B$, so $A \equiv B$ as desired.

(c):
We can classify the equivalence classes by matrix rank. That is, each equivalence class is of the form

$$[J_r] = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \text{rank}A = r\}.$$

The possible ranks of $m \times n$ matrices range from 0 to $\min\{n, m\}$, so there are $\min\{n, m\} + 1$ different values of r. We will verify that these equivalence classes are exhaustive and disjoint. Every $m \times n$ matrix must have a rank, so it belongs to at least one of the classes, but at the same time, a matrix can possibly only have one rank, so it necessarily belongs to exactly one equivalence class.

$\square$

Question 10.

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, and $T : V \to W$ a linear map with rank$T = 2$. Set $n = \dim V$, $m = \dim W$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}^n$ be two non-parallel vectors. Prove there exists bases β, γ of V, W respectively, so that $[T]_\beta^\gamma = (\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{0} \ \cdots \ \mathbf{0})$. (Hint: use problems 7,8.)

Proof. By the Dimension Theorem, null$(T) = n - 2$. Let $\alpha = \{v_1, ..., v_{n-2}\}$ be a basis for $N(T)$ and extend this basis into a basis $\{a, b, v_1, ..., v_{n-2}\}$ which we set as γ.

□

Question 11.

Let $T : V \to V$ be linear. We say that a subspace $W \subseteq V$ is "T-invariant" if $T(W) \subseteq W$. For example, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is counter-clockwise rotation around the z-axis by angle θ, then $P_{xy} = \{(x, y, 0) \in \mathbb{R}^3\}$ is T-invariant, as is L_z (the z-axis).

(a) Verify the claims made above, by showing that P_{xy} and L_z are T-invariant.

(b) Show that $\mathbb{R}^3 = P_{xy} \oplus L_z$ by finding a basis $\beta = \beta_1 \cup \beta_2$ for $\mathbb{R}^3$ so that β_1 is a basis for P_{xy} and β_2 is a basis for L_z.

(c) Using your basis β from (b), find $[T]_\beta$.

Proof.
(a):
We begin by finding an expression for T. Notice that

$$\begin{aligned} T(e_1) &= (\cos\theta, \sin\theta, 0) \\ T(e_2) &= (-\sin\theta, \cos\theta, 0) \\ T(e_3) &= (0, 0, 1) \end{aligned}$$

In the case of e_1, e_2, the projection onto the xy-plan lies on the unit circle, and thus each vector is rotated θ and $\theta + \frac{\pi}{2}$ radians respectively (relative to the point $(0, 1)$). Thus we have that

$$\begin{aligned} T(x, y, z) &= x(\cos\theta, \sin\theta, 0) + y(-\sin\theta, \cos\theta, 0) + z(0, 0, 1) \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \end{aligned}$$

Now, let $(x, y, 0) \in P_{xy}$. Then

$$T(x, y, 0) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, 0) \in P_{xy}$$

Additionally, let $(0, 0, z) \in L_z$. Then

$$T(0, 0, z) = (0, 0, z) \in L_z$$

Thus P_{xy} and L_z are T-invariant subspaces.

(b):
Let $\beta_1 = \{e_1, e_2\}, \beta_2 = \{e_3\}$. It is clear that β_1 is a basis for the xy-plane and β_2 is a basis for the z-axis. Then $\beta = \{e_1, e_2, e_3\}$ is the standard ordered basis for $\mathbb{R}^3$, which was what we wanted to show.

(c):
We have already found all we need from the previous parts:

$$[T]_\beta = \begin{pmatrix} [T(e_1)]_\beta & [T(e_2)]_\beta & [T(e_3)]_\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

□

Question 12.

Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$, and $W_1 \subseteq V$ a T-invariant subspace with basis β_1. Set $k = \dim W_1$.
We will generalize what we saw in #11c.

(a) Extend β_1 to a basis β of V. Show that $[T]_\beta = \begin{pmatrix} A & C \\ O_{n-k,k} & B \end{pmatrix}$, where A is $k \times k$, B is $(n-k) \times (n-k)$, and C is $k \times (n-k)$.

(b) Suppose that W_2 is a subspace so that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$, where β_2 is any basis for W_2.

Prove that if W_2 is T-invariant, then $[T]_\beta = \begin{pmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{pmatrix}$ is block diagonal.

(c) Is the converse of (b) true or false? Justify your answer.

Question 13.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

(a) Let $\beta = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{F}^n$, and $\gamma = \{v_1, \ldots, v_n\}$ a basis for $\mathbb{F}^n$. Then there exists a sequence of row operations that takes β to γ. (That is, v_i is obtained from e_i using the same row operations for all i.)

(b) Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. If β, γ are bases for V so that $[T]_\beta^\gamma = I_n$, then $T = I_V$.

(c) Let V be a finite dimensional vector space over $\mathbb{F}$ and $S, T : V \to V$ linear maps. If $\operatorname{rank} T = \operatorname{rank} S$, then there exist bases $\beta, \beta', \gamma, \gamma'$ for V so that $[S]_\beta^\gamma = [T]_{\beta'}^{\gamma'}$.

(d) Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$. If $A^2 \sim B^2$, then $A \sim B$.

Proof.
(a):
This statement is true. Consider the linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ defined by $T(e_i) = v_i$ for all $i \in \{1, ..., n\}$. Notice that T is surjective, as $\text{span}(T(\beta)) = \text{span}(\gamma) = \mathbb{F}^n$. It follows that T is invertible, so $[T]_\beta^\gamma$ is invertible, so it can be decomposed into a number of elementary matrices and thus represent a sequence of row operations. But notice that $[T]_\beta^\gamma$ is exactly the matrix that maps e_i to v_i, so we have what we wanted.

(b):
This is false. Let $V = \mathbb{F}^n$. Take β as the standard basis. As well, let $\gamma = \{e_1, e_2, ..., e_{n-1}, -e_n\}$, so that γ is just β only with the last basis vector multiplied by -1. Consider the linear map T defined in the previous part. Then $T(e_n) = -e_n$, so T is not the identity map, but it is not hard to see that $[T]_\beta^\gamma = I_n$.

(c):
This is true. Let $k = \text{rank } T = \text{rank } S$. Take k vectors $e_1, ..., e_k$. By Question 10, there exists bases $\beta, \gamma, \beta', \gamma'$ such that

$$[T]_\beta^\gamma = \left(\begin{array}{c|c} I_k & O \\ O & O \end{array}\right) = [S]_{\beta'}^{\gamma'}$$

as desired.

(d):
This is false. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = 0_n$, the zero matrix. Then

$$A^2 = B^2$$

But $\text{rank} A = 1$ and $\text{rank} B = 0$, so for any invertible matrix P,

$$\text{rank} PAP^{-1} = 1$$

so A is not similar to B.

□