You are not allowed to post the assignment questions anywhere; however, you are allowed to search the internet (just cite your resources if you find any). You are also allowed to bounce ideas off classmates and TAs, but at the end, you must write your own solutions.

If you use AI tools, please mention the name of the tool and the prompts you used.

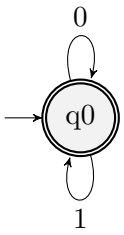
Q1. (18 pts)

Show with proof which of the following languages is regular and which is not. If you show that a language is regular, you are expected to provide a DFA that accepts the language.

Let $\Sigma = \{0, 1\}$.

a) **(3 pts)** Σ^*

Proof. For this language, define the DFA $\mathcal{D} = (\{q_0\}, \Sigma, \delta, q_0, \{q_0\})$, where $\delta : \{q_0\} \times \Sigma \to \{q_0\}$ is a function that always maps to q_0.



It will be shown that $\mathcal{D}$ is correct. For $w \in \Sigma^*$, define the state invariant

$$P_{q_0}(w) : \delta(q_0, w) = q_0 \iff w \in \Sigma^*$$

It will be shown using induction that for all $w \in \Sigma^*$, $P_{q_0}(w)$ is true.

Base Case. Let $w = \varepsilon$. Both $\delta(q_0, \varepsilon) = q_0$ and $\varepsilon \in \Sigma^*$ are true, so $P_{q_0}(\varepsilon)$ is true.

Induction Hypothesis. Let $q \in \mathcal{Q}$, $w \in \Sigma^*$, and $\sigma \in \Sigma$. Since $\mathcal{Q} = \{q_0\}$, $q = q_0$. Suppose that $P_{q_0}(w)$ is true.

Induction Step. For all σ, $\delta(q_0, \sigma) = q_0$. It is obvious that $wz \in \Sigma^*$, so $P_{q_0}(wz)$ holds true.

Thus the state invariants are correct.

Now, let $w \in \Sigma^*$. Then by the state invariant, $\delta(q_0, w) = q_0$, which is an accepting state, so w is accepted.

A string that is not in Σ^* cannot be found so the backwards direction is vacuously true.

Therefore $\mathcal{D}$ is correct.

□

b) **(3 pts)** $\Sigma^* \backslash K, K = \{01, 101, 010\}$

Proof. 

□

c) **(3 pts)** $\{w \mid w \text{ is a palindrome}\}$

Proof. This is not regular. Suppose for contradiction that this language was regular. Then there would be a DFA associated with this language. Let q be the number of states in this DFA, and let q_0 denote the initial state. Define the following strings:

$$w_i = 0^q 1 0^i, \text{ for } i \in \{0, 1, 2, ..., q\}$$

Notice that there are $q+1$ strings in total. By the pigeonhole principle, at least 2 w_i, w_j will be in the same state q_c, $i < j$, that is, $\delta(q_0, w_i) = \delta(q_0, w_j) = q_c$. It follows that $\delta(q_0, w_i 0^{q-i}) = \delta(q_0, w_j 0^{q-i})$. But notice that

$$w_i = 0^q 1 0^q \text{ and } w_j = 0^q 1 0^{q+j-i}.$$

It can be seen that w_i is a palindrome and should be accepted, but because $q+j-i \neq q$, w_j is not a palindrome, which contradicts the fact that w_i and w_j are in the same state.

□

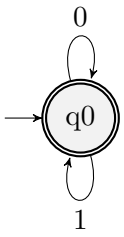
d) **(3 pts)** $\{ww \mid w \in \Sigma^*\}$

e) **(3 pts)** $\{w \mid ww \in \Sigma^*\}$

Proof. Denote the language above as A. It will be shown that the language above is equivalent to Σ^* using double subset inclusion.

Trivially, $A \subseteq \Sigma^*$. Conversely, let $w \in \Sigma^*$. Then since $ww \in \Sigma^*$, $w \in A$, proving the claim. Therefore, from part a), it can be concluded that $A = \Sigma^*$ is a regular language. The same DFA given in part a) accepts A. The same proof in part a) is listed here.

For this language, define the DFA $\mathcal{D} = (\{q_0\}, \Sigma, \delta, q_0, \{q_0\})$, where $\delta : \{q_0\} \times \Sigma \to \{q_0\}$ is a function that always maps to q_0.



It will be shown that $\mathcal{D}$ is correct. For $w \in \Sigma^*$, define the state invariant

$$P_{q_0}(w) : \delta(q_0, w) = q_0 \iff w \in \Sigma^*$$

It will be shown using induction that for all $w \in \Sigma^*$, $P_{q_0}(w)$ is true.

Base Case. Let $w = \varepsilon$. Both $\delta(q_0, \varepsilon) = q_0$ and $\varepsilon \in \Sigma^*$ are true, so $P_{q_0}(\varepsilon)$ is true.

Induction Hypothesis. Let $q \in \mathcal{Q}$, $w \in \Sigma^*$, and $\sigma \in \Sigma$. Since $\mathcal{Q} = \{q_0\}$, $q = q_0$. Suppose that $P_{q_0}(w)$ is true.

Induction Step. For all σ, $\delta(q_0, \sigma) = q_0$. It is obvious that $wz \in \Sigma^*$, so $P_{q_0}(wz)$ holds true.

Thus the state invariants are correct.

Now, let $w \in \Sigma^*$. Then by the state invariant, $\delta(q_0, w) = q_0$, which is an accepting state, so w is accepted.

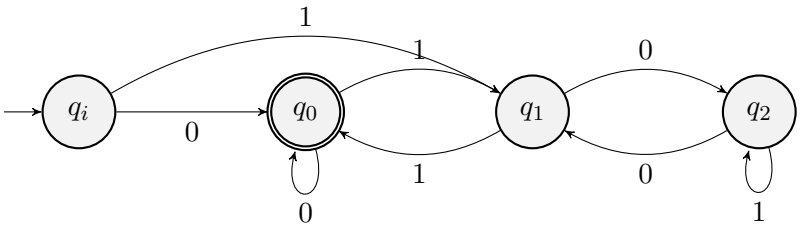
A string that is not in Σ^* cannot be found so the backwards direction is vacuously true.

Therefore $\mathcal{D}$ is correct.

□

f) **(3 pts)** $\{w \mid w$ is a binary representation of a multiple of 3$\}$

Proof. Consider the following DFA:



It will be shown that this DFA correctly represents the language above. For all $w \in \Sigma^*$, denote n_w to be the number that w represents.

Define the following state invariants:

$$P_i(w) : \delta(q_i, w) = q_i \iff w = \varepsilon$$

$$P_k(w) : \delta(q_i, w) = q_k \iff n_w \equiv k \bmod 3$$

It will be proven that $P(w) : P_i(w) \wedge P_k(w)$ for all $w \in \Sigma^*$ using structural induction.

Base Case. Let $w = \varepsilon$. Since q_i is not the end state of any transition from any other state and ε does not represent any binary number, $P_k(w)$ is vacuously true. To show that $P_i(w)$, it is sufficient to say that $\delta(q_i, w) = q_i$, which is obviously true. Thus $P(\varepsilon)$ is true.

Induction Hypothesis. Suppose that $P(w)$ is true, for some $w \in \Sigma^*$. The goal is to show that $P(w\sigma)$ is true for all $\sigma \in \Sigma^*$.

Induction Step. Consider the case when $\sigma = 0$. □

Q2. (10 pts)

Implementations of regular expressions often allow you to also have a complement operation. For example, in python [^a] means a string that does not have the character 'a' in it.

More formally, the *complement* of L is the language $\overline{L} = \{x \in \Sigma^* \mid x \notin L\}$.

Then in regular expressions, if r is a regular expression matching $\mathcal{L}(r)$, then $\overline{r}$ is a regular expression, with $\mathcal{L}(\overline{r}) = \overline{\mathcal{L}(r)}$.

Prove that regular expressions that also have access to complement can still only express the same class of languages (i.e., the class of regular languages) as regular expressions without the complement operation.

Proof. It suffices to show that if r is a regex, then $\overline{r}$ is a regex as well, meaning that $\mathcal{L}(\overline{r})$ matches a regular language.

Let Σ represent the alphabet. Suppose that $\mathcal{L}(r)$ matches a regular language. Then there exists a DFA $D = (\Sigma, \delta, \mathcal{Q}, s, F)$ that accepts $\mathcal{L}(r)$, where δ is the transition function, $\mathcal{Q}$ is the set of states, $s \in \mathcal{Q}$ is the starting state, and $F \subseteq Q$ is the set of accepting states.

Construct a new DFA $D' = (\Sigma, \delta, \mathcal{Q}, s, \mathcal{Q} \setminus F)$. It will be shown that D' accepts the language $\mathcal{L}(\overline{r}) = \overline{\mathcal{L}(r)}$.

Let $w \in \overline{\mathcal{L}(r)}$. This is equivalent to saying that $w \in \Sigma^* \setminus \mathcal{L}(r)$. It follows that D rejects w, that is, $\delta(s, w) \notin F \implies \delta(s, w) \in \mathcal{Q} \setminus F$. However, by construction, when D' reads w, it will end on the state $\delta(s, w)$. This is an accepting state in D', so w is accepted by D'.

Conversely, suppose that $w \in \mathcal{L}(r)$. This implies that D accepts w, so $\delta(s, w) \in F \implies \delta(s, w) \notin \mathcal{Q} \setminus F$. Similar to the previous case, when w is ran through D', the ending state is $\delta(s, w)$, which is not an accepting state. Thus w is rejected by D'.

It has been proven that D' accepts $\mathcal{L}(\overline{r})$, which shows that r is a regex, completing the proof.

□

Q3. (15 pts)

Counter-free languages are a subset of regular languages that satisfy the condition: $\exists n \in \mathbb{N}, \forall x, y, z \in \Sigma^*, \forall m \geq n, [xy^m z \in L \iff xy^n z \in L]$

A known result in formal language theory is that counter-free languages are equivalent to the languages that can be expressed by **star-free regular expressions**. **Star-free regular expressions** are regular expressions without the Kleene star, but with complementation.

a) **(5 pts)** Prove that $(ab)^*$ can be matched with a star-free regular expression, $\Sigma = \{a, b\}$

Proof. Define the regex $r = \overline{(b\overline{\varnothing}) + (\overline{\varnothing}bb\overline{\varnothing}) + (\overline{\varnothing}aa\overline{\varnothing}) + (\overline{\varnothing}a)}$. Simplifying,

$$\mathcal{L}(r) = \mathcal{L}\left(\overline{(b\overline{\varnothing}) + (\overline{\varnothing}bb\overline{\varnothing}) + (\overline{\varnothing}aa\overline{\varnothing}) + (\overline{\varnothing}a)}\right) = \overline{\mathcal{L}((b\overline{\varnothing}) + (\overline{\varnothing}bb\overline{\varnothing}) + (\overline{\varnothing}aa\overline{\varnothing}) + (\overline{\varnothing}a))}$$

$$= \overline{\mathcal{L}(b\overline{\varnothing}) \cup \mathcal{L}(\overline{\varnothing}bb\overline{\varnothing}) \cup \mathcal{L}(\overline{\varnothing}aa\overline{\varnothing}) \cup \mathcal{L}(\overline{\varnothing}a)}$$

$$= \overline{\mathcal{L}(b)\overline{\mathcal{L}(\varnothing)} \cup \overline{\mathcal{L}(\varnothing)}\mathcal{L}(bb)\overline{\mathcal{L}(\varnothing)} \cup \overline{\mathcal{L}(\varnothing)}\mathcal{L}(aa)\overline{\mathcal{L}(\varnothing)} \cup \overline{\mathcal{L}(\varnothing)}\mathcal{L}(a)}$$

$$= \overline{\{b\}\overline{\varnothing} \cup \overline{\varnothing}\{bb\}\overline{\varnothing} \cup \overline{\varnothing}\{aa\}\overline{\varnothing} \cup \overline{\varnothing}\{a\}} = \overline{\{b\}\Sigma^* \cup \Sigma^*\{bb\}\Sigma^* \cup \Sigma^*\{aa\}\Sigma^* \cup \Sigma^*\{a\}}$$

$$= \overline{\{b\}\Sigma^*} \cap \overline{\Sigma^*\{bb\}\Sigma^*} \cap \overline{\Sigma^*\{aa\}\Sigma^*} \cap \overline{\Sigma^*\{a\}}$$

Let $w \in (ab)^*$. Then $w = (ab)^k$, where $k \in \mathbb{N}$. Notice that w can never start with a b, contain two identical consecutive symbols, nor end with an a, so $w \in \overline{\{b\}\Sigma^*}$, $w \in \overline{\Sigma^*\{bb\}\Sigma^*}$, $w \in \overline{\Sigma^*\{aa\}\Sigma^*}$, and $w \in \overline{\Sigma^*\{a\}}$. Thus $w \in \mathcal{L}(r)$.

Now, suppose that $w \in \mathcal{L}(r)$. First, consider the case when $w = \varepsilon$. Since $\varepsilon = (ab)^0$, then $w \in (ab)*$.

If $|w| > 0$, note the following observations:

w must start with an a, because if not then $w \in \{b\}\Sigma^*$.

Also, w must alternate. Otherwise, $w \in \overline{\Sigma^*\{bb\}\Sigma^*}$ or $w \in \overline{\Sigma^*\{aa\}\Sigma^*}$.

Finally, w must end in a b because $w \in \overline{\Sigma^*\{a\}}$.

This means that $w = ab...ab$. It is obvious that $w = (ab)^n$ for some $n \in \mathbb{N}^+$. Thus $w \in (ab)*$, proving that $(ab)^*$ can be matched by a star-free expression.

□

b) **(5 pts)** Prove that $(ab)^*$ is a counter-free language, $\Sigma = \{a, b\}$.

Proof. Let $n = 1$. Fix $x, y, z \in \Sigma^*$, and let m be a natural number such that $m \geq n$.

Suppose that $xy^m z \in (ab)^*$. Then $xy^m z = (ab)^l$, for some $l \in \mathbb{N}$. Consider the case where x ends with the symbol a. x must be in the form $(ab)^i a$ for some $i \in \mathbb{N}$. Then y must begin with a b and alternate symbols, for if not, then $xy^m z \notin L$. Thus y can be rewritten as $y = (ba)^j$, for some natural l. It follows that z starts with a b, and can be written as $b(ab)^k$, for some natural k. Therefore

$$xy^n z = (ab)^i a(ba)^l b(ab)^k = (ab)^{i+l+k+1} \in L$$

Now, consider the case where x does not end with an a. Using a similar argument to the previous case, $x = (ab)^i$, $y = (ab)^j$, and $z = (ab)^k$, for different constants $i, j, k \in \mathbb{N}$. Then

$$xy^n z = (ab)^i (ab)^j (ab)^k = (ab)^{i+j+k} \in L$$

Conversely, suppose that $xyz = xy^n z \in L$. Again, either $x = (ab)^i a$, $y = (ba)^j$, and $z = b(ab)^k$ or $x = (ab)^i$, $y = (ab)^j$, and $z = (ab)^k$, for naturals i, j, k. Then $xy^m z$ is equal to either

$$xy^m z = (ab)^i a((ba)^j)^m b(ab)^k = (ab)^{i+jm+k+1}$$

or

$$(ab)^i((ab)^j)^m(ab)^k = (ab)^{i+jm+k}.$$

Regardless, $xy^m z \in L$. Since both directions have been shown, the proof is complete.

□

c) **(5 pts)** Prove that $(aa)^*$ is not a counter-free language, $\Sigma = \{a\}$

Proof. First, notice that in order for a string w to be a member of $\mathcal{L}((aa)^*)$, $w = (a)^{2k}$ for some natural k. Let $n \in \mathbb{N}$. Let $x = \varepsilon$, $y = a$, $z = \varepsilon$. Let $m = n + 1 \geq n$. Then $xy^m z = a^{n+1}$, and $xy^n z = a^n$. The goal is to show that either $a^{n+1} \in L \wedge a^n \notin L$ or $a^{n+1} \notin L \wedge a^n \in L$. Indeed, exactly one of $n + 1$ or n are even, it must be true that exactly one of a^n or a^{n+1} are in $\mathcal{L}((aa)^*)$, and the conclusion follows promptly.

□

(Continued on the next page)

Q4. (10 pts + 3 bonus)

The textbook introduces non-determinism on page 78 by highlighting the difference in DFAs and NFAs for language $L = \{w \mid$ the third last character of w is $1\}$.

Prove that for any $k \in \mathbb{N}^+$, the language $L = \{w \mid$ the kth to last character of w is $1\}$ has the following attributes:

a) **(5 pts)** A DFA that accepts L has to have at least 2^k number of states.

Proof. Suppose for contradiction that there is a DFA $\mathcal{D}$ with less than 2^k states that accepts L.

Consider the language $M = \{w \in \Sigma^* : |w| = k\}$. Notice that $|M| = 2^k$ because each index can be either a 0 or a 1.

Since $\mathcal{D}$ has less than 2^k states, at least two strings $w, x \in M$ end in the same state. Since these strings are distinct, they differ at some index i. Namely, one string contains a 0 in the ith index and the other contains a 1 in the ith index. Note that the ith index is the same as the $k - i$th last index.

Consider the strings $w0^i$ and $x0^i$. Notice that in both in these strings, the kth last character is exactly the character in the ith index. Since w and x were in the same state, it follows that $w0^i$ and $x0^i$ should also be in the same state and should both be accepted or both be rejected, but the character in the kth index is different, so only one of them would be accepted. This is a contradiction. A DFA that accepts L must have at least 2^k number of states.

□

b) **(5 pts)** The smallest NFA that accepts L has to have exactly $k + 1$ number of states.

Proof. First, it will be shown that there is no NFA that accepts L in less than $k + 1$ states. Then, an NFA with $k + 1$ states that accepts L will be provided.

Suppose for contradiction that there is an NFA that accepts L with less than $k + 1$ states. Let $w = 10^{k-2}$. For $i \in \{1, ..., k\}$, define

$$sdf\,sdf$$

Notice that $w0$ is a string that should be accepted.

Now, define an NFA $\mathcal{N}$ as follows. Let $\mathcal{Q} = \{q_0, ..., q_k\}$, $s = q_0$, $F = \{q_k\}$. Define $\delta : \mathcal{Q} \times \Sigma \to \mathcal{P}(\mathcal{Q})$ by

$$\delta(q_0, 0) = \{q_0\}$$

$$\delta(q_0, 1) = \{q_0, q_1\}$$

$$\delta(q_i, \sigma) = \{q_{i+1}\},\ \forall \sigma \in \Sigma,\ 0 < i < k$$

$$\delta(q_k, \sigma) = \{q_0\},\ \forall \sigma \in \Sigma$$

It will be shown that $\mathcal{N} = (\Sigma, \delta, Q, s, F)$ is correct.

For a positive integer i, define $q(i)$ to be the state of the machine before reading the ith symbol.

Let $w \in L$. Then the kth last symbol is a 1. The correct path is to first transition to q_0 for every symbol before the th last symbol. Thus before $\mathcal{N}$ reads the kth last symbol, the machine is in state q_0. Then, transition to q_1 when reading the kth last symbol. After this, the machine will always transition to the next state. Specifically, this means that for $i \in \{1, ..., k-1\}$,, if $\mathcal{N}$ is in state q_i before reading a symbol, it will transition to q_{i+1} after reading it.

Since there are $k-1$ symbols remaining to be read after the kth last symbol was read, then after the very last symbol is read, $\mathcal{N}$ will be in state q_k, which is an accepting state. Therefore $\mathcal{N}$ accepts w.

Conversely, suppose that $w \notin L$. Every path that w can take will be examined more closely. However, since there is only one transition that leads to multiple states, that is the only place that needs to be considered.

Firstly, if the machine always chooses the transition $q_0 \xrightarrow{1} q_0$, it will never reach the accepting state q_k. This case can be disregarded.

Now, let A be the set of indices such that $i \in A$ implies that $w(i) = 1$. Notice that $|w| - k \notin A$. As well, A is a finite set, so its elements can be enumerated in increasing order as

$$i_1 < i_2 < ... < i_m, \text{ for some } m \in \mathbb{N}$$

It will be proven using backwards induction on n that there is no path where $\mathcal{N}$ accepts w. In particular, the goal is to show that for all integers n so that $1 \leq n \leq m$, taking the transition $q_0 \xrightarrow{1} q_1$ when reading the i_nth symbol will result in w being rejected.

Base Case. Let $n = m$. Suppose the machine chooses the path that always transitions to q_0 until $\mathcal{N}$ reads $w(n)$, where it chooses $q_0 \xrightarrow{1} q_1$. Here, there are two cases to consider.

Case 1: $i_n > |w| - k + 1$.

Since $k - 1 > |w| - i_n$, there are less than $k-1$ symbols that remain to be read after transitioning to q_1. Thus the machine will end up in the state $q_{|w|-i_n}$, which is not an accepting state.

Case 2: $i_n < |w| - k + 1$.

On the other hand, now there are more than $k-1$ symbols to be read. After reading $k-1$ symbols, the machine will be in state q_k. Reading one more character will transition back to q_0. Since i_m is the maximal element in A, there are no more 1's to read, so w is rejected.

Induction Hypothesis. Let $l \in \mathbb{Z}$ such that $1 < l \leq m$, and suppose that for all $l \leq n \leq m$, the machine rejects w when it chooses to transition $q_0 \xrightarrow{1} q_1$ when reading $w(i_n)$. It will be shown that the same applies when the machine transitions $q_0 \xrightarrow{1} q_1$ when reading $w(i_{l-1})$.

Induction Step. Choosing the transition $q_0 \xrightarrow{1} q_1$, the two cases from the base case apply in the same way. The only difference is that in case 2, there are more 1's to be read. However, the index i_j of those 1's is an element of A and $l < j$, so it is covered by the induction hypothesis.

If the transition $q_0 \xrightarrow{1} q_0$ is taken, w can never be accepted. On the other hand, if the transition $q_0 \xrightarrow{1} q_1$ is taken for some $w(i_j)$, then w is rejected by the induction hypothesis.

By the principle of complete induction, it can be concluded that there is no path where $\mathcal{N}$ accepts $w \notin L$.

Therefore, the NFA $\mathcal{N}$ correctly accepts L, completing the proof.

□

c) **(3 pts)** *Bonus*: The smallest DFA that accepts L has to have exactly $2^{k+1} - 1$ number of states. lol

Q5. (10 pts) Solve question 4 on page 83 in the textbook.

Prove by induction that every finite language can be represented by a regular expression.

Proof. Let $\mathcal{L}$ be a finite language, and let $n \in \mathbb{N}$ represent the number of strings in $\mathcal{L}$. The claim will be proven using simple induction on n.

Base Case. Let $n = 0$. Then $\mathcal{L}$ is the empty set $\varnothing$, which is matched by the regex $\varnothing$.

Let $n = 1$. Then $\mathcal{L}$ contains only one string w. It can be shown using induction on $k = |w|$ that languages with one string can be expressed using some regular expression.

Base Case. Let $k = 0$. Then $w = \varepsilon$, which is matched by the regex ε.

Induction Hypothesis. Suppose that all strings of length m can be represented by a regex. The goal is to show that any string of length $m + 1$ can also be represented by a regex.

Induction Step. Assume that $|w| = m + 1$. Suppose that the last symbol of w is b, for some $b \in \Sigma$. Notice that $w = xb$, where x is a string with m characters. By the induction hypothesis, x is represented by some regex r. As well, b can be represented by the regex b.

Consider the regex rb. It follows that

$$\mathcal{L}(rb) = \mathcal{L}(r)\mathcal{L}(b)$$

which matches only $xb = w$.

By the principle of induction, it is true that all strings can be represented by a regex.

Thus finite languages of size 0 and 1 can be represented by a regex.

Induction Hypothesis. Let $l \in \mathbb{N}$. Suppose that any finite language of size l can be represented by a regex.

Induction Step. Let $\mathcal{L}$ be a language such that $|\mathcal{L}| = l+1$. Since $l+1 > 0$, $\mathcal{L}$ is non-empty, so there exists some string $w \in \mathcal{L}$. Consider the language $\mathcal{L} \setminus \{w\}$. The number of elements

in this set is l, so by the induction hypothesis, it can be represented as a regex r_1. As well, it was proved in the base case that any language with one string can be represented by a regex. In this case, let r_2 be the regex that represents the language $\{w\}$. It can be verified that the regex $r_1 + r_2$ represents $\mathcal{L}$, since

$$\mathcal{L}(r_1 + r_2) = \mathcal{L}(r_1) \cup \mathcal{L}(r_2) = \mathcal{L} \setminus \{w\} \cup \{w\} = \mathcal{L}$$

Thus by the principle of induction all finite languages can be represented as a regex.

□

End of questions