

Question 30.

Let  $U \subseteq \mathbf{R}^n$  be an open set in  $\mathbf{R}^n$ , and let  $K$  be a compact subset of  $U$ . Prove that there exists an *infinitely differentiable* function  $\varphi : \mathbf{R}^n \rightarrow [0, 1]$  such that  $\varphi(p) = 1$  for all  $p \in K$ , and  $\varphi(p) = 0$  for all  $p \in \mathbf{R}^n \setminus U$ . This is called a **bump function** supported on  $U$ .

(For a function  $f : U \rightarrow Y$ , the  *$n$ th total derivative*  $f^{(n)}$  is defined as follows: for  $n = 0$ , we set  $f^{(0)} = f$ ; for  $n \geq 1$ , if  $f^{(n-1)}$  is totally differentiable, we set  $f^{(n)} = (f^{(n-1)})'$ . We say that  $f$  is **infinitely differentiable** if  $f^{(n)}$  exists for all  $n \geq 0$ .)

*Proof.* First, we notice that the bump function in Big List #4 can be generalised to arbitrary intervals by simply performing horizontal translations. Now, we show that bump functions for closed rectangles within open rectangles in  $\mathbb{R}^n$  can be constructed.

Let  $R = \prod_{i=1}^n [a_i, b_i]$  be a closed rectangle that is inside an open rectangle  $S = \prod_{i=1}^n (c_i, d_i)$  in  $\mathbb{R}^n$ . This implies that for all  $i$ ,  $c_i < a_i \leq b_i < d_i$ . Considering this as an interval in  $\mathbb{R}$ , we can find a bump function  $\varphi_i$  such that  $\varphi_i([a_i, b_i]) = \{1\}$  and  $\varphi_i(\mathbb{R} \setminus (c_i, d_i)) = \{0\}$ . Notice that this is a function from  $\mathbb{R}$  to  $\mathbb{R}$ . We define  $\alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\alpha_i(x) = \varphi_i(x_i)$ . It will be shown that the bump function supported in  $S$  is

$$\beta(x) = \prod_{i=1}^n \alpha_i(x)$$

First, note that each  $\alpha_i$  is infinitely differentiable, so  $\beta$  is infinitely differentiable as well. If  $p = (p_1, \dots, p_n) \in R$ , then for all  $i \in \{1, \dots, n\}$ ,  $p_i \in [a_i, b_i]$ , so  $\alpha_i(p) = 1$ . We have

$$\beta(p) = \prod_{i=1}^n \alpha_i(p) = 1$$

Using a similar argument, if  $p \in \mathbb{R}^n \setminus S$ , there is at least one component of  $p$  such that  $p_i \notin (c_i, d_i)$ , and so  $\alpha_i(p) = 0$ , which implies that

$$\beta(p) = 0$$

as desired.

Additionally, if there exists bump functions  $\alpha_1$  and  $\alpha_2$  supported on open sets  $U_1$  and  $U_2$  respectively, where the compact sets are  $K_1 \subseteq U_1$  and  $K_2 \subseteq U_2$ , then there exists a bump function  $\varphi$  supported on  $U_1 \cup U_2$  for the compact set  $K_1 \cup K_2$ , which is defined by

$$\varphi(x) = \alpha_1(x) + \alpha_2(x) - \alpha_1(x)\alpha_2(x).$$

Verifying this, we see that if  $x \in K_1 \cup K_2$ , then either  $x \in K_1$  or  $x \in K_2$ . Assuming without loss of generality that  $x \in K_1$ , we have that

$$\varphi(x) = 1 + \alpha_2(x) - \alpha_2(x) = 1$$

As well, if  $x \in \mathbb{R}^n \setminus (U_1 \cup U_2)$ , then  $\varphi(x) = 0$ , showing that  $\varphi$  is indeed a bump function.

Next, let  $\mathbb{R}^n$  be equipped with the max-norm. We claim that  $\prod_{i=1}^n (p_i - r, p_i + r) \subseteq B(p, r)$ , for  $p \in \mathbb{R}^n$  and  $r > 0$ .

Let  $x \in \prod_{i=1}^n (p_i - r, p_i + r)$ . For all  $i$ ,  $|p_i - x_i| < r$ , so  $\|x - p\|_{\max} < r$ . Hence  $x \in B(p, r)$ .



Question 31.

(a) Let  $A \in M_n(\mathbf{R})$  be a symmetric matrix and let  $Q(\vec{x}) = \vec{x}^T A \vec{x}$  be the corresponding quadratic form. Prove that the following two statements are equivalent:

- (i)  $Q(\vec{x}) > 0$  for all  $\vec{x} \neq \vec{0}$ .
- (ii) All eigenvalues of  $A$  are strictly positive.

In this case, we say that  $Q$  is a **positive definite** quadratic form, and that  $A$  is a **positive definite** matrix.

- (b) Prove the following “stay away” lemma, which you will need for part (c): if  $Q : \mathbf{R}^n \rightarrow \mathbf{R}$  is a positive definite quadratic form, then there exists a constant  $\eta > 0$  such that  $Q(\vec{x}) \geq \eta \|\vec{x}\|^2$  for all  $\vec{x} \in \mathbf{R}^n$ .
- (c) Let  $U \subseteq \mathbf{R}^n$  be an open set, let  $f : U \rightarrow \mathbf{R}$  be a twice continuously differentiable function, and let  $p_0 \in U$  be a point at which  $\nabla f(p_0) = \vec{0}$ . Prove that if the Hessian matrix  $Hf(p_0)$  is positive definite, then  $f$  achieves a **local minimum** at  $p_0$ ; i.e.  $p_0$  has an open neighborhood  $U_0$ , contained in  $U$ , such that  $f(p) \geq f(p_0)$  for all  $p \in U_0$ .

*Proof.* (a):

Suppose that  $Q(\vec{x}) > 0$  for all  $\vec{x} \neq 0$ . Since  $A$  is symmetric, it is orthogonally diagonalizable, so there exists an orthogonal matrix  $P \in M_n(\mathbb{R})$  such that  $B = P^T A P$  is diagonal. Letting  $Q_B(\vec{x}) = \vec{x}^T B \vec{x}$ , we have that

$$Q(\vec{x}) = Q_B \circ P(\vec{x})$$

We will show that  $Q_B(\vec{x}) > 0$  for all  $\vec{x} \neq 0$ .

Let  $\vec{x} \in \mathbb{R}^n$  so that  $\vec{x} \neq 0$ . Since  $Q(\vec{x}) > 0$ , it follows that  $Q_B(\vec{x}) > 0$  as well, which implies that its eigenvalues are strictly positive. Since  $A$  and  $B$  share the same eigenvalues, we are done with this direction.

□