Question 36.

In Handout #7, we defined differentiability for functions on open sets. Now we give a definition that works over arbitrary sets. For this problem, you will need to read Piazza Post @274 and use Theorem 1.1.

Let $A \subseteq \mathbf{R}^n$ be an arbitrary set, let $f : A \to \mathbf{R}$ be a function, and let $p \in A$ be a point. We say that f is **differentiable** at p if there exists an open neighborhood U of p and a function $\widehat{f} : U \to \mathbf{R}$ such that $\widehat{f}$ is differentiable at p (in the sense of Handout #7) and $\widehat{f}|_{U \cap A} = f|_{U \cap A}$.

(a) Prove that f is differentiable at every point of A if and only if f extends to a differentiable function defined on an open set containing A.

(b) Suppose further that A is closed. Prove that f is differentiable at every point of A if and only if f extends to a differentiable function on $\mathbf{R}^n$.

Proof.

(a):

Suppose that f extends to a differentiable function $\hat{f}$ on an open set $U \supseteq A$. That is, $\hat{f}|_A = f$. Let $x \in A$. Since U is open, we can find an open ball such that $B(x, \varepsilon) \subseteq U$. Immediately, we get that the function $\hat{f}|_{B(x,\varepsilon)}$ is the desired extension of f at x, as $\hat{f}$ is differentiable at x and $\hat{f}|_{B(x,\varepsilon) \cap A} = f_{B(x,\varepsilon) \cap A}$.

Conversely, suppose that f is differentiable at every point of A. For all $p_a \in A$, there exists an open neighborhood U_a of p_a and a function $f_a : U_a \to \mathbb{R}$ that is differentiable at p_a and $f_a|_{U_a \cap A} = f|U \cap A$. Notice that $U = \{U_a\}_{a \in I}$ forms an open cover of A. Thus we can find a partition of unity $\{\varphi_a : \mathbb{R}^n \to [0,1]\}_{a \in I}$ subordinate to U. In particular,

- $\varphi_a \in C^\infty$,
- $\operatorname{supp}(\varphi_a) \cap U \subseteq U_a$,
- $\{\operatorname{supp}(\varphi_a)\}$ is locally finite,
- $\sum\limits_{a \in I} \varphi_a = 1$.

We claim that $\hat{f} = \sum_{a \in I} \varphi_a \cdot f_a$ is the differentiable function that extends f to $\{U_a\}_{a \in I}$. First, note that this sum is well defined because the partition of unity is locally finite, so the sum is finite at every point, and as well, any point outside the domain of f_a implies that $\varphi_a = 0$, so $\varphi_a \cdot f_a = 0$.

Moreover, for all $x \in U$, since $\{\varphi_a\}_{a \in I}$ is locally finite, for some open neighborhood V of x, for some finite sequence of $a_n \in I$,

$$\hat{f}(y) = \sum_{n=1}^{N} \varphi_{a_n}(y) \cdot f_{a_n}(y), \text{ where } y \in V.$$

Since each term is differentiable, $\hat{f}$ is differentiable at x. Thus $\hat{f}$ is differentiable on the entirety of U. Additionally, for each a_n, $x \in U_{a_n}$, so by our initial assumption,

$$\sum_{n=1}^{N} \varphi_{a_n}(x) \cdot f_{a_n}(x) = f(x) \cdot \sum_{n=1}^{N} \varphi_{a_n}(x) = f(x)$$

which shows that $\hat{f}|_A = f$ as needed.

(b):

Suppose that A is a closed set.

The converse of this statement is exactly the same as the previous part, so we will omit the proof.

Suppose that f is differentiable at every point of A. By the previous part, f extends to a differentiable function $\widetilde{f}$ on some open set U that contains A. We want to find an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$. We see that the set $V = \left\{ x \in \mathbb{R}^n : \operatorname{dist}_A(x) < \frac{\operatorname{dist}(A, U^c)}{2} \right\}$ is sufficient. It is pretty immediate that $A \subseteq V$, as any element of A has distance 0 from A. As well, if we let $v \in \overline{V}$, since v is a limit point of V we can find $v' \in V$ such that $\|v - v'\| < \frac{\operatorname{dist}(A, U^c)}{4}$. Furthermore, by the definition of infimum we can find $a \in A$ such that $\|v' - a\| < \operatorname{dist}_A(x) + \frac{\operatorname{dist}(A, U^c)}{4} < \frac{3\operatorname{dist}(A, U^c)}{4}$. Thus we have that

$$\|a - v\| \leq \|a - v'\| + \|v' - v\| < \operatorname{dist}(A, U^c)$$

For all $w \in U^c$,

$$\|w - v\| \geq \|w - a\| - \|a - v\| > \operatorname{dist}(A, U^c) - \operatorname{dist}(A, U^c) = 0$$

which implies that $v \notin U^c$, so $v \in U$ and $\overline{V} \subseteq U$.

Now, notice that A^c is open and $A^c \cup V = \mathbb{R}^n$. Thus $\{A^c, V\}$ form an open cover of $\mathbb{R}^n$, so we can find a partition of unity $\{\varphi_{A^c}, \varphi_V\}$ subordinate to $\{A^c, V\}$. We claim that our desired function $\widehat{f}$ is given by

$$\widehat{f} = \widetilde{f} \cdot \varphi_V.$$

This function is well defined, as outside of the domain of $\widetilde{f}$, it is identitally equal to 0.

First, we will show that $\widehat{f}$ is differentiable on $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$, and begin by considering the case when $p \in \overline{V} \subseteq U$. Then for some small enough neighborhood W of p, for all $x \in W$,

$$\widehat{f}(x) = \widetilde{f}(x) \cdot \varphi_V(x)$$

so $\widehat{f}$ is differentiable at p.

If $p \in \mathbb{R}^n \setminus \overline{V}$, then it is also true that $p \notin V$, which is outside of $\operatorname{supp}(\varphi_V)$. Thus, using the fact that $\mathbb{R}^n \setminus \overline{V}$ is open, we get an open neighborhood W' of p such that

$$\widehat{f}(x) = 0, \text{ for } x \in W'.$$

so $\widehat{f}$ is differentiable at p.

Now, we show that it is a valid extension of f. For all $x \in A$, notice that $\varphi_{A^c}|_A \equiv 0$, so it must be true that $\varphi_V|_A \equiv 1$. Then it follows that

$$\widehat{f}|_A = \widetilde{f}|_A = f|_A$$

and the proof is complete.

$\square$

Question 37.

The following set is called the n**-simplex**:

$$\Delta_n := \{\vec{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \cdots + x_n \leq 1\}.$$

You can assume, without proof, that Δ_n is Jordan measurable.
Find, with proof, an explicit formula for $\mu(\Delta_n)$ in terms of n.

Proof. First, we show that Δ_n is the same as the set

$$S = \left\{(x_1, ..., x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, ..., 0 \leq x_n \leq 1 - \sum_{i=1}^{n-1} x_i\right\}$$

Let $x \in \Delta_n$. We want to show that $0 \leq x_i \leq 1 - \sum_{j=1}^{i-1} x_j$. We get that $x_i \geq 0$ immediately. As well, since $\sum_{j=1}^{n} x_j \leq 1$ and every component is non-negative, we have that

$$x_i \leq 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^{n} x_j \leq 1 - \sum_{j=1}^{i-1} x_j$$

which shows that $\Delta_n \subseteq S$.
Now, let $x \in S$. We know every x_i is non-negative and additionally

$$x_n \leq 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^{n} x_i \leq 1$$

so $S \subseteq \Delta_n$.
Now, we proceed to find $\mu(S) = \mu(\Delta_n)$. Using Fubini's Theorem, we get

$$\mu(S) = \int_S 1 = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 \, dx_n \cdots dx_2 \, dx_1$$

Let $I : \mathbb{N} \times [0,1] \to \mathbb{R}$ be defined recursively as follows:

$$\begin{aligned} I(1, \alpha) &= \int_0^{1-\alpha} 1 \, dt \\ I(k, \alpha) &= \int_0^{1-\alpha} I(k-1, \alpha + t) \, dt, && \text{for } > 1. \end{aligned}$$

Notice that if we continue applying the definition, we get that

$$I(n, 0) = \mu(S)$$

Now, we will prove using induction on n that for all $\alpha \in [0,1]$, $I(n, \alpha) = \dfrac{1}{n!}(1-\alpha)^n$.
Let $n = 1$. Then

$$I(1, \alpha) = \int_0^{1-\alpha} 1 \, dt = 1 - \alpha$$

Now, suppose that $I(k, \alpha) = \frac{1}{k!}(1-\alpha)^k$ holds for all $\alpha \in [0,1]$ and some $k \in \mathbb{N}$. We want to show that the same holds for $k+1$ as well. For an arbitrary α, we get

$$\begin{aligned} I(k+1, \alpha) &= \int_0^{1-\alpha} I(k, \alpha + t)\ dt \\ &= \int_0^{1-\alpha} \frac{1}{k!}(1-\alpha-t)^k\ dt \\ &= -\frac{1}{(k+1)!}(1-\alpha-t)^{k+1}\Big|_0^{1-\alpha} \\ &= \frac{1}{(k+1)!}(1-\alpha)^{k+1} \end{aligned}$$

as desired. Thus we get that

$$\mu(S) = I(n, 0) = \frac{1}{n!}$$

$\square$