## Question 42

Let  $M \subseteq \mathbf{R}^N$  be a smooth *n*-manifold (with or without boundary!).

- (a) Show that if n < N, then M is a Lebesgue null set.
- (b) Show that if n = N and M is closed and its boundary is nonempty, then  $\partial M$  coincides with the usual topological boundary (as defined on Handout #2).
- (c) Show that if M is compact and its boundary is nonempty, then M is Jordan measurable.

Proof.

(a):

We begin by proving a number of lemmas:

**Lemma 1:** An open cover of any subset  $M \subseteq \mathbb{R}^n$  has a countable subcover.

We know that  $\mathbb{R}^n$  is separable, so M is also separable. Let C be a countable dense subset of M. Let  $\mathcal{U}$  be an open cover for M. We construct the countable subcover  $\hat{U}$  as follows. For each  $q \in C$  and  $k \in \mathbb{Q}$ , consider the open ball B(q,k). If there exists a  $U_{qk} \in \mathcal{U}$  such that  $B(q,k) \in U_{qk}$ , include it in  $\hat{U}$ . Notice that  $\hat{U}$  is at most countable. We claim that it is also an open cover.

Let  $x \in M$ . Then it is contained in some open set  $U \in \mathcal{U}$ . As well, we can find an open ball such that  $B(x, \delta) \in U$ . Since C is dense, we can find  $q \in C$  such that  $q \in B(x, \frac{\delta}{4})$ . Let  $k \in \mathbb{Q}$  such that  $\frac{\delta}{4} < k < \frac{\delta}{2}$ . Then  $x \in B(q, k) \subseteq B(x, \delta)$ , because for all  $y \in B(q, k)$ ,

$$||x - y|| \le ||x - q|| + ||q - y|| < \frac{\delta}{4} + \frac{\delta}{2} < \delta$$

It follows that  $B(q, k) \in U$ , so it is guaranteed that some  $U_{qk}$  from our construction exists. Thus  $x \in U_{qk} \in U$  so U is indeed an open cover and we are done.

Lemma 2: A countable union of sets with Jordan measure 0 is a Lebesgue null set.

Let  $E = \bigcup_{i \geq 1} E_i$ , where  $\mu(E_i) = 0$ . Let  $\varepsilon > 0$ . For each  $E_i$ , we can find a finite union of boxes  $B_i$  such that  $B_i \supseteq E_i$  and  $\operatorname{vol}(B_i) < \frac{\varepsilon}{2^i}$ . We see that  $\bigcup_{i \geq 1} B_i$  is a countable union of boxes,  $E \subseteq \bigcup_{i \geq 1} B_i$ , and

$$\sum_{i=1}^{\infty} \operatorname{vol}(B_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2(1 - \frac{1}{2})} = \varepsilon$$

as desired

Now, we prove the problem at hand. Let  $M \subseteq \mathbb{R}^N$  be a smooth n-manifold with n < N. Let  $\{(U_i, \varphi)\}_{i \in I}$  be an atlas for M. By Lemma 1, we can assume without loss of generality that the atlas is countable. We can also assume that each  $U_i$  is bounded, for if not, we can take a countable union of open balls that cover the unbounded  $U_i$ , and restrict the embedding to each ball.

For each chart  $(U_i, \varphi_i)$ , assume that the domain  $\hat{U}$  is measurable.

(b):

Let p be a point in the topological boundary of M. We will show that  $p \in \partial M$ . Suppose for contradiction that p is in  $M^{\circ}$ , meaning it is contained in a chart  $(U, \varphi)$  that is diffeomorphic

to an open set  $\hat{U}$  in  $\mathbb{R}^N$ . Note that  $\varphi$  is a diffeomorphism with domain  $\hat{U}$  and codomain U. Then it must be true that  $\varphi(\hat{U}) = U$  is an open subset of  $\mathbb{R}^N$ . But this is a contradiction, as that would imply that the boundary point p is in the interior of M. Therefore  $p \in \partial M$ . Next, let  $p \in \partial M$  and again suppose for contradiction that p is not in the topological boundary of M. Then it must be true that p is in the topological interior of M. Recall that  $p \in \partial M$  implies that it is contained in a chart  $(U, \varphi)$  that is diffeomorphic to  $\overline{\mathbb{H}^n}$  and  $p \in \mathrm{bd}(\mathbb{H}^n)$ . Since p is in the topological interior of M, we can find an open ball  $B(p, r) \subseteq M$  which is also open in  $\mathbb{R}^N$ . Then  $\varphi^{-1}(B(p, r))$  should also be open in  $\mathbb{R}^N$ . But this implies that for small enough  $\delta$ ,  $\varphi(p) - (0, ..., \delta) \in \overline{\mathbb{H}^N}$ , which cannot happen.

Therefore we can conclude that  $\partial M$  coincides with the topological boundary of M.

(c):

First, we prove that a compact Lebesgue null set E has Jordan measure 0. It suffices to show that the upper measure  $\mu^*(E) = 0$ .

Let  $\varepsilon > 0$ . By definition, we can find a countable union of boxes  $B = \bigcup_{i=1}^{\infty} B_i$  such that  $E \subseteq B$  and  $vol(B) < \varepsilon$ . But since E is compact, it can be covered by finitely many boxes  $B_{n_i}$ ,  $0 < i \le N$ . Thus

$$\operatorname{vol}\left(\bigcup_{i=1}^{N} B_{n_i}\right) = \sum_{i=1}^{N} \operatorname{vol}(B_{n_i}) \le \operatorname{vol}(B) < \varepsilon.$$

Since  $\varepsilon$  was chosen arbitrarily, we can conclude that  $\mu^*(E) = 0$ , and E has Jordan measure 0.

Now, suppose that M is compact and its boundary is nonempty. If  $\dim M < N$ , M is a Lebesgue null set and has Jordan measure 0, and therefore measurable. Otherwise, if  $\dim M = N$ , since the boundary of M is non-empty, the topological boundary of M is actually a smooth manifold of dimension (N-1), and therefore a Lebesgue null set. As well, the topological boundary of M is compact, so it is Jordan measure 0, which implies that M is Jordan measurable.

## Question 34

Let  $\Phi: \mathbf{R}^n \to \mathbf{R}^m$  be a  $C^1$  mapping.

- (a) Suppose that n > m = 1. Show that  $\Phi$  cannot be injective
- (b) Suppose that n < m. Show that if  $K \subseteq \mathbf{R}^n$  is a compact set, then  $\Phi(K)$  is a Jordan measurable set, and has Jordan measure zero.

Proof.

(a):

Suppose for contradiction that n > m = 1 and  $\Phi$  is a  $C^1$  injective function. Since  $\Phi$  cannot be a constant function, by the results of Big List #26, there is a  $p \in \mathbb{R}^n$  so that  $\nabla \Phi(p) \neq 0$ . In particular, we will say that  $\frac{\partial \Phi}{\partial x_j} \neq 0$ . Define  $\alpha : \mathbb{R}^n \to \mathbb{R}$  by  $\alpha(x) = \Phi(x) - \Phi(p)$ . Injectivity is translation-invariant, so  $\alpha$  is injective. Notice that  $\alpha(p) = 0$ . We can apply the implicit function theorem to obtain an open set  $W \subseteq \mathbb{R}^{n-1}$  that contains  $p' = (p_1, ..., p_{j-1}, p_{j+1}, ..., p_n)$  and a  $C^1$  function  $\Psi : W \to \mathbb{R}$  such that for all  $x = (x_1, ..., x_{n-1}) \in W$ ,

$$\alpha(x_1, ..., x_{i-1}, \Psi(x), x_i, ..., x_{n-1}) = 0$$

Then, since W is open and contains p', we can find another distinct point  $q \in W$ . We have

$$\alpha(p_1, ..., p_{j-1}, \Psi(p'), p_{j+1}, ..., p_n) = 0 = \alpha(q_1, ..., q_{j-1}, \Psi(q), q_j, ..., q_{n-1})$$

which contradicts the fact that  $\alpha$  is injective.

(b):

Since K is compact, and thus bounded, we can enclose it in a closed box  $B = [-L, L]^n$ , for some positive L. It suffices to show that  $\Phi(B)$  has measure 0, as we can apply the monotonicity of measure to conclude that  $\Phi(K)$  has measure 0.

First, let  $\hat{\Phi}: \mathbb{R}^m \to \mathbb{R}^m$  be defined by  $\hat{\Phi}(x) = \Phi(\pi_{\mathbb{R}^n}(x))$ . That is,  $\hat{\Phi}$  first projects vectors in  $\mathbb{R}^m$  onto  $\mathbb{R}^n$  and then composes it with  $\Phi$ . Let  $\hat{B} = B \times \{0\}^{m-n}$ . Then note that  $\hat{\Phi}(\hat{B}) = \Phi(B)$ . Trivially,  $\hat{B}$  has measure 0. We now show that  $\Phi(B)$  also has measure 0. Let  $\varepsilon > 0$ . Since  $\hat{\Phi}$  is  $C^1$ , its derivative is continuous. By the extreme value theorem, each

Let  $\varepsilon > 0$ . Since  $\Phi$  is  $C^*$ , its derivative is continuous. By the extreme value theorem, each component derivative attains a maximum on  $\hat{B}$ . Let  $\alpha$  a positive number greater than all the maximums. Since  $\hat{B}$  has measure 0, we can find a finite number of cubes  $B_1, ..., B_k$  with side length d such that

$$\hat{B} \subseteq \bigcup_{i=1}^k B_i \text{ and } \sum_{i=1}^k \operatorname{vol}(B_i) < \frac{\varepsilon}{m^m \alpha^m}$$

Consider some cube  $B_i = \prod_{j=1}^m [a_{ij}, a_{ij} + d]$ . For each component  $\hat{\Phi}_j : \mathbb{R}^m \to \mathbb{R}$ , it must be true that  $\hat{\Phi}_j$  attains a maximum  $\hat{\Phi}_j(M_{ij})$  and minimum  $\hat{\Phi}_{ij}(m_{ij})$ , for some  $M_{ij}, m_{ij} \in B_i$ . Define  $g: [0,1] \to \mathbb{R}$  by

$$q(t) = \hat{\Phi}_i(tM_{ij} + (1-t)m_{ij})$$

By the Mean Value Theorem, there exists some  $c \in (0,1)$  such that

$$g(1) - g(0) = g'(c) \implies \hat{\Phi}_i(M) - \hat{\Phi}_i(m) = \hat{\Phi}'_i(cM_{ij} + (1-c)m_{ij})(M_{ij} - m_{ij})$$

Taking the 2-norm of both sides, it follows that

$$\hat{\Phi}_j(M_{ij}) - \hat{\Phi}_j(m_{ij}) \le \|\hat{\Phi}_j'(cM_{ij} + (1 - c)m_{ij})\| \|M_{ij} - m_{ij}\| \le (\sqrt{m}\alpha)(\sqrt{m}d) = m\alpha d$$

Note that we used the fact that each component of  $\hat{\Phi}'_j(cM_{ij} + (1-c)m_{ij})$  is bounded by a and  $||M_{ij} - m_{ij}||$  can be no bigger than the diagonal of  $B_i$ .

Finally, we see that the box  $C_i = \prod_{j=1}^m [\hat{\Phi}_j(m_{ij}), \hat{\Phi}_j(M_{ij})]$  covers  $\hat{\Phi}(B_i)$ , and its volume is given by

$$\prod_{j=1}^{m} (\hat{\Phi}_j(M_{ij}) - \hat{\Phi}_j(m_{ij})) \le \prod_{j=1}^{m} m\alpha d = m^m \alpha^m \text{vol}(B_i)$$

From here, we know that the union  $\bigcup_{i=1}^k C_i \supseteq \hat{\Phi}(\hat{B})$  but

$$\sum_{i=1}^{k} C_i < m^m \alpha^m \sum_{i=1}^{k} \operatorname{vol}(B_i) < \varepsilon.$$

Therefore  $\Phi(B)$  has measure 0, so  $\Phi(K)$  has measure 0 and we are done.