Let V be a vector space over the field \mathbb{F} , and S a (non-empty) set. Let $\mathcal{F}(S,V) = \{f : S \to V\}$ be the set of V-valued functions.

We define addition and scaling on $\mathcal{F}(S, V)$ pointwise:

$$(f+g)(s) = f(s) + g(s)$$
$$(cf)(s) = cf(s)$$

We will verify some of the vector space axioms required to prove that $\mathcal{F}(S,V)$ is a vector space over \mathbb{F} .

- (a) Why do these operations make sense?
- (b) Prove (using only the definitions above, and the fact that V is a vector space) that c(f+g)=cf+cg for all $f,g\in\mathcal{F}(S,V)$ and $c\in\mathbb{F}$.
- (c) Prove that for all $f \in \mathcal{F}(S, V)$ there exists $g \in \mathcal{F}(S, V)$, so that f + g = 0. (Here $0: S \to V$ is the constant function defined by $0(s) = 0_V$ for $s \in S$.)

Proof.

(a):

To show that these operations are well defined, notice that $f(s), g(s) \in V$, so by the axiom of closure on V we have that

$$(f+g)(s) = f(s) + g(s) \in V \text{ and } (cf)(s) = cf(s) \in V$$

(b):

We will do this by showing that for all $s \in S$, we have c(f(s) + g(s)) = cf(s) + cg(s). Fix $s \in S$. It follows that $f(s), g(s) \in V$, so by the axiom of distributivity in V, we have that

$$c(f(s) + g(s)) = cf(s) + cg(s)$$

(c)

Let $f \in \mathcal{F}(S, V)$. Choose $g = (-1 \cdot f)$. Then for all $s \in S$,

$$f(s) + g(s) = f(s) + (-f(s)) = 0$$

as needed.

 \Box

Let
$$W = \left\{ (x, y, z, w) \in \mathbb{Q}^4 \middle| \begin{array}{l} x + 5w = y + 5z \\ y = 4w - 3z \\ x + y + z = 3w \end{array} \right\}$$

Do not use Q3 to solve this problem. This problem is a "warm up" for Q3.

- (a) Rearrange the equations defining W to show that W is the set of solutions to a homogeneous system of equations.
- (b) Solve the system using row-reduction and express the general solution as a linear combination of the "basic solutions".
- (c) Show that $W = \operatorname{span} S$, for some set $S \subseteq \mathbb{Q}^4$.
- (d) Deduce that W is a subspace of \mathbb{Q}^4 .

Proof.

(a):

Rearranging, the equations become

$$\begin{cases} x - y - 5z + 5w = 0 \\ y + 3z - 4w = 0 \\ x + y + z - 3w = 0 \end{cases}$$

(b):

The augmented matrix associated with this system of equations is

$$\begin{pmatrix}
1 & -1 & -5 & 5 & 0 \\
0 & 1 & 3 & -4 & 0 \\
1 & 1 & 1 & -3 & 0
\end{pmatrix}$$

Row reducing this, we get

$$\begin{pmatrix}
1 & -1 & -5 & 5 & 0 \\
0 & 1 & 3 & -4 & 0 \\
1 & 1 & 1 & -3 & 0
\end{pmatrix}
\xrightarrow{r_3 \to r_3 - r_1}
\begin{pmatrix}
1 & -1 & -5 & 5 & 0 \\
0 & 1 & 3 & -4 & 0 \\
0 & 2 & 6 & -8 & 0
\end{pmatrix}
\xrightarrow{r_1 \to r_1 + r_2}
\begin{pmatrix}
1 & 0 & -2 & 1 & 0 \\
0 & 1 & 3 & -4 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

We parameterize z and w to obtain that

$$x = 2s - t$$

$$y = -3s + 4t$$

$$z = s$$

$$x = t$$

so the general solution of this system of equations is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

Let
$$S = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$$
. Let $\vec{v} \in \text{span}S$. It follows that

$$\vec{v} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

for some $s, t \in \mathbb{Q}$. But notice that this is actually a solution to the system in W, so $\vec{v} \in W$. Now let $\vec{w} \in W$, so \vec{w} solves the system of equations in W, but this means that we can write \vec{w} as a linear combination of the vectors in S, so $\vec{w} \in \text{span}S$, so W = spanS.

(d):

By the previous part, W is actually a spanning set, and we know that all spanning sets are subspaces, so we conclude that W is a subspace of \mathbb{Q}^4 .

 \Box

We now generalize Q2. Consider a linear system with m equations and n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

We saw in Week 3 that any solution $x = (x_1, x_2, ..., x_n) \in \mathbb{F}^n$ can be expressed as $x = \sum_{i=1}^k t_i x_i$, where $t_i \in \mathbb{F}$ are the parameters, and $x_i \in \mathbb{F}^n$ are the "basic solutions". Let W be the set of solutions to this system.

- (a) Prove that $W = \operatorname{span} S$ for some set S, and hence that W is a subspace of \mathbb{F}^n .
- (b) Prove that the set $\{x_1, x_2, \dots, x_k\}$ is linearly independent. (Hint: Think about the variables which correspond to the choice of parameters. There is exactly one vector for each such parameter. Use the corresponding entry to show that if $t_1x_1 + t_2x_2 + \dots + t_kx_k = 0$ then $t_i = 0$ for each i.)
- (c) Find a basis for W.

Proof.

(a):

Let $S = \{x_1, ..., x_k\}$. We note that any linear combination of the vectors in S form a solution to the system, but not only that, any solution to the system can be written as a linear combination of these vectors, so span S = W.

(b)

Suppose that $\sum_{i=1}^{k} t_i x_i = 0$.

(c):

From part (a) and part (b), the set $S = \{x_1, ..., x_k\}$ spans W and is linearly independent, which by definition forms a basis of W.

Is the set $S = \{e_1 + 2e_2 - 3e_3, e_1 + e_2 - e_3, e_2 - e_3\} \subseteq \mathbb{Q}^3$ a basis for \mathbb{Q}^3 ? Justify your answer

Proof. We claim that S is indeed a basis for \mathbb{Q}^3 . For convenience, denote the vectors in S by v_1, v_2, v_3 respectively. Notice that

$$e_1 = v_2 - v_3, \ e_2 = -v_1 + v_2 + 2v_3, \ e_3 = -v_1 + v_2 + v_3.$$

Thus, for any $x \in \mathbb{Q}^3$, since $\{e_1, e_2, e_3\}$ is a basis for \mathbb{Q}^3 , for some $a, b, c \in \mathbb{Q}$, we have that

$$x = ae_1 + be_2 + ce_3 = a(v_2 - v_3) + b(-v_1 + v_2 + 2v_3) + c(-v_1 + v_2 + v_3)$$

$$\implies x = (-b - c)v_1 + (a + b + c)v_2 + (-a + c)v_3$$

which shows that S spans \mathbb{Q}^3

Now, for constants $p, q, r \in \mathbb{Q}$, suppose that

$$0 = pv_1 + qv_2 + rv_3$$

SUbstituting back our values, we get that

$$0 = p(e_1 + 2e_2 - 3e_3) + q(e_1 + e_2 - e_3) + r(e_2 - e_3)$$

= $(p+q)e_1 + (2p+q+r)e_2 + (-3p-q-r)e_3$

By the linear independence of the standard vectors, we have that

$$p+q = 0$$

$$2p+q+r = 0$$

$$-3p-q-r = 0$$

We can solve for p, q, r to get that p = q = r = 0.

Thus we can conclude that S is a basis for \mathbb{Q}^3 .

Г

Let V be a finite dimensional vector space over a field \mathbb{F} .

- (a) Prove that if $W \subseteq V$ is a subspace with basis β_W , then there exists a linearly independent set α so that $\beta = \beta_W \cup \alpha$ is a basis for V. (We say that β "extends" β_W . So you are proving that "every basis of a subspace W can be extended to a basis of V".)
- (b) Prove that for any linearly independent set I and spanning set S, we have $|I| \leq \dim V \leq |S|$.

Proof.

(a):

Let γ be a basis for V. Since β_W is linearly independent, we apply the Replacement Theorem to get that there exists a subset $\alpha \subseteq \gamma$ such that $\beta_W \cup \alpha$ is a basis for V, and we are done.

(b):

Let I be a linearly independent set. Then W = span(I) is a subspace of V with basis I. By part (a), we can extend I to a basis β of V, where $\beta = I \cup \alpha$. Since $|\alpha| \ge 0$, we have that

$$\dim V = |\beta| = |I| + |\alpha| \ge |I|$$

Now, let S be a spanning set of V. If dim V = 0, then $V = \{0\}$ and its basis is $\beta = \emptyset$. S must be either \emptyset or $\{0\}$, so dim $V \leq |S|$.

If dim V > 0, it contains a non-zero vector, so S also contains a non-zero vector. Pick $s_0 \in S$, and note that $S_0 = \{s_0\}$ is linearly independent.

If there are no elements w_i in S such that $\{s_1, w_i\}$ is linearly independent, that is, $s = c_i w_i$ for some $c_i \in \mathbb{F}$, then for all $v \in V$, because S is a spanning set, for m vectors in S we have that

$$v = \sum_{i=1}^{m} a_i w_i = \sum_{i=1}^{m} a_i \cdot c_i s_0$$

which implies that $\operatorname{span}(\{s_0\}) = V$, and the result that we want follows immediately after. If not, we can find another non-zero vector s_1 so that $S_1 = \{s_0, s_1\}$ is linearly independent. We repeat this process until we have a linearly independent set $S_n = \{s_0, ..., s_n\}$, where $n = \dim V$. We claim that S_n is a basis for V, and it suffices to show that S_n spans V. First, if $S_n = S$, then our result is immediate. Otherwise, let $s_j \in S$ so that $s_j \notin S_n$. Consider the set $S_n \cup \{s_j\}$, whose number of elements is greater than dim V. By the first half of this proof, we know that no linearly independent set can have a size larger than dim V, so it must be true that $S_n \cup \{s_j\}$ is linearly dependent. In particular, for constants $a_j, a_{ij} \in \mathbb{F}$ not all zero,

$$0 = a_j s + \sum_{i=1}^n a_{ij} s_i.$$

Notice that $a_i \neq 0$, for if not, then we get that

$$0 = \sum_{i=1}^{n} a_{ij} s_i$$

which implies that every $a_{ij} = 0$: a contradiction. Thus we can rearrange for s to obtain

$$s = a_j^{-1} \sum_{i=1}^n a_{ij} s_i$$

Now, let $v \in V$. Since S spans V, we have that for m vectors $s_i \in S$ and non-zero $c_i \in \mathbb{F}$,

$$v = \sum_{j=1}^{m} c_j s_j = \sum_{j=1}^{m} \left(c_j \cdot a_j^{-1} \sum_{i=1}^{n} a_{ij} s_i \right)$$

which shows that v is a linear combination of the vectors in S_n , so S_n is a linearly independent spanning set for V. Therefore we can conclude that S_n is a basis for V and we are done.

П

Consider a matrix $M \in \mathcal{M}_{n \times n}(\mathbb{F})$. Given $p \in \{1, \dots, n\}$ we can split M into "blocks":

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A is $k \times k$, B is $k \times (n-k)$, C is $(n-k) \times k$ and D is $(n-k) \times (n-k)$. For example, if n=5 and k=2, then such a block matrix would be of the form

$$M = \begin{pmatrix} 1 & 2 & 3 & 2 & 3 \\ -5 & 3 & 3 & 1 & 1 \\ \hline 1 & 2 & 0 & -1 & 1 \\ 3 & 1 & 3 & -1 & 7 \\ 1 & 0 & -1 & 3 & 5 \end{pmatrix}$$

where
$$A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$$
, $B = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 7 \\ -1 & 3 & 5 \end{pmatrix}$.

Prove that if $M = \begin{pmatrix} A & B \\ \hline C & D \end{pmatrix}$ and $N = \begin{pmatrix} A' & B' \\ \hline C' & D' \end{pmatrix}$, then

$$\alpha M + N = \left(\begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array} \right)$$

Proof.

Let $W = \left\{ A \in \mathcal{M}_{2n \times 2n}(\mathbb{F}) \mid A = \left(\frac{X - X^T}{O_n} \middle| \frac{O_n}{X + X^T} \right) \text{ with } X \in \mathcal{M}_{n \times n}(\mathbb{F}) \right\}$. (Assume char(\mathbb{F}) \neq 2.)

- (a) Let n=2. Find a basis for W.
- (b) Now generalize to arbitrary n. Find a basis for W, and use it to compute dim W.

Proof.

(a):

Let n=2. We claim that a basis for W is given by

For convenience, we label these matrices in order as f_1 , f_2 , f_3 , and f_4 . It is pretty clear that β is linear independent. To show that β is spanning, let $A \in W$. Then

$$A = \begin{pmatrix} X - X^T & 0_n \\ 0_n & X + X^T \end{pmatrix}, \text{ for some } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{n \times n}(\mathbb{F}).$$

We substitute a, b, c, d to get

$$A = \begin{pmatrix} 0 & b-c & 0 & 0\\ c-b & 0 & 0 & 0\\ 0 & 0 & 2a & b+c\\ 0 & 0 & b+c & 2d \end{pmatrix} = 2af_1 + bf_2 + cf_3 + 2df_4$$

so we can conclude that W is spanned by β

(b):

Let $n \in \mathbb{N}$. We define e_{ij} to be the $n \times n$ matrix with all its entries equal to 0 except the entry at the *i*th row and *j*th column. We claim that

$$\beta = \left\{ \left(\frac{e_{ij} - e_{ij}^T \mid 0_n}{0_n \mid e_{ij} - e_{ij}^T} \right), 1 \le i, j \le n \right\}$$

is a basis for W.

Similarly to the previous part, define each matrix associated with e_{ij} by f_{ij} . Let $A \in W$, so

$$A = \begin{pmatrix} X - X^T & 0_n \\ 0_n & X + X^T \end{pmatrix}, \text{ for an } n \times n \text{ matrix } X.$$

We can write X in terms of the basis for $\mathcal{M}_{n\times n}(\mathbb{F})$ to obtain that

$$X = \sum_{i,j=1}^{n} c_{ij} e_{ij}.$$

By the linearity of matrix transposition, it follows that

$$A = \begin{pmatrix} \frac{\sum_{i,j=1}^{n} c_{ij} e_{ij} - \left(\sum_{i,j=1}^{n} c_{ij} e_{ij}\right)^{T} & 0_{n} \\ 0_{n} & \sum_{i,j=1}^{n} c_{ij} e_{ij} + \left(\sum_{i,j=1}^{n} c_{ij} e_{ij}\right)^{T} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sum_{i,j=1}^{n} c_{ij} e_{ij} - \sum_{i,j=1}^{n} c_{ij} e_{ij}^{T} & 0_{n} \\ 0_{n} & \sum_{i,j=1}^{n} c_{ij} e_{ij} + \sum_{i,j=1}^{n} c_{ij} e_{ij}^{T} \end{pmatrix}$$

$$= \sum_{i,j=1}^{n} c_{ij} \begin{pmatrix} e_{ij} - e_{ij}^{T} & 0_{n} \\ 0_{n} & e_{ij} + e_{ij}^{T} \end{pmatrix}$$

$$= \sum_{i,j=1}^{n} c_{ij} f_{ij}$$

so W is spanned by β .

To show independence, for constants $c_{ij} \in \mathbb{F}$, let

$$\sum_{i,j=1}^{n} c_{ij} f_{ij} = 0$$

We get that

$$\sum_{i,j=1}^{n} c_{ij} \left(\frac{e_{ij} - e_{ij}^{T} \mid 0_{n}}{0_{n} \mid e_{ij} + e_{ij}^{T}} \right) = 0$$

In order for this to be true, we must have that

$$\sum_{i,j=1}^{n} c_{ij} (e_{ij} - e_{ij}^{T}) = 0$$

and

$$\sum_{i,j=1}^{n} c_{ij} (e_{ij} + e_{ij}^{T}) = 0$$

We add both equations together to see that

$$2\sum_{i,j=1}^{n} c_{ij}e_{ij} = 0.$$

Since the e_{ij} 's form a basis on $\mathcal{M}_{n\times n}(\mathbb{F})$, it follows that $c_{ij}=0$ for all i,j, and we are done.

- (a) Prove that if $W_1, W_2 \subseteq V$ are subspaces, then $W_1 + W_2$ is a subspace.
- (b) Let $W_1 = \{(x, y, x + y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$. Find two subspaces W_2, W_3 so that:
 - $W_1 + W_2 = \mathbb{F}^3$ but $\mathbb{F}^3 \neq W_1 \oplus W_2$.
 - $W_1 \oplus W_3 = \mathbb{F}^3$.
- (c) Find another subspace $U \subseteq \mathbb{F}^3$ so that $W_1 \oplus U = \mathbb{F}^3$

Proof.

(a):

Let W_1, W_2 be subspaces of V. We verify that $W_1 + W_2$ is also a subspace of V. First, note that $0 \in W_1, W_2$, so $0 + 0 = 0 \in W_1 + W_2$. Next, let $c \in \mathbb{F}$, $\vec{v}, \vec{w} \in W_1 + W_2$. Then $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{w} = \vec{w}_1 + \vec{w}_2$, for some $\vec{v}_1, \vec{w}_1 \in W_1$ and $\vec{v}_2, \vec{w}_2 \in W_2$. Since W_1, W_2 are subspaces, it is true that

$$c\vec{v_1} + \vec{w_1} \in W_1 \text{ and } c\vec{v_2} + \vec{w_2} \in W_2$$

which implies that

$$\vec{v} + \vec{w} = (c\vec{v_1} + \vec{w_1}) + (c\vec{v_2} + \vec{w_2}) \in W_1 + W_2,$$

verifying that $W_1 + W_2$ is indeed a subspace

(b):

Let $W_2 = \mathbb{F}^3$, $W_3 = \operatorname{span}\{e_3\}$. We start by showing that $\mathbb{F}^3 = W_1 + W_2$. The backward direction is instant so we will only show that $\mathbb{F}^3 \subseteq W_1 + W_2$.

Let $x \in \mathbb{F}^3$. We know that x is the same as 0 + x, and $0 \in W_1$ and $x \in W_2$, so $x \in W_1 + W_2$. However, $W_1 \subseteq W_2$, so $W_1 \cap W_2 = W_1 \neq \{0\}$, so \mathbb{F}^3 is not a direct sum of W_1 and W_2 .

Now, we will show that $W_1 \oplus W_3 = \mathbb{F}^3$. Again, the fact that $W_1 + W_3 \subseteq \mathbb{F}^3$ is obvious. To show that $\mathbb{F}^3 \subseteq W_1 + W_3$, let $(x, y, z) \in \mathbb{F}^3$. Notice that

$$(x, y, z) = (x, y, x + y) + (0, 0, z - x - y)$$

and

$$(x, y, x + y) \in W_1$$
 and $(0, 0, z - x - y) \in W_3$

so $(x, y, z) \in W_1 + W_3$.

Now let $(a, b, c) \in W_1 \cap W_3$. Then we have that a = b = 0, but this implies that c = a + b = 0, so (a, b, c) = 0. Thus $W_1 \cap W_3 = \{0\}$ and we can conclude that $\mathbb{F}^3 = W_1 \oplus W_2$.

(c):

Let $U = \text{span}\{(0,1,2)\}$. We follow the same structure as before in part (b). Let $(x,y,z) \in \mathbb{F}^3$. We rewrite

$$(x,y,z) = (x,x+2y-z,2x+2y-z) + (0,-x-y+z,2(-x-y+z))$$

and note that $(x, x + 2y - z, 2x + 2y - z) \in W_1$ and $(0, -x - y + z, 2(-x - y + z)) \in W_2$, so $(x, y, z) \in W_1 + W_2$.

Now, let $(a, b, c) \in W_1 \cap W_2$. Since $(a, b, c) \in W_2$, it must be true that a = 0 and 2b = c. At the same time, since $(a, b, c) \in W_1$ we have that a + b = c, which implies that b = c, which can only be true if b = c = 0, so (a, b, c) = 0, thus showing that $W_1 \oplus W_2 = \mathbb{F}^3$.

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Let V be a finite dimensional vector space over \mathbb{F} , and $W_1, W_2 \subseteq V$ subspaces with mutually disjoint bases β_1, β_2 respectively. Prove that $V = W_1 \oplus W_2$ if and only if $\beta = \beta_1 \cup \beta_2$ is a basis for V.

Proof. Let $m = |\beta_1|, k = |\beta_2|.$

Suppose that $V = W_1 \oplus W_2$. We will show that $\beta = \beta_1 \cup \beta_2$ is a basis for V.

Let $x \in V$. By our assumption, $x = w_1 + w_2$, for some $w_1 \in W_1$ and $w_2 \in W_2$. These vectors can in turn be written as

$$w_1 = \sum_{i=1}^{m} a_i v_i \text{ and } w_2 = \sum_{i=1}^{k} b_i w_i$$

where $v_i \in \beta_1$ and $w_i \in \beta_2$. Thus x can be written as a linear combination of vectors in β :

$$x = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{k} b_i w_i$$

so β spans V.

To show that β is linearly independent, suppose that

$$\sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{k} b_i w_i = 0$$

We put the vectors of each subspace on each side to get

$$\sum_{i=1}^{m} a_i v_i = -\sum_{i=1}^{k} b_i w_i$$

By the closure property of subspaces, $\sum_{i=1}^{m} a_i v_i \in W_1$ and $\sum_{i=1}^{k} b_i w_i \in W_2$, but since they are equal, it must be true that $\sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i \in W_1 \cap W_2 = \{0\}$, so $\sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i = 0$. Since β_1, β_2 are linearly independent, it must be true that $a_i = 0$ and $b_i = 0$, which was what we wanted to show. Therefore β is indeed a basis for V.

Conversely, suppose that β is a basis for V. We want to show that $V = W_1 \oplus W_2$. It is obvious that $W_1 + W_2 \subseteq V$, so it suffices to prove that $V \subseteq W_1 \oplus W_2$ and $W_1 \cap W_2 = \{0\}$. Let $x \in V$. Then since β is a basis, we have that

$$x = \sum_{j=1}^{m} a_i v_i + \sum_{j=1}^{k} b_i w_i$$
, for $a_i, b_i \in \mathbb{F}, v_i \in \beta_1$, and $w_i \in \beta_2$.

By closure, we have that $\sum_{j=1}^{m} a_i v_i \in W_1$ and $\sum_{j=1}^{k} b_i w_i \in W_2$, so we see that $x \in W_1 + W_2$. Thus $V = W_1 + W_2$.

To show that $W_1 \cap W_2 = \{0\}$, it suffices to show that if $x \in W_1 \cap W_2$, then it must be true that x = 0. Indeed, if $x \in W_1 \cap W_2$, we can write it as a two linear combinations of vectors in either β_1 or β_2 :

$$x = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{k} b_i w_i$$

$$\implies \sum_{i=1}^{m} a_i v_i - \sum_{i=1}^{k} b_i w_i = 0$$

By the linear independence of β , we have that $a_i = b_i = 0$, for all i, which means that x = 0 as desired, and the proof is complete.

Question 10.

Let $J = \begin{pmatrix} O & -I_2 \\ \hline I_2 & O \end{pmatrix}$ and $\mathbb{F} = \mathbb{C}$.

- (a) Verify that $J^2 = -I_4$.
- (b) Find all $X \in \mathcal{M}_{4\times 4}(\mathbb{F})$ so that XJ = JX.
- (c) Show that $\mathfrak{sp}_4 = \{X \in \mathcal{M}_{4\times 4}(\mathbb{F}) | XJ = JX\}$ is a subspace of $\mathcal{M}_{4\times 4}(\mathbb{F})$.
- (d) Find dim \mathfrak{sp}_4 by finding a basis for \mathfrak{sp}_4 .

Proof.

(a):

Indeed, we have that

$$J^{2} = \left(\begin{array}{c|c} O & -I_{2} \\ \hline I_{2} & O \end{array}\right) \left(\begin{array}{c|c} O & -I_{2} \\ \hline I_{2} & O \end{array}\right) = \left(\begin{array}{c|c} O^{2} + -I_{2}^{2} & O(-I_{2}) + -I_{2}O \\ \hline OI_{2} + I_{2}O & -I_{2}^{2} + O^{2} \end{array}\right) = \left(\begin{array}{c|c} -I_{2} & O \\ \hline O & -I_{2} \end{array}\right) = -I_{4}$$

(b):

For $A, B, C, C \in \mathcal{M}_2(\mathbb{F})$, let $X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathcal{M}_4(\mathbb{F})$ and suppose that XJ = JX. Then we have that

$$\begin{pmatrix} A & B \\ \hline C & D \end{pmatrix} \begin{pmatrix} O & -I_2 \\ \hline I_2 & O \end{pmatrix} = \begin{pmatrix} O & -I_2 \\ \hline I_2 & O \end{pmatrix} \begin{pmatrix} A & B \\ \hline C & D \end{pmatrix}$$

$$\implies \begin{pmatrix} AO + BI_2 & A(-I_2) + BO \\ \hline CO + DI_2 & C(-I_2) \end{pmatrix} = \begin{pmatrix} OA + -I_2C & OB - I_2D \\ \hline I_2A + OC & I_2B + OD \end{pmatrix}$$

$$\implies \begin{pmatrix} B & -A \\ \hline D & -C \end{pmatrix} = \begin{pmatrix} -C & -D \\ \hline A & B \end{pmatrix}$$

$$\implies B = -C \text{ and } A = -D$$

Therefore all X that satisfy this equation are of the form

$$X = \begin{pmatrix} P & -Q \\ \hline -Q & P \end{pmatrix}$$
, where $P, Q \in M_2(\mathbb{F})$.

(c):

Notice that $X \in \mathfrak{sp}_4$ if and only if X can be decomposed into a block matrix such that

$$X = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right)$$

for some matrices $P, Q \in M_2(\mathbb{F})$.

We know that $0 \in \mathfrak{sp}_4$ because as a 4×4 matrix, $0 = \left(\begin{array}{c|c} O & O \\ \hline O & O \end{array} \right)$, which is clearly satisfies the condition outlined above

Now, let $c \in \mathbb{F}$ and $X, Y \in \mathfrak{sp}_4$. Then for some 2×2 matrices P, Q, R, S,

$$X = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right)$$
 and $Y = \left(\begin{array}{c|c} R & S \\ \hline -S & -R \end{array}\right)$

It follows that

$$cX + Y = c\left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right) + \left(\begin{array}{c|c} R & S \\ \hline -S & -R \end{array}\right) = \left(\begin{array}{c|c} cP + R & cQ + S \\ \hline -cQ - S & -cP - R \end{array}\right)$$

so $cX + Y \in \mathfrak{sp}_4$. Thus we can conclude that \mathfrak{sp}_4 is indeed a subspace.

(d):

For $1 \leq i, j \leq 2$, let e_{ij} be the (i, j)th standard basis vector for $M_2(\mathbb{F})$. We claim that a basis for \mathfrak{sp}_4 is given by

$$\beta = \left\{ X \in \mathcal{M}_4(\mathbb{F}) : X = \left(\begin{array}{c|c} e_{ij} & O \\ \hline O & -e_{ij} \end{array} \right) \text{ or } X = \left(\begin{array}{c|c} O & e_{ij} \\ \hline -e_{ij} & O \end{array} \right) \right\}.$$

Let $A \in \mathfrak{sp}_4$. Then for matrices $P, Q \in \mathcal{M}_2(\mathbb{R})$,

$$A = \left(\begin{array}{c|c} P & Q \\ \hline -Q & -P \end{array}\right)$$

We can rewrite P, Q as linear combinations of basis vectors:

$$A = \left(\frac{\sum_{i,j=1}^{2} p_{ij} e_{ij}}{-\sum_{i,j=1}^{2} p_{ij} e_{ij}} \left| \sum_{i,j=1}^{2} q_{ij} e_{ij} \right| - \sum_{i,j=1}^{2} q_{ij} e_{ij}} \right) = \sum_{i,j=1}^{2} p_{ij} \left(\frac{e_{ij}}{O} \left| \frac{O}{-e_{ij}} \right| \right) + \sum_{i,j=1}^{2} q_{ij} \left(\frac{O}{-e_{ij}} \left| \frac{O}{O} \right| \right) \right)$$

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Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

- (a) Let V be a finite dimensional vector space over \mathbb{F} . If $I \subseteq V$ is a linearly independent set so that for any $x \in V \setminus I$, the set $I \cup \{x\}$ is linearly dependent, then I is a basis for V.
- (b) Let V be a finite dimensional vector space over \mathbb{F} . If $S \subseteq V$ is a spanning set so that $|S| = \dim V$, then S is a basis for V.
- (c) Let V be a finite dimensional vector space over \mathbb{F} . If $W \subseteq V$ a subspace, then there exists a unique subspace $U \subseteq V$ so that $V = W \oplus U$.

Proof.

(a):

Let V be a finite dimensional vector space over \mathbb{F} . Suppose $I \subseteq V$ is linearly independent and that adding any vector in $V \setminus I$ will result in the set no longer being linearly independent, and note that the same also applies when choosing a vector that is in I. Then for any $x \in V$, we have that for some vectors $v_1, ..., v_n \in I$,

$$cx + \sum_{i=1}^{n} c_i v_i = 0$$

for $c, c_i \in \mathbb{F}$ not all zero. We make the important note that it is necessary for $c \neq 0$, because if not, then

$$\sum_{i=1}^{n} c_i v_i = 0$$

which implies that all coefficients are zero by independence, which contradicts our claim that not all coefficients were zero. It follows that c has an inverse c^{-1} and

$$x = \sum_{i=1}^{n} -c^{-1}c_i v_i.$$

Since every $x \in V$ is a linear combination of vectors in I, it follows that I is indeed a basis for V.

(b):

Suppose for contradiction that S is not a basis for V. Then S is not linearly independent, that is, for some $s \in S$, $c_i \in \mathbb{F}$, $s_i \in S \setminus \{s\}$,

$$s = \sum_{i=1}^{n} c_i s_i$$

This means that $S \setminus \{s\}$ is also a spanning set. But $|S \setminus \{s\}| < \dim V$, which is a contradiction by Question 5. Thus S is a basis for V.

(c):

Suppose that W is a subspace of V. Then it has a basis β_W , which is linearly independent. By Question 5, we can extend β_W to a basis for V by combining it with some linearly independent set α . We claim that our desired subspace is $U = \operatorname{span} \alpha$. Since β_W is disjoint from α , by Question 9, $V = W \oplus U$.

Now, suppose that U_1, U_2 are subspaces such that $V = W \oplus U_1 = W \oplus U_2$.

Let $u \in U_1$. It follows that $u \in V = W \oplus U_2$, so u = w + v, for some $w \in W$ and $v \in U_2$.

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