NAME (PRINT): _____

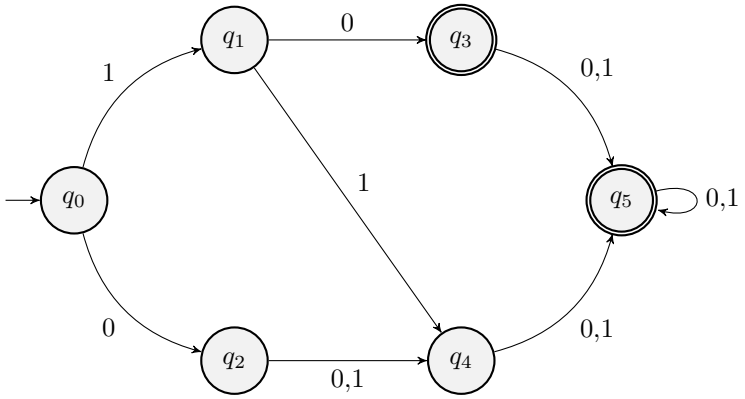
STUDENT NUMBER (PRINT): _____

University of Toronto Mississauga
FALL 2024 MOCK FINAL EXAMINATION
Introduction to Theory Computation
Macho Man (m^2)
Duration - ~~3 hours~~ 10 minutes
Aids : お前のお母さん:
https://youtu.be/3UHR_XWrGhc?si=kO03tfL7xwKDE3-o

This is a mock exam designed for studying CSC236. Any and all similarities with the Fall 2024 CSC236 final examination are purely coincidence.

This examination is meant to be done in 3 hours, but it is a little bit long, so do not worry about finishing. Part marks are very easy to get!

Q1. (9 points) Consider the DFA below:



a) (1 point) Describe the language accepted by the following DFA.

b) (3 points) Convert this DFA into a minimal NFA (i.e., there is no smaller NFA that accepts this language). Give a brief justification.

c) (2 points) Provide a DFA that accepts the language matched by $(a + ab)^*$.

d) (3 points) Prove the correctness of the DFA you provided.

Q2. (10 points) For each of the statements below, decide whether it is **true or false** and provide a proof justifying your answer.

a) (2 points) Let $\Sigma = \{0, 1\}$. The language $L = \{w \in \Sigma^* : |w| = 3\}$ is regular.

b) (2 points) Let $\Sigma = \{0, 1\}$. $L = \{1^i 0^j : i, j \in \mathbb{N}, i \geq j\}$ is a regular language.

c) (2 points) Let L, M be regular languages. The language $L \cap M$ is regular.

d) (2 points) Let $\Sigma = \{a, b, c\}$. The language of all strings whose characters are alphabetically ordered is regular.

e) (2 points) Let L, M be languages. If L is not regular, then $L \cup M$ is not regular.

Q3. (6 points)

a) (2 points) State the CLRS version of master theorem. Define all variables and state their conditions.

b) (4 points) Let $f : \mathbb{N} \to \mathbb{R}$ be a nonnegative function. Prove that if $f \in \Theta(n^k)$ for some $k > 0$, then the regularity condition holds true.

Q4. (7 points) Consider the program below:

```
1     def binary_search(x: int, lst: list[int]):
2         l ← 0, r ← len(lst) − 1
3         while(r − l > 0):
4             mid ← (l+r) // 2
5             if(lst[mid] = x):
6                 return mid
7             elif(lst[mid] < x):
8                 r ← mid
9             else:
10                l ← mid + 1
11        return l
```

a) (2 points) State the preconditions and postconditions of this program.

b) (5 points) Prove that this program is correct.

Q5. (6 points) Let A be a set of functions defined recursively as follows:

- $\sqrt{x} \in A$
- If $f \in A$, then $\frac{1}{f} - f \in A$

a) (4 points) Let $P(f)$ be a predicate on A and suppose you have managed to prove that

- $P(\sqrt{x})$ is true,
- $P(f) \implies P(\frac{1}{f} - f)$ for all $f \in A$.

Prove that $\forall f \in A, P(f)$. You may only assume that the principle of induction holds for the natural numbers, NOT the set A.

b) (2 points) Prove that $\forall f \in A$, $f\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$.

Q6. (7 points) Let $\Sigma = \{0, 1\}$. For any language $L \in \Sigma^*$, define

$$S = \{w \in \Sigma^* : 0w1 \in L\}$$

Given that L is a regular language, prove that S is a regular language.

Now, this is where a normal exam would end. However, this is not a normal exam.

Q7. (9 points) Recall that a segment tree is a data structure effective for querying and updating information about a range of values in a list, such as the minimum element in a provided range. In fact, we will examine an implementation of a segment tree that does this.

Suppose that we have a list of numbers `lst`. In this implementation, the segment tree can be thought of as a binary tree, with each node storing a range of indices `[l, r]`, a value which represents the minimum number in `lst[l:r]` (inclusive of `l`, exclusive of `r`), and pointers to a left and right child. **Note that a node does not have a left and right child if and only if it is true that** `l = r`. Otherwise, the left child will keep track of the range `[l,` $\left\lfloor \frac{l+r}{2} \right\rfloor$`]` and the right child will track `[`$\left\lfloor \frac{l+r}{2} \right\rfloor$ `+ 1, r]`

As an example, consider the following pseudocode for `update`, which updates the segment tree to correctly return queries about `lst` after setting `lst[i] = x`:

```
1        def update(root: Node, x: int, i: int):
2            if i not in [root.l, root.r] then return
3            else if root.l = i = root.r
4                then root.val ← x
5            update(root.left_child, x, i)
6            update(root.right_child, x, i)
7            root.val ← min(root.left_child.val, root.right_child.val)
```

a) (4 points) Prove that this method is correct.

b) (3 points) Find the recursive worst-case runtime of `update`.

c) (2 points) Find the tight asymptotic bound for the runtime of `update`.

Q8. (15 points) Recall the statement of the *pumping lemma:*

Let L be a regular language associated with some alphabet Σ. Then $\exists p \in \mathbb{N}^+, \forall w \in L$, where $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ so that

- $|y| \geq 1$
- $|xy| \leq p$
- $\forall n \in \mathbb{N}, xy^n z \in L$

a) (2 points) Show why the statement must have the condition $|xy| \leq p$. (Hint: Consider the language $\{w \in \{a, b\}^* : w = a^i b^j, i \leq j\}$)

b) (2 points) Find with proof the value of p for the language $\mathcal{L}(((ab)^*a)^*)$

c) (1 point) Explain why every finite language can be pumped.

Now, assume that L, M are regular languages that satisfy the pumping lemma, that is, they each have a value p_L and p_M that make the statement of the pumping lemma true. We say that a language can be **pumped** if it satisfies this property.

d) (2 points) Prove that $L \cup M$ can also be pumped.

e) (2 points) Prove that M^* can be pumped (You would want to consider when M is finite or when M is infinite).

f) (3 points) Prove that LM can be pumped.

g) (3 points) Prove the pumping lemma.

Q9. Given two bits of either 0 or 1, the XOR operator returns 1 if and only if exactly one of the bits is 1. This operator can be generalised to be used with binary strings by comparing each individual bit. It is denoted by the symbol ^. When given integer inputs, the operator will automaticall convert them to binary, perform the operation, and output the result as a decimal number. This operator can be quite useful, as it will be illustrated in this question.

Consider a list with with the property that there is only one number whose number of occurences in the list is odd. You can also think of it as every number having an even number of occurences except for one number. Take a look at the method below:

```
1           def find_odd_number(A):
2               '''
3   ------------Pre:-A-is-a-list-with-x-being-the-only-number-having-an
4   ------------odd-number-of-occurences
5   ------------Post:-Returns-the-odd-element-x
6   ------------'''
7               m ← 0
8               i ← 0
9               while i < len(lst):
10                  m ← m^lst[i]
11                  i ← i + 1
12              return m
```

Prove the correctness of this program.