Exercise 13.14

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Recall the statement of Theorem 13.6 and the definition of a simple mapping:
Theorem 13.6. Let $U, V \subseteq \mathbf{R}^n$ be two open sets and let $\Phi : U \to V$ be a diffeomorphism. Let $E \subseteq U$ be a measurable set. Then $\Phi(E)$ is also measurable, and its measure is

$$\mu(\Phi(E)) = \int_E |\det J\Phi|.$$

Definition 13.12. Let $\Phi : U \to \mathbf{R}^n$ be a function defined on some set $U \subseteq \mathbf{R}^n$, and let $\varphi_1, \ldots, \varphi_n$ be the component functions of Φ:

$$\Phi(p) = (\varphi_1(p), \ldots, \varphi_n(p)) \quad \text{for all } p \in U.$$

If the ith component function is just x_i — *i.e.* $\varphi_i(x_1, \ldots, x_n) = x_i$ for some i — then we say that Φ is x_i**-simple**. We say that Φ is **simple** if it is x_i-simple for all but one i.
Exercise 13.14. Show that Theorem 13.6 holds if Φ is simple.

Proof. Let $U, V \subseteq \mathbb{R}^n$ be open, and let $\Phi : U \to V$ be a simple diffeomorphism. Without loss of generality, let the only non-simple component of Φ be φ_n (I will explain why this reduction is valid below!), and let $E = \prod_{i=1}^n [a_i, b_i] \subseteq U$ be a closed box. Since Φ is a diffeomorphism, this implies that $\Phi(E)$ is measurable, and as well, because Φ is simple,

$$\Phi(E) = \left\{ (x_1, ..., x_{n-1}, \varphi(x)) \in V : x = (x_1, ..., x_n) \in \prod_{i=1}^n [a_i, b_i] \right\}$$

Let $E' = \prod_{i=1}^{n-1} [a_i, b_i]$, and define $\Psi : E' \to \mathbb{R}$ by

$$\Psi(x') = \int_{\varphi(x', [a_n, b_n])} 1 \; dx_n.$$

Φ is a diffeomorphism, so the x'-slice $\varphi(x', \cdot)$ is C^1 and injective. It follows that the image $\varphi(x', [a_n, b_n])$ is an interval. Now, we want to apply single-variable change of variables with $\varphi(x', u) = x_n$. Notice that

$$dx_n = \frac{\partial \varphi}{\partial u}(x', u) du$$

and

$$\det J\Phi = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi}{\partial x_1}(x', u) & \frac{\partial \varphi}{\partial x_2}(x', u) & \cdots & \frac{\partial \varphi}{\partial u}(x', u) \end{vmatrix} = \frac{\partial \varphi}{\partial u}(x', u)$$

so it follows that

$$\Psi(x') = \int_{\varphi(x', [a_n, b_n])} 1 \; dx_n = \int_{a_n}^{b_n} |\det J\Phi|$$

Finally, using Fubini's Theorem, we get that

$$\mu(\Phi(E)) = \int_{\Phi(E)} 1 = \int_{E'} \Psi = \int_{E'} \int_{a_n}^{b_n} |\det J\Phi| = \int_E |\det J\Phi|$$

as needed.
Explanation of Reduction. It, in fact, does not matter in which component Φ is simple in.
Let $\Phi : U \to V$ be a simple diffeomorphism, where the ith component is non-simple, and we will denote this component as the function $\varphi : U \to \mathbb{R}$. Let L be the linear map that swaps the ith and nth component. Notice that $L = L^{-1}$ and $\det L = \det L^{-1} = 1$ (this will be important later). We would like to apply the

theorem to $L \circ \Phi$, but this is not simple! However, we have a workaround for this issue. Let $\Psi : U \to V$ be the function defined by $\Psi(x_1, ..., x_n) = (x_1, ..., x_{n-1}, \varphi \circ L(x_1, ..., x_n))$. We claim that $L \circ \Phi(E) = \Psi(E)$. Let $x \in E$ and consider $L \circ \Phi(x) = (x_1, ..., x_n, ..., \varphi(x))$. Let $x' = L(x)$. We have that

$$\Psi(x') = \Psi(x_1, ..., x_n, ..., x_i) = (x_1, ..., x_n, ..., \varphi(x)) = L \circ \Phi(x)$$

We will not show the converse because it is the same argument. As well, convince yourself that $\det J\Phi = \det J\Psi$. Now, we can apply the statement to the simple function Ψ. Here is everything we need:

- $\mu(L(R)) = |\det L|\mu(R)$: This is Exercise 13.11/12.18
- $\det L = 1$
- $L \circ \Phi(E) = \Psi(E)$
- The exercise above
- $\det J\Psi = \det J\Phi$

We put it all together and see that

$$\mu(\Phi(E)) = \mu(L^{-1} \circ L \circ \Phi(E)) = |\det L^{-1}|\mu(L \circ \Phi(E)) = \mu(\Psi(E)) = \int_E |\det J\Psi| = \int_E |\det J\Phi|$$

$\square$