Lemma 6.44

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Let X be a vector space with finite dimension n equipped with an arbitrary norm $\|\cdot\|$. Define the linear isomorphism $\Phi : X \to \mathbb{R}^n$ by

$$\Phi(\vec{x}) = (x_1, ..., x_n), \text{ where } \vec{x} = \sum_{i=1}^{n} x_i \vec{b_i},\ \{b_1, ..., b_n\} \text{ is a basis for } X.$$

Define $\|\cdot\|_1$ on X as

$$\|\vec{x}\|_1 = \|\Phi(\vec{x})\|_1$$

Note that the norm on the right hand side is the 1-norm in $\mathbb{R}^n$. Next, we introduce a lemma that has already been proven.

Lemma 6.43. The closed unit ball of $(X, \|\cdot\|_1)$ is compact.

Corollary. The unit circle in $(X, \|\cdot\|_1)$ is compact.

This follows from the fact that the unit circle is a closed subset of the closed unit ball.

Now, we prove the following lemma:

Lemma 6.44. There exists a constant $m > 0$ such that $m\|\vec{x}\|_1 \leq \|\vec{x}\|$ for all $\vec{x} \in X$.

Proof. Let C' denote the unit circle in $(X, \|\cdot\|_1)$. Define a function $f : C' \to \mathbb{R}$ by

$$f(\vec{x}) = \frac{\|\vec{x}\|}{\|\vec{x}\|_1}.$$

Since C' is compact, by the generalized EVT, f attains a minumum m. Notice that since norms are positive, $f(\vec{x}) > 0$, so $m > 0$. We claim that this m is the value we are looking for. That is, $m\|\vec{x}\|_1 \leq \|\vec{x}\|$ is true for all $\vec{x} \in X$.

If $\vec{x} = \vec{0}$, then the inequality follows immediately.

Otherwise, for $\vec{x} \in X \setminus \{\vec{0}\}$, notice that $\dfrac{\vec{x}}{\|x\|_1} \in C'$, so

$$f\left(\frac{\vec{x}}{\|x\|_1}\right) = \frac{\left\|\dfrac{\vec{x}}{\|x\|_1}\right\|}{\left\|\dfrac{\vec{x}}{\|x\|_1}\right\|_1} \geq m \implies \frac{\frac{1}{\|\vec{x}\|_1}\|\vec{x}\|}{\frac{1}{\|\vec{x}\|_1}\|\vec{x}\|_1} \geq m \implies \|\vec{x}\| \geq m\|\vec{x}\|_1$$

and the proof is complete. $\square$