Question 39.

Let $M \subseteq \mathbf{R}^N$ be a set and let $\varphi : U \to M$ be a smooth parametrization of M, defined on some open set $U \subseteq \mathbf{R}^n$. We say that φ is a **regular parametrization** if φ is a homeomorphism, and if $\varphi'(p)$ has rank n at every point $p \in U$. Note that we require φ^{-1} to be continuous, but not necessarily smooth!

(a) Prove that if φ is a regular parametrization of M, then φ^{-1} is also smooth.

(b) Prove that if $\varphi : U \to M$ and $\psi : V \to M$ are two regular parametrizations of M, then $\mathrm{vol}_\varphi(M) = \mathrm{vol}_\psi(M)$.

(c) Use regular parametrizations to find the surface area of a sphere of radius r in $\mathbf{R}^4$.

Proof.
(a):
Suppose φ is a smooth regular parametrization of M.

(b):
Suppose that $\varphi : U \to M$ and $\psi : V \to M$ are regular parametrizations of M. We can rewrite

$$\psi = \varphi \circ \left(\varphi^{-1} \circ \psi\right).$$

Let $\Phi : V \to U$ be defined by $\Phi = \varphi^{-1} \circ \psi$. We claim that Φ is a diffeomorphim. ψ is smooth, and by part (a), we know that φ^{-1} is smooth, so Φ is also smooth. Furthermore, $\Phi^{-1} = \psi^{-1} \circ \varphi$, which is smooth for similar reasons. Thus Φ is indeed a diffeomorphism. This being the case, we know that parametrizations are diffeomorphism-invariant (done in class), so we can conclude that

$$\mathrm{vol}_\varphi(M) = \mathrm{vol}_\psi(M).$$

□