Question 25.

Let $\varphi : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be the function given by $\varphi(A) = A^2$. For each $A \in M_n(\mathbf{R})$, find a linear approximation $L_A : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ to φ at A. Give an explicit formula for $L_A(B)$ as a function of B, a proof that L_A is a bounded linear mapping, and a proof that L_A is a linear approximation to φ at A.

Proof. First, we supply a lemma.
Lemma. For all $B \in M_n(\mathbb{R})$, $\|B^2\| \leq K\|B\|^2$, for some positive constant K.
Define the isomorphism $\Phi : M_n(\mathbb{R}) \to B(\mathbb{R}^n, \mathbb{R}^n)$ as mapping a matrix representation of a linear mapping to the original linear mapping. We define the operator norm on $M_n(\mathbb{R})$ by $\|A\|_{\text{op}} = \|\Phi(A)\|_{\text{op}}$, where the right hand side is the operator norm on $B(\mathbb{R}^n, \mathbb{R}^n)$.
Since all norms are equivalent on $M_n(\mathbb{R})$, there are constants $M, N > 0$ so that for any norm $\|\cdot\|$,

$$M\|A\| \leq \|A\|_{\text{op}} \leq N\|A\|$$

From this, using the subnormality of bounded linear operators, it follows that

$$\|B^2\| \leq \frac{1}{M}\|\Phi(B^2)\|_{\text{op}} = \frac{1}{M}\|\Phi(B) \circ \Phi(B)\|_{\text{op}} \leq \frac{1}{M}\|\Phi(B)\|_{\text{op}}^2 \leq \frac{N^2}{M}\|B\|$$

Since $M, N > 0$, we have what we wanted.
We claim that for $A \in M_n(\mathbb{R})$, $L_A(B) = BA + AB$. For $C, D \in M_n(\mathbb{R})$, $k \in \mathbb{R}$,

$$L_A(kC + D) = (kC + D)A + A(kC + D) = k(CA + AC) + DA + AD = kL_A(C) + L_A(D)$$

so L_A is linear. As well, we get that L_A is bounded for free because we are working in a finite dimensional vector space. Finally, we have that

$$0 \leq \frac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} = \frac{\|(A+B)^2 - A^2 - (BA + AB)\|}{\|B\|}$$

$$\frac{\|A^2 + AB + BA + B^2 - A^2 - BA - AB\|}{\|B\|} = \frac{\|B^2\|}{\|B\|} \leq K\|B\|$$

$$\implies 0 \leq \frac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} \leq K\|B\|$$

By the Squeeze Theorem, $\lim_{h\to 0} \frac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} = 0$ and we are done.

□

Question 25.

Let X be a finite-dimensional normed vector space, let U be an open convex subset of X, and let $f : U \to \mathbf{R}^m$ be a totally differentiable function. (Note: a set $C \subseteq X$ is called **convex** if $tx + (1-t)y \in C$ for all $x, y \in C$ and $t \in [0,1]$.) Let $f : U \to \mathbf{R}^m$ be a totally differentiable function.

(a) Suppose that there exists a constant $C \geq 0$ such that $\|f'(p)\|_{\text{op}} \leq C$ for all $p \in U$. Prove that

$$\|f(p) - f(q)\| \leq C\|p - q\| \quad \text{for all } p, q \in U.$$

Conclude that f is uniformly continuous.

(b) Prove that $f'(p) = 0$ for all $p \in U$ if and only if f is a constant function.

(c) Assume $U = X$ and suppose that f is **twice totally differentiable** — meaning that $f' : X \to B(X, Y)$ itself is differentiable at every point of X, with total derivative $f'' = (f')'$. Show that $f'' = 0$ if and only if f is **affine-linear**: there exists a bounded linear mapping $M : X \to Y$ and a vector $b \in Y$ such that

$$f(p) = M(p) + b \quad \text{for all } p \in X.$$

(Compare with the formula $y = mx + b$ from single-variable calculus.)

Proof. (a):
Fix $p, q \in U$. If $f(p) = f(q)$, the inequality is trivially true. Otherwise, since U is open, there is some open ball $B(p, \varepsilon_1) \subseteq U$ and $B(q, \varepsilon_2) \subseteq U$. Let $\delta = \dfrac{\min\{\varepsilon_1, \varepsilon_2\}}{\|q - p\|}$. For all $t \in (-\delta, 1 + \delta)$, we have that
If $t \in [0, 1]$,

$$tq + (1-t)p \in U$$

since U is convex. If $t \in (-\delta, 0)$, then

$$\|(tq + (1-t)p) - p\| = \|t(q - p)\| = |t|\|q - p\| < \varepsilon_1$$
$$\implies tq + (1-t)p \in B(p, \varepsilon_1) \subseteq U$$

If $t \in (1 + \delta)$, then

$$\|(tq + (1-t)p) - q\| = \|(t-1)q - (t-1)p\| = |t-1|\|q - p\| < \varepsilon_2$$
$$\implies tq + (1-t)p \in B(q, \varepsilon_2) \subseteq U$$

Thus we can construct a function $\alpha : (-\delta, 1 + \delta) \to U$ defined by $\alpha(t) = tq + (1-t)p$. We will show that $\alpha'(t)(\varphi) = \varphi(q - p)$:

$$\lim_{h \to 0} \frac{\|((t+h)q + (1-t-h)p) - (tq + (1-t)p) - h(q-p)\|}{\|h\|}$$
$$= \lim_{h \to 0} \frac{\|hq - hp - h(q-p)\|}{\|h\|} = 0$$

Since α is totally differentiable and f is totally differentiable, by the chain rule, we can say that

$$(f \circ \alpha)'(t) = f'(a(t))(h(q-p)) = hf'(a(t))(q-p)$$

Now, we construct a function from $(-\delta, 1+\delta)$ to $\mathbb{R}$ by

$$g(t) = (f(q) - f(p)) \cdot f(\alpha(t))$$

Now, we will show that for $t \in (-\delta, 1+\delta)$,

$$g'(t) = (f(q) - f(p)) \cdot f'(\alpha(t))(q-p)$$

First, we see that

$$\lim_{h \to 0} \frac{|g(t+h) - g(p) - (f(q) - f(p)) \cdot f'(\alpha(t))(q-p)|}{|h|}$$

$$= \lim_{h \to 0} \frac{|(f(q) - f(p)) \cdot f(\alpha(t+h)) - (f(q) - f(p)) \cdot f(\alpha(t)) - (f(q) - f(p)) \cdot f'(\alpha(t))(q-p)}{|h|}$$

By the linearity of the dot product,

$$\lim_{h \to 0} \left| (f(q) - f(p)) \cdot \frac{f(\alpha(t+h)) - f(\alpha(t)) - f'(\alpha(t))(q-p)}{|h|} \right| = 0$$

Using this, consider the limit

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}.$$

Adding and subtracting $h(f(q) - f(p)) \cdot f'(\alpha(t))(q-p)$,

$$= \lim_{h \to 0} \frac{[g(t+h) - g(t) - h(f(q) - f(p)) \cdot f'(\alpha(t))(q-p)] + h(f(q) - f(p)) \cdot f'(\alpha(t))(q-p)}{h}$$

The first sum is exactly the limit that was computed above and is equal to zero. The limit then reduces to

$$\lim_{h \to 0} \frac{h(f(q) - f(p)) \cdot f'(\alpha(t))(q-p)}{h} = (f(q) - f(p)) \cdot f'(\alpha(t))(q-p)$$

which shows that g is differentiable on its domain and is equal to $(f(q)-f(p)) \cdot f'(\alpha(t))(q-p)$. In particular, we have continuity on $[0,1]$ and differentiability on $(0,1)$. Applying the Mean Value Theorem, there is some $c \in (0,1)$ so that

$$g(1) - g(0) = g'(c) \implies (f(q)-f(p)) \cdot f(q) - (f(q)-f(p)) \cdot f(p) = (f(q)-f(p)) \cdot f'(a(c))(q-p)$$

$$\implies (f(q) - f(p)) \cdot (f(q) - f(p)) = (f(q) - f(p)) \cdot f'(\alpha(c))(q-p)$$

$$\implies \|f(q) - f(p)\|^2 = \|f(q) - f(p)\| \|f'(\alpha(c))(q-p)\| \cos(\theta)$$

where θ is the angle between the vectors $f(q) - f(p)$ and $f'(\alpha(c))(q-p)$. Thus

$$\|f(q) - f(p)\|^2 \le \|f(q) - f(p)\| \|f'(\alpha(c))(q-p)\| \le \|f(q) - f(p)\| \|f'(\alpha(c))\|_{\text{op}} \|q-p\|$$

$$\implies \|f(q)-f(p)\|^2 \le \|f(q)-f(p)\|C\|q-p\|$$
$$\implies \|f(q)-f(p)\| \le C\|q-p\|$$

which is the desired inequality.

With this, the uniform continuity of f is easy to show.

For $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{C}$. For all $p, q \in X$ so that $\|q-p\| < \delta$,

$$\|f(q)-f(p)\| \le C\|q-p\| < \varepsilon$$

(b):

Suppose that $f'(p) = 0$ for all $p \in U$. Then $\|f'(p)\|_{\text{op}} \le 0 = C$, so by part (a), for all $a, b \in U$,

$$\|f(a)-f(b)\| \le 0 \implies \|f(a)-f(b)\| = 0 \implies f(a) = f(b)$$

so f is constant.

Conversely, suppose that f is a constant function. To show that $f'(p) = 0$, notice that

$$\lim_{h\to 0} \frac{\|f(p+h)-f(p)\|}{\|h\|} = 0$$

Thus $f'(p) = 0$ for all $p \in U$.

(c):

We know that X is convex because it is a vector space. Suppose that $f'' = 0$. Then by part (b), f' is a constant function. We will denote $f' = L \in B(X, Y)$. We claim that $M = L$ and $b = f(0)$, that is, $f(p) = L(p) + f(0)$ for all $p \in U$.

Define the function $g : U \to \mathbb{R}^m$ by $g(p) = f(p) - f(0) - L(p)$. For all $p \in U$, we have that

$$\lim_{h\to 0} \frac{\|g(p+h)-g(p)\|}{\|h\|} = \lim_{h\to 0} \frac{\|(f(p+h)-f(0)-L(p+h)) - (f(p)-f(0)-L(p))}{\|h\|}$$
$$= \lim_{h\to 0} \frac{\|f(p+h)-f(p)-L(h)\|}{\|h\|} = 0$$

where the last step is from the fact that $f'(p) = L$. We can see that $g'(p) = 0$ for all $p \in U$. From part (b), it follows that g is a constant function. Finally, notice that

$$g(0) = f(0) - f(0) - L(0) = 0,$$

so $g = 0$. Therefore for all $p \in U$,

$$f(p) - f(0) - L(p) = 0 \implies f(p) = L(p) + f(0)$$

as desired.

Conversely, suppose that f is affline-linear. Then for some $M \in B(X, \mathbb{R}^m)$, $b \in \mathbb{R}^m$,

$$f(p) = M(p) + b.$$

We claim that $f'(p) = M$ using the definition. Immediately, we have that

$$\lim_{h\to 0} \frac{\|(M(p+h)+b) - (M(p)+b) - M(h)\|}{\|h\|} = 0$$

which verifies our claim. Since $f'(p) = M$ for all $p \in U$, f' is constant, and from part (b), this implies that $f'' = 0$, which completes the proof.

$\square$