Exercise 16.7. Tangent space to a graph

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Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be some function (that is smooth). Let M be the graph of f, so that M is a smooth manifold in $\mathbb{R}^{n+1}$.
We attempt to find the tangent space T_pM of some point p.
Since M is a graph, we can cover M with a single chart (U, φ), where $\varphi(x) = (x, f(x))$. Fix $p \in M$ and write $p = \varphi(q)$, for some $q \in U$. From exercise 16.3, a vector v is in the tangent space T_pM if v is in the image of $\varphi'(q)$. That is, the tangent space coincides exactly with the image of $\varphi(q)$. Thus it suffices to find $\varphi(q)(\mathbb{R}^n)$.
We have that

$$J\varphi(q) = \left(\frac{I_n}{\nabla f(q)} \right)$$

So for any $h \in \mathbb{R}^n$,

$$\varphi'(q)(h) = (h, \nabla f(q)h)$$

and we can conclude that

$$T_pM = \left\{ (p, (h, \nabla f(q)h)) \in \mathbb{R}^{n+1}_p : p = (q, f(q)), h \in \mathbb{R}^n \right\}.$$