

Question 35.

Perturbing the roots of a polynomial.

Let $f(x) = \sum_{i=0}^n a_i x^i$ be a **monic** polynomial with **no repeated real roots**. This means that $a_n = 1$, and that all real roots of f have multiplicity 1.

- (a) Let r be a root of $f(x)$. Prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that: if $g(x) = \sum_{i=0}^n b_i x^i$ is a monic polynomial with coefficients b_i satisfying $|a_i - b_i| < \delta$, then $g(x)$ has at least one root in the interval $(r - \varepsilon, r + \varepsilon)$.

This shows that slight perturbations of the coefficients results in slight perturbations of the roots.

- (b) Suppose that f has fewer than n real roots. Prove that number of real roots of f does not change under small perturbation of the coefficients.

Proof.

(a):

First, we prove the following lemma:

Lemma. Let $r \in \mathbb{R}$ be a root of a polynomial p . Then r is a repeated root if and only if $p'(r) = 0$.

Suppose that p has a repeated root. Then we can factor p as $(x - r)^k q(x)$, for some $k > 1$ and $q \in \mathbb{P}(\mathbb{R})$. We can take the derivative of this and get that

$$\begin{aligned} p'(x) &= k(x - r)^{k-1} q(x) + (x - r)^k q'(x) \\ \implies p'(r) &= 0 \end{aligned}$$

Conversely, suppose that $p'(r) = 0$, for some $r \in \mathbb{R}$. We can write

$$p'(x) = (x - r) \sum_{i=0}^m c_i x^i, \text{ for constants } c_0, \dots, c_m$$

We can integrate both sides to get that

$$\begin{aligned} p(x) &= \int (x - r) \sum_{i=0}^m c_i x^i dx = \int x \sum_{i=0}^m c_i x^i - r \sum_{i=0}^m c_i x^i dx \\ &= \sum_{i=0}^m \frac{c_i}{i+2} x^{i+2} - r \sum_{i=0}^m \frac{c_i}{i+1} x^{i+1} + C \end{aligned}$$

In order for r to be a root of p , we must have that

$$\begin{aligned} p(r) &= \sum_{i=0}^m \left(\frac{c_i}{i+2} r^{i+2} - \frac{c_i}{i+1} r^{i+2} \right) + C = 0 \\ \implies C &= \sum_{i=0}^m \left(\frac{c_i}{i+1} r^{i+2} - \frac{c_i}{i+2} r^{i+2} \right) \end{aligned}$$

