

Question 32.

Let M be a subset of \mathbf{R}^n , let $p_0 \in M$ be a point, and let $\vec{v} \in \mathbf{R}^n$ be a vector. We say that \vec{v} is a **tangent vector** to M at p_0 if there exists $\delta > 0$ and a C^1 function $\alpha : (-\delta, \delta) \rightarrow M$ such that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. In other words, \vec{v} is the velocity vector of a curve through M .

- (a) Suppose now that M is the *zero set* of some C^1 function $f : U \rightarrow \mathbf{R}$, where U is an open set in \mathbf{R}^n : thus

$$M = \{p \in U : f(p) = 0\}.$$

Suppose that $p_0 \in M$ is a point such that $\nabla f(p_0) \neq \vec{0}$, and let $\vec{v} \in \mathbf{R}^n$ be a vector. Show that \vec{v} is a tangent vector to M at p_0 if and only if $\nabla f(p_0) \cdot \vec{v} = 0$.

- (b) Let E be the ellipsoid in \mathbf{R}^3 defined by the following equation:

$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$

Find the equation of the tangent plane to M at the point $p_0 = (1, 2, 0)$.

Hint: Define an appropriate function f , then find two vectors which are orthogonal to $\nabla f(p_0)$. By (a), these two vectors span the tangent plane. I recommend using graphing software to confirm your result.

Proof. (a):

Suppose that \vec{v} is a tangent vector to M at p_0 . Then there exists a function $\alpha : (-\delta, \delta) \rightarrow M$ so that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. Define $g : (-\delta, \delta) \rightarrow \mathbf{R}$ by $g(t) = f(\alpha(t))$. For all $t \in (-\delta, \delta)$, $\alpha(t) \in M$, so $g(t) = 0$. It follows that

$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

Substituting $t = 0$ yields

$$\nabla f(p_0) \cdot \vec{v} = 0$$

as needed.

Conversely, suppose that $\nabla f(p_0) \cdot \vec{v} = 0$. Since $\nabla f(p_0) \neq 0$, $\frac{\partial f}{\partial x_i}(p_0) \neq 0$ for some $i \in \{0, \dots, n\}$. Define the C^1 function $g : \mathbf{R}^n \rightarrow \mathbf{R}$ as the function that swaps the i th and n th coordinate. That is,

$$g(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_n, x_{i+1}, \dots, x_i)$$

Let p' be the vector in \mathbf{R}^{n-1} whose components are the same as p_0 except that its i th component is p_n . In particular,

$$p' = (p_1, \dots, p_{i-1}, p_n, p_{i+1}, \dots, p_{n-1}).$$

Notice that $\frac{\partial g}{\partial x_n}(p', p_i) = \frac{\partial f}{\partial x_i}(p_0) \neq 0$. Applying the Implicit Function Theorem with $k = 1$, there exists an open set $W \subseteq \mathbf{R}^{n-1}$ that contains p' and a continuously differentiable function $\psi : W \rightarrow \mathbf{R}$ such that for all $x' \in W$, $\psi(p') = p_i$ and

$$g(x', \psi(x')) = f(x_1, \dots, x_{i-1}, \psi(x'), x_{i+1}, \dots, x_{n-1}) = 0$$

Question 33.

- (a) Let $g : U \rightarrow \mathbf{R}$ be a C^1 function defined on an open set $U \subseteq \mathbf{R}^n$, and let M be its zero set:

$$M = \{p \in U : g(p) = 0\}.$$

Suppose that we have a C^1 function $f : U \rightarrow \mathbf{R}$, defined on an open set $U \subseteq \mathbf{R}^n$ which contains M , and we wish to find the maximum of f on M . Assume that M is compact, and that f achieves its maximum on M at some point $p_0 \in M$. Prove that there exists a real number $\lambda \in \mathbf{R}$ such that

$$\nabla f(p_0) = \lambda \nabla g(p_0).$$

This number λ is known as the **Lagrange multiplier**.

- (b) Use Lagrange multipliers to solve the following optimization problem: *Find the point(s) on the ellipsoid $x^2 + yz + y^2 - xy - xz + z^2 = 3$ which are **closest** and **furthest** from the origin.*

Proof.

(a):

Let $h : U \times \mathbf{R} \rightarrow \mathbf{R}$ be the C^1 function defined by

$$h(x, y) = yg(x) + y - f(x)$$

Notice that $h(p_0, f(p_0)) = f(p_0) \cdot g(p_0) + f(p_0) - f(p_0) = 0$ and $\frac{\partial h}{\partial y}(p_0, f(p_0)) = g(p_0) + 1 = 1 \neq 0$. By the implicit function theorem, there exists an open set $W \subseteq \mathbf{R}^n$ and C^1 function $\psi : W \rightarrow \mathbf{R}$ such that for all $x \in W$,

$$h(x, \psi(x)) = \psi(x)g(x) + \psi(x) - f(x) = 0$$

Taking the derivative of both sides at p_0 with respect to x , we see that

$$g(p_0)\nabla\psi(p_0) + \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

$$\implies \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

Also note that $\frac{\partial\psi}{\partial x_i}(p_0) = \frac{\frac{\partial h}{\partial x_i}(p_0, f(p_0))}{\frac{\partial h}{\partial y}(p_0, f(p_0))} = \frac{\partial g}{\partial x_i}(p_0) - \frac{\partial f}{\partial x_i}(p_0)$. Thus our equation becomes

$$(\psi(p_0) + 1)\nabla g(p_0) - 2\nabla f(p_0) = 0 \implies \nabla f(p_0) = \frac{\psi(p_0) + 1}{2}\nabla g(p_0)$$

Therefore the value $\lambda = \frac{\psi(p_0) + 1}{2}$ is the one that we needed.

(b):

Define

$$g(x, y, z) = x^2 + yz + y^2 - xy - xz + z^2 - 3.$$

Question 34.

Let $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a C^1 mapping.

- (a) Suppose that $n > m = 1$. Show that Φ cannot be injective.
- (b) Suppose that $n < m$. Show that if $K \subseteq \mathbf{R}^n$ is a compact set, then $\Phi(K)$ is a Jordan measurable set, and has Jordan measure zero.

Proof.

(a):

Suppose for contradiction that $n > m = 1$ and Φ is a C^1 injective function. Since Φ cannot be a constant function, by the results of Big List #26, there is a $p \in \mathbf{R}^n$ so that $\nabla\Phi(p) \neq 0$. In particular, we will say that $\frac{\partial\Phi}{\partial x_j} \neq 0$. Define $\alpha(x) = \Phi(x) - \Phi(p)$. Injectivity is translation-invariant, so α is injective. Notice that $\alpha(p) = 0$. We can apply the implicit function theorem to obtain an open set $W \subseteq \mathbf{R}^{n-1}$ that contains $p' = (p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n)$ and a C^1 function $\Psi : W \rightarrow \mathbf{R}$ such that for all $x = (x_1, \dots, x_{n-1}) \in W$,

$$\alpha(x_1, \dots, x_{j-1}, \Psi(x), x_{j+1}, \dots, x_{n-1}) = 0$$

Then, since W is open and contains p' , we can find another distinct point $q \in W$. We have

$$\alpha(p_1, \dots, p_{j-1}, \Psi(p'), p_{j+1}, \dots, p_n) = 0 = \alpha(q_1, \dots, q_{j-1}, \Psi(q), q_{j+1}, \dots, q_{n-1})$$

which contradicts the fact that α is injective.

(b):

□