Question 38

Basel Problem. Here you will use multivariable calculus to establish the following famous equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

To do it, you will evaluate the (improper) double integral $\int_U \frac{1}{1-xy}$ in two ways. Let $f:(0,1)^2\to \mathbf{R}$ be the function given by $f(x,y)=\frac{1}{1-xy}$, and let K_N denote the closed box $\left[\frac{1}{N},1-\frac{1}{N}\right]^2$.

- (a) Evaluate $\int_{K_N} f$ using Fubini's theorem.
- (b) Evaluate $\int_{K_N} f$ using the Change of Variables formula twice: first using the linear diffeomorphism (x, y) = (u + v, u v), then using the polar coordinates transform.
- (c) Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof.

(a):

By Fubini's:

$$\int_{K_N} f = \int_{\frac{1}{N}}^{1 - \frac{1}{N}} \int_{\frac{1}{N}}^{1 - \frac{1}{N}} \frac{1}{1 - xy} \, dy \, dx$$

$$= \int_{\frac{1}{N}}^{1 - \frac{1}{N}} -\frac{1}{x} \ln(1 - xy) \Big|_{\frac{1}{N}}^{1 - \frac{1}{N}} \, dx$$

$$= \int_{\frac{1}{N}}^{1 - \frac{1}{N}} -\frac{1}{x} \left(\ln\left(1 - \left(1 - \frac{1}{N}\right)x\right) - \ln\left(1 - \frac{1}{N}x\right) \right) \, dx$$

Notice that $-1 < -\left(1 - \frac{1}{N}\right), -\frac{1}{N} < 1$, so we can use the power series expansion of $\ln(1+t)$

to get that

$$\int_{K_N} f = \int_{\frac{1}{N}}^{1 - \frac{1}{N}} - \frac{1}{x} \left(\sum_{n=1}^{\infty} \frac{\left(1 - \frac{1}{N}\right)^n x^n}{n} - \sum_{n=1}^{\infty} \frac{\left(\frac{1}{N}\right)^n x^n}{n} \right) dx$$

$$= \int_{\frac{1}{N}}^{1 - \frac{1}{N}} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \left(\left(\frac{1}{N}\right)^n - \left(1 - \frac{1}{N}\right)^n \right) dx$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \left(\left(\frac{1}{N}\right)^n - \left(1 - \frac{1}{N}\right)^n \right) \Big|_{\frac{1}{N}}^{1 - \frac{1}{N}}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left(\frac{1}{N}\right)^n - \left(1 - \frac{1}{N}\right)^n \right)^2$$

(b):

Let $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by

$$\Phi(x,y) = (x+y, x-y).$$

We want to find E, such that $\Phi(E) = K_N$. We will do this by finding $\Phi^{-1}(K_N)$. Notice that $\Phi^{-1}(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$. Thus

$$\Phi^{-1}(K_N) = \left[\frac{1}{N}, 1 - \frac{1}{N}\right] \times \left[\frac{1}{N} - \frac{1}{2}, \frac{1}{2} - \frac{1}{N}\right]$$

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