Question 32.

Let M be a subset of $\mathbf{R}^n$, let $p_0 \in M$ be a point, and let $\vec{v} \in \mathbf{R}^n$ be a vector. We say that $\vec{v}$ is a **tangent vector** to M at p_0 if there exists $\delta > 0$ and a C^1 function $\alpha : (-\delta, \delta) \to M$ such that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. In other words, $\vec{v}$ is the velocity vector of a curve through M.

(a) Suppose now that M is the *zero set* of some C^1 function $f : U \to \mathbf{R}$, where U is an open set in $\mathbf{R}^n$: thus
$$M = \{p \in U : f(p) = 0\}.$$
Suppose that $p_0 \in M$ is a point such that $\nabla f(p_0) \neq \vec{0}$, and let $\vec{v} \in \mathbf{R}^n$ be a vector. Show that $\vec{v}$ is a tangent vector to M at p_0 if and only if $\nabla f(p_0) \cdot \vec{v} = 0$.

(b) Let E be the ellipsoid in $\mathbf{R}^3$ defined by the following equation:
$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$
Find the equation of the tangent plane to M at the point $p_0 = (1, 2, 0)$.

Proof. (a):
Suppose that $\vec{v}$ is a tangent vector to M at p_0. Then there exists a function $\alpha : (-\delta, \delta) \to M$ so that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. Define $g : (-\delta, \delta) \to \mathbb{R}$ by $g(t) = f(\alpha(t))$. For all $t \in (-\delta, \delta)$, $\alpha(t) \in M$, so $g(t) = 0$. It follows that
$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$
Substituting $t = 0$ yields
$$\nabla f(p_0) \cdot \vec{v} = 0$$
as needed.

Conversely, suppose that $\nabla f(p_0) \cdot \vec{v} = 0$. Since $\nabla f(p_0) \neq 0$, $\dfrac{\partial f}{\partial x_i}(p_0) \neq 0$ for some $i \in \{0, ..., n\}$. Define the C^1 function $g : \mathbb{R}^n \to \mathbb{R}$ as the function that swaps the ith and nth coordinate. That is,
$$g(x_1, ..., x_n) = f(x_1, ..., x_{i-1}, x_n, x_{i+1}, ..., x_i)$$
Let p' be the vector in $\mathbb{R}^{n-1}$ whose components are the same as p_0 except that its ith component is p_n. In particular,
$$p' = (p_1, ..., p_{i-1}, p_n, p_{i+1}, ..., p_{n-1}).$$
Notice that $\dfrac{\partial g}{\partial x_n}(p', p_i) = \dfrac{\partial f}{\partial x_i}(p_0) \neq 0$. Applying the Implicit Function Theorem with $k = 1$, there exists an open set $W \subseteq \mathbb{R}^{n-1}$ that contains p' and a continuously differentiable function $\psi : W \to \mathbb{R}$ such that for all $x' \in W$, $\psi(p') = p_i$ and
$$g(x', \psi(x')) = f(x_1, ..., x_{i-1}, \psi(x'), x_{i+1}, ..., x_{n-1}) = 0$$
Since W is open, there exists $\delta > 0$ so that for all $\|t\| < \delta$, $p' + t\pi_{\mathbb{R}^{n-1}}(\vec{v}) \in W$.

Let $\hat{v}$ be $\vec{v}$ with swapped ith and nth components. Let $\alpha : (-\delta, \delta) \to M$ be defined by

$$\alpha(t) = (p_1 + tv_1, ..., p_{i-1} + tv_{i-1}, \psi(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1)), p_{i+1} + tv_{i+1}, ..., p_n + tv_n).$$

We can see that $\alpha(0) = p_0$. Next, we want to show that $\alpha'(0) = \vec{v}$. We first find the total derivative of $\psi(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1))$. For $j \in \{1, ..., n-1\}$, we know that the jth partial derivative is given by

$$\frac{\frac{\partial g}{\partial x_j}(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1)) \cdot v_i \cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1)) \cdot (-v_i)} = -\frac{\frac{\partial g}{\partial x_j}(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1)) \cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p' + tv_i(\pi_{\mathbb{R}^{n-2}}(\hat{v}), -1))}$$

substituting $t = 0$ into this expression gives us

$$-\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial g}{\partial x_n}(p')} = -\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)}$$

For $j \neq i$,

$$-\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)} = -\frac{\frac{\partial f}{\partial x_j}(p_0) \cdot v_j}{\frac{\partial f}{\partial x_i}(p_0)}.$$

For $j = i$,

$$-\frac{\frac{\partial g}{\partial x_j}(p') \cdot \hat{v}_j}{\frac{\partial f}{\partial x_i}(p_0)} = -\frac{\frac{\partial f}{\partial x_n}(p_0) \cdot v_n}{\frac{\partial f}{\partial x_i}(p_0)}$$

Thus the total derivative is given by

$$-\frac{1}{\frac{\partial f}{\partial x_i}(p_0)} \sum_{j \in \{1,...,i-1,i+1,...,n\}} \frac{\partial f}{\partial x_j}(p_0) \cdot v_j$$

Recall that $\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p_0) \cdot v_j = \nabla f(p_0) \cdot \vec{v} = 0$. Using this, our expression for the total derivative at $t = 0$ becomes

$$-\frac{1}{\frac{\partial f}{\partial x_i}(p_0)} \cdot \left(-\frac{\partial f}{\partial x_i}(p_0) \cdot v_i\right) = v_i$$

Therefore, we conclude that

$$\alpha'(0) = (v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n) = \vec{v},$$

Which verifies that $\vec{v}$ is indeed a tangent vector to M at p_0, finishing the proof.

(b):

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$$f(x, y, z) = x^2 + yz + y^2 - xy - xz + z^2 - 3$$

Its zero set is exactly M and its gradient is

$$\nabla f(x, y, z) = (2x - y - z, z + 2y - x, y - x + 2z).$$

Substituting p_0, we get $\nabla f(p_0) = (0, 3, 1) \neq 0$. Let $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (0, 1, -3)$. Notice that

$$\nabla f(p_0) \cdot \vec{v}_1 = \nabla f(p_0) \cdot \vec{v}_2 = 0$$

By the results of part (a), $\vec{v}_1$ and $\vec{v}_2$ are tangent vectors to M at p_0. The equation of the plane spanned by $\vec{v}_1$ and $\vec{v}_2$ is given by

$$3y + z = 0$$

We shift this equation to p_0, giving us

$$3y + z = 6$$

$\square$

Question 33.

(a) Let $g : U \to \mathbf{R}$ be a C^1 function defined on an open set $U \subseteq \mathbf{R}^n$, and let M be its zero set:

$$M = \{p \in U : g(p) = 0\}.$$

Suppose that we have a C^1 function $f : U \to \mathbf{R}$, defined on an open set $U \subseteq \mathbf{R}^n$ which contains M, and we wish to find the maximum of f on M. Assume that M is compact, and that f achieves its maximum on M at some point $p_0 \in M$. Prove that there exists a real number $\lambda \in \mathbf{R}$ such that

$$\nabla f(p_0) = \lambda \nabla g(p_0).$$

This number λ is known as the **Lagrange multiplier**.

(b) Use Lagrange multipliers to solve the following optimization problem: *Find the point(s) on the ellipsoid $x^2 + yz + y^2 - xy - xz + z^2 = 3$ which are **closest** and **furthest** from the origin.*

Proof.
(a):
Let $h : U \times \mathbb{R} \to \mathbb{R}$ be the C^1 function defined by

$$h(x, y) = yg(x) + y - f(x)$$

Notice that $h(p_0, f(p_0)) = f(p_0) \cdot g(p_0) + f(p_0) - f(p_0) = 0$ and $\dfrac{\partial h}{\partial y}(p_0, f(p_0)) = g(p_0) + 1 = 1 \neq 0$. By the implicit function theorem, there exists an open set $W \subseteq \mathbb{R}^n$ and C^1 function $\psi : W \to \mathbb{R}$ such that for all $x \in W$,

$$h(x, \psi(x)) = \psi(x)g(x) + \psi(x) - f(x) = 0$$

Taking the derivative of both sides at p_0 with respect to x, we see that

$$g(p_0)\nabla\psi(p_0) + \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

$$\implies \psi(p_0)\nabla g(p_0) + \nabla\psi(p_0) - \nabla f(p_0) = 0$$

Also note that $\dfrac{\partial \psi}{\partial x_i}(p_0) = \dfrac{\frac{\partial h}{\partial x_i}(p_0, f(p_0))}{\frac{\partial h}{\partial y}(p_0, f(p_0))} = \dfrac{\partial g}{\partial x_i}(p_0) - \dfrac{\partial f}{\partial x_i}(p_0)$. Thus our equation becomes

$$(\psi(p_0) + 1)\nabla g(p_0) - 2\nabla f(p_0) = 0 \implies \nabla f(p_0) = \frac{\psi(p_0) + 1}{2} g(p_0)$$

Therefore the value $\lambda = \dfrac{\psi(p_0) + 1}{2}$ is the one that we needed.

(b):
Define

$$g(x, y, z) = x^2 + yz + y^2 - xy - xz + z^2 - 3.$$

Its zero set M is defined by the ellipsoid

$$x^2 + yz + y^2 - xy - xz + z^2 - 3 = 0.$$

We will attempt to minimize and maximize $f(x,y,z) = x^2 + y^2 + z^2$ on M. We know that if f achieves a maximum or minimum, by part (a), there exists $\lambda \in \mathbb{R}$ so that

$$\nabla f(p_0) = \lambda \nabla g(p_0) \implies (2x, 2y, 2z) = \lambda(2x - y - z, z + 2y - x, y - x + 2z)$$

We simply evaluate f at these points and check which ones are maximums, minimums, or neither. From the above equality, we end up with the system of equations

$$\begin{aligned} 2(1-\lambda)x + \quad y + \quad z &= 0 \\ x + 2(1-\lambda)y - \quad z &= 0 \\ x - \quad y + 2(1-\lambda)z &= 0 \end{aligned}$$

Consider if $\lambda = \frac{3}{2}$. The equations collect and become

$$x = y + z$$

From the ellipsoid, we have

$$x^2 + yz + y^2 - xy - xz + z^2 = 3 \implies f(x,y,z) = x(y+z) - yz + 3,$$

but at the same time, we have

$$x^2 + yz + y^2 - xy - xz + z^2 = 3 \implies x^2 + yz + y^2 - (y+z)y - (y+z)z + z^2 = 3$$

Expand the brackets and simplify to get

$$x^2 - yz = 3 \implies x^2 - yz + 3 = 6.$$

Thus

$$f(x,y,z) = 6, \text{ when } \lambda = \frac{3}{2}$$

Otherwise, we take the first and second equation in the original system and add them together to obtain

$$\begin{aligned} (2(1-\lambda)+1)x + (2(1-\lambda)+1)y &= 0 \\ \implies (2(1-\lambda)+1)(x+y) &= 0 \qquad (1) \end{aligned}$$

Likewise, we can subtract the third equation from the second to get

$$(2(1-\lambda)+1)y - (2(1-\lambda)+1)z = 0$$

$$(2(1-\lambda)+1)(y-z) = 0 \qquad (2)$$

Since $(2(1-\lambda)+1) \neq 0$, from equations (1) and (2), we get that $z = y = -x$. We substitute every variable into the original equation for x and see that

$$x^2 + x^2 + x^2 + x^2 + x^2 + x^2 = 3 \implies x = \frac{1}{\sqrt{2}},\ y = z = -\frac{1}{\sqrt{2}}$$

so

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$$

Thus the points on the ellipsoid that are furthest from the origin satisfy the equation $x = y + z$, and the point that is closest to the origin is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

$\square$

Question 34.

Let $\Phi : \mathbf{R}^n \to \mathbf{R}^m$ be a C^1 mapping.

(a) Suppose that $n > m = 1$. Show that Φ cannot be injective.

(b) Suppose that $n < m$. Show that if $K \subseteq \mathbf{R}^n$ is a compact set, then $\Phi(K)$ is a Jordan measurable set, and has Jordan measure zero.

Proof.
(a):
Suppose for contradiction that $n > m = 1$ and Φ is a C^1 injective function. Since Φ cannot be a constant function, by the results of Big List #26, there is a $p \in \mathbb{R}^n$ so that $\nabla\Phi(p) \neq 0$. In particular, we will say that $\frac{\partial \Phi}{\partial x_j} \neq 0$. Define $\alpha : \mathbb{R}^n \to \mathbb{R}$ by $\alpha(x) = \Phi(x) - \Phi(p)$. Injectivity is translation-invariant, so α is injective. Notice that $\alpha(p) = 0$. We can apply the implicit function theorem to obtain an open set $W \subseteq \mathbb{R}^{n-1}$ that contains $p' = (p_1, ..., p_{j-1}, p_{j+1}, ..., p_n)$ and a C^1 function $\Psi : W \to \mathbb{R}$ such that for all $x = (x_1, ..., x_{n-1}) \in W$,

$$\alpha(x_1, ..., x_{j-1}, \Psi(x), x_j, ..., x_{n-1}) = 0$$

Then, since W is open and contains p', we can find another distinct point $q \in W$. We have

$$\alpha(p_1, ..., p_{j-1}, \Psi(p'), p_{j+1}, ..., p_n) = 0 = \alpha(q_1, ..., q_{j-1}, \Psi(q), q_j, ..., q_{n-1})$$

which contradicts the fact that α is injective.

(b):
Let R be a closed rectangle. $\Phi(R)$ is compact. Claim that $\Phi(R) \subseteq \overline{\Phi(R)^C}$.
Let $x \in \Phi(R)$.

$\square$