Question 30.

Let $U \subseteq \mathbf{R}^n$ be an open set in $\mathbf{R}^n$, and let K be a compact subset of U. Prove that there exists an *infinitely differentiable* function $\varphi : \mathbf{R}^n \to [0,1]$ such that $\varphi(p) = 1$ for all $p \in K$, and $\varphi(p) = 0$ for all $p \in \mathbf{R}^n \setminus U$. This is called a **bump function** supported on U.
(For a function $f : U \to Y$, the n**th total derivative** $f^{(n)}$ is defined as follows: for $n = 0$, we set $f^{(0)} = f$; for $n \geq 1$, if $f^{(n-1)}$ is totally differentiable, we set $f^{(n)} = (f^{(n-1)})'$. We say that f is **infinitely differentiable** if $f^{(n)}$ exists for all $n \geq 0$.)

Proof. First, we notice that the bump function in Big List #4 can be generalised to arbitrary intervals by simply performing horizontal translations. Now, we show that bump functions for closed rectangles within open rectangles in $\mathbb{R}^n$ can be constructed.
Let $R = \prod_{i=1}^n [a_i, b_i]$ be a closed rectangle that is inside an open rectangle $S = \prod_{i=1}^n (c_i, d_i)$ in $\mathbb{R}^n$. This implies that for all i, $c_i < a_i \leq b_i < d_i$. Considering this as an interval in $\mathbb{R}$, we can find a bump function φ_i such that $\varphi_i([a_i, b_i]) = \{1\}$ and $\varphi_i(\mathbb{R} \setminus (c_i, d_i)) = \{0\}$. Notice that this is a function from $\mathbb{R}$ to $\mathbb{R}$. We define $\alpha_i : \mathbb{R}^n \to \mathbb{R}$ by $\alpha_i(x) = \varphi_i(x_i)$. It will be shown that the bump function supported in S is

$$\beta(x) = \prod_{i=1}^n \alpha_i(x)$$

First, note that each α_i is infinitely differentiable, so β is infinitely differentiable as well. If $p = (p_1, ..., p_n) \in R$, then for all $i \in \{1, ..., n\}$, $p_i \in [a_i, b_i]$, so $\alpha_i(p) = 1$. We have

$$\beta(p) = \prod_{i=1}^n \alpha_i(p) = 1$$

Using a similar argument, if $p \in \mathbb{R}^n \setminus S$, there is at least one component of p such that $p_i \notin (c_i, d_i)$, and so $\alpha_i(p) = 0$, which implies that

$$\beta(p) = 0$$

as desired.
Additionally, if there exists bump functions α_1 and α_2 supported on open sets U_1 and U_2 respectively, where the compact sets are $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$, then there exists a bump function φ supported on $U_1 \cup U_2$ for the compact set $K_1 \cup K_2$, which is defined by

$$\varphi(x) = \alpha_1(x) + \alpha_2(x) - \alpha_1(x)\alpha_2(x).$$

Verifying this, we see that if $x \in K_1 \cup K_2$, then either $x \in K_1$ or $x \in K_2$. Assuming without loss of generality that $x \in K_1$, we have that

$$\varphi(x) = 1 + \alpha_2(x) - \alpha_2(x) = 1$$

As well, if $x \in \mathbb{R}^n \setminus (U_1 \cup U_2)$, then $\varphi(x) = 0$, showing that φ is indeed a bump function.
Next, let $\mathbb{R}^n$ be equipped with the max-norm. We claim that $\prod_{i=1}^n (p - r, p + r) \subseteq B(p, r)$, for $p \in \mathbb{R}^n$ and $r > 0$.
Let $x \in \prod_{i=1}^n (p - r, p + r)$. For all i, $|p_i - x_i| < r$, so $\|x - p\|_{\max} < r$. Hence $x \in B(p, r)$.

Thus $\prod_{i=1}^{n}(p-r,p+r) \subseteq B(p,r)$, and moreover by taking the closure of both sets, $\prod_{i=1}^{n}[p-r,p+r] \subseteq \overline{B}(p,r)$.
Now, we can proceed proving the main result.
Let U be an open subset of $\mathbb{R}^n$, and let $K \subseteq U$ be compact. For every $y_i \in K$, there exists an open ball centered around y_i so that $B(y_i, \delta_i) \subseteq U$. Furthermore, we have that

$$\prod_{i=1}^{n}\left[y_i - \frac{\delta_i}{2}, y_i + \frac{\delta_i}{2}\right] \subseteq \prod_{i=1}^{n}(y_i - \delta_i, y_i + \delta_i) \subseteq B(y_i, \delta_i) \subseteq U$$

Let $R_i = \prod_{i=1}^{n}\left[y_i - \frac{\delta_i}{2}, y_i + \frac{\delta_i}{2}\right]$ and $S_i = \prod_{i=1}^{n}(y_i - \delta_i, y_i + \delta_i)$. Notice that $\{R_i^\circ\}$ forms an open cover on K, and by compactness, there is an finite subcover $\{R_{i_n}^\circ\}_{n \leq N}$. Note that if we take the closure of each set, we get the closed cover $\{R_{i_n}\}$.
For each closed rectangle R_{i_n}, we can find a bump function supported on S_i. Since we have a finite number of rectangles, we can “merge” all the bump functions to obtain a bump function γ supported on $\bigcup_{n=1}^{N} S_{i_n}$ for the compact set $\bigcup_{n=1}^{N} K_{i_n}$. We claim that this bump function is also a bump function support on U for the compact set K.
If $x \in K$, then it is also true that $x \in \bigcup_{n=1}^{N} K_{i_n}$. It follows that $\gamma(x) = 1$.
In a similar fashion, if $x \in \mathbb{R}^n \setminus U$, then $x \in \mathbb{R}^n \setminus \bigcup_{n=1}^{N} S_{i_n}$, from which it follows that $\gamma(x) = 0$.
Therefore we have found a bump function as desired and the proof is complete.

$\square$

Question 31.

(a) Let $A \in M_n(\mathbf{R})$ be a symmetric matrix and let $Q(\vec{x}) = \vec{x}^T A\vec{x}$ be the correpsonding quadratic form. Prove that the following two statements are equivalent:

(i) $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.

(ii) All eigenvalues of A are strictly positive.

In this case, we say that Q is a **positive definite** quadratic form, and that A is a **positive definite** matrix.

(b) Prove the following "stay away" lemma, which you will need for part (c): if $Q : \mathbf{R}^n \to \mathbf{R}$ is a positive definite quadratic form, then there exists a constant $\eta > 0$ such that $Q(\vec{x}) \geq \eta \|\vec{x}\|^2$ for all $\vec{x} \in \mathbf{R}^n$.

(c) Let $U \subseteq \mathbf{R}^n$ be an open set, let $f : U \to \mathbf{R}$ be a twice continuously differentiable function, and let $p_0 \in U$ be a point at which $\nabla f(p_0) = \vec{0}$. Prove that if the Hessian matrix $Hf(p_0)$ is positive definite, then f achieves a **local minimum** at p_0; *i.e.* p_0 has an open neighborhood U_0, contained in U, such that $f(p) \geq f(p_0)$ for all $p \in U_0$.

Proof. (a):
Suppose that $Q(\vec{x}) > 0$ for all $\vec{x} \neq 0$. Since A is symmetric, it is orthogonally diagonalizable, so there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ such that $B = P^\top AP$ is diagonal. Letting $Q_B(\vec{x}) = \vec{x}^\top B\vec{x}$, we have that

$$Q(\vec{x}) = Q_B \circ P(\vec{x})$$

We will show that $Q_B(\vec{x}) > 0$ for all $\vec{x} \neq 0$.
Let $\vec{x} \in \mathbb{R}^n$ so that $\vec{x} \neq 0$. Since $Q(\vec{x}) > 0$, it follows that $Q_B(\vec{x}) > 0$ as well, which implies that its eigenvalues are strictly positive. Since A and B share the same eigenvalues, we are done.
Conversely, suppose that all the eigenvalues of A are strictly positive. By a similar argument as before, the eigenvalues of B are positive, so $Q_B(\vec{x}) > 0$ for all $\vec{x} \neq 0$. The same applies to Q and we are done.

(b):
First, equip $\mathbb{R}^n$ with the 2-norm.
Let Q be a positive definite quadratic form, so $Q(\vec{x}) = \vec{x}^\top A\vec{x}$, for some diagonalizable matrix A, where all its eigenvalues λ_i are strictly positive.
Let $\eta = \min_{1\leq i\leq n}\{\lambda_i\}$. Let B be the diagonal matrix such that $A = P^\top BP$ for an orthogonal matrix P. Denote $P(\vec{x}) = (p_1, ..., p_n)$. We have

$$Q(\vec{x}) = Q_B \circ P(\vec{x}) = (P(\vec{x}))^\top B(P(\vec{x})) = \sum_{i=1}^{n} \lambda_i \cdot p_i^2 \geq \eta \sum_{i=1}^{n} p_i^2 = \eta \|P(\vec{x})\|_2$$

Since P preserves distance, we have that

$$Q(\vec{x}) \geq \eta \|P(\vec{x})\|_2^2 = \eta \|\vec{x}\|_2^2$$

as desired.

(c):

Let $U \subseteq \mathbb{R}^n$ be an open set, let $f : U \to \mathbb{R}$ be a C^2 function. Let $p_0 \in U$ be a point such that $\nabla f(p_0) = 0$. Suppose that $Hf(p_0)$ is positive definite. We will show that $Q(x) = \frac{1}{2}x^T Hf(p_0)x$ is a quadratic approximation to f at p_0. That is,

$$\lim_{h \to 0} \frac{f(p_0 + h) - f(p_0) - Q(h)}{\|h\|^2} = 0$$

Let $\varepsilon > 0$. Since U is open, there is a δ_o so that $p_0 + h \in U$, where $\|h\| < \delta_o$. Additionally, by the continuity of Hf, there exists δ_c so that for $\|h\| < \delta_c$, $|Hf(p_0 + h) - Hf(p_0) < 2\varepsilon$. Let $\delta = \min\{\delta_o, \delta_c\}$. Fix $h \in \mathbb{R}^n$ such that $0 < \|h\| < \delta$.

Since U is open, there exists $r > 0$ so that $p_0 + th \in U$ for $t \in (-r, 1 + r)$. Define $g(t) = f(p_0 + th)$. Applying Taylor's quadratic remainder formula to g, there exists $\theta \in (0, 1)$ so that

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(\theta) \implies f(p_0 + h) = f(p_0) + f'(p_0)h + \frac{1}{2}f''(p_0 + \theta h)(h)h$$

Since $\nabla f(p_0) = 0$, we have

$$f(p_0 + h) = f(p_0) + \frac{1}{2}h^T Hf(p_0 + \theta h)h$$

Thus we have

$$|f(p_0 + h) - f(p_0) - Q(h)| = \left|\frac{1}{2}h^T Hf(p_0 + \theta h)h - \frac{1}{2}h^T Hf(p_0)h\right|$$

$$\leq \frac{1}{2}\|h^T\|_c \|Hf(p_0 + \theta h) - Hf(p_0)\| \|h\|$$

where $\| \cdot \|_c$ is a norm on row vectors defined by $\|x\|_c = \|x^T\|$. Automatically we have that

$$|f(p_0 + h) - f(p_0) - Q(h)| \leq \frac{1}{2}\|Hf(p_0 + \theta h) - Hf(p_0)\| \|h\|^2 < \varepsilon \|h\|^2$$

$$\implies \frac{f(p_0 + h) - f(p_0) - Q(h)}{\|h\|^2} < \varepsilon$$

as desired. Thus Q is indeed a quadratic approximation for f at p_0. It remains to show that f attains a local minimum at p_0. Suppose that this is not the case. In particular, for every open neighborhood around p_0 there is a point q where $f(q) < f(p_0)$. We write $q = p_0 + s$, where $s = q - p_0$. Since $Hf(p_0)$ is positive definite, Q is positive definite. Using the quadratic approximation, there is a $\delta > 0$ so that for all $0 < \|h\| < \delta$,

$$|f(p_0 + h) - f(p_0) - Q(h)| < \eta \|h\|^2,$$

where η comes from part (b) applied to Q. Find a suitable s such that $f(p_0 + s) < f(p_0)$. By part (a), $Q(s) > 0$. This implies that $f(p_0 + s) - f(p_0) - Q(s) < 0$, so

$$f(p_0) - f(p_0 + s) + Q(s) < \eta \|h\|^2 \implies f(p_0) + Q(s) - \eta \|h\|^2 < f(p_0 + s).$$

From part (b), we know that $Q(s) - \eta\|h\|^2 \geq 0$, so

$$f(p_0) < f(p_0 + s)$$

which is a contradiction.
Therefore we have shown that f attains a local minimum at p_0 and we are done.

$\square$