# 5 Homework 5

**13.** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed vector spaces and let $T : X \to Y$ be a linear mapping.

(a) Prove that $T$ is continuous if and only if $T$ is a bounded linear mapping.

*Proof.* Suppose that $T$ is continuous. Then at $x_0 = 0$, there exists a $\delta > 0$ such that

$$\|x\|_X < \delta \implies \|T(x)\|_2 \le 1$$

We claim that our $M = \frac{2}{\delta}$.Let $x \in X$. Notice that $\left\| \dfrac{\delta \cdot x}{2\|x\|_X} \right\|_X <= \delta$. By the continuity of $T$ we have that

$$\left\| T\left(\frac{\delta \cdot x}{2\|x\|_X}\right) \right\|_Y \le 1 \implies \frac{\delta}{2 \cdot \|x\|_X} \|T\left(x\right)\|_Y \le 1 \implies \|T\left(x\right)\|_Y \le \frac{2}{\delta}\|x\|_X$$

Thus $T$ is a bounded linear mapping.

Next, suppose that $T$ is a bounded linear mapping. Then there is an $M > 0$ such that for all $x \in X$,

$$\|T(x)\|_Y \le M\|x\|_X$$

We will show that $T$ is continuous everywhere. Fix $a \in X$. Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{M}$. Let $x \in X$ and suppose that $\|x - a\|_X < \delta$. Then

$$\|T(x) - T(a)\|_Y = \|T(x - a)\|_Y \le M\|x - a\|_X < \varepsilon$$

and we are done. □

(b) Suppose that $(Y, \|\cdot\|_Y) = (\mathbf{R}^n, \|\cdot\|_2)$. Prove that $T$ is continuous if and only if $\ker(T)$ is closed.

*Proof.* Suppose that $T$ is continuous. Let $a$ be a limit point for $\ker(T)$. We want to show that $T(a) = 0$. By definition, $T$ being continuous implies that

$$\lim_{x \to a} T(x) = T(a)$$

or more formally, letting $\varepsilon$ be arbitrary, there exists a $\delta$ such that

$$\|x - a\|_X < \delta \implies \|T(x) - T(a)\|_2$$

Since $a$ is a limit point of $\ker(T)$, there exists $x \in \ker(T)$ such that

$$x \in B(a, \delta) \implies \|x - a\|_X \implies \|T(x) - T(a)\|_2 < \varepsilon \implies \|T(a)\|_2 < \varepsilon$$

By properties of norms $\|T(a)\|_2 \ge 0$, so we must have that $T(a) = 0 \implies a \in \ker(T)$. Thus $\ker(T)$ is closed.

I have no clue how to prove the converse so yeah.

□

**14.** Let $T : X \to Y$ be a bounded linear mapping between complete normed vector spaces $X$ and $Y$. Is it necessarily true that $T^{-1}$ is also bounded?

*Proof.* $T^{-1}$ is indeed bounded. Let $T$ be a bounded linear mapping. For some $M > 0$,

$$\|T(x)\|_Y \leq M\|x\|_X \text{for all } x \in X$$

$\square$

**16.** Let $(X, \|\cdot\|)$ be a normed vector space, let $B(X) = B(X, X)$ be the space of bounded linear operators on $X$, and equip $B(X)$ with the operator norm $\|\cdot\|_{\text{op}}$. Let $\text{GL}(X)$ be the set of invertible bounded linear operators:

$$\text{GL}(X) = \{T \in B(X) : T \text{ is invertible}\}.$$

(The notation GL means "general linear group.")

Prove that if $(X, \|\cdot\|)$ is complete, then $\text{GL}(X)$ is an open subset of $(B(X), \|\cdot\|_{\text{op}})$.

*Proof.* First, we prove a couple lemmas.

**Lemma 1.** For all $S, T \in B(X, X)$,

$$\|ST\|_{\text{op}} \leq \|S\|_{\text{op}}\|T\|_{\text{op}}$$

To prove this, let $x \in X$ so that $\|x\|_X \leq 1$. Then

$$\|S(T(x))\|_X = \left\|S\left(\frac{\|T(x)\|_X}{\|T(x)\|_X}T(x)\right)\right\|_X = \|T(x)\|_X \left\|S\left(\frac{T(x)}{\|T(x)\|_X}\right)\right\|_X$$

Notice that $\left\|\dfrac{T(x)}{\|T(x)\|_X}\right\|_X = 1$. We obtain that

$$\|S(T(x))\|_X = \|T(x)\|_X \left\|S\left(\frac{T(x)}{\|T(x)\|_X}\right)\right\|_X \leq \|T\|_{\text{op}}\|S\|_{\text{op}}$$

Since $\|T\|_{\text{op}}\|S\|_{\text{op}}$ is an upper bound for $\|S(T(x))\|_X$ when $\|x\|_X \leq 1$, by definition of supremum we can conclude that $\|ST\|_{\text{op}} \leq \|S\|_{\text{op}}\|T\|_{\text{op}}$, proving the first lemma.

Next, the following lemma will help in proving Lemma 3.

**Lemma 2.** If $T, U \in B(X)$, then for $x \in X$, $\|T(x) - U(x)\|_X \leq \|x\|_X\|T - U\|_{\text{op}}$.

Let $T, U \in B(X)$. Then

$$\|(T - U)(x)\|_X = \left\|\|x\|_X(T - U)\left(\frac{x}{\|x\|_X}\right)\right\| = \|x\|_X \left\|(T - U)\left(\frac{x}{\|x\|_X}\right)\right\|$$

Since $\dfrac{x}{\|x\|_X} = 1$,

$$\|x\|_X \left\|(T - U)\left(\frac{x}{\|x\|_X}\right)\right\| \leq \|x\|_X\|T - U\|_{\text{op}}$$

which is what we wanted.

We will use the proceeding lemma to help us prove Lemma 4.

**Lemma 3.** If $X$ is complete then $B(X, X)$ is complete.

For every Cauchy sequence $(T_n)$ in $B(X, X)$, the sequence $(T_n(x_i))$, where $x_i$ is an arbitrary element in $X$, is a Cauchy sequence in $X$. By the completeness of $X$, $(T_n(x_i))$ converges to some $L_i$. Let $T$ be a linear operator on $X$ defined by $T(x_i) = L_i$. We can see that this is indeed a linear operator because

$$cT(x_i) + T(x_j) = cL_i + L_j = c\lim_{n\to\infty} T_n(x_i) + \lim_{n\to\infty} T_n(x_j) = \lim_{n\to\infty} cT_n(x_i) + T_n(x_j) = \lim_{n\to\infty} T_n(cx_i + x_j)$$

$X$ is closed, so $cx_i + x_j$ is equal to some $x_k \in X$. Thus

$$cT(x_i) + T(x_j) = \lim_{n\to\infty} T_n(cx_i + x_j) = \lim_{n\to\infty} T_n(x_k) = L_k = T(x_k)$$

which implies that $T$ is a linear operator.

To show that $T$ is bounded, we know that $T_n$ is bounded, so $\|T_n(x)\|_X < M_n\|x\|_X$, for some $M_n > 0$. As well, we have that $T_n$ is uniformly continuous. Thus for a sufficiently large $n \in \mathbb{N}$, for all $x \in X$, using Lemma 2,

$$\begin{aligned}\|T(x)\|_X = \lim_{m\to\infty} \|T_m(x)\|_X \leq \lim_{m\to\infty} \|T_m(x) - T_n(x)\|_X + \|T_n(x)\|_X < \lim_{m\to\infty} \|T_n - T_m\|_{\text{op}}\|x\|_X + \|T_n(x)\|_X \\ \leq \|x\|_X + M_n\|x\|_X\end{aligned}$$

We get the last inequality because $T_n$ is Cauchy and bounded as well. Thus $T \in B(X)$. To show that $T_n$ converges to $T$, fix $\varepsilon > 0$. For all $x \in X$ such that $\|x\|_X \leq 1$, there exists a sufficiently large $N$ such that for all $x \in X$, if $n > N$ then

$$\|T(x) - T_n(x)\|_X < \varepsilon \implies \|T - T_n\|_{\text{op}} < \varepsilon$$

Thus we can conclude that $B(X, X)$ is complete.

**Lemma 4.** If $T \in B(X, X)$ such that $\|I - T\|_{\text{op}} < 1$, then $T$ is invertible and

$$T^{-1} = \sum_{i=0}^{\infty} (I - T)^i$$

Suppose that $T \in B(X, X)$ and $\|I - T\|_{\text{op}} < 1$. Let $(a_n)_{n \geq 1}$ be a Cauchy sequence in $B(X, X)$ defined by

$$a_n = \sum_{i=0}^{n} (I - T)^i$$

We verify that this sequence is Cauchy. Let $\varepsilon > 0$. Since $\|I - T\|_{\text{op}} < 1$, there exists natural $N$ such that

$$\frac{\|I - T\|_{\text{op}}^N}{1 - \|I - T\|_{\text{op}}} < \varepsilon$$

Let $m, n > N$ and suppose that $m < n$. Then

$$\|a_n - a_m\|_{\text{op}} = \left\| \sum_{i=0}^{n} (I - T)^i - \sum_{i=0}^{m} (I - T)^i \right\|_{\text{op}} = \left\| \sum_{i=m+1}^{n} (I - T)^i \right\|_{\text{op}}$$

By the triangle inequality and Lemma 1,

$$\left\| \sum_{i=m+1}^{n} (I - T)^i \right\| \leq \sum_{i=m+1}^{n} \left\| (I - T)^i \right\| \leq \sum_{i=m+1}^{n} \|I - T\|^i = \|I - T\|^{m+1} \sum_{i=0}^{n-m-1} \|I - T\|^i$$

$$= \|I - T\|^{m+1} \frac{1 - \|I - T\|^{n-m}}{1 - \|I - T\|}$$

$1 - \|I - T\|^{n-m} < 1$, so

$$\|I - T\|^{m+1} \frac{1 - \|I - T\|^{n-m}}{1 - \|I - T\|} < \frac{\|I - T\|^{m+1}}{1 - \|I - T\|} < \frac{\|I - T\|^N}{1 - \|I - T\|} < \varepsilon$$

We see that $(a_n)$ is indeed a Cauchy sequence.

By Lemma 4, $B(X, X)$ is complete, so $(a_n)$ converges, meaning that $\sum_{i=0}^{\infty} (I - T)^i$ exists. Now, notice that

$$T \left( \sum_{i=0}^{\infty} (I - T)^i \right) = (I - (I - T)) \left( \sum_{i=0}^{\infty} (I - T)^i \right) = \sum_{i=0}^{\infty} (I - T)^i - \sum_{i=0}^{\infty} (I - T)^{i+1} = \sum_{i=0}^{\infty} \left( (I - T)^i - (I - T)^{i+1} \right)$$

This is a telescoping series. As $i \to \infty$, $(I - T)^i \to 0$. Thus

$$T \left( \sum_{i=0}^{\infty} (I - T)^i \right) = \sum_{i=0}^{\infty} \left( (I - T)^i - (I - T)^{i+1} \right) = I$$

Thus $\sum_{i=0}^{\infty} (I - T)^i$ is the inverse of $T$, which implies that $T^{-1}$ exists.

Now we can show that $B(X)$ is open.

Let $T \in B(X)$. Consider the open ball $B(T, \frac{1}{\|T^{-1}\|_{\text{op}}})$. For all $S \in B(T, \frac{1}{\|T^{-1}\|_{\text{op}}})$,

$$\|T - S\|_{\text{op}} < \frac{1}{\|T^{-1}\|_{\text{op}}} \implies \|T - S\|_{\text{op}} \|T^{-1}\|_{\text{op}} < 1$$

By Lemma 1, we have that

$$1 > \|T - S\|_{\text{op}} \|T^{-1}\|_{\text{op}} \geq \|(T - S)T^{-1}\|_{\text{op}} = \|I - ST^{-1}\|_{\text{op}}$$

By Lemma 4, $ST^{-1}$ is invertible. If we let $S^{-1} = T^{-1}(ST^{-1})^{-1}$, we see that

$$SS^{-1} = S(T^{-1}(ST^{-1})^{-1}) = (ST^{-1})(ST^{-1})^{-1} = I$$

Thus $S$ is invertible, which means that $S \in B(X)$. Therefore $B(X)$ is open.

$\square$

**17.** *The magic number lemma.*

Let $(X, d)$ be a metric space and let $\{U_i\}_{i\in I}$ be an open cover of $X$; this means that each $U_i$ is an open subset of $X$, and that $X = \bigcup_{i\in I} U_i$. A **magic number** for $\{U_i\}_{i\in I}$ is a number $\delta > 0$ with the following property: if $A \subseteq X$ is a set with $\operatorname{diam}(A) < \delta$, then $A \subseteq U_i$ for at least one index $i \in I$.

Suppose that $(X, d)$ is a clustering metric space. Prove that every open cover has a magic number.

*Proof.* Suppose that $(X, d)$ is a clustering metric space. Suppose for the sake of contradiction that there exists an open cover $\{U_i\}_{i\in I}$ that doesnt have a magic number.

For $n \in \mathbb{N}$, there is $A_n \subseteq X$ with $\operatorname{diam} A_n < \frac{1}{n}$ so that $A_n \nsubseteq U_i$ for all indices $i \in I$. Since $\operatorname{diam} A_n < \frac{1}{n}$, we can cover $A_n$ with an open ball $B(a_n, \frac{1}{n})$, where $a_n$ is some element in $X$.

Define a sequence $(a_n)_{n\in\mathbb{N}}$ in $X$ such that $a_n$ is equal to the one above.

By the clustering property of $X$, $(a_n)$ has a convergent subsequence, which will be redefined as $(a_n)$. Denote the limit of $(a_n)$ as $p$.

$p$ is an element of $X$, so it is contained in some $U_i$ in the open cover. Since $U_i$ is open, we can find $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U_i$. As well, since $(a_n)$ converges to $p$ we can find infinitely many entries of the sequence within the open ball $B(p, \frac{\varepsilon}{2})$. Thus we can find a large enough $n$ such that $n > \frac{2}{\varepsilon}$, which gives $\frac{1}{n} < \frac{\varepsilon}{2}$, and still have that $a_n$ is $\frac{\varepsilon}{2}$-close to $p$.

Now, consider the open ball $B(a_n, \frac{1}{n})$. We will show that $B(a_n, \frac{1}{n}) \subseteq B(p, \varepsilon)$.

Let $x \in B(a_n, \frac{1}{n})$. Then $d(x, a_n) < \frac{1}{n} < \frac{\varepsilon}{2}$, so we have

$$d(x, p) \le d(x, a_n) + d(a_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies x \in B(p, \varepsilon)$$

which is what we wanted.

Recall that the set $A_n$ is covered by $B(a_n, \frac{1}{n})$. Then we have

$$A_n \subseteq B(a_n, \frac{1}{n}) \subseteq B(p, \varepsilon) \subseteq U_i$$

contradicting the fact that $A_n \nsubseteq U_i$. Thus every open cover has a magic number.

□

**18.** The sequence space $\ell^1$ has two norms: the 1-norm $\|\cdot\|_1$, and the sup norm $\|\cdot\|_\infty$ inherited from $\ell^\infty$.

(a) Is $(\ell^1, \|\cdot\|_\infty)$ separable?

(b) Is $(\ell^1, \|\cdot\|_1)$ separable?

*Proof.* Yes to both. We will show part (b) first and notice that part (a) follows right after.

Let $D$ be the set of all sequences of rational numbers that are eventually 0. This set is obviously a subset of $\ell^1$. We can write $D$ as the countable union of rational sequences that terminate after the $n$th term, which countable since there is a bijection to $\mathbb{Q}^n$. It follows that $D$ is countable. It remains to show that $D$ is dense in $\ell^1$ with respect to the 1-norm.

Let $x \in \ell^1$ and $\varepsilon > 0$. Since $x$ is absolutely summable, every partial sum is strictly less than the limit of the series, denoted as $L$. Also, for all $n$ greater than some $N > 0$,

$$\sum_{i=N+1}^{n} |x_i| = \sum_{i=0}^{n} |x_i| - \sum_{i=0}^{N} |x_i| < L - \sum_{i=0}^{N} |x_i| < L - \left(L - \frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2}$$

When $1 \le i \le N$, by the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find a $q_i \in \mathbb{Q}$ such that $\|x_i - q_i\| < \frac{\varepsilon}{2N}$.

Let $q = \begin{cases} q_i, & \text{if } 1 \le i \le N; \\ 0, & \text{otherwise.} \end{cases}$. We see that

$$\|x - q\|_1 = \sum_{i=0}^{\infty} |x_i - q_i| = \sum_{i=0}^{N} |x_i - q_i| + \sum_{i=N+1}^{\infty} |x_i| < \frac{N\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

Thus $D$ is dense, implying that $\ell^1$ is separable with respect to the 1-norm.

Note that for all $x \in \ell^1$, $\|x\|_\infty \le \|x\|_1$. it follows immediately that $(\ell^1, \|\cdot\|_\infty)$ is separable as well.

$\square$