6 HOMEWORK 6 HAAHHAHAHAHAHAHAA

Question 19.

Let X be a metric space and let $A \subseteq X$. A **compact exhaustion** for A is a sequence of compact sets $K_1, K_2, K_3, \ldots$ such that $U = \bigcup_{i \geq 1} K_i$ and $K_i \subseteq K_{i+1}^{\circ}$.

(a) Let $U \subseteq \mathbf{R}^n$ be a bounded open set. Show that U has a compact exhaustion.

(b) Now show that every open set $U \subseteq \mathbf{R}^n$ has a compact exhaustion.

Proof. (b):

□

Question 20.

Let $x, y \in \ell^\infty$ be two sequences. Let us say that y is **dominated** by x, denoted $x \geq y$, if $|x_n| \geq |y_n|$ for all $n \in \mathbf{N}$. Let D_x denote the set of all sequences which are dominated by x:

$$D_x = \{y \in \ell^\infty : |y_n| \leq |x_n| \text{ for all } n \in \mathbf{N}\}.$$

Prove that D_x is compact if and only if $x_n \to 0$.

Proof. Suppose that D_x is compact. Suppose for contradiction that $x_n \nrightarrow 0$. For some $\varepsilon > 0$, $|x_{N_k}| \geq \varepsilon$ for an infinite number of N_k. Consider the open cover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \in I}$, which is the collection of $\frac{\varepsilon}{2}$-balls centered around every $\vec{y}_i \in D_x$. By compactness of D_x, there is a finite subcover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \leq m}$. Now, we construct a $y \in D_x$ as follows:
For every sequence $\vec{y}_i$, let

$$y_{N_i} = \begin{cases} \varepsilon, & \text{if } (\vec{y}_i)_{N_i} < \dfrac{\varepsilon}{2}; \\ 0, & \text{if } (\vec{y}_i)_{N_i} \geq \dfrac{\varepsilon}{2}; \end{cases}$$

For all other terms in y, make it 0. Notice that for all $B(\vec{y}_i, \frac{\varepsilon}{2})$,

$$\|y - \vec{y}_i\|_\infty \geq |y_{N_i} - (\vec{y}_i)_{N_i}| \geq \frac{\varepsilon}{2} \implies y \notin D_x$$

which is a contradiction.
Conversely, suppose that $x_n \to 0$. We will show that D_x is complete and totally bounded, which is equivalent to compactness.
To show completeness, notice that the ambient space ℓ^∞ is complete. Thus if we can show that D_x is closed, it will follow that D_x is complete.
Let $a \notin D_x$. We will show that a is not a limit point of D_x, which means that D_x is closed. We know that there is a $k \in \mathbb{N}$ such that $|a_k| > |x_k|$. Define $\varepsilon = |a_k| - |x_k|$. Fix $y \in D_x$. Then

$$\|a_k - y_k\|_\infty \geq |a_k - y_k| \geq |a_k| - |y_k| \geq |a_k| - |x_k| = \varepsilon$$

which implies that $y \notin B(a_k, \varepsilon)$, so a is not a limit point of D_x. Thus D_x is closed. It follows that D_x is complete.
To show that D_x is totally bounded, first let $\varepsilon > 0$. Since $x_n \to 0$, there is a large enough N such that for $n > N$, $|x_n| < \frac{\varepsilon}{2}$. For all $y \in D_x$, $|y_n| < |x_n| < \frac{\varepsilon}{2}$. Consider the set of elements in D_x such that their terms are 0 for $n > N$. This set is totally bounded, so there is a ε-ball cover $\{B(y_i, \varepsilon)\}_{i \leq n}$. We show that this collection also covers D_x.
For $y \in D_x$, there is an open ball $B(y_i, \varepsilon)$ such that $\sup_{n \leq N} |y_n - (y_i)_n| < \varepsilon$. But also notice that for $n > N$, $|(y_i)_n - y_n| \leq |(y_i)_n| + |y_n| < |y_n| < \frac{\varepsilon}{2}$. Thus $\|y - y_i\|_\infty < \varepsilon$ so $y \in B(y_i, \varepsilon)$. Thus D_x is totally bounded.
Since D_x is both complete and totally bounded, we can conclude that D_x is compact.

$\square$