Question 42.

Let $M \subseteq \mathbf{R}^N$ be a smooth n-manifold (with or without boundary!).

(a) Show that if $n < N$, then M is a *Lebesgue null set*.

(b) Show that if $n = N$ and M is closed and its boundary is nonempty, then ∂M coincides with the usual topological boundary (as defined on Handout #2).

(c) Show that if M is compact and its boundary is nonempty, then M is Jordan measurable.

Proof.
(a):
We begin by proving a number of lemmas:
Lemma 1: An open cover of any subset $M \subseteq \mathbb{R}^n$ has a countable subcover.
We know that $\mathbb{R}^n$ is separable, so M is also separable. Let C be a countable dense subset of M. Let $\mathcal{U}$ be an open cover for M. We construct the countable subcover $\hat{U}$ as follows. For each $q \in C$ and $k \in \mathbb{Q}$, consider the open ball $B(q,k)$. If there exists a $U_{qk} \in \mathcal{U}$ such that $B(q,k) \in U_{qk}$, include it in $\hat{U}$. Notice that $\hat{U}$ is at most countable. We claim that it is also an open cover.
Let $x \in M$. Then it is contained in some open set $U \in \mathcal{U}$. As well, we can find an open ball such that $B(x,\delta) \in U$. Since C is dense, we can find $q \in C$ such that $q \in B(x, \frac{\delta}{4})$. Let $k \in \mathbb{Q}$ such that $\frac{\delta}{4} < k < \frac{\delta}{2}$. Then $x \in B(q,k) \subseteq B(x,\delta)$, because for all $y \in B(q,k)$,

$$\|x - y\| \leq \|x - q\| + \|q - y\| < \frac{\delta}{4} + \frac{\delta}{2} < \delta$$

It follows that $B(q,k) \in U$, so it is guaranteed that some U_{qk} from our construction exists. Thus $x \in U_{qk} \in U$ so U is indeed an open cover and we are done.

Lemma 2: A countable union of sets with Jordan measure 0 is a Lebesgue null set.
Let $E = \bigcup_{i \geq 1} E_i$, where $\mu(E_i) = 0$. Let $\varepsilon > 0$. For each E_i, we can find a finite union of boxes B_i such that $B_i \supseteq E_i$ and $\text{vol}(B_i) < \frac{\varepsilon}{2^i}$. We see that $\bigcup_{i \geq 1} B_i$ is a countable union of boxes, $E \subseteq \bigcup_{i \geq 1} B_i$, and

$$\sum_{i=1}^{\infty} \text{vol}(B_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2(1 - \frac{1}{2})} = \varepsilon$$

as desired.

Now, we prove the problem at hand. Let $M \subseteq \mathbb{R}^N$ be a smooth n-manifold with $n < N$. Let $\{(U_i, \varphi)\}_{i \in I}$ be an atlas for M. By Lemma 1, we can assume without loss of generality that the atlas is countable. We can also assume that each U_i is bounded, for if not, we can take a countable union of open balls that cover the unbounded U_i, and restrict the embedding to each ball.
For each chart (U_i, φ_i), assume that the

(b):
Let p be a point in the topological boundary of M. We will show that $p \in \partial M$. Suppose for contradiction that p is in M°, meaning it is contained in a chart (U, φ) that is diffeomorphic

to an open set $\hat{U}$ in $\mathbb{R}^N$. Note that φ is a diffeomorphism with domain $\hat{U}$ and codomain U. Then it must be true that $\varphi(\hat{U}) = U$ is an open subset of $\mathbb{R}^N$. But this is a contradiction, as that would imply that the boundary point p is in the interior of M. Therefore $p \in \partial M$.
Next, let $p \in \partial M$ and again suppose for contradiction that p is not in the topological boundary of M. Then it must be true that p is in the topological interior of M. Recall that $p \in \partial M$ implies that it is contained in a chart (U, φ) that is diffeomorphic to $\overline{\mathbb{H}^n}$ and $p \in \mathrm{bd}(\mathbb{H}^n)$. Since p is in the topological interior of M, we can find an open ball $B(p,r) \subseteq M$ which is also open in $\mathbb{R}^N$. Then $\varphi^{-1}(B(p,r))$ should also be open in $\mathbb{R}^N$. But this implies that for small enough δ, $\varphi(p) - (0, ..., \delta) \in \overline{\mathbb{H}^N}$, which cannot happen.

Therefore we can conclude that ∂M coincides with the topological boundary of M.

(c):
First, we prove that a compact Lebesgue null set E has Jordan measure 0. It suffices to show that the upper measure $\mu^*(E) = 0$.
Let $\varepsilon > 0$. By definition, we can find a countable union of boxes $B = \bigcup_{i=1}^{\infty} B_i$ such that $E \subseteq B$ and $\mathrm{vol}(B) < \varepsilon$. But since E is compact, it can be covered by finitely many boxes B_{n_i}, $0 < i \leq N$. Thus

$$\mathrm{vol}\left(\bigcup_{i=1}^{N} B_{n_i}\right) = \sum_{i=1}^{N} \mathrm{vol}(B_{n_i}) \leq \mathrm{vol}(B) < \varepsilon.$$

Since ε was chosen arbitrarily, we can conclude that $\mu^*(E) = 0$, and E has Jordan measure 0.
Now, suppose that M is compact and its boundary is nonempty. If $\dim M < N$, M is a Lebesgue null set and has Jordan measure 0, and therefore measurable. Otherwise, if $\dim M = N$, since the boundary of M is non-empty, the topological boundary of M is actually a smooth manifold of dimension $(N-1)$, and therefore a Lebesgue null set. As well, the topological boundary of M is compact, so it is Jordan measure 0, which implies that M is Jordan measurable.

$\square$