Question 38.

Basel Problem. Here you will use multivariable calculus to establish the following famous equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

To do it, you will evaluate the (improper) double integral $\int_U \frac{1}{1-xy}$ in two ways. Let $f : (0,1)^2 \to \mathbf{R}$ be the function given by $f(x,y) = \frac{1}{1-xy}$, and let K_N denote the closed box $\left[\frac{1}{N}, 1-\frac{1}{N}\right]^2$.

(a) Evaluate $\int_{K_N} f$ using Fubini's theorem.

(b) Evaluate $\int_{K_N} f$ using the Change of Variables formula twice: first using the linear diffeomorphism $(x,y) = (u+v, u-v)$, then using the polar coordinates transform.

(c) Conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof.
(a):
By Fubini's:

$$\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{\frac{1}{N}}^{1-\frac{1}{N}} \frac{1}{1-xy}\, dy\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x} \ln(1-xy)\Big|_{\frac{1}{N}}^{1-\frac{1}{N}}\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x}\left(\ln\left(1-\left(1-\frac{1}{N}\right)x\right) - \ln\left(1-\frac{1}{N}x\right)\right)\, dx
\end{aligned}$$

Notice that $-1 < -\left(1-\frac{1}{N}\right), -\frac{1}{N} < 1$, so we can use the power series expansion of $\ln(1+t)$

to get that

$$\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{\left(1-\frac{1}{N}\right)^n x^n}{n} - \sum_{n=1}^{\infty}\frac{\left(\frac{1}{N}\right)^n x^n}{n}\right)\,dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)\,dx \\
&= \sum_{n=1}^{\infty}\frac{x^n}{n^2}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)\Bigg|_{\frac{1}{N}}^{1-\frac{1}{N}} \\
&= -\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)^2
\end{aligned}$$

(b):
Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the diffeomorphism defined by

$$\Phi(x,y) = (x+y, x-y).$$

We want to find E, such that $\Phi(E) = K_N$. We will do this by finding $\Phi^{-1}(K_N)$. Notice that $\Phi^{-1}(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$. Fix $u, v \in \left[\frac{1}{N}, 1-\frac{1}{N}\right]$ and let $x = \frac{u+v}{2}$. We have that $x \in \left[\frac{1}{N}, 1-\frac{1}{N}\right]$. Thus $\frac{u-v}{2} = x - v \in \left[x - 1 + \frac{1}{N}, x - \frac{1}{N}\right]$ and we can write

$$\Phi^{-1}(K_N) = \left\{(x,y) : \frac{1}{N} \le x \le 1-\frac{1}{N}, x-1+\frac{1}{N} \le y \le x - \frac{1}{N}\right\} = E.$$

Therefore we have that

$$\begin{aligned}
\int_{K_N} f &= \int_E f\circ\Phi \cdot |\det J\Phi| \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{x-1+\frac{1}{N}}^{x-\frac{1}{N}} \frac{1}{1-(x+y)(x-y)} \cdot |-2|\;dy\;dx \\
&= 2\int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{x-1+\frac{1}{N}}^{x-\frac{1}{N}} \frac{1}{1-x^2+y^2}\;dy\;dx
\end{aligned}$$

Now, let $\Psi : (0,\infty)\times(-\frac{\pi}{2},\frac{\pi}{2}) \to \mathbb{R}^2 \setminus (-\infty,0]\times\mathbb{R}$ be the diffeomorphism defined by $\Psi(r,\theta) = (r\cos\theta, r\sin\theta)$. We use a similar strategy to find F, where $\Psi(F) = E$. We see that $\Psi^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)$. Let $r = \sqrt{x^2+y^2}$. Based on our definition of $\Phi^{-1}(K_N)$, we have that $\frac{1}{N} \le x \le 1-\frac{1}{N}$ and $\frac{2}{N} - 1 \le y \le 1 - \frac{2}{N}$, so $\frac{\frac{2}{N}-1}{1-\frac{1}{N}} \le \frac{y}{x} \le N\left(1-\frac{2}{N}\right)$. Thus

$$\arctan\left(\frac{2-N}{N-1}\right) \le \theta \le \arctan(N-2).$$

Additionally, we see that $\sqrt{x^2+y^2} = |x|\sqrt{1+\left(\frac{y}{x}\right)} = x\sqrt{1+\tan^2\theta} = x\sec\theta$ (we can remove absolute values because $x>0$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$). Therefore we can write

$$F = \Psi^{-1}(E) = \left[(r,\theta) : \arctan\left(\frac{2-N}{N-1}\right) \leq \theta \leq \arctan(N-2), \frac{1}{N}\sec\theta \leq r \leq \left(1-\frac{1}{N}\right)\sec\theta\right].$$

We use change of variables on the integral once again to obtain

$$\begin{aligned}
\int_{K_N} &= 2\int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{x-1+\frac{1}{N}}^{x-\frac{1}{N}} \frac{1}{1-x^2+y^2}\, dy\ dx \\
&= 2\int_{\arctan\left(\frac{2-N}{N-1}\right)}^{\arctan(N-2)} \int_{\frac{1}{N}\sec\theta}^{\left(1-\frac{1}{N}\right)\sec\theta} \frac{r}{1-r^2(\cos^2\theta-\sin^2\theta)}\, dr\ d\theta \\
&= 2\int_{\arctan\left(\frac{2-N}{N-1}\right)}^{\arctan(N-2)} -\frac{1}{2(\cos^2\theta-\sin^2\theta)}\ln(1-r^2(\cos^2\theta-\sin^2\theta))\Big|_{\frac{1}{N}\sec\theta}^{\left(1-\frac{1}{N}\right)\sec\theta} d\theta \\
&= -\int_{\arctan\left(\frac{2-N}{N-1}\right)}^{\arctan(N-2)} \frac{\ln(1-\left(1-\frac{1}{N}\right)^2(1-\tan^2\theta)) - \ln(1-\frac{1}{N^2}(1-\tan^2\theta))}{\cos^2\theta-\sin^2\theta}\, d\theta \\
&= -\int_{\arctan\left(\frac{2-N}{N-1}\right)}^{\arctan(N-2)} \frac{\ln(1-\left(1-\frac{1}{N}\right)^2(1-\tan^2\theta)) - \ln(1-\frac{1}{N^2}(1-\tan^2\theta))}{1-\tan^2\theta}\cdot\sec^2\theta\ d\theta
\end{aligned}$$

Let

$\square$