Question 23.

Let S^2 denote the unit sphere in $\mathbf{R}^3$. Let $N = (0, 0, 1)$ denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to $\mathbf{R}^2$. To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbf{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in $\mathbf{R}^3$.

(a) Given $P = (x, y, z)$, find an explicit formula for $\Phi(P)$ in terms of x, y, z.

(b) Deduce that Φ is continuous.

(c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.

(d) Deduce that Φ is a homeomorphism.

Proof. (a):
Let $P = (x, y, z)$. First, we find the equation of the line that passes P and N. Consider the equation of the line $L(t) = (tx, ty, (z-1)t + 1)$. Notice that $L(0) = N$ and $L(1) = P$, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when $(z-1)t + 1 = 0$. Solving for t gives $t = \dfrac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Thus

$$\Phi(P) = \frac{1}{1-z}(x, y).$$

(b):
(c):
Let $p = (s, t) \in \mathbb{R}^2$. Our goal is to find $(x, y, z) \in S^2 \setminus \{N\}$ such that $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s, t)$. Immediately, we obtain the following system of equations:

$$\frac{x}{1-z} = s,$$
$$\frac{y}{1-z} = t,$$
$$x^2 + y^2 + z^2 = 1$$

We also have the restriction $z \neq 1$ because $(x, y, z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$
$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^2(1-z)^2 + t^2(1-z)^2 + z^2 = 1 \implies (s^2 + t^2 + 1)z^2 - 2(s^2 + t^2)z + s^2 + t^2 - 1 = 0$$

We can replace the term $t^2 + s^2$ with $\|p\|_2^2$, and the equation becomes

$$(\|p\|_2^2 + 1)z^2 - 2\|p\|_2^2 z + \|p\|_2^2 - 1 = 0$$

Using the quadratic formula:

$$z = \frac{2(\|p\|_2^2) \pm \sqrt{4(\|p\|_2^2)^2 - 4(\|p\|_2^2 + 1)(\|p\|_2^2 - 1)}}{2(\|p\|_2^2 + 1)}$$

$$\implies z = \frac{\|p\|_2^2 \pm \sqrt{\|p\|_2^4 - (\|p\|_2^4 - 1)}}{\|p\|_2^2 + 1}$$

$$z = \frac{\|p\|_2^2 \pm 1}{\|p\|_2^2 + 1}$$

Notice that we cannot use the positive solution, for then

$$z = \frac{\|p\|_2^2 + 1}{\|p\|_2^2 + 1} = 1$$

Thus it must be true that

$$z = \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}$$

$$x = s(1 - z) = \frac{2s}{\|p\|_2^2 + 1}$$

$$y = \frac{2t}{\|p\|_2^2 + 1}$$

It can be verified that these values of x, y, z result in $\Phi(x, y, z) = (s, t)$. In fact, using this, we obtain that the formula for Φ^{-1} is

$$\Phi^{-1}(s, t) = \left(\frac{2s}{\|p\|_2^2 + 1}, \frac{2t}{\|p\|_2^2 + 1}, \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1} \right)$$

$\square$

Question 24.

Let X be a normed vector space. Prove that the following statements are equivalent.

(i) X is finite-dimensional.

(ii) The unit ball $\overline{B}(\vec{0}, 1)$ is compact.

(iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that $\overline{U}$ is compact.

Proof. It will be proven that (i) $\implies$ (ii) $\iff$ (iii) $\implies$ (i).
(i) $\implies$ (ii):
Suppose that X has finite dimension n. Then there is a continuous linear isomorphism Φ between X and $\mathbb{R}^n$. Since $\overline{B}(0,1)$ is closed and bounded, $\Phi(\overline{B}(0,1))$ is also closed and bounded in $\mathbb{R}^n$, so the set is compact. Since homeomorphisms preserve compactness, we can conclude that the closed unit ball in X is compact.
(ii) $\implies$ (iii):
Suppose that the unit ball $\overline{B}(\vec{0}, 1)$ is compact. Let $p \in X$. We claim that $U = B(p, 1)$. Consider $\overline{U} = \overline{B}(p, 1)$. There is an isometry Φ from $\overline{B}(0,1)$ to $\overline{B}(p,1)$ defined by $\Phi(x) = p+x$. Since the closed unit ball is compact, it follows that $\overline{B}(p,1)$ is compact. Thus X is locally compact.
(iii) $\implies$ (ii):
Suppose that X is locally compact. Then $\vec{0}$ is contained in an open set U such that $\overline{U}$ is compact. Since U is open, $B(0,\varepsilon) \subseteq U$ for some ε-ball centered around 0. It follows that $\overline{B}(0,\varepsilon)$ is compact, as it is a closed subset of $\overline{U}$. Since there is a homeomorphism from this closed ball to $\overline{B}(0,1)$, the closed unit ball is also compact, as desired.
(iii) $\implies$ (i):
Suppose that X is locally compact. We know previously that this implies that the closed unit ball is compact. Firstly, a quick lemma will be proven.
Lemma 1. $\forall x \in X$, $r > 0$, $B(x,r) = \{x\} + B(0,r)$.
Let $y \in B(x,r)$. Notice that $y - x \in B(0,r)$. Thus $y = x + (y-x)$ so $y \in \{x\} + B(0,r)$.
Then, let $y \in \{x\} + B(0,r)$, so $y = x + s$, for some $s \in B(0,r)$. Since $\|y - x\| = \|s\| < r$, it follows that $y \in B(x,r)$ and we are done.
Moving on, we construct a finite set of vectors in the following way:
Since $\overline{B}(0,1)$ is totally bounded, we can find a finite set of vectors β such that $\bigcup_{x\in\beta} B\left(x, \frac{1}{2}\right)$.
Lemma 2. $B(0,1) \subseteq \operatorname{span}(\beta) + 2^{-n}B(0,1)$, $\forall n \in \mathbb{N}$.
To prove this, let $n \in \mathbb{N}$ and $y \in B(0,1)$. It follows that $y \in B(x_1, \frac{1}{2})$ for some $x_1 \in \beta$. Using Lemma 1,

$$y \in \{x_1\} + B\left(0, \frac{1}{2}\right) = \{x_1\} + \frac{1}{2}B(0,1)$$

We can repeatedly use this argument to obtain that

$$y \in \left\{x_1 + \frac{1}{2}x_2 + \frac{1}{2^2}x_3 + \cdots + \frac{1}{2^{n-1}}x_n\right\} + \frac{1}{2^n}B(0,1)$$

Note that x_i are not necessarily distinct. This simplifies down to

$$y \in \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1) \implies B(0,1) \subseteq \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1)$$

which is what we wanted.
Finally, we claim that β is the basis for X.
Let $v \in X$. Since $\dfrac{v}{2\|v\|} \in B(0,1)$, by Lemma 2,

$$\frac{v}{2\|v\|} \in \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1), \text{ for all } n \in \mathbb{N}$$

Then we construct a sequence $\{v_n\}_{n \geq 1}$ in $\operatorname{span}(\beta)$ by

$\square$