Exercise 18.15

Recall the following definitions:

(a) F is **exact** if $F = \nabla f$ for some smooth function $f : U \to \mathbf{R}$. The function f is called a **scalar potential function** for F.

(b) F is **conservative** if $\displaystyle\oint_C F \cdot d\vec{x} = 0$ for every loop C contained in U. (A **loop** is the image of a piecewise smooth map $\gamma : [a,b] \to \mathbf{R}^n$ such that γ is a regular embedding on $[a,b)$, and $\gamma(a) = \gamma(b)$. Or, you could think of it as a smooth, oriented curve C which is closed as a subset of $\mathbf{R}^n$, and such that $\partial C = \varnothing$.)

In this exercise, we will show that a conservative function is also exact.

Proof. Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be conservative. For each connected component U_i of U, we pick an arbitrary $q_i \in U_i$. Define a function $f : U \to \mathbb{R}$ as follows. For each $p \in U$, it is contained in some U_i. Let $\gamma : [0,1] \to U_i$ parametrize a path from p to p_i, that is, γ is smooth, regular, and satisfies $\gamma(0) = p$, $\gamma(1) = p_i$. For this γ, we define

$$f(p) = \oint_{\gamma[0,1]} F \cdot d\vec{x} = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\ dt.$$

We assert that this function is well defined, that is to say that $f(p)$ is independent of the choice of the path parametrized by γ. Let $\hat{\gamma} : [0,1] \to U$ be a parametrization of another path between p and p_0. We craft a piecewise function $\varphi : [0,2] \to U$ defined by

$$\varphi(t) = \begin{cases} \gamma(t), & \text{if } t \in [0,1]; \\ \hat{\gamma}(2-t), & \text{if } t \in (1,2]. \end{cases}$$

Notice that $\varphi(0) = \gamma(0) = p = \hat{\gamma}(0) = \varphi(2)$, so φ actually parametrizes a loop. So from our assumption, we have that

$$\oint_{\varphi[0,2]} F \cdot d\vec{x} = \int_0^2 F(\varphi(t)) \cdot \varphi'(t)\ dt = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\ dt + \int_1^2 F(\hat{\gamma}(2-t)) \cdot \hat{\gamma}'(2-t)\ dt = 0$$

$$\implies f(p) = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\ dt = -\int_1^2 F(\hat{\gamma}(2-t)) \cdot \hat{\gamma}'(2-t) = -\int_0^1 F(\hat{\gamma}(t)) \cdot \hat{\gamma}(t)\ dt$$

□