Question 38.

Basel Problem. Here you will use multivariable calculus to establish the following famous equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

To do it, you will evaluate the (improper) double integral $\displaystyle\int_U \frac{1}{1-xy}$ in two ways. Let $f : (0,1)^2 \to \mathbf{R}$ be the function given by $f(x,y) = \dfrac{1}{1-xy}$, and let K_N denote the closed box $\left[\dfrac{1}{N}, 1 - \dfrac{1}{N}\right]^2$.

(a) Evaluate $\displaystyle\int_{K_N} f$ using Fubini's theorem.

(b) Evaluate $\displaystyle\int_{K_N} f$ using the Change of Variables formula twice: first using the linear diffeomorphism $(x,y) = (u+v, u-v)$, then using the polar coordinates transform.

(c) Conclude that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

Proof.
(a):
By Fubini's:

$$\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \int_{\frac{1}{N}}^{1-\frac{1}{N}} \frac{1}{1-xy}\, dy\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x} \ln(1-xy)\Big|_{\frac{1}{N}}^{1-\frac{1}{N}}\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x}\left(\ln\left(1-\left(1-\frac{1}{N}\right)x\right) - \ln\left(1 - \frac{1}{N}x\right)\right)\, dx
\end{aligned}$$

Notice that $-1 < -\left(1-\frac{1}{N}\right), -\frac{1}{N} < 1$, so we can use the power series expansion of $\ln(1+t)$

to get that

$$
\begin{aligned}
\int_{K_N} f &= \int_{\frac{1}{N}}^{1-\frac{1}{N}} -\frac{1}{x}\left(\sum_{n=1}^{\infty} \frac{\left(1-\frac{1}{N}\right)^n x^n}{n} - \sum_{n=1}^{\infty} \frac{\left(\frac{1}{N}\right)^n x^n}{n}\right)\, dx \\
&= \int_{\frac{1}{N}}^{1-\frac{1}{N}} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)\, dx \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n^2}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)\Bigg|_{\frac{1}{N}}^{1-\frac{1}{N}} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\left(\frac{1}{N}\right)^n - \left(1-\frac{1}{N}\right)^n\right)^2
\end{aligned}
$$

(b):

$\square$