Question 36.

In Handout #7, we defined differentiability for functions on open sets. Now we give a definition that works over arbitrary sets. For this problem, you will need to read Pinzza Post.

Let $A \subseteq \mathbf{R}^n$ be an arbitrary set, let $f: A \to \mathbf{R}$ be a function, and let $p \in A$ be a point. We say that f is **differentiable** at p if there exists an open neighborhood U of p and a function $\hat{f}: U \to \mathbf{R}$ such that \hat{f} is differentiable at p (in the sense of Handout #7) and $\hat{f}|_{U \cap A} = f|_{U \cap A}$.

- (a) Prove that f is differentiable at every point of A if and only if f extends to a differentiable function defined on an open set containing A.
- (b) Suppose further that A is closed. Prove that f is differentiable at every point of A if and only if f extends to a differentiable function on \mathbf{R}^n .

Question 37.

The following set is called the n-simplex:

$$\Delta_n := \{ \vec{x} = (x_1, \dots, x_n) \in \mathbf{R}^n : x_1, \dots, x_n \ge 0 \text{ and } x_1 + \dots + x_n \le 1 \}.$$

You can assume, without proof, that Δ_n is Jordan measurable. Find, with proof, an explicit formula for $\mu(\Delta_n)$ in terms of n.

Proof. First, we show that Δ_n is the same as the set

$$S = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n : 0 \le x_1 \le 1, 0 \le x_2 \le 1 - x_1, ..., 0 \le x_n \le 1 - \sum_{i=1}^{n-1} x_i \right\}$$

Let $x \in \Delta_n$. We want to show that $0 \le x_i \le 1 - \sum_{j=1}^{i-1} x_j$. We get that $x_i \ge 0$ immediately As well, since $\sum_{j=1}^{n} x_j \le 1$ and every component is non-negative, we have that

$$x_i \le 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^n x_j \le 1 - \sum_{j=1}^{i-1} x_j$$

which shows that $\Delta_n \subseteq S$.

Now, let $x \in S$. We know every x_i is non-negative and additionally

$$x_n \le 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^n x_i \le 1$$

so $S \subseteq \Delta_n$.

Now, we proceed to find $\mu(S) = \mu(\Delta_n)$. Using Fubini's Theorem, we get

$$\mu(S) = \int_{S} 1 = \int_{0}^{1} \int_{0}^{1-x_{1}} \cdots \int_{0}^{1-\sum_{i=1}^{n-1} x_{i}} 1 \ dx_{n} \cdots dx_{2} \ dx_{1}$$

Let $I: \mathbb{N} \times [0,1] \to \mathbb{R}$ be defined recursively as follows:

$$I(1,\alpha) = \int_0^{1-\alpha} 1 \ dt$$

$$I(k,\alpha) = \int_0^{1-\alpha} I(k-1,\alpha+t) \ dt, \qquad \text{for } > 1$$

Notice that if we continue applying the definition, we get that

$$I(n,0) = \mu(S)$$

Now, we will prove using induction on n that for all $\alpha \in [0,1]$, $I(n,\alpha) = \frac{1}{n!}(1-\alpha)^n$. Let n = 1. Then

$$I(1,\alpha) = \int_0^{1-\alpha} 1 \ dt = 1 - \alpha$$

Now, suppose that $I(k,\alpha) = \frac{1}{k!}(1-\alpha)^k$ holds for all $\alpha \in [0,1]$ and some $k \in \mathbb{N}$. We want to show that the same holds for k+1 as well. For an arbitrary α , we get

$$I(k+1,\alpha) = \int_0^{1-\alpha} I(k,\alpha+t) dt$$

$$= \int_0^{1-\alpha} \frac{1}{k!} (1-\alpha-t)^k dt$$

$$= -\frac{1}{(k+1)!} (1-\alpha-t)^{k+1} \Big|_0^{1-\alpha}$$

$$= \frac{1}{(k+1)!} (1-\alpha)^{k+1}$$

as desired. Thus we get that

$$\mu(S) = I(n,0) = \frac{1}{n!}$$