

Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that O_n is a smooth manifold, and find its dimension.

Proof. First, we note that $O_n(\mathbb{R})$ is the zero set of the function $f : M_n(\mathbb{R}) \rightarrow S^n$ defined by

$$f(A) = A^t A - I_n$$

where S^n is the set of symmetric $n \times n$ matrices. Notice that f is smooth as it is constructed by smooth functions. Additionally, we show that $Jf(X)(h) = X^t h + h^t X$. Indeed,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(X+h) - f(X) - X^t h - h^t X}{\|h\|} &= \lim_{h \rightarrow 0} \frac{(X+h)^t(X+h) - X^t X - X^t h - h^t X}{\|h\|} \\ &= \lim_{h \rightarrow 0} \frac{h^t h}{\|h\|} \\ &= 0 \end{aligned}$$

Next, we want to show that $\text{rank } Jf(X) = \frac{1}{2}n(n+1)$ for all $X \in O_n(\mathbb{R})$. It suffices to show that $Jf(X)$ is surjective to S^n .

Let $Y \in S^n$. Let $h = \frac{1}{2}XY$. We see that

$$\begin{aligned} Jf(X)(h) &= X^t \left(\frac{1}{2}XY \right) + \left(\frac{1}{2}XY \right)^t X = \frac{1}{2} (X^t XY + Y^t X^t X) \\ &= \frac{1}{2} (Y + Y^t) && (X \text{ is orthogonal}) \\ &= Y && (Y \text{ is symmetric}) \end{aligned}$$

Thus $R(Jf(X)) = S^n$ so $\text{rank } Jf(X) = \dim S^n = \frac{1}{2}n(n+1)$.

We now prove that $O_n(\mathbb{R})$ is a smooth manifold of dimension $\frac{1}{2}n(n+1)$. Let $p \in O_n(\mathbb{R})$. Then $f(p) = 0$ and $Jf(p)$ has the maximal rank of $\frac{1}{2}n(n+1)$. We write

$$Jf(p) = (A \mid B)$$

where A is a $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n-1)$ matrix and B is a $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ matrix and assume without loss of generality that B is an invertible submatrix of $Jf(p)$. We can do this because we can swap the components of f , and therefore columns of $Jf(p)$ without affecting the conclusion of the statement (because manifolds are invariant under diffeomorphisms). Thus, we write $p = (a, b)$ for $a \in \mathbb{R}^{\frac{1}{2}n(n-1)}$, $b \in \mathbb{R}^{\frac{1}{2}n(n+1)}$ and apply the Implicit Function Theorem and obtain an open set $\hat{U} \subseteq \mathbb{R}^{\frac{1}{2}n(n-1)}$ containing a and a C^∞ function $\Phi : \hat{U} \rightarrow \mathbb{R}^{\frac{1}{2}n(n+1)}$ so that

$$f(x, \Phi(x)) = 0$$

for all $x \in \hat{U}$. We claim that $\varphi : \hat{U} \rightarrow \Phi(\hat{U})$ defined by

$$\varphi(x) = (x, \Phi(x))$$

