

Question 36.

In Handout #7, we defined differentiability for functions on open sets. Now we give a definition that works over arbitrary sets. For this problem, you will need to read Piazza Post 6274 and use Theorem 1.1.

Let  $A \subseteq \mathbf{R}^n$  be an arbitrary set, let  $f : A \rightarrow \mathbf{R}$  be a function, and let  $p \in A$  be a point. We say that  $f$  is **differentiable** at  $p$  if there exists an open neighborhood  $U$  of  $p$  and a function  $\hat{f} : U \rightarrow \mathbf{R}$  such that  $\hat{f}$  is differentiable at  $p$  (in the sense of Handout #7) and  $\hat{f}|_{U \cap A} = f|_{U \cap A}$ .

- Prove that  $f$  is differentiable at every point of  $A$  if and only if  $f$  extends to a differentiable function defined on an open set containing  $A$ .
- Suppose further that  $A$  is closed. Prove that  $f$  is differentiable at every point of  $A$  if and only if  $f$  extends to a differentiable function on  $\mathbf{R}^n$ .

*Proof.*

(a):

Suppose that  $f$  extends to a differentiable function  $\hat{f}$  on an open set  $U \supseteq A$ . That is,  $\hat{f}|_A = f$ . Let  $x \in A$ . Since  $U$  is open, we can find an open ball such that  $B(x, \varepsilon) \subseteq U$ . Immediately, we get that the function  $\hat{f}|_{B(x, \varepsilon)}$  is the desired extension of  $f$  at  $x$ , as  $\hat{f}$  is differentiable at  $x$  and  $\hat{f}|_{B(x, \varepsilon) \cap A} = f|_{B(x, \varepsilon) \cap A}$ .

Conversely, suppose that  $f$  is differentiable at every point of  $A$ . For all  $p_a \in A$ , there exists an open neighborhood  $U_a$  of  $p_a$  and a function  $f_a : U_a \rightarrow \mathbf{R}$  that is differentiable at  $p_a$  and  $f_a|_{U_a \cap A} = f|_{U_a \cap A}$ . Notice that  $U = \{U_a\}_{a \in I}$  forms an open cover of  $A$ . Thus we can find a partition of unity  $\{\phi_a : \mathbf{R}^n \rightarrow [0, 1]\}_{a \in I}$ . In particular,

- $\phi_a \in C^\infty$ ,
- $\text{supp}(\phi_a) \cap U \subseteq U_a$ ,
- $\{\text{supp}(\phi_a)\}$  is locally finite,
- $\sum_{a \in I} \phi_a = 1$ .

We claim that  $\hat{f} = \sum_{a \in I} \phi_a \cdot f_a$  is the function that extends  $f$  to  $\{U_a\}_{a \in I}$ . First, note that this sum is well defined because the partition of unity is locally finite, so the sum is finite at every point, and as well, any point outside the domain of  $f_a$  implies that  $\phi_a = 0$ , so  $\phi_a \cdot f_a = 0$ .

Moreover, for all  $x \in U$ , since  $\{\phi_a\}_{a \in I}$  is locally finite, for some open neighborhood  $V$  of  $x$ , for some finite sequence of  $a_n \in I$ ,

$$\hat{f}(y) = \sum_{n=1}^N \phi_{a_n}(y) \cdot f_{a_n}(y), \text{ where } y \in V.$$

Since each term is differentiable,  $\hat{f}$  is differentiable at  $x$ . Thus  $\hat{f}$  is differentiable on the



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The following set is called the  $n$ -**simplex**:

$$\Delta_n := \{\vec{x} = (x_1, \dots, x_n) \in \mathbf{R}^n : x_1, \dots, x_n \geq 0 \text{ and } x_1 + \dots + x_n \leq 1\}.$$

You can assume, without proof, that  $\Delta_n$  is Jordan measurable.

Find, with proof, an explicit formula for  $\mu(\Delta_n)$  in terms of  $n$ .

*Proof.* First, we show that  $\Delta_n$  is the same as the set

$$S = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, \dots, 0 \leq x_n \leq 1 - \sum_{i=1}^{n-1} x_i \right\}$$

Let  $x \in \Delta_n$ . We want to show that  $0 \leq x_i \leq 1 - \sum_{j=1}^{i-1} x_j$ . We get that  $x_i \geq 0$  immediately. As well, since  $\sum_{j=1}^n x_j \leq 1$  and every component is non-negative, we have that

$$x_i \leq 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^n x_j \leq 1 - \sum_{j=1}^{i-1} x_j$$

which shows that  $\Delta_n \subseteq S$ .

Now, let  $x \in S$ . We know every  $x_i$  is non-negative and additionally

$$x_n \leq 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^n x_i \leq 1$$

so  $S \subseteq \Delta_n$ .

Now, we proceed to find  $\mu(S) = \mu(\Delta_n)$ . Using Fubini's Theorem, we get

$$\mu(S) = \int_S 1 = \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 \, dx_n \dots dx_2 \, dx_1$$

Let  $I : \mathbb{N} \times [0, 1] \rightarrow \mathbb{R}$  be defined recursively as follows:

$$\begin{aligned} I(1, \alpha) &= \int_0^{1-\alpha} 1 \, dt \\ I(k, \alpha) &= \int_0^{1-\alpha} I(k-1, \alpha+t) \, dt, \end{aligned} \quad \text{for } k > 1.$$

Notice that if we continue applying the definition, we get that

$$I(n, 0) = \mu(S)$$

Now, we will prove using induction on  $n$  that for all  $\alpha \in [0, 1]$ ,  $I(n, \alpha) = \frac{1}{n!}(1-\alpha)^n$ .

Let  $n = 1$ . Then

$$I(1, \alpha) = \int_0^{1-\alpha} 1 \, dt = 1 - \alpha$$

