6 HOMEWORK 6 HAAHHAHAHAHAHAHAA

Question 19.

Let X be a metric space and let $A \subseteq X$. A **compact exhaustion** for A is a sequence of compact sets $K_1, K_2, K_3, \ldots$ such that $U = \bigcup_{i \geq 1} K_i$ and $K_i \subseteq K_{i+1}^\circ$.

(a) Let $U \subseteq \mathbf{R}^n$ be a bounded open set. Show that U has a compact exhaustion.

(b) Now show that every open set $U \subseteq \mathbf{R}^n$ has a compact exhaustion.

Proof. Suppose Question 19 is true. Then the result immediately follows. $\square$

Question 19.

Let $x, y \in \ell^\infty$ be two sequences. Let us say that y is **dominated** by x, denoted $x \geq y$, if $|x_n| \geq |y_n|$ for all $n \in \mathbf{N}$. Let D_x denote the set of all sequences which are dominated by x:

$$D_x = \{y \in \ell^\infty : |y_n| \leq |x_n| \text{ for all } n \in \mathbf{N}\}.$$

Prove that D_x is compact if and only if $x_n \to 0$.

Proof. Suppose that D_x is compact. Suppose for contradiction that $x_n \nrightarrow 0$. For some $\varepsilon > 0$, $|x_{N_k}| \geq \varepsilon$ for an infinite number of N_k. Consider the open cover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \in I}$, which is the collection of $\frac{\varepsilon}{2}$-balls centered around every $\vec{y}_i \in D_x$. By compactness of D_x, there is a finite subcover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \leq m}$. Now, we construct a $y \in D_x$ as follows:
For every sequence $\vec{y}_i$, let

$$y_{N_i} = \begin{cases} \varepsilon, & \text{if } (\vec{y}_i)_{N_i} < \dfrac{\varepsilon}{2}; \\ 0, & \text{if } (\vec{y}_i)_{N_i} \geq \dfrac{\varepsilon}{2}; \end{cases}$$

For all other terms in y, make it 0. Notice that for all $B(\vec{y}_i, \frac{\varepsilon}{2})$,

$$\|y - \vec{y}_i\|_\infty \geq |y_{N_i} - (\vec{y}_i)_{N_i}| \geq \frac{\varepsilon}{2} \implies y \notin D_x$$

which is a contradiction.
Conversely, suppose that $x_n \to 0$. Let $\{U_i\}_{i \in I}$ be an open cover of D_x.

$\square$