Question 1.

Let V be a vector space over the field $\mathbb{F}$, and S a (non-empty) set. Let $\mathcal{F}(S, V) = \{f : S \to V\}$ be the set of V-valued functions.
We define addition and scaling on $\mathcal{F}(S, V)$ pointwise:

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

We will verify some of the vector space axioms required to prove that $\mathcal{F}(S, V)$ is a vector space over $\mathbb{F}$.

(a) Why do these operations make sense?

(b) Prove (using only the definitions above, and the fact that V is a vector space) that $c(f + g) = cf + cg$ for all $f, g \in \mathcal{F}(S, V)$ and $c \in \mathbb{F}$.

(c) Prove that for all $f \in \mathcal{F}(S, V)$ there exists $g \in \mathcal{F}(S, V)$, so that $f + g = 0$. (Here $0 : S \to V$ is the constant function defined by $0(s) = 0_V$ for $s \in S$.)

Proof.
(a):
These operations make sense because they allow us to use the properties of the sets underlying $\mathcal{F}(S, V)$.

(b):
We will do this by showing that for all $s \in S$, we have $c(f(s) + g(s)) = cf(s) + cg(s)$.
Fix $s \in S$. It follows that $f(s), g(s) \in V$, so by the axiom of distributivity in V, we have that

$$c(f(s) + g(s)) = cf(s) + cg(s)$$

(c):
Let $f \in \mathcal{F}(S, V)$. Choose $g = (-1 \cdot f)$. Then for all $s \in S$,

$$f(s) + g(s) = f(s) + (-f(s)) = 0$$

as needed.

$\square$

Question 2.

Let $W = \left\{ (x,y,z,w) \in \mathbb{Q}^4 \,\middle|\, \begin{array}{l} x + 5w = y + 5z \\ y = 4w - 3z \\ x + y + z = 3w \end{array} \right\}$.

Do not use Q3 to solve this problem. This problem is a "warm up" for Q3.

(a) Rearrange the equations defining W to show that W is the set of solutions to a homogeneous system of equations.

(b) Solve the system using row-reduction and express the general solution as a linear combination of the "basic solutions".

(c) Show that $W = \operatorname{span} S$, for some set $S \subseteq \mathbb{Q}^4$.

(d) Deduce that W is a subspace of $\mathbb{Q}^4$.

Proof.
(a):
Rearranging, the equations become

$$\begin{cases} x - y - 5z + 5w = 0 \\ \phantom{x - {}} y + 3z - 4w = 0 \\ x + y + z - 3w = 0 \end{cases}$$

(b):
The augmented matrix associated with this system of equations is

$$\left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right)$$

Row reducing this, we get

$$\left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right) \xrightarrow{r_3 \to r_3 - r_1} \left(\begin{array}{cccc|c} 1 & -1 & -5 & 5 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 2 & 6 & -8 & 0 \end{array}\right) \xrightarrow[r_3 \to r_3 - 2r_2]{r_1 \to r_1 + r_2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We parameterize z and w to obtain that

$$\begin{aligned} x &= 2s - t \\ y &= -3s + 4t \\ z &= s \\ w &= t \end{aligned}$$

so the general solution of this system of equations is given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

(c):

Let $S = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}$. Let $\vec{v} \in \text{span} S$. It follows that

$$\vec{v} = s \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

for some $s, t \in \mathbb{Q}$. But notice that this is actually a solution to the system in W, so $\vec{v} \in W$. Now let $\vec{w} \in W$, so $\vec{w}$ solves the system of equations in W, but this means that we can write $\vec{w}$ as a linear combination of the vectors in S, so $\vec{w} \in \text{span} S$, so $W = \text{span} S$.

(d):

By the previous part, W is actually a spanning set, and we know that all spanning sets are subspaces, so we conclude that W is a subspace of $\mathbb{Q}^4$.

$\square$

Question 3.

We now generalize Q2. Consider a linear system with m equations and n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

We saw in Week 3 that any solution $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ can be expressed as $x = \sum_{i=1}^{k} t_i x_i$, where $t_i \in \mathbb{F}$ are the parameters, and $x_i \in \mathbb{F}^n$ are the "basic solutions".
Let W be the set of solutions to this system.

(a) Prove that $W = \operatorname{span} S$ for some set S, and hence that W is a subspace of $\mathbb{F}^n$.

(b) Prove that the set $\{x_1, x_2, \ldots, x_k\}$ is linearly independent.

(Hint: Think about the variables which correspond to the choice of parameters. There is exactly one vector for each such parameter. Use the corresponding entry to show that if $t_1x_1 + t_2x_2 + \cdots + t_kx_k = 0$ then $t_i = 0$ for each i.)

(c) Find a basis for W.

Proof.
(a):
Let $S = \{x_1, ..., x_k\}$. We note that any linear conbination of the vectors in S form a solution to the system, but not only that, any solution to the system can be written as a linear combination of these vectors, so $\operatorname{span} S = W$.

(b):
Suppose that $\sum_{i=1}^{k} t_i x_i = 0$.

(c):
From part (a) and part (b), the set $S = \{x_1, ..., x_k\}$ spans W and is linearly independent, which by definition forms a basis of W.

$\square$

Question 4.

Is the set $S = \{e_1 + 2e_2 - 3e_3, e_1 + e_2 - e_3, e_2 - e_3\} \subseteq \mathbb{Q}^3$ a basis for $\mathbb{Q}^3$? Justify your answer.

Question 5.

Let V be a finite dimensional vector space over a field $\mathbb{F}$.

1. Prove that if $W \subseteq V$ is a subspace with basis β_W, then there exists a linearly independent set α so that $\beta = \beta_W \cup \alpha$ is a basis for V. (We say that β "extends" β_W. So you are proving that "every basis of a subspace W can be extended to a basis of V".)
2. Prove that for any linearly independent set I and spanning set S, we have $|I| \leq \dim V \leq |S|$.

Question 6.

Consider a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{F})$. Given $p \in \{1, \ldots, n\}$ we can split M into "blocks":

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A is $k \times k$, B is $k \times (n-k)$, C is $(n-k) \times k$ and D is $(n-k) \times (n-k)$.
For example, if $n = 5$ and $k = 2$, then such a block matrix would be of the form

$$M = \left(\begin{array}{cc|ccc} 1 & 2 & 3 & 2 & 3 \\ -5 & 3 & 3 & 1 & 1 \\ \hline 1 & 2 & 0 & -1 & 1 \\ 3 & 1 & 3 & -1 & 7 \\ 1 & 0 & -1 & 3 & 5 \end{array}\right)$$

where $A = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 7 \\ -1 & 3 & 5 \end{pmatrix}$.

Prove that if $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $N = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right)$, then

$$\alpha M + N = \left(\begin{array}{c|c} \alpha A + A' & \alpha B + B' \\ \hline \alpha C + C' & \alpha D + D' \end{array}\right)$$

Question 7.

Let $W = \left\{ A \in \mathcal{M}_{2n\times 2n}(\mathbb{F}) \mid A = \left(\begin{array}{c|c} \frac{X-X^t}{O_n} & \frac{O_n}{X+X^t} \end{array}\right) \text{ with } X \in \mathcal{M}_{n\times n}(\mathbb{F}) \right\}$.
(Assume $\operatorname{char}(\mathbb{F}) \neq 2$.)

1. Let $n = 2$. Find a basis for W.
2. Now generalize to arbitrary n. Find a basis for W, and use it to compute $\dim W$.

Question 8.

(a) Prove that if $W_1, W_2 \subseteq V$ are subspaces, then $W_1 + W_2$ is a subspace.

(b) Let $W_1 = \{(x, y, x+y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$. Find two subspaces W_2, W_3 so that:

- $W_1 + W_2 = \mathbb{F}^3$ but $\mathbb{F}^3 \neq W_1 \oplus W_2$.
- $W_1 \oplus W_3 = \mathbb{F}^3$.

(c) Find another subspace $U \subseteq \mathbb{F}^3$ so that $W_1 \oplus U = \mathbb{F}^3$.

Proof.
(a):
Let W_1, W_2 be subspaces of V. We verify that $W_1 + W_2$ is also a subspace of V.
First, note that $0 \in W_1, W_2$, so $0 + 0 = 0 \in W_1 + W_2$. Next, let $c \in \mathbb{F}$, $\vec{v}, \vec{w} \in W_1 + W_2$. Then $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{w} = \vec{w}_1 + \vec{w}_2$, for some $\vec{v}_1, \vec{w}_1 \in W_1$ and $\vec{v}_2, \vec{w}_2 \in W_2$. Since W_1, W_2 are subspaces, it is true that

$$c\vec{v}_1 + \vec{w}_1 \in W_1 \text{ and } c\vec{v}_2 + \vec{w}_2 \in W_2$$

which implies that

$$\vec{v} + \vec{w} = (c\vec{v}_1 + \vec{w}_1) + (c\vec{v}_2 + \vec{w}_2) \in W_1 + W_2,$$

verifying that $W_1 + W_2$ is indeed a subspace.

(b):
Let $W_2 = \mathbb{F}^3$, $W_3 = \text{span}\{e_3\}$.

□

Question 9.

Let V be a finite dimensional vector space over $\mathbb{F}$, and $W_1, W_2 \subseteq V$ subspaces with bases β_1, β_2 respectively. Prove that $V = W_1 \oplus W_2$ if and only if $\beta = \beta_1 \cup \beta_2$ is a basis for V.

Question 10.

Let $J = \left(\begin{array}{c|c} O & -I_2 \\ \hline I_2 & O \end{array}\right)$ and $\mathbb{F} = \mathbb{C}$.

1. Verify that $J^2 = -I_4$.
2. Find all $X \in \mathcal{M}_{4\times 4}(\mathbb{F})$ so that $XJ = JX$.
3. Show that $sp_4 = \{X \in \mathcal{M}_{4\times 4}(\mathbb{F}) | XJ = JX\}$ is a subspace of $\mathcal{M}_{4\times 4}(\mathbb{F})$.
4. Find dim sp_4 by finding a basis for sp_4.

Question 11.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

(a) Let V be a finite dimensional vector space over $\mathbb{F}$. If $I \subseteq V$ is a linearly independent set so that for any $x \in V \setminus I$, the set $I \cup \{x\}$ is linearly dependent, then I is a basis for V.

(b) Let V be a finite dimensional vector space over $\mathbb{F}$. If $S \subseteq V$ is a spanning set so that $|S| = \dim V$, then S is a basis for V.

(c) Let V be a finite dimensional vector space over $\mathbb{F}$. If $W \subseteq V$ a subspace, then there exists a unique subspace $U \subseteq V$ so that $V = W \oplus V$.