# Fubini's Theorem + Exercise 12.15

**Solvers:** Ethan Let $E$ be a closed rectangle in $\mathbb{R}^2$, so $E = [a, b] \times [c, d]$. Let $f : E \to \mathbb{R}$ be a continuous function. Then

$$\int_E f = \int_c^d \int_a^b f(x, y)\ dx\ dy$$

*Proof.* We make the quick note that $f(\cdot, y)$ is integrable because it is continuous. Moreover, it is uniformly continuous because it is defined on the closed interval $[a, b]$. Define $g : [c, d] \to \mathbb{R}$ by

$$g(y) = \int_a^b f(x, y)\ dx.$$

We can argue that $g$ is continuous because for any $y$,

$$|g(y + h) - g(y)| = \left| \int_a^b f(x, y + h)\ dx - \int_a^b f(x, y)\ dx \right| < \int_a^b |f(x, y + h) - f(x, y)|\ dx,$$

which we can make arbitrarily small using the uniform continuity of $f$.
Let $P$ partition $[a, b]$ and $Q$ partition $[c, d]$, and let $n, m$ represent the number of intervals in $P$ and $Q$ respectively. Define

$$M_{ij} = \sup\{f(x, y) : (x, y) \in [p_{i-1}, p_i] \times [q_{j-1}, q_j]\}.$$

We claim that for all $j$, $y \in [q_{j-1}, q_j]$,

$$\sum_{i=1}^{n} M_{ij}(p_i - p_{i-1}) \geq U(f(\cdot, y), P).$$

Define $M_{iy} = \sup\{f(x, y) : x \in [p_{i-1}, p_i]\}$. We notice that the set in $M_{iy}$ is a subset of the one in $M_{ij}$, so we immediately have that $M_{iy} < M_{ij}$. Thus

$$U(f(\cdot, y), P) = \sum_{i=1}^{n} M_{iy}(p_{i-1}, p_i) \leq \sum_{i=1}^{n} M_{ij}(p_{i-1}, p_i).$$

From this, we get that for any $j$,

$$g(y_j) = \int_a^b f(x, y)\ dx \leq \sum_{i=1}^{n} M_{ij}(p_{i-1}, p_i), \text{ where } y_j \in [q_{j-1}, q_j],$$

which implies that

$$\sum_{i=1}^{n} M_{ij}(p_{i-1}, p_i) \geq M_j \coloneqq \sup\{g(y) : y \in [q_{j-1}, q_j]\}.$$

It follows that

$$U(f, (P, Q)) = \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ij}(p_i - p_{i-1})(q_j - q_{j-1}) \geq \sum_{j=1}^{m} M_j(q_j - q_{j-1}) = U(g, Q) \geq \int_c^d g(y)\ dy,$$

$$\implies U(f, (P, Q)) \geq \int_c^d \int_a^b f(x, y)\ dx\ dy.$$

We can follow the same steps from above but reversing the inequality signs to get that

$$L(f, (P, Q)) \leq \int_c^d \int_a^b f(x, y)\ dx\ dy.$$

Since we did this with arbitrary partitions $P, Q$, we actually know that

$$L(f) \leq \int_c^d \int_a^b f(x,y)\ dx\ dy \leq U(f),$$

but $L(f) = U(f)$ because of the integrability of $f$, so we conclude that

$$\int_E f = \int_c^d \int_a^b f(x,y)\ dx\ dy.$$

$\square$

**Example:** Integrate $f(x,y) = x^y$ on $E = [0,1]^2$.
Note that $f$ is undefined at $(0,0)$, but we can construct a function $g$ that is the same as $f$ but with $g(0,0) = 1$ to solve this issue. It is also continuous, which I invite you to do as an exercise.
Using Fubini's Theorem, we have that

$$\int_E f = \int_0^1 \int_0^1 x^y\ dx\ dy = \int_0^1 \frac{x^{y+1}}{y+1}\Big|_0^1\ dy = \int_0^1 \frac{1}{y+1}\ dy = \ln|y+1|\Big|_0^1 = \ln 2.$$

Doing this from the definition is kind of painful, so having Fubini in your closet is quite useful.