

Question 43.

**Transverse intersection.** Let  $M, N$  be two smooth surfaces in  $\mathbf{R}^3$ . We say that  $M$  and  $N$  intersect transversally if  $T_p M \neq T_p N$  for all  $p \in M \cap N$ .

- (a) Prove that if  $M, N$  intersect transversally, then  $M \cap N$  is a smooth curve in  $\mathbf{R}^3$ .
- (b) Show by example that the conclusion of (a) fails without the assumption of transverse intersection.

*Proof.*

(a):

Suppose that  $M$  and  $N$  intersect transversally. We will show that  $M \cap N$  is a smooth 1-manifold in  $\mathbf{R}^3$ . Let  $p \in M \cap N$ . Then there is some relatively open neighborhood  $U$  of  $M$  and  $V$  of  $N$  that is the zero set of some smooth functions  $f, g : \mathbf{R}^3 \rightarrow \mathbf{R}$ , that is,  $U = Z(f)$  and  $V = Z(g)$ . Let  $\Phi : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  be defined by  $\Phi = (f, g)$ . We claim that  $(U \cap V, \Phi)$  is the desired chart containing  $p$ . We first start by verifying that  $U \cap V$  is relatively open to  $M \cap N$ . This is quick, as we know that  $U$  and  $V$  are relatively open to  $M$  and  $N$  respectively, for each point in  $U \cap V$ , we can choose two open balls with radii  $r_1, r_2$  that stays within  $M$  and  $N$  respectively. We then take the lesser of the radii as our radius.

Moving on, we see that  $\Phi$  is smooth, and  $Z(\Phi) = U \cap V$ . It remains to show that  $J\Phi(q)$  has rank 2 for all  $q \in U \cap V$ . The Jacobian of  $\Phi$  is a  $2 \times 3$  matrix given by

$$J\Phi(q) = \begin{pmatrix} \nabla f(q) \\ \nabla g(q) \end{pmatrix}.$$

It necessarily has rank at most 2 and at least 1 (because  $\nabla f, \nabla g$  have at least rank 1). Suppose for contradiction that  $\text{rank } J\Phi(q) = 1$ . Then  $\nabla f(q) = c\nabla g(q)$  for some non-zero constant  $c$ . We know that the tangent space  $T_q M$  is given by the set of tagged vectors orthogonal to  $\nabla f(q)$ . Likewise,  $T_q N$  consists of tagged vectors orthogonal to  $\nabla g(q)$ . But notice that for  $v \in T_q M$ , we have  $\nabla f(q) \cdot v = 0$  but also  $c\nabla f(q) \cdot v = \nabla g(q) \cdot v = 0$ . If we additionally apply the same argument to  $u \in T_q N$ , we can see that  $T_q M = T_q N$  which is a contradiction. Thus  $J\Phi(q)$  must have rank 2. From here, it follows that  $\Phi^{-1}(\{0\}) = U \cap V$  is a smooth manifold with dimension 1, so we can conclude that  $M \cap N$  is a smooth curve.

(b):

Let  $M$  be the  $xy$ -plane, and let  $N$  be the graph of  $f(x, y) = x^2 + y^2$ . Then  $M \cap N$  is simply the origin, which is not a smooth curve.

□