Question 1.

Suppose $\widetilde{p}$ must approximate p with relative error at most 10^{-3}. Find the largest interval in which $\widetilde{p}$ must lie if $p = 900$.

Proof. Since we want the relative error to be at most 10^{-3}, we set

$$\frac{|\widetilde{p} - p|}{|p|} \leq 10^{-3}$$

Substitute $p = 900$ to get

$$\frac{|\widetilde{p} - 900|}{900} \leq 10^{-3} \implies |\widetilde{p} - 900| \leq \frac{9}{10} \implies 900 - \frac{9}{10} \leq \widetilde{p} \leq 900 + \frac{9}{10}.$$

Thus $\widetilde{p}$ lies within the interval $\left[900 - \frac{9}{10}, 900 + \frac{9}{10}\right]$.

$\square$

Question 2.

Compute the absolute error and relative error of the following approximation of e:

$$\sum_{n=0}^{5} \frac{1}{n!}$$

Using Octave, we find that the absolute error is

$$\left| e - \sum_{n=0}^{5} \frac{1}{n!} \right| = \left| e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \right) \right| = \left| e - \frac{326}{120} \right| \approx \texttt{1.615161792378306e-03}$$

and the relative error is

$$\left| \frac{e - \sum_{n=0}^{5} \frac{1}{n!}}{e} \right| = \left| \frac{e - \frac{326}{120}}{e} \right| \approx \texttt{5.941848175815963e-04}$$

Question 3.

Find the second Taylor polynomial, $P_2(x)$, $f(x) = e^x\cos(x)$ about $x_0 = 0$.

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound on the error $|f(0.5) - P_2(0.5)|$ using the remainder term and compare it to the actual error.

First, we find $P_2(x)$. We calculate that

$$\begin{aligned} f(x_0) &= 1 \\ f'(x_0) &= e^{x_0}(\cos(x_0) - \sin(x_0)) = 1 \\ f''(x_0) &= -2e^{x_0}\sin(x_0) = 0 \\ f^{(3)}(x) &= -2e^x(\sin(x) - \cos(x)) \end{aligned}$$

Thus $P_2(x) = 1 + x$, so $f(0.5) \approx P_2(0.5) = 1.5$. The error term is

$$|R_2(0.5)| = \left|\frac{f^{(3)}(\xi)}{3!}(0.5)^3\right| = \left|\frac{e^\xi(\sin(\xi) - \cos(\xi))}{24}\right|, \text{ for } \xi \in (0, 0.5).$$

Since $e^\xi < e^{0.5}$, $\sin(\xi), \cos(\xi) \leq 1$, we get that

$$|R_2(0.5)| < \frac{e^{0.5}}{12} \approx 0.1374.$$

and the actual absolute error is

$$\left|e^{0.5}\cos(0.5) - 1.5\right| \approx 0.053111$$

(b) Find a bound on the error $|f(x) - P_2(x)|$ good on the interval $[0, 1]$.

Similar to the previous part, the error term is

$$|R_2(x)| = \left|\frac{e^\xi(\sin(\xi) - \cos(\xi))}{3}x^3\right|, \text{ for } \xi \in (0, x)$$

We know that $e^\xi < e^x$, so

$$|R_2(x)| < \frac{2}{3}x^3e^x.$$

(c) Approximate $\int_0^1 f(x)\,dx$ by calculating $\int_0^1 P_2(x)\,dx$ instead.

We have that

$$\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx = \int_0^1 1 + x\,dx = x + \frac{x^2}{2}\bigg|_0^1 = 1.5$$

(d) Find an upper bound for the error in (c) using $\int_0^1 |R(x)|\,dx$ and compare the bound to the actual error.

From part (b), the error term $|R_2(x)$ is bounded above by $\dfrac{2}{3}x^3e^x$. Thus the error for the computation is

$$\int_0^1 |R_2(x)|\,dx < \int_0^1 \frac{2}{3}x^3e^x\,dx = \frac{2}{3}\left(x^3e^x - 3x^2e^x + 6xe^x - 6e^x\right)\bigg|_0^1$$

$$= \frac{2}{3}(-2e + 6) \approx 0.3756.$$

and the actual error is

$$\left|\int_0^1 f(x)\ dx - 1.5\right| = \left|\int_0^1 e^x \cos(x)\ dx - 1.5\right| = \left|\frac{1}{2}e^x(\cos(x) + \sin(x))\Big|_0^1 - 1.5\right|$$

$$= \left|\frac{e(\cos(1) + \sin(1))}{2} - \frac{1}{2} - 1.5\right| \approx 0.1220.$$

Question 4.

Find a theoretical upper bound, as a function of x, for the absolute error in using $T_4(x)$ to approximate $f(x) = \frac{10}{x} + \sin(10x)$; $x_0 = \pi$.

We only need to find the remainder term $R_4(x)$, so we will just compute the 5th derivative of f by hand:

$$\begin{aligned} f'(x) &= -\frac{10}{x^2} + 10\cos(10x) \\ f''(x) &= \frac{20}{x^3} - 100\sin(10x) \\ f'''(x) &= -\frac{60}{x^4} - 1000\cos(10x) \\ f^{(4)}(x) &= \frac{240}{x^5} + 10000\sin(10x) \\ f^{(5)}(x) &= \frac{1200}{x^6} + 100000\cos(10x) \end{aligned}$$

Thus we have that

$$\begin{aligned} R_4(x) &= \frac{f^{(5)}(\xi)}{5!}(x-\pi)^5 = \frac{\frac{1200}{\xi^5} + 100000\cos(10\xi)}{120}(x-\pi)^5 \\ &= \left(\frac{10}{\xi^5} + \frac{2500}{3}\cos(10\xi)\right)(x-\pi)^5 \end{aligned}$$

where ξ is a number between π and x. Let $m(x) = \min\{\pi, x\}$. Then

$$R_4(x) \leq \left(\frac{10}{m(x)} + \frac{2500}{3}\right)(x-\pi)^5$$

which is a desired upper bound.

Question 5.

Let $(p_n) = \left\langle \frac{3n^5-5n}{1-n^5} \right\rangle \to -3$ Find the (fastest) rate of convergence of the form $\mathcal{O}\left(\frac{1}{n^p}\right)$ or $\mathcal{O}\left(\frac{1}{a^n}\right)$ for each. If this is not possible, suggest a reasonable rate of convergence.
We claim that (p_n) converges to $p = 3$ with rate of convergence $\mathcal{O}\left(\frac{1}{n^4}\right)$. To prove this, first let $n_0 = 1$ and $\lambda = 7$. We see that for all $n > n_0$,

$$|p_n - p| = \left|\frac{3n^5-5n}{1-n^5} + 3\right| = \left|\frac{3-5n}{(1-n)(1+n+n^2+n^3+n^4)}\right|$$

$$= \left|\frac{5-5n-2}{(1-n)(1+n+n^2+n^3+n^4)}\right| = \left|\frac{5}{1+n+n^2+n^3+n^4} + \frac{2}{(n-1)(1+n+n^2+n^3+n^4)}\right|$$

We can remove the absolute values because the term inside is positive. Since $n-1>1$, we get

$$\frac{5}{1+n+n^2+n^3+n^4} + \frac{2}{(n-1)(1+n+n^2+n^3+n^4)} \leq \frac{5}{n^4} + \frac{2}{n^4} = \frac{7}{n^4}$$

which is what we wanted.
To show that this is indeed the fastest rate of convergence, we can show that the sequence $\left(\frac{1}{n^4}\right)$ converges to 0 with rate of convergence $\mathcal{O}(p_n+3)$. To do this, let $n_0 = 1$, $\lambda = 1$. Let $n > n_0$. From the work we did in the previous part, we know that

$$|p_n - p| = \frac{5}{1+n+n^2+n^3+n^4} + \frac{2}{(n-1)(1+n+n^2+n^3+n^4)}$$

Since $\dfrac{2}{(n-1)(1+n+n^2+n^3+n^4)} > 0$ and $1 < n < n^2 < n^3 < n^4$, we get

$$|p_n - p| > \frac{5}{1+n+n^2+n^3+n^4} > \frac{5}{n^4+n^4+n^4+n^4+n^4} = \frac{1}{n^4}$$

as needed.

Question 6.

(a) Suppose you are trying to find the root of $f(x) = x - e^{-x}$ using the bisection method. Find an integer a such that the interval $[a, a+2]$ is an appropriate one in which to start the search.

Let $a = 0$. We see that $f(0) = -1 < 0$, $f(2) = 2 - \frac{1}{e^2} > 0$, so the interval [0,2] satisfies the conditions to use the bisection method.

(b) Use the bisection method to find the root in your interval in (a), accurate to 10^{-4}. Provide the Octave code you used to produce your result.

Using the method, the root was found at $x \approx 0.5672$.

Below is the Octave code used, along with the command used in command line:

```
function [m,M,i]=bisection(a,b,f,tol)
N=ceil((log(b-a)-log(tol))/log(2));
L=f(a);
for i=1:N
m=(a+b)/2;
M=f(m);
if (M==0)
return;
end%if
if (L*M<0)
b=m;
else
a=m;
L=M;
end%if
end%for
i=N;
end%function
```

```
>> [m,M,i] = bisection(0, 2, @(x) x-e^(-x), 0.0001)
```