Question 1.

Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$. Use row and column operations on A to obtain a matrix B of the form in Theorem 53. Use that work to find invertible matrices P, Q so that $B = PAQ$.

Proof. We perform the following row and column operations:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Define this matrix we obtained as B. We will perform the same row and column operations above on I_3 and I_4, respectively in order to define P and Q. We have that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 \to r_3 - 2r_1]{r_2 \to r_2 - r_1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$
$$\xrightarrow{r_3 \to r_3 - r_2} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[c_4 \to c_4 - c_1 + c_2]{c_3 \to c_3 + c_1 - 2c_2} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $P = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We see that

$$PAQ = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= B
\end{aligned}
$$

as required. $\square$

Question 2.

Let $A = \begin{pmatrix} 1 & -2 & -4 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

(a) Verify that A is invertible, by row-reducing the augmented matrix $(A|I_3)$.

(b) Use (a) to find A^{-1}.

(c) Express A as a product of elementary matrices.

Proof.
(a): We see that

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_2 \to r_2 - r_1, r_3 \to r_3 - r_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{r_1 \to r_1 + r_3, r_2 \to r_2 - r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{r_3 \to r_3 - 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array}\right)$$
$$\xrightarrow{r_1 \to r_1 + \frac{1}{3} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & -2 & 3 \end{array}\right) \xrightarrow{r_3 \to \frac{1}{3} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right)$$

Since A can be row reduced into the identity matrix, A is invertible.

(b):

By our row reductions above, we know that $A^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 1 & -1 \\ -\frac{1}{3} & -\frac{2}{3} & 1 \end{pmatrix}$.

(c):
To express A is a product of elementary matrices, we can apply the opposite row operations to the identity matrix in reverse order. That is,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

□

Question 3.

Find the explicit formula for the linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^3$ which satisfies:

$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad T\begin{pmatrix}2\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Proof. Notice that

$$\beta = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\}$$

is a basis for $\mathbb{Q}^4$. We attempt to find the general form for a vector $(x, y, z, w) \in \mathbb{Q}^4$ in terms of these vectors. By inspection, we see that

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = (x - 2y + z)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (y - z)\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + (z - w)\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + w\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Thus

$$\begin{aligned} T\begin{pmatrix}x\\y\\z\\w\end{pmatrix} &= (x - 2y + z)T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (y - z)T\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + (z - w)T\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + wT\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \\ &= (x - 2y + z)\begin{pmatrix}1\\2\\3\end{pmatrix} + (y - z)\begin{pmatrix}0\\1\\1\end{pmatrix} + (z - w)\begin{pmatrix}0\\0\\1\end{pmatrix} + w\begin{pmatrix}0\\0\\0\end{pmatrix} \\ &= \begin{pmatrix}x - 2y + z\\2x - 3y + z\\3x - 5y + 3z - w\end{pmatrix} \end{aligned}$$

□

Question 4.

Let $\mathbb{F} = \mathbb{Q}$ and $V = \mathcal{M}_{2\times 2}(\mathbb{F})$. Consider the linear map $T : \mathcal{M}_{2\times 2}(\mathbb{F}) \to \mathcal{M}_{2\times 2}(\mathbb{F})$ given by $T(A) = A^T$. Set $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\gamma = \{E_{11}, E_{22}, E_{12} + E_{21}, E_{12} - E_{21}\}$.

(a) Find P - the change of coordinate matrix from γ to β coordinates.

We have

$$\begin{aligned} P &= \begin{pmatrix} [E_{11}]_\beta & [E_{22}]_\beta & [E_{12} + E_{21}]_\beta & [E_{12} - E_{21}]_\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

(b) Find P^{-1} - the change of coordinate matrix from β to γ coordinates.

Similarly,

$$\begin{aligned} P^{-1} &= \begin{pmatrix} [E_{11}]_\gamma & [E_{12}]_\gamma & [E_{21}]_\gamma & [E_{22}]_\gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{aligned}$$

(c) Find $A = [T]_\beta$.

We see that

$$\begin{aligned} A &= \begin{pmatrix} [T(E_{11})]_\beta & [T(E_{12})]_\beta & [T(E_{21})]_\beta & [T(E_{22})]_\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(d) Find $B = [T]_\gamma$.

Once again,

$$\begin{aligned} B &= \begin{pmatrix} [T(E_{11})]_\gamma & [T(E_{22})]_\gamma & [T(E_{12} + E_{21})]_\gamma & [T(E_{12} - E_{21})]_\gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

(e) Confirm that $A = PBP^{-1}$ using (a)-(d).

$$
\begin{aligned}
PBP^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= A
\end{aligned}
$$

as expected.

Question 5.

Let $T : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathcal{M}_{n\times n}(\mathbb{F})$ be the linear map given by $T(A) = A + A^T$.

(a) Find $N(T)$ and $\dim N(T)$.

We claim that $N(T)$ is the set of all skew symmetric matrices with zeroes on the diagonal, which has dimension $\frac{1}{2}n(n-1)$.

Set $T(A) = A + A^T = 0$. We have that $A_{ij} + A_{ji} = 0$ for each $0 < i, j \leq n$. In particular, we have that $A_{ij} = 0$ if $i = j$ and $A_{ij} = -A_{ji}$ otherwise. But this describes exactly all skew symmetric matrices with zeroes on the diagonal. The basis for this set is

$$\beta = \{E_{ij} - E_{ji} : 0 < i < j \leq n\}$$

and there are $\frac{1}{2}n(n-1)$ vectors in this set, so $\dim N(T) = \frac{1}{2}n(n-1)$.

(b) What is $\operatorname{im}(T)$?

We claim that $\operatorname{im}(T)$ is the set of all symmetric matrices S_n. We see that

$$(A + A^t)_{ij} = A_{ij} + A^t_{ij} = A_{ij} + A_{ji} = A_{ji} + A^t_{ji} = (A + A^t)_{ji}$$

so $\operatorname{im}(T) \subseteq S_n$. To show set equality, suppose that B is a symmetric matrix. Let $A = \frac{1}{2}B$ then

$$T(A) = \frac{1}{2}T(B) = \frac{1}{2}(B + B^t) = B$$

Thus $\operatorname{im}(T) = S_n$ and has basis

$$\gamma = \{E_{ij} : 0 < i \leq j \leq n\}.$$

and is dimension $\frac{1}{2}n(n+1)$.

(c) Is $\mathcal{M}_{n\times n}(\mathbb{F}) = \operatorname{im}(T) \oplus N(T)$?

Yes.

To show this, notice that $\beta \cap \gamma = \varnothing$, so $\operatorname{im}(T) \oplus N(T)$ has basis $\alpha = \beta \cup \gamma$. But notice that $|\alpha| = \frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$, which is the dimension of $\mathcal{M}_n(\mathbb{F})$. Therefore α is actually a basis for $\mathcal{M}_n(\mathbb{F})$ and thus $\mathcal{M}_n(\mathbb{F}) = \operatorname{im}(T) \oplus N(T)$.

Question 6.

Let V, W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$ a linear map. Prove that T is injective if and only if $N(T) = \{\mathbf{0}_V\}$. (Make no assumption here about $\dim V, \dim W$.)

Proof. Suppose that T is injective. Let $T(x) = 0$, for some $x \in V$. Recall that $T(0) = 0$ for any linear map. Therefore by injectivity $x = 0$, so $N(T) = \{0\}$.
Conversely, suppose that $N(T) = \{0\}$. Let $x, y \in V$ such that $T(x) = T(y)$. By linearity, we have that $T(x - y) = 0$, but this implies that $x - y = 0$, so $x = y$ and T is injective. $\square$

Question 7.

Let V, W be vector spaces over a field $\mathbb{F}$, and $T : V \to W$ a linear map. Find a condition on T which is equivalent to "$T(S)$ spans W for any spanning set $S \subseteq V$ of V".
(Hint: Write down the definition of $T(S)$ is spanning to get started.)

Proof. We claim that this statement is equivalent to saying that T is surjective.
Suppose that for any set $S \subseteq V$ that spans V, $T(S)$ spans W. We prove that T is surjective. Let $w \in W$. We can write w as a linear combination of some number of vectors in $T(S)$. That is, for some $k \in \mathbb{N}$ and $s_i \in S$, $c_i \in \mathbb{F}$, $i \in \{1, ..., k\}$,

$$w = \sum_{i=1}^{k} c_i T(s_i) = T\left(\sum_{i=1}^{k} c_i s_i\right)$$

so T is surjective.
Conversely, suppose that T is surjective. Let S be a spanning set of V. We will show that $T(S)$ spans W. Let $w \in W$. By surjectivity, there exists $v \in V$ so that $T(v) = w$. We can rewrite

$$v = \sum_{i=1}^{k} c_i s_i$$

for some number of vectors $s_i \in S$ and $c_i \in \mathbb{F}$. Then

$$T\left(\sum_{i=1}^{k} c_i s_i\right) = w \implies \sum_{i=1}^{k} c_i T(s_i) = w$$

Notice that $T(s_i) \in T(S)$, from which it follows that $T(S)$ spans W, and the proof is complete.

$\square$

Question 8.

Let $P \in \mathcal{M}_{n\times n}(\mathbb{F})$. Prove the following three conditions are equivalent.

(a) P is invertible.

(b) There exists bases β, γ of $\mathbb{F}^n$ so that $P = [\mathrm{I}_{\mathbb{F}^n}]_\beta^\gamma$.

(c) For any n-dimensional vector space V over $\mathbb{F}$, there exists bases β, γ of V so that $P = [\mathrm{I}_V]_\beta^\gamma$.

Proof. Suppose (a). We prove (b) and (c) at the same time.
Let $\beta = \{v_1, ..., v_n\}, \beta' = \{v_1', ..., v_n'\}$ be bases for $\mathbb{F}^n$ and V respectively. For $i = 1, ..., n$, let

$$u_i = \sum_{j=1}^n P_{ji} v_j \text{ and } u_i' = \sum_{j=1}^n P_{ji} v_j'.$$

Define $\gamma = \{u_1, ..., u_n\}, \gamma' = \{u_1', ..., u_n'\}$. We claim that they are bases for $\mathbb{F}^n$ and V. We will only show that is the case for γ, because the argument is the same for γ'.
It suffices to show that γ is linearly independent, as it is a set of n vectors, from which it will follow that γ is a basis for $\mathbb{F}^n$. For constants $c_i \in \mathbb{F}$, let

$$\sum_{i=1}^n c_i \sum_{j=1}^n P_{ji} v_j = 0$$

By the linear independence of β, for each j,

$$\sum_{i=1}^n c_i P_{ji} = 0$$

But this is the same as saying

$$P \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0}$$

Since P^{-1} exists, we perform left multiplication by P^{-1} to see that

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$$

which means that $c_i = 0$ for all i, thus showing that γ is linearly independent and indeed a basis for $\mathbb{F}^n$, and by the same argument, γ' is also a basis for V.
Finally, notice that for each u_i, $[u_i]_\beta$ is equal to the ith row of P which confirms that $P = [I_{\mathbb{F}^n}]_\gamma^\beta$. The same applies for u_i' so $P = [I_V]_{\gamma'}^{\beta'}$.
Consider the linear transformation $T_P : \mathbb{F}^n \to \mathbb{F}^n$. Let β be an ordered basis for $\mathbb{F}^n$. We will show that $\gamma = T_P(\beta)$ is also an ordered basis for $\mathbb{F}^n$. Since P is invertible, T_P has an

inverse $(T_P)^{-1} = T_{P^{-1}}$, so T_P is surjective and $\text{span}(T_P(\beta)) = \mathbb{F}^n$. Since $|T_P(\beta)| = n$, $T_P(\beta)$ is indeed an ordered basis. Thus we can conclude that P is a change of basis matrix from β to γ.

Suppose (c). We prove (a).
For some bases β, γ, $P = [I_V]_\beta^\gamma$. We claim that $P^{-1} = [I_V]_\gamma^\beta$. Indeed,

$$PP^{-1} = [I_V]_\beta^\gamma [I_V]_\gamma^\beta = [I_V]_\gamma = I_n$$

This covers all the equivalences and we are done.

$\square$

Question 9.

Consider the relation $\equiv$ on $\mathcal{M}_{m\times n}(\mathbb{F})$ defined by $A \equiv B$ if $A \to B$ using a combination of row and/or column operations.

(a) Prove that $\equiv$ is an equivalence relation on $\mathcal{M}_{m\times n}(\mathbb{F})$.

(b) Find a condition on A, B which is equivalent to $A \equiv B$. (Hint: Theorem 53.)

(c) Classify the equivalence classes for this relation, and prove that there are exactly $1 + \min\{n, m\}$ such classes.

Proof.
(a):
We show reflexivity, symmetry, and transitivity in that order.
Reflexivity: Since $IA = A$, and I is considered a row operation, $A \equiv A$.
Symmetry: Suppose that $A \equiv B$ then for some invertible matrices P, Q we have that $PAQ = B$. But at the same time this means that $P^{-1}BQ^{-1} = A$ so $B \equiv A$.
Transitivity: Suppose that $A \equiv B$ and $B \equiv C$. Then for invertible matrices P, Q, R, S, $PAQ = B$ and $RBS = C$, so $(RP)A(QS) = R(PAQ)S = RBS = C$. Since RP, QS are also invertible, we have that $A \equiv C$.

(b):
We claim that an equivalent condition is $\operatorname{rank} A = \operatorname{rank} B$. Suppose that $A \equiv B$. Then $PAQ = B$ for some invertible matrices P, Q, but it is known that rank is preserved by multiplication with invertible matrices, so $\operatorname{rank} A = \operatorname{rank} PAQ = \operatorname{rank} B$.
Conversely, suppose that $r := \operatorname{rank} A = \operatorname{rank} B$. By Theorem 53, there exist row/column operations so that

$$A, B \to \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

We denote this matrix by J_r. that is, for invertible matrices P, Q, R, S, $PAQ = I' = RBS$. It follows that $R^{-1}PAQS^{-1} = B$, so $A \equiv B$ as desired.

(c):
We can classify the equivalence classes by matrix rank. That is, each equivalence class is of the form

$$[J_r] = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \operatorname{rank} A = r\}.$$

The possible ranks of $m \times n$ matrices range from 0 to $\min\{n, m\}$, so there are $\min\{n, m\} + 1$ different values of r. We will verify that these equivalence classes are exhaustive and disjoint. Every $m \times n$ matrix must have a rank, so it belongs to at least one of the classes, but at the same time, a matrix can possibly only have one rank, so it necessarily belongs to exactly one equivalence class.

$\square$

Question 10.

Let V, W be finite dimensional vector spaces over $\mathbb{F}$, and $T : V \to W$ a linear map with $\text{rank}T = 2$. Set $n = \dim V$, $m = \dim W$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}^n$ be two non-parallel vectors. Prove there exists bases β, γ of V, W respectively, so that $[T]_\beta^\gamma = (\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{0}\ \cdots\ \mathbf{0})$. (Hint: use problems 7,8.)

Proof. Define

$$A = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & 0 & \cdots & 0 \end{pmatrix}$$

rank $A = 2$ because there are 2 linearly independent rows. Fix two arbitrary bases β', γ' and consider $[T]_{\beta'}^{\gamma'}$, which is rank 2 by assumption. By question 9, we can perform a sequence of row and column operations to turn $[T]_{\beta'}^{\gamma'}$ into A. In particular, for invertible matrices $P \in \mathcal{M}_m(\mathbb{F}), Q \in \mathcal{M}_n(\mathbb{F})$,

$$P[T]_{\beta'}^{\gamma'}Q = A$$

Using question 8, and noting that the choice of β was arbitrary in its proof, we can say that $P = [I_W]_{\gamma'}^{\gamma}$ and $Q^{-1} = [I_V]_{\beta'}^{\beta}$ for some basis γ of W and β of V. Thus we have that

$$A = P[T]_{\beta'}^{\gamma'}Q = [I_W]_{\gamma'}^{\gamma}[T]_{\beta'}^{\gamma'}[I_V]_{\beta}^{\beta'} = [T]_{\beta}^{\gamma}$$

which is what we wanted.

$\square$

Question 11.

Let $T : V \to V$ be linear. We say that a subspace $W \subseteq V$ is "T-invariant" if $T(W) \subseteq W$. For example, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is counter-clockwise rotation around the z-axis by angle θ, then $P_{xy} = \{(x, y, 0) \in \mathbb{R}^3\}$ is T-invariant, as is L_z (the z-axis).

(a) Verify the claims made above, by showing that P_{xy} and L_z are T-invariant.

(b) Show that $\mathbb{R}^3 = P_{xy} \oplus L_z$ by finding a basis $\beta = \beta_1 \cup \beta_2$ for $\mathbb{R}^3$ so that β_1 is a basis for P_{xy} and β_2 is a basis for L_z.

(c) Using your basis β from (b), find $[T]_\beta$.

Proof.
(a):
We begin by finding an expression for T. Notice that

$$\begin{aligned} T(e_1) &= (\cos\theta, \sin\theta, 0) \\ T(e_2) &= (-\sin\theta, \cos\theta, 0) \\ T(e_3) &= (0, 0, 1) \end{aligned}$$

In the case of e_1, e_2, the projection onto the xy-plan lies on the unit circle, and thus each vector is rotated θ and $\theta + \frac{\pi}{2}$ radians respectively (relative to the point $(0, 1)$). Then we have that

$$\begin{aligned} T(x, y, z) &= x(\cos\theta, \sin\theta, 0) + y(-\sin\theta, \cos\theta, 0) + z(0, 0, 1) \\ &= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) \end{aligned}$$

Now, let $(x, y, 0) \in P_{xy}$. Then

$$T(x, y, 0) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, 0) \in P_{xy}$$

Additionally, let $(0, 0, z) \in L_z$. Then

$$T(0, 0, z) = (0, 0, z) \in L_z$$

Thus P_{xy} and L_z are T-invariant subspaces.

(b):
Let $\beta_1 = \{e_1, e_2\}, \beta_2 = \{e_3\}$. It is clear that β_1 is a basis for the xy-plane and β_2 is a basis for the z-axis. Then $\beta = \{e_1, e_2, e_3\}$ is the standard ordered basis for $\mathbb{R}^3$, which was what we wanted to show.

(c):
We have already found all we need from the previous parts:

$$[T]_\beta = \big([T(e_1)]_\beta \quad [T(e_2)]_\beta \quad [T(e_3)]_\beta\big) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\square$

Question 12.

Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in \mathcal{L}(V)$, and $W_1 \subseteq V$ a T-invariant subspace with basis β_1. Set $k = \dim W_1$.
We will generalize what we saw in #11c.

(a) Extend β_1 to a basis β of V. Show that $[T]_\beta = \left(\begin{array}{c|c} A & C \\ \hline O_{n-k,k} & B \end{array}\right)$, where A is $k \times k$, B is $(n-k) \times (n-k)$, and C is $k \times (n-k)$.

(b) Suppose that W_2 is a subspace so that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$, where β_2 is any basis for W_2.

Prove that if W_2 is T-invariant, then $[T]_\beta = \left(\begin{array}{c|c} A & O_{k,n-k} \\ \hline O_{n-k,k} & B \end{array}\right)$ is block diagonal.

(c) Is the converse of (b) true or false? Justify your answer.

Proof.
(a):
Let $n = \dim V$. Let $\beta_1 = \{v_1, ..., v_k\}$ be a basis for W_1. Let $\beta = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ be the basis obtained from extending β_1. To show that $[T]_\beta = \left(\begin{array}{c|c} A & C \\ \hline O_{n-k,k} & B \end{array}\right)$, we are unconcerned about the matrices on the right hand side, so we only focus on the left side of the matrix. Consider the ith column of $[T]_\beta$, where $1 \leq i \leq k$. This is equal to $[T(v_i)]_\beta$. Since W_1 is T-invariant, $T(v_i) \in W_1$, so it can be expressed as a linear combination of exclusively vectors in β_1. Thus

$$[T(v_i)]_\beta = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $a_{ji} \in \mathbb{F}$ and the bottom $n-k$ entries are 0. Since this is true for all colummns 1 to k, the statement is proven.

(b):
Now take W_2 to be a T-invariant subspace so that $V = W_1 \oplus W_2$. Let $\beta = \beta_1 \cup \beta_2$, where β_1, β_2 are bases for W_1, W_2 respectively. To show that $\left(\begin{array}{c|c} A & O_{k,n-k} \\ \hline O_{n-k,k} & B \end{array}\right)$, it suffices to show the right sides are equivalent, as we have proved the left side in the previous part. The argument is very similar to before, so we will be brief. For each $T(v_i)$, where $v_i \in W_2$, the representation as a linear combination does not use any vectors from W_1, so the first k entries of $[T(v_i)]_\beta$ are guaranteed to be 0 as required.

(c):

The converse is true. Suppose that $[T]_\beta = \left(\begin{array}{c|c} A & O_{k,n-k} \\ \hline O_{n-k,k} & B \end{array}\right)$ is block diagonal. Let $v \in W_2$. Write $v = \sum_{i=k+1}^{n} a_i v_i$ as a linear combination of vectors in β_2. It follows that

$$T(v) = \sum_{i=k+1}^{n} a_i T(v_i) = \sum_{i=k+1}^{n} a_i \sum_{j=k+1}^{n} b_{ij} v_j \in W_2$$

thus verifying that W_2 is T-invariant.

$\square$

Question 13.

Determine if the statements below are true or false. If true, give a proof. If false, explain why, and/or provide a counterexample.

(a) Let $\beta = \{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{F}^n$, and $\gamma = \{v_1, \ldots, v_n\}$ a basis for $\mathbb{F}^n$. Then there exists a sequence of row operations that takes β to γ. (That is, v_i is obtained from e_i using the same row operations for all i.)

(b) Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \to V$ a linear map. If β, γ are bases for V so that $[T]_\beta^\gamma = I_n$, then $T = I_V$.

(c) Let V be a finite dimensional vector space over $\mathbb{F}$ and $S, T : V \to V$ linear maps. If $\operatorname{rank} T = \operatorname{rank} S$, then there exist bases $\beta, \beta', \gamma, \gamma'$ for V so that $[S]_\beta^\gamma = [T]_{\beta'}^{\gamma'}$.

(d) Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$. If $A^2 \sim B^2$, then $A \sim B$.

Proof.
(a):
This statement is true. Consider the linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ defined by $T(e_i) = v_i$ for all $i \in \{1, ..., n\}$. Notice that T is surjective, as $\operatorname{span}(T(\beta)) = \operatorname{span}(\gamma) = \mathbb{F}^n$. It follows that T is invertible, so $[T]_\beta^\gamma$ is invertible, so it can be decomposed into a number of elementary matrices and thus represent a sequence of row operations. But notice that $[T]_\beta^\gamma$ is exactly the matrix that maps e_i to v_i, so we have what we wanted.

(b):
This is false. Let $V = \mathbb{F}^n$. Take β as the standard basis. As well, let $\gamma = \{e_1, e_2, ..., e_{n-1}, -e_n\}$, so that γ is just β only with the last basis vector multiplied by -1. Consider the linear map T defined in the previous part. Then $T(e_n) = -e_n$, so T is not the identity map, but it is not hard to see that $[T]_\beta^\gamma = I_n$.

(c):
This is true. Let $k = \operatorname{rank} T = \operatorname{rank} S$. Take k vectors $e_1, ..., e_k$. By Question 10, there exists bases $\beta, \gamma, \beta', \gamma'$ such that

$$[T]_\beta^\gamma = \left(\begin{array}{c|c} I_k & O \\ O & O \end{array}\right) = [S]_{\beta'}^{\gamma'}$$

as desired.

(d):
This is false. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = 0_n$, the zero matrix. Then

$$A^2 = B^2$$

But $\operatorname{rank} A = 1$ and $\operatorname{rank} B = 0$, so for any invertible matrix P,

$$\operatorname{rank} PAP^{-1} = 1$$

so A is not similar to B.

□