

Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that O_n is a smooth manifold, and find its dimension.

Proof. First, we will prove the Regular Level-Set Theorem:

Let X, Y be normed vector spaces with dimensions n, m and ordered bases α, β respectively, where $n > m$. Let $F : X \rightarrow Y$ be a smooth function. Define $M = F^{-1}(0_Y)$. If $F'(p)$ is surjective for all $p \in M$, then M is a smooth manifold of dimension $n - m$. That is, if $\phi_\alpha : X \rightarrow \mathbb{R}^n$ is the coordinate isomorphism corresponding to α , then M is a smooth k -manifold if $\phi_\alpha(M)$ is a smooth k -manifold in the usual sense.

This problem reduces to trying to prove that $N := \phi_\alpha(M)$ is a smooth manifold of dimension $n - m$. Notice that $N = \phi_\alpha(F^{-1}(0_Y)) = \phi_\alpha(F^{-1}(\phi_\beta(0_{\mathbb{R}^m})))$. Since ϕ_α, ϕ_β are isomorphisms, we have that $N = \phi_\alpha \circ F^{-1} \circ \phi_\beta(0_{\mathbb{R}^m})$. Let $\hat{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function defined by $\hat{F} = \phi_\alpha \circ F^{-1} \circ \phi_\beta$, and notice that N is the zero set of \hat{F} . For any $p \in N$, notice that $\hat{F}'(p)$ is surjective, because

$$\hat{F}'(p) = (\phi_\beta \circ F \circ \phi_\alpha^{-1})'(p) = \phi_\beta \circ F'(\phi_\alpha^{-1}) \circ \phi_\alpha^{-1}$$

is just $F'(\phi_\alpha^{-1})$ —a surjective map—composed with linear isomorphisms. This implies that $R(J\hat{F}(p)) = \mathbb{R}^m$ and $\text{rank } J\hat{F}(p) = m$. We can write

$$J\hat{F}(p) = (A \mid B)$$

where A is a $m \times n - m$ matrix and B is a $m \times m$ matrix, and assume without loss of generality that B is invertible, for if not, $J\hat{F}(p)$ is still guaranteed to contain an invertible $m \times m$ submatrix, and we can perform column swaps to move the matrix to the right, which does not change the conclusion of our statement.

Recall that since N is the zero set of \hat{F} , $\hat{F}(p) = 0$. Thus, we write $p = (a, b)$ for $a \in \mathbb{R}^{n-m}$, $b \in \mathbb{R}^m$ and apply the Implicit Function Theorem and obtain an open set $\hat{U} \subseteq \mathbb{R}^{n-m}$ containing a and a C^∞ function $\Phi : \hat{U} \rightarrow \mathbb{R}^m$ so that

$$\hat{F}(x, \Phi(x)) = 0$$

for all $x \in \hat{U}$. We claim that $\varphi : \hat{U} \rightarrow \Phi(\hat{U})$ defined by

$$\varphi(x) = (x, \Phi(x))$$

is our desired smooth regular embedding. It is fairly clear that φ is smooth. Additionally, $J\varphi(x) = \begin{pmatrix} I_{n-m} \\ J\Phi \end{pmatrix}$ is a $(n - m) \times n$ matrix and has at least $n - m$ linearly independent rows, so φ is regular. Finally, if we let $\varphi(x) = \varphi(y)$, we have that $(x, \Phi(x)) = (y, \Phi(y))$, from which we get $x = y$, so φ is injective, and therefore bijective to its image. In addition, we can explicitly find $\varphi^{-1}(y) = \pi_{\mathbb{R}^{n-m}}(y)$, which is continuous because it is a linear map. Therefore

Question 41.

Let $0 < a < b$. In the xz -plane, draw a circle of radius a centered at the point $(b, 0, 0)$; rotate this circle about the z -axis. The resulting subset of \mathbf{R}^3 is called a **torus**, denoted by $\mathbf{T} = \mathbf{T}_{a,b}$.

- (a) Find a smooth function $f : U \rightarrow \mathbf{R}$, defined on some open set $U \subseteq \mathbf{R}^3$, so that \mathbf{T} is equal to the zero set of f .
- (b) Show that \mathbf{T} is a smooth manifold.
- (c) Find the surface area of \mathbf{T} , in terms of a and b .

Proof.

(a):

Notice that in cylindrical coordinates, the torus can be defined by

$$T = \{(r, \theta, z) : (r - b)^2 + z^2 = a^2\}.$$

If we map the polar part of the set back to cartesian coordinates, we see that T is actually the zero set of the function

$$f(x, y, z) = (\sqrt{x^2 + y^2} - b)^2 + z^2 - a^2$$

This function is smooth everywhere except for when $x = 0 = y$, so we let $U = \mathbf{R}^3 \setminus \{(x, y, z) : x = y = 0\}$.

(b):

Notice that for all $p = (x, y, z) \in T$, $f'(p)$ is rank 1, because $f'(p)$ is a 1×3 matrix, so its rank is at most 1, but it cannot be rank 0 because

$$\frac{\partial f}{\partial x}(x, y, z) = 2 \frac{\sqrt{x^2 + y^2} - b}{\sqrt{x^2 + y^2}} = 2 - \frac{2b}{\sqrt{x^2 + y^2}}$$

and

$$\frac{\partial f}{\partial z}(x, y, z) = 2z$$

so being rank 0 implies that

$$f'(p) = \left(2 - \frac{2b}{\sqrt{x^2 + y^2}}, 2 - \frac{2b}{\sqrt{x^2 + y^2}}, 2z \right) = (0, 0, 0) \implies \sqrt{x^2 + y^2} = b \text{ and } z = 0$$

but if this is the case, $f(p) = -a^2 \neq 0$, so $p \notin T$, which is a contradiction.

Thus $Jf(p)$ is always rank 1, so according to the Regular Level-Set Theorem, T is a smooth manifold of dimension $3 - 1 = 2$.

□