

### Question 1.

Find the error term for the derivative approximation:

$$f''(x_0) \approx \frac{2f(x_0 - h) - 3f(x_0) + f(x_0 + 2h)}{3h^2}.$$

We write the polynomial expansion for each term on the right:

$$f(x_0 - h) = f(x_0) - f'(x_0)h + f''(x_0)h^2 - \frac{f'''(\xi_1)}{6}h^3$$

$$f(x_0) = f(x_0)$$

$$f(x_0 + 2h) = f(x_0) + 2f'(x_0)h + 4f''(x_0)h^2 + \frac{4f'''(\xi_2)}{3}h^3$$

Then

$$2f(x_0 - h) - 3f(x_0) + f(x_0 + 2h) = 6f''(x_0)h^2 - \frac{2f'''(\xi_1)}{6}h^3 + \frac{4f'''(\xi_2)}{3}h^3$$

$$\frac{2f(x_0 - h) - 3f(x_0) + f(x_0 + 2h)}{3h^2} = 2f''(x_0)h^2 - \frac{1}{9}f'''(\xi_1)h + \frac{4}{9}f'''(\xi_2)h$$

so the error term is

$$f''(x_0) - \left[ 2f''(x_0)h^2 + \frac{1}{9}f'''(\xi_1)h - \frac{4}{9}f'''(\xi_2)h \right] = -f''(x_0)h^2 + \frac{1}{9}f'''(\xi_1)h - \frac{4}{9}f'''(\xi_2)h$$

### Question 2.

Find the error term for the quadrature method, and state its degree of precision.

$$\int_{x_0}^{x_0+2h} f(x) dx \approx \frac{h}{2} \left[ 3f\left(x_0 + \frac{4}{3}h\right) + f(x_0) \right].$$

We expand the left hand side:

$$\begin{aligned} & \int_{x_0}^{x_0+2h} f(x) dx \\ &= \int_{x_0}^{x_0+2h} f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{6}(x - x_0)^3 + \frac{f^{(4)}(\xi_1)}{24}(x - x_0)^4 dx \\ &= 2f(x_0)h + 2f'(x_0)h^2 + \frac{4}{3}f''(x_0)h^3 + \frac{2}{3}f'''(x_0)h^4 + \frac{4}{15}f^{(4)}(\xi_1)h^5 \end{aligned}$$

Now we expand each term on the right hand side:

$$f\left(x_0 + \frac{4}{3}h\right) = f(x_0) + \frac{4}{3}f'(x_0)h + \frac{8}{9}f''(x_0)h^2 + \frac{32}{81}f'''(x_0)h^3 + \frac{32}{243}f^{(4)}(\xi_2)h^4.$$

Thus

$$\frac{h}{2} \left[ 3f \left( x_0 + \frac{4}{3}h \right) + f(x_0) \right] = 2f(x_0)h + 2f'(x_0)h^2 + \frac{4}{3}f''(x_0)h^3 + \frac{16}{27}f'''(x_0)h^4 + \frac{16}{81}f^{(4)}(\xi_2)h^5.$$

and the error term is

$$\int_{x_0}^{x_0+2h} f(x) dx - \frac{h}{2} \left[ 3f \left( x_0 + \frac{4}{3}h \right) + f(x_0) \right] = \frac{2}{27}f'''(x_0)h^4 + \frac{4}{15}f^{(4)}(\xi_1)h^5 - \frac{16}{81}f^{(4)}(\xi_2)h^5.$$

### Question 3.

Consider the integral  $\int_1^7 \cos(x^2) dx$

- Use the composite Simpson's rule to approximate the value of this integral using  $n = 3$  intervals.
- Determine the number of intervals  $n$  needed to guarantee an error of at most  $10^{-4}$ .

### Question 4.

Consider the IVP:

$$2\dot{y} + y = t^4 + 1, \quad y(1) = 2.$$

Apply the second degree Taylor method with  $h = 0.5$  to this ODE to approximate  $y(2)$ . Show the details in each step.

### Question 5.

Derive an ODE solver based on the stencil and corresponding integration formula.

$$\begin{array}{ccccccc} & x_0 & & x_0 + \frac{1}{3}h & & x_0 + \frac{2}{3}h & & x_0 + h \\ \leftarrow & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow \end{array} \quad \text{Formula: } \frac{h}{4} \left( 3f \left( x_0 + \frac{1}{3}h \right) + f(x_0 + h) \right) + O(h^4)$$