Question 25.

Let $\varphi : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ be the function given by $\varphi(A) = A^2$. For each $A \in M_n(\mathbf{R})$, find a linear approximation $L_A : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ to φ at A. Give an explicit formula for $L_A(B)$ as a function of B, a proof that L_A is a bounded linear mapping, and a proof that L_A is a linear approximation to φ at A.

Proof. First, we supply a lemma.
Lemma. For all $B \in M_n(\mathbb{R})$, $\|B^2\| \leq K\|B\|^2$, for some positive constant K.
Define the isomorphism $\Phi : M_n(\mathbb{R}) \to B(\mathbb{R}^n, \mathbb{R}^n)$ as mapping a matrix representation of a linear mapping to the original linear mapping. We define the operator norm on $M_n(\mathbb{R})$ by $\|A\|_{\text{op}} = \|\Phi(A)\|_{\text{op}}$, where the right hand side is the operator norm on $B(\mathbb{R}^n, \mathbb{R}^n)$.
Since all norms are equivalent on $M_n(\mathbb{R})$, there are constants $M, N > 0$ so that for any norm $\|\cdot\|$,

$$M\|A\| \leq \|A\|_{\text{op}} \leq N\|A\|$$

From this, using the subnormality of bounded linear operators, it follows that

$$\|B^2\| \leq \frac{1}{M}\|\Phi(B^2)\|_{\text{op}} = \frac{1}{M}\|\Phi(B) \circ \Phi(B)\|_{\text{op}} \leq \frac{1}{M}\|\Phi(B)\|_{\text{op}}^2 \leq \frac{N^2}{M}\|B\|$$

Since $M, N > 0$, we have what we wanted.
We claim that for $A \in M_n(\mathbb{R})$, $L_A(B) = BA + AB$. For $C, D \in M_n(\mathbb{R})$, $k \in \mathbb{R}$,

$$L_A(kC + D) = (kC + D)A + A(kC + D) = k(CA + AC) + DA + AD = kL_A(C) + L_A(D)$$

so L_A is linear. As well, we get that L_A is bounded for free because we are working in a finite dimensional vector space. Finally, we have that

$$0 \leq \frac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} = \frac{\|(A+B)^2 - A^2 - (BA + AB)\|}{\|B\|}$$

$$\frac{\|A^2 + AB + BA + B^2 - A^2 - BA - AB\|}{\|B\|} = \frac{\|B^2\|}{\|B\|} < K\|B\|$$

$$\implies 0 \leq \frac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} \leq K\|B\|$$

By the Squeeze Theorem, $\lim_{h\to 0} \dfrac{\|\varphi(A+B) - \varphi(A) - L_A(B)\|}{\|B\|} = 0$ and we are done. $\square$

Question 25.

Let X be a finite-dimensional normed vector space, let U be an open convex subset of X, and let $f : U \to \mathbf{R}^m$ be a totally differentiable function. (Note: a set $C \subseteq X$ is called **convex** if $tx + (1-t)y \in C$ for all $x, y \in C$ and $t \in [0,1]$.) Let $f : U \to \mathbf{R}^m$ be a totally differentiable function.

(a) Suppose that there exists a constant $C \geq 0$ such that $\|f'(p)\|_{\text{op}} \leq C$ for all $p \in U$. Prove that

$$\|f(p) - f(q)\| \leq C\|p - q\| \quad \text{for all } p, q \in U.$$

Conclude that f is uniformly continuous.

(b) Prove that $f'(p) = 0$ for all $p \in U$ if and only if f is a constant function.

(c) Assume $U = X$ and suppose that f is **twice totally differentiable** — meaning that $f' : X \to B(X, Y)$ itself is differentiable at every point of X, with total derivative $f'' = (f')'$. Show that $f'' = 0$ if and only if f is **affine-linear**: there exists a bounded linear mapping $M : X \to Y$ and a vector $b \in Y$ such that

$$f(p) = M(p) + b \quad \text{for all } p \in X.$$

(Compare with the formula $y = mx + b$ from single-variable calculus.)

Proof. (a):
Fix $p, q \in U$. If $p = q$, the inequality is trivially true. Otherwise, construct a function $\alpha : (-\frac{1}{2}, \frac{3}{2}) \to U$ defined by $\alpha(t) = tq + (1-t)p$. We will show that $a'(t)(\varphi) = \varphi(q - p)$:

$$\lim_{h \to 0} \frac{\|((t+h)q + (1-t-h)p) - (tq + (1-t)p) - h(q-p)\|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{\|hq - hp - h(q-p)\|}{\|h\|} = 0$$

Thus $\alpha'(t)(\varphi) = \varphi(q - p)$. Next, we want to show that α is uniformly continuous.
For all $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{\|q-p\|}$. Then for all $s, t \in [0,1]$ such that $\|s - t\| < \delta$,

$$\|\alpha(s) - \alpha(t)\| = \|sq + (1-s)p - (tq + (1-t)p)\| = \|(s-t)q - (s-t)p\|$$

$$\|(s-t)(q-p)\| < \varepsilon$$

(b):
Suppose that $f'(p) = 0$ for all $p \in U$. Then $\|f'(p)\|_{\text{op}} \leq 0 = C$, so by part (a), for all $a, b \in U$,

$$\|f(a) - f(b)\| \leq 0 \implies \|f(a) - f(b)\| = 0 \implies f(a) = f(b)$$

so f is constant.
Conversely, suppose that f is a constant function. To show that $f'(p) = 0$, notice that

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p)\|}{\|h\|} = 0$$

Thus $f'(p) = 0$ for all $p \in U$.
(c):
We know that X is convex because it is a vector space. Suppose that $f'' = 0$. Then by part (b), f' is a constant function. We will denote $f' = L \in B(X, Y)$. We claim that $M = L$ and $b = f(0)$, that is, $f(p) = L(p) + f(0)$ for all $p \in U$.
Define the function $g : U \to \mathbb{R}^m$ by $g(p) = f(p) - f(0) - L(p)$. For all $p \in U$, we have that

$$\lim_{h\to 0} \frac{\|g(p+h) - g(p)\|}{\|h\|} = \lim_{h\to 0} \frac{\|(f(p+h) - f(0) - L(p+h)) - (f(p) - f(0) - L(p))}{\|h\|}$$

$$= \lim_{h\to 0} \frac{\|f(p+h) - f(p) - L(h)\|}{\|h\|} = 0$$

where the last step is from the fact that $f'(p) = L$. We can see that $g'(p) = 0$ for all $p \in U$. From part (b), it follows that g is a constant function. Finally, notice that

$$g(0) = f(0) - f(0) - L(0) = 0,$$

so $g = 0$. Therefore for all $p \in U$,

$$f(p) - f(0) - L(p) = 0 \implies f(p) = L(p) + f(0)$$

as desired.
Conversely, suppose that f is affine-linear. Then for some $M \in B(X, \mathbb{R}^m)$, $b \in \mathbb{R}^m$,

$$f(p) = M(p) + b.$$

We claim that $f'(p) = M$ using the definition. Immediately, we have that

$$\lim_{h\to 0} \frac{\|(M(p+h) + b) - (M(p) + b) - M(h)\|}{\|h\|} = 0$$

which verifies our claim. Since $f'(p) = M$ for all $p \in U$, f' is constant, and from part (b), this implies that $f'' = 0$, which completes the proof.

$\square$