Use the general Principle of Inclusion-Exclusion to obtain a formula involving a summation for the number of ways to place n identical balls into k labeled bins so that no bin is left empty, where n and k are positive integers. Note that you may need to consider different possible cases for n, k as necessary; make sure you show all your steps.

Proof. Fix the numbers $n, k \in \mathbb{N}$.
Label the bins from 1 to k. Let A_i be the ways to place the n balls into the k bins such that bin i has no balls. We want to count

$$\left| A_1^C \cap \cdots \cap A_n^C \right|$$

using PIE. We first find the number of ways the place the balls into the bins so that we guarantee that certain bins are empty. Say that we want to avoid placing any balls into j bins that we have already determined beforehand. The question reduces to placing the n balls into $k - j$ bins that were not chosen. We use the standard balls and bins formula to get that number is

$$\binom{n + k - j - 1}{n}$$

which applies that any choice of j bins to keep empty. This means that for any subset $S \subseteq \{1, ..., k\}$ we have that

$$\left| \bigcap_{i \in S} A_i \right| = \binom{n + k - |S| - 1}{n}$$

As well, we can count the universe U and get that

$$|U| = \binom{n + k - 1}{n}$$

Thus, by PIE, we have that

$$\left| A_1^C \cap \cdots \cap A_n^C \right| = |U| - \sum_{S \subseteq \{1, ..., k\}} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

$$= \binom{n + k - 1}{n} - \sum_{S \subseteq \{1, ..., k\}} (-1)^{|S|} \binom{n + k - |S| - 1}{n}$$

For every $i \in \{1, ..., k\}$, there are $\binom{k}{i}$ sets with cardinality i, so we can simplify the sum to get

$$\binom{n + k - 1}{n} - \sum_{i=1}^{k} (-1)^i \binom{k}{i} \binom{n + k - i - 1}{n}$$

$\square$