

Question 36.

In Handout #7, we defined differentiability for functions on open sets. Now we give a definition that works over arbitrary sets. [For this problem, you will need to read Piazza Post #274 and use Theorem 1.1.](#)

Let $A \subseteq \mathbf{R}^n$ be an arbitrary set, let $f : A \rightarrow \mathbf{R}$ be a function, and let $p \in A$ be a point. We say that f is **differentiable** at p if there exists an open neighborhood U of p and a function $\hat{f} : U \rightarrow \mathbf{R}$ such that \hat{f} is differentiable at p (in the sense of Handout #7) and $\hat{f}|_{U \cap A} = f|_{U \cap A}$.

- Prove that f is differentiable at every point of A if and only if f extends to a differentiable function defined on an open set containing A .
- Suppose further that A is closed. Prove that f is differentiable at every point of A if and only if f extends to a differentiable function on \mathbf{R}^n .

Question 37.

The following set is called the n -**simplex**:

$$\Delta_n := \{\vec{x} = (x_1, \dots, x_n) \in \mathbf{R}^n : x_1, \dots, x_n \geq 0 \text{ and } x_1 + \dots + x_n \leq 1\}.$$

You can assume, without proof, that Δ_n is Jordan measurable.

Find, with proof, an explicit formula for $\mu(\Delta_n)$ in terms of n .

Proof. First, we show that Δ_n is the same as the set

$$S = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1, \dots, 0 \leq x_n \leq 1 - \sum_{i=1}^{n-1} x_i \right\}$$

Let $x \in \Delta_n$. We want to show that $0 \leq x_i \leq 1 - \sum_{j=1}^{i-1} x_j$. We get that $x_i \geq 0$ immediately. As well, since $\sum_{j=1}^n x_j \leq 1$ and every component is non-negative, we have that

$$x_i \leq 1 - \sum_{j=1}^{i-1} x_j - \sum_{j=i+1}^n x_j \leq 1 - \sum_{j=1}^{i-1} x_j$$

which shows that $\Delta_n \subseteq S$.

Now, let $x \in S$. We know every x_i is non-negative and additionally

$$x_n \leq 1 - \sum_{i=1}^{n-1} x_i \implies \sum_{i=1}^n x_i \leq 1$$

so $S \subseteq \Delta_n$.

Now, we proceed to find $\mu(S) = \mu(\Delta_n)$. Using Fubini's Theorem, we get

$$\mu(S) = \int_S 1 = \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 \, dx_n \dots dx_2 \, dx_1$$

