

Assignment 3: Due March 16, 2025 (Sunday) at 11:59PM ET

Answers must be submitted via Crowdmark using your personalized submission link (which will be sent to your email). Late submissions will not be accepted.

If you used Octave to assist you in solving a problem, also attach to the solution to that problem the script and commands you used and the corresponding outputs obtained.

A proper selection of questions from the ones below will be graded.

Question 1.

(5.1, #18) The Bézier curve

$$x(t) = 11t^3 - 18t^2 + 3t + 5$$

$$y(t) = t^3 + 1$$

has control points $(5, 1)$, $(6, 1)$, and $(1, 2)$. Find the fourth control point.

Notice that $(x(0), y(0)) = (5, 1)$ and $(x(1), y(1)) = (1, 2)$, so $(5, 1), (1, 2)$ are the first and last control points. Let us suppose that (a, b) is the missing control point, and it is ordered second.

Question 2.

Sketch a stencil for the following data, picking appropriate x_0 and h , that can be used to approximate $f'(1)$. You do not need to derive the formula.

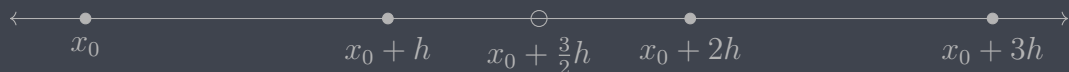
x	-3	-1.5	0	2
$f(x)$	3.11	3.05	3.4	3.7

We let $x_0 = -3$, $h = 1$. Then the stencil looks like this:



Question 3.

(4.1, #13c) Derive an approximation formula for the second derivative f'' over the stencil:



using a direct method as discussed in Section 4.1.

We begin by finding the interpolating polynomial of least degree for this stencil. Using

Lagrange's interpolation formula:

$$\begin{aligned}
L(x_0 + \theta h) &= \frac{(\theta - 1)(\theta - 2)(\theta - 3)}{(-1)(-2)(-3)}f(x_0) + \frac{\theta(\theta - 2)(\theta - 3)}{(1)(-1)(-2)}f(x_0 + h) \\
&\quad + \frac{\theta(\theta - 1)(\theta - 3)}{(2)(1)(-1)}f(x_0 + 2h) + \frac{\theta(\theta - 1)(\theta - 2)}{(3)(2)(1)}f(x_0 + 3h) \\
&= -\frac{\theta^3 - 6\theta^2 + 11\theta - 6}{6}f(x_0) + \frac{\theta^3 - 5\theta^2 + 6\theta}{2}f(x_0 + h) \\
&\quad - \frac{\theta^3 - 4\theta^2 + 3\theta}{2}f(x_0 + 2h) + \frac{\theta^3 - 3\theta^2 + 2\theta}{6}f(x_0 + 3h)
\end{aligned}$$

We take the derivative twice with respect to θ to get that

$$\begin{aligned}
h^2 L''(x_0 + \theta h) &= -\frac{6\theta - 12}{6}f(x_0) + \frac{6\theta - 10}{2}f(x_0 + h) - \frac{6\theta - 8}{2}f(x_0 + 2h) + \frac{6\theta - 6}{6}f(x_0 + 3h) \\
&= (2 - \theta)f(x_0) + (3\theta - 5)f(x_0 + h) + (4 - 3\theta)f(x_0 + 2h) + (\theta - 1)f(x_0 + 3h)
\end{aligned}$$

Substituting $\theta = \frac{3}{2}$ gives us the required approximation formula:

$$f''\left(x_0 + \frac{3}{2}h\right) \approx \frac{1}{h^2} \left(\frac{1}{2}f(x_0) - \frac{1}{2}f(x_0 + h) - \frac{1}{2}f(x_0 + 2h) + \frac{1}{2}f(x_0 + 3h) \right)$$

Question 4.

(4.2, #3c) Over the same stencil as in the previous question, derive an approximation formula for the second derivative f'' but now using undetermined coefficients.

Let our formula be of the form

$$a_0 f(x_0) + a_1 f(x_0 + h) + a_2 f(x_0 + 2h) + a_3 f(x_0 + 3h).$$

For $i = 0, 1, 2, 3$, the formula is exact for functions $f_i(x) = (x - x_0)^i$. Thus using each function, we have the system of equations

$$\begin{aligned}
0 &= a_0 + a_1 + a_2 + a_3 \\
0 &= ha_1 + 2ha_2 + 3ha_3 \\
2 &= h^2a_1 + 4h^2a_2 + 9h^2a_3 \\
9h &= h^3a_1 + 8h^3a_2 + 27h^3a_3
\end{aligned}$$

which we can solve by row reducing the matrix

$$\begin{aligned} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & h & 2h & 3h & 0 \\ 0 & h^2 & 4h^2 & 9h^2 & 2 \\ 0 & h^3 & 8h^3 & 27h^3 & 9h \end{array} \right) &\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 2h^2 & 6h^2 & 2 \\ 0 & 0 & 6h^2 & 24h^2 & 9 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 2h^2 & 6h^2 & 2 \\ 0 & 0 & 0 & 6h^2 & 3 \end{array} \right) \\ &\rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2h^2} \\ 0 & 0 & 0 & 1 & \frac{1}{2h^2} \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2h^2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2h^2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2h^2} \\ 0 & 0 & 0 & 1 & \frac{1}{2h^2} \end{array} \right) \end{aligned}$$

Therefore our approximating formula for $f''(x_0 + \frac{3}{2}h)$ is

$$f''\left(x_0 + \frac{3}{2}h\right) \approx \frac{1}{2h^2}f(x_0) - \frac{1}{2h^2}f(x_0 + h) - \frac{1}{2h^2}f(x_0 + 2h) + \frac{1}{2h^2}f(x_0 + 3h)$$

which is consistent with the formula we obtained in the previous question.

Question 5.

(Based on 4.1, #19) Let $x_0, x_0 + \alpha h, x_0 + 2h$ be nodes where $\alpha \neq 0, 2$.

- a. Write the first Taylor polynomials of $f(x_0 + \alpha h)$ and $f(x_0 + 2h)$ around $x = x_0$, then use these to show that:

$$f'(x_0) = \frac{1}{2h} \left[-\frac{2+\alpha}{\alpha}f(x_0) + \frac{4}{\alpha(2-\alpha)}f(x_0 + \alpha h) - \frac{\alpha}{2-\alpha}f(x_0 + 2h) \right].$$

Assuming f is analytic, the Taylor series expansion of f around $x = x_0$ is

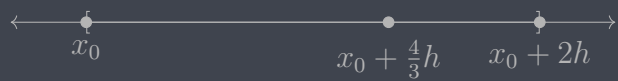
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

so

$$\begin{aligned} f(x_0 + \alpha h) &\approx f(x_0) + f'(x_0)\alpha h \text{ and } f(x_0 + 2h) \approx f(x_0) + f'(x_0)2h \\ \implies f'(x_0) &\approx \frac{1}{\alpha h}(f(x_0 + \alpha h) - f(x_0)) \text{ and } f'(x_0) \approx \frac{1}{2h}(f(x_0 + 2h) - f(x_0)) \end{aligned}$$

Question 6.

(4.2, #4i) Use undetermined coefficients to derive an approximation formula for the integral over the stencil:



First