Exercise 12.8

Solvers: Ethan, Ali, Mitya
Writeup: Ethan
Lemma 12.4. Let P be a polybox in $\mathbb{R}^n$. Then

1. P is a finite disjoint union of boxes.

2. If $\{E_1, ..., E_k\}$ and $\{F_1, ..., F_m\}$ are two pairwise disjoint collections of boxes, and $\bigcup_{i=1}^k E_i = \bigcup_{j=1}^m F_j$, then
$$\sum_{i=1}^k \mathrm{vol}(E_i) = \sum_{j=1}^m \mathrm{vol}(F_j)$$

Proof. We start with a proof of part (a). Since P is a polybox, we can write it as a finite union of boxes not necessarily disjoint. We have
$$P = \bigcup_{i=1}^p P_i, \text{ where } P_i \text{ is a box in } \mathbb{R}^n$$
We proceed using induction on p to prove (a). When $p = 1$, P itself is a box and is considered a finite union of disjoint boxes.

Suppose that the claim holds for some $p - 1$. We can rewrite P as
$$P = P_p \cup \bigcup_{i=1}^{p-1} P_i$$
From our assumption, the polybox $\bigcup_{i=1}^{p-1} P_i$ can be expressed as a finite union of disjoint boxes, which we will denote as B_i, so we have
$$P = P_p \cup \bigcup_{i=1}^{q} B_i = \bigcup_{i=1}^{q} (B_i \cup P_p)$$
Consider each pair of boxes B_i and P_p. By Lemma 12.2,

1. $P_p \cap B_i$ is a box

2. $P_p \setminus B_i$ and $B_i \setminus P_p$ are finite unions of disjoint boxes.

Additionally, all these sets are disjoint from each other. We can take the union of all the sets above and see that $(P_p \cap B_i) \cup (P_p \setminus B_i) \cup (B_i \setminus P_p) = B_i \cup P_p$ is also a finite union of disjoint boxes. Thus P is a finite union of disjoint boxes.

By induction, part (a) has been proven and we move on to prove part (b).

Let $\{E_1, ..., E_k\}, \{F_1, ..., F_m\}$ be pairwise disjoint collections of boxes such that
$$P = \bigcup_{i=1}^k E_i = \bigcup_{j=1}^m F_j$$
For $i \in \{1, ..., k\}, j \in \{1, ..., m\}$, define $A_{ij} = E_i \cap F_j$. We will show that $A \coloneqq \bigcup_{i=1}^k \bigcup_{j=1}^m A_{ij} = P$. It is clear that $A \subseteq P$, so only $P \subseteq A$ will be shown.

Let $x \in P$. Then x must belong to some boxes E_i and F_j, so $x \in A_{ij} \subseteq A$. This shows that A is also a valid disjoint decomposition of P.

Finally, notice that for all E_i, F_j,
$$E_i = \bigcup_{j=1}^m A_{ij} \text{ and } F_j = \bigcup_{i=1}^k A_{ij}$$

By part (b) of Lemma 12.2, we have

$$\mathrm{vol}(E_i) = \sum_{j=1}^{m} \mathrm{vol}(A_{ij}) \text{ and } \mathrm{vol}(F_j) = \sum_{i=1}^{k} \mathrm{vol}(A_{ij})$$

Therefore

$$\sum_{i=1}^{k} \mathrm{vol}(E_i) = \sum_{i=1}^{k}\sum_{j=1}^{m} \mathrm{vol}(A_{ij}) = \sum_{j=1}^{m}\sum_{i=1}^{k} \mathrm{vol}(A_{ij}) = \sum_{j=1}^{m} \mathrm{vol}(F_j)$$

what was what we wanted.

$\square$