Question 40.

Let $O_n(\mathbf{R})$ be the set of all $n \times n$ real orthogonal matrices:

$$O_n(\mathbf{R}) = \{A \in M_n(\mathbf{R}) : A^t A = I_n\}.$$

Show that O_n is a smooth manifold, and find its dimension.

Proof. First, we note that $O_n(\mathbb{R})$ is the zero set of the function $f : M_n(\mathbb{R}) \to S^n$ defined by

$$f(A) = A^t A - I_n$$

where S^n is the set of symmetric $n \times n$ matrices. Notice that f is smooth as it is constructed by smooth functions. Additionally, we show that $Jf(X)(h) = X^t h + h^T X$. Indeed,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(X+h) - f(X) - X^t h - h^t X}{\|h\|} &= \lim_{h\to 0} \frac{(X+h)^t(X+h) - X^t X - X^t h - h^t X}{\|h\|} \\
&= \lim_{h\to 0} \frac{h^t h}{\|h\|} \\
&= 0
\end{aligned}$$

Next, we want to show that $\operatorname{rank} Jf(X) = \frac{1}{2}n(n+1)$ for all $X \in O_n(\mathbb{R})$. It suffices to show that $Jf(X)$ is surjective to S^n.
Let $Y \in S^n$. Let $h = \frac{1}{2}XY$. We see that

$$\begin{aligned}
Jf(X)(h) = X^t\left(\frac{1}{2}XY\right) + \left(\frac{1}{2}XY\right)^t X &= \frac{1}{2}\left(X^t XY + Y^t X^t X\right) \\
&= \frac{1}{2}(Y + Y^t) && (X \text{ is orthogonal}) \\
&= Y && (Y \text{ is symmetric})
\end{aligned}$$

Thus $R(Jf(X)) = S^n$ so $\operatorname{rank} Jf(X) = \dim S^n = \frac{1}{2}n(n+1)$.
We now prove that $O_n(\mathbb{R})$ is a smooth manifold of dimension $\frac{1}{2}n(n+1)$. Let $p \in O_n(\mathbb{R})$. Then $f(p) = 0$ and $Jf(p)$ has the maximal rank of $\frac{1}{2}n(n+1)$. We write

$$Jf(p) = \big(A \mid B\big)$$

where A is a $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n-1)$ matrix and B is a $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ matrix and assume without loss of generality that B is an invertible submatrix of $Jf(p)$. We can do this because we can swap the components of f, and therefore columns of $Jf(p)$ without affecting the conclusion of the statement (because manifolds are invariant under diffeomorphims). Thus, we write $p = (a, b)$ for $a \in \mathbb{R}^{\frac{1}{2}n(n-1)}$, $b \in \mathbb{R}^{\frac{1}{2}n(n+1)}$ and apply the Implicit Function Theorem and obtain an open set $\hat{U} \subseteq \mathbb{R}^{\frac{1}{2}n(n-1)}$ containing a and a C^∞ function $\Phi : \hat{U} \to \mathbb{R}^{\frac{1}{2}n(n+1)}$ so that

$$f(x, \Phi(x)) = 0$$

for all $x \in \hat{U}$. We claim that $\varphi : \hat{U} \to \Phi(\hat{U})$ defined by

$$\varphi(x) = (x, \Phi(x))$$

is our desired smooth regular embedding. It is fairly clear that φ is smooth. Additionally, $J\varphi(x) = \begin{pmatrix} I_{\frac{1}{2}n(n-1)} \\ \hline J\Phi \end{pmatrix}$ is a $n^2 \times \frac{1}{2}n(n-1)$ matrix and has at least $\frac{1}{2}n(n-1)$ linearly independent rows, so φ is regular. Finally, if we let $\varphi(x) = \varphi(y)$, we have that $(x, \Phi(x)) = (y, \Phi(y))$, from which we get $x = y$, so φ is injective, and therefore bijective to its image. In addition, we can explicitly find $\varphi^{-1}(y) = \pi_{\mathbb{R}^{\frac{1}{2}n(n-1)}}(y)$, which is continuous because it is a linear map. Therefore φ is a homeomorphism onto its image, and we are done.

$\square$