Question 42.

Let $M \subseteq \mathbf{R}^N$ be a smooth n-manifold (with or without boundary!).

(a) Show that if $n < N$, then M is a *Lebesgue null set.*

(b) Show that if $n = N$ and M is closed and its boundary is nonempty, then ∂M coincides with the usual topological boundary (as defined on Handout #2).

(c) Show that if M is compact and its boundary is nonempty, then M is Jordan measurable.

Proof.
(a):
We begin by proving a number of lemmas:
Lemma 1: An open cover of any subset $M \subseteq \mathbb{R}^n$ has a countable subcover.
We know that $\mathbb{R}^n$ is separable, so M is also separable. Let C be a countable dense subset of M. Let $\mathcal{U}$ be an open cover for M. We construct the countable subcover $\hat{U}$ as follows. For each $q \in C$ and $k \in \mathbb{Q}$, consider the open ball $B(q, k)$. If there exists a $U_{qk} \in \mathcal{U}$ such that $B(q, k) \in U_{qk}$, include it in $\hat{U}$. Notice that $\hat{U}$ is at most countable. We claim that it is also an open cover.
Let $x \in M$. Then it is contained in some open set $U \in \mathcal{U}$. As well, we can find an open ball such that $B(x, \delta) \in U$. Since C is dense, we can find $q \in C$ such that $q \in B(x, \frac{\delta}{4})$. Let $k \in \mathbb{Q}$ such that $\frac{\delta}{4} < k < \frac{\delta}{2}$. Then $x \in B(q, k) \subseteq B(x, \delta)$, because for all $y \in B(q, k)$,

$$\|x - y\| \leq \|x - q\| + \|q - y\| < \frac{\delta}{4} + \frac{\delta}{2} < \delta$$

It follows that $B(q, k) \in U$, so it is guaranteed that some U_{qk} from our construction exists. Thus $x \in U_{qk} \in U$ so U is indeed an open cover and we are done.

Lemma 2: A countable union of sets with Jordan measure 0 is a Lebesgue null set.
Let $E = \bigcup_{i \geq 1} E_i$, where $\mu(E_i) = 0$. Let $\varepsilon > 0$. For each E_i, we can find a finite union of boxes B_i such that $B_i \supseteq E_i$ and $\text{vol}(B_i) < \frac{\varepsilon}{2^i}$. We see that $\bigcup_{i \geq 1} B_i$ is a countable union of boxes, $E \subseteq \bigcup_{i \geq 1} B_i$, and

$$\sum_{i=1}^{\infty} \text{vol}(B_i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2(1 - \frac{1}{2})} = \varepsilon$$

as desired.

Lemma 3: If K is a Jordan measurable set with a compact exhaustion K_n, then $\mu(K) = \lim_{n \to \infty} \mu(K_n)$.
Let $\varepsilon > 0$. Since K is Jordan measurable, we can find a closed polybox $I \subseteq K$ such that

$$\mu(I) > \mu(K) - \varepsilon$$

Notice that I is covered by $\{K_n\}_{n \in \mathbb{N}}$ and is compact, so there exists $N \in \mathbb{N}$ such that $I \subseteq K_n$ for all $n > N$. Thus

$$\mu(K_n) > \mu(I) > \mu(K) - \varepsilon \implies |\mu(K) - \mu(K_n)| < \varepsilon$$

as needed.

Now, we prove the problem at hand. Let $M \subseteq \mathbb{R}^N$ be a smooth n-manifold with $n < N$. Let $\{(U_i, \varphi)\}_{i \in I}$ be an atlas for M. By Lemma 1, we can assume without loss of generality that the atlas is countable. We can also assume that each U_i is bounded, for if not, we can take a countable union of open balls that cover the unbounded U_i, and restrict the embedding to each ball.
For each chart (U_i, φ_i), assume that the domain $\hat{U}$ is $\mathbb{R}^n$ or $\overline{\mathbb{H}^n}$, depending on if the chart includes the manifold boundary. Regardless, we can take the compact exhaustion $K_n = B(0, n)$ and see that $\varphi_i(K_n)$ is a compact exhaustion for U_i. Additionally, since φ_i maps from lower dimension to higher dimension, we know from a previous Big List question that $\mu(\varphi_i(K_n)) = 0$, so by Lemma 3, $\mu(U_i) = 0$. This is true for each chart U_i, so we apply Lemma 2 to conclude that $\cup_{i \geq 1} U_i = M$ is a Lebesgue null set.

(b):
Let p be a point in the topological boundary of M. We will show that $p \in \partial M$. Suppose for contradiction that p is in M°, meaning it is contained in a chart (U, φ) that is diffeomorphic to an open set $\hat{U}$ in $\mathbb{R}^N$. Note that φ is a diffeomorphism with domain $\hat{U}$ and codomain U. Then it must be true that $\varphi(\hat{U}) = U$ is an open subset of $\mathbb{R}^N$. But this is a contradiction, as that would imply that the boundary point p is in the interior of M. Therefore $p \in \partial M$.
Next, let $p \in \partial M$ and again suppose for contradiction that p is not in the topological boundary of M. Then it must be true that p is in the topological interior of M. Recall that $p \in \partial M$ implies that it is contained in a chart (U, φ) that is diffeomorphic to $\overline{\mathbb{H}^n}$ and $p \in \mathrm{bd}(\mathbb{H}^n)$. Since p is in the topological interior of M, we can find an open ball $B(p, r) \subseteq M$ which is also open in $\mathbb{R}^N$. Then $\varphi^{-1}(B(p, r))$ should also be open in $\mathbb{R}^N$. But this implies that for small enough δ, $\varphi(p) - (0, ..., \delta) \in \overline{\mathbb{H}^N}$, which cannot happen.
Therefore we can conclude that ∂M coincides with the topological boundary of M.

(c):
First, we prove that a compact Lebesgue null set E has Jordan measure 0. It suffices to show that the upper measure $\mu^*(E) = 0$.
Let $\varepsilon > 0$. By definition, we can find a countable union of boxes $B = \bigcup_{i=1}^\infty B_i$ such that $E \subseteq B$ and $\mathrm{vol}(B) < \varepsilon$. But since E is compact, it can be covered by finitely many boxes B_{n_i}, $0 < i \leq N$. Thus

$$\mathrm{vol}\left(\bigcup_{i=1}^N B_{n_i}\right) = \sum_{i=1}^N \mathrm{vol}(B_{n_i}) \leq \mathrm{vol}(B) < \varepsilon.$$

Since ε was chosen arbitrarily, we can conclude that $\mu^*(E) = 0$, and E has Jordan measure 0.
Now, suppose that M is compact and its boundary is nonempty. If $\dim M < N$, M is a Lebesgue null set and has Jordan measure 0, and therefore measurable. Otherwise, if $\dim M = N$, since the boundary of M is non-empty, the topological boundary of M is actually a smooth manifold of dimension $(N-1)$, and therefore a Lebesgue null set. As well, the topological boundary of M is compact, so it is Jordan measure 0, which implies that M is Jordan measurable.

$\square$

Question 34.

Let $\Phi : \mathbf{R}^n \to \mathbf{R}^m$ be a C^1 mapping.

(a) Suppose that $n > m = 1$. Show that Φ cannot be injective.

(b) Suppose that $n < m$. Show that if $K \subseteq \mathbf{R}^n$ is a compact set, then $\Phi(K)$ is a Jordan measurable set, and has Jordan measure zero.

Proof.
(a):
Suppose for contradiction that $n > m = 1$ and Φ is a C^1 injective function. Since Φ cannot be a constant function, by the results of Big List #26, there is a $p \in \mathbb{R}^n$ so that $\nabla\Phi(p) \neq 0$. In particular, we will say that $\dfrac{\partial \Phi}{\partial x_j} \neq 0$. Define $\alpha : \mathbb{R}^n \to \mathbb{R}$ by $\alpha(x) = \Phi(x) - \Phi(p)$. Injectivity is translation-invariant, so α is injective. Notice that $\alpha(p) = 0$. We can apply the implicit function theorem to obtain an open set $W \subseteq \mathbb{R}^{n-1}$ that contains $p' = (p_1, ..., p_{j-1}, p_{j+1}, ..., p_n)$ and a C^1 function $\Psi : W \to \mathbb{R}$ such that for all $x = (x_1, ..., x_{n-1}) \in W$,

$$\alpha(x_1, ..., x_{j-1}, \Psi(x), x_j, ..., x_{n-1}) = 0$$

Then, since W is open and contains p', we can find another distinct point $q \in W$. We have

$$\alpha(p_1, ..., p_{j-1}, \Psi(p'), p_{j+1}, ..., p_n) = 0 = \alpha(q_1, ..., q_{j-1}, \Psi(q), q_j, ..., q_{n-1})$$

which contradicts the fact that α is injective.

(b):
Since K is compact, and thus bounded, we can enclose it in a closed box $B = [-L, L]^n$, for some positive L. It suffices to show that $\Phi(B)$ has measure 0, as we can apply the monotonicity of measure to conclude that $\Phi(K)$ has measure 0.
First, let $\hat{\Phi} : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $\hat{\Phi}(x) = \Phi(\pi_{\mathbb{R}^n}(x))$. That is, $\hat{\Phi}$ first projects vectors in $\mathbb{R}^m$ onto $\mathbb{R}^n$ and then composes it with Φ. Let $\hat{B} = B \times \{0\}^{m-n}$. Then note that $\hat{\Phi}(\hat{B}) = \Phi(B)$. Trivially, $\hat{B}$ has measure 0. We now show that $\Phi(B)$ also has measure 0.
Let $\varepsilon > 0$. Since $\hat{\Phi}$ is C^1, its derivative is continuous. By the extreme value theorem, each component derivative attains a maximum on $\hat{B}$. Let α a positive number greater than all the maximums. Since $\hat{B}$ has measure 0, we can find a finite number of cubes $B_1, ..., B_k$ with side length d such that

$$\hat{B} \subseteq \bigcup_{i=1}^{k} B_i \text{ and } \sum_{i=1}^{k} \operatorname{vol}(B_i) < \frac{\varepsilon}{m^m \alpha^m}$$

Consider some cube $B_i = \prod_{j=1}^{m}[a_{ij}, a_{ij} + d]$. For each component $\hat{\Phi}_j : \mathbb{R}^m \to \mathbb{R}$, it must be true that $\hat{\Phi}_j$ attains a maximum $\hat{\Phi}_j(M_{ij})$ and minimum $\hat{\Phi}_{ij}(m_{ij})$, for some $M_{ij}, m_{ij} \in B_i$. Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \hat{\Phi}_j(tM_{ij} + (1-t)m_{ij})$$

By the Mean Value Theorem, there exists some $c \in (0,1)$ such that

$$g(1) - g(0) = g'(c) \implies \hat{\Phi}_j(M) - \hat{\Phi}_j(m) = \hat{\Phi}_j'(cM_{ij} + (1-c)m_{ij})(M_{ij} - m_{ij})$$

Taking the 2-norm of both sides, it follows that

$$\hat{\Phi}_j(M_{ij}) - \hat{\Phi}_j(m_{ij}) \leq \|\hat{\Phi}_j'(cM_{ij} + (1-c)m_{ij})\|\|M_{ij} - m_{ij}\| \leq (\sqrt{m}\alpha)(\sqrt{m}d) = m\alpha d$$

Note that we used the fact that each component of $\hat{\Phi}_j'(cM_{ij} + (1-c)m_{ij})$ is bounded by α and $\|M_{ij} - m_{ij}\|$ can be no bigger than the diagonal of B_i.
Finally, we see that the box $C_i = \prod_{j=1}^{m}[\hat{\Phi}_j(m_{ij}), \hat{\Phi}_j(M_{ij})]$ covers $\hat{\Phi}(B_i)$, and its volume is given by

$$\prod_{j=1}^{m}(\hat{\Phi}_j(M_{ij}) - \hat{\Phi}_j(m_{ij})) \leq \prod_{j=1}^{m} m\alpha d = m^m\alpha^m \text{vol}(B_i)$$

From here, we know that the union $\bigcup_{i=1}^{k} C_i \supseteq \hat{\Phi}(\hat{B})$ but

$$\sum_{i=1}^{k} C_i < m^m\alpha^m \sum_{i=1}^{k} \text{vol}(B_i) < \varepsilon.$$

Therefore $\Phi(B)$ has measure 0, so $\Phi(K)$ has measure 0 and we are done.

$\square$