6 HOMEWORK 6 HAAHHAHAHAHAHAA

Question 19.

Let X be a metric space and let $A \subseteq X$. A **compact exhaustion** for A is a sequence of compact sets $K_1, K_2, K_3, \ldots$ such that $U = \bigcup_{i \geq 1} K_i$ and $K_i \subseteq K_{i+1}^{\circ}$.

(a) Let $U \subseteq \mathbf{R}^n$ be a bounded open set. Show that U has a compact exhaustion.

(b) Now show that every open set $U \subseteq \mathbf{R}^n$ has a compact exhaustion.

Proof. (a):
Let $U \subseteq \mathbb{R}^n$ be a bounded open set. Let N be smallest natural number such that the set $S = \{x \in U : \operatorname{dist}(\{x\}, U^c) \geq \frac{1}{N}\}$ is non-empty. Define the sequence of sets $(S_n)_{n \geq 1}$ by

$$S_n = \left\{ x \in U : \operatorname{dist}(\{x\}, U^c) \geq \frac{1}{n+N} \right\}$$

We will show that S_n is closed. Let s be limit point of S_n. Then for $y \in U^c$,

$$d(s,y) \geq d(x,y) - d(x,s) \implies d(s,y) \geq \sup_{x \in S_n} (d(x,y) - d(x,s))$$

We will show that $\sup_{x \in S_n} (d(x,y) - d(x,s)) = \frac{1}{N+n}$. For all $\varepsilon > 0$, there is an $x \in S_n$ such that $d(x,s) < \varepsilon$. Thus

$$d(x,y) - d(x,s) > \frac{1}{N+n} - \varepsilon$$

which means that $\sup_{x \in S_n} (d(x,y) - d(x,s)) = \frac{1}{N+n}$. Therefore we have that

$$d(s,y) \geq \frac{1}{N+n} \implies s \in S_n$$

so it follows that S_n is closed. Since $S_n \subseteq U$, it is also bounded, so because we are working in $\mathbb{R}^n$, S_n is compact.
As well, we need to have that $S_n \subseteq S_{n+1}^{\circ}$. This result is quite quick, as we can notice that

$$S_{n+1}^{\circ} = \left\{ x \in U : \operatorname{dist}(\{x\}, U^c) > \frac{1}{N+n} \right\}$$

From this, the inclusion follows nicely.
Then, as n tends to infinity, $S_n = \{x \in U : \operatorname{dist}(\{x\}, U^c) > 0\}$ which is exactly $(U^c)^c = U$. Therefore U has compact exhaustion.
(b):
Now, let $U \subseteq \mathbb{R}^n$ be open. Define the sequence of bounded open sets $A_n = U \cap B(\vec{0}, n)$. As n tends to infinity, $\bigcup_{n=1}^{\infty} A_n = U$. By the previous part, A_n has a compact exhaustion $(K_{nk})_{k \geq 1}$. Take the sequence $(K_k)_{k \geq 1}$ to be $K_k = \bigcup_{n=1}^{\infty} K_{nk}$. This sequence of sets satisfies the conditions for a compact exhaustion. Now we attempt to prove that the sequence converges to U. We have

$$\bigcup_{k=1}^{\infty} K_k = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} K_{nk}$$

We swap the order of union and get that

$$\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty}K_{nk}=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}K_{nk}=\bigcup_{n=1}^{\infty}A_n=U$$

Thus every open subset of $\mathbb{R}^n$ has a compact exhaustion.

$\square$

Question 20.

Let $x, y \in \ell^\infty$ be two sequences. Let us say that y is **dominated** by x, denoted $x \geq y$, if $|x_n| \geq |y_n|$ for all $n \in \mathbf{N}$. Let D_x denote the set of all sequences which are dominated by x:

$$D_x = \{y \in \ell^\infty : |y_n| \leq |x_n| \text{ for all } n \in \mathbf{N}\}.$$

Prove that D_x is compact if and only if $x_n \to 0$.

Proof. Suppose that D_x is compact. Suppose for contradiction that $x_n \nrightarrow 0$. For some $\varepsilon > 0$, $|x_{N_k}| \geq \varepsilon$ for an infinite number of N_k. Consider the open cover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \in I}$, which is the collection of $\frac{\varepsilon}{2}$-balls centered around every $\vec{y}_i \in D_x$. By compactness of D_x, there is a finite subcover $\{B(\vec{y}_i, \frac{\varepsilon}{2})\}_{i \leq m}$. Now, we construct a $y \in D_x$ as follows:
For every sequence $\vec{y}_i$, let

$$y_{N_i} = \begin{cases} \varepsilon, & \text{if } (\vec{y}_i)_{N_i} < \dfrac{\varepsilon}{2}; \\ 0, & \text{if } (\vec{y}_i)_{N_i} \geq \dfrac{\varepsilon}{2}; \end{cases}$$

For all other terms in y, make it 0. Notice that for all $B(\vec{y}_i, \frac{\varepsilon}{2})$,

$$\|y - \vec{y}_i\|_\infty \geq |y_{N_i} - (\vec{y}_i)_{N_i}| \geq \frac{\varepsilon}{2} \implies y \notin D_x$$

which is a contradiction.
Conversely, suppose that $x_n \to 0$. We will show that D_x is complete and totally bounded, which is equivalent to compactness.
To show completeness, notice that the ambient space ℓ^∞ is complete. Thus if we can show that D_x is closed, it will follow that D_x is complete.
Let $a \notin D_x$. We will show that a is not a limit point of D_x, which means that D_x is closed. We know that there is a $k \in \mathbb{N}$ such that $|a_k| > |x_k|$. Define $\varepsilon = |a_k| - |x_k|$. Fix $y \in D_x$. Then

$$\|a_k - y_k\|_\infty \geq |a_k - y_k| \geq |a_k| - |y_k| \geq |a_k| - |x_k| = \varepsilon$$

which implies that $y \notin B(a_k, \varepsilon)$, so a is not a limit point of D_x. Thus D_x is closed. It follows that D_x is complete.
To show that D_x is totally bounded, first let $\varepsilon > 0$. Since $x_n \to 0$, there is a large enough N such that for $n > N$, $|x_n| < \frac{\varepsilon}{2}$. For all $y \in D_x$, $|y_n| < |x_n| < \frac{\varepsilon}{2}$. Consider the set of elements in D_x such that their terms are 0 for $n > N$. This set is totally bounded, so there is a ε-ball cover $\{B(y_i, \varepsilon)\}_{i \leq n}$. We show that this collection also covers D_x.
For $y \in D_x$, there is an open ball $B(y_i, \varepsilon)$ such that $\sup_{n \leq N} |y_n - (y_i)_n| < \varepsilon$. But also notice that for $n > N$, $|(y_i)_n - y_n| \leq |(y_i)_n| + |y_n| < |y_n| < \frac{\varepsilon}{2}$. Thus $\|y - y_i\|_\infty < \varepsilon$ so $y \in B(y_i, \varepsilon)$. Thus D_x is totally bounded.
Since D_x is both complete and totally bounded, we can conclude that D_x is compact.

□