Question 1.

(Based on 2.2, #24) Let $g(x) = \left(\frac{1}{2}\right)^x + \left(\frac{1}{5}\right)^x - 10^{-5}$.

a. Show that if g has a zero at p, then the function $f(x) = x + cg(x)$ has a fixed point at p.

   Suppose that g has a zero at p. Then $g(p) = 0$. It follows immediately that $f(p) = p + cg(p) = p$, so f has a fixed point at p.

b. Find a value of c for which fixed point iteration of $f(x)$ will successfully converge for any starting value, p_0, in the interval $[16, 17]$. (*Note: You don't need to include the graphs.)

   To guarantee convergence, we will find c such that $|f'(x)| < 1$ for all $x \in [16, 17]$. First, we rule out $c = 0$, as despite $f(x) = x$ converging to a fixed point everywhere, it is unable to tell us about the roots of g. Now, we compute that

$$f'(x) = 1 + c\left(\left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right) + \left(\frac{1}{5}\right)^x \cdot \ln\left(\frac{1}{5}\right)\right) = 1 - c\left(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5\right)$$

   We note that if $c < 0$, then $-c\left(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5\right) > 0$, so $f'(x) > 1$, which is not what we want. If $c > 0$, f' is an increasing function. Since $16 \leq x \leq 17$ we get that

$$1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) \leq f'(x) \leq 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5)$$

   We solve for c in the following inequality:

$$\begin{aligned} 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) < 1 &\implies c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) > 0 \\ &\implies c > 0 \end{aligned}$$

   We also want the lower bound of $f'(x)$ to be -1:

$$\begin{aligned} 1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) > -1 &\implies c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) < 2 \\ &\implies c < \frac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5} \end{aligned}$$

   Thus any value of c between 0 and $\dfrac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}$ will work, so we can just pick $c = \dfrac{1}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}$.

c. Use the function from part (b) with the value of c you have determined to find a root of $g(x)$ accurate to within 10^{-4}. State the value you used for p_0 and show the last three iterations. How many iterations did it take?

   We will use fixed point iteration on $f(x) = x + cg(x)$ with $p_0 = 16.5$. Below is the Octave code, input, and output:

```
function [m] = fixedpoint(f,x,N,tol)
  for j=1:N
    m = f(x);
```

```
    disp(["Value at iteration number " num2str(j) ": " num2str(
      m)]);
    if abs(m - x) <= tol
    disp(["Fixed point within given tolerance found in "
       num2str(j) " iterations."])
    return;
    else
    x = m;
    end%if
  end%for
  disp("Method failed. Max iterations exceeded.")
end%function

>> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5)
    )*(1/2^x + 1/5^x - 10^(-5)), 16.5, 1000, 10^(-4))
 Value at iteration number 1: 16.5747
 Value at iteration number 2: 16.5979
 Value at iteration number 3: 16.6056
 Value at iteration number 4: 16.6083
 Value at iteration number 5: 16.6092
 Value at iteration number 6: 16.6095
 Value at iteration number 7: 16.6096
 Value at iteration number 8: 16.6096
 Fixed point within given tolerance found in 8 iterations.
m = 16.610
```

We found a fixed point for f around $x = 16.610$, which implies that g has a root around that point as well.

d. Now repeat part (c) and find a root of g accurate to within 10^{-7}, using potentially other values for c as necessary. Explain your process and how you picked an appropriate c and x_0.

We continue using fixed point iteration, keeping the value of c the same. We know that our fixed point is close to $x = 16.610$, so that will be where we start the next fixed point iteration. Below is the Octave commands used and the output:

```
>> format long
>> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5)
    )*(1/2^x + 1/5^x - 10^(-5)), 16.610, 1000, 10^(-7))
 Value at iteration number 1: 16.6098
 Value at iteration number 2: 16.6097
 Value at iteration number 3: 16.6097
 Value at iteration number 4: 16.6096
 Value at iteration number 5: 16.6096
 Value at iteration number 6: 16.6096
 Value at iteration number 7: 16.6096
```

```
Value at iteration number 8: 16.6096
Value at iteration number 9: 16.6096
Fixed point within given tolerance found in 9 iterations.
m = 16.60964085351603
```

Question 2.

(2.3, #9) The function $g(x) = \sqrt[3]{5-3x}$ satisfies the hypotheses of Proposition 5 over the interval $[1, 1.3]$.
Find a bound on the number of iterations required to find the fixed point to within 10^{-5} accuracy starting with initial value x_0 of your choice.
Let $\hat{x}$ be the fixed point of g within the interval $[1, 1.3]$. To find the M value in proposition 5, we take the derivative of g:

$$g'(x) = \frac{-1}{(5-3x)^{\frac{2}{3}}}, \text{ where } x \in [1, 1.3].$$

Since $1 \leq x \leq 1.3$, we have

$$1.1 \leq 5 - 3x \leq 2 \implies \frac{1}{5-3x} \leq \frac{1}{1.1} < 1$$

Since the function $t^{\frac{2}{3}}$ is increasing for positive t, we have that

$$\frac{1}{(5-3x)^{\frac{2}{3}}} \leq \frac{1}{1.1^{\frac{2}{3}}} < 1$$

Thus $|g'(x)| \leq \frac{1}{1.1^{\frac{2}{3}}} < 1$, so $M = \frac{1}{1.1^{\frac{2}{3}}} \approx 0.9384$ is our desired value.
Let $x_0 = 1$. Notice that $1 \leq \hat{x} \leq 1.3$. By proposition 5, we have that

$$|x_{165} - \hat{x}| \leq M^{165}|1 - \hat{x}| = M^{165}(\hat{x} - 1) \leq M^{165}(1.3 - 1) = 0.3 \cdot M^{165} \approx 8.393 \cdot 10^{-6} < 10^{-5}$$

Thus the upper bound on the number of iterations is 165.

Question 3.

Consider the function $g(x) = \ln(\sin x + 1.5)$.
Find an initial value x_0 (to four decimal places) so that Newton's method fails at the second iteration. That is, Newton's method finds x_1 but cannot find x_2.

Solution.
We know that Newton's method fails to find x_2 if $g'(x_1) = \dfrac{\cos x_1}{\sin x_1 + 1.5} = 0$. Let's pick $x_1 = \dfrac{\pi}{2}$, which is a root of g'. We want to try and obtain x_1 using Newton's method, that is, we attempt to solve for x_0 in the equation

$$\frac{\pi}{2} = x_0 - \frac{g(x_0)}{g'(x_0)}.$$

which is equivalent to trying to find a root of the function

$$h(x) = x - \frac{g(x)}{g'(x)} - \frac{\pi}{2} = x - (\sin x + 1.5) \cdot \frac{\ln(\sin x + 1.5)}{\cos x} - \frac{\pi}{2}.$$

Notice that h is defined for all values of x except for those of the form $x = \frac{\pi}{2} + n\pi$, for $n \in \mathbb{Z}$. Since $-\frac{\pi}{2} < -1.5 < -1 < \frac{\pi}{2}$, we know that h is defined on the entire interval $[-1.3, -1]$. Moreover, h is continuous on this interval, and $h(-1.5) = 1.81 > 0$ and $h(-1) = -2.06 < 0$, so the conditions to use the bisection method are met. Below is the Octave code used to find the root:

```
function [m,M,i]=bisection(a,b,f,tol)
      N=ceil((log(b-a)-log(tol))/log(2));
      L=f(a);
      for i=1:N
            m=(a+b)/2;
            M=f(m);
            if (M==0)
            return;
            end%if
            if (L*M<0)
            b=m;
            else
            a=m;
            L=M;
            end%if
      end%for
      i=N;
end%function
>>f = @(x) x - (sin(x)+1.5)*log(sin(x)+1.5)/cos(x) - pi/2;
>> [m,M,i] = bisection(-1.5, -1, f, 0.00001)
m = -1.4567
M = 2.287e-05
 i = 16
```

Therefore, h has a root at around $x = -1.4567$. We choose x_0 to be this value, from which it follows that $x_1 = \frac{\pi}{2}$, and thus since $g'(x_1) = 0$, x_2 cannot be found.

Question 4.

Let $g(x) = \cos x - e^{-x/2} + 1.0005$, which has one negative root in $[-1, 0]$. Using $x_0 = -1$ and $x_1 = 0$, determine x_2 and x_3 when using:

a. the bracketed Newton's method, and

b. the bracketed secant method.

Show the results of your computation in a table and explain your steps.

(a):

Let l denote the left endpoint of the current interval and r denote the right endpoint of the current interval. To begin, we set $l = x_0$ and $r = x_1$. For each row, we evaluate the candidate for the subsequent term in the sequence x_{k+1}, which is $c = x_{k-1} - \frac{g(x_k)}{g'(x_k)}$. If $c \in [l, r]$, then we set $x_{k+1} = c$. Otherwise, we take $x_{k+1} = \frac{l+r}{2}$, the midpoint of the interval. For the next iteration, we choose the new left and right endpoints such that x_{k+1} is one of the new endpoints and we choose between the previous endpoints for our second endpoint such that the function evaluated at the chosen endpoint has opposite sign to $f(x_{k+1})$. Below is the table of values computed:

k	l	r	$g(l)$	$g(r)$	$g'(x_k)$	Candidate	x_{k+1}	$g(x_{k+1})$
1	-1	0	-0.1079	1.0005	0.5	-2.001	-0.5	0.5941
2	-1	-0.5	-0.1079	0.5941	1.121	-1.030	-0.75	0.2772

Thus, using the bracketed Newton's method, we have that $x_2 = -0.5$ and $x_3 = -0.75$.

(b):

The method for calculating the values is similar to the previous part, only that our candidate is now given by $c_{k+1} = x_k - g(x_k)\dfrac{x_k - x_{k-1}}{g(x_k) - g(x_{k-1})}$. Below are the values calculated using the bracketed secant method:

k	l	r	$g(l)$	$g(r)$	Candidate	x_{k+1}	$g(x_{k+1})$
1	-1	0	-0.1079	1.0005	-0.9026	-0.9026	0.0497
2	-1	-0.9026	-0.1079	0.0497	-0.9333	0.9333	0.0010

Thus $x_2 = -0.9026$ and $x_3 = 0.9333$.

Question 5.

Let $g(x) = \cos x - e^{-x/2} + 1.0005$.

Using any of the root-finding methods discussed in Chapter 2, find all of its positive roots to within 10^{-4}. Explain how you know you've found all of them.

Solution.

First, we notice that for $x > \ln 1000$

Question 6.

(3.2, #11, 12) A Lagrange interpolating polynomial is constructed for the function $f(x) = (\sqrt{2})^x$ using $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

a. If this polynomial is used to approximate $f(1.5)$, find a bound on the error in this approximation.

Let P_3 be the interpolating polynomial of least degree that passes through the points $(0,1), (1,\sqrt{2}), (2,2)$ and $(3, 2\sqrt{2})$. We know that there exists $\xi \in (0,3)$ such that the absolute error

$$|f(1.5) - P(1.5)| = \left|\frac{f^{(4)}(\xi)}{4!}1.5(1.5-1)(1.5-2)(1.5-3)\right| = \frac{5625}{24000}\left|\left(\frac{\ln 2}{2}\right)^4(\sqrt{2})^\xi\right|$$

f is increasing, so it follows that

$$|f(1.5) - P(1.5)| \leq \frac{5625 \ln 2}{24000 \cdot 2^4}(\sqrt{2})^3 = \frac{5625 \ln 2}{24000 \cdot 2^3}\sqrt{2} \approx 0.2871$$

b. Find the Lagrange interpolating polynomial, and use it to approximate $f(1.5)$. Then calculate the actual error in approximation.

First, we will find p_1, p_2, p_3, and p_4:

$$p_1(x) = f(0) \cdot \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3)$$

$$p_2(x) = f(1) \cdot \frac{x(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \sqrt{2}x(x-2)(x-3)$$

$$p_3(x) = f(2) \cdot \frac{x(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -x(x-1)(x-3)$$

$$p_4(x) = f(3) \cdot \frac{x(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{\sqrt{2}}{3}x(x-1)(x-2)$$

Thus the Lagrange interpolating polynomial is

$$P_3(x) = p_1(x) + p_2(x) + p_3(x) + p_4(x)$$

$$= -\frac{1}{6}(x-1)(x-2)(x-3) + \sqrt{2}x(x-2)(x-3) - x(x-1)(x-3) + \frac{\sqrt{2}}{3}x(x-1)(x-2)$$