

Question 32.

Let M be a subset of \mathbf{R}^n , let $p_0 \in M$ be a point, and let $\vec{v} \in \mathbf{R}^n$ be a vector. We say that \vec{v} is a **tangent vector** to M at p_0 if there exists $\delta > 0$ and a C^1 function $\alpha : (-\delta, \delta) \rightarrow M$ such that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. In other words, \vec{v} is the velocity vector of a curve through M .

- (a) Suppose now that M is the *zero set* of some C^1 function $f : U \rightarrow \mathbf{R}$, where U is an open set in \mathbf{R}^n : thus

$$M = \{p \in U : f(p) = 0\}.$$

Suppose that $p_0 \in M$ is a point such that $\nabla f(p_0) \neq \vec{0}$, and let $\vec{v} \in \mathbf{R}^n$ be a vector. Show that \vec{v} is a tangent vector to M at p_0 if and only if $\nabla f(p_0) \cdot \vec{v} = 0$.

- (b) Let E be the ellipsoid in \mathbf{R}^3 defined by the following equation:

$$x^2 + yz + y^2 - xy - xz + z^2 = 3.$$

Find the equation of the tangent plane to M at the point $p_0 = (1, 2, 3)$.

Hint: Define an appropriate function f , then find two vectors which are orthogonal to $\nabla f(p_0)$. By (a), these two vectors span the tangent plane. I recommend using graphing software to confirm your result.

Proof. (a):

Suppose that \vec{v} is a tangent vector to M at p_0 . Then there exists a function $\alpha : (-\delta, \delta) \rightarrow M$ so that $\alpha(0) = p_0$ and $\alpha'(0) = \vec{v}$. Define $g : (-\delta, \delta) \rightarrow \mathbf{R}$ by $g(t) = f(\alpha(t))$. For all $t \in (-\delta, \delta)$, $\alpha(t) \in M$, so $g(t) = 0$. It follows that

$$0 = g'(t) = \nabla f(\alpha(t)) \cdot \alpha'(t)$$

Substituting $t = 0$ yields

$$\nabla f(p_0) \cdot \vec{v} = 0$$

as needed.

Conversely, suppose that $\nabla f(p_0) \cdot \vec{v} = 0$.

(b):

Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ be a function defined by

$$f(x, y, z) = x^2 + yz + y^2 - xy - xz + z^2 - 3$$

The gradient of f is

$$\nabla f(x, y, z) = (2x - y - z, z + 2y - x, y - x + 2z)$$

and its zero set is exactly defined by the set of solutions of the equation

$$f(x, y, z) = 0$$

Substituting p_0 into this gradient, we have

$$\nabla f(1, 2, 3) = (-3, 6, 7)$$

Question 33.

- (a) Let $g : U \rightarrow \mathbf{R}$ be a C^1 function defined on an open set $U \subseteq \mathbf{R}^n$, and let M be its zero set:

$$M = \{p \in U : g(p) = 0\}.$$

Suppose that we have a C^1 function $f : U \rightarrow \mathbf{R}$, defined on an open set $U \subseteq \mathbf{R}^n$ which contains M , and we wish to find the maximum of f on M . Assume that M is compact, and that f achieves its maximum on M at some point $p_0 \in M$. Prove that there exists a real number $\lambda \in \mathbf{R}$ such that

$$\nabla f(p_0) = \lambda \nabla g(p_0).$$

This number λ is known as the **Lagrange multiplier**.

- (b) Use Lagrange multipliers to solve the following optimization problem: *Find the point(s) on the ellipsoid $x^2 + yz + y^2 - xy - xz + z^2 = 3$ which are **closest** and **furthest** from the origin.*