

NAME (PRINT): \_\_\_\_\_

STUDENT NUMBER (PRINT): \_\_\_\_\_

**University of Toronto Mississauga  
FALL 2024 MOCK FINAL EXAMINATION  
Introduction to Theory Computation**

**Macho Man ( $m^2$ )**

**Duration - ~~3 hours~~ 10 minutes**

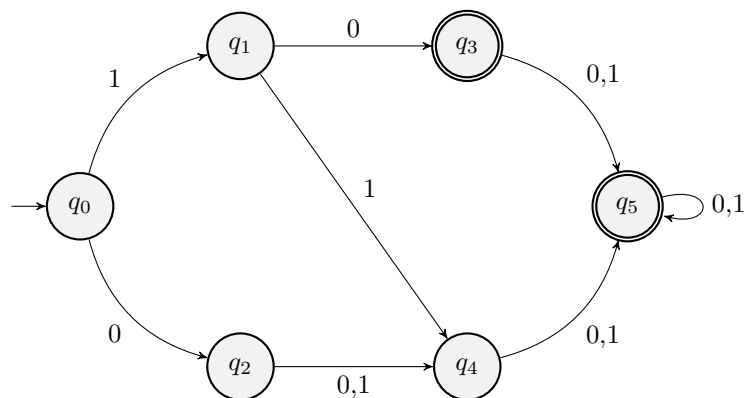
**Aids : お前のお母さん:**

[https://youtu.be/3UHR\\_XWrGhc?si=k003tfl7xwKDE3-o](https://youtu.be/3UHR_XWrGhc?si=k003tfl7xwKDE3-o)

This is a mock exam designed for studying CSC236. Any and all similarities with the Fall 2024 CSC236 final examination are purely coincidence.

This examination is meant to be done in 3 hours, but it is a little bit long, so do not worry about finishing. Part marks are very easy to get!

**Q1. (9 points)** Consider the DFA below:



- a) (1 point) Describe the language accepted by the following DFA.
- b) (3 points) Convert this DFA into a minimal NFA (i.e., there is no smaller NFA that accepts this language). Give a brief justification.

c) (2 points) Provide a DFA that accepts the language matched by  $(a + ab)^*$ .

d) (3 points) Prove the correctness of the DFA you provided.

**Q2. (10 points)** For each of the statements below, decide whether it is **true or false** and provide a proof justifying your answer.

a) (2 points) Let  $\Sigma = \{0, 1\}$ . The language  $L = \{w \in \Sigma^* : |w| = 3\}$  is regular.

b) (2 points) Let  $\Sigma = \{0, 1\}$ .  $L = \{1^{n^2} : n \in \mathbb{N}\}$  is a regular language.

c) (2 points) Let  $L, M$  be regular languages. The language  $L \cap M$  is regular.

d) (2 points) Let  $\Sigma = \{a, b, c\}$ . The language of all strings whose characters are alphabetically ordered is regular.

e) (2 points) Let  $L, M$  be languages. If  $L$  is not regular, then  $L \cup M$  is not regular.

**Q3. (6 points)**

- a) (2 points) State the CLRS version of master theorem. Define all variables and state their conditions.

- b) (4 points) Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be a nonnegative function. Prove that if  $f \in \Theta(n^k)$  for some  $k > 0$ , then the regularity condition holds true.

**Q4. (7 points)** Consider the program below:

```
1      def binary_search(x: int, lst: list[int]):
2          l ← 0, r ← len(lst) - 1
3          while(r - l > 0):
4              mid ← (l+r) // 2
5              if(lst[mid] = x):
6                  return mid
7              elif(lst[mid] < x):
8                  r ← mid
9              else:
10                 l ← mid + 1
11         return l
```

a) (2 points) State the preconditions and postconditions of this program.

b) (5 points) Prove that this program is correct.

**Q5. (6 points)** Let  $A$  be a set of functions defined recursively as follows:

- $\sqrt{x} \in A$
- If  $f \in A$ , then  $\frac{1}{f} - f \in A$

a) (4 points) Let  $P(f)$  be a predicate on  $A$  and suppose you have managed to prove that

- $P(\sqrt{x})$  is true,
- $P(f) \implies P(\frac{1}{f} - f)$  for all  $f \in A$ .

Prove that  $\forall f \in A, P(f)$ . You may only assume that the principle of induction holds for the natural numbers, NOT the set  $A$ .

b) (2 points) Prove that  $\forall f \in A, f\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$ .



**Q6. (7 points)** Let  $\Sigma = \{0, 1\}$ . For any language  $L \in \Sigma^*$ , define

$$S = \{w \in \Sigma^* : 0w1 \in L\}$$

Given that  $L$  is a regular language, prove that  $S$  is a regular language.

Now, this is where a normal exam would end. However, this is not a normal exam.

**Q7. (9 points)** Recall that a segment tree is a data structure effective for querying and updating information about a range of values in a list, such as the minimum element in a provided range. In fact, we will examine an implementation of a segment tree that does this.

Suppose that we have a list of numbers `lst`. In this implementation, the segment tree can be thought of as a binary tree, with each node storing a range of indices  $[l, r]$ , a value which represents the minimum number in `lst[l:r]` (inclusive of  $l$ , exclusive of  $r$ ), and pointers to a left and right child. **Note that a node does not have a left and right child if and only if it is true that  $l = r$ .** Otherwise, the left child will keep track the range  $[l, \lfloor \frac{l+r}{2} \rfloor]$  and the right child will track  $[\lfloor \frac{l+r}{2} \rfloor + 1, r]$

As an example, consider the following pseudocode for `update`, which updates the segment tree to correctly return queries about `lst` after setting `lst[i] = x`:

```
1      def update(root: Node, x: int, i: int):
2          if i not in [root.l, root.r] then return
3          else if root.l = i = root.r
4              then root.val ← min(x, root.val)
5          update(root.l, x, i)
6          update(root.r, x, i)
7          root.val ← min(root.l.val, root.r.val)
```

a) (4 points) Prove that this method is correct.

b) (3 points) Find the recursive worst-case runtime of `update`.

c) (2 points) Find the tight asymptotic bound for the runtime of `update`.