Question 22

For a normed vector space $(X, \|\cdot\|)$, let $X^* = B(X, \mathbf{R})$ denote the set of bounded linear mappings from X to \mathbf{R} . Here X^* is equipped with the operator norm $\|\cdot\|_{\text{op}}$, and the normed vector space $(X^*, \|\cdot\|_{\text{op}})$ is called the **topological dual** of X.

Let c_0 denote the set of sequences converging to zero. Prove that $c_0^* \equiv \ell^1$.

Proof. Equip ℓ^1 with the 1-norm and c_0 with the sup-norm. Define the function $f:\ell^1\to c_0^*$ by $f(\vec{x})(\vec{a})=\sum_{i=1}^\infty x_i\cdot a_i$. We claim that this is a bijective isometry.

Let $A = \{ |f(x)(a) - f(x)(a)| : a \in c_0, ||a||_{\infty} \le 1 \}.$

To show that this function is an isometry, let $x, y \in \ell^1$. It will be shown that $||x-y||_1 = \sup A$ Let $a \in c_0$ such that $||a||_{\infty} \le 1$. Then

$$|f(x)(a) - f(y)(a)| = \left| \sum_{i=1}^{\infty} (x_i \cdot a_i - y_i \cdot a_i) \right| = \left| \sum_{i=1}^{\infty} a_i (x_i - y_i) \right| \le \sum_{i=1}^{\infty} |a_i| |x_i - y_i| \le \sum_{i=1}^{\infty} |x_i - y_i|$$

This means that $||x - y||_1$ is an upper bound for A.

Question 23.

Let S^2 denote the unit sphere in \mathbb{R}^3 . Let N = (0,0,1) denote the "north pole". In this problem, you will show that $S^2 \setminus \{N\}$ is homeomorphic to \mathbb{R}^2 . To do this, we define a function $\Phi : S^2 \setminus \{N\} \to \mathbb{R}^2$ known as the **stereographic projection**: given a point P in $S^2 \setminus \{N\}$, draw a line between P and N, and let $\Phi(P)$ denote the point where this line intersects the xy-plane in \mathbb{R}^3 .

- (a) Given P = (x, y, z), find an explicit formula for $\Phi(P)$ in terms of x, y, z.
- (b) Deduce that Φ is continuous.
- (c) Prove that Φ is a bijection; in fact, given $p = (s, t) \in \mathbf{R}^2$, find an explicit formula for $\Phi^{-1}(p)$.
- (d) Deduce that Φ is a homeomorphism.

Proof. (a):

Let P = (x, y, z). First, we find the equation of the line that passes P and N. Consider the equation of the line L(t) = (tx, ty, (z-1)t+1). Notice that L(0) = N and L(1) = P, so L satisfies what we were looking for. Now we find the point where L intersects with the xy-plane. This happens exactly when (z-1)t+1=0. Solving for t gives $t=\frac{1}{1-z}$. This value is always defined as $z \neq 1$. As a result, it turns out that

$$L\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$

Thus

$$\Phi(P) = \frac{1}{1-z} (x, y).$$

(b):

(c)

Let $p = (s, t) \in \mathbb{R}^2$. Our goal is to find $(x, y, z) \in S^2 \setminus \{N\}$ such that $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (s, t)$. Immediately, we obtain the following system of equations:

$$\frac{x}{1-z} = s,$$

$$\frac{y}{1-z} = t,$$

$$x^{2} + y^{2} + z^{2} = 1$$

We also have the restriction $z \neq 1$ because $(x, y, z) \neq N$. Isolating for x and y yields

$$x = s(1-z)$$

$$y = t(1-z)$$

Then we substitute this into the third equation and get

$$s^{2}(1-z)^{2} + t^{2}(1-z)^{2} + z^{2} = 1 \implies (s^{2} + t^{2} + 1)z^{2} - 2(s^{2} + t^{2})z + s^{2} + t^{2} - 1 = 0$$

We can replace the term $t^2 + s^2$ with $||p||_2^2$, and the equation becomes

$$(\|p\|_2^2 + 1)z^2 - 2\|p\|_2^2z + \|p\|_2^2 - 1 = 0$$

Using the quadratic formula:

$$z = \frac{2(\|p\|_2^2) \pm \sqrt{4(\|p\|_2^2)^2 - 4(\|p\|_2^2 + 1)(\|p\|_2^2 - 1)}}{2(\|p\|_2^2 + 1)}$$

$$\implies z = \frac{\|p\|_2^2 \pm \sqrt{\|p\|_2^4 - (\|p\|_2^4 - 1)}}{\|p\|_2^2 + 1}$$

$$z = \frac{\|p\|_2^2 \pm 1}{\|p\|_2^2 + 1}$$

Notice that we cannot use the positive solution, for then

$$z = \frac{\|p\|_2^2 + 1}{\|p\|_2^2 + 1} = 1$$

Thus it must be true that

$$z = \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}$$
$$x = s(1 - z) = \frac{2s}{\|p\|_2^2 + 1}$$
$$y = \frac{2t}{\|p\|_2^2 + 1}$$

It can be verified that these values of x, y, z result in $\Phi(x, y, z) = (s, t)$. In fact, using this, we obtain that the formula for Φ^{-1} is

$$\Phi^{-1}(s,t) = \left(\frac{2s}{\|p\|_2^2 + 1}, \frac{2t}{\|p\|_2^2 + 1}, \frac{\|p\|_2^2 - 1}{\|p\|_2^2 + 1}\right)$$

 \vdash

Question 24

Let X be a normed vector space. Prove that the following statements are equivalent.

- (i) X is finite-dimensional.
- (ii) The unit ball $\overline{B}(\vec{0},1)$ is compact
- (iii) X is **locally compact**: each point $p \in X$ is contained in some open set U such that \overline{U} is compact.

Proof. It will be proven that (i) \implies (ii) \iff (iii) \implies (i).

 $(i) \implies (ii)$

Suppose that X has finite dimension n. Then there is a continuous linear isomorphism Φ between X and \mathbb{R}^n . Since $\overline{B}(0,1)$ is closed and bounded, $\Phi(\overline{B}(0,1))$ is also closed and bounded in \mathbb{R}^n , so the set is compact. Since homeomorphisms preserve compactness, we can conclude that the closed unit ball in X is compact.

 $(ii) \implies (iii)$

Suppose that the unit ball $\overline{B}(0,1)$ is compact. Let $p \in X$. We claim that U = B(p,1). Consider $\overline{U} = \overline{B}(p,1)$. There is an isometry Φ from $\overline{B}(0,1)$ to $\overline{B}(p,1)$ defined by $\Phi(x) = p + x$. Since the closed unit ball is compact, it follows that $\overline{B}(p,1)$ is compact. Thus X is locally compact.

 $(iii) \implies (ii)$:

Suppose that X is locally compact. Then $\overline{0}$ is contained in an open set U such that \overline{U} is compact. Since U is open, $B(0,\varepsilon) \subseteq U$ for some ε -ball centered around 0. It follows that $\overline{B}(0,\varepsilon)$ is compact, as it is a closed subset of \overline{U} . Since there is a homeomorphism from this closed ball to $\overline{B}(0,1)$, the closed unit ball is also compact, as desired.

 $(iii) \implies (i)$:

Suppose that X is locally compact. We know previously that this implies that the closed unit ball is compact. Firstly, a quick lemma will be proven.

Lemma 1. $\forall x \in X, r > 0, B(x, r) = \{x\} + B(0, r).$

Let $y \in B(x, r)$. Notice that $y - x \in B(0, r)$. Thus y = x + (y - x) so $y \in \{x\} + B(0, r)$. Then, let $y \in \{x\} + B(0, r)$, so y = x + s, for some $s \in B(0, r)$. Since ||y - x|| = ||s|| < r, it

follows that $y \in B(x,r)$ and we are done.

Moving on, we construct a finite set of vectors in the following way:

Since $\overline{B}(0,1)$ is totally bounded, we can find a finite set of vectors β such that $\bigcup_{x\in\beta} B\left(x,\frac{1}{2}\right)$.

Lemma 2. $B(0,1) \subseteq \operatorname{span}(\beta) + 2^{-n}B(0,1), \forall n \in \mathbb{N}.$

To prove this, let $n \in \mathbb{N}$ and $y \in B(0,1)$. It follows that $y \in B(x_1, \frac{1}{2})$ for some $x_1 \in \beta$. Using Lemma 1,

$$y \in \{x_1\} + B\left(0, \frac{1}{2}\right) = \{x_1\} + \frac{1}{2}B(0, 1)$$

We can repeatedly use this argument to obtain that

$$y \in \left\{ x_1 + \frac{1}{2}x_2 + \frac{1}{2^2}x_3 + \dots + \frac{1}{2^{n-1}}x_n \right\} + \frac{1}{2^n}B(0,1)$$

Note that x_i are not necessarily distinct. This simplifies down to

$$y \in \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1) \implies B(0,1) \subseteq \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1)$$

which is what we wanted.

Finally, we claim that β is the basis for X.

Let $v \in X$. Since $\frac{v}{2||v||} \in B(0,1)$, by Lemma 2.

$$\frac{v}{2\|v\|} \in \operatorname{span}(\beta) + \frac{1}{2^n} B(0,1), \text{ for all } n \in \mathbb{N}$$

Then we construct a sequence $\{v_n\}_{n\geq 1}$ in $\operatorname{span}(\beta)$ where $v_n\in\operatorname{span}(\beta)$ such that $y=v_n+p$, for some $p\in\frac{1}{2n}B(0,1)$. Notice that

$$||y - v_n|| = ||p|| < \frac{1}{2^n}$$

which can become arbitrarily small. This implies that y is a limit point for span(β). However, this set is closed because it is finite dimensional, so we also have that $y \in \text{span}(\beta)$. Therefore $X \subseteq \text{span}(\beta)$, and since X is spanned by a finite set, it is finite dimensional. These implications are sufficient for proving that (i), (ii), (iii) are equivalent.