

# Question 1.

(Based on 2.2, #24) Let  $g(x) = \left(\frac{1}{2}\right)^x + \left(\frac{1}{5}\right)^x - 10^{-5}$ .

- a. Show that if  $g$  has a zero at  $p$ , then the function  $f(x) = x + cg(x)$  has a fixed point at  $p$ .

Suppose that  $g$  has a zero at  $p$ . Then  $g(p) = 0$ . It follows immediately that  $f(p) = p + cg(p) = p$ , so  $f$  has a fixed point at  $p$ .

- b. Find a value of  $c$  for which fixed point iteration of  $f(x)$  will successfully converge for any starting value,  $p_0$ , in the interval  $[16, 17]$ . (\*Note: You don't need to include the graphs.)

To guarantee convergence, we will find  $c$  such that  $|f'(x)| < 1$  for all  $x \in [16, 17]$ . First, we rule out  $c = 0$ , as despite  $f(x) = x$  converging to a fixed point everywhere, it is unable to tell us about the roots of  $g$ . Now, we compute that

$$f'(x) = 1 + c \left( \left(\frac{1}{2}\right)^x \cdot \ln \left(\frac{1}{2}\right) + \left(\frac{1}{5}\right)^x \cdot \ln \left(\frac{1}{5}\right) \right) = 1 - c(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5)$$

We note that if  $c < 0$ , then  $-c(2^{-x} \cdot \ln 2 + 5^{-x} \cdot \ln 5) > 0$ , so  $f'(x) > 1$ , which is not what we want. If  $c > 0$ ,  $f'$  is an increasing function. Since  $16 \leq x \leq 17$  we get that

$$1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) \leq f'(x) \leq 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5)$$

We solve for  $c$  in the following inequality:

$$\begin{aligned} 1 - c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) < 1 &\implies c(2^{-17} \cdot \ln 2 + 5^{-17} \cdot \ln 5) > 0 \\ &\implies c > 0 \end{aligned}$$

We also want the lower bound of  $f'(x)$  to be -1:

$$\begin{aligned} 1 - c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) > -1 &\implies c(2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5) < 2 \\ &\implies c < \frac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5} \end{aligned}$$

Thus any value of  $c$  between 0 and  $\frac{2}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}$  will work, so we can just pick

$$c = \frac{1}{2^{-16} \cdot \ln 2 + 5^{-16} \cdot \ln 5}.$$

- c. Use the function from part (b) with the value of  $c$  you have determined to find a root of  $g(x)$  accurate to within  $10^{-4}$ . State the value you used for  $p_0$  and show the last three iterations. How many iterations did it take?

We will use fixed point iteration on  $f(x) = x + cg(x)$  with  $p_0 = 16.5$ . Below is the Octave code, input, and output:

```
1 function [m] = fixedpoint(f,x,N,tol)
2   for j=1:N
3     m = f(x);
```

```

4     disp(["Value at iteration number " num2str(j) ": " num2str(m)]);
5     if abs(m - x) <= tol
6         disp(["Fixed point within given tolerance found in "
              num2str(j) " iterations."])
7     return;
8     else
9         x = m;
10    end%if
11 end%for
12 disp("Method failed. Max iterations exceeded.")
13 end%function
14
15 >> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5))
16    *(1/2^x + 1/5^x - 10^(-5)), 16.5, 1000, 10^(-4))
17 Value at iteration number 1: 16.5747
18 Value at iteration number 2: 16.5979
19 Value at iteration number 3: 16.6056
20 Value at iteration number 4: 16.6083
21 Value at iteration number 5: 16.6092
22 Value at iteration number 6: 16.6095
23 Value at iteration number 7: 16.6096
24 Value at iteration number 8: 16.6096
25 Fixed point within given tolerance found in 8 iterations.
26 m = 16.610

```

We found a fixed point for  $f$  around  $x = 16.610$ , which implies that  $g$  has a root around that point as well.

- d. Now repeat part (c) and find a root of  $g$  accurate to within  $10^{-7}$ , using potentially other values for  $c$  as necessary. Explain your process and how you picked an appropriate  $c$  and  $x_0$ .

We continue using fixed point iteration, keeping the value of  $c$  the same. We know that our fixed point is close to  $x = 16.610$ , so that will be where we start the next fixed point iteration. Below is the Octave commands used and the output:

```

1 >> format long
2 >> [m] = fixedpoint(@(x) x + 1/(2^(-16)*log(2) + 5^(-16)*log(5))
3    *(1/2^x + 1/5^x - 10^(-5)), 16.610, 1000, 10^(-7))
4 Value at iteration number 1: 16.6098
5 Value at iteration number 2: 16.6097
6 Value at iteration number 3: 16.6097
7 Value at iteration number 4: 16.6096
8 Value at iteration number 5: 16.6096
9 Value at iteration number 6: 16.6096
10 Value at iteration number 7: 16.6096

```



Question 2.

(2.3, #9) The function  $g(x) = \sqrt[3]{5 - 3x}$  satisfies the hypotheses of Proposition 5 over the interval  $[1, 1.3]$ .

Find a bound on the number of iterations required to find the fixed point to within  $10^{-5}$  accuracy starting with initial value  $x_0$  of your choice.

Let's take the derivative of  $g$ :

$$g'(x) = \frac{-1}{(5 - 3x)^{\frac{2}{3}}}$$

Question 3.

Consider the function  $g(x) = \ln(\sin x + 1.5)$ .

Find an initial value  $x_0$  (to four decimal places) so that Newton's method fails at the second iteration. That is, Newton's method finds  $x_1$  but cannot find  $x_2$ .

Question 4.

Let  $g(x) = \cos x - e^{-x/2} + 1.0005$ , which has one negative root in  $[-1, 0]$ . Using  $x_0 = -1$  and  $x_1 = 0$ , determine  $x_2$  and  $x_3$  when using:

- the bracketed Newton's method, and
- the bracketed secant method.

Show the results of your computation in a table and explain your steps.

Question 5.

Let  $g(x) = \cos x - e^{-x/2} + 1.0005$ .

Using any of the root-finding methods discussed in Chapter 2, find all of its positive roots to within  $10^{-4}$ . Explain how you know you've found all of them.

Question 6.

(3.2, #11, 12) A Lagrange interpolating polynomial is constructed for the function  $f(x) = (\sqrt{2})^x$  using  $x_0 = 0$ ,  $x_1 = 1$ ,  $x_2 = 2$ ,  $x_3 = 3$ .

- If this polynomial is used to approximate  $f(1.5)$ , find a bound on the error in this approximation.

