

Question 1.

Use row operations on the matrix  $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix}$  to obtain an upper triangular matrix, then use Theorem 59 to find  $\det A$ . (You will get no credit for using a row/column expansion.)

We have

$$\begin{aligned} \det A &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & -2 & 6 & 3 \\ 2 & 4 & -6 & -2 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 4 & 2 \\ 0 & 4 & -10 & -4 \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & -12 & -6 \end{pmatrix} \\ &= -6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix} \\ &= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{pmatrix} \\ &= 6 \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \\ &= 6(1)(2)(2) \left( \frac{1}{2} \right) \\ &= 12 \end{aligned}$$

Question 2.

Let  $T = T_A : \mathbb{Q}^5 \rightarrow \mathbb{Q}^5$  where  $A = \begin{pmatrix} 1 & 0 & 1 & -2 & 0 \\ 3 & 0 & 1 & 0 & -2 \\ 2 & 0 & 0 & 2 & -2 \\ 2 & 0 & 0 & 1 & -2 \\ 2 & 0 & 1 & -2 & -1 \end{pmatrix}$ .

(a) Find  $C_T$  and the eigenvalues of  $T$ .

We have

$$\begin{aligned} C_T(\lambda) &= \det(\lambda I - T) \\ &= \det \begin{pmatrix} \lambda - 1 & 0 & -1 & 2 & 0 \\ -3 & \lambda & -1 & 0 & 2 \\ -2 & 0 & \lambda & -2 & 2 \\ -2 & 0 & 0 & \lambda - 1 & 2 \\ -2 & 0 & -1 & 2 & \lambda + 1 \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} \lambda - 1 & -1 & 2 & 0 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda - 1 & 2 \\ -2 & -1 & 2 & \lambda + 1 \end{pmatrix} \\ &= -\lambda \det \begin{pmatrix} \lambda + 1 & 0 & 0 & -\lambda - 1 \\ -2 & \lambda & -2 & 2 \\ -2 & 0 & \lambda - 1 & 2 \\ -2 & -1 & 2 & \lambda + 1 \end{pmatrix} \\ &= -\lambda \left( (\lambda + 1) \det \begin{pmatrix} \lambda & -2 & 2 \\ 0 & \lambda - 1 & 2 \\ -1 & 2 & \lambda + 1 \end{pmatrix} + (\lambda + 1) \det \begin{pmatrix} -2 & \lambda & -2 \\ -2 & 0 & \lambda - 1 \\ -2 & -1 & 2 \end{pmatrix} \right) \\ &= \lambda(\lambda + 1) (-(\lambda - 1)(\lambda(\lambda + 1) + 2) + 2(2\lambda - 2) + -2(2\lambda - 2) + (\lambda - 1)(2 + 2\lambda)) \\ &= \lambda(\lambda + 1) ((\lambda - 1)(2 - \lambda - \lambda^2 - 2 + 2\lambda)) \\ &= \lambda(\lambda + 1)(\lambda - 1)(\lambda - \lambda^2) \\ &= -\lambda^2(\lambda - 1)^2(\lambda + 1) \end{aligned}$$

The eigenvalues are the roots of  $C_T$ , which are  $\lambda = 0, 1, -1$ .

(b) For each eigenvalue, find a basis for the corresponding eigenspace.

For  $\lambda = 0$ , we solve the equation  $Ax = 0$  via row reduction:

$$\left( \begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 1 & 0 & -2 & 0 \\ 2 & 0 & 0 & 2 & -2 & 0 \\ 2 & 0 & 0 & 1 & -2 & 0 \\ 2 & 0 & 1 & -2 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccccc|c} 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 6 & -2 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \end{array} \right) \rightarrow \left( \begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right)$$





Let  $x_5 = t$ . The general solution is

$$x = t \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So a basis for  $E_{-1}(T)$  is  $\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ .

- (c) Determine if  $T$  is diagonalizable, and if so, find a basis  $\beta$  so that  $[T]_\beta$  is diagonal.

Since the dimension of each eigenspace matches the algebraic multiplicity of each corresponding eigenvalue,  $T$  is diagonalizable and the basis  $\beta$  that makes  $[T]_\beta$  diagonal is exactly the basis consisting of the basis vectors of each eigenspace. In particular,

$$\beta = \left\{ e_2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

### Question 3.

- (a) Read the proof of Theorem 58 from the additional file in the Week 10 Readings on the course page.
- (b) Prove Part 1 of Theorem 59 using a strategy similar to the proof of Theorem 58. (You cannot use other parts of Theorem 59 in this proof.)

Let  $A \in M_n(\mathbb{F})$  with  $n \geq 2$ . If  $A$  has a row of 0's, then  $\det A = 0$ .

*Proof.* Write  $A = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$ , where  $r_j$  represents the rows of  $A$ . Suppose that  $r_i = \vec{0}$ . If  $i = 1$ , the result is immediate by cofactor expansion. Otherwise, if  $i > 1$ , we do induction on  $n$ . Let  $n = 2$ . The only possibility is  $i = 2$ , so denote

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

From here, it is easy to see that  $\det A = 0$ .

Now, suppose that this is true for some  $n$ . We will show that it is true for  $n + 1$ . Define  $\tilde{r}_{j,k}$  to be the row obtained by deleting the  $k$ th entry of  $r_j$ . Using cofactor expansion along the first row, we have

$$\det A = \sum_{k=1}^{n+1} A_{1k} \det \tilde{A}_{1k}$$

Observe that  $\tilde{A}_{1k}$  are  $n \times n$  matrices, and since the  $i$ th row was 0 in the original matrix (and  $i > 1$ ), the  $i - 1$ th row in  $\tilde{A}_{1k}$  is 0, so by the induction hypothesis  $\det \tilde{A}_{1k} = 0$ , thus  $\det A = 0$  and we are done. □

#### Question 4.

Assume that Parts 1 and 2 of Theorem 59 have been proved. You cannot use Parts 4 through 7 of Theorem 59 in the following problem.

- (a) Prove Part 3 using induction on  $n$ . (Check  $n = 1, 2$  by hand, then in the inductive step assume  $n + 1 \geq 3$ .)
- (b) Prove Part 4 using row-swapping matrices and properties of determinants.

*Proof.*

(a):

We prove Part 3 using induction on  $n$ .

Let  $n = 1$ . The statement is vacuously true, as  $A$  cannot have 2 identical rows.

Let  $n = 2$ . Then it must be true that

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \text{ for } a, b \in \mathbb{F}.$$

Then  $\det A = ab - ab = 0$  as expected.

Now, suppose that this statement is true for some  $n \in \mathbb{N}$ , where  $n > 1$ . We will show it also holds for  $n + 1$ . Let  $r_i, r_j$  be the identical rows. Since  $n + 1 > 2$ , we are guaranteed to have one other row  $r_k$  that is not  $r_i$  or  $r_j$ . We perform a row  $k$  expansion of  $\det A$  and see that

$$\det A = \sum_{l=1}^{n+1} A_{kl} \det \tilde{A}_{kl}$$

Notice that  $\tilde{A}_{kl}$  is a  $n \times n$  matrix, and contain both  $r_i$  and  $r_j$  with the  $l$ th entry deleted. But these rows are still identical because the same entry got deleted. By the induction hypothesis,

$$\det A = \sum_{l=1}^{n+1} A_{kl} 0 = 0$$

which was what we wanted.

(b):

Suppose that  $B$  is obtained from  $A$  by swapping row  $i$  and row  $j$ . Denote these rows as  $r_i, r_j$  respectively. Using linearity in one row of the determinant, and the previous result we proved,

$$\begin{aligned}
0 &= \det \begin{pmatrix} r_1 \\ \vdots \\ r_i + r_j \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix} = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i + r_j \\ \vdots \\ r_n \end{pmatrix} \\
&\implies 0 = \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} + \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_j \\ \vdots \\ r_n \end{pmatrix} \\
&\implies 0 = \det A + 0 + \det B + 0 \\
&\implies \det B = -\det A
\end{aligned}$$

as needed. □

### Question 5.

Prove that if  $U \in M_{n \times n}(F)$  is upper triangular, then  $\det U = \prod_{i=1}^n U_{ii}$ .

*Proof.* We proceed using induction on  $n$ . If  $n = 1$ , the result is immediate. Suppose that the statement holds for some  $n \in \mathbb{N}$ . We will show the same is the case for  $n + 1$ .

Let  $U \in M_{n+1}(\mathbb{F})$  be upper triangular. We have

$$\det U = \sum_{j=1}^{n+1} U_{1j} \det \tilde{U}_{1j}.$$

For  $j \neq 1$ , notice that the entries of the first column of  $\tilde{U}_{1j}$  are 0, as  $(\tilde{U}_{1j})_{i1} = U_{(i+1)1} = 0$ . Thus

$$\det \tilde{U}_{1j} = \det \tilde{U}_{1j}^t = 0$$

as the transpose has a row of 0's. The cofactor expansion of  $\det U$  reduces to

$$\det U = U_{11} \det \tilde{U}_{11}$$

but  $\tilde{U}_{11} \in M_n(\mathbb{F})$  is upper triangular, so

$$\det U = U_{11} \prod_{i=1}^n \tilde{U}_{ii} = U_{11} \prod_{i=2}^{n+1} U_{ii} = \prod_{i=1}^{n+1} U_{ii}.$$

which completes the proof. □

### Question 6.

Let  $V$  be a vector space over  $F$ , and  $T : V \rightarrow V$  a linear map. If  $W \subseteq V$  is a  $T$ -invariant subspace, then we can restrict  $T$  to  $W$ , to obtain a map  $T_W : W \rightarrow W$ . We call  $T_W$  the restriction map.

(a) Let  $\beta_W$  be a basis for  $W$ . In HW#3 we proved that if  $\beta = \beta_W \beta_1$  is an extension of  $\beta_W$  to a basis for  $V$ , then  $[T]_\beta = \left( \begin{array}{c|c} A & B \\ \hline O & C \end{array} \right)$ . Prove that  $A = [T_W]_{\beta_W}$ .

(b) Let  $M = \left( \begin{array}{c|c} A & B \\ \hline O & C \end{array} \right)$ . Prove that  $\det M = \det A \det C$ .

*Proof.*

(a):

Denote  $\beta = \{w_1, \dots, w_n\}$ . Then the  $j$ th column of  $\left( \begin{array}{c} A \\ \hline O \end{array} \right)$  is  $[T(w_j)]_\beta$ . □

### Question 7.

Deduce from Question 6 that if  $W$  is a  $T$ -invariant subspace, then  $C_{T_W}$  divides  $C_T$ .



### Question 8.

Let  $V$  be a finite-dimensional vector space over a field  $F$ , and  $W_1, W_2 \subseteq V$  such that  $V = W_1 \oplus W_2$ . Define the projection maps  $P_i : V \rightarrow V$  by  $P(x) = x_i$  where  $x = x_1 + x_2$  with  $x_1 \in W_1$  and  $x_2 \in W_2$ .

1. Prove that  $P_i$  is linear.
2. Prove that  $P_i^2 = P_i$ .
3. Prove that each  $W_j$  is  $P_i$ -invariant.
4. Determine if  $P_i$  is diagonalizable and justify your answer.

### Question 9.

Define the direct sum for more than two subspaces. Let  $W_1, \dots, W_k \subseteq V$  be subspaces such that  $V = W_1 \oplus \dots \oplus W_k$ .

1. Prove that every basis  $\beta$  for  $V$  gives a direct sum decomposition  $V = W_1 \oplus \dots \oplus W_n$  where  $\dim W_i = 1$ .
2. Prove the converse: If  $V = W_1 \oplus \dots \oplus W_n$  with  $\dim W_i = 1$ , then choosing non-zero  $w_i \in W_i$  forms a basis for  $V$ .
3. Let  $T : V \rightarrow V$  be linear. Show that  $[T]_\beta$  is block diagonal.

### Question 10.

Let  $W_1, \dots, W_k \subseteq V$  with bases  $\beta_1, \dots, \beta_k$ . Prove that  $V = W_1 \oplus \dots \oplus W_k$  if and only if  $\beta = \beta_1 \cup \dots \cup \beta_k$  is a basis for  $V$ .

### Question 11.

Determine whether the following statements are true or false. Justify your answers.

1. If  $V = W_1 \oplus W_2$  and  $T_{W_1}, T_{W_2}$  are diagonalizable, then  $T$  is diagonalizable.
2. If  $W_i \cap W_j = \{0\}$  for  $i \neq j$  and  $V = W_1 + W_2 + W_3$ , then  $V = W_1 \oplus W_2 \oplus W_3$ .
3. If  $\dim V = 7$ ,  $\dim N(T) = 3$ , and  $\text{rank}(T - I) = 4$ , then  $T$  is diagonalizable.