Eigencentrality of a Specialized Graph

Let $G = (V, E)$ be a graph with adjacency matrix M let $B \subset V$ be a structural set and let $\mathcal{S}(G)$ and $\hat{M}$ represent the specialization of G over B and it's adjacency matrix respectively. Without loss of generality we can write

$$M = \begin{bmatrix} U & W \\ Y & Z \end{bmatrix}$$

where U is the adjacency matrix of $G|_B$ and Z is the adjacency matrix of $G|_{\bar{B}}$. Assume that Z is strongly connected. Then we can write

$$\hat{M} = \begin{bmatrix} U & \begin{bmatrix} W_1 \cdots W_1 \end{bmatrix} & \cdots & \begin{bmatrix} W_m \cdots W_m \end{bmatrix} \\ \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} & \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix} & & \\ \vdots & & \ddots & \\ \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} & & & \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix} \end{bmatrix}$$

Where each Y_j and W_i has a single entry and

$$\sum_{j=1}^{k} Y_j = Y \qquad \sum_{i=1}^{m} W_i = W$$

by the definition of specialization.

Assume that $Mv = \rho v$ where ρ is the spectral radius of M. Since specialization preserves spectrum, we know that there exists $u \neq 0$ such that $\hat{M}u = \rho u$. Write $v = \begin{bmatrix} v_b \\ v_z \end{bmatrix}$ so that

$$Mv = \begin{bmatrix} U & W \\ Y & Z \end{bmatrix} \begin{bmatrix} v_b \\ v_z \end{bmatrix} = \begin{bmatrix} Uv_b + Wv_z \\ Yv_b + Zv_z \end{bmatrix} = \begin{bmatrix} \rho v_b \\ \rho v_z \end{bmatrix}$$

and

$$\rho v_b = Uv_b + Wv_z \tag{1}$$

$$\rho v_z = Yv_b + Zv_z \tag{2}$$

Write

$$u = \begin{bmatrix} u_b & \begin{bmatrix} u_{w_1,y_1} & u_{w_1,y_2} & \cdots & u_{w_1,y_k} \end{bmatrix} & \cdots & \begin{bmatrix} u_{w_m,y_1} & u_{w_m,y_2} & \cdots & u_{w_m,y_k} \end{bmatrix} \end{bmatrix}^T$$

so that

$$\hat{M}u = \begin{bmatrix} U & \begin{bmatrix} W_1 \cdots W_1 \end{bmatrix} & \cdots & \begin{bmatrix} W_m \cdots W_m \end{bmatrix} \\ \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} & \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix} & & \\ \vdots & & \ddots & \\ \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} & & & \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_b \\ \begin{bmatrix} u_{w_1,y_1} \\ \vdots \\ u_{w_1,y_k} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} u_{w_m,y_1} \\ \vdots \\ u_{w_m,y_k} \end{bmatrix} \end{bmatrix}^T = \begin{bmatrix} \rho u_b \\ \begin{bmatrix} \rho u_{w_1,y_1} \\ \vdots \\ \rho u_{w_1,y_k} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \rho u_{w_m,y_1} \\ \vdots \\ \rho u_{w_m,y_k} \end{bmatrix} \end{bmatrix}^T$$

producing the equations:

$$\rho u_b = U u_b + \sum_{i=1}^{m} \sum_{j=1}^{k} W_i u_{w_i,y_j} \tag{3}$$

$$\rho u_{w_i,y_j} = Y_j u_b + Z u_{w_i,y_j} \tag{4}$$

Summing (4) over j produces

$$\rho \sum_{j=1}^{k} u_{w_i,y_j} = \sum_{j=1}^{k} (Y_j u_b + Z u_{w_i,y_j}) = \sum_{j=1}^{k} Y_j u_b + Z \sum_{j=1}^{k} u_{w_i,y_j}$$

$$\rho(\sum_{j=1}^{k} u_{w_i,y_j}) = Y u_b + Z(\sum_{j=1}^{k} u_{w_i,y_j})$$

Subtracting (2) from this produces

$$\rho(\sum_{j=1}^{k} u_{w_i,y_j} - v_z) = Z(\sum_{j=1}^{k} u_{w_i,y_j} - v_z)$$

Then $\sum_{j=1}^{k} u_{w_i,y_j} - v_z = 0$ because ρ is not an eigenvalue of Z. (The spectral radius of a subgraph is strictly smaller than the spectral radius of the graph.) Thus $\sum_{j=1}^{k} u_{w_i,y_j} = v_z$. This means that the sum of the vectors in each sub-bracket of u is equal to the eigenvector centrality of the nodes in Z in the original graph.

Next we consider equation (3).

$$\rho u_b = U u_b + \sum_{i=1}^{m} \sum_{j=1}^{k} W_i u_{w_i,y_j} = U u_b + \sum_{i=1}^{m} W_i \sum_{j=1}^{k} u_{w_i,y_j} = U u_b + (\sum_{i=1}^{m} W_i) v_z = U u_b + W v_z$$

subtracting (1) from this gives

$$\rho(u_b - v_b) = U(u_b - v_b)$$

implying that $u_b = v_b$. Then the eigenvector for the base set remains the same.

To solve for each u_{w_i,y_j} explicitly using v_b of we use equation (4).

$$\rho u_{w_i,y_j} = Y_j u_b + Z u_{w_i,y_j}$$

$$(\rho I - Z) u_{w_i,y_j} = Y_j v_b$$

$$u_{w_i,y_j} = (\rho I - Z)^{-1} Y_j v_b$$

The matrix $(\rho I - Z)^{-1}$ is invertible since ρ is not an eigenvalue of Z.

Specialization in this manner produces km copies of Z. For a particular node in Z one might be interested in how it's centrality in G relates to the centrality of it's copies in $\mathcal{S}(G)$. Each u_{w_i,y_j} represents the centralities of the nodes in a particular copy of Z. Thus, the sum,

$$\sum_{i=1}^{m} \sum_{j=1}^{k} u_{w_i,y_j} = m v_z$$

tells us the the sum of the centrality of the copies is m times the centrality of the node in the original graph where m is the number of entries in W and is the out degree of the component Z.

Finally by equation (4) for any u_{w_i,y_j}, u_{w_s,y_j} with $i \neq s$ we have,

$$\rho u_{w_i,y_j} = Y_j u_b + Z u_{w_i,y_j}$$

$$\rho u_{w_s,y_j} = Y_j u_b + Z u_{w_s,y_j}$$

Subtracting these two equations gives,

$$\rho(u_{w_i,y_j} - u_{w_s,y_j}) = Z(u_{w_i,y_j} - u_{w_s,y_j})$$

Implying that, $u_{w_i,y_j} = u_{w_s,y_j}$ because ρ is not an eigenvalue of Z. Thus, each u_{w_i,y_j} depends only on j.