Centrality and Specialization

Definition (Incoming branch)

Given a graph $G = (V, E, \omega)$, a set $B \subset V$ and a strongly connected component Z of $G|_{\overline{B}}$, let $S_1, S_2 \cdots S_k$ be strongly connected components of $G|_{\overline{B}}$. Let H represent the subgraph $G|_B$.

If there exist edges $e_0 \cdots e_m$ such that,

(i) e_0 is an edge from H to S_1

(ii) e_j is an edge from S_j to S_{j+1} for $1 \leq j \leq k-1$

(iii) e_k is an edge from S_k to Z

The we call the ordered set $\alpha = \{H,\, e_0,\, S_1,\, e_1,\, S_2, ..., S_k,\, e_k,\, Z\}$ an incoming branch from B to Z. We let $In(H, Z)$ denote the set of all incoming branches from H to Z in G.

Definition (Outgoing branch)

Given a graph $G = (V, E, \omega)$, a set $B \subset V$ and a strongly connected component Z of $G|_B$, let $S_1, S_2 \cdots S_k$ are strongly connected components of $G|_B$. Let H represent the subgraph $G|_B$.

If there exist edges $e_0 \cdots e_m$ such that,

(i) e_0 is an edge from Z to S_1

(ii) e_j is an edge from S_j to S_{j+1} for $1 \leq j \leq k-1$

(iii) e_k is an edge from S_k to H

The we call the ordered set $\gamma = \{Z,\, e_0,\, S_1,\, e1,\, S_2, ..., S_k,\, e_k,\, H\}$ an outgoing branch from Z to H. We let $Out(Z, H)$ denote the set of all outgoing branches from Z to H.

With these definitions established we can make a few observations.

Proposition

Let $G(V, E)$ be a strongly connected graph. Let $B \subset V$ and $H = G|_B$. Let Z be a strongly connected component of $G|_{\overline{B}}$. By definition of specialization, the subgraph $H = G|_B$ is also a subgraph of $\mathcal{S}_B(G)$. Then,

(1) There is a nonempty set $C(Z) = \{Z_1,...,Z_k\}$ of copies of Z in the specialized graph $\mathcal{S}_B(G)$.

(2) For each $Z_i \in C(Z)$, the sets $In(H, Z_i)$ and $Out(Z_i, H)$ for the graph $\mathcal{S}_B(G)$ contain one element.

Proof

(1) Since G is strongly connected, there is a path from vertices in B to Z and a path from Z to vertices in B. Then, there must be component branches of G that contain Z. By definition of specialization, there is a copy of Z in $\mathcal{S}_B(G)$ for each component branch of G containing Z. We let $C(Z)$ be the set of all copies of Z in $\mathcal{S}_B(G)$.

(2) Let $Z_i \in C(G)$. Then by definition of specialization, Z_i corresponds with a unique component branch β in G containing Z and Z_i is contained in exactly one component branch $\hat{\beta} = \{v_j,\, e_0,\, S_1,\, e_1,\, S_2,...,S_k,\, e_k,\, v_p\}$ of $\mathcal{S}_B(G)$. Therefore, Z_i must have exactly one incoming branch (the first half of $\hat{\beta}$) and exactly one outgoing branch (the second half of $\hat{\beta}$).

Definition

Let $G = (V, E, \omega)$ have a centrality vector $\mathbf{v}$ and let S be a subgraph of G. We denote the restriction of $\mathbf{v}$ to nodes in S by $\mathbf{v}_S$.

Proposition 1

Let $G = (V,\, E,\, \omega)$ be strongly connected and let $B \subset V$ with $H = G|_B$. Assume Z is a strongly connected component of $G|_{\overline{B}}$ and let $C(Z)$ be the set of copies of Z in $\mathcal{S}_B(G)$. Let $\mathbf{u}$ be the centrality vector of $\mathcal{S}|_B(G)$. Then, if $Z_i, Z_j \in C(Z)$ have the same incoming branch, then $\mathbf{u}_{Z_i} = \mathbf{u}_{Z_j}$. That is, the centralities of nodes in Z_i are the same as the centralities of analogous nodes in Z_j.

Proof

For clarity we let the variable denoting We may write the adjacency matrix of $\mathcal{S}_B(G)$ as follows.

$$M = \begin{bmatrix} H & & & & & W \\ Y_1 & L & & & & \\ & Y_2 & Z_i & & & \\ Y_1 & & & L & & \\ & & & Y_2 & Z_j & \\ Y & & Y_i & & Y_j & X \end{bmatrix}$$

For clarity, we allow an abuse of notation, and let H, Z_i,and Z_j represent the adjacency matrixes of their respective subgraphs in $\mathcal{S}_B(G)$. The matrixes

Y_1, Y_2, Y_i,and Y_j each contain a single entry. The matrix X is the adjacency matrix for the rest of the graph and Y is the adjacency matrix for all links from nodes in B to nodes in X. W represents links from nodes in X to nodes in B. The matrix L is of the form,

$$L = \begin{bmatrix} S_1 & & & & \\ Y_{21} & S_2 & & & \\ & Y_{32} & \ddots & & \\ & & Y_{n(n-1)} & S_n & \end{bmatrix}$$

where each S_k is an adjacency matrix for a strongly connected component in the in-going path to Z_i, and each $Y_{k(k-1)}$ is a matrix with a single entry corresponding to the edge from S_{k-1} to S_k.

We can write M in this form for two reasons. First, because Z_i and Z_j have the same in-going path. Second, because any non-zero entries above the diagonal must be contained in W. Otherwise, they would violate the strongly connected component structure of $\mathcal{S}_B(G)$

To complete the proof, we let $\mathbf{u}$ represent the leading eigenvector of M with associated eigenvalue ρ. Such an eigenvector exists by the Perron-Frobenius theorem. Note that ρ is the spectral radius of M. The vector $\mathbf{u}$ is the centralities of nodes in $\mathcal{S}_B(G)$ and we can partition $\mathbf{u}$ into $\mathbf{u} = \begin{bmatrix} \mathbf{u}_B & \mathbf{u}_L & \mathbf{u}_{Z_i} & \mathbf{u}_L & \mathbf{u}_{Z_j} & \mathbf{u}_X & \end{bmatrix}^T$. This produces the eigenvector equation

$$M\mathbf{u} = \begin{bmatrix} B & & & & & W \\ Y_1 & L & & & & \\ & Y_2 & Z_i & & & \\ Y_1 & & & L & & \\ & & & Y_2 & Z_j & \\ Y & & Y_i & & Y_j & X \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_L \\ \mathbf{u}_{Z_i} \\ \mathbf{u}_L \\ \mathbf{u}_{Z_j} \\ \mathbf{u}_X \end{bmatrix} = \rho \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_L \\ \mathbf{u}_{Z_i} \\ \mathbf{u}_L \\ \mathbf{u}_{Z_j} \\ \mathbf{u}_X \end{bmatrix}$$

Solving for $\mathbf{u}_{Z_i}$ produces,

$$Y_2\mathbf{u}_L + Z_i\mathbf{u}_{Z_i} = \rho\mathbf{u}_{Z_i}$$

$$\mathbf{u}_{Z_i} = (\rho I - Z_i)^{-1} Y_2 \mathbf{u}_L$$

We know that $rhoI - Z$ is invertable because if it is not invertible, there exists a nonzero vector $\mathbf{e}$ such that $(rhoI - Z)\mathbf{e} = \mathbf{0}$. Then, $\rho\mathbf{e} = Z\mathbf{e}$ and ρ is an eigenvalue for Z. However, in a strongly connected graph the spectral radius of a subgraph is strictly smaller than the spectral radius of the graph. Since ρ is the spectral radius of G and Z is a subgraph of G, ρ cannot be an eigenvalue for Z. Thus $\rho I - Z$ is invertable.

Solving for $\mathbf{u}_L$ produces,

$$Y_1\mathbf{u}_B + L\mathbf{u}_L = \rho\mathbf{u}_L$$

$$\mathbf{u}_L = (\rho I - L)^{-1} Y_1 \mathbf{u}_B$$

Thus,

$$\mathbf{u}_{Z_i} = (\rho I - Z_i)^{-1} Y_2 (\rho I - L)^{-1} Y_1 \mathbf{u}_B$$

Solving for $\mathbf{u}_{Z_j}$ similarly produces,

$$\mathbf{u}_{Z_j} = (\rho I - Z_j)^{-1} Y_2 (\rho I - L)^{-1} Y_1 \mathbf{u}_B$$

Since $Z_i = Z_j$, $\mathbf{u}_{Z_i} = \mathbf{u}_{Z_j}$.

Definition (Centrality Transfer Matrix)

Let $G = (V, E, \omega)$ with $B \subset V$ and Z a strongly connected component of $G|_{\overline{B}}$. If α =$\{B$, e_0, S_1, e_1, $S_2, ..., S_k$, e_k, $Z\}\in In(B, Z)$, then the subgraph generated by $alpha$ is the

Definition (Centrality Transfer Matrix)

Let $G = (V, E, \omega)$ have a centrality vector and associated spectral radius ρ, let $B \subset V$ and let Z be a strongly connected component of $G|_B$. If $\alpha = \{B$, e_0, S_1, $e1$, S_2,...,S_k, e_k, Z $\}$ is an incoming path from B to Z then

$$P(\alpha) = (\rho I - Z)^{-1} Y_k (\rho I - S_k)^{-1} Y_{k-1} \cdots (\rho I - S_1)^{-1} Y_0 (\rho I - B)^{-1}$$

is a centrality transfer matrix of α, if

$$\begin{bmatrix} B & & & & & \\ Y_0 & S_1 & & & & \\ & Y_1 & S_2 & & & \\ & & & \ddots & & \\ & & & Y_{k-1} & S_k & \\ & & & & Y_k & Z \end{bmatrix}$$

is an adjacency matrix for the subgraph generated by α.

Lemma

Assume G $= (V, E, \omega)$ is not strongly connected. Let If $\rho > \max\{|\lambda| : \lambda \in \sigma(G)\}$ then,

$$(\rho I - A)^{-1} = \begin{bmatrix} (\rho I - S_1)^{-1} & & & & \\ X_{21} & (\rho I - S_2)^{-1} & & & \\ X_{31} & X_{32} & (\rho I - S_3)^{-1} & & \\ \vdots & \vdots & \ddots & \ddots & \\ X_{k1} & X_{k2} & & X_{kk-1} & (\rho I - S_k)^{-1} \end{bmatrix}$$

where A is an adjacency matrix of G, S_1, S_2,...,S_k are all adjacency matrices of the strongly connected components of G, and each

$$X_{ij} = \sum_{\alpha \in In(S_j,S_i)} P(\alpha).$$

Proof

Since G is not strongly connected, there is a block lower triangular adjacency matrix, A of G such that,

$$A = \begin{bmatrix} S_1 & & & \\ Y_{21} & S_2 & & \\ \vdots & & \ddots & \\ Y_{k1} & \dots & Y_{kk-1} & S_k \end{bmatrix}$$

Since A is block lower triangular, $(\rho I - A)$ and $(\rho I - A)^{-1}$ are also block lower triangular. Since we can write,

$$(\rho I - A) = \begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix}$$

we can write $(\rho I - A)^{-1}$ as,

$$\begin{bmatrix} C_1 & & & \\ X_{21} & C_2 & & \\ \vdots & & \ddots & \\ X_{k1} & \dots & X_{kk-1} & C_k \end{bmatrix}.$$

where for each $i, j \in \{1, 2, ..., k\}$ the matricesX_{ij} and C_i have the same dimensions as Y_{ij} and $(\rho I - S_i)$ respectively. Then it must be the case that,

$$\begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix} \begin{bmatrix} C_1 & & & \\ X_{21} & C_2 & & \\ \vdots & & \ddots & \\ X_{k1} & \dots & X_{kk-1} & C_k \end{bmatrix} = \begin{bmatrix} I_1 & & & \\ & I_2 & & \\ & & \ddots & \\ & & & I_k \end{bmatrix}$$

Where each I_i is the identity matrix with the same dimensions as S_i. Let

$j \in \{1, 2, ...,\text{k}\}$. Then,

$$\begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_j \\ X_{(j+1)j} \\ \vdots \\ X_{kj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying the jth row of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$ produces

$$(\rho I - S_j)C_r = I_j$$
$$C_j = (\rho I - S_j)^{-1} \tag{1}$$

We will show by induction that $X_{ij} = \sum_{\alpha \in In(S_j, S_i)} P(\alpha)$ when $i > j$. As a base case, consider $X_{(j+1)j}$. By multiplying row $(j + 1)$ of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$, we obtain the equations:

$$-Y_{(j+1)j}(\rho I - S_j)^{-1} + (\rho I - S_{(j+1)})X_{(j+1)j} = 0$$

$$X_{(j+1)j} = (\rho I - S_{(j+1)})^{-1} Y_{(j+1)j}(\rho I - S_j)^{-1} \tag{2}$$

Let m be the number of nonzero entries in $Y_{(j+1)j}$ then we can write,

$$Y_{(j+1)j} = \sum_{k=1}^{m} Y_{(j+1)j}^{(k)}$$

where each $Y_{(j+1)j}^{(k)}$ has one non zero entry and that entry is equal to a non-zero entry of $Y_{(j+1)j}$. Thus,

$$X_{(j+1)j} = (\rho I - S_{j+1})^{-1} \sum_{k=1}^{m} Y_{(j+1)j}^{(k)} (\rho I - S_j)^{-1}$$

$$X_{(j+1)j} = \sum_{k=1}^{m} (\rho I - S_{j+1})^{-1} Y_{(j+1)j}^{(k)} (\rho I - S_j)^{-1}$$

For each k, the matrix

$$\begin{bmatrix} S_j & \\ Y_{(j+1)j}^{(k)} & S_{j+1} \end{bmatrix}$$

is an adjacency matrix for $\alpha^{(k)} = \{S_j, e^{(k)}, S_{j+1}\}$ where $e^{(k)}$ is an edge from S_j to S_{j+1}. Then $\alpha^{(k)} \in In(S_j, S_{j+1})$ and

$$X_{(j+1)j} = \sum_{k=1}^{m} P(\alpha^{(k)})$$

We assert that $\cup_{k=1}^{n}\{\alpha^{(k)}\} = In(S_j, S_{j+1})$. Let $\alpha \in In(S_j, S_{j+1})$. Then $\alpha = \{S_j, e, S_{j+1}\}$ because if α contained any other strongly connected component S_l, it would imply that a path exists from S_j to S_l to S_{j+1} and because of the structure of A, it must be the case that $l < j$ or $j+1 < l$. If an edge existed from S_j to S_l to S_{j+1} the matrix A would have an entry above the diagonal this is a contradiction. Thus,

$$X_{(j+1)j} = \sum_{\alpha \in In(S_j, S_{j+1})} P(\alpha)$$

By induction hypothesis assume that when $i < n$,

$$X_{(j+i)j} = \sum_{\alpha \in In(S_j, S_{j+i})} P(\alpha).$$

Consider $X_{(j+n)j}$. By multiplying the $j+n$th row of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$ we obtain the equations,

$$-Y_{(j+n)j}(\rho I - S_j)^{-1} - Y_{(j+n)(j+1)}X_{(j+1)j} \cdots - Y_{(j+n)(j+n-1)}X_{(j+n-1)j} + (\rho I - S_{j+n})X_{(j+n)j} = 0$$

$$X_{(j+n)j} = (\rho I - S_{j+n})^{-1}Y_{(j+n)j}(\rho I - S_j)^{-1} + \sum_{i=1}^{n-1}(\rho I - S_{j+n})^{-1}Y_{(j+n)(j+i)}X_{(j+i)j} \tag{3}$$

As shown in the base case, the first term can be broken up into a sum of centrality transfer matrices . Let m_0 be the number of non-zero entries in $Y_{(j+n)j}$. If we define $Y^{(k)}_{(j+n)j}$ so that each $Y^{(k)}_{(j+1)j}$ has a single nonzero entry that is equal to a distinct non-zero entry of $Y_{(j+1)j}$ and

$$Y_{(j+n)j} = \sum_{k=1}^{m_0} Y^{(k)}_{(j+n)j}$$

then,

$$(\rho I - S_{j+n})^{-1}Y_{(j+n)j}(\rho I - S_j)^{-1} = \sum_{k=1}^{m_0}(\rho I - S_{j+n})^{-1}Y^{(k)}_{(j+n)j}(\rho I - S_j)^{-1}.$$

For each $1 \leq k \leq m_0$, $(\rho I - S_{j+n})^{-1}Y^{(k)}_{(j+n)j}(\rho I - S_j)^{-1}$ is the centrality transfer matrix for a distinct branch α in $In(S_{j+n}, S_j)$ that does not contain any strongly connected components except $S_j + n$ and S_j. Let D_0 denote the set of all such branches. By definition of an adjacency matrix, D_0 contains one branch for each non zero entry in $Y_{(j+n)j}$. Thus,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j} (\rho I - S_j)^{-1} = \sum_{\beta \in D_0} P(\beta) \tag{4}$$

We consider the other terms in the sum (3). Let $1 \leq i \leq n-1$. By hypothesis,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i} X_{(j+i)j} = \sum_{\alpha \in In(S_j, S_{j+i})} (\rho I - S_{j+n})^{-1} Y_{(j+n)j+i} P(\alpha)$$

Let m_i represent the number of nonzero entries in $Y_{(j+n)j+i}$. As before we write $Y_{(j+n)j+i}$ as a sum of m_i single entry matrices. $Y_{(j+n)j+i} = \sum_{k=1}^{m_i} Y_{(j+n)j+i}^{(k)}$. Then,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i} X_{(j+i)j} = \sum_{\alpha \in In(S_j, S_{j+i})} \sum_{k=1}^{m_i} (\rho I - S_{j+n})^{-1} Y_{(j+n)j+i}^{(k)} P(\alpha)$$

It is clear that for each α and k, the term

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i}^{(k)} P(\alpha)$$

is a centrality transfer matrix for some incoming branch $\gamma \in In(S_j, S_j + n)$, because the matrix $Y_{(j+n)j+i}^{(k)}$ is non zero if and only if an edge exists from S_{j+i} to S_{j+n}. If $In(S_j, S_{j+i})$ is non empty, there is a branch from S_j to S_{j+i}, implying that there must be a branch from S_j to S_{j+n} with centrality transfer matrix $(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i}^{(k)} P(\alpha)$.

What we see here is that for a given i, the term

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i} X_{(j+i)j}$$

is equal to the sum of all centrality transfer matrices for the branches in $In(S_j, S_{j+n})$ that pass through S_{j+i} immediately before reaching S_{j+n}. Let D_i denote the set of all such branches. Then,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)j+i} X_{(j+i)j} = \sum_{\gamma \in D_i} P(\gamma) \tag{5}$$

Putting (3), (4), and (5) together gives,

$$X_{j+n,j} = \sum_{\beta \in D_0} P(\beta) + \sum_{i=1}^{n-1} \sum_{\gamma \in D_i} P(\gamma)$$

Let $D = \cup_{i=0}^{n-1} D_i$. Then

$$X_{j+n,j} = \sum_{\alpha \in D} P(\alpha)$$

We assert that $D = In(S_j, S_{j+n})$. Clearly, $D \subset In(S_j, S_{j+n})$. Let $\alpha \in In(S_j, S_{j+n})$. If α has only two components, then $\alpha = \{S_j, e, S_{j+n}\}$ for some edge e and $\alpha \in D_0 \subset D$ by definition of D_0. If α has more then two components, then it has a second to last component, S_{j+i} where $1 \leq i \leq n-1$. By definition of D_i, $\alpha \in D_i$. Thus,

$$X_{j+n,j} = \sum_{\alpha \in In(S_j, S_{j+n})} P(\alpha)$$

This concludes the proof.

Proposition 2

Let $G = (V, E, \omega)$ be strongly connected and let $B \subset V$. Assume Z is a strongly connected component of $G|_{\overline{B}}$. If $C = \{Z_1, Z_2, \cdots Z_k\}$ is the set of all copies of Z in $\mathcal{S}_B(G)$ with the same outgoing branch, then

$$\sum_{Z_i \in C} \mathbf{u}_{Z_i} = \mathbf{v}_Z$$

That is, if we sum together the centralities of each copy of Z with the same outgoing branch, it is equal to the centrality of Z in the original network.

Proof

We write the adjacency matrix for G as follows.

$$A = \begin{bmatrix} B & \begin{bmatrix} & W & \end{bmatrix} \\ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_4 \end{bmatrix} & \begin{bmatrix} T & & \\ Y_3 & Z & \\ Y_5 & Y_6 & X \end{bmatrix} \end{bmatrix}$$

Where T is of the form,

$$T = \begin{bmatrix} S_1 & & & \\ Y_{21} & S_2 & & \\ \vdots & & \ddots & \\ Y_{k1} & \dots & Y_{kk-1} & S_k \end{bmatrix}$$

And each S_i is the adjacency matrix of a strongly connected component of $G|_{\overline{B}}$. We can write A in this form, because there are no edges from Z to componenets in T, or from the rest of the graph X to Z or T because this would violate the assumed strongly connected component structure. Let $\mathbf{v}$ be an eigenvector for ρ. for Consider the eigenvalue equation,

$$A\mathbf{v} = \begin{bmatrix} B & \begin{bmatrix} & W & \end{bmatrix} \\ \begin{bmatrix} Y_1 \\ Y_2 \\ Y_4 \end{bmatrix} & \begin{bmatrix} T & & \\ Y_3 & Z & \\ Y_5 & Y_6 & X \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_T \\ \mathbf{v}_Z \\ \mathbf{v}_X \end{bmatrix} = \rho \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_T \\ \mathbf{v}_Z \\ \mathbf{v}_X \end{bmatrix}$$

Solving for $\mathbf{v}_Z$ produces,

$$\mathbf{v}_Z = (\rho I - Z)^{-1} Y_2 \mathbf{v}_B + (\rho I - Z)^{-1} Y_3 (\rho I - T)^{-1} Y_1 \mathbf{v}_B$$

$$\mathbf{v}_Z = (\rho I - Z)^{-1} \sum_{k=1}^{n_1} Y_2^{(k)} \mathbf{v}_B + (\rho I - Z)^{-1} \sum_{l=1}^{n_3} Y_3^{(l)} (\rho I - T)^{-1} \sum_{m=1}^{n_2} Y_1^{(m)} \mathbf{v}_B$$

$$\mathbf{v}_Z = \sum_{k=1}^{n_1} (\rho I - Z)^{-1} Y_2^{(k)} \mathbf{v}_B + \sum_{l=1}^{n_3} \sum_{m=1}^{n_2} (\rho I - Z)^{-1} Y_3^{(l)} (\rho I - T)^{-1} Y_1^{(m)} \mathbf{v}_B$$

We can show that for each k,

$$(\rho I - Z)^{-1} Y_2^{(k)} \mathbf{v}_B = \mathbf{v}_{Z_i}$$

for some $Z_i \in C$, and for fixed l and m, by lemma,

$$(\rho I - Z)^{-1} Y_3^{(l)} (\rho I - T)^{-1} Y_1^{(m)} \mathbf{v}_B = \mathbf{v}_{Z_i}$$

for some $Z_i \in C$. Showing that every unique $\mathbf{v}_{Z_i}$ is accounted for in the sum will conclude the proof.