MATH 165
Linear Algebra & Diff. Equation
Final
Notes with Examples

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Contents

Chapter 1

Linear Transformations

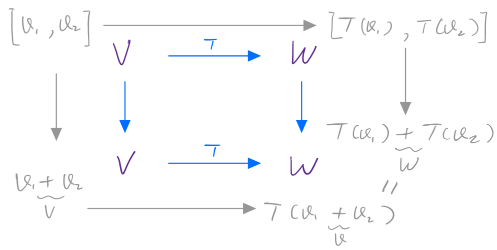

Figure 1.1: Linear Transformation Diagram

1.1 Lecture 17 & 18: Linear Transformation, Kernel & Range

This lecture covers:

- *6.1 Definition of Linear Transformations*
- *6.2 Transformations of* $\mathbb{R}^2$
- *6.3 The Kernel and Range of a Linear Transformation*

1.1.1 Definition of Linear Transformations

Definition

Definition 1.1.1: Mapping

Let V and W be vector spaces. A **mapping** T from V to W is a rule that assigns to each vector $\vec{v}$ in V precisely one vector $\vec{w} = T(\vec{v})$. We write $T : V \to W$.

Linear Transformation is a kind of mapping that preserves the operations of vector addition and scalar multiplication.

Definition 1.1.2: Linear Transformation

Let V and W be vector spaces over the same field. A mapping $T : V \to W$ is a **linear transformation** if for all $\vec{v}_1, \vec{v}_2 \in V$ and all scalars c:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in V$

In the above equations, the operations on the left of the equal signs are the ones defined in the domain V and the ones on the right of the equal signs are the ones defined in the codomain W.

1.1.2 How to prove a transformation is linear

Example.

Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.

Proof. We need to show that $T(p+q) = T(p) + T(q)$ and $T(cp) = cT(p)$ for all $p, q \in P_2$ and $c \in \mathbb{R}$.

Let $p, q \in P_2$ and $c \in \mathbb{R}$.

$$T(p+q) = x^2(p+q)(x) = x^2 p(x) + x^2 q(x) = T(p) + T(q)$$
$$T(cp) = x^2(cp)(x) = cx^2 p(x) = cT(p)$$

Thus, T is linear.

□

Here's a short-cut to test if a transformation is linear:

Theorem 1.1.3

Let V, W be vector spaces over field F. A mapping $T : V \to W$ is a linear transformation if and only if for all $\lambda_1, \lambda_2 \in F$ and all $\vec{v}_1, \vec{v}_2 \in V$:

$$T(\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2) = \lambda_1 T(\vec{v}_1) + \lambda_2 T(\vec{v}_2)$$

Example.

Which of the following are linear transformations?

1. $T : P_2 \to P_1$ given by $T(p) = p'$
2. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, 2x)$
3. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, x^2)$

Proof. We can use the definition of linear transformations to check each case:

1. $T(p) = p'$ is linear since $T(\lambda_1 p_1 + \lambda_2 p_2) = (\lambda_1 p_1 + \lambda_2 p_2)' = \lambda_1 p_1' + \lambda_2 p_2' = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
2. $T(x) = (x, 2x)$ is linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, 2(\lambda_1 x_1 + \lambda_2 x_2)) = \lambda_1(x_1, 2x_1) + \lambda_2(x_2, 2x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$
3. $T(x) = (x, x^2)$ is not linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, (\lambda_1 x_1 + \lambda_2 x_2)^2) \neq \lambda_1(x_1, x_1^2) + \lambda_2(x_2, x_2^2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$

□

Example.

> $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ given by $T(a, b) = a + bx$ is linear.
>
> *Proof.* We want to show: $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$
>
> $T(\lambda_1(a_1, b_1)+\lambda_2(a_2, b_2)) = T(\lambda_1 a_1+\lambda_2 a_2, \lambda_1 b_1+\lambda_2 b_2) = (\lambda_1 a_1+\lambda_2 a_2)+(\lambda_1 b_1+\lambda_2 b_2)x$
> $\lambda_1 T(a_1, b_1)+\lambda_2 T(a_2, b_2) = \lambda_1(a_1+b_1 x)+\lambda_2(a_2+b_2 x) = (\lambda_1 a_1+\lambda_2 a_2)+(\lambda_1 b_1+\lambda_2 b_2)x$
>
> Therefore, $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$
> Thus, T is linear.
>
> $\square$

Three very important examples of linear transformations

Example.

> **1:** A transformation given by a matrix: $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ where A is a fixed $m \times n$ matrix.
>
> **Remark.**
>
> > Note that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$ – we reverse the order in which the dimensions of A are listed.
>
> Check that T is linear:
>
> ***Proof.*** We want to show: $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$
> Let $v_1, v_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$T(\lambda_1 v_1 + \lambda_2 v_2) = A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A v_1 + \lambda_2 A v_2 = \lambda_1 T(v_1) + \lambda_2 T(v_2)$$
>
> Therefore, T is linear. $\square$

Example.

> **2:** Differentiation: $D : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $D(f) = f'$.
> Check that D is linear:
>
> ***Proof.*** We want to show: $D(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 D(f_1) + \lambda_2 D(f_2)$
> Let $f_1, f_2 \in C^1(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$D(\lambda_1 f_1 + \lambda_2 f_2) = (\lambda_1 f_1 + \lambda_2 f_2)' = \lambda_1 f_1' + \lambda_2 f_2' = \lambda_1 D(f_1) + \lambda_2 D(f_2)$$
>
> Therefore, D is linear. $\square$

Example.

> **3:** The Identity map: $I : V \to V$ is defined by $I(v) = v$. For example when $V = \mathbb{R}$ the identity map is $I(x) = x$.
> Check that I is linear:
>
> ***Proof.*** We want to show: $I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 I(v_1) + \lambda_2 I(v_2)$
> Let $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 I(v_1) + \lambda_2 I(v_2)$$
>
> Therefore, I is linear. □

More example of proofing a transformation is linear:

Example.

> Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.
>
> *Proof.* We want to show that $T(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
> Let $p_1, p_2 \in P_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$T(\lambda_1 p_1 + \lambda_2 p_2) = x^2(\lambda_1 p_1 + \lambda_2 p_2)(x) = \lambda_1 x^2 p_1(x) + \lambda_2 x^2 p_2(x) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$$
>
> Therefore, T is linear. □

An example of a transformation that is not linear:

Remark.

> Notice that when disproving a transformation is linear, we only need to find one counterexample. (i.e. one pair of vectors and one scalar that does not satisfy the properties of linearity)

Example.

Is $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(M) = \det(M)$ linear?

Proof. We can choose either disproving using addition or multiplication:
Addition:
proof by contradiction: let A $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$
$T(A+B) = \det(A+B) = \det(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}) = 3(9) - 5(7) = \boxed{-8}$
$T(A) + T(B) = \det(A) + \det(B) = 4 - 6 + 10 - 12 = \boxed{-4}$

$$\boxed{-8 \neq -4}$$

Since $T(A+B) \neq T(A) + T(B)$, T is not linear.
Multiplication:
proof by contradiction: Let A $= I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $\lambda \in \mathbb{R}$ be 2. $T(\lambda A) = \det(\lambda A) = \det(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 2(2) - 0(0) = \boxed{4}$
$\lambda T(A) = \lambda \det(A) = 2(1) - 0(0) = \boxed{2}$

$$\boxed{4 \neq 2}$$

Since $T(\lambda A) \neq \lambda T(A)$, T is not linear.

□

Strategies for proving a transformation is linear:
Guess whether it is linear or not linear, and use that as the hypothesis. If you are trying to find a counter-example and you can't find one, you may want to try proving it is linear. On the other hand, if you want to prove it is linear, you may find the counter-example while trying to prove it is linear.

Example for the notation:

Remark.

Notation:
D: Take the derivative.
D^2: Take the derivative twice.
I: The identity, meaning to return the input.

Example.

> Evaluate $(D^2 + D - 3I)(e^{rt})$
>
> *Proof.*
> $$(D^2 + D - 3I)(e^{rt}) = D^2(e^{rt}) + D(e^{rt}) - 3I(e^{rt})$$
> $$= D(re^{rt}) + re^{rt} - 3e^{rt} = r^2e^{rt} + re^{rt} - 3e^{rt} = (r^2 + r - 3)e^{rt}$$
> $$= (r^2 + r - 3)e^{rt}$$
>
> □

Remark.

> Here, I is a transformation rule or instructions that returns the input. It is important because each term in the transformation has a rule to transform it like the D^2 and D terms. Sometimes, the I term is not written out, but it is always there.

1.1.3 Properties of Linear Transformations

Theorem 1.1.4

For any linear map from any V to any W,
$T(\vec{0_V}) = \vec{0_W}$

Theorem 1.1.5

For any linear map from any V to any W,
$T(-\vec{v}) = -T(\vec{v})$

Proof. Let $T : V \to W$ be a linear transformation.
1. $T(\vec{0_V}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0_W}$
2. $T(\vec{0_V}) = T(\vec{v} + (-\vec{v})) = T(\vec{v}) + T(-\vec{v}) = \vec{0_W}$

□

1.1.4 The Kernel and Range of a Linear Transformation

Definition 1.1.6: Kernel

Let $T : V \to W$ be a linear transformation. The **kernel** of T, denoted by $\ker(T)$, is the set of all vectors in V that are mapped to $\vec{0_W}$ by T.

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0_W}\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then $\ker(T) = \text{null}(A)$

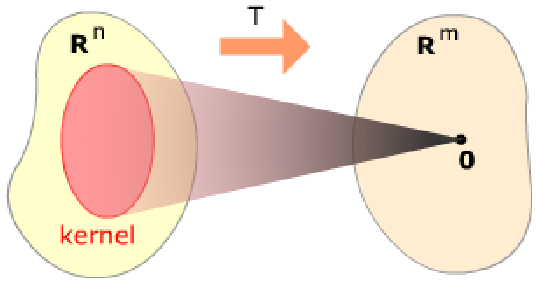

Figure 1.2: Kernel Diagram

Definition 1.1.7

Let $T : V \to W$ be a linear transformation. The **range** of T, denoted by range(T), is the set of all vectors in W that are mapped to by T.

$$\text{range}(T) = \{T(\vec{v}) \in W : \vec{v} \in V\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then range$(T) = \text{col}(A)$

Remark.

> Domain is the set of all possible inputs, codomain is the set of all possible outputs, and range is the set of all actual outputs.

Example.

> $D^2 + D = 3I : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
> Find the kernel and range of $D^2 + D$.
>
> $$\ker(D^2 + D) = \{y \in C^\infty(\mathbb{R}) : (D^2 + D)(y) = 0\}$$
>
> $$= \{y \in C^\infty(\mathbb{R}) : y'' + y' = 0\}$$

Therefore, a homogenous linear differential equation is an example of a kernel.

Example.

Let $T : P_2 \to P_1$ be defined as follow:

$$T(ax^2 + bx + c) = (a + b) + (b - c)x$$

Find the kernel and range of T.

Kernel:

We want $(a + b) + (b - c)x = 0$ in P_2

Which is $0 + 0x$ in P_1

Therefore, $(a + b)$ and $(b - c)$ must be 0.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$a = -c, b = c, c = c$$
$$\ker(T) = \{-cx^2 + cx + c | c \in \mathbb{R}\}$$

Write it as a spanned set:

$$\ker(T) = \text{span}\{-x^2 + x + 1\}$$

Range:

We want $(a + b) + (b - c)x$ in P_1

Let $b + ex$ be a generic element in P_1

Therefore, $d = a + b$ and $e = b - c$

We want to solve for a, b, c in terms of d, e

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & d \\ 0 & 1 & -1 & e \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & d - e \\ 0 & 1 & -1 & e \end{array}\right]$$

$$a = d - e - c, b = e + c, c \text{ is free}$$

$$(a + b) + (b - c)x = (d - e + e + c) + (e + c - c)x = d + ex$$

$$\text{range}(T) = \{d + ex | d, e \in \mathbb{R}\}$$

So the range is all of $P_1(\mathbb{R})$.

Theorem 1.1.8

Let $T : V \to W$ be a linear transformation. Then:

1. $\ker(T)$ is a subspace of V
2. $\text{range}(T)$ is a subspace of W

Proof. Let $T : V \to W$ be a linear transformation.

1. $\ker(T)$ is a subspace of V

1. $\vec{0_V} \in \ker(T)$ since $T(\vec{0_V}) = \vec{0_W}$

2. Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$ and $\lambda \in \mathbb{R}$.
   Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0_W} + \vec{0_W} = \vec{0_W}$
   Therefore, $\vec{v}_1 + \vec{v}_2 \in \ker(T)$

3. Also, $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{0_W} = \vec{0_W}$
   Therefore, $\lambda\vec{v}_1 \in \ker(T)$
   Thus, $\ker(T)$ is a subspace of V.

2. $\text{range}(T)$ is a subspace of W

   1. $\vec{0_W} \in \text{range}(T)$ since $T(\vec{0_V}) = \vec{0_W}$

   2. Let $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$ and $\lambda \in \mathbb{R}$.
      Then there exists $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$.
      Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$
      Therefore, $\vec{w}_1 + \vec{w}_2 \in \text{range}(T)$

   3. Also, $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{w}_1$
      Therefore, $\lambda\vec{w}_1 \in \text{range}(T)$
      Thus, $\text{range}(T)$ is a subspace of W.

□

Example.

> Find the basis for the range and kernel of: $D^2 : P_3 \to P_3$ given by $D^2(p) = p''$

Proof.

$$p(x) = ax^3 + bx^2 + cx + d.$$

Basis for the Kernel:

The kernel of a linear transformation consists of all elements that map to the zero element of the codomain. For the transformation D^2, we need to find all $p \in P_3$ such that $D^2(p) = 0$. Applying D^2 to the general polynomial:

$$p''(x) = (ax^3 + bx^2 + cx + d)'' = (3ax^2 + 2bx + c)'' = 6ax + 2b.$$

Setting $p''(x) = 0$ yields:

$$6ax + 2b = 0.$$

For this to be true for all x, we must have $a = 0$ and $b = 0$. Therefore, $p(x)$ reduces to:

$$p(x) = cx + d.$$

Polynomials of the form $cx + d$ clearly form the kernel of D^2, and are elements of P_1 (the space of all polynomials of degree at most 1). Hence, a basis for the kernel of D^2 consists of the polynomials:

$$\boxed{\{1, x\}}$$

□

Proof. **Basis for the Range:**
The range of D^2 consists of all possible outputs $D^2(p)$ for $p \in P_3$. From the calculation above, we found:

$$D^2(p) = 6ax + 2b.$$

This expression tells us that the output of D^2 can be any polynomial of the form $6ax+2b$, which can be rewritten as a linear combination of the basis $\{x, 1\}$ (or equivalently, $\{6x, 2\}$).

$$\boxed{\{6x, 2\}}$$

□

Theorem 1.1.9: Genearl Rank-Nullity Theorem

Let $T : V \to W$ be a linear transformation. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$$

Example.

Let $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transformation with the rule $S(A) = AA^T$. Find a basis for its kernel and the dimension of its range. Confirm the general rank/nullity theorem based on what we found about its kernel and range previously.

Proof. **Kernel:**
We want to find A such that $AA^T = \vec{0}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2+b^2 = 0, ac+bd = 0, c^2+d^2 = 0$$

$$a = b = c = d = 0$$

Therefore, the kernel is $\{0\}$ and the dimension is 0.
Range:
We want to find A such that $AA^T = B$ for some $B \in M_2(\mathbb{R})$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

The basis for it will be:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore, the range is all of $M_2(\mathbb{R})$ and the dimension is 4.
Rank-Nullity Theorem:

$$\dim(M_2(\mathbb{R})) = \dim(\ker(S)) + \dim(\text{range}(S))$$
$$\boxed{4 \neq 0 + 3}$$

□

This is not a linear transformation because it does not satisfy the properties of linearity. Therefore, the rank-nullity theorem does not hold.

Example.

Consider the linear transformation $T : M_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - b + d) + (-a + b - d)x^2$$

Determine range(T). What is the dimension of ker(T)? Determine a basis for ker(T).

Step 1: Verify it is linear:

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$T(A{+}B) = T\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right) = (a_1{+}a_2{-}b_1{-}b_2{+}d_1{+}d_2){+}({-}a_1{-}a_2{+}b_1{+}b_2{-}d_1{-}d_2)x^2$

$T(A){+}T(B) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right){+}T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = (a_1{-}b_1{+}d_1){+}({-}a_1{+}b_1{-}d_1)x^2{+}(a_2{-}b_2{+}d_2){+}({-}a_2{+}b_2{-}d_2)$

$$(a_1 + a_2 - b_1 - b_2 + d_1 + d_2) + (-a_1 - a_2 + b_1 + b_2 - d_1 - d_2)x^2$$

$$T(A + B) = T(A) + T(B)$$

Therefore, T is linear.

1.1.5 Linear transformation and basis

If we have a matrix A that's m by n, we get a free transformation. $T : f_n \to f_m$ given by $T(x) = Ax$ if V and W are finite-dimensional, then $T : V \to W$ can be accomplished by a matrix multiplication. That is there exist a matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in V$.

Theorem 1.1.10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists an $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

We have seen that, given an $m \times n$ matrix A, we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(v) = Av$. (Remember that an $m \times n$ matrix maps $\mathbb{R}^n$ to $\mathbb{R}^m$.)

Now, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we identify an $m \times n$ matrix A such that $T(v) = Av$.

This is possible, because if we know how T acts on a basis $\{b_1, \cdots, b_n\}$ of V, then we know how T acts on the whole of V, since if $v = c_1b_1 + \cdots + c_kb_k$, then

$$T(v) = T(c_1b_1 + \cdots + c_kb_k) = c_1T(b_1) + \cdots + c_kT(b_k)$$

In particular, if $T : \mathbb{R}^n \to \mathbb{R}^m$ and if $v = (c_1, c_2, \cdots, c_n) = c_1e_1 + c_2e_2 + \cdots + c_ne_n$, then

$$T(v) = T(c_1e_1 + c_2e_2 + \cdots + c_ne_n) = \boxed{c_1T(e_1) + c_2T(e_2) + \cdots + c_nT(e_n)}$$

By the definition of matrix multiplication, the boxed expression is equal to

$$\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then $T(v) = Av$ where $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$.

Example.

Find A. $T_1\mathbb{R}^2 \to \mathbb{R}^2$

$$T(x) = \beta$$

$$\beta = \{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$$

$$T(x) = (a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n)$$

$$= a_1T(\vec{e_1}) + a_2T(\vec{e_2}) + \cdots + a_nT(\vec{e_n})$$

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$$

If I take the matrix A and apply it to the coefficient vector, I get the transformation.

$$A \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= a_1T(\vec{e_1}) + a_2T(\vec{e_2}) + \cdots + a_nT(\vec{e_n})$$

$$= T(a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n})$$

Example.

Find a matrix A such that $T(x) = A(x)$ where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(a,b,c) = (c, a+b)$.

Since this is from $\mathbb{R}^3$ to $\mathbb{R}^2$, the matrix A will be 2×3.

$$T(a,b,c) = (c, a+b)$$

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix}$$

$$A = \begin{bmatrix} T(1,0,0) & T(0,1,0) & T(0,0,1) \end{bmatrix}$$

$$= A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c \\ a+b \end{bmatrix}$$

$$\boxed{= (c, a+b)}$$

Chapter 2

Eigenvalues and Eigenvectors

2.1 Lecture 20: Eigenspaces & Eigenbase

This lecture covers:

- *7.1 The Eigenvalue/Eigenvector Problem*
- *7.2 General Results on Eigenvalues and Eigenvectors*
- *7.3 Diagonalization*

Chapter 3

Systems of Linear Differential Equations

3.1 Lecture 24: First Order Linear Systems

This lecture covers:

- *9.1 First Order Linear Systems*
- *9.2 Vector Formulation*
- *9.3 General Results for First Order Linear Systems*

Remark.

> Today we are going to cover 9.1 to 9.3. We are going to skip many theorems.
> Next week we are going to do some examples for chapter 8.

3.1.1 Chapter 9.1: First Order Linear Systems

An example of the first-order linear system is:

Example.

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + 2x_2$$
$$x_1(0) = 1, x_2(0) = 0$$

If $b_i(t) = 0$, the system is homogenous.

A **solution** is an ordered n-tuple of functions $x_1(t), x_2(t), \ldots, x_n(t)$ that satisfies the system of equations.

Example.

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 + 2x_2$$
$$x_1(0) = 1, x_2(0) = 0$$
$$x_1(t) = e^t, x_2(t) = e^t$$

> is a solution.

There's a trick to solving the 2 by 2 system, however, we are going to skip it.

Remark.

> The first-order linear system is restrictive. However, we can transform a higher-order system by renaming functions.

Example.

$$\frac{d^2x}{dt^2} + 4e^t \frac{dx}{dt} - 9t^2 x = 7t^2$$

Strataegy: Let $x_1 = x$, $x_1' = x_2$

$$x_1' = x_2$$

$$x_2' = 9t^2 x + 4e^t x_2 - 7t^2$$

(Chapter 9.1)

3.1.2 Chapter 9.2: Vector Formulation

Remark.

Chapter 9.2: How to transform the system into a matrix form.

Example.

Formulate with vectors.

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

$$\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix}$$

Coefficient matrix:

$$\vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdots \\ b_n(t) \end{bmatrix}$$

Theorem 3.1.1

$V_n(I)$ is a column vector of n functions defined on an interval I.

Example.

$$\begin{bmatrix} e^{3t} \\ 2 \\ e^{7t} \end{bmatrix} \in V_3(\mathbb{R})$$

for any fixed n, I, $V_n(I)$ is a vector space.

Wronskian

Remark.

> We do not test this

Wronskan of a set of n column vectors in $V_n(I)$

Wronskien of $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$
$W(t) = \text{Wronskian} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$

Theorem 3.1.2

If $W(t) \neq 0$ for all $t \in I$, then $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is linearly independent.

Theorem 3.1.3

If $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is linearly independent, then $W(t) \neq 0$ for all $t \in I$.

3.1.3 Chapter 9.3: Genearl results for first-order linear systems

Theorem 3.1.4: Initial value problem

$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{b}(t)$ and $\vec{x}(t_0) = \vec{x}_0$ has a unique solution on an interval I containing t_0 if $A(t)$ and $\vec{b}(t)$ are continuous on I.

Theorem 3.1.5: 9.3.2

The set of solutions to the homogenous system $\vec{x}'(t) = A(t)\vec{x}(t)$ is a vector space of dimension n.

Fundamental solution set

The fundamental solution set is basically a basis for the solution space.

Theorem 3.1.6: The fundamental solution set

$$S = \{x_1, x_2, \cdots, x_n\}$$

S is a set of solutions to the homogenous system $\vec{x}'(t) = A(t)\vec{x}(t)$ that are linearly independent.

The non-homogenous case:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n + \boxed{\vec{x}_p}$$

We add a single solution to the homogenous system to the solution of the non-homogenous system.

Simplifying assumptions:

$$\vec{x}'(t) = A\vec{x}(t)$$

is homogenous and A is a matrix of constants and A is non-defective.

Remark.

> Recall: Non-defective means that the matrix has n linearly independent eigenvectors.

Theorem 3.1.7

Let A be a $n \times n$ matrix of real constants, and let λ be a real eigenvalue of corresponding to the eigenvector $\vec{v}$.

Then

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a solution to the homogenous system $\vec{x}'(t) = A\vec{x}(t)$.

Remark.

> Recall: Means that $A\vec{v} = \lambda\vec{v}$

Proof. If $\vec{x}(t) = e^{\lambda t}\vec{v}$, then

$$A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v} = \lambda e^{\lambda t}\vec{v}$$

On the other side, if I take the derivative of $\vec{x}(t)$, I get

$$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$$

We notice that the two sides are equal. □

And we are going to have enough solutions to form a basis for the solution space.

Theorem 3.1.8

If A has n real eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ with corresponding real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then

$$\vec{x}_k = e^{\lambda_k t}\vec{v}_k$$

is a solution to the homogenous system $\vec{x}'(t) = A\vec{x}(t)$.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

The eigenvalue is:

$$|A - \lambda I| = (1 - \lambda) \cdot (-2 - \lambda) - 4 = 0$$
$$\lambda = 2, -3$$

nullspace of $A - 2I$ is $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

nullspace of $A + 3I$ is $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

What's new today:
Solution =

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$x_1 = 2c_1 e^{2t} - c_2 e^{-3t}$$
$$x_2 = c_1 e^{2t} + 2c_2 e^{-3t}$$

Initial values:

$$x_1(0) = 2c_1 - c_2 = 1$$

$$x_2(0) = c_1 + 2c_2 = 0$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -1/5 \end{array}\right]$$

Example.

$$x' = A\vec{x}$$

$$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

$$x_1' = 4x_2$$

$$x_2' = x_1$$

Step 1: Find the eigenvalues and eigenvectors

$$|A - \lambda I| = \begin{bmatrix} -\lambda & 4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 4 = 0$$

$$\lambda_1 = 2, \lambda_2 = -2$$

For $\lambda_1 = 2$

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

For $\lambda_2 = -2$

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Step 2: Write the general solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Remark.

If you can not zero out the second role, it means that you are wrong.

Let's do a 3 by 3 example. We will leave the complex case for the next lecture.s

Example.

$$A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 1: Find the eigenvalues and eigenvectors

$$|A - \lambda I| = \begin{bmatrix} 0-\lambda & -3 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

Characteristic polynomial:

$$(2-\lambda)(\lambda-3)(\lambda-2) = 0$$

$$\lambda_1 = 2, \lambda_2 = -3, \lambda_3 = 2$$

For $\lambda_1 = 2$

$$\begin{bmatrix} -2 & -3 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 are free

$$-2x_1 = 3x_2 - x_3$$

$$x_1 = -\frac{3}{2}x_2 + \frac{1}{2}x_3$$

$$\vec{v}_1 = (3, -2, 0)$$

$$\vec{v}_2 = (1, 0, 2)$$

For $\lambda_2 = -3$

$$\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free

$$x_1 = x_2$$

$$x_3 = 0$$

$$\vec{v}_3 = (1, 1, 0)$$

Step 2: Write the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$