MATH 165
Linear Algebra & Diff. Equation
Final
Notes with Examples

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Contents

Chapter 1

Linear Transformations

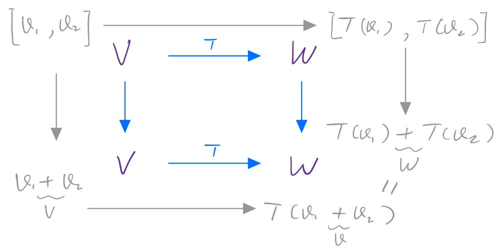

Figure 1.1: Linear Transformation Diagram

1.1 Lecture 17 & 18: Linear Transformation, Kernel & Range

This lecture covers:

- *6.1 Definition of Linear Transformations*
- *6.2 Transformations of* $\mathbb{R}^2$
- *6.3 The Kernel and Range of a Linear Transformation*

1.1.1 Definition of Linear Transformations

Definition

Definition 1.1.1: Mapping

Let V and W be vector spaces. A **mapping** T from V to W is a rule that assigns to each vector $\vec{v}$ in V precisely one vector $\vec{w} = T(\vec{v})$. We write $T : V \to W$.

Linear Transformation is a kind of mapping that preserves the operations of vector addition and scalar multiplication.

Definition 1.1.2: Linear Transformation

Let V and W be vector spaces over the same field. A mapping $T : V \to W$ is a **linear transformation** if for all $\vec{v}_1, \vec{v}_2 \in V$ and all scalars c:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in V$

In the above equations, the operations on the left of the equal signs are the ones defined in the domain V and the ones on the right of the equal signs are the ones defined in the codomain W.

1.1.2 How to prove a transformation is linear

Example.

Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.

Proof. We need to show that $T(p+q) = T(p) + T(q)$ and $T(cp) = cT(p)$ for all $p, q \in P_2$ and $c \in \mathbb{R}$.

Let $p, q \in P_2$ and $c \in \mathbb{R}$.

$$T(p+q) = x^2(p+q)(x) = x^2 p(x) + x^2 q(x) = T(p) + T(q)$$
$$T(cp) = x^2(cp)(x) = cx^2 p(x) = cT(p)$$

Thus, T is linear.

□

Here's a short-cut to test if a transformation is linear:

Theorem 1.1.3

Let V, W be vector spaces over field F. A mapping $T : V \to W$ is a linear transformation if and only if for all $\lambda_1, \lambda_2 \in F$ and all $\vec{v}_1, \vec{v}_2 \in V$:

$$T(\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2) = \lambda_1 T(\vec{v}_1) + \lambda_2 T(\vec{v}_2)$$

Example.

Which of the following are linear transformations?

1. $T : P_2 \to P_1$ given by $T(p) = p'$
2. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, 2x)$
3. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, x^2)$

Proof. We can use the definition of linear transformations to check each case:

1. $T(p) = p'$ is linear since $T(\lambda_1 p_1 + \lambda_2 p_2) = (\lambda_1 p_1 + \lambda_2 p_2)' = \lambda_1 p_1' + \lambda_2 p_2' = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
2. $T(x) = (x, 2x)$ is linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, 2(\lambda_1 x_1 + \lambda_2 x_2)) = \lambda_1(x_1, 2x_1) + \lambda_2(x_2, 2x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$
3. $T(x) = (x, x^2)$ is not linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, (\lambda_1 x_1 + \lambda_2 x_2)^2) \neq \lambda_1(x_1, x_1^2) + \lambda_2(x_2, x_2^2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$

□

Example.

$T : \mathbb{R}^2 \to P_1(\mathbb{R})$ given by $T(a, b) = a + bx$ is linear.

Proof. We want to show: $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$

$$T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = T(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 b_1 + \lambda_2 b_2) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x$$
$$\lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2) = \lambda_1(a_1 + b_1 x) + \lambda_2(a_2 + b_2 x) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x$$

Therefore, $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$
Thus, T is linear.

□

Three very important examples of linear transformations

Example.

1: A transformation given by a matrix: $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ where A is a fixed $m \times n$ matrix.

Remark.

Note that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$ – we reverse the order in which the dimensions of A are listed.

Check that T is linear:

Proof. We want to show: $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$
Let $v_1, v_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$T(\lambda_1 v_1 + \lambda_2 v_2) = A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A v_1 + \lambda_2 A v_2 = \lambda_1 T(v_1) + \lambda_2 T(v_2)$$

Therefore, T is linear. □

Example.

2: Differentiation: $D : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $D(f) = f'$.
Check that D is linear:

Proof. We want to show: $D(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 D(f_1) + \lambda_2 D(f_2)$
Let $f_1, f_2 \in C^1(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$D(\lambda_1 f_1 + \lambda_2 f_2) = (\lambda_1 f_1 + \lambda_2 f_2)' = \lambda_1 f_1' + \lambda_2 f_2' = \lambda_1 D(f_1) + \lambda_2 D(f_2)$$

Therefore, D is linear. □

Example.

> **3:** The Identity map: $I : V \to V$ is defined by $I(v) = v$. For example when $V = \mathbb{R}$ the identity map is $I(x) = x$.
> Check that I is linear:
>
> ***Proof.*** We want to show: $I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 I(v_1) + \lambda_2 I(v_2)$
> Let $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 I(v_1) + \lambda_2 I(v_2)$$
>
> Therefore, I is linear. $\square$

More example of proofing a transformation is linear:
Example.

> Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.
>
> *Proof.* We want to show that $T(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
> Let $p_1, p_2 \in P_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$T(\lambda_1 p_1 + \lambda_2 p_2) = x^2(\lambda_1 p_1 + \lambda_2 p_2)(x) = \lambda_1 x^2 p_1(x) + \lambda_2 x^2 p_2(x) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$$
>
> Therefore, T is linear. $\square$

An example of a transformation that is not linear:
Remark.

> Notice that when disproving a transformation is linear, we only need to find one counterexample. (i.e. one pair of vectors and one scalar that does not satisfy the properties of linearity)

Example.

Is $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(M) = \det(M)$ linear?

Proof. We can choose either disproving using addition or multiplication:
Addition:
proof by contradiction: let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$
$T(A+B) = \det(A+B) = \det(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}) = 3(9) - 5(7) = \boxed{-8}$
$T(A) + T(B) = \det(A) + \det(B) = 4 - 6 + 10 - 12 = \boxed{-4}$

$$\boxed{-8 \neq -4}$$

Since $T(A+B) \neq T(A) + T(B)$, T is not linear.
Multiplication:
proof by contradiction: Let A = $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $\lambda \in \mathbb{R}$ be 2. $T(\lambda A) = \det(\lambda A) = \det(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 2(2) - 0(0) = \boxed{4}$
$\lambda T(A) = \lambda \det(A) = 2(1) - 0(0) = \boxed{2}$

$$\boxed{4 \neq 2}$$

Since $T(\lambda A) \neq \lambda T(A)$, T is not linear.

$\square$

Strategies for proving a transformation is linear:
Guess whether it is linear or not linear, and use that as the hypothesis. If you are trying to find a counter-example and you can't find one, you may want to try proving it is linear. On the other hand, if you want to prove it is linear, you may find the counter-example while trying to prove it is linear.

Example for the notation:

Remark.

Notation:
D: Take the derivative.
D^2: Take the derivative twice.
I: The identity, meaning to return the input.

Example.

> Evaluate $(D^2 + D - 3I)(e^{rt})$
>
> *Proof.*
> $$(D^2 + D - 3I)(e^{rt}) = D^2(e^{rt}) + D(e^{rt}) - 3I(e^{rt})$$
> $$= D(re^{rt}) + re^{rt} - 3e^{rt} = r^2e^{rt} + re^{rt} - 3e^{rt} = (r^2 + r - 3)e^{rt}$$
> $$= (r^2 + r - 3)e^{rt}$$
>
> □

Remark.

> Here, I is a transformation rule or instructions that returns the input. It is important because each term in the transformation has a rule to transform it like the D^2 and D terms. Sometimes, the I term is not written out, but it is always there.

1.1.3 Properties of Linear Transformations

Theorem 1.1.4

For any linear map from any V to any W,
$T(\vec{0_V}) = \vec{0_W}$

Theorem 1.1.5

For any linear map from any V to any W,
$T(-\vec{v}) = -T(\vec{v})$

Proof. Let $T : V \to W$ be a linear transformation.
1. $T(\vec{0_V}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0_W}$
2. $T(\vec{0_V}) = T(\vec{v} + (-\vec{v})) = T(\vec{v}) + T(-\vec{v}) = \vec{0_W}$

□

1.1.4 The Kernel and Range of a Linear Transformation

Definition 1.1.6: Kernel

Let $T : V \to W$ be a linear transformation. The **kernel** of T, denoted by $\ker(T)$, is the set of all vectors in V that are mapped to $\vec{0_W}$ by T.

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0_W}\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then $\ker(T) = \text{null}(A)$

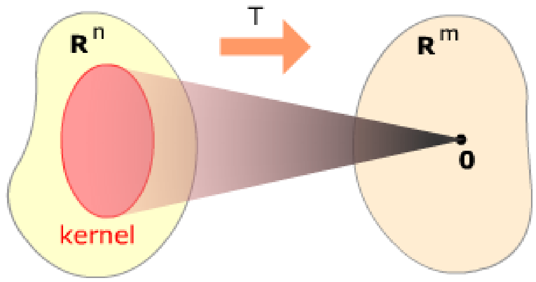

Figure 1.2: Kernel Diagram

Definition 1.1.7

Let $T : V \to W$ be a linear transformation. The **range** of T, denoted by range(T), is the set of all vectors in W that are mapped to by T.

$$\text{range}(T) = \{T(\vec{v}) \in W : \vec{v} \in V\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then $\text{range}(T) = \text{col}(A)$

Remark.

> Domain is the set of all possible inputs, codomain is the set of all possible outputs, and range is the set of all actual outputs.

Example.

> $D^2 + D = 3I : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
> Find the kernel and range of $D^2 + D$.
>
> $$\ker(D^2 + D) = \{y \in C^\infty(\mathbb{R}) : (D^2 + D)(y) = 0\}$$
>
> $$= \{y \in C^\infty(\mathbb{R}) : y'' + y' = 0\}$$

Therefore, a homogenous linear differential equation is an example of a kernel.

Example.

Let $T : P_2 \to P_1$ be defined as follow:

$$T(ax^2 + bx + c) = (a + b) + (b - c)x$$

Find the kernel and range of T.

Kernel:

We want $(a + b) + (b - c)x = 0$ in P_2

Which is $0 + 0x$ in P_1

Therefore, $(a + b)$ and $(b - c)$ must be 0.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$a = -c, b = c, c = c$$
$$\ker(T) = \{-cx^2 + cx + c | c \in \mathbb{R}\}$$

Write it as a spanned set:

$$\ker(T) = \text{span}\{-x^2 + x + 1\}$$

Range:

We want $(a + b) + (b - c)x$ in P_1

Let $b + ex$ be a generic element in P_1

Therefore, $d = a + b$ and $e = b - c$

We want to solve for a, b, c in terms of d, e

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & d \\ 0 & 1 & -1 & e \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & d - e \\ 0 & 1 & -1 & e \end{array}\right]$$

$$a = d - e - c, b = e + c, c \text{ is free}$$

$$(a + b) + (b - c)x = (d - e + e + c) + (e + c - c)x = d + ex$$

$$\text{range}(T) = \{d + ex | d, e \in \mathbb{R}\}$$

So the range is all of $P_1(\mathbb{R})$.

Theorem 1.1.8

Let $T : V \to W$ be a linear transformation. Then:

1. $\ker(T)$ is a subspace of V
2. $\text{range}(T)$ is a subspace of W

Proof. □

Example.

Theorem 1.1.9: Genearl Rank-Nullity Theorem

Let $T : V \to W$ be a linear transformation. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$$

Example.

Example.

1.1.5 Linear transformation and basis