

**MATH 165**  
**Linear Algebra & Diff. Equation**  
**Final**  
**Notes with Examples**

**Professor Kalyani Madhu**

by Ethan

University of Rochester

Spring 2024

# Contents

<b>1</b>	<b>Linear Transformations</b>	<b>2</b>
1.1	Lecture 17 & 18: Linear Transformation, Kernel & Range . . . . .	3
1.1.1	Definition of Linear Transformations . . . . .	3
1.1.2	How to prove a transformation is linear . . . . .	4
1.1.3	Properties of Linear Transformations . . . . .	8
1.1.4	The Kernel and Range of a Linear Transformation . . . . .	8
1.1.5	Linear transformation and basis . . . . .	14
<b>2</b>	<b>Eigenvalues and Eigenvectors</b>	<b>17</b>
2.1	Lecture 19: Kernal Range and Eigenvalues . . . . .	18
2.1.1	Eigenvalues and Eigenvectors . . . . .	18
2.1.2	Complex Eigenvalues . . . . .	21
2.1.3	Properties and Theorems . . . . .	22
2.2	Lecture 20: Eigenspaces & Eigenbase . . . . .	23
2.2.1	Eigenspaces and Non-defective Matrices . . . . .	23
<b>3</b>	<b>Linear Differential Equations of Order n</b>	<b>32</b>
3.1	Lecture 21: Linear Differential Equations . . . . .	33
3.1.1	General Theory of Linear Differential Equations . . . . .	33
3.1.2	Constant Coefficient Homogeneous Linear Differential Equations . . .	35
3.2	Lecture 22: Annihilators . . . . .	41
3.3	Lecture 23: Oscillations of a Mechanical System . . . . .	42
<b>4</b>	<b>Systems of Linear Differential Equations</b>	<b>43</b>
4.1	Lecture 24: First Order Linear Systems . . . . .	44
4.1.1	Chapter 9.1: First Order Linear Systems . . . . .	44
4.1.2	Chapter 9.2: Vector Formulation . . . . .	45
4.1.3	Chapter 9.3: Genearl results for first-order linear systems . . . . .	46
4.2	Lecture 25: . . . . .	52
4.2.1	Complex-Valued Solutions . . . . .	52
<b>5</b>	<b>Review</b>	<b>56</b>
5.1	S15 FQ6 . . . . .	57

## Chapter 1

# Linear Transformations

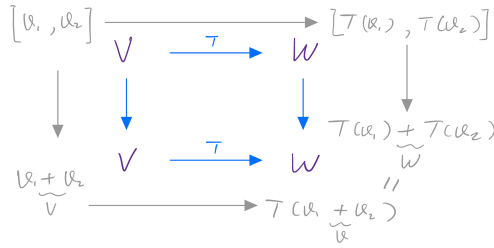


Figure 1.1: Linear Transformation Diagram

## 1.1 Lecture 17 & 18: Linear Transformation, Kernel & Range

*This lecture covers:*

- 6.1 Definition of Linear Transformations
- 6.2 Transformations of  $\mathbb{R}^2$
- 6.3 The Kernel and Range of a Linear Transformation

### 1.1.1 Definition of Linear Transformations

**Definition**

#### Definition 1.1.1: Mapping

Let  $V$  and  $W$  be vector spaces. A **mapping**  $T$  from  $V$  to  $W$  is a rule that assigns to each vector  $\vec{v}$  in  $V$  precisely one vector  $\vec{w} = T(\vec{v})$ . We write  $T : V \rightarrow W$ .

Linear Transformation is a kind of mapping that preserves the operations of vector addition and scalar multiplication.

#### Definition 1.1.2: Linear Transformation

Let  $V$  and  $W$  be vector spaces over the same field. A mapping  $T : V \rightarrow W$  is a **linear transformation** if for all  $\vec{v}_1, \vec{v}_2 \in V$  and all scalars  $c$ :

1.  $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$  for all  $\vec{u}, \vec{v} \in V$
2.  $T(c\vec{v}) = cT(\vec{v})$  for all  $\vec{v} \in V$

In the above equations, the operations on the left of the equal signs are the ones defined in the domain  $V$  and the ones on the right of the equal signs are the ones defined in the codomain  $W$ .

### 1.1.2 How to prove a transformation is linear

**Example.**

Show  $T : P_2 \rightarrow P_4$  given by  $T(p) = x^2p(x)$  is linear.

*Proof.* We need to show that  $T(p+q) = T(p) + T(q)$  and  $T(cp) = cT(p)$  for all  $p, q \in P_2$

and  $c \in \mathbb{R}$ .

Let  $p, q \in P_2$  and  $c \in \mathbb{R}$ .

$$\begin{aligned} T(p+q) &= x^2(p+q)(x) = x^2p(x) + x^2q(x) = T(p) + T(q) \\ T(cp) &= x^2(cp)(x) = cx^2p(x) = cT(p) \end{aligned}$$

Thus,  $T$  is linear. □

Here's a short-cut to test if a transformation is linear:

#### Theorem 1.1.3

Let  $V, W$  be vector spaces over field  $F$ . A mapping  $T : V \rightarrow W$  is a linear transformation if and only if for all  $\lambda_1, \lambda_2 \in F$  and all  $\vec{v}_1, \vec{v}_2 \in V$ :

$$T(\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2) = \lambda_1T(\vec{v}_1) + \lambda_2T(\vec{v}_2)$$

**Example.**

Which of the following are linear transformations?

1.  $T : P_2 \rightarrow P_1$  given by  $T(p) = p'$
2.  $T : \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $T(x) = (x, 2x)$
3.  $T : \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $T(x) = (x, x^2)$

*Proof.* We can use the definition of linear transformations to check each case:

1.  $T(p) = p'$  is linear since  $T(\lambda_1p_1 + \lambda_2p_2) = (\lambda_1p_1 + \lambda_2p_2)' = \lambda_1p_1' + \lambda_2p_2' = \lambda_1T(p_1) + \lambda_2T(p_2)$
2.  $T(x) = (x, 2x)$  is linear since  $T(\lambda_1x_1 + \lambda_2x_2) = (\lambda_1x_1 + \lambda_2x_2, 2(\lambda_1x_1 + \lambda_2x_2)) = \lambda_1(x_1, 2x_1) + \lambda_2(x_2, 2x_2) = \lambda_1T(x_1) + \lambda_2T(x_2)$
3.  $T(x) = (x, x^2)$  is not linear since  $T(\lambda_1x_1 + \lambda_2x_2) = (\lambda_1x_1 + \lambda_2x_2, (\lambda_1x_1 + \lambda_2x_2)^2) \neq \lambda_1(x_1, x_1^2) + \lambda_2(x_2, x_2^2) = \lambda_1T(x_1) + \lambda_2T(x_2)$

□

**Example.**

$T : \mathbb{R}^2 \rightarrow P_1(\mathbb{R})$  given by  $T(a, b) = a + bx$  is linear.

*Proof.* We want to show:  $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$

$$T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = T(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 b_1 + \lambda_2 b_2) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x \\ \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2) = \lambda_1(a_1 + b_1 x) + \lambda_2(a_2 + b_2 x) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x$$

Therefore,  $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$

Thus,  $T$  is linear. □

**Three very important examples of linear transformations****Example.**

**1:** A transformation given by a matrix:  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by  $T(x) = Ax$  where  $A$  is a fixed  $m \times n$  matrix.

**Remark.**

Note that the domain of  $T$  is  $\mathbb{R}^n$  and the codomain is  $\mathbb{R}^m$  – we reverse the order in which the dimensions of  $A$  are listed.

Check that  $T$  is linear:

**Proof.** We want to show:  $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$

Let  $v_1, v_2 \in \mathbb{R}^n$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

$$T(\lambda_1 v_1 + \lambda_2 v_2) = A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A v_1 + \lambda_2 A v_2 = \lambda_1 T(v_1) + \lambda_2 T(v_2)$$

Therefore,  $T$  is linear. □

**Example.**

**2:** Differentiation:  $D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R})$  given by  $D(f) = f'$ .

Check that  $D$  is linear:

**Proof.** We want to show:  $D(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 D(f_1) + \lambda_2 D(f_2)$

Let  $f_1, f_2 \in C^1(\mathbb{R})$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

$$D(\lambda_1 f_1 + \lambda_2 f_2) = (\lambda_1 f_1 + \lambda_2 f_2)' = \lambda_1 f_1' + \lambda_2 f_2' = \lambda_1 D(f_1) + \lambda_2 D(f_2)$$

Therefore,  $D$  is linear. □

**Example.**

**3:** The Identity map:  $I : V \rightarrow V$  is defined by  $I(v) = v$ . For example when  $V = \mathbb{R}$  the identity map is  $I(x) = x$ .

Check that  $I$  is linear:

**Proof.** We want to show:  $I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 I(v_1) + \lambda_2 I(v_2)$   
Let  $v_1, v_2 \in V$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

$$I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 I(v_1) + \lambda_2 I(v_2)$$

Therefore,  $I$  is linear. □

More example of proofing a transformation is linear:

**Example.**

Show  $T : P_2 \rightarrow P_4$  given by  $T(p) = x^2 p(x)$  is linear.

*Proof.* We want to show that  $T(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$

Let  $p_1, p_2 \in P_2$  and  $\lambda_1, \lambda_2 \in \mathbb{R}$ .

$$T(\lambda_1 p_1 + \lambda_2 p_2) = x^2(\lambda_1 p_1 + \lambda_2 p_2)(x) = \lambda_1 x^2 p_1(x) + \lambda_2 x^2 p_2(x) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$$

Therefore,  $T$  is linear. □

An example of a transformation that is not linear:

**Remark.**

Notice that when disproving a transformation is linear, we only need to find one counterexample. (i.e. one pair of vectors and one scalar that does not satisfy the properties of linearity)

**Example.**

Is  $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$  given by  $T(M) = \det(M)$  linear?

*Proof.* We can choose either disproving using addition or multiplication:

**Addition:**

proof by contradiction: let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

$$T(A + B) = \det(A + B) = \det\left(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}\right) = 3(9) - 5(7) = \boxed{-8}$$

$$T(A) + T(B) = \det(A) + \det(B) = 4 - 6 + 10 - 12 = \boxed{-4}$$

$$\boxed{-8 \neq -4}$$

Since  $T(A + B) \neq T(A) + T(B)$ ,  $T$  is not linear.

**Multiplication:**

proof by contradiction: Let  $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , let  $\lambda \in \mathbb{R}$  be 2.  $T(\lambda A) = \det(\lambda A) =$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 2(2) - 0(0) = \boxed{4}$$

$$\lambda T(A) = \lambda \det(A) = 2(1) - 0(0) = \boxed{2}$$

$$\boxed{4 \neq 2}$$

Since  $T(\lambda A) \neq \lambda T(A)$ ,  $T$  is not linear.

□

**Strategies for proving a transformation is linear:**

Guess whether it is linear or not linear, and use that as the hypothesis. If you are trying to find a counter-example and you can't find one, you may want to try proving it is linear. On the other hand, if you want to prove it is linear, you may find the counter-example while trying to prove it is linear.

Example for the notation:

**Remark.**

Notation:

$D$ : Take the derivative.

$D^2$ : Take the derivative twice.

$I$ : The identity, meaning to return the input.



**Example.**

Evaluate  $(D^2 + D - 3I)(e^{rt})$

*Proof.*

$$\begin{aligned} (D^2 + D - 3I)(e^{rt}) &= D^2(e^{rt}) + D(e^{rt}) - 3I(e^{rt}) \\ &= D(re^{rt}) + re^{rt} - 3e^{rt} = r^2e^{rt} + re^{rt} - 3e^{rt} = (r^2 + r - 3)e^{rt} \\ &= (r^2 + r - 3)e^{rt} \end{aligned}$$

□

**Remark.**

Here,  $I$  is a transformation rule or instructions that returns the input. It is important because each term in the transformation has a rule to transform it like the  $D^2$  and  $D$  terms. Sometimes, the  $I$  term is not written out, but it is always there.

**1.1.3 Properties of Linear Transformations****Theorem 1.1.4**

For any linear map from any  $V$  to any  $W$ ,  
 $T(\vec{0}_V) = \vec{0}_W$

**Theorem 1.1.5**

For any linear map from any  $V$  to any  $W$ ,  
 $T(-\vec{v}) = -T(\vec{v})$

**Proof.** Let  $T : V \rightarrow W$  be a linear transformation.

1.  $T(\vec{0}_V) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0}_W$
2.  $T(\vec{0}_V) = T(\vec{v} + (-\vec{v})) = T(\vec{v}) + T(-\vec{v}) = \vec{0}_W$

□

**1.1.4 The Kernel and Range of a Linear Transformation****Definition 1.1.6: Kernel**

Let  $T : V \rightarrow W$  be a linear transformation. The **kernel** of  $T$ , denoted by  $\ker(T)$ , is the set of all vectors in  $V$  that are mapped to  $\vec{0}_W$  by  $T$ .

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}_W\}$$

Note: If  $T(\vec{v}) = A\vec{v}$ , then  $\ker(T) = \text{null}(A)$

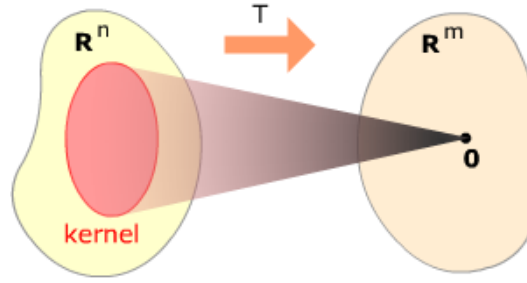


Figure 1.2: Kernel Diagram

**Definition 1.1.7**

Let  $T : V \rightarrow W$  be a linear transformation. The **range** of  $T$ , denoted by  $\text{range}(T)$ , is the set of all vectors in  $W$  that are mapped to by  $T$ .

$$\text{range}(T) = \{T(\vec{v}) \in W : \vec{v} \in V\}$$

Note: If  $T(\vec{v}) = A\vec{v}$ , then  $\text{range}(T) = \text{col}(A)$

**Remark.**

Domain is the set of all possible inputs, codomain is the set of all possible outputs, and range is the set of all actual outputs.

**Example.**

$$D^2 + D = 3I : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$$

Find the kernel and range of  $D^2 + D$ .

$$\begin{aligned} \ker(D^2 + D) &= \{y \in C^\infty(\mathbb{R}) : (D^2 + D)(y) = 0\} \\ &= \{y \in C^\infty(\mathbb{R}) : y'' + y' = 0\} \end{aligned}$$

Therefore, a homogenous linear differential equation is an example of a kernel.

**Example.**

Let  $T : P_2 \rightarrow P_1$  be defined as follow:

$$T(ax^2 + bx + c) = (a + b) + (b - c)x$$

Find the kernel and range of  $T$ .

**Kernel:**

We want  $(a + b) + (b - c)x = 0$  in  $P_2$

Which is  $0 + 0x$  in  $P_1$

Therefore,  $(a + b)$  and  $(b - c)$  must be 0.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$$\begin{array}{l} a = -c, b = c, c = c \\ \ker(T) = \{-cx^2 + cx + c | c \in \mathbb{R}\} \end{array}$$

Write it as a spanned set:

$$\ker(T) = \text{span}\{-x^2 + x + 1\}$$

**Range:**

We want  $(a + b) + (b - c)x$  in  $P_1$

Let  $b + ex$  be a generic element in  $P_1$

Therefore,  $d = a + b$  and  $e = b - c$

We want to solve for  $a, b, c$  in terms of  $d, e$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 0 & d \\ 0 & 1 & -1 & e \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 1 & d - e \\ 0 & 1 & -1 & e \end{array} \right]$$

$$a = d - e - c, b = e + c, c \text{ is free}$$

$$(a + b) + (b - c)x = (d - e + e + c) + (e + c - c)x = d + ex$$

$$\text{range}(T) = \{d + ex | d, e \in \mathbb{R}\}$$

So the range is all of  $P_1(\mathbb{R})$ .

**Theorem 1.1.8**

Let  $T : V \rightarrow W$  be a linear transformation. Then:

1.  $\ker(T)$  is a subspace of  $V$
2.  $\text{range}(T)$  is a subspace of  $W$

**Proof.** Let  $T : V \rightarrow W$  be a linear transformation.

1.  $\ker(T)$  is a subspace of  $V$

1.  $\vec{0}_V \in \ker(T)$  since  $T(\vec{0}_V) = \vec{0}_W$
  2. Let  $\vec{v}_1, \vec{v}_2 \in \ker(T)$  and  $\lambda \in \mathbb{R}$ .  
 Then  $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0}_W + \vec{0}_W = \vec{0}_W$   
 Therefore,  $\vec{v}_1 + \vec{v}_2 \in \ker(T)$
  3. Also,  $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{0}_W = \vec{0}_W$   
 Therefore,  $\lambda\vec{v}_1 \in \ker(T)$   
 Thus,  $\ker(T)$  is a subspace of  $V$ .
2.  $\text{range}(T)$  is a subspace of  $W$ 
    1.  $\vec{0}_W \in \text{range}(T)$  since  $T(\vec{0}_V) = \vec{0}_W$
    2. Let  $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$  and  $\lambda \in \mathbb{R}$ .  
 Then there exists  $\vec{v}_1, \vec{v}_2 \in V$  such that  $T(\vec{v}_1) = \vec{w}_1$  and  $T(\vec{v}_2) = \vec{w}_2$ .  
 Then  $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$   
 Therefore,  $\vec{w}_1 + \vec{w}_2 \in \text{range}(T)$
    3. Also,  $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{w}_1$   
 Therefore,  $\lambda\vec{w}_1 \in \text{range}(T)$   
 Thus,  $\text{range}(T)$  is a subspace of  $W$ .

□

**Example.**

Find the basis for the range and kernel of:  $D^2 : P_3 \rightarrow P_3$  given by  $D^2(p) = p''$

**Proof.**

$$p(x) = ax^3 + bx^2 + cx + d.$$

**Basis for the Kernel:**

The kernel of a linear transformation consists of all elements that map to the zero element of the codomain. For the transformation  $D^2$ , we need to find all  $p \in P_3$  such that  $D^2(p) = 0$ . Applying  $D^2$  to the general polynomial:

$$p''(x) = (ax^3 + bx^2 + cx + d)'' = (3ax^2 + 2bx + c)'' = 6ax + 2b.$$

Setting  $p''(x) = 0$  yields:

$$6ax + 2b = 0.$$

For this to be true for all  $x$ , we must have  $a = 0$  and  $b = 0$ . Therefore,  $p(x)$  reduces to:

$$p(x) = cx + d.$$

Polynomials of the form  $cx + d$  clearly form the kernel of  $D^2$ , and are elements of  $P_1$  (the space of all polynomials of degree at most 1). Hence, a basis for the kernel of  $D^2$  consists of the polynomials:

$$\boxed{\{1, x\}}$$

□

**Proof. Basis for the Range:**

The range of  $D^2$  consists of all possible outputs  $D^2(p)$  for  $p \in P_3$ . From the calculation above, we found:

$$D^2(p) = 6ax + 2b.$$

This expression tells us that the output of  $D^2$  can be any polynomial of the form  $6ax + 2b$ , which can be rewritten as a linear combination of the basis  $\{x, 1\}$  (or equivalently,  $\{6x, 2\}$ ).

$$\boxed{\{6x, 2\}}$$

□

**Theorem 1.1.9: General Rank-Nullity Theorem**

Let  $T : V \rightarrow W$  be a linear transformation. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$$

**Example.**

Let  $S : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$  be the transformation with the rule  $S(A) = AA^T$ . Find a basis for its kernel and the dimension of its range. Confirm the general rank/nullity theorem based on what we found about its kernel and range previously.

**Proof. Kernel:**

We want to find  $A$  such that  $AA^T = \vec{0}$ .

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2 + b^2 = 0, ac + bd = 0, c^2 + d^2 = 0$$

$$a = b = c = d = 0$$

Therefore, the kernel is  $\{0\}$  and the dimension is 0.

**Range:**

We want to find  $A$  such that  $AA^T = B$  for some  $B \in M_2(\mathbb{R})$ .

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

The basis for it will be:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore, the range is all of  $M_2(\mathbb{R})$  and the dimension is 4.

**Rank-Nullity Theorem:**

$$\dim(M_2(\mathbb{R})) = \dim(\ker(S)) + \dim(\text{range}(S))$$

$4 \neq 0 + 3$

□

This is not a linear transformation because it does not satisfy the properties of linearity. Therefore, the rank-nullity theorem does not hold.

**Example.**

Consider the linear transformation  $T : M_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a - b + d) + (-a + b - d)x^2$$

Determine  $\text{range}(T)$ . What is the dimension of  $\ker(T)$ ? Determine a basis for  $\ker(T)$ .

Step 1: Verify it is linear:

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$T(A+B) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = (a_1+a_2-b_1-b_2+d_1+d_2) + (-a_1-a_2+b_1+b_2-d_1-d_2)x^2$$

$$T(A)+T(B) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = (a_1-b_1+d_1) + (-a_1+b_1-d_1)x^2 + (a_2-b_2+d_2) + (-a_2+b_2-d_2)x^2$$

$$(a_1 + a_2 - b_1 - b_2 + d_1 + d_2) + (-a_1 - a_2 + b_1 + b_2 - d_1 - d_2)x^2$$

$$T(A+B) = T(A) + T(B)$$

Therefore,  $T$  is linear.

### 1.1.5 Linear transformation and basis

If we have a matrix  $A$  that's  $m$  by  $n$ , we get a linear transformation.  $T : f_n \rightarrow f_m$  given by  $T(x) = Ax$  if  $V$  and  $W$  are finite-dimensional, then  $T : V \rightarrow W$  can be accomplished by a matrix multiplication. That is there exist a matrix  $A$  such that  $T(\vec{v}) = A\vec{v}$  for all  $\vec{v} \in V$ .

#### Theorem 1.1.10

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then there exists an  $m \times n$  matrix  $A$  such that  $T(v) = Av$  for all  $v \in \mathbb{R}^n$ .

We have seen that, given an  $m \times n$  matrix  $A$ , we can define a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  by  $T(v) = Av$ . (Remember that an  $m \times n$  matrix maps  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .)

Now, given a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we identify an  $m \times n$  matrix  $A$  such that  $T(v) = Av$ .

This is possible, because if we know how  $T$  acts on a basis  $\{b_1, \dots, b_n\}$  of  $V$ , then we know how  $T$  acts on the whole of  $V$ , since if  $v = c_1b_1 + \dots + c_nb_n$ , then

$$T(v) = T(c_1b_1 + \dots + c_nb_n) = c_1T(b_1) + \dots + c_nT(b_n)$$

In particular, if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and if  $v = (c_1, c_2, \dots, c_n) = c_1e_1 + c_2e_2 + \dots + c_ne_n$ , then

$$T(v) = T(c_1e_1 + c_2e_2 + \dots + c_ne_n) = \boxed{c_1T(e_1) + c_2T(e_2) + \dots + c_nT(e_n)}$$

By the definition of matrix multiplication, the boxed expression is equal to

$$\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then  $T(v) = Av$  where  $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$ .

**Example.**

Find  $A$ .  $T_1 \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(x) = \beta$$

$$\beta = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$$

$$T(x) = (a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n)$$

$$= a_1 T(\vec{e}_1) + a_2 T(\vec{e}_2) + \cdots + a_n T(\vec{e}_n)$$

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

If I take the matrix  $A$  and apply it to the coefficient vector, I get the transformation.

$$\begin{aligned} A \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ &= a_1 T(\vec{e}_1) + a_2 T(\vec{e}_2) + \cdots + a_n T(\vec{e}_n) \\ &= T(a_1 \vec{e}_1 + a_2 \vec{e}_2 + \cdots + a_n \vec{e}_n) \end{aligned}$$



**Example.**

Find a matrix  $A$  such that  $T(x) = A(x)$  where  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is given by  $T(a, b, c) = (c, a + b)$ .

Since this is from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ , the matrix  $A$  will be  $2 \times 3$ .

$$T(a, b, c) = (c, a + b)$$

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)]$$

$$A = [T(1, 0, 0) \quad T(0, 1, 0) \quad T(0, 0, 1)]$$

$$= A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c \\ a + b \end{bmatrix}$$

$$\boxed{= (c, a + b)}$$

## Chapter 2

# Eigenvalues and Eigenvectors

## 2.1 Lecture 19: Kernel Range and Eigenvalues

*This lecture covers:*

- 7.2 The Results for Eigenvalues and Eigenvectors

### 2.1.1 Eigenvalues and Eigenvectors

#### Definition 2.1.1: Eigenvalues

Let  $A$  be an  $n \times n$  matrix. A scalar  $\lambda$  is called an **eigenvalue** of  $A$  if there exists a nontrivial (non-zero) vector  $\vec{x}$  such that

$$A\vec{x} = \lambda\vec{x}$$

The corresponding vector  $\vec{x}$  is called an **eigenvector** of  $A$ .

The eigenvalue  $\lambda$  will have infinitely many corresponding eigenvectors  $\vec{v}$  that satisfy:

1.  $\vec{v} \neq \vec{0}$
2.  $A\vec{v} = \lambda\vec{v}$

Notice that the  $\vec{x}$  and  $\vec{v}$  here are interchangeable.

#### Definition 2.1.2: Characteristic Polynomials

The characteristic polynomial of an  $n \times n$  matrix  $A$  is defined as

$$p(\lambda) = \det(A - \lambda I)$$

where  $I$  is the  $n \times n$  identity matrix.

If  $\lambda_i$  is a root of the characteristic polynomial, that is  $p(\lambda_i) = 0$ , then  $A - \lambda_i I_n$  will have a nontrivial null space.

This means there is a non-zero vector  $\vec{x}$  such that  $(A - \lambda_i I_n)\vec{x} = \vec{0}$ .

Hence  $\lambda_i$  is an eigenvalue of  $A$  with corresponding eigenvector  $\vec{v}_i$ .

All the **nonzero** elements of the null space of  $A - \lambda_i I_n$  are eigenvectors of  $A$  corresponding to  $\lambda_i$ .

**Example.**

Find all the eigenvalues and corresponding eigenvectors of the matrix  $A$ :

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

**Step 1:** Find the Eigenvalues:

We start with the equation:

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ &= A\vec{x} - \lambda\vec{x} = \vec{0} \\ &= (A - \lambda I)\vec{x} = \vec{0} \end{aligned}$$

This is a homogeneous system of equations. We can solve for  $\lambda$  by finding the determinant of  $A - \lambda I$  and setting it equal to zero.

**Remark.**

The reason this works is because we always have a solution that  $\vec{x} = \vec{0}$ . However, we are looking for solutions where  $\vec{x} \neq \vec{0}$  (nontrivial solutions). When  $\det A - \lambda I = 0$  it means that we have more than one solutions that make the system consistent. This means we have free variables at the end of the row reduction, which makes the matrix non-invertible and hence the determinant is zero.

$$\det(A - \lambda I) = \det \left( \begin{bmatrix} 3 - \lambda & -1 \\ -5 & -1 - \lambda \end{bmatrix} \right) = 0$$

The characteristic polynomial is:

$$p(\lambda) = (3 - \lambda)(-1 - \lambda) - (-1)(-5) = \lambda^2 - 2\lambda - 8 = 0$$

We can solve for  $\lambda$  by factoring:

$$\begin{aligned} (\lambda - 4)(\lambda + 2) &= 0 \\ \lambda &= 4, -2 \end{aligned}$$

**Example.****Step 2:** Find the Eigenvectors:We plugin the eigenvalues into the equation  $(A - \lambda I)\vec{x} = \vec{0}$  to find the eigenvectors.For  $\lambda = 4$ :

$$A - 4I = \begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

**Remark.**

Notice that if the second row can not be reduced to all zeros, that means you made a mistake in the previous steps.

$$x_1 = -x_2$$

Let  $x_2 = 1$ , then  $x_1 = -1$ .The eigenvector corresponding to  $\lambda = 4$  is:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For  $\lambda = -2$ :

$$\left[ \begin{array}{cc|c} 5 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = \frac{1}{5}x_2$$

Let  $x_2 = 5$ , then  $x_1 = 1$ .The eigenvector corresponding to  $\lambda = -2$  is:

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

**Remark.**

Notice that the eigenvectors are not unique. We can scale the eigenvectors by any scalar and they will still be eigenvectors. In other words, for each eigenvalue, there are infinitely many eigenvectors.

### 2.1.2 Complex Eigenvalues

#### Definition 2.1.3: Complex Conjugate

The complex number (notation  $\mathbb{C}$ ) have the form:

$$z = a + bi, \text{ where } a, b \in \mathbb{R}, \text{ and } i^2 = -1$$

The **complex conjugate** of  $z = a + bi$  is  $a - bi$ . It is denoted as  $\bar{z}$ .

**Properties of complex conjugates:** If  $A$  is a real matrix and  $\vec{v}$  is an eigenvector with eigenvalue  $\lambda$  then its complex conjugate  $\bar{\vec{v}}$  is also an eigenvector with eigenvalue  $\bar{\lambda}$ . Justification:

$$A\vec{v} = \lambda\vec{v} \implies A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

Consider the complex conjugate of the entire equation. Recall that taking the complex conjugate of a complex number  $z$  is denoted  $\bar{z}$ , and for any complex vector  $\vec{v}$ , the conjugate  $\bar{\vec{v}}$  is the vector of the conjugates of the components of  $\vec{v}$ .

Given  $A\vec{v} = \lambda\vec{v}$ , we conjugate both sides:

$$\overline{A\vec{v}} = \overline{\lambda\vec{v}}$$

Since  $A$  is a real matrix, the elements of  $A$  remain unchanged when conjugated, so  $\bar{A} = A$ . This applies to every component of the matrix-vector product, resulting in:

$$A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

**Example.**

(Complex eigenvalues) Find all the eigenvalues and corresponding eigenvectors of  $A$ .

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

**Step 1:** Find the Eigenvalues:

$$\det(A - \lambda I) = \det \left( \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \right) = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1 \implies \lambda = \pm i$$

**Step 2:** Find the Eigenvectors:

For  $\lambda = i$ :

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array} \right]$$

$$x_1 = -ix_2$$

Let  $x_2 = 1$ , then  $x_1 = -i$ .

The eigenvector corresponding to  $\lambda = i$  is:

$$\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Using the properties of complex conjugates, we can find the eigenvector corresponding to  $\lambda = -i$ :

$$\begin{bmatrix} i \\ 1 \end{bmatrix}$$

### 2.1.3 Properties and Theorems

- 0 is an eigenvalue of  $A$  if and only if  $A$  has non-trivial null space. (So 0 can be an eigenvalue but not an eigenvector)
- If  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , then  $k\vec{v}$  is also an eigenvector of  $A$  with eigenvalue  $\lambda$  for any scalar  $k \neq 0$ .
- If  $A$  is invertible and  $\vec{v}$  is an eigenvector with eigenvalue  $\lambda$ , then  $\lambda \neq 0$  and  $\vec{v}$  is an eigenvector of  $A^{-1}$  with eigenvalue  $\lambda^{-1}$ .
- If  $\vec{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , then  $\vec{v}$  is an eigenvector of  $A^k$  with eigenvalue  $\lambda^k$  for any positive integer  $k$ .

## 2.2 Lecture 20: Eigenspaces & Eigenbase

*This lecture covers:*

- 7.2 General Results on Eigenvalues and Eigenvectors
- 7.3 Diagonalization

### 2.2.1 Eigenspaces and Non-defective Matrices

#### Definition 2.2.1: Defective Matrices

A matrix is defective if it does not have a complete set of eigenvectors.

The matrix  $A$  is defective if the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity.

This is a common simple example that illustrate the concept of defective matrices.

**Example.**

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Here, we can find the eigenvalues by solving the characteristic equation  $\det(A - \lambda I) = 0$ , where  $I$  is the identity matrix.

1. Subtract  $\lambda$  from the diagonal entries of  $A$  and set the determinant to zero:

$$\begin{vmatrix} 1 - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(1 - \lambda) - 0 \cdot 1 = (1 - \lambda)^2 = 0$$

2. Solving  $(1 - \lambda)^2 = 0$ , we find that the only eigenvalue is  $\lambda = 1$ , with algebraic multiplicity 2 (because it is a double root of the characteristic equation).

3. Next, to find the eigenvectors corresponding to  $\lambda = 1$ , solve  $(A - I)x = 0$ :

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This simplifies to  $0 \cdot x_1 + 1 \cdot x_2 = 0$ , which implies  $x_2 = 0$ . So, the eigenvectors are of the form  $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ , where  $x_1$  can be any non-zero value.

4. Thus, we have only one linearly independent eigenvector,  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , despite having an eigenvalue with algebraic multiplicity 2. This means the geometric multiplicity of  $\lambda = 1$  is 1, which is less than its algebraic multiplicity.

Since the algebraic and geometric multiplicities do not match, the matrix  $A$  is defective and cannot be diagonalized.



**Example.**

Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}$$

whose characteristic polynomial is  $p(\lambda) = -(\lambda - 2)^2(\lambda + 1)$ .

**Solution:** The eigenvalues are  $\lambda_1 = 2$  and  $\lambda_2 = -1$ . For  $\lambda_1 = 2$ , we have

$$A - 2I = \left[ \begin{array}{ccc|c} 3 & 12 & -6 & 0 \\ -3 & -12 & 6 & 0 \\ -3 & -12 & 6 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

which gives us the eigenvector  $x_1 = -4x_2 + 2x_3$ .

$$\vec{x}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

For  $\lambda_2 = -1$ , we have

$$A + I = \left[ \begin{array}{ccc|c} 6 & 12 & -6 & 0 \\ -3 & -9 & 6 & 0 \\ -3 & -12 & 9 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

which gives us the eigenvector  $x_1 = -2x_3$  and  $x_2 = 3x_3$ .

$$\vec{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We can plug in the eigenvectors into the matrix to check if they are correct.

**Example.**

**Check:**

$$A\vec{v} = \lambda\vec{v}$$

For  $\lambda_1 = 2$  and  $\vec{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ , we have

$$A\vec{v}_1 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

For  $\lambda_2 = 2$  and  $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ , we have

$$A\vec{v}_2 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

For  $\lambda_2 = -1$  and  $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ , we have

$$A\vec{v}_2 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

### Definition 2.2.2: Algebraic Multiplicity

Any polynomial with coefficients in  $\mathbb{R}$  can be written as follows:

$$p(t) = a(t - r_1)^{m_1}(t - r_2)^{m_2} \cdots (t - r_k)^{m_k}$$

The  $r_i \in \mathbb{C}$  are the distinct roots of the polynomial. The exponent  $m_i$  is the **algebraic multiplicity** of the root  $r_i$ .

**Example.**

In the previous example, the algebraic multiplicity of  $\lambda_1 = 2$  is 2 and the algebraic multiplicity of  $\lambda_2 = -1$  is 1.

**Definition 2.2.3: Eigenspaces**

Let  $A \in M_n(\mathbb{R})$ . For any given Eigenvalues  $\lambda_i$ , define the **Eigenspace** associate to each  $\lambda_i$  as follows:

$$E_i = \{x \in \mathbb{C}^n | A\vec{v} = \lambda_i \vec{v}\} = \text{Null}(A - \lambda_i I_n) = \{\text{all Eigenvectors of } \lambda_i\} \cup \{0_n\}$$

The dimension  $\dim(E_i) = n_i$  is called the **geometric multiplicity** of  $\lambda_i$ .

**Example.**

In the previous example, the geometric multiplicity of  $\lambda_1 = 2$  is 2 and the geometric multiplicity of  $\lambda_2 = -1$  is 1.

**Remark.****Algebraic Multiplicity**

The **algebraic multiplicity** of an eigenvalue is the number of times that eigenvalue appears as a root of the characteristic polynomial of the matrix. In other words, it's the exponent of the eigenvalue in the factorized form of the characteristic polynomial.

For example, if the characteristic polynomial is  $(\lambda - 1)^2(\lambda + 2)$ , the eigenvalue  $\lambda = 1$  has an algebraic multiplicity of 2 because it appears twice as a root (i.e., it's squared in the polynomial).

**Geometric Multiplicity**

The **geometric multiplicity** of an eigenvalue is the number of linearly independent eigenvectors associated with that eigenvalue. This is technically the dimension of the eigenspace corresponding to the eigenvalue. The eigenspace is the null space of the matrix  $A - \lambda I$ , where  $A$  is the matrix in question,  $\lambda$  is the eigenvalue, and  $I$  is the identity matrix of the same size as  $A$ .

For the same example, if for  $\lambda = 1$ , we can only find one linearly independent eigenvector despite the algebraic multiplicity being 2, then the geometric multiplicity of  $\lambda = 1$  is 1.

**Relationship and Properties**

1. The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.
2. If the geometric multiplicity of every eigenvalue equals its algebraic multiplicity, the matrix is said to be diagonalizable, meaning there exists a basis of eigenvectors for the matrix, and the matrix can be expressed as  $PDP^{-1}$  where  $D$  is a diagonal matrix of eigenvalues and  $P$  is the matrix whose columns are the corresponding eigenvectors.

**Theorem 2.2.4**

Let  $\lambda_i$  be an eigenvalue of  $A$  of algebraic multiplicity  $m_i$ , and  $E_i$  the corresponding eigenspace. Then:

1.  $E_i$  is a subspace of  $\mathbb{C}^n$ .
2.  $\dim(E_i) \leq m_i$ .

**Theorem 2.2.5: The independence of Eigenvectors**

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. The union of linearly independent elements of **distinct** eigenspace  $E_1, E_2, \dots, E_k$  is a linearly independent set. (This means if we take the union of basis of distinct eigenspaces, we get an independent set.)

**Fact 2.2.6**

1. If  $A$  has  $n$  distinct eigenvalues, then  $A$  is non-defective.
2.  $A$  is **non-defective** if and only if the **geometric multiplicity** of each eigenvalue equals its **algebraic multiplicity**. (That way, the sum of the dimensions of the eigenspaces equals  $n$ .)

**Example.**

1. Show the following matrix is non-defective:

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

**Step 1:** Find the Eigenvalues

The characteristic polynomial is  $\lambda^2 - 3\lambda + 2$ . The eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = 2$ .

**Step 2:** Find the Eigenvectors

For  $\lambda_1 = 1$ , we have

$$A - I = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$$

which gives us the eigenvector  $2x_1 = -3x_2$ .

$$\vec{x}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

For  $\lambda_2 = 2$ , we have

$$A - 2I = \begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix}$$

which gives us the eigenvector  $x_1 = -x_2$ .

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

**Step 3:** Check the Independence of Eigenvectors

The eigenvectors corresponding to different eigenvalues are always linearly independent.

Here, the eigenvectors are:

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

**Conclusion** For  $\lambda_1 = 1$ , we have algebraic multiplicity 1 and geometric multiplicity 1. For  $\lambda_2 = 2$ , we have algebraic multiplicity 1 and geometric multiplicity 1. Since the geometric multiplicity of each eigenvalue equals its algebraic multiplicity, the matrix  $A$  is non-defective.

**Example.**

2. Find a basis of eigenvectors of  $A$  for  $\mathbb{R}^2$

**Forming a Basis:**

$$P = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

The determinant of  $P$  is:

$$\det(P) = -3 \cdot 1 - (-1) \cdot 2 = -3 + 2 = -1$$

Since the determinant is non-zero, the vectors are linearly independent.

**Spanning:**

$$\left[ \begin{array}{cc|c} -3 & -1 & x \\ 2 & 1 & y \end{array} \right]$$

RREF:

$$\left[ \begin{array}{cc|c} 1 & 0 & -x - y \\ 0 & 1 & 2x + 3y \end{array} \right]$$

The determinant is non-zero, so the vectors span  $\mathbb{R}^2$ . Thus, the vectors form a basis of  $\mathbb{R}^2$ .

**Example.**

3. Let  $\vec{v} = (-2, 1) \in \mathbb{R}^2$  with the standard basis. Express  $\vec{v}$  as a linear combination of the eigenvectors of  $A$ .

**Solution:**

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving the system of equations, we get  $c_1 = 1$  and  $c_2 = -1$ . Thus,  $\vec{v}$  can be expressed as a linear combination of the eigenvectors of  $A$ .

**Example.**

4. Find  $A^2\vec{v}$  and  $A^{-1}\vec{v}$ .

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

and a vector  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ .

**Step 1:**  $A^2\vec{v}$

Now, compute  $A^2\vec{v}$  by multiplying  $A$  again by  $A\vec{v}$ :

$$A^2\vec{v} = A(A\vec{v}) = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -x - 3y \\ 2x + 4y \end{bmatrix}$$

Simplifying the multiplication:

$$A^2\vec{v} = \begin{bmatrix} (-1)(-x - 3y) + (-3)(2x + 4y) \\ (2)(-x - 3y) + (4)(2x + 4y) \end{bmatrix}$$

$$A^2\vec{v} = \begin{bmatrix} x + 3y - 6x - 12y \\ -2x - 6y + 8x + 16y \end{bmatrix}$$

$$A^2\vec{v} = \begin{bmatrix} -5x - 9y \\ 6x + 10y \end{bmatrix}$$

**Step 2:**  $A^{-1}\vec{v}$

To compute  $A^{-1}\vec{v}$ , we first need to find the inverse of the matrix  $A$ .

Inverse of a  $2 \times 2$  Matrix

For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , the inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying this formula:

$$A^{-1} = \frac{1}{(-1)(4) - (-3)(2)} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4 + 6} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$$

Now compute  $A^{-1}\vec{v}$ :

$$A^{-1}\vec{v} = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A^{-1}\vec{v} = \begin{bmatrix} 2x + \frac{3}{2}y \\ -x - \frac{1}{2}y \end{bmatrix}$$

**Example.**

Example: Is the following matrix defective? Find a basis for the span of its eigenvectors.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

**Step 1:** find the Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & -1 \\ 1 & -1 & 2 - \lambda \end{vmatrix}$$

$$(3 - \lambda)((2 - \lambda)(2 - \lambda) - 1) = 0$$

$$\lambda_1 = 3(Mul : 2), \lambda_2 = 1(Mul : 1)$$

**Step 2:** Find the Eigenvectors

For  $\lambda_1 = 3$ , we have

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$x_1 = x_2 + x_3$$

$$x_2 = -x_3$$

let  $x_2 = t$ , then  $x_3 = -t$ ,  $x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

For  $\lambda_2 = 1$ , we have

$$A - I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = x_3$$

let  $x_2 = t$ , then  $x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We can actually stop by  $\lambda_1$  since we only find one eigenvector but the algebraic multiplicity is 2. Thus, the matrix is defective.



## Chapter 3

# Linear Differential Equations of Order $n$

### 3.1 Lecture 21: Linear Differential Equations

*This lecture covers:*

- 8.1 General Theory of Linear Differential Equations
- 8.2 Constant Coefficient Homogeneous Linear Differential Equations

#### 3.1.1 General Theory of Linear Differential Equations

##### Definition 3.1.1: Linear Differential Equation

A linear Differential Equation is an equation of the form

$$y^{(n)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

##### Definition 3.1.2: Homogeneous equation

We have **Homogeneous equation**  $F(x) = 0$  and **Non-homogeneous equation**  $F(x) \neq 0$

##### Fact 3.1.3

The Differential operator  $D$  is a linear Transformations

$$D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R})$$

given by  $D(f) = f'$ , and we extrapolated to higher order derivatives. We define  $L : C^n(\mathbb{R}) \rightarrow C(\mathbb{R})$ :

$$L = D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)I$$

called the differential operator of order  $n$ . (the book leaves out the  $I$  term)

##### Example.

Consider  $y'' + 2y' + e^x y = 0$ . Then a function  $y$  is a solution if and only if  $y$  satisfies  $Ly = 0$ , where

$$L = D^2 + 2D + e^x I$$

$$\text{Solution Set} = \{y | L(y) = 0\} = \ker(L)$$

The solution set looks like this:

If  $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$  is a homogeneous linear differential equation of order  $n$ , then  $y = c_1y_1 + \cdots + c_ny_n$  is a general solution, where  $y_1, \dots, y_n$  are linearly independent solutions.

**Theorem 3.1.4**

**The set of solutions of a homogeneous linear differential equation of order  $n$  is a vector space of dimension  $n$ .** It's the kernel of the linear map  $L$ . We call it the **solution space**.

Any  $n$  linearly independent solutions  $y_1, \dots, y_n$  form a basis of the solution space. Every solution  $y$  can be expressed as a linear combination of basis vectors

$$y = c_1 y_1 + \dots + c_n y_n$$

The above expression is still called the general solution to the homogeneous equation.

**Example.**

Note that both  $y_1 = \cos x$  and  $y_2 = \sin x$  satisfy  $y'' + y = 0$ . Now we add information from linear algebra:

- We are talking about function spaces, so the vectors are functions (and the functions are vectors.)
- $\{\cos x, \sin x\}$  is independent. (How can we show this?) - Wronskian
- Any 2 independent functions in a 2-dimensional function space constitute a basis for the space.
- Any function in the space is a linear combination of basis vectors.

So we get the general solution,  $y = c_1 \cos x + c_2 \sin x$ . All solutions have this form.

**Example.**

Homogeneous, non-constant coefficients: Using trial solutions of the form  $y = x^r$ , find a basis for the solution space of

$$2x^2y'' + 5xy' + y = 0$$

on the interval  $x > 0$  and the general solution.

Trail solution:  $y = x^r$

Goal: Figure out  $r$

Order: 2

Substitute  $y = x^r$  into the equation:

$$y' = rx^{r-1} \quad y'' = r(r-1)x^{r-2}$$

$$2x^2r(r-1)x^{r-2} + 5xrx^{r-1} + x^r = 0$$

$$2r(r-1)x^r + 5rx^r + x^r = 0$$

$$x^r(2r^2 + 3r + 1) = 0$$

$$(2r+1)(r+1) = 0$$

$$r = -1/2, -1$$

$$y_1 = x^{-1/2}, \quad y_2 = x^{-1}$$

$$\boxed{y = c_1x^{-1/2} + c_2x^{-1}} \quad \text{General Solution}$$

**Remark.**

Note that this is a very specific case that shows when the sum of terms of  $L$  is in the form of  $c_ix^iD^i$ .

### 3.1.2 Constant Coefficient Homogeneous Linear Differential Equations

We build up a technique for solving order  $n$  linear homogeneous diffeqs with constant coefficients.

1. We start small by letting  $L = D - aI$ . Then  $L(y) = y' - ay = 0$  has general solution  $y = ce^{at}$ .
2. When  $L$  has the form  $L = (D - aI)(D - bI)$ , the factors commute.
3. If  $y$  is a solution to  $(D - a_iI)$ , then

$$(D - a_1I)(D - a_2I) \cdots (D - a_iI) \cdots (D - a_nI)y = 0,$$

because the factors commute.

4. Since the solution space is closed under addition and scalar multiplication, linear combinations of solutions to each factor are again solutions.

5. We'll call  $p(r) = (r - a_1)(r - a_2) \cdots (r - a_n)$  the auxiliary polynomial.

### Case 1: Distinct Real Roots

$$p(r) = (r - a_1)(r - a_2) \cdots (r - a_n)$$

#### Fact 3.1.5

$y_1 = e^{r_1 x}, \dots, y_n = e^{r_n x}$  are linearly independent solutions to  $L(y) = 0$ .

#### Theorem 3.1.6: More about Wronskian:

Let  $y_1, y_2, \dots, y_n$  be solutions to the regular  $n$ -th order differential equation  $Ly = 0$  on the interval  $I$ , and let  $W$  denote their Wronskian. If  $W(x_0) = 0$  at some point  $x_0 \in I$ , then  $\{y_1, y_2, \dots, y_n\}$  is linearly dependent on  $I$ .

So, when the functions of interest are solutions to  $Ly = 0$ , **the Wronskian completely determines the independence of the functions.**

The general solution to the differential equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

We use **initial conditions** to find the constants  $c_1, c_2, \dots, c_n$ .

**Example.**

Solve the initial value problem:

$$y'' - 3y' - 4y = 0, \quad y(0) = 0 \quad y'(0) = 2$$

**Step 1:** Find the general solution to the differential equation.

We first write the Differential Equation in the form  $L(y) = 0$ :

$$D^2 - 3D - 4I = 0$$

**Remark.**

Another way to think of this is to let  $y = e^{rt}$ .  
The characteristic polynomial is

$$P(r) = r^2 - 3r - 4 = (r - 4)(r + 1) = 0$$

We set  $P(r) = 0$  to find the root. The root is

$$r_1 = 4, \quad r_2 = -1$$

each with multiplicity 1. So the general solution is

$$y = c_1 e^{4t} + c_2 e^{-t}$$

**Step 2:** Use the initial conditions to find the constants  $c_1$  and  $c_2$ .

We have  $y(0) = 0$  and  $y'(0) = 2$ . We substitute these into the general solution:

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = 4c_1 - c_2 = 2$$

We solve this system of equations to find  $c_1$  and  $c_2$ .

$$c_1 = \frac{2}{5}, \quad c_2 = -\frac{2}{5}$$

**Remark.**

For this step, we can also write the equation into matrix form and solve it using matrix algebra. e.g.

$$\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \left[ \begin{array}{cc|c} 1 & 0 & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} \end{array} \right]$$

So the solution to the initial value problem is

$$y = \frac{2}{5}e^{4t} - \frac{2}{5}e^{-t}$$

### Case 2: Repeated Real Roots

If  $p(r)$  has repeated roots (some  $r_i$  has multiplicity greater than 1), then solutions of the form  $e^{r_i t}$  are not sufficient, as we will not get  $n$  independent ones.

Note that  $(D - aI)c_1 t e^{at} = c_1 e^{at}$ . So  $(D - aI)^2(c_1 t e^{at} + c_2 e^{at}) = 0$ , and  $\{e^{at}, t e^{at}\}$  is an independent set.

Technique: If a real root  $a$  appears with multiplicity  $m$ , then include the term

$$c_1 e^{at} + c_2 t e^{at} + c_3 t^2 e^{at} + \cdots + c_m t^{m-1} e^{at}$$

in the general solution.

#### Example.

Find the general solution to the diffeq

$$y^{(4)} - 2y^{(3)} + y'' = 0.$$

**Step 1:** Find the general solution to the differential equation.

$$P(r) = r^4 - 2r^3 + r^2 = r^2(r^2 - 2r + 1) = r^2(r - 1)^2$$

$$r_1 = 0(m = 2), \quad r_2 = 1(m = 2)$$

The general solution is

$$y = c_1 + c_2 t + c_3 e^t + c_4 t e^t$$

### Case 3: Complex Roots

#### Definition 3.1.7: Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

In particular

$$e^{ibt} = \cos(bt) + i \sin(bt), \text{ and } e^{-ibt} \text{ is the complex conjugate of } e^{ibt}$$

Suppose the (real) auxiliary polynomial  $P(r)$  has complex roots  $r = a \pm bi$ . Then the complex conjugate  $\overline{a + ib} = a - ib$  is also a root of  $P(r)$ .

**Fact 3.1.8**

The two complex valued functions  $e^{(a \pm ib)t} = e^{at}e^{\pm ibt} = e^{at}(\cos(bt) \pm i \sin(bt))$  are two linearly independent solutions to the differential equation.

These are complex-valued functions. We can obtain from them a pair of linearly independent, real functions. We set

$$y_1 = \frac{e^{(a+ib)t} + e^{(a-ib)t}}{2} = e^{at} \cos(bt) \quad \text{and} \quad y_2 = \frac{e^{(a+ib)t} - e^{(a-ib)t}}{2i} = e^{at} \sin(bt).$$

We conclude that the general real solution contains the term

$$c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)$$

**Example.**

Solve the initial value problem  $y'' - 6y' + 25y = 0, y(0) = 0, y'(0) = 1$

$$P(r) = r^2 - 6r + 25 = (r - 3)^2 + 16 = 0$$

The roots are  $r = 3 \pm 4i$ . The general solution is

$$y = c_1 e^{3t} \cos(4t) + c_2 e^{3t} \sin(4t)$$

We substitute the initial conditions to find  $c_1$  and  $c_2$ .

$$y(0) = c_1 = 0$$

$$y'(0) = 3c_1 + 4c_2 = 1$$

$$c_2 = \frac{1}{4}$$

So the solution to the initial value problem is

$$y = \frac{1}{4} e^{3t} \sin(4t)$$

**Case 4: Repeated Complex Roots**

(Analogous to repeated real roots—multiply by powers of  $x$ .)

Technique: If a (pair of) complex roots  $a \pm ib$  appears with multiplicity  $m$ , then include

$$c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt) + c_3 t e^{at} \cos(bt) + c_4 t e^{at} \sin(bt) + \cdots + c_{2m-1} t^{m-1} e^{at} \cos(bt) + c_{2m} t^{m-1} e^{at} \sin(bt)$$

in the general solution.



**Example.**

Find the general solution:  $(D^2 + 4)^2(D + 1)y = 0$ .

$$P(r) = (r^2 + 4)^2(r + 1) = (r^2 + 4)(r^2 + 4)(r + 1)$$

The roots are  $r = 2i(m = 2), -2i(m = 2), -1(m = 1)$ . The general solution is

$$y = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t \cos(2t) + c_4 t \sin(2t) + c_5 e^{-t}$$

## 3.2 Lecture 22: Annihilators

*This lecture covers:*

- 8.3 The Method of Undetermined Coefficients: Annihilators
- 8.4 Complex-Valued Trial Solutions

### 3.3 Lecture 23: Oscillations of a Mechanical System

*This lecture covers:*

- 8.5 Oscillations of a Mechanical System

## Chapter 4

# Systems of Linear Differential Equations

## 4.1 Lecture 24: First Order Linear Systems

*This lecture covers:*

- 9.1 First Order Linear Systems
- 9.2 Vector Formulation
- 9.3 General Results for First Order Linear Systems

**Remark.**

Today we are going to cover 9.1 to 9.3. We are going to skip many theorems.

Next week we are going to do some examples for chapter 8.

### 4.1.1 Chapter 9.1: First Order Linear Systems

$$\begin{bmatrix} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\ \vdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t) \end{bmatrix}$$

The  $b_i$  is the consists of the non-homogenous part. If  $b_i(t) = 0$ , the system is homogenous.

An example of the first-order linear system is:

**Example.**

$$x_1' = x_1 + 2x_2$$

$$x_2' = 2x_1 + 2x_2$$

$$x_1(0) = 1, x_2(0) = 0$$

The **Initial value** here is  $x_1(0) = 1, x_2(0) = 0$ .

A **solution** is an ordered n-tuple of functions  $x_1(t), x_2(t), \dots, x_n(t)$  that satisfies the system of equations.

The solution will be in the form of:

**Example.**

$$x_1(t) = \text{Some function}, x_2(t) = \text{Some function}$$

Which is a vector of functions.

There's a clever trick to solving the 2 by 2 system using the derivative, however, it is not that fast and no one use it in the exam so we are going to skip it.

**Remark.**

The first-order linear system is restrictive. However, we can transform a higher-order system by renaming functions.

**Example.**

$$\frac{d^2x}{dt^2} + 4e^t \frac{dx}{dt} - 9t^2x = 7t^2$$

Strategy: Let  $x_1 = x$ ,  $x'_1 = x_2$

$$\begin{aligned} x'_1 &= x_2 \\ x'_2 &= 9t^2x + 4e^tx_2 - 7t^2 \end{aligned}$$

(Chapter 9.1)

## 4.1.2 Chapter 9.2: Vector Formulation

**Remark.**

Chapter 9.2: How to transform the system into a matrix form.

**Example.**

Formulate with vectors.

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{bmatrix}$$

$$\vec{x}'(t) = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ \dots \\ x'_n(t) \end{bmatrix}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}$$

Coefficient matrix:

$$\vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \dots \\ b_n(t) \end{bmatrix}$$

### Theorem 4.1.1

$V_n(I)$  is a column vector of  $n$  functions defined on an interval  $I$ .

**Example.**

$$\begin{bmatrix} e^{3t} \\ 2 \\ e^{7t} \end{bmatrix} \in V_3(\mathbb{R})$$

for any fixed  $n$ ,  $I$ ,  $V_n(I)$  is a vector space.

### Wronskian

**Remark.**

We do not test this

Wronskian of a set of  $n$  column vectors in  $V_n(I)$

Wronskien of  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$

$$W(t) = \text{Wronskian} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

#### Theorem 4.1.2

If  $W(t) \neq 0$  for all  $t \in I$ , then  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  is linearly independent.

#### Theorem 4.1.3

If  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  is linearly independent, then  $W(t) \neq 0$  for all  $t \in I$ .

### 4.1.3 Chapter 9.3: General results for first-order linear systems

#### Theorem 4.1.4: Initial value problem

$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{b}(t)$  and  $\vec{x}(t_0) = \vec{x}_0$  has a unique solution on an interval  $I$  containing  $t_0$  if  $A(t)$  and  $\vec{b}(t)$  are continuous on  $I$ .

#### Theorem 4.1.5: 9.3.2

The set of solutions to the homogenous system  $\vec{x}'(t) = A(t)\vec{x}(t)$  is a vector space of dimension  $n$ .

### Fundamental solution set

The fundamental solution set is basically a basis for the solution space.

**Theorem 4.1.6: The fundamental solution set**

$$S = \{x_1, x_2, \dots, x_n\}$$

$S$  is a set of solutions to the homogenous system  $\vec{x}'(t) = A(t)\vec{x}(t)$  that are linearly independent.

The non-homogenous case:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n + \boxed{\vec{x}_p}$$

We add a single solution to the homogenous system to the solution of the non-homogenous system.

Simplifying assumptions:

$$\vec{x}'(t) = A\vec{x}(t)$$

is homogenous and  $A$  is a matrix of constants and  $A$  is non-defective.

**Remark.**

Recall: Non-defective means that the matrix has  $n$  linearly independent eigenvectors.

**Theorem 4.1.7**

Let  $A$  be a  $n \times n$  matrix of real constants, and let  $\lambda$  be a real eigenvalue of corresponding to the eigenvector  $\vec{v}$ .

Then

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a solution to the homogenous system  $\vec{x}'(t) = A\vec{x}(t)$ .

**Remark.**

Recall: Means that  $A\vec{v} = \lambda\vec{v}$

**Proof.** If  $\vec{x}(t) = e^{\lambda t}\vec{v}$ , then

$$A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v} = \lambda e^{\lambda t}\vec{v}$$

On the other side, if I take the derivative of  $\vec{x}(t)$ , I get

$$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$$

We notice that the two sides are equal. □

And we are going to have enough solutions to form a basis for the solution space.



**Theorem 4.1.8**

If  $A$  has  $n$  real eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  with corresponding real eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then

$$\vec{x}_k = e^{\lambda_k t} \vec{v}_k$$

is a solution to the homogenous system  $\vec{x}'(t) = A\vec{x}(t)$ .

**Example.**

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

The eigenvalue is:

$$|A - \lambda I| = (1 - \lambda) \cdot (-2 - \lambda) - 4 = 0$$

$$\lambda = 2, -3$$

$$\text{nullspace of } A - 2I \text{ is } \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\text{nullspace of } A + 3I \text{ is } \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

**What's new today:**

Solution =

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$x_1 = 2c_1 e^{2t} - c_2 e^{-3t}$$

$$x_2 = c_1 e^{2t} + 2c_2 e^{-3t}$$

Initial values:

$$x_1(0) = 2c_1 - c_2 = 1$$

$$x_2(0) = c_1 + 2c_2 = 0$$

$$\left[ \begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 2 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -1/5 \end{array} \right]$$

**Example.**

$$\begin{aligned} \vec{x}' &= A\vec{x} \\ A &= \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \\ x_1' &= 4x_2 \\ x_2' &= x_1 \end{aligned}$$

**Step 1: Find the eigenvalues and eigenvectors**

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 = 0 \\ \lambda_1 &= 2, \lambda_2 = -2 \end{aligned}$$

**For  $\lambda_1 = 2$**

$$\begin{aligned} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\ \vec{v}_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

**For  $\lambda_2 = -2$**

$$\begin{aligned} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{aligned}$$

**Step 2: Write the general solution**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

**Remark.**

If you can not zero out the second row, it means that you are wrong.

Let's do a 3 by 3 example. We will leave the complex case for the next lecture.s

**Example.**

$$A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

**Step 1: Find the eigenvalues and eigenvectors**

$$|A - \lambda I| = \begin{vmatrix} 0 - \lambda & -3 & 1 \\ -2 & -1 - \lambda & 1 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = 0$$

Characteristic polynomial:

$$(2 - \lambda)(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda_1 = 2, \lambda_2 = -3, \lambda_3 = 2$$

**For  $\lambda_1 = 2$**

$$\begin{bmatrix} -2 & -3 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2, x_3$  are free

$$-2x_1 = 3x_2 - x_3$$

$$x_1 = -\frac{3}{2}x_2 + \frac{1}{2}x_3$$

$$\vec{v}_1 = (3, -2, 0)$$

$$\vec{v}_2 = (1, 0, 2)$$

**For  $\lambda_2 = -3$**

$$\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2$  is free

$$x_1 = x_2$$

$$x_3 = 0$$

$$\vec{v}_3 = (1, 1, 0)$$

**Step 2: Write the general solution**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

## 4.2 Lecture 25:

*This lecture covers:*

- 

### 4.2.1 Complex-Valued Solutions

#### **Theorem 4.2.1**

Let  $\vec{u}(t)$  and  $\vec{v}(t)$  be real-valued vector functions. If

$$\vec{w}_1(t) = \vec{u}(t) + \vec{v}(t) \quad \text{and} \quad \vec{w}_2(t) = \vec{u}(t) - \vec{v}(t)$$

are complex conjugate solutions to  $x' = Ax$ , then

$$x_1(t) = \vec{u}(t) \text{ and } x_2(t) = \vec{v}(t)$$

are themselves real-valued solutions to  $x' = Ax$ .

**Example.**

Let  $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ . Find the general solution to  $x' = Ax$ .

**Step 1: Find the Eigenvalues and Eigenvectors**

The characteristic equation is  $\lambda^2 + 4 = 0$ , so  $\lambda = \pm 2i$ .

For  $\lambda = 2i$ , we have

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

RREF:

$$\left[ \begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

So,  $\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$  or  $\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$  is an eigenvector corresponding to  $\lambda = \pm 2i$ .

**Step 2: Find the General Solution**

The general solution is

$$\begin{aligned} x(t) &= c_1 e^{2it} \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{-2it} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= c_1 \begin{bmatrix} -i \cos(2t) - \sin(2t) \\ \cos(2t) - i \sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(2t) - i \sin(2t) \\ i \cos(2t) - \sin(2t) \end{bmatrix} \end{aligned}$$

Here, we want it to be in the form of  $\vec{u}(t) + i\vec{v}(t)$ . So, we can rewrite the general solution as

$$\begin{aligned} &= \begin{bmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ c_2 \cos(2t) - c_1 \sin(2t) \end{bmatrix} + i \begin{bmatrix} c_2 \cos(2t) - c_1 \sin(2t) \\ c_1 \cos(2t) + c_2 \sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ c_2 \cos(2t) - c_1 \sin(2t) \end{bmatrix} + i \begin{bmatrix} c_2 \cos(2t) - c_1 \sin(2t) \\ c_1 \cos(2t) + c_2 \sin(2t) \end{bmatrix} \end{aligned}$$

**Example.**

**Example 1:**  $(D^2 + 2D + 10)^2$  The first step is to find the order of this, which is 2 inside from  $D^2$  and squared it to get  $\boxed{4}$ .

$$p(r) = r^2 + 2r + 10 = 0$$

Which doesn't obviously factor, so we check:

$$b^2 - 4ac = 4 - 40 = \boxed{-36}$$

Which is negative, so we have complex roots.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = \boxed{-1 \pm 3i}$$

The multiplicity of the roots is 2 for each, so we have to use the formula:

$$e^{rt} = e^{-t}(\cos(3t) \pm i \sin(3t))$$

The general solution is:

$$\boxed{y = c_1 e^{-t} \cos(3t) + c_2 e^{-t} (\sin(3t)) + c_3 t e^{-t} (\cos(3t)) + c_4 t e^{-t} (\sin(3t))}$$

**Example.**

**Example 2:**  $y''' - y'' + y' - y = 0$

$$\begin{aligned} p(r) &= r^3 - r^2 + r - 1 = 0 \\ &= r^2(r - 1) + 1(r - 1) \\ &= (r + 1)(r - 1)^2 = 0 \end{aligned}$$

Rational root Theorem:

Trial:

$$r = \pm 1$$

try  $r = 1$ :

$$1 - 1 + 1 - 1 = 0$$

So, we have a root of 1.

$$\boxed{y = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)}$$

**Example.**

**Example 3:**  $y'' + y = 6xe^x$

$y_c$  = General solution to  $y'' + y = 0$

$y_p$  = Trial solution

$y_c$ :

$$p(r) = r^2 + 1 = 0$$

$$r = \pm i$$

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

$$y(p) = Ae^x + Bxe^x$$

Apply  $(D^2 + I)y_p$

$$D(Ae^x + Bxe^x)$$

$$= Ae^x + Be^x + Bxe^x$$

$$= (A + B)e^x + Bxe^x$$

$$D^2(Ae^x + Bxe^x) = (A + B)e^x + Be^x + Bxe^x$$

$$= (A + B)e^x + 2Be^x + Bxe^x$$

$$y = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)$$

$$(D^2 + I)y_p = (A + 2B)e^x + Bxe^x + Ae^x + Bxe^x$$

$$= (2A + 2B)e^x + 2Bxe^x$$

$$= 6xe^x$$

$$2A + 2B = 0$$

$$2B = 6$$

$$B = 3$$

$$A = -3$$

$$y = c_1$$



## Chapter 5

## Review

## 5.1 S15 FQ6

Let  $M_{2 \times 2}(\mathbb{R})$  be the vector space of  $2 \times 2$  real matrices. Consider the linear transformation  $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$  defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

**Kernel:**

$$a + d = 0 \quad b - c = 0 \quad a - c = 0 \quad b + d = 0$$

$$a = c = b = -d$$

Therefore, the basis for the kernel is:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So the nullity of  $T$  is 1.

**Range:**

$$\begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, these four vectors are not linearly independent:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Therefore, the basis for the range is:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and the rank of  $T$  is 3.

**Conclusion:** This aligned with the Rank-Nullity Theorem, as the rank of  $T$  is 3 the nullity is 1, and the sum of the rank and nullity is 4, which is the dimension of the domain of  $T$ .