MATH 165
Linear Algebra & Diff. Equation
Final
Notes with Examples

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Contents

Chapter 1

Linear Differential Equations of Order n

1.1 Lecture 21: Linear Differential Equations

This lecture covers:

- *8.1 General Theory of Linear Differential Equations*
- *8.2 Constant Coefficient Homogeneous Linear Differential Equations*

1.1.1 General Theory of Linear Differential Equations

Definition 1.1.1: Linear Differential Equation

A linear Differential Equation is an equation of the form

$$y^{(n)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

Definition 1.1.2: Homogeneous equation

We have **Homogeneous equation** $F(x) = 0$ and **Non-homogeneous equation** $F(x) \neq 0$

Fact 1.1.3

The Differential operator D is a linear Transformations

$$D : C^1(\mathbb{R}) \to C(\mathbb{R})$$

given by $D(f) = y'$, and we extapolated to higher order derivatives. We define $L : C^n(\mathbb{R}) \to C(\mathbb{R})$:

$$L = D^n + a_1(x)D^{n-1} + \cdots + a_{n-1}(x)D + a_n(x)I$$

called the differential operator of order n. (the book leaves out the I term)

Example.

> Consider $y'' + 2y' + e^x y = 0$. Then a function y is a solution if and only if y satisfies $Ly = 0$, where
>
> $$L = D^2 + 2D + e^x I$$
>
> $$\text{Solution Set} = \{y | L(y) = 0\} = \ker(L)$$
>
> The solution set looks like this:
> If $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ is a homogeneous linear differential equation of order n, then $y = c_1 y_1 + \cdots + c_n y_n$ is a general solution, where $y_1, \cdots, y_n$ are linearly independent solutions.

Theorem 1.1.4

The set of solutions of a homogeneous linear differential equation of order n is a vector space of dimension n. It's the kernel of the linear map L. We call it the **solution space**.

Any n linearly independent solutions $y_1, \cdots, y_n$ form a basis of the solution space. Every solution y can be expressed as a linear combination of basis vectors

$$y = c_1 y_1 + \cdots + c_n y_n$$

The above expression is still called the general solution to the homogeneous equation.

Example.

Note that both $y_1 = \cos x$ and $y_2 = \sin x$ satisfy $y'' + y = 0$. Now we add information from linear algebra:

- We are talking about function spaces, so the vectors are functions (and the functions are vectors.)
- $\{\cos x, \sin x\}$ is independent. (How can we show this?) - Wronskian
- Any 2 independent functions in a 2-dimensional function space constitute a basis for the space.
- Any function in the space is a linear combination of basis vectors.

So we get the general solution, $y = c_1 \cos x + c_2 \sin x$. All solutions have this form.

Example.

Homogeneous, non-constant coefficients: Using trial solutions of the form $y = x^r$, find a basis for the solution space of

$$2x^2y'' + 5xy' + y = 0$$

on the interval $x > 0$ and the general solution.

Trail solution: $y = x^r$

Goal: Figure out r

Order: 2

Substitute $y = x^r$ into the equation:

$$y' = rx^{r-1} \quad y'' = r(r-1)x^{r-2}$$

$$2x^2r(r-1)x^{r-2} + 5xrx^{r-1} + x^r = 0$$

$$2r(r-1)x^r + 5rx^r + x^r = 0$$

$$x^r(2r^2 + 3r + 1) = 0$$

$$(2r+1)(r+1) = 0$$
$$r = -1/2, -1$$

$$y_1 = x^{-1/2}, \quad y_2 = x^{-1}$$

$$\boxed{y = c_1x^{-1/2} + c_2x^{-1}} \quad \text{General Solution}$$

Remark.

Note that this is a very specific case that shows when the sum of terms of L is in the form of $c_ix^iD^i$.

1.1.2 Constant Coefficient Homogeneous Linear Differential Equations

We build up a technique for solving order n linear homogeneous diffeqs with constant coefficients.

1. We start small by letting $L = D - aI$. Then $L(y) = y' - ay = 0$ has general solution $y = ce^{at}$.

2. When L has the form $L = (D - aI)(D - bI)$, the factors commute.

3. If y is a solution to $(D - a_iI)$, then

$$(D - a_1I)(D - a_2I)\cdots(D - a_iI)\cdots(D - a_nI)y = 0,$$

because the factors commute.

4. Since the solution space is closed under addition and scalar multiplication, linear combinations of solutions to each factor are again solutions.

5. We'll call $p(r) = (r - a_1)(r - a_2) \cdots (r - a_n)$ the auxiliary polynomial.

Case 1: Distinct Real Roots

$$p(r) = (r - a_1)(r - a_2) \cdots (r - a_n)$$

Fact 1.1.5

$y_1 = e^{r_1 x}, \ldots, y_n = e^{r_n x}$ are linearly independent solutions to $L(y) = 0$.

Theorem 1.1.6: More about Wronskian:

Let $y_1, y_2, \ldots, y_n$ be solutions to the regular n-th order differential equation $Ly = 0$ on the interval I, and let W denote their Wronskian. If $W(x_0) = 0$ at some point $x_0 \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on I.

So, when the functions of interest are solutions to $Ly = 0$, **the Wronskian completely determines the independence of the functions**.

The general solution to the differential equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

We use **initial conditions** to find the constants $c_1, c_2, \ldots, c_n$.

Example.

Solve the initial value problem:

$$y'' - 3y' - 4y = 0, \quad y(0) = 0 \quad y'(0) = 2$$

Step 1: Find the general solution to the differential equation.
We first write the Differential Equation in the form $L(y) = 0$:

$$D^2 - 3D - 4I = 0$$

Remark.

Another way to think of this is to let $y = e^{rt}$.
The characteristic polynomial is

$$P(r) = r^2 - 3r - 4 = (r - 4)(r + 1) = 0$$

We set $P(r) = 0$ to find the root. The root is

$$r_1 = 4, \quad r_2 = -1$$

each with multiplicity 1. So the general solution is

$$y = c_1 e^{4t} + c_2 e^{-t}$$

Step 2: Use the initial conditions to find the constants c_1 and c_2.
We have $y(0) = 0$ and $y'(0) = 2$. We substitute these into the general solution:

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = 4c_1 - c_2 = 2$$

We solve this system of equations to find c_1 and c_2.

$$c_1 = \frac{2}{5}, \quad c_2 = -\frac{2}{5}$$

Remark.

For this step, we can also write the equation into matrix form and solve it using matrix algebra. e.g.

$$\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{5} \\ 0 & 1 & -\frac{2}{5} \end{array}\right]$$

So the solution to the initial value problem is

$$\boxed{y = \frac{2}{5}e^{4t} - \frac{2}{5}e^{-t}}$$

Case 2: Repeated Real Roots

If $p(r)$ has repeated roots (some r_i has multiplicity greater than 1), then solutions of the form e^{rit} are not sufficient, as we will not get n independent ones.

Note that $(D-aI)c_1te^{at} = c_1e^{at}$. So $(D-aI)^2(c_1te^{at}+c_2e^{at}) = 0$, and $\{e^{at}, te^{at}\}$ is an independent set.

Technique: If a real root a appears with multiplicity m, then include the term

$$c_1e^{at} + c_2te^{at} + c_3t^2e^{at} + \cdots + c_mt^{m-1}e^{at}$$

in the general solution.

Example.

> Find the general solution to the diffeq
>
> $$y^{(4)} - 2y^{(3)} + y'' = 0.$$
>
> **Step 1:** Find the general solution to the differential equation.
>
> $$P(r) = r^4 - 2r^3 + r^2 = r^2(r^2 - 2r + 1) = r^2(r-1)^2$$
>
> $$r_1 = 0(m=2), \quad r_2 = 1(m=2)$$
>
> The general solution is
>
> $$\boxed{y = c_1 + c_2t + c_3e^t + c_4te^t}$$

Case 3: Complex Roots

Definition 1.1.7: Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

In particular

$$e^{ibt} = \cos(bt) + i\sin(bt), \text{ and } e^{-ibt} \text{ is the complex conjugate of } e^{ibt}$$

Suppose the (real) auxiliary polynomial $P(r)$ has complex roots $r = a \pm bi$. Then the complex conjugate $\overline{a+ib} = a - ib$ is also a root of $P(r)$.

Fact 1.1.8

The two complex valued functions $e^{(a\pm ib)t} = e^{at}e^{\pm ibt} = e^{at}(\cos(bt) \pm i\sin(bt))$ are two linearly independent solutions to the differential equation.
These are complex-valued functions. We can obtain from them a pair of linearly independent, real functions. We set

$$y_1 = \frac{e^{(a+ib)t} + e^{(a-ib)t}}{2} = e^{at}\cos(bt) \quad \text{and} \quad y_2 = \frac{e^{(a+ib)t} - e^{(a-ib)t}}{2i} = e^{at}\sin(bt).$$

We conclude that the general real solution contains the term

$$\boxed{c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)}$$

Example.

Solve the initial value problem $y'' - 6y' + 25y = 0, y(0) = 0, y'(0) = 1$

$$P(r) = r^2 - 6r + 25 = (r-3)^2 + 16 = 0$$

The roots are $r = 3 \pm 4i$. The general solution is

$$y = c_1e^{3t}\cos(4t) + c_2e^{3t}\sin(4t)$$

We substitute the initial conditions to find c_1 and c_2.

$$y(0) = c_1 = 0$$

$$y'(0) = 3c_1 + 4c_2 = 1$$
$$c_2 = \frac{1}{4}$$

So the solution to the initial value problem is

$$\boxed{y = \frac{1}{4}e^{3t}\sin(4t)}$$

Case 4: Repeated Complex Roots

(Analogous to repeated real roots–multiply by powers of x.)
Technique: If a (pair of) complex roots $a \pm ib$ appears with multiplicity m, then include

$$c_1e^{at}\cos(bt)+c_2e^{at}\sin(bt)+c_3te^{at}\cos(bt)+c_4te^{at}\sin(bt)+\cdots+c_{2m-1}t^{m-1}e^{at}\cos(bt)+c_{2m}t^{m-1}e^{at}\sin(bt)$$

in the general solution.

Example.

Find the general solution: $(D^2+4)^2(D+1)y=0$.

$$P(r) = (r^2+4)^2(r+1) = (r^2+4)(r^2+4)(r+1)$$

The roots are $r = 2i(m=2), -2i(m=2), -1(m=1)$. The general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) + c_3 t\cos(2t) + c_4 t\sin(2t) + c_5 e^{-t}$$