MATH 165
Linear Algebra & Diff. Equation
Final
Notes with Examples

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Contents

Chapter 1

Linear Transformations

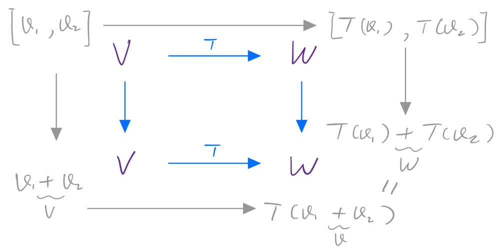

Figure 1.1: Linear Transformation Diagram

1.1 Lecture 17 & 18: Linear Transformation, Kernel & Range

This lecture covers:

- *6.1 Definition of Linear Transformations*
- *6.2 Transformations of* $\mathbb{R}^2$
- *6.3 The Kernel and Range of a Linear Transformation*

1.1.1 Definition of Linear Transformations

Definition

Definition 1.1.1: Mapping

Let V and W be vector spaces. A **mapping** T from V to W is a rule that assigns to each vector $\vec{v}$ in V precisely one vector $\vec{w} = T(\vec{v})$. We write $T : V \to W$.

Linear Transformation is a kind of mapping that preserves the operations of vector addition and scalar multiplication.

Definition 1.1.2: Linear Transformation

Let V and W be vector spaces over the same field. A mapping $T : V \to W$ is a **linear transformation** if for all $\vec{v}_1, \vec{v}_2 \in V$ and all scalars c:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in V$

In the above equations, the operations on the left of the equal signs are the ones defined in the domain V and the ones on the right of the equal signs are the ones defined in the codomain W.

1.1.2 How to prove a transformation is linear

Example.

Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.

Proof. We need to show that $T(p+q) = T(p) + T(q)$ and $T(cp) = cT(p)$ for all $p, q \in P_2$ and $c \in \mathbb{R}$.

Let $p, q \in P_2$ and $c \in \mathbb{R}$.

$$T(p+q) = x^2(p+q)(x) = x^2p(x) + x^2q(x) = T(p) + T(q)$$
$$T(cp) = x^2(cp)(x) = cx^2p(x) = cT(p)$$

Thus, T is linear.

□

Here's a short-cut to test if a transformation is linear:

Theorem 1.1.3

Let V, W be vector spaces over field F. A mapping $T : V \to W$ is a linear transformation if and only if for all $\lambda_1, \lambda_2 \in F$ and all $\vec{v}_1, \vec{v}_2 \in V$:

$$T(\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2) = \lambda_1 T(\vec{v}_1) + \lambda_2 T(\vec{v}_2)$$

Example.

Which of the following are linear transformations?

1. $T : P_2 \to P_1$ given by $T(p) = p'$
2. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, 2x)$
3. $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, x^2)$

Proof. We can use the definition of linear transformations to check each case:

1. $T(p) = p'$ is linear since $T(\lambda_1 p_1 + \lambda_2 p_2) = (\lambda_1 p_1 + \lambda_2 p_2)' = \lambda_1 p_1' + \lambda_2 p_2' = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
2. $T(x) = (x, 2x)$ is linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, 2(\lambda_1 x_1 + \lambda_2 x_2)) = \lambda_1(x_1, 2x_1) + \lambda_2(x_2, 2x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$
3. $T(x) = (x, x^2)$ is not linear since $T(\lambda_1 x_1 + \lambda_2 x_2) = (\lambda_1 x_1 + \lambda_2 x_2, (\lambda_1 x_1 + \lambda_2 x_2)^2) \neq \lambda_1(x_1, x_1^2) + \lambda_2(x_2, x_2^2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$

□

Example.

$T : \mathbb{R}^2 \to P_1(\mathbb{R})$ given by $T(a, b) = a + bx$ is linear.

Proof. We want to show: $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$

$$T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = T(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 b_1 + \lambda_2 b_2) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x$$
$$\lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2) = \lambda_1(a_1 + b_1 x) + \lambda_2(a_2 + b_2 x) = (\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2)x$$

Therefore, $T(\lambda_1(a_1, b_1) + \lambda_2(a_2, b_2)) = \lambda_1 T(a_1, b_1) + \lambda_2 T(a_2, b_2)$
Thus, T is linear.

□

Three very important examples of linear transformations

Example.

1: A transformation given by a matrix: $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(x) = Ax$ where A is a fixed $m \times n$ matrix.

Remark.

Note that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$ – we reverse the order in which the dimensions of A are listed.

Check that T is linear:

Proof. We want to show: $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$
Let $v_1, v_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$T(\lambda_1 v_1 + \lambda_2 v_2) = A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A v_1 + \lambda_2 A v_2 = \lambda_1 T(v_1) + \lambda_2 T(v_2)$$

Therefore, T is linear. □

Example.

2: Differentiation: $D : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $D(f) = f'$.
Check that D is linear:

Proof. We want to show: $D(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 D(f_1) + \lambda_2 D(f_2)$
Let $f_1, f_2 \in C^1(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$D(\lambda_1 f_1 + \lambda_2 f_2) = (\lambda_1 f_1 + \lambda_2 f_2)' = \lambda_1 f_1' + \lambda_2 f_2' = \lambda_1 D(f_1) + \lambda_2 D(f_2)$$

Therefore, D is linear. □

Example.

> **3:** The Identity map: $I : V \to V$ is defined by $I(v) = v$. For example when $V = \mathbb{R}$ the identity map is $I(x) = x$.
> Check that I is linear:
>
> ***Proof.*** We want to show: $I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 I(v_1) + \lambda_2 I(v_2)$
> Let $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$I(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 I(v_1) + \lambda_2 I(v_2)$$
>
> Therefore, I is linear. □

More example of proofing a transformation is linear:
Example.

> Show $T : P_2 \to P_4$ given by $T(p) = x^2 p(x)$ is linear.
>
> *Proof.* We want to show that $T(\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$
> Let $p_1, p_2 \in P_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
>
> $$T(\lambda_1 p_1 + \lambda_2 p_2) = x^2(\lambda_1 p_1 + \lambda_2 p_2)(x) = \lambda_1 x^2 p_1(x) + \lambda_2 x^2 p_2(x) = \lambda_1 T(p_1) + \lambda_2 T(p_2)$$
>
> Therefore, T is linear. □

An example of a transformation that is not linear:
Remark.

> Notice that when disproving a transformation is linear, we only need to find one counterexample. (i.e. one pair of vectors and one scalar that does not satisfy the properties of linearity)

Example.

Is $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(M) = \det(M)$ linear?

Proof. We can choose either disproving using addition or multiplication:
Addition:
proof by contradiction: let A $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$
$T(A+B) = \det(A+B) = \det(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}) = 3(9) - 5(7) = \boxed{-8}$
$T(A) + T(B) = \det(A) + \det(B) = 4 - 6 + 10 - 12 = \boxed{-4}$

$$\boxed{-8 \neq -4}$$

Since $T(A+B) \neq T(A) + T(B)$, T is not linear.
Multiplication:
proof by contradiction: Let A $= I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $\lambda \in \mathbb{R}$ be 2. $T(\lambda A) = \det(\lambda A) = \det(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 2(2) - 0(0) = \boxed{4}$
$\lambda T(A) = \lambda \det(A) = 2(1) - 0(0) = \boxed{2}$

$$\boxed{4 \neq 2}$$

Since $T(\lambda A) \neq \lambda T(A)$, T is not linear.

□

Strategies for proving a transformation is linear:
Guess whether it is linear or not linear, and use that as the hypothesis. If you are trying to find a counter-example and you can't find one, you may want to try proving it is linear. On the other hand, if you want to prove it is linear, you may find the counter-example while trying to prove it is linear.

Example for the notation:

Remark.

Notation:
D: Take the derivative.
D^2: Take the derivative twice.
I: The identity, meaning to return the input.

Example.

> Evaluate $(D^2 + D - 3I)(e^{rt})$
>
> *Proof.*
> $$(D^2 + D - 3I)(e^{rt}) = D^2(e^{rt}) + D(e^{rt}) - 3I(e^{rt})$$
> $$= D(re^{rt}) + re^{rt} - 3e^{rt} = r^2e^{rt} + re^{rt} - 3e^{rt} = (r^2 + r - 3)e^{rt}$$
> $$= (r^2 + r - 3)e^{rt}$$
>
> □

Remark.

> Here, I is a transformation rule or instructions that returns the input. It is important because each term in the transformation has a rule to transform it like the D^2 and D terms. Sometimes, the I term is not written out, but it is always there.

1.1.3 Properties of Linear Transformations

Theorem 1.1.4

For any linear map from any V to any W,
$T(\vec{0_V}) = \vec{0_W}$

Theorem 1.1.5

For any linear map from any V to any W,
$T(-\vec{v}) = -T(\vec{v})$

Proof. Let $T : V \to W$ be a linear transformation.
1. $T(\vec{0_V}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0_W}$
2. $T(\vec{0_V}) = T(\vec{v} + (-\vec{v})) = T(\vec{v}) + T(-\vec{v}) = \vec{0_W}$

□

1.1.4 The Kernel and Range of a Linear Transformation

Definition 1.1.6: Kernel

Let $T : V \to W$ be a linear transformation. The **kernel** of T, denoted by $\ker(T)$, is the set of all vectors in V that are mapped to $\vec{0_W}$ by T.

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0_W}\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then $\ker(T) = \text{null}(A)$

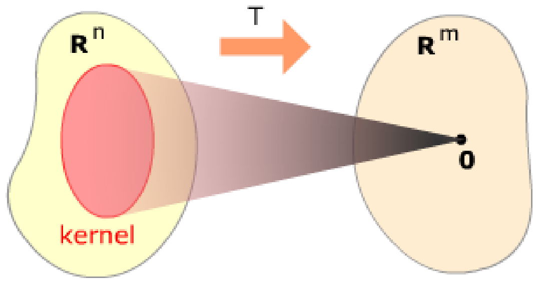

Figure 1.2: Kernel Diagram

Definition 1.1.7

Let $T : V \to W$ be a linear transformation. The **range** of T, denoted by range(T), is the set of all vectors in W that are mapped to by T.

$$\text{range}(T) = \{T(\vec{v}) \in W : \vec{v} \in V\}$$

Note: If $T(\vec{v}) = A\vec{v}$, then range$(T) = \text{col}(A)$

Remark.

> Domain is the set of all possible inputs, codomain is the set of all possible outputs, and range is the set of all actual outputs.

Example.

> $D^2 + D = 3I : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
> Find the kernel and range of $D^2 + D$.
>
> $$\ker(D^2 + D) = \{y \in C^\infty(\mathbb{R}) : (D^2 + D)(y) = 0\}$$
>
> $$= \{y \in C^\infty(\mathbb{R}) : y'' + y' = 0\}$$

Therefore, a homogenous linear differential equation is an example of a kernel.

Example.

Let $T : P_2 \to P_1$ be defined as follow:

$$T(ax^2 + bx + c) = (a + b) + (b - c)x$$

Find the kernel and range of T.

Kernel:

We want $(a + b) + (b - c)x = 0$ in P_2

Which is $0 + 0x$ in P_1

Therefore, $(a + b)$ and $(b - c)$ must be 0.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$a = -c, b = c, c = c$$
$$\ker(T) = \{-cx^2 + cx + c | c \in \mathbb{R}\}$$

Write it as a spanned set:

$$\ker(T) = \text{span}\{-x^2 + x + 1\}$$

Range:

We want $(a + b) + (b - c)x$ in P_1

Let $b + ex$ be a generic element in P_1

Therefore, $d = a + b$ and $e = b - c$

We want to solve for a, b, c in terms of d, e

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & d \\ 0 & 1 & -1 & e \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & d - e \\ 0 & 1 & -1 & e \end{array}\right]$$

$$a = d - e - c, b = e + c, c \text{ is free}$$

$$(a + b) + (b - c)x = (d - e + e + c) + (e + c - c)x = d + ex$$

$$\text{range}(T) = \{d + ex | d, e \in \mathbb{R}\}$$

So the range is all of $P_1(\mathbb{R})$.

Theorem 1.1.8

Let $T : V \to W$ be a linear transformation. Then:

1. $\ker(T)$ is a subspace of V
2. $\text{range}(T)$ is a subspace of W

Proof. Let $T : V \to W$ be a linear transformation.

1. $\ker(T)$ is a subspace of V

1. $\vec{0_V} \in \ker(T)$ since $T(\vec{0_V}) = \vec{0_W}$
2. Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$ and $\lambda \in \mathbb{R}$.
   Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0_W} + \vec{0_W} = \vec{0_W}$
   Therefore, $\vec{v}_1 + \vec{v}_2 \in \ker(T)$
3. Also, $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{0_W} = \vec{0_W}$
   Therefore, $\lambda\vec{v}_1 \in \ker(T)$
   Thus, $\ker(T)$ is a subspace of V.

2. range(T) is a subspace of W

1. $\vec{0_W} \in \text{range}(T)$ since $T(\vec{0_V}) = \vec{0_W}$
2. Let $\vec{w}_1, \vec{w}_2 \in \text{range}(T)$ and $\lambda \in \mathbb{R}$.
   Then there exists $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$.
   Then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$
   Therefore, $\vec{w}_1 + \vec{w}_2 \in \text{range}(T)$
3. Also, $T(\lambda\vec{v}_1) = \lambda T(\vec{v}_1) = \lambda\vec{w}_1$
   Therefore, $\lambda\vec{w}_1 \in \text{range}(T)$
   Thus, range(T) is a subspace of W.

□

Example.

> Find the basis for the range and kernel of: $D^2 : P_3 \to P_3$ given by $D^2(p) = p''$

Proof.

$$p(x) = ax^3 + bx^2 + cx + d.$$

Basis for the Kernel:
The kernel of a linear transformation consists of all elements that map to the zero element of the codomain. For the transformation D^2, we need to find all $p \in P_3$ such that $D^2(p) = 0$. Applying D^2 to the general polynomial:

$$p''(x) = (ax^3 + bx^2 + cx + d)'' = (3ax^2 + 2bx + c)'' = 6ax + 2b.$$

Setting $p''(x) = 0$ yields:

$$6ax + 2b = 0.$$

For this to be true for all x, we must have $a = 0$ and $b = 0$. Therefore, $p(x)$ reduces to:

$$p(x) = cx + d.$$

Polynomials of the form $cx + d$ clearly form the kernel of D^2, and are elements of P_1 (the space of all polynomials of degree at most 1). Hence, a basis for the kernel of D^2 consists of the polynomials:

$$\boxed{\{1, x\}}$$

□

Proof. **Basis for the Range:**
The range of D^2 consists of all possible outputs $D^2(p)$ for $p \in P_3$. From the calculation above, we found:

$$D^2(p) = 6ax + 2b.$$

This expression tells us that the output of D^2 can be any polynomial of the form $6ax+2b$, which can be rewritten as a linear combination of the basis $\{x, 1\}$ (or equivalently, $\{6x, 2\}$).

$$\boxed{\{6x, 2\}}$$

□

Theorem 1.1.9: Genearl Rank-Nullity Theorem

Let $T : V \to W$ be a linear transformation. Then:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$$

Example.

Let $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transformation with the rule $S(A) = AA^T$. Find a basis for its kernel and the dimension of its range. Confirm the general rank/nullity theorem based on what we found about its kernel and range previously.

Proof. **Kernel:**
We want to find A such that $AA^T = \vec{0}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a^2+b^2 = 0, ac+bd = 0, c^2+d^2 = 0$$

$$a = b = c = d = 0$$

Therefore, the kernel is $\{0\}$ and the dimension is 0.
Range:
We want to find A such that $AA^T = B$ for some $B \in M_2(\mathbb{R})$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$$

The basis for it will be:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore, the range is all of $M_2(\mathbb{R})$ and the dimension is 4.
Rank-Nullity Theorem:

$$\dim(M_2(\mathbb{R})) = \dim(\ker(S)) + \dim(\text{range}(S))$$
$$\boxed{4 \neq 0 + 3}$$

$\square$

This is not a linear transformation because it does not satisfy the properties of linearity. Therefore, the rank-nullity theorem does not hold.

Example.

Consider the linear transformation $T : M_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-b+d) + (-a+b-d)x^2$$

Determine range(T). What is the dimension of $\ker(T)$? Determine a basis for $\ker(T)$.

Step 1: Verify it is linear:

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$T(A+B) = T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) = (a_1+a_2-b_1-b_2+d_1+d_2)+(-a_1-a_2+b_1+b_2-d_1-d_2)x^2$

$T(A)+T(B) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)+T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = (a_1-b_1+d_1)+(-a_1+b_1-d_1)x^2+(a_2-b_2+d_2)+(-a_2+b_2-d_2)x$

$$(a_1+a_2-b_1-b_2+d_1+d_2)+(-a_1-a_2+b_1+b_2-d_1-d_2)x^2$$

$$T(A+B) = T(A)+T(B)$$

Therefore, T is linear.

1.1.5 Linear transformation and basis

If we have a matrix A that's m by n, we get a free transformation. $T : f_n \to f_m$ given by $T(x) = Ax$ if V and W are finite-dimensional, then $T : V \to W$ can be accomplished by a matrix multiplication. That is there exist a matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in V$.

Theorem 1.1.10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists an $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

We have seen that, given an $m \times n$ matrix A, we can define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(v) = Av$. (Remember that an $m \times n$ matrix maps $\mathbb{R}^n$ to $\mathbb{R}^m$.)

Now, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we identify an $m \times n$ matrix A such that $T(v) = Av$.

This is possible, because if we know how T acts on a basis $\{b_1, \cdots, b_n\}$ of V, then we know how T acts on the whole of V, since if $v = c_1b_1 + \cdots + c_kb_k$, then

$$T(v) = T(c_1b_1 + \cdots + c_kb_k) = c_1T(b_1) + \cdots + c_kT(b_k)$$

In particular, if $T : \mathbb{R}^n \to \mathbb{R}^m$ and if $v = (c_1, c_2, \cdots, c_n) = c_1e_1 + c_2e_2 + \cdots + c_ne_n$, then

$$T(v) = T(c_1e_1 + c_2e_2 + \cdots + c_ne_n) = \boxed{c_1T(e_1) + c_2T(e_2) + \cdots + c_nT(e_n)}$$

By the definition of matrix multiplication, the boxed expression is equal to

$$\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then $T(v) = Av$ where $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$.

Example.

> Find A. $T_1 \mathbb{R}^2 \to \mathbb{R}^2$
>
> $$T(x) = \beta$$
>
> $$\beta = \{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$$
>
> $$T(x) = (a_1 \cdot e_1 + a_2 \cdot e_2 + \cdots + a_n \cdot e_n)$$
>
> $$= a_1 T(\vec{e_1}) + a_2 T(\vec{e_2}) + \cdots + a_n T(\vec{e_n})$$
>
> $$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$$
>
> If I take the matrix A and apply it to the coefficient vector, I get the transformation.
>
> $$A \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$
>
> $$= a_1 T(\vec{e_1}) + a_2 T(\vec{e_2}) + \cdots + a_n T(\vec{e_n})$$
>
> $$= T(a_1 \vec{e_1} + a_2 \vec{e_2} + \cdots + a_n \vec{e_n})$$

Example.

Find a matrix A such that $T(x) = A(x)$ where $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(a, b, c) = (c, a + b)$.

Since this is from $\mathbb{R}^3$ to $\mathbb{R}^2$, the matrix A will be 2×3.

$$T(a, b, c) = (c, a + b)$$

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix}$$

$$A = \begin{bmatrix} T(1, 0, 0) & T(0, 1, 0) & T(0, 0, 1) \end{bmatrix}$$

$$= A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} c \\ a + b \end{bmatrix}$$

$$\boxed{= (c, a + b)}$$

Chapter 2

Eigenvalues and Eigenvectors

2.1 Lecture 19: Kernal Range and Eigenvalues

This lecture covers:

- *7.2 The Results for Eigenvalues and Eigenvectors*

2.1.1 Eigenvalues and Eigenvectors

Definition 2.1.1: Eigenvalues

Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if there exists a nontrivial (non-zero) vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}$$

The corresponding vector $\vec{x}$ is called an **eigenvector** of A.

The eigenvalue λ will have infinitely many corresponding eigenvectors $\vec{v}$ that satisfy:

1. $\vec{v} \neq \vec{0}$
2. $A\vec{v} = \lambda\vec{v}$

Notice that the $\vec{x}$ and $\vec{v}$ here are interchangeable.

Definition 2.1.2: Characteristic Polynomials

The characteristic polynomial of an $n \times n$ matrix A is defined as

$$p(\lambda) = \det(A - \lambda I)$$

where I is the $n \times n$ identity matrix.
If λ_i is a root of the characteristic polynomial, that is $p(\lambda_i) = 0$, then $A - \lambda_i I_n$ will have a nontrivial null space.
This means there is a non-zero vector $\vec{x}$ such that $(A - \lambda_i I_n)\vec{x} = \vec{0}$.
Hence λ_i is an eigenvalue of A with corresponding eigenvector $\vec{v}_i$.
All the **nonzero** elements of the null space of $A - \lambda_i I_n$ are eigenvectors of A corresponding to λ_i.

Example.

Find all the eigenvalues and corresponding eigenvectors of the matrix A:

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

Step 1: Find the Eigenvalues:
We start with the equation:

$$A\vec{x} = \lambda\vec{x}$$

$$= A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$= (A - \lambda I)\vec{x} = \vec{0}$$

This is a homogeneous system of equations. We can solve for λ by finding the determinant of $A - \lambda I$ and setting it equal to zero.

Remark.

The reason this works is because we always have a solution that $\vec{x} = \vec{0}$. However, we are looking for solutions where $\vec{x} \neq \vec{0}$ (nontrivial solutions). When $\det A - \lambda I = 0$ it means that we have more than one solutions that make the system consistent. This means we have free variables at the end of the row reduction, which makes the matrix non-invertible and hence the determinant is zero.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 - \lambda & -1 \\ -5 & -1 - \lambda \end{bmatrix}\right) = 0$$

The characteristic polynomial is:

$$p(\lambda) = (3 - \lambda)(-1 - \lambda) - (-1)(-5) = \lambda^2 - 2\lambda - 8 = 0$$

We can solve for λ by factoring:

$$(\lambda - 4)(\lambda + 2) = 0$$
$$\boxed{\lambda = 4, -2}$$

Example.

> **Step 2:** Find the Eigenvectors:
> We plugin the eigenvalues into the equation $(A - \lambda I)\vec{x} = \vec{0}$ to find the eigenvectors.
> For $\lambda = 4$:
>
> $$A - 4I = \begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix}$$
>
> $$\begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
>
> $$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
>
> **Remark.**
>
> > Notice that if the second row can not be reduced to all zeros, that means you made a mistake in the previous steps.
>
> $$x_1 = -x_2$$
>
> Let $x_2 = 1$, then $x_1 = -1$.
> The eigenvector corresponding to $\lambda = 4$ is:
>
> $$\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$
>
> For $\lambda = -2$:
>
> $$\left[\begin{array}{cc|c} 5 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$
>
> $$x_1 = \frac{1}{5}x_2$$
>
> Let $x_2 = 5$, then $x_1 = 1$.
> The eigenvector corresponding to $\lambda = -2$ is:
>
> $$\boxed{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}$$
>
> **Remark.**
>
> > Notice that the eigenvectors are not unique. We can scale the eigenvectors by any scalar and they will still be eigenvectors. In other words, for each eigenvalue, there are infinitely many eigenvectors.

2.1.2 Complex Eigenvalues

Definition 2.1.3: Complex Conjugate

The complex number (notation $\mathbb{C}$) have the form:

$$z = a + bi, \text{ where } a, b \in \mathbb{R}, \text{and } i^2 = -1$$

The **complex conjugate** of $z = a + bi$ is $a - bi$. It is denoted as $\overline{z}$.

Properties of complex conjugates: If A is a real matrix and $\vec{v}$ is an eigenvector with eigenvalue λ then its complex conjugate $\overline{\vec{v}}$ is also an eigenvector with eigenvalue $\overline{\lambda}$. Justification:

$$A\vec{v} = \lambda\vec{v} \implies A\overline{\vec{v}} = \overline{\lambda\vec{v}}$$

Consider the complex conjugate of the entire equation. Recall that taking the complex conjugate of a complex number z is denoted $\overline{z}$, and for any complex vector $\vec{v}$, the conjugate $\overline{\vec{v}}$ is the vector of the conjugates of the components of $\vec{v}$.

Given $A\vec{v} = \lambda\vec{v}$, we conjugate both sides:

$$\overline{A\vec{v}} = \overline{\lambda\vec{v}}$$

Since A is a real matrix, the elements of A remain unchanged when conjugated, so $\overline{A} = A$. This applies to every component of the matrix-vector product, resulting in:

$$A\overline{\vec{v}} = \overline{\lambda\vec{v}}$$

Example.

(Complex eigenvalues) Find all the eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Step 1: Find the Eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1 \implies \lambda = \pm i$$

Step 2: Find the Eigenvectors:
For $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$x_1 = -ix_2$$

Let $x_2 = 1$, then $x_1 = -i$.
The eigenvector corresponding to $\lambda = i$ is:

$$\boxed{\begin{bmatrix} -i \\ 1 \end{bmatrix}}$$

Using the properties of complex conjugates, we can find the eigenvector corresponding to $\lambda = -i$:

$$\boxed{\begin{bmatrix} i \\ 1 \end{bmatrix}}$$

2.1.3 Properties and Theorems

- 0 is an eigenvalue of A if and only if A has non-trivial null space. (So 0 can be an eigenvalue but not an eigenvector)
- If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $k\vec{v}$ is also an eigenvector of A with eigenvalue λ for any scalar $k \neq 0$.
- If A is invertible and $\vec{v}$ is an eigenvector with eigenvalue λ, then $\lambda \neq 0$ and $\vec{v}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.
- If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\vec{v}$ is an eigenvector of A^k with eigenvalue λ^k for any positive integer k.

2.2 Lecture 20: Eigenspaces & Eigenbase

This lecture covers:

- *7.2 General Results on Eigenvalues and Eigenvectors*
- *7.3 Diagonalization*

2.2.1 Eigenspaces and Non-defective Matrices

Definition 2.2.1: Defective Matrices

A matrix is defective if it does not have a complete set of eigenvectors.
The matrix A is defective if the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity.

This is a common simple example that illustrate the concept of defective matrices.

Example.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Here, we can find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = 0$, where I is the identity matrix.

1. Subtract λ from the diagonal entries of A and set the determinant to zero:

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 0 \cdot 1 = (1-\lambda)^2 = 0$$

2. Solving $(1-\lambda)^2 = 0$, we find that the only eigenvalue is $\lambda = 1$, with algebraic multiplicity 2 (because it is a double root of the characteristic equation).

3. Next, to find the eigenvectors corresponding to $\lambda = 1$, solve $(A - I)x = 0$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This simplifies to $0 \cdot x_1 + 1 \cdot x_2 = 0$, which implies $x_2 = 0$. So, the eigenvectors are of the form $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, where x_1 can be any non-zero value.

4. Thus, we have only one linearly independent eigenvector, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, despite having an eigenvalue with algebraic multiplicity 2. This means the geometric multiplicity of $\lambda = 1$ is 1, which is less than its algebraic multiplicity.

Since the algebraic and geometric multiplicities do not match, the matrix A is defective and cannot be diagonalized.

Example.

Find all ythe eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}$$

whose characteristic polynomial is $p(\lambda) = -(\lambda - 2)^2(\lambda + 1)$.

Solution: The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. For $\lambda_1 = 2$, we have

$$A - 2I = \left[\begin{array}{ccc|c} 3 & 12 & -6 & 0 \\ -3 & -12 & 6 & 0 \\ -3 & -12 & 6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives us the eigenvector $x_1 = -4x_2 + 2x_3$.

$$\vec{x}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$, we have

$$A + I = \left[\begin{array}{ccc|c} 6 & 12 & -6 & 0 \\ -3 & -9 & 6 & 0 \\ -3 & -12 & 9 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives us the eigenvector $x_1 = -2x_3$ and $x_2 = 3x_3$.

$$\vec{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

We can plug in the eigenvectors into the matrix to check if they are correct.

Example.

Check:

$$A\vec{v} = \lambda\vec{v}$$

For $\lambda_1 = 2$ and $\vec{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$, we have

$$A\vec{v}_1 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, we have

$$A\vec{v}_2 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, we have

$$A\vec{v}_2 = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Definition 2.2.2: Algebraic Multiplicity

Any polynomial with coefficients in $\mathbb{R}$ can be written as follows:

$$p(t) = a(t - r_1)^{m_1}(t - r_2)^{m_2} \cdots (t - r_k)^{m_k}$$

The $r_i \in \mathbb{C}$ are the distinct roots of the polynomial. The exponent m_i is the **algebraic multiplicity** of the root r_i.

Example.

In the previous example, the algebraic multiplicity of $\lambda_1 = 2$ is 2 and the algebraic multiplicity of $\lambda_2 = -1$ is 1.

Definition 2.2.3: Eigenspaces

Let $A \in M_n(\mathbb{R})$. For any given Eigenvalues λ_i, define the **Eigenspace** associate to each λ_i as follows:

$$E_i = \{x \in \mathbb{C}^n | A\vec{v} = \lambda_i \vec{v}\} = \text{Null}(A - \lambda_i I_n) = \{\text{all Eigenvectors of } \lambda_i\} \cup \{0_n\}$$

The dimension $\dim(E_i) = n_i$ is called the **geometric multiplicity** of λ_i.

Example.

> In the previous example, the geometric multiplicity of $\lambda_1 = 2$ is 2 and the geometric multiplicity of $\lambda_2 = -1$ is 1.

Remark.

> **Algebraic Multiplicity**
> The **algebraic multiplicity** of an eigenvalue is the number of times that eigenvalue appears as a root of the characteristic polynomial of the matrix. In other words, it's the exponent of the eigenvalue in the factorized form of the characteristic polynomial.
>
> For example, if the characteristic polynomial is $(\lambda - 1)^2(\lambda + 2)$, the eigenvalue $\lambda = 1$ has an algebraic multiplicity of 2 because it appears twice as a root (i.e., it's squared in the polynomial).
>
> **Geometric Multiplicity**
> The **geometric multiplicity** of an eigenvalue is the number of linearly independent eigenvectors associated with that eigenvalue. This is technically the dimension of the eigenspace corresponding to the eigenvalue. The eigenspace is the null space of the matrix $A - \lambda I$, where A is the matrix in question, λ is the eigenvalue, and I is the identity matrix of the same size as A.
>
> For the same example, if for $\lambda = 1$, we can only find one linearly independent eigenvector despite the algebraic multiplicity being 2, then the geometric multiplicity of $\lambda = 1$ is 1.
>
> **Relationship and Properties**
>
> 1. The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.
> 2. If the geometric multiplicity of every eigenvalue equals its algebraic multiplicity, the matrix is said to be diagonalizable, meaning there exists a basis of eigenvectors for the matrix, and the matrix can be expressed as PDP^{-1} where D is a diagonal matrix of eigenvalues and P is the matrix whose columns are the corresponding eigenvectors.

Theorem 2.2.4

Let λ_i be an eigenvalue of A of algebraic multiplicity m_i, and E_i the corresponding eigenspace. Then:

1. E_i is a subspace of $\mathbb{C}^n$.
2. $\dim(E_i) \leq m_i$.

Theorem 2.2.5: The independence of Eigenvectors

1. Eigenvectors corresponding to distinct eigenvalues are linearly independent.
2. The union of linearly independent elements of **distinct** eigenspace $E_1, E_2, \cdots, E_k$ is a linearly independent set. (This means if we take the union of basis of distinct eigenspaces, we get an independent set.)

Fact 2.2.6

1. If A has n distinct eigenvalues, then A is non-defective.
2. A is **non-defective** if and only if the **geometric multiplicity** of each eigenvalue equals its **algebraic multiplicity**. (That way, the sum of the dimensions of the eigenspaces equals n.)

Example.

1. Show the following matrix is non-defective:

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

Step 1: Find the Eigenvalues
The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.
Step 2: Find the Eigenvectors
For $\lambda_1 = 1$, we have

$$A - I = \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix}$$

which gives us the eigenvector $2x_1 = -3x_2$.

$$\vec{x}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 2$, we have

$$A - 2I = \begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix}$$

which gives us the eigenvector $x_1 = -x_2$.

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Step 3: Check the Independence of Eigenvectors
The eigenvectors corresponding to different eigenvalues are always linearly independent. Here, the eigenvectors are:

$$\begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Conclusion For $\lambda_1 = 1$, we have algebraic multiplicity 1 and geometric multiplicity 1. For $\lambda_2 = 2$, we have algebraic multiplicity 1 and geometric multiplicity 1. Since the geometric multiplicity of each eigenvalue equals its algebraic multiplicity, the matrix A is non-defective.

Example.

2. Find a basis of eigenvectors of A for $\mathbb{R}^2$

Forming a Basis:

$$P = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

The determinant of P is:

$$\det(P) = -3 \cdot 1 - (-1) \cdot 2 = -3 + 2 = -1$$

Since the determinant is non-zero, the vectors are linearly independent.

Spanning:

$$\left[\begin{array}{cc|c} -3 & -1 & x \\ 2 & 1 & y \end{array}\right]$$

RREF:

$$\left[\begin{array}{cc|c} 1 & 0 & -x - y \\ 0 & 1 & 2x + 3y \end{array}\right]$$

The determinant is non-zero, so the vectors span $\mathbb{R}^2$. Thus, the vectors form a basis of $\mathbb{R}^2$.

Example.

3. Let $\vec{v} = (-2, 1) \in \mathbb{R}^2$ with the standard basis. Express $\vec{v}$ as a linear combination of the eigenvectors of A.

Solution:

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving the system of equations, we get $c_1 = 1$ and $c_2 = -1$. Thus, $\vec{v}$ can be expressed as a linear combination of the eigenvectors of A.

Example.

4. Find $A^2\vec{v}$ and $A^{-1}\vec{v}$.

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

and a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Step 1: $A^2\vec{v}$

Now, compute $A^2\vec{v}$ by multiplying A again by $A\vec{v}$:

$$A^2\vec{v} = A(A\vec{v}) = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -x - 3y \\ 2x + 4y \end{bmatrix}$$

Simplifying the multiplication:

$$A^2\vec{v} = \begin{bmatrix} (-1)(-x - 3y) + (-3)(2x + 4y) \\ (2)(-x - 3y) + (4)(2x + 4y) \end{bmatrix}$$

$$A^2\vec{v} = \begin{bmatrix} x + 3y - 6x - 12y \\ -2x - 6y + 8x + 16y \end{bmatrix}$$

$$A^2\vec{v} = \begin{bmatrix} -5x - 9y \\ 6x + 10y \end{bmatrix}$$

Step 2: $A^{-1}\vec{v}$

To compute $A^{-1}\vec{v}$, we first need to find the inverse of the matrix A.
Inverse of a 2×2 Matrix

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying this formula:

$$A^{-1} = \frac{1}{(-1)(4) - (-3)(2)} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-4 + 6} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix}$$

Now compute $A^{-1}\vec{v}$:

$$A^{-1}\vec{v} = \begin{bmatrix} 2 & \frac{3}{2} \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A^{-1}\vec{v} = \begin{bmatrix} 2x + \frac{3}{2}y \\ -x - \frac{1}{2}y \end{bmatrix}$$

Example.

Example: Is the following matrix defective? Find a basis for the span of its eigenvectors.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Step 1: find the Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$(3-\lambda)((2-\lambda)(2-\lambda)-1) = 0$$

$$\lambda_1 = 3(Mul:2), \lambda_2 = 1(Mul:1)$$

Step 2: Find the Eigenvectors

For $\lambda_1 = 3$, we have

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$x_1 = x_2 + x_3$$

$$x_2 = -x_3$$

let $x_2 =$ t, then $x_3 = -t$, $x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 1$, we have

$$A - I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = x_3$$

let $x_2 = t$, then $x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We can actually stop by λ_1 since we only find one eigenvector but the algebraic multiplicity is 2. Thus, the matrix is defective.

Chapter 3

Linear Differential Equations of Order n

3.1 Lecture 21: Linear Differential Equations

This lecture covers:

- *8.1 General Theory of Linear Differential Equations*
- *8.2 Constant Coefficient Homogeneous Linear Differential Equations*

3.2 Lecture 22: Annihilators

This lecture covers:

- *8.3 The Method of Undetermined Coefficients: Annihilators*
- *8.4 Complex-Valued Trial Solutions*

3.3 Lecture 23: Oscillations of a Mechanical System

This lecture covers:

- *8.5 Oscillations of a Mechanical System*

Chapter 4

Systems of Linear Differential Equations

4.1 Lecture 24: First Order Linear Systems

This lecture covers:

- *9.1 First Order Linear Systems*
- *9.2 Vector Formulation*
- *9.3 General Results for First Order Linear Systems*

Remark.

> Today we are going to cover 9.1 to 9.3. We are going to skip many theorems.
> Next week we are going to do some examples for chapter 8.

4.1.1 Chapter 9.1: First Order Linear Systems

$$\begin{bmatrix} x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\ x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\ \cdots \\ x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t) \end{bmatrix}$$

The b_i is the consists of the non-homogenous part. If $b_i(t) = 0$, the system is homogenous.

An example of the first-order linear system is:

Example.

> $$x_1' = x_1 + 2x_2$$
> $$x_2' = 2x_1 + 2x_2$$
> $$x_1(0) = 1, x_2(0) = 0$$

The **Initial value** here is $x_1(0) = 1, x_2(0) = 0$.

A **solution** is an ordered n-tuple of functions $x_1(t), x_2(t), \ldots, x_n(t)$ that satisfies the system of equations.

The solution will be in the form of:

Example.

> $$x_1(t) = \text{Some function}, x_2(t) = \text{Some function}$$

Which is a vector of functions.

There's a clever trick to solving the 2 by 2 system using the derivative, however, it is not that fast and no one use it in the exam so we are going to skip it.

Remark.

> The first-order linear system is restrictive. However, we can transform a higher-order system by renaming functions.

Example.

$$\frac{d^2x}{dt^2} + 4e^t \frac{dx}{dt} - 9t^2 x = 7t^2$$

Strataegy: Let $x_1 = x$, $x_1' = x_2$

$$x_1' = x_2$$
$$x_2' = 9t^2 x + 4e^t x_2 - 7t^2$$

(Chapter 9.1)

4.1.2 Chapter 9.2: Vector Formulation

Remark.

Chapter 9.2: How to transform the system into a matrix form.

Example.

Formulate with vectors.

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

$$\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix}$$

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

Coefficient matrix:

$$\vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdots \\ b_n(t) \end{bmatrix}$$

Theorem 4.1.1

$V_n(I)$ is a column vector of n functions defined on an interval I.

Example.

$$\begin{bmatrix} e^{3t} \\ 2 \\ e^{7t} \end{bmatrix} \in V_3(\mathbb{R})$$

for any fixed n, I, $V_n(I)$ is a vector space.

Wronskian

Remark.

We do not test this

Wronskan of a set of n column vectors in $V_n(I)$

Wronskien of $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$
$W(t) = \text{Wronskian} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$

Theorem 4.1.2

If $W(t) \neq 0$ for all $t \in I$, then $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is linearly independent.

Theorem 4.1.3

If $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is linearly independent, then $W(t) \neq 0$ for all $t \in I$.

4.1.3 Chapter 9.3: Genearl results for first-order linear systems

Theorem 4.1.4: Initial value problem

$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{b}(t)$ and $\vec{x}(t_0) = \vec{x}_0$ has a unique solution on an interval I containing t_0 if $A(t)$ and $\vec{b}(t)$ are continuous on I.

Theorem 4.1.5: 9.3.2

The set of solutions to the homogenous system $\vec{x}'(t) = A(t)\vec{x}(t)$ is a vector space of dimension n.

Fundamental solution set

The fundamental solution set is basically a basis for the solution space.

Theorem 4.1.6: The fundamental solution set

$$S = \{x_1, x_2, \cdots, x_n\}$$

S is a set of solutions to the homogenous system $\vec{x}'(t) = A(t)\vec{x}(t)$ that are linearly independent.

The non-homogenous case:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n + \boxed{\vec{x}_p}$$

We add a single solution to the homogenous system to the solution of the non-homogenous system.

Simplifying assumptions:

$$\vec{x}'(t) = A\vec{x}(t)$$

is homogenous and A is a matrix of constants and A is non-defective.

Remark.

> Recall: Non-defective means that the matrix has n linearly independent eigenvectors.

Theorem 4.1.7

Let A be a $n \times n$ matrix of real constants, and let λ be a real eigenvalue of corresponding to the eigenvector $\vec{v}$.

Then

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a solution to the homogenous system $\vec{x}'(t) = A\vec{x}(t)$.

Remark.

> Recall: Means that $A\vec{v} = \lambda\vec{v}$

Proof. If $\vec{x}(t) = e^{\lambda t}\vec{v}$, then

$$A(e^{\lambda t}\vec{v}) = e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v} = \lambda e^{\lambda t}\vec{v}$$

On the other side, if I take the derivative of $\vec{x}(t)$, I get

$$\vec{x}'(t) = \lambda e^{\lambda t}\vec{v}$$

We notice that the two sides are equal. □

And we are going to have enough solutions to form a basis for the solution space.

Theorem 4.1.8

If A has n real eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ with corresponding real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then

$$\vec{x}_k = e^{\lambda_k t}\vec{v}_k$$

is a solution to the homogenous system $\vec{x}'(t) = A\vec{x}(t)$.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

The eigenvalue is:

$$|A - \lambda I| = (1-\lambda)\cdot(-2-\lambda) - 4 = 0$$

$$\lambda = 2, -3$$

nullspace of $A - 2I$ is $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

nullspace of $A + 3I$ is $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

What's new today:
Solution =

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$x_1 = 2c_1 e^{2t} - c_2 e^{-3t}$$

$$x_2 = c_1 e^{2t} + 2c_2 e^{-3t}$$

Initial values:

$$x_1(0) = 2c_1 - c_2 = 1$$

$$x_2(0) = c_1 + 2c_2 = 0$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -1/5 \end{array}\right]$$

Example.

$$x' = \vec{A}\vec{x}$$

$$A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

$$x_1' = 4x_2$$

$$x_2' = x_1$$

Step 1: Find the eigenvalues and eigenvectors

$$|A - \lambda I| = \begin{bmatrix} -\lambda & 4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 4 = 0$$

$$\lambda_1 = 2, \lambda_2 = -2$$

For $\lambda_1 = 2$

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

For $\lambda_2 = -2$

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Step 2: Write the general solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Remark.

If you can not zero out the second role, it means that you are wrong.

Let's do a 3 by 3 example. We will leave the complex case for the next lecture.s

Example.

$$A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 1: Find the eigenvalues and eigenvectors

$$|A - \lambda I| = \begin{bmatrix} 0-\lambda & -3 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

Characteristic polynomial:

$$(2-\lambda)(\lambda-3)(\lambda-2) = 0$$

$$\lambda_1 = 2, \lambda_2 = -3, \lambda_3 = 2$$

For $\lambda_1 = 2$

$$\begin{bmatrix} -2 & -3 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 are free

$$-2x_1 = 3x_2 - x_3$$

$$x_1 = -\frac{3}{2}x_2 + \frac{1}{2}x_3$$

$$\vec{v}_1 = (3, -2, 0)$$

$$\vec{v}_2 = (1, 0, 2)$$

For $\lambda_2 = -3$

$$\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free

$$x_1 = x_2$$

$$x_3 = 0$$

$$\vec{v}_3 = (1, 1, 0)$$

Step 2: Write the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

4.2 Lecture 25:

This lecture covers:

-

4.2.1 Complex-Valued Solutions

Theorem 4.2.1

Let $\vec{u}(t)$ and $\vec{v}(t)$ be real-valued vector functions. If

$$\vec{w}_1(t) = \vec{u}(t) + \vec{v}(t) \quad \text{and} \quad \vec{w}_2(t) = \vec{u}(t) - \vec{v}(t)$$

are complex conjugate solutions to $x' = Ax$, then

$$x_1(t) = \vec{u}(t) \text{and} \quad x_2(t) = \vec{v}(t)$$

are themselves real-valued solutions to $x' = Ax$.

Example.

Let $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Find the general solution to $x' = Ax$.

Step 1: Find the Eigenvalues and Eigenvectors

The characteristic equation is $\lambda^2 + 4 = 0$, so $\boxed{\lambda = \pm 2i}$.

For $\lambda = 2i$, we have

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

RREF:

$$\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, $\boxed{\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}}$ or $\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}}$ is an eigenvector corresponding to $\lambda = \pm 2i$.

Step 2: Find the General Solution

The general solution is

$$\begin{aligned} x(t) &= c_1 e^{2it} \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{-2it} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= c_1 \begin{bmatrix} -i\cos(2t) - \sin(2t) \\ \cos(2t) - i\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(2t) - i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{bmatrix} \end{aligned}$$

Here, we want it to be in the form of $\vec{u}(t)+i\vec{v}(t)$. So, we can rewrite the general solution as

$$\begin{aligned} &= \begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ c_2\cos(2t) - c_1\sin(2t) \end{bmatrix} + i \begin{bmatrix} c_2\cos(2t) - c_1\sin(2t) \\ c_1\cos(2t) + c_2\sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} c_1\cos(2t) + c_2\sin(2t) \\ c_2\cos(2t) - c_1\sin(2t) \end{bmatrix} + i \begin{bmatrix} c_2\cos(2t) - c_1\sin(2t) \\ c_1\cos(2t) + c_2\sin(2t) \end{bmatrix} \end{aligned}$$

Example.

Example 1: $(D^2+2D+10)^2$ The first step is to find the order of this, which is 2 inside from D^2 and squared it to get $\boxed{4}$.

$$p(r) = r^2 + 2r + 10 = 0$$

Which doesn't obviously factor, so we check:

$$b^2 - 4ac = 4 - 40 = \boxed{-36}$$

Which is negative, so we have complex roots.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = \boxed{-1 \pm 3i}$$

The multiplicity of the roots is 2 for each, so we have to use the formula:

$$e^{rt} = e^{-t}(\cos(3t) \pm i\sin(3t))$$

The general solution is:

$$\boxed{y = c_1e^{-t}\cos(3t) + c_2e^{-t}(\sin(3t)) + c_3te^{-t}(\cos(3t)) + c_4te^{-t}(\sin(3t))}$$

Example.

Example 2: $y''' - y'' + y' - y = 0$

$$p(r) = r^3 - r^2 + r - 1 = 0$$
$$= r^2(r-1) + 1(r-1)$$
$$= (r+1)(r-1)^2 = 0$$

Rational root Theorem:

Trial:

$$r = \pm 1$$

try $r = 1$:

$$1 - 1 + 1 - 1 = 0$$

So, we have a root of 1.

$$\boxed{y = c_1e^t + c_2\cos(t) + c_3\sin(t)}$$

Example.

Example 3: $y'' + y = 6xe^x$
$y_c =$ General solution to $y'' + y = 0$
$y_p =$ Trial solution
y_c:

$$p(r) = r^2 + 1 = 0$$

$$r = \pm i$$

$$y_c = c_1 \cos(x) + c_2 \sin(x)$$

$$y(p) = Ae^x + Bxe^x$$

Apply $(D^2 + I)y_p$

$$D(Ae^x + Bxe^x)$$

$$= Ae^x + Be^x + Bxe^x$$

$$= (A + B)e^x + Bxe^x$$

$$D^2(Ae^x + Bxe^x) = (A + B)e^x + Be^x + Bxe^x$$

$$= (A + B)e^x + 2Be^x + Bxe^x$$

$$y = c_1 e^t + c_2 \cos(t) + c_3 \sin(t)$$

$$(D^2 + I)y_p = (A + 2B)e^x + Bxe^x + Ae^x + Bxe^x$$

$$= (2A + 2B)e^x + 2Bxe^x$$

$$= 6xe^x$$

$$2A + 2B = 0$$

$$2B = 6$$

$$B = 3$$

$$A = -3$$

$$y = c_1$$

Chapter 5

Review

5.1 S15 FQ6

Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 real matrices. Consider the linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix}.$$

Kernel:

$$a+d=0 \quad b-c=0 \quad a-c=0 \quad b+d=0$$

$$a=c=b=-d$$

Therefore, the basis for the kernel is:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So the nullity of T is 1.
Range:

$$\begin{bmatrix} a+d & b-c \\ a-c & b+d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, these for vectors are not linearly independent:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Therefore, the basis for the range is:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and the rank of T is 3.
Conclusion: This aligned with the Rank-Nullity Theorem, as the rank of T is 3 the nullity is 1, and the sum of the rank and nullity is 4, which is the dimension of the domain of T.