MATH 165
Linear Algebra & Diff. Equation
Midterm II
Notes with Examples

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Contents

0.1 Lecture 14: Spanning, Linear Independence & Basis

This lecture covers:

- *4.5 Linear Dependence and Linear Independence*
- *4.6 Basis and Dimension*

Remark.

> The intuitive meaning of spanning is that if I want to go from point A to point B in the vector land, I can go along the set of vectors to go there.

0.1.1 Spanning Set

Using determinant to show a set span of a particular vector space

Show a set span of a particular vector space:

Example.

Let

$$S = \{(1,3),(2,-1)\}, S \subset \mathbb{R}^2$$

We want to know is $\mathbb{R}^2 = \text{span}(S)$? Are there scalars C_1, C_2 such that any vector in $\mathbb{R}^2$ $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written as a linear combination of $(1,3)$ and $(2,-1)$?

If A $= \begin{bmatrix} \vec{c_1} & \vec{c_2} & \cdots & \vec{c_n} \end{bmatrix}$ If x $= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

Then $A\vec{x} = \lambda_1\vec{c_1} + \lambda_2\vec{c_2} + \cdots + \lambda_n\vec{c_n}$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \vec{\lambda_1} \\ \vec{\lambda_2} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solve:

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 3 & -1 & b \end{array}\right]$$

Is there a solution no matter a, b is? What is the determinant of the matrix?

$$\det \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = -1 - 6 = -7 \neq 0$$

A solution exists for any a, b.
Therefore, no matter what a, b I choose in $\mathbb{R}^2$, I can always find c_1, c_2 so that a, b is a linear combination of (1,3) and (2,-1).
Therefore, $\mathbb{R}^2 = \text{span}(S)$.

The idea is to show the determinant of the matrix composed by the given vector is not zero. This means the rank of the matrix equals n(Theorem 1.2.2). Therefore, they are able to span the vector space.

Remark.

> The vector of the spanning set became the **column** vector of the matrix.

Example.

> Show the set $S = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a spanning set for $\mathbb{R}^3$.
> Step 1: Write the vectors in the matrix form.
>
> $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
>
> Step 2: Find the determinant of the matrix.
>
> $$\det A = 1 \neq 0$$
>
> Step 3: Therefore, the set S is a spanning set for $\mathbb{R}^3$.
> Which means for any possible $\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ there's a unique solution using the linear combination of the given vectors.

Using the augmented matrix to find the linear combination of the given vectors

Now we know how to use the given vector to prove if they are a spanning set for the vector space. We want to know the specific weight constant that brings us to a specific vector in the vector space.

The idea is to stick to the equation $Ax = b$ and find the solution. For example, we want to find the linear combination of the given vectors that gives us a specific vector $\vec{v}$:

$$\vec{v} = \lambda_1 \vec{c_1} + \lambda_2 \vec{c_2} + \cdots + \lambda_n \vec{c_n}$$

Where lambda is the weight constant we want to find and c is the given vectors. We can write this in the matrix form:

$$\begin{bmatrix} \vec{c_1} & \vec{c_2} & \cdots & \vec{c_n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \vec{v}$$

We can use the augmented matrix to find $\vec{x}$ (which is the matrix with λ here).

$$\left[\begin{array}{cccc|c} \vec{c_1} & \vec{c_2} & \cdots & \vec{c_n} & \vec{v} \end{array} \right]$$

When we change the matrix into the row echelon form, we are actually solving the equation. The solution is the weight constant we want to find. (I don't know how exactly the question holds during the transformation, but the matrix after transformation means the weight applies to each original vector.)

Example.

> Express $\vec{v} = (1, 3, 7)$ as a linear combination of $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$.
> Step 1: Write the vectors in the matrix form.
> The equation here is $Ax = b$.
> A is the matrix of the given vectors.
>
> $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
>
> x is the vector of the weights we want to find. $x = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$
>
> b is the vector we want to express. $b = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$
>
> So the equation becomes:
>
> $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$
>
> To solve it, we can write it in the augmented matrix form.
>
> $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$
>
> RREF:
>
> $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 7 \end{array}\right]$$
>
> This means the first vector times -2, the second vector times -10, and the third vector times 7 will give us the vector we want to express.
> Therefore, $\boxed{\vec{v} = 4(1, 0, 0) - 10(1, 1, 0) + 7(1, 1, 1)}$.
>
> Let's check if the solution is correct.
>
> $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$
>
> Which is indeed correct.

To prove it span a set, we can also use the same way but make vector b to an arbitrary vector $\begin{bmatrix} a \\ b \\ \vdots \\ n \end{bmatrix}$

We can prove the following example does not span $\mathbb{R}^3$ by either showing the determinant of the matrix is zero or showing the augmented matrix is inconsistent when a, b, c are arbitrary. They mean the same.

Example.

> Determine whether the vectors $S = \{v_1 = (1, 1, 4), v_2 = (2, 1, 3), v_3 = (4, 3, 5)\}$ span $\mathbb{R}^3$.
> Step 1: Write the vectors in the matrix form.
> The equation here is $Ax = b$.
> A is the matrix of the given vectors.
>
> $$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 4 & 3 & 5 \end{bmatrix}$$
>
> x is the vector of the weights we want to find. $x = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$
>
> b is the vector we want to express. $b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
>
> The equation can be written as:
>
> $$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
>
> or
>
> $$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 1 & 1 & 3 & b \\ 4 & 3 & 5 & c \end{array}\right]$$
>
> REF:
>
> $$\left[\begin{array}{ccc|c} 1 & 2 & 4 & a \\ 0 & 1 & -1 & -(a+b) \\ 0 & 0 & 0 & 7a + 11b + c \end{array}\right]$$
>
> Which is inconsistent when $7a + 11b + c \neq 0$.
> Therefore, the given vectors are unable to span $\mathbb{R}^3$ since not every arbitrary vector can be expressed as a linear combination of the given vectors. (The vector must meet the requirement that $7a + 11b + c = 0$).

Remark.

> We can't say that the given vectors only span $\mathbb{R}^3$ if $7a+11b+c=0$. The given vectors above span $S=\{(a,b,c)\mid a,b,c\in\mathbb{R}, 7a+11b+c=0\}$. However, the definition of $\mathbb{R}^3$ is $\mathbb{R}^3=\{(a,b,c)\mid a,b,c\in\mathbb{R}\}$. $S\neq\mathbb{R}^3$. Therefore, the given vectors are unable to span $\mathbb{R}^3$ in any case.

Spanning set for the null space

Remark.

> Null space is defined as the set of solutions to the homogeneous equation $A\vec{x}=\vec{0}$.

Therefore, what we need to do is to change the vector $\vec{b}$ in the general form to the zero vector.

Example.

> $A=\begin{bmatrix}1 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. Find two vectors that span the null space of A.
>
> $$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$
>
> This means $1\lambda_1-2\lambda_2=0$.
>
> $$\lambda_1=2\lambda_2$$
>
> $$\lambda_2=\lambda_2$$
>
> $$\lambda_3=\lambda_3$$
>
> Let $\lambda_2=s, \lambda_3=t$.
>
> $$\text{null}(A)=\{(2s,s,t)|s,t\in\mathbb{R}\}$$
>
> This can be also written as:
>
> $$\text{null}(A)=\{s(2,1,0)+t(0,0,1)|s,t\in\mathbb{R}\}$$
>
> Therefore, the two vectors that span the null space of A are $(2,1,0)$ and $(0,0,1)$.

Spanning set for matrices

Example.

Find a spanning set for the subspace of symmetric matrices of $M_{2\times 2}(\mathbb{R})$.
The symmetric matrix is the matrix that is equal to its transpose.
Step 1: Write the generic symmetric matrix in the matrix form.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Step 2: Change each element in the matrix to 1 (others 0). The set of matrices is the spanning set for the subspace of symmetric matrices.

$$\boxed{S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}$$

Remark.

This is also the way we find the basis for matrices.

Spanning set for polynomials

Example.

Find a spaning set for P_2, the polynomial of degree at most 2.
Step 1: Write the generic P_2:

$$P_2 = ax^2 + bx + c$$

Step 2: Change each element in the polynomial to 1 (others 0). The set of polynomials is the spanning set for P_2.

$$\boxed{S = \{1, x, x^2\}}$$

Now we know how to:

- Show a set span of a particular vector space.
- Find the linear combination of the given spanning set.
- Find the spanning set for the null space.
- Find the spanning set for matrices.
- Find the spanning set for polynomials.

0.1.2 Linear Independence

Let $S = \{(1,0),(0,1),(1,3)\} \subset \mathbb{R}^2$. Find span(S)

$$\boxed{\text{span}(S) = \{(1,0),(0,1)\}}$$

Did we need the (1, 3)? It turns out that we would like a minimal spanning set for V . So we formalize the notion of an 'extra' or 'redundant' vector.

Definition 0.1.1: Linear Independence

Let V be a vector space and let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a set of vectors in V. The set S is **linearly independent** if the only solution to the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$.
If there exists a solution where at least one of the c is not zero, then the set is **linearly dependent**.

Example.

> The set $S = \{(1,0),(0,1),(1,3)\} \subset \mathbb{R}^2$ is linearly dependent because:
>
> $$1 \cdot (1,0) + 3 \cdot (0,1) - 1 \cdot (1,3) = (0,0)$$
>
> and the coefficients $1, 3, -1$ are not all zero.

Fact 0.1.2

An equivalent definition would be that V is linearly independent if and only if no vector in S can be expressed as a linear combination of the others. That is, no vector is a member of the span of the other vectors.

Example.

> Dependent: $\{(1,0),(2,0)\}$.
> $2 \cdot (1,0) = (2,0)$, so $(2,0)$ is in the span of $\{(1,0)\}$.
> Independent: $\{(1,0),(2,1)\}$.
> If $(2,1)$ is in the span of $\{(1,0)\}$, it must be a scalar multiple of $(1,0)$, which it is not.

Example.

Let $S = \{(1,0,0,0),(1,2,0,0),(1,2,3,0),(1,2,3,4)\} \subset \mathbb{R}^4$. Determine whether or not S is linearly dependent. Then determine whether or not S is a spanning set for $\mathbb{R}^4$.
Step 1: Write the vectors in the matrix form.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Step 2: Find the RREF

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Find the determinant of the matrix.

$$\det A = 1 \neq 0$$

Therefore, the set S is linearly independent.

Remark.

- The determinant will only work in square matrices.
- A linearly independent spanning set is called a **basis**.

Technique: If S is a subset of $\mathbb{R}^n$, then S is linearly independent if and only if the matrix A whose column vectors are the elements of S has a unique solution for $A\vec{x} = \vec{0}$. If this is true and also S has n vectors, then it spans $\mathbb{R}^n$.

0.2 Lecture 15: Linear Independence & Basis and dimension

0.3 Lecture 16: Basis, Row and Columns Spaces

0.4 Lecture 17: The Rank-Nullity Theorem