MATH 165
Linear Algebra & Diff. Equation
Midterm II
Review Note with Examples

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Contents

Chapter 1

Determinants

1.1 Lecture 9: Def. of Determinants & it's calculation

This lecture covers:

- *3.1 The Definition of the Determinant*
- *3.3 Cofactor Expansions (partly)*

1.1.1 What is Determinanrts

Definition 1.1.1: Determinants

The **determinants** of a square matrix A, denoted $\det(A)$, is a number associated with the matrix A that is *designed* to carry information about the invertibility (among other things) of the matrix A. We also use the notation $|A|$ to denote the determinant of A.

The way we calculate determinants is derived from the fact of changing the matrix into RREF (Reduce-Row-Echelon-Form) and seeing if the matrix is invertible. If the matrix is invertible, then the determinant is not zero. If the matrix is not invertible, then the determinant is zero. The way we calculate it is based on observation. (P.196)

1.1.2 How to calculate Determinants

1 by 1 matrix The determinant of a 1×1 matrix $[a11]$ is a_{11}.

Example.

> **Calculate the determinant of the matrix $A = [3]$.**
>
> $$\begin{vmatrix} 3 \end{vmatrix} = 3$$
>
> **What is the rank of the matrix A?**
> The rank of the matrix A is 1.

Example.

Calculate the determinant of the matrix $A = [0]$.

$$\begin{vmatrix} 0 \end{vmatrix} = 0$$

What is the rank of the matrix A?
The rank of the matrix A is 0. (Since the determinant is 0, the matrix is not invertible.)

2 by 2 matrix The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ad = bc$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The way we calculate it is by taking the product of the diagonal elements and subtracting the product of the off-diagonal elements.

Example.

Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

What is the rank of the matrix A?
The rank of the matrix A is 2.
Is the matrix invertible?
Yes, the matrix is invertible, since $-2 \neq 0$.

Example.

Calculate the determinant of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \cdot 6 - 3 \cdot 4 = 12 - 12 = 0$$

What is the rank of the matrix A?
The rank of the matrix A is 1.
Is the matrix invertible?
No, the matrix is not invertible, since $0 = 0$.

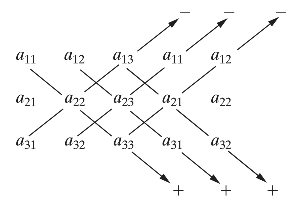

Figure 1.1: Determinant of a 3×3 matrix

3 by 3 matrix The determinant of a 3×3 matrixhas a similar 'diagonals'-type definition:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

We can use a clever trick with arrows by repeating the first two columns to calculate the determinant of a 3×3 matrix.

Example.

> **Calculate the determinant of the matrix** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$**.**
>
> $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 2 \cdot 4 \cdot 9 - 1 \cdot 6 \cdot 8 = 0$$
>
> **What is the rank of the matrix** A**?**
> The rank of the matrix A is 2.
> **Is the matrix invertible?**
> No, the matrix is not invertible, since $0 = 0$.

Remark.

> If the dimension of a matrix is greater than 3×3, we won't be able to find the determinant in one step, as the sub-matrices will have dimension at least 3×3.

Larger matrix Another more common way to find the determinant of a 3×3 matrix is to use the **cofactor expansion** method. The cofactor expansion method is a way to calculate the determinant of a matrix by breaking it down into smaller matrices. This can also apply to **larger matrices**.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Theorem 1.1.2: Cofactor Expansion

We may expandalong row i:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

We may expand along column j:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

The way you do so is to choose a row or a column (typically, a row or column with the most zeros) and expand the determinant along that row or column. If the matrix after expansion is still not a 2×2 matrix, you can expand it again.

Example.

Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$**.**

Let's choose the first row to expand the determinant.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 1\begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} - 2\begin{vmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{vmatrix} + 3\begin{vmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{vmatrix} - 4\begin{vmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{vmatrix}$$

$$= 1 \times 6\begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} - 2 \times 7\begin{vmatrix} 9 & 12 \\ 13 & 16 \end{vmatrix} + 3 \times 8\begin{vmatrix} 9 & 11 \\ 13 & 15 \end{vmatrix} - 4 \times 5\begin{vmatrix} 10 & 11 \\ 14 & 15 \end{vmatrix} ... = 0$$

What is the rank of the matrix A**?**
The rank of the matrix A is 2.

Is the matrix invertible?
No, the matrix is not invertible, since $0 = 0$.

1.1.3 Inverse of 2 by 2 matrix

Proposition 1.1.3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. If $\det(A) \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. If $\det(A) \neq 0$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

$$A \times A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{1}{ad-bc}(a \times d + b \times (-c)) = \frac{ad-bc}{ad-bc} = 1$$

$$\frac{1}{ad-bc}(a \times (-b) + b \times a) = \frac{-ab+ab}{ad-bc} = 0$$

$$\frac{1}{ad-bc}(c \times d + d \times (-c)) = \frac{cd-cd}{ad-bc} = 0$$

$$\frac{1}{ad-bc}(c \times (-b) + d \times a) = \frac{-bc+ad}{ad-bc} = 1$$

$$A \times A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus, $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$\square$

Example.

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$**.**

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$$

Since $\det(A) = -2 \neq 0$, we can find the inverse of A.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

1.1.4 Determinat of matrix functions

Given a matrix function $M(t)$, its determinant $\det(M(t))$ can be found using exactly the same techniques as that of a numerical matrix. The only difference is that the determinant is a function of t.

Example.

Find

$$\begin{vmatrix} \cos(t) & -\sin(t) \\ \cos(t) & \sin(t) \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} \cos(t) & -\sin(t) \\ \cos(t) & \sin(t) \end{vmatrix} &= \cos(t) \cdot \sin(t) - (-\sin(t) \cdot \cos(t)) \\ &= \sin(t) \cdot \cos(t) + \sin(t) \cdot \cos(t) \\ &= 2\sin(t) \cdot \cos(t) \\ &\boxed{= \sin(2t)} \end{aligned}$$

1.1.5 Geometry Application: areas and volumes

Areas Suppose O is the origin in the xy-plane. Let P be a parallelogram with vertices $O = (0,0)$,$A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$.

Fact 1.1.4

The area of the parallelogram P is given by

$$\text{Area}(P) = \left| \det \left(\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right) \right|$$

Since the area is always positive, we take the absolute value of the determinant.

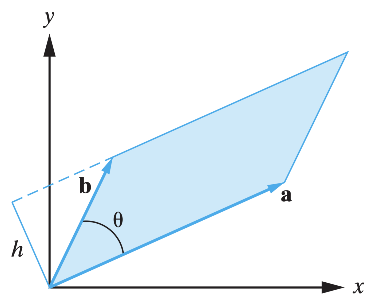

Figure 1.2: Area of a parallelogram

Volumes Similarly, the volume of a parallelepiped in $\mathbb{R}^3$ is given by the absolute value of the determinant of the matrix whose rows are the vectors representing the edges of the parallelepiped.

Fact 1.1.5

The volume of the parallelepiped determined by sides OA, OB, and OC where

$$A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3)$$

is given by

$$\text{Volume} = \left| \det \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right) \right|$$

Proof. The area of the parallelogram is Area = (length of base) × (perpendicular height). This can be written as

$$\text{Area } = ||a||h = ||a||||b|| \sin(\theta)||a \times b||$$

Since the k components of a and b are both zero (since the vectors lie in thexy-plane), substitution from Equation yields

$$\text{Area} = ||(a_1b_2 - a_2b_1)k|| = |a_1b_2 - a_2b_1| = |\det(A)|.$$

□

Example.

> **Find the area of the triangle with vertices** $(1,1),(2,3),(6,9)$**.**
>
> $$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \cdot 3 - 1 \cdot 2 = 3 - 2 = \boxed{1}$$
>
> Note: Since the vector $(6,9)$ is a multiple of the vector $(2,3)$ (linearly dependent), we can ignore the third vertex $(6,9)$.

1.2 Lecture 10: Rank, Invertibility, Elementary Row Operations & additional properties of determinants

This lecture covers:

- *3.2 Properties of Determinants*

1.2.1 Determinants, Rank, Invertibility

Theorem 1.2.1

Let A be an upper (or lower) triangular square matrix. Then $\det(A)$ is the product of the diagonal entries of A.

Proof. Let A be an upper triangular matrix. Then A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdots a_{nn}$$

$\square$

Since the possible effect on the determinant of putting a matrix into row-echelon form can only be multiplied by a non-zero constant, for an $n \times n$ matrix A, $\det(A) = 0$ if and only if the product of the diagonal entries of a row-echelon form of A is zero. This happens if and only if the rank of A is less than n.

Example.

Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$**.**

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24$$

What is the rank of the matrix A**?**
The rank of the matrix A is 3.
Is the matrix invertible?
Yes, the matrix is invertible, since $24 \neq 0$.

The following theorem is based on the previous theorem.

Remark.

> **Rank** is the number of linearly independent rows (or columns) of a matrix, which is also equal to the number of leading 1's in the RREF of the matrix.

Theorem 1.2.2

Let A be an $n \times n$ matrix. Then $\det(A) \neq 0$ if and only if $\operatorname{rank}(A) = n$.

This means that whether or not a square matrix A has a non-zero determinant tells us whether or not A is invertible and whether or not $Ax = 0$ has a unique solution.

Example.

> **Let** $A = \begin{bmatrix} 5 & 9 & 6 \\ 0 & 3 & 7 \\ 0 & 0 & 8 \end{bmatrix}$. **Find the rank of** A.
>
> $$\det(A) = 5 \ldots 3 \ldots 8 = 120 \neq 0$$
>
> Since $\det(A) \neq 0$, A is a 3×3 matrix,the rank of A is 3.

Since it's easy to find the determinant of row-echelon matrices, and also as a step to proving Thm. 3.2.5, we are interested in the effect of EROs on determinants.

1.2.2 Elementary Row Operations

Theorem 1.2.3

Multiplying a row of a matrix by a scalar k multiplies the determinant of the matrix by k.

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ kd & ke & kf \\ g & h & i \end{bmatrix}$.

Then

$$\mid A \mid = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$\mid B \mid = -kd \begin{vmatrix} b & c \\ h & i \end{vmatrix} + ke \begin{vmatrix} a & c \\ g & i \end{vmatrix} - kf \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

Therefore, $\mid B \mid = k \mid A \mid$.

Example.

Find $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & -1 \end{vmatrix}$ by "factoring" a 2 from the second row.

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

$$= 2 \times \left(2 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \right)$$

$$= 2 \times (2 \cdot (-2 - (-1)) + 1 \cdot (-1 - 0)) = 2 \times (-2) = \boxed{-4}$$

Note: Since we are multiplying the entire matrix by the scalar, we are actually raising each row by the scalar. Therefore, the determinant of the matrix should be raised by the scalar to the power of the number of rows.

Corollary 1.2.4

If A is a square matrix, then $\det(kA) = k^n \det(A)$.

Example.

Suppose A is a 4×4 metaix and $\det(A) = 3$. What is the determinant of $2A$?

$$\det(2A) = 2^4 \det(A) = 16 \cdot 3 = \boxed{48}$$

Theorem 1.2.5

Exchanging two rows of a matrix changes the sign of the determinant of the matrix.

Example.

Find $\begin{vmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix}$

$$\begin{vmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix} = - \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \boxed{-abc}$$

Theorem 1.2.6

Adding a multiple of one row to another row **does not change the determinant** of the matrix.

Example.

Find $\begin{vmatrix} a+2c & b+2d \\ c & d \end{vmatrix}$

$$\begin{vmatrix} a+2c & b+2d \\ c & d \end{vmatrix} \xrightarrow{R_1-2R_2} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \boxed{ad-bc}$$

1.2.3 Additional Properties of Determinants

Proposition 1.2.7

If $C = \begin{bmatrix} r_1 \\ \vdots \\ r_{i-1} \\ a+b \\ r_{i+1} \\ \vdots \\ r_n \end{bmatrix}$, and $A = \begin{bmatrix} r_1 \\ \vdots \\ r_{i-1} \\ a \\ r_{i+1} \\ \vdots \\ r_n \end{bmatrix}$, and $B = \begin{bmatrix} r_1 \\ \vdots \\ r_{i-1} \\ b \\ r_{i+1} \\ \vdots \\ r_n \end{bmatrix}$, then $\det(C) = \det(A) + \det(B)$.

Remark.

This property does **NOT** mean $\det(A+B) = \det(A) + \det(B)$

Example.

Let $A = \begin{bmatrix} 2 & 4 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix}$. Find the determinant of these three matrices and relate them to the property.

$$\det(A) = 2 \cdot 0 - 4 \cdot (-1) = \boxed{4}$$

$$\det(B) = 1 \cdot 0 - (-1) \cdot (-1) = \boxed{-1}$$

$$\det(C) = 3 \cdot 0 - 3 \cdot (-1) = \boxed{3}$$

Notice that the second column of these three matrix are the same. Therefore, we can use the property to find the determinant of C.

$$\det(C) = \det(A) + \det(B) = 4 + -1 = \boxed{3}$$

Proposition 1.2.8

If A is a square matrix, then $\det(A^T) = \det(A)$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then $\det(A) = ad - bc$ and $\det(A^T) = ad - bc$. Therefore, $\det(A) = \det(A^T)$. $\square$

Proposition 1.2.9

The effects on the determinant of elementary row operations still hold when the word row is replaced by the word column.

Proposition 1.2.10

Since the rank of a square $n \times n$ matrix is n if and only if the determinant is non-zero,

- If a matrix has a row (or column) of zeros, then it has determinant zero.
- If a matrix has two identical rows (or columns), then it has determinant zero.
- If a row (or column) of a matrix is a multiple of another, then it has determinant zero.

Proposition 1.2.11

Determinet is multiplicative. That is, if A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

To get an idea of why this might be true, suppose A and B are upper-triangular matrices. Then AB is also upper-triangular.

Example.

Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ **and** $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$**. Find** $\det(AB)$ **and** $\det(A)\det(B)$**.**

$$\det(AB) = \det \begin{pmatrix} a \cdot d + b \cdot 0 & a \cdot e + b \cdot f \\ 0 \cdot d + c \cdot 0 & 0 \cdot e + c \cdot f \end{pmatrix} = \det \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix} = ad \cdot cf = \boxed{acdf}$$

Why can we extend this to non-upper-triangular matrices?

If the determinant is multiplicative, what can we say about $\det(A^{-1})$**?**
Let $\det(A) = k$.

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

$$\det(A^{-1}) = \frac{1}{k}$$

Chapter 2

Vector Spaces

2.1 Lecture 11: Vector Spaces, Zero-Vectors, Dimentions, basis, & Linear Combinations

2.2 Lecture 12: Vector Subspace & Proof

2.3 Lecture 13: Spanning Set

2.4 Lecture 14: Spanning, Linear Independence & Basis

2.5 Lecture 15: Linear Independence & Basis and dimension

2.6 Lecture 16: Basis, Row and Columns Spaces