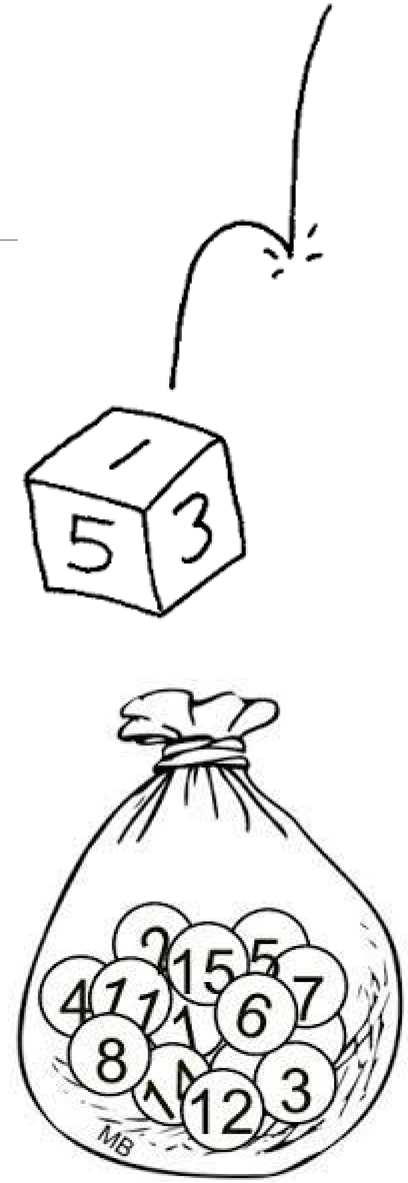
Lecture 09

- Independent Random Variables
- PDF of Functions of R.V.s



Independent discrete R.V.s

Recall the definition of independent events E and F: $P(EF) = P(E)P(F)$

Two discrete random variables X and Y are independent if:

for all x, y:

$$F(x,y) = F_X(x) \cdot F_Y(y)$$

$$P\left(X = x_i, Y = y_j\right) = P(X = x_i) \cdot P\left(Y = y_j\right)$$

Different notation, same meaning:

$$p_{X,Y}\left(x_i, y_j\right) = p_X(x_i) \cdot p_Y(y_j)$$

$$p_{ij} = p_{i\cdot} \cdot p_{\cdot j}$$

- **Intuition**: knowing value of X tells us nothing about the distribution of Y (and vice versa).
- If two variables are not independent, they are dependent.

2-Dimensional

$F(x,y)$ $\quad\rightleftarrows\quad$ $f(x,y)$

$$f(x,y)=\frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$F(x,y)=\int_{-\infty}^{y}\left[\int_{-\infty}^{x} f(u,v)du\right]dv$$

$F(x,y)$
$= F_X(x)\cdot F_Y(y)$
(X,Y are independent)

Let $y \to +\infty$,
$F(x,+\infty)=F_X(x)$
Let $x \to +\infty$,
$F(+\infty,y)=F_Y(y)$

$f(x,y)$
$= f_X(x)\cdot f_Y(y)$
(X,Y are independent)

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)dy$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)dx$$

$F_X(x), F_Y(y)$ $\quad\rightleftarrows\quad$ $f_X(x), f_Y(y)$

$$f(x)=\frac{dF(x)}{dx}$$

$$F(x)=\int_{-\infty}^{x} f(t)\,dt$$

1-Dimensional

Independent continuous R.V.s

Two continuous random variables X and Y are independent if:

for all x, y:

$$F(x,y) = F_X(x) \cdot F_Y(y)$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

- Independent: (独立) $f(x,y) = f_X(x) \cdot f_Y(y)$ for all x, y
- Uncorrelated: (不相关) $E(XY) = E(X)E(Y)$
- Orthogonal: (正交) $E(XY) = 0$

to be covered in a few weeks.

Simple dice rolling example

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

Let: D_1 and D_2 be the outcomes of two rolls

$S = D_1 + D_2$ be the sum of two rolls

- Each roll of a 6-sided die is an independent trial.
- Random variables D_1 and D_2 are independent.

1. Are events $(D_1 = 1)$ and $S = 7$ independent? ✅

$$P(D_1 = 1) \cdot P(S = 7)$$
$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = p(D_1 = 1, S = 7)$$

2. Are events $(D_1 = 1)$ and $S = 5$ independent? ❌

$$P(D_1 = 1) \cdot P(S = 5)$$
$$= \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54} \neq p(D_1 = 1, S = 5)$$

3. Are random variables D_1 and S independent?

All events $X = x, Y = y$ must be independent!

Quick check

Ex. Assume X and Y are independent, fill the following table.

$X \backslash Y$	y_1	y_2	y_3	$p_{i\cdot}$
x_1		1/8		
x_2	1/8			
$p_{\cdot j}$	1/6			1

Quick check

Ex. Assume X and Y are independent, fill the following table.

$X \backslash Y$	y_1	y_2	y_3	$p_{i\cdot}$
x_1	1/24	1/8	1/12	1/4
x_2	1/8	3/8	1/4	3/4
$p_{\cdot j}$	1/6	1/2	1/3	1

(Ex. 2c P249) A man and a woman decide to meet at a train station. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 p.m., find the probability that the first to arrive has to wait longer than 10 minutes.

Sol. X and Y denote the time past 12 noon that the man and the woman arrive. $X, Y \sim U(0,60)$.

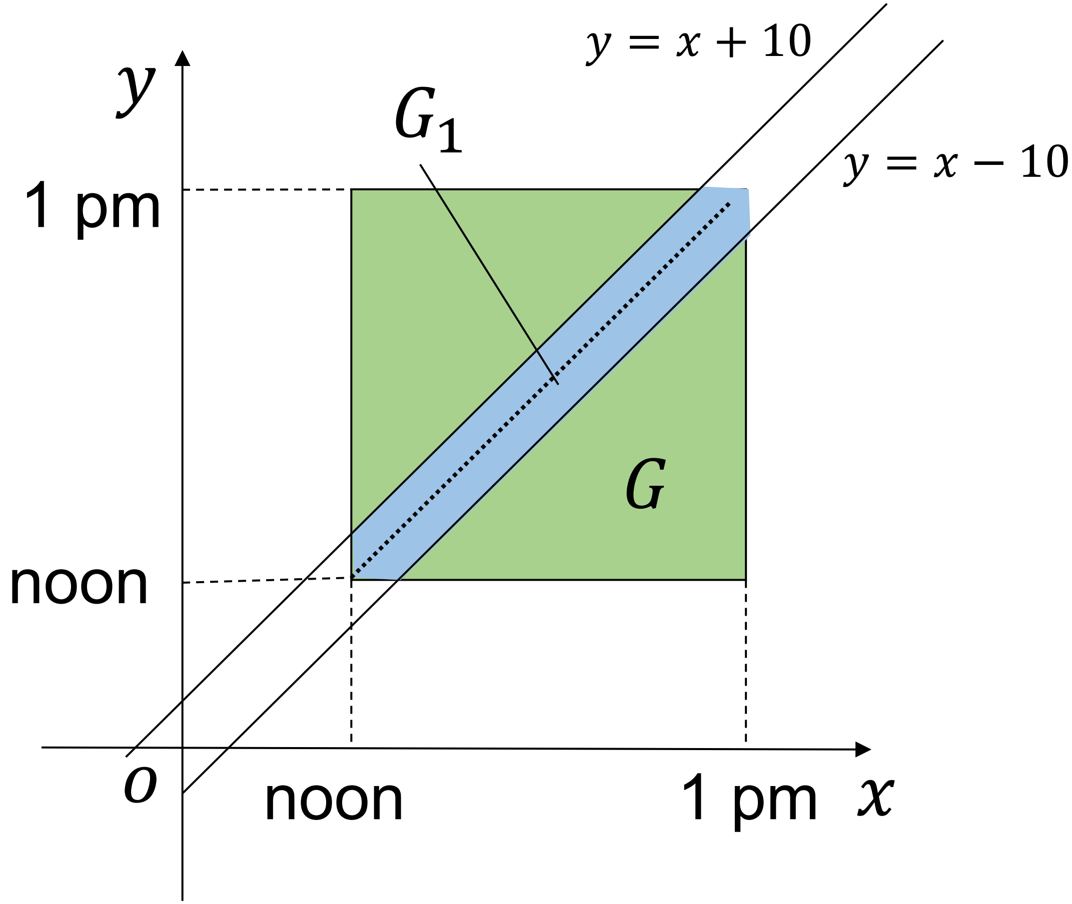
(Ex. 2c P249) A man and a woman decide to meet at a train station. If each of them independently arrives at a time uniformly distributed between 12 noon and 1 p.m., find the probability that the first to arrive has to wait longer than 10 minutes.

Sol. X and Y denote the time past 12 noon that the man and the woman arrive. $X, Y \sim U(0,60)$.

$$P\{X+10<Y\}+P\{Y+10<X\} \xRightarrow{\text{symmetric}} 2\cdot P\{X+10<Y\}$$

$$= 2\iint\limits_{x+10<y} f(x,y)dxdy \xRightarrow{\text{independent}} 2\iint\limits_{x+10<y} f_X(x)\, f_Y(y)\, dxdy$$

$$= 2\int_{10}^{60}\int_{0}^{y-10}\left(\frac{1}{60}\right)^2 dx\, dy = \frac{2}{60^2}\int_{10}^{60}(y-10)dy = \frac{25}{36}$$



$$\text{Area of } G = 2\left[\frac{1}{2} \times \left(1 - \frac{1}{6}\right)^2\right] = \frac{25}{36}$$

Ex. Given the joint PDF of R.V.s X and Y

1) $f(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$

2) $f(x,y) = \begin{cases} 24xy & 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0 & \text{otherwise} \end{cases}$

Evaluate the independency of X and Y. Hint: $\int e^{ax}dx = \frac{1}{a}e^{ax}$

Ex. Given the joint PDF of R.V.s X and Y

1) $f(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$

Evaluate the independency of X and Y. Hint: $\int e^{ax}dx = \frac{1}{a}e^{ax}$

Sol. Yes, since $f(x,y) = f_X(x) \cdot f_Y(y)$.

$f_X(x) = \int_0^\infty f(x,y)dy$
$= 6e^{-2x}\int_0^\infty e^{-3y}dy = 2 \cdot e^{-2x}, 0 < x < \infty,$

$f_Y(y) = \int_0^\infty f(x,y)\,dx$
$= 6e^{-3y}\int_0^\infty e^{-2x}dx = 3 \cdot e^{-3y}, 0 < y < \infty.$

Ex. Given the joint PDF of R.V.s X and Y

2) $f(x,y) = \begin{cases} 24xy & 0 < x < 1, 0 < y < 1, 0 < x + y < 1 \\ 0 & \text{otherwise} \end{cases}$

Evaluate the independency of X and Y.

Sol. No, the defined ranges of x and y are not separable.

The values of x and y are dependent.

For example, if $x = 0.9999$, $P(y > 0.0001) = 0$

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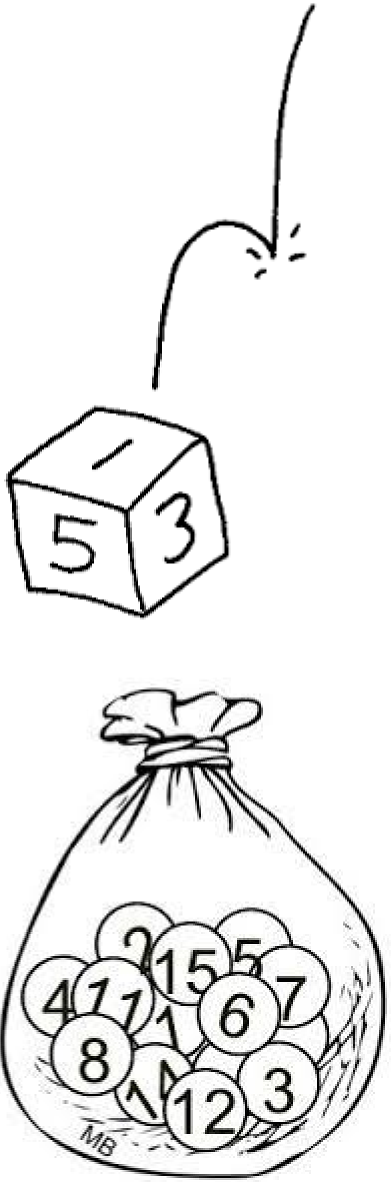
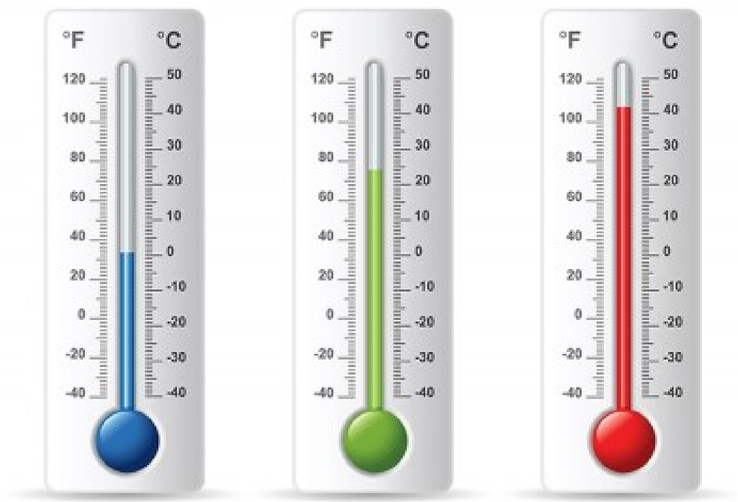


Review: PDF/PMF of a function of a R.V.

Known: PDF/PMF of X; Want: PDF/PMF of $Y = g(x)$.

How do we measure temperature?



Cold Comfort Hot

Height of the mercury bar (x)
⇒Temperature (y)

$$Y = g(X) = a \cdot X + b$$

Known: PDF of X; Want: PDF of Y.

What if we consider the relationship between multiple R.V.s

⇒ PDF/PMF of Functions of R.V.s

$$Z = g(X, Y)$$

PMF of functions of discrete R.V.s

Ex. Given the PMF of (X,Y)

$X \backslash Y$	0	1	2
-1	0.2	0.3	0.1
2	0.1	0.1	0.2

Find the PMF of (1) $Z = X + Y$, (2) $Z = XY$.

Sol.

(X,Y)	(-1,0)	(-1,1)	(-1,2)	(2,0)	(2,1)	(2,2)
$Z = X + Y$	-1	0	1	2	3	4
$Z = XY$	0	-1	-2	0	2	4
$p(z)$	0.2	0.3	0.1	0.1	0.1	0.2

$Z = XY$	-2	-1	0	2	4
$p(z)$	0.1	0.3	0.3	0.1	0.2

PDF of functions of continuous R.V.s

Assume the PDF of (X, Y) is $f(x, y)$, find the PDF of $Z = g(X, Y)$.

$$
\begin{aligned}
F_Z(z) &= P\{Z \leq z\} \\
&= P\{g(X, Y) \leq z\} \\
&= P\{(X, Y) \in D_Z\} \\
&= \iint_{D_z} f(x, y) dx dy
\end{aligned}
$$

$$f_Z(z) = F_Z'(z)$$

Check P224 or slide06 for the single variable case $Y = g(X)$ (monotonically increasing):

$$F_Y(y) = P\{g(x) \leq y\}$$

$$= P\{X \leq g^{-1}(y)\} = F_X\left(g^{-1}(y)\right)$$

$$F_Y(y) = P\{Y \leq y\} = P\{g(x) \leq y\}$$

$$= P\{X \in D_y\} = \int_{D_y} f(x)\, dx$$

PDF of functions of continuous R.V.s

PDF of $Z = X + Y$

$$F_Z(z) = \iint\limits_{D_Z} f(x,y)dxdy = \iint\limits_{x+y\leq z} f(x,y)dxdy$$

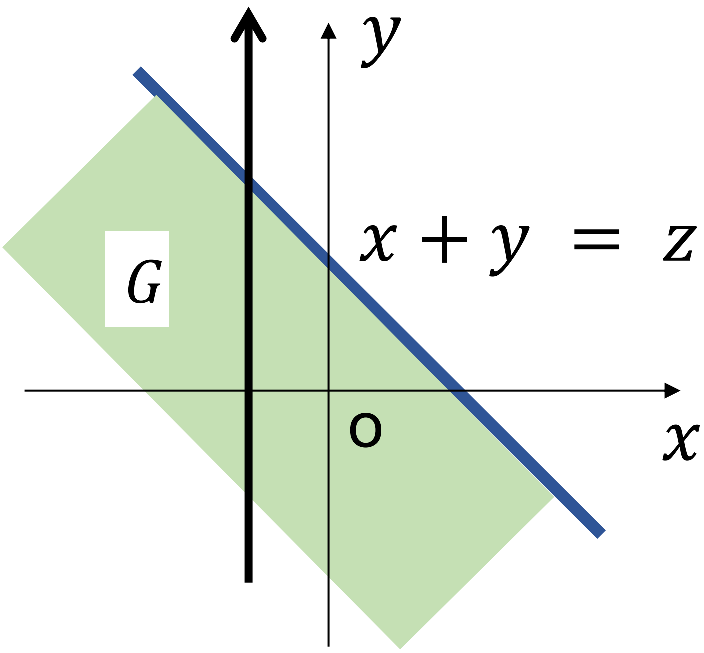
$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z-x} f(x,y)dy$$

$y = u - x$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z} f(x,u-x)du$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{+\infty} f(x,u-x)dx\,du$$

$$\boxed{f_Z(z) = \int_{-\infty}^{+\infty} f(x,z-x)dx}$$



Note: The derivation process is for your reference only.

PDF of functions of continuous R.V.s

PDF of $Z = X + Y$ is given as

$$f_Z(z) = \int_{-\infty}^{+\infty} f\,(x, z - x)dx$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f\,(z - y, y)dy$$

If X and Y are independent

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z - y)\, f_Y(y)dy$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z - x)dx$$

Ex. Let $X \sim \mathcal{N}(0,1), Y \sim \mathcal{N}(0,1)$. X and Y are independent, find the PDF of $Z = X + Y$.

Sol. $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, (-\infty < x, y < +\infty)$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}} dx \quad = \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{+\infty} e^{-\left(x-\frac{z}{2}\right)^2} dx$$

$t = x - \frac{z}{2}$

$$f_Z(z) = \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{+\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{z^2}{4}} (-\infty < z < +\infty)$$

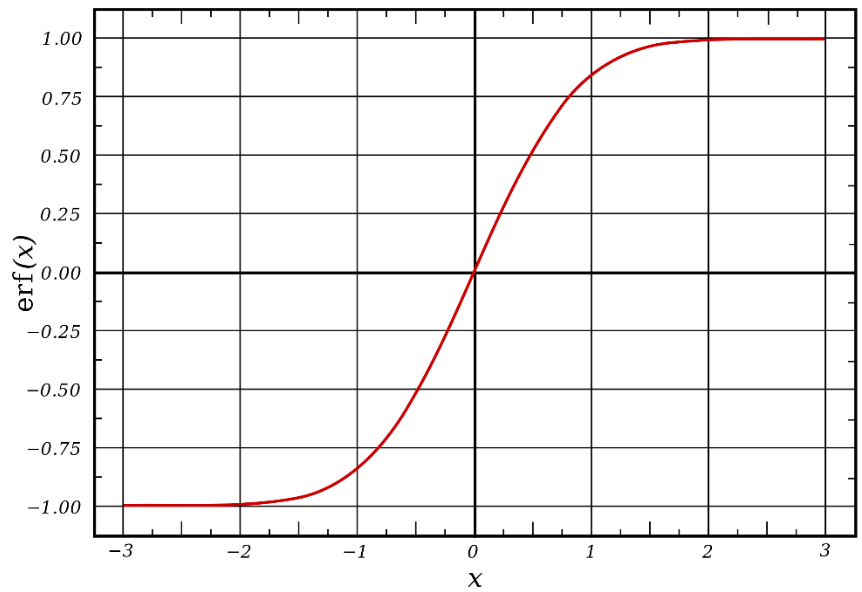
$$Z = X + Y \sim \mathcal{N}(0,2)$$

About error function, erf

$$\int e^{-ax^2}dx = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}(\sqrt{a}x)$$

$$\Longrightarrow$$

$$\int e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$$



In mathematics, the **error function** (also called the **Gauss error function**), often denoted by **erf**, is a complex function of a complex variable defined as:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

----[Wikipedia]

- If $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and X, Y are independent, then $X + Y$ follows normal distribution

$$X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), (i = 1,2 \cdots, n)$ and X_i are independent, then

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i{}^2\right)$$

- The linear combination of finite number of independent normal variables follows normal distribution

$$\sum_{i=1}^{n} c_i X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} c_i \mu_i , \sum_{i=1}^{n} c_i^2 \sigma_i{}^2\right)$$

Review: Linear transformation of Normal R.V.s

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$. Linear transformations of X are also Normal.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

$$F_Y(y) = P\{Y \leq y\}$$
$$= P\{aX + b \leq y\}$$
$$= P\left\{X \leq \frac{y-b}{a}\right\}$$
$$= F_X\left(\frac{y-b}{a}\right)$$

$\xrightarrow{\text{Differentiation}}$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$
$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\}$$
$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2(a\sigma)^2}\right\}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

PDF/CDF of functions of continuous R.V.s

$\max\{X, Y\}$ and $\min\{X, Y\}$, where X, Y are independent.

$$F_{\max}(z) = P\{\max\{X, Y\} \le z\} = P\{X \le z, Y \le z\}$$
$$= P\{X \le z\} \cdot P\{Y \le z\}$$

$$F_{\max}(z) = F_X(z) \cdot F_Y(z)$$

$$F_{\min}(z) = P\{\min\{X, Y\} \le z\} = 1 - P\{\min\{X, Y\} > z\}$$
$$= 1 - P\{X > z, Y > z\} = 1 - P\{X > z\} \cdot P\{Y > z\}$$

$$F_{\min}(z) = 1 - [1 - F_X(z)] \cdot [1 - F_Y(z)]$$

For $X_1, \dots, X_n$, n i.i.d. R.V.s with CDF $F(x)$, we have
$F_{\max}(z) = [F(z)]^n, F_{\min}(z) = 1 - [1 - F(z)]^n$

Ex. If the system L consists of two sub-systems with lifetime X and Y following PDFs

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0, \\ 0, & x \le 0. \end{cases} \qquad f_Y(y) = \begin{cases} \beta e^{-\beta y}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

where $\alpha > 0, \beta > 0, \alpha \neq \beta$. Find the PDFs of system lifetimes under the following connections.

(1) series. (2) parallel. (3) backup.

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where $\alpha > 0, \beta > 0, \alpha \neq \beta$. Find the PDFs of system lifetimes under the following connections.

(1) series. (2) parallel. (3) backup.

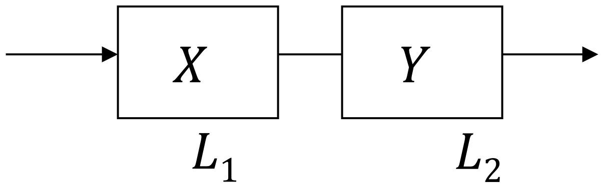
$$f_Z(z) = \int_{-\infty}^{+\infty} f\,(x, z-x)dx, Z = X + Y$$

$$F_{\max}(z) = F_X(z) \cdot F_Y(z)$$

$$F_{\min}(z) = 1 - [1 - F_X(z)] \cdot [1 - F_Y(z)]$$

Sol.

(1) Under series connection, we have $Z = \min\{X, Y\}$



$$F_{\min}(z) = 1 - [1 - F_X(z)] \cdot [1 - F_Y(z)]$$

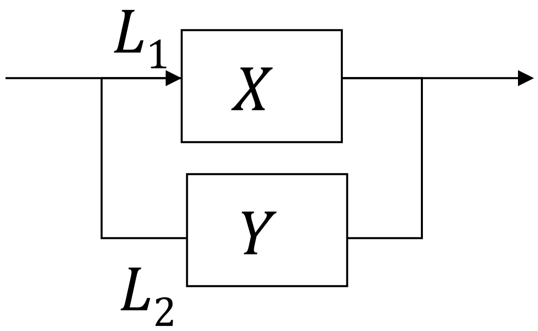
$$F_X(x) = \begin{cases} 1 - e^{-\alpha x}, & x > 0, \\ 0, & x \leq 0. \end{cases} \qquad F_Y(y) = \begin{cases} 1 - e^{-\beta y}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

$$F_Z(z) = \begin{cases} 1 - e^{-(\alpha+\beta)z}, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

$$f_Z(z) = F_Z'(z) = \begin{cases} (\alpha + \beta)e^{-(\alpha+\beta)z}, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

(2) Under parallel connection, we have $Z = \max\{X, Y\}$.

$$\boxed{F_{\max}(z) = F_X(z) \cdot F_Y(z)}$$

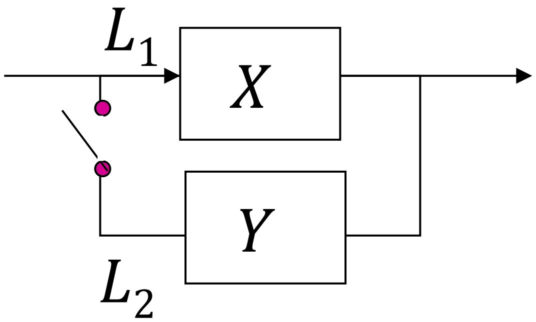


$$F_X(x) = \begin{cases} 1 - e^{-\alpha x}, & x > 0, \\ 0, & x \le 0. \end{cases} \qquad F_Y(y) = \begin{cases} 1 - e^{-\beta y}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

$$F_Z(z) = \begin{cases} (1 - e^{-\alpha z})(1 - e^{-\beta z}), & z > 0 \\ 0, & z \le 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \alpha e^{-\alpha z} + \beta e^{-\beta z} - (\alpha + \beta) e^{-(\alpha+\beta) z}, & z > 0 \\ 0, & z \le 0 \end{cases}$$

(3) Under backup connection, we have $Z = X + Y$.



$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

$$f_Y(y) = \begin{cases} \beta e^{-\beta y}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

$$f_Z(z) = \begin{cases} \int_{-\infty}^{+\infty} \alpha e^{-\alpha x} \cdot \beta e^{-\beta(z-x)} dx, & 0 < x < z, \\ 0, & \text{otherwise} \end{cases}$$

Quiz 4

Date: 4-Nov-2024

Scope: Multi-dimensional R.V.s

Open-book exam, with fill-in-the-blank and multiple-choice questions.

Time: ~45 mins

Case study

Text Classification and Naïve Bayes

The Task of Text Classification

CS 124: From Languages to Information
https://web.stanford.edu/class/cs124/