Chapter 1 homework

1. Find the real part, imaginary part, module, argument and complex conjugate of the following complex numbers:
   (a) $2 + 2i$;
   (b) $i + \dfrac{1-i}{1+i}$;
   (c) $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^{100} + \left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{100}$;
   (d) $i^{10} - 4i^{15} + i$.
2. Convert the following complex numbers into triangular expressions and exponential expressions:
   (a) $-6 - 4i$;
   (b) $1 + i\tan\theta$;
   (c) $1 - \cos\varphi + i\sin\varphi, \quad 0 \le \varphi \le \pi$;
   (d) $\dfrac{(\cos 3\varphi + i\sin 3\varphi)^3}{(\cos 2\varphi - i\sin 2\varphi)^{10}}$.
3. Point out the relationship between complex numbers z and iz.
4. Find the values of the following formulas:
   (a) $\left(\dfrac{1+i}{1-i}\right)^8$;
   (b) $(\sqrt{3} + i)^4$;
   (c) $\sqrt[6]{-1}$;
   (d) $(1-i)^{\frac{1}{3}}$.
5. Convert the following coordinate transformation formula into complex number form:
   (a) Translation formula $\begin{cases} x = x_1 + a_1 \\ y = y_1 + b_1 \end{cases}$;

(b) Rotation formula $\begin{cases} x = x_1 \cos\alpha - y_1 \sin\alpha \\ y = x_1 \sin\alpha + y_1 \cos\alpha \end{cases}$.

6. Let ω be the n-th root of 1, but $\omega \neq 1$, prove that ω satisfies the equation $1 + w + w^2 + \cdots + w^{n-1} = 0$.

7. Compute the curves represented by the following equations (where t is a real parameter):

   (a) $z = t + \dfrac{1}{t}i \quad (t \neq 0)$;

   (b) $z = a + re^{it} \quad$ ($r > 0$ is a real constant and a is a complex number).

8. Find the curves represented by the following equations:

   (a) $|z - 2| = 4$;

   (b) $\arg(z - i) = \dfrac{\pi}{4}$;

   (c) $z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} = b\overline{b} \quad$ (a and b are complex constants).

9. Draw the graphs for the set of points z that satisfy the following inequalities, and indicate whether these point sets are bounded or unbounded, simply connected or multi-connected.

   (a) $|z - 3| > 4$;

   (b) $\left|\dfrac{z-3}{z-2}\right| \geq 1$;

   (c) $|z - 2| - |z + 2| > 3$;

   (d) $z\overline{z} - (2 + i)z - (2 - i)\overline{z} \leq 4$.

10. There is a function $w = f(z) = \dfrac{1}{z}$. Find what are the curves on the w-plane when we apply the mapping of f to the following functions on the z-plane.

    (a) $x^2 + y^2 = 3$;

    (b) $y = -x$.

11. Let
$$f(z) = \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) \quad (z \neq 0)$$
and prove that when $z \to 0$, the limit of $f(z)$ does not exist.