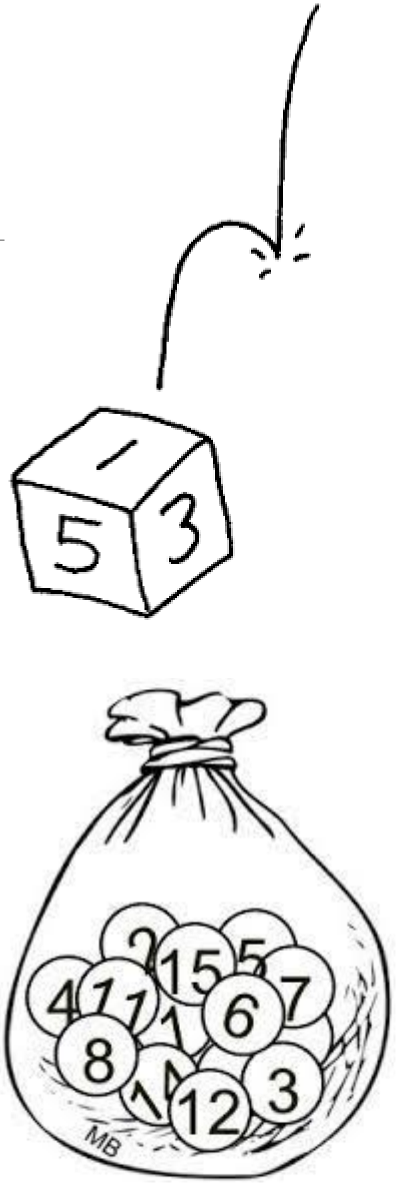
Lecture 04

- Random Variables
- Probability Mass Function (PMF)
- Some Common PMFs
  (Bernoulli, Binomial, and Poisson)
- Cumulative Distribution Functions
  (CDF)



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Variables in different domains

type name value

int a = **5**;

double b = **4.2**;

bit c = **1**;

Variables in CS

A is the number of courses taken by a student.

$$A \in \{1, 2, \dots, 6\}$$

B is the amount of student in a class.

$$B \in \mathbb{N}$$

C is 1 *if* we successfully pass an exam. 0 otherwise.

$$C \in \{0,1\}$$

Random variables in probability

Variables in different domains

$$x + 2 = 6$$

$$x^2 - 2x + 1 = 0$$

- Usually written in lower case letters.
- The value of x is not random.

Variables in algebra

A is the number of courses taken by a student.

$$A \in \{1, 2, \dots, 6\}$$

B is the amount of student in a class.

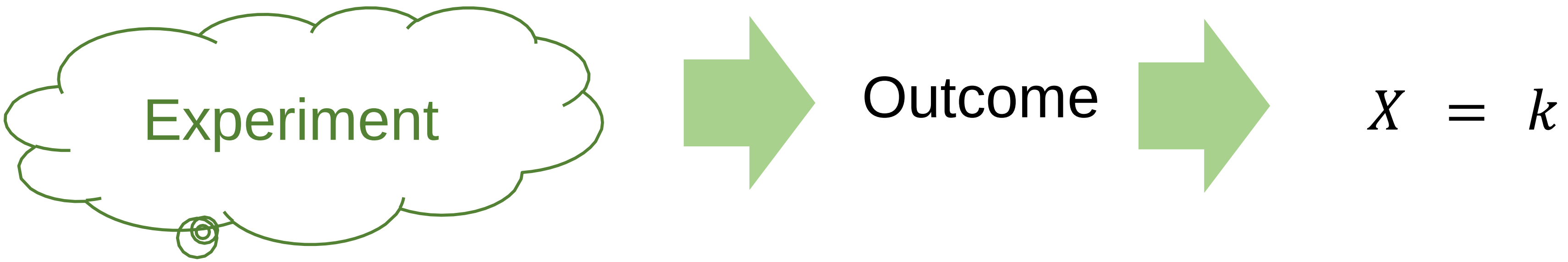
$$B \in \mathbb{N}$$

- Usually written in capital letters.
- The values can be randomly chosen from a set.

Random variables in probability

Random Variable

Def. A random variable is a real-valued function on a sample space.



Example:

3 coins are flipped.
Let $X = \#$ of heads.
X is a random variable.

simple and clear!

1. The value of X for the outcomes.

- (H,H,H)
- (H,T,H)

2. What is the event (set of outcomes) where $X = 2$?

3. What is $P(X = 2)$?

Random Variable vs. Event

It is confusing that random variables and events use the same notation.

- Random variables ≠ events.
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let $X = \#$ of heads.
X is a random variable.

$X = 2$
event

$P(X = 2)$
Probability

Random Variable vs. Event

It is confusing that random variables and events use the same notation.

- Random variables $\neq$ events.
- We can define an event to be a particular assignment of a random variable.

Example:

3 coins are flipped.
Let $X = \#$ of heads.
X is a random variable.

A full list.

$X = k$	Set of outcomes	$P(X = k)$
$X = 0$	{(T,T,T)}	1/8
$X = 1$	{(H, T, T), (T, H, T), (T, T, H)}	3/8
$X = 2$	{(H, H, T), (H, T, H), (T, H, H)}	3/8
$X = 3$	{(H, H, H)}	1/8
$X \geq 4$	$\emptyset$	0

wants: A unified definition for this list!

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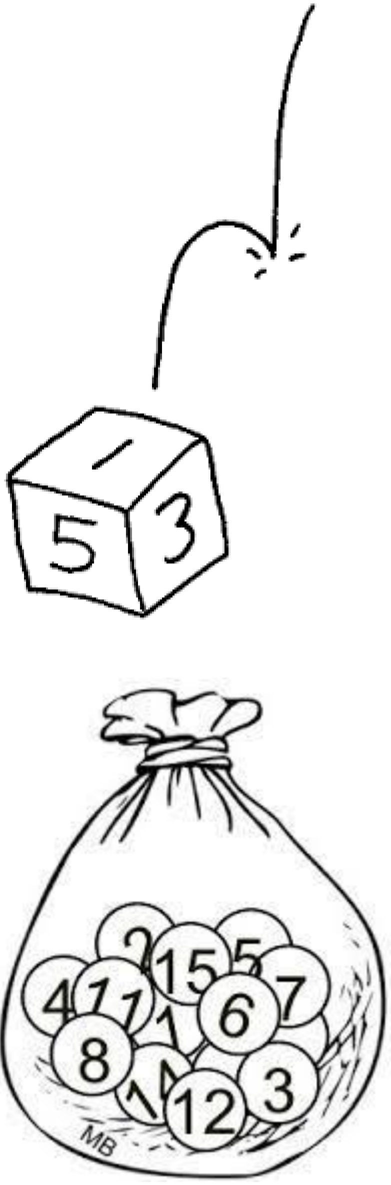


Discrete R.V.s and Probability Mass Functions

A random variable X is discrete (离散的) if it takes on countably many values.

- $X = x$, where $x \in \{x_1, x_2, x_3, \dots\}$

The PMF of a discrete random variable is

$$P(X = x) = \underbrace{p(x) = p_X(x)}_{\text{shorthand notation}}$$

- PMF of discrete R.V.s in a table

X	x_1	x_2	$\dots$	x_k	$\dots$
$P(x_k)$	p_1	p_2	$\dots$	p_k	$\dots$

- Important Properties:

1. $0 \leq p_k \leq 1, (k = 1,2, \dots)$
2. $\sum_{k=1}^{\infty} p_k = 1$

Probability Mass Function (PMF, 分布律)

3 coins are flipped. Let X = # of heads.

$X = k$	0	1	2	3
$P(X = k)$	1/8	3/8	3/8	1/8

parameter input

A function of k

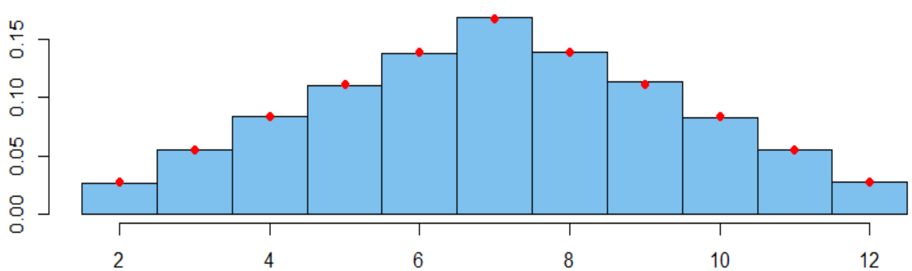
with range [0,3]

$$P(X = k)$$

return value/output
number between
1/8 and 3/8

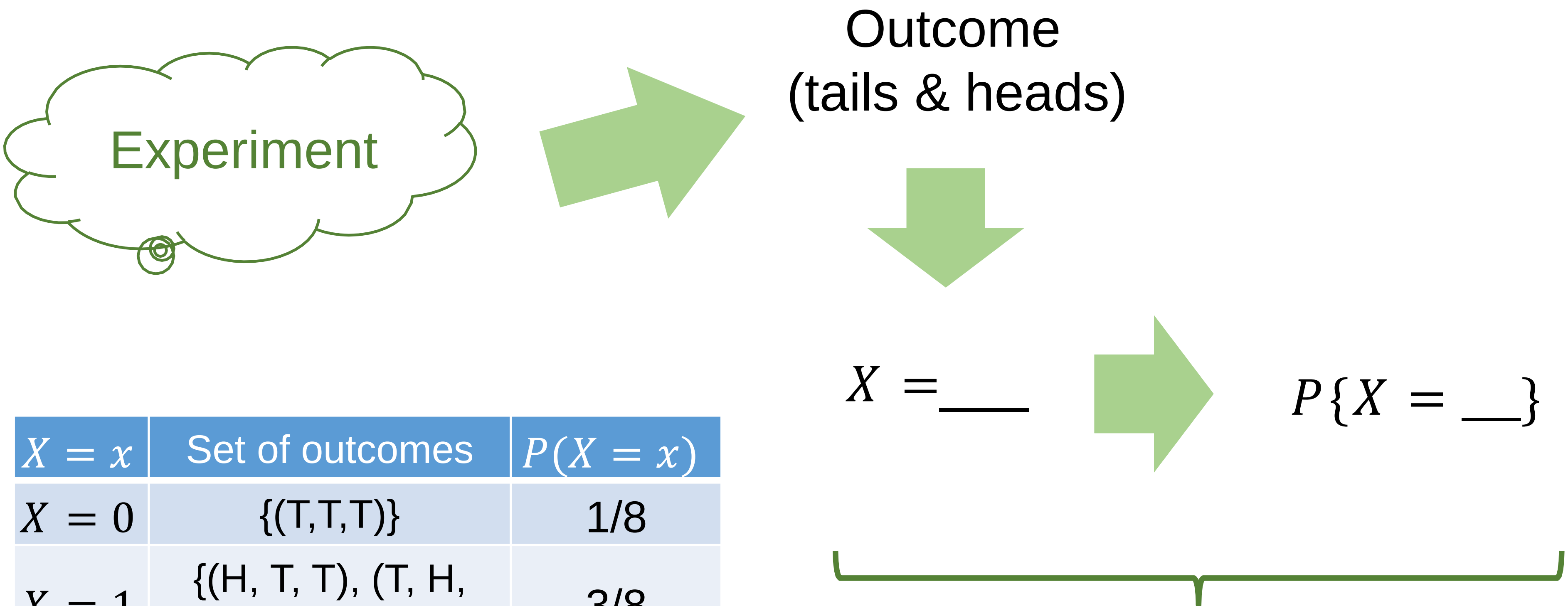
The probability of the event that a R.V. X takes on the value k!

For a discrete random variable, this is a probability mass function.



PMF of rolling
a pair of dice.

A short summary



$X = x$	Set of outcomes	$P(X = x)$
$X = 0$	{(T,T,T)}	1/8
$X = 1$	{(H, T, T), (T, H, T), (T, T, H)}	3/8
$X = 2$	{(H, H, T), (H, T, H), (T, H, H)}	3/8
$X = 3$	{(H, H, H)}	1/8
$X \geq 4$	∅	0

Can we find some regular patterns from the last step?

Seems like it might be more convenient than listing a table!

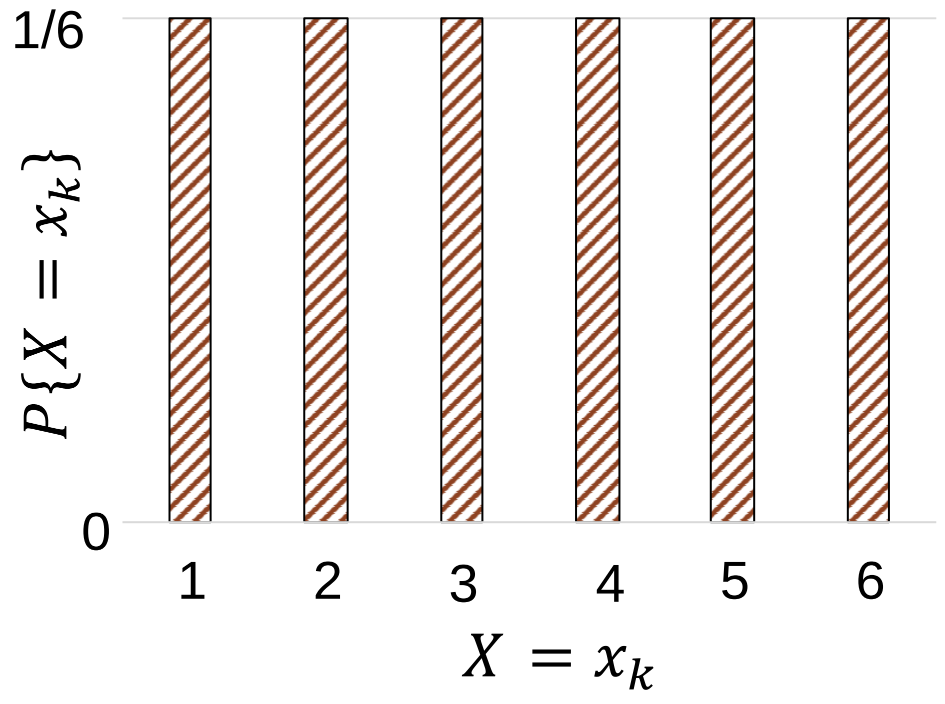
PMF for a single 6-sided die

Let X be a random variable that represents the result of a single dice roll.

- Support of X: $\{1, 2, 3, 4, 5, 6\}$
- Therefore, X is a discrete random variable.

PMF of X:

$$p(x) = \begin{cases} 1/6, & x \in \{1, \dots, 6\} \\ 0, & \text{otherwise} \end{cases}$$



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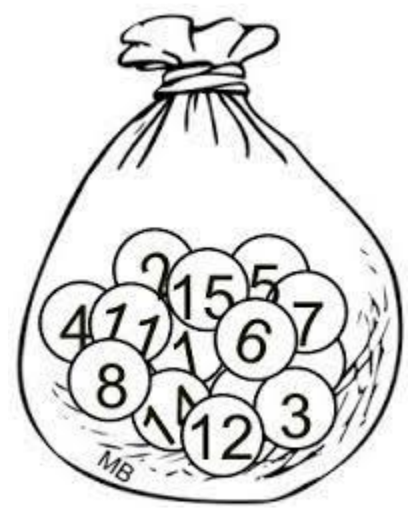


Some common PMFs

Bernoulli distribution (0-1分布)

$$p(x) = \begin{cases} p, & \text{if } x \text{ is } 1 \\ 1-p, & \text{if } x \text{ is } 0 \end{cases}$$

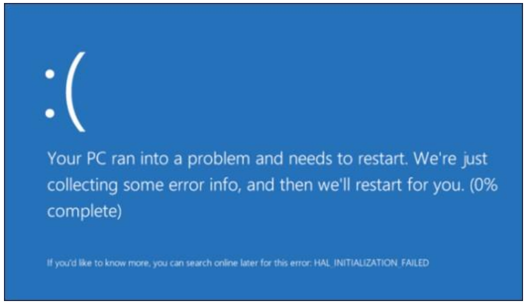
X	**0**	**1**
$p(x)$	$1-p$	p

Or $$P\{X=k\} = p^k(1-p)^{1-k}, k = 0,1 \ (0 < p < 1)$$

written as $X \sim \mathrm{Ber}(p)$

- The simplest distribution for discrete R.V.s.
- Used to model an experiment with only two possible outcomes.
- {Head, Tail}, {Hit, Miss}, {Pass, Fail}, etc.

Example of Bernoulli distribution



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crashes

$$X \sim \mathrm{Ber}(p)$$

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

Serve an ad.

- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \mathrm{Ber}(\_\_\_)$$

$$P(X = 1) = \_\_\_\_\_\_\_\_$$

$$P(X = 0) = \_\_\_\_\_\_\_\_$$

Some common PMFs

Consider n independent trials of Ber(p) random variables.

- Binomial distribution (二项分布), $X \sim b(n, p)$ or $X \sim \mathrm{Bin}(n, p)$

$$P(X = k) = C_n^k p^k (1 - p)^{n-k}, \qquad k = 0,1,2, \ldots, n$$

- $n = 1$, Binomial distribution = Bernoulli distribution.
- models the number of occurrence when someone performs the same experiment n times.

Ex. The probability of getting exactly one '6' when someone rolls a fair die three times can be modelled by

$$P\{X = 1\} = C_3^1\, p\, (1 - p)^2, \text{ with } p = 1/6$$

- # of 1's in randomly generated length n bit string, with $p = 1/2$
- # of disk drives crashed in a cluster with 1000 computers.

Notations: Bernoulli vs. Binomial distribution

$$X \sim \mathrm{Ber}(p)$$

The random variable

is modeled as

a Bernoulli distribution

with parameter p

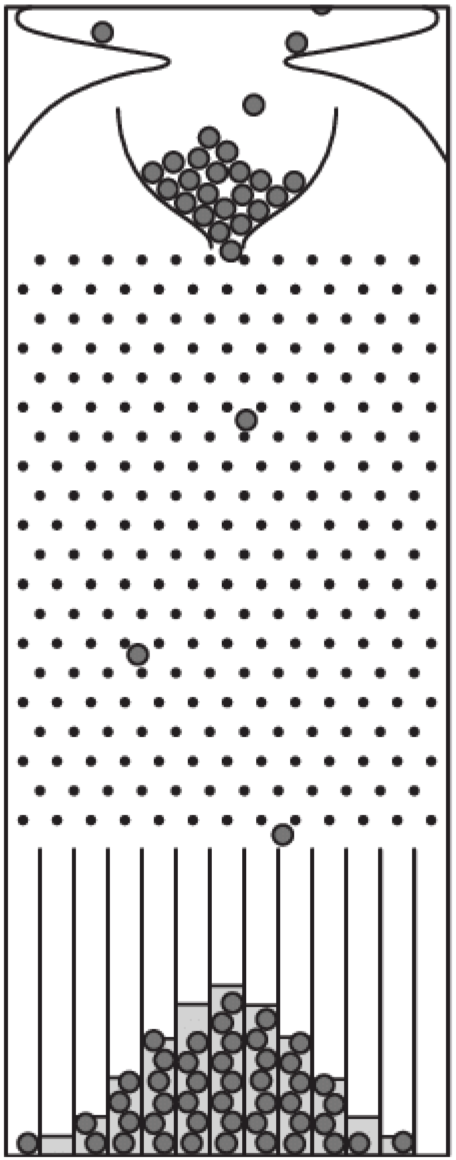
$$Y \sim b(n, p)$$

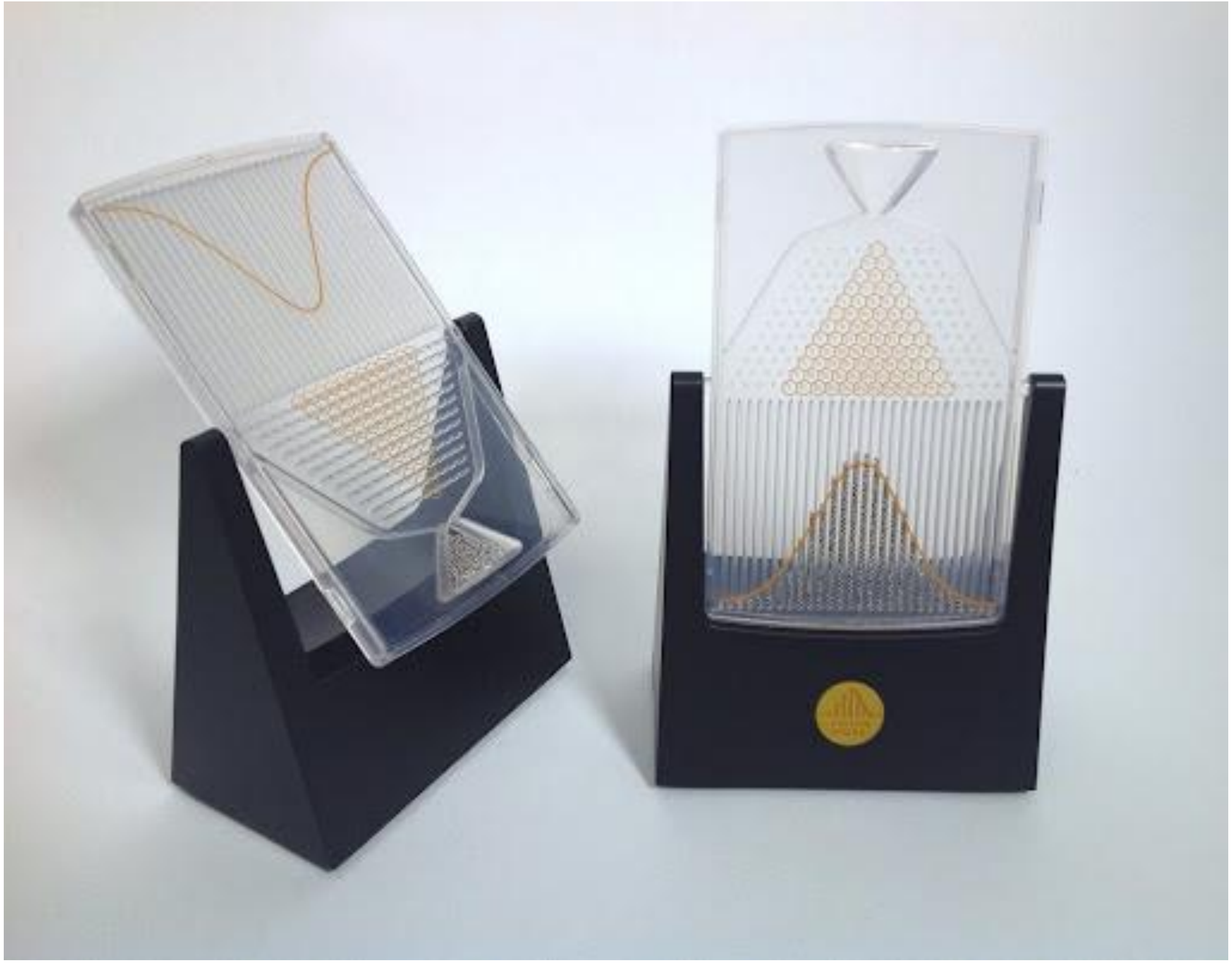
Identify as a function of a random variable.

Remember our example in slide P7:

$X = k$	0	1	2	3
$P(X = k)$	1/8	3/8	3/8	1/8

$$P(X = k) = C_3^k\, 0.5^k 0.5^{1-k}$$

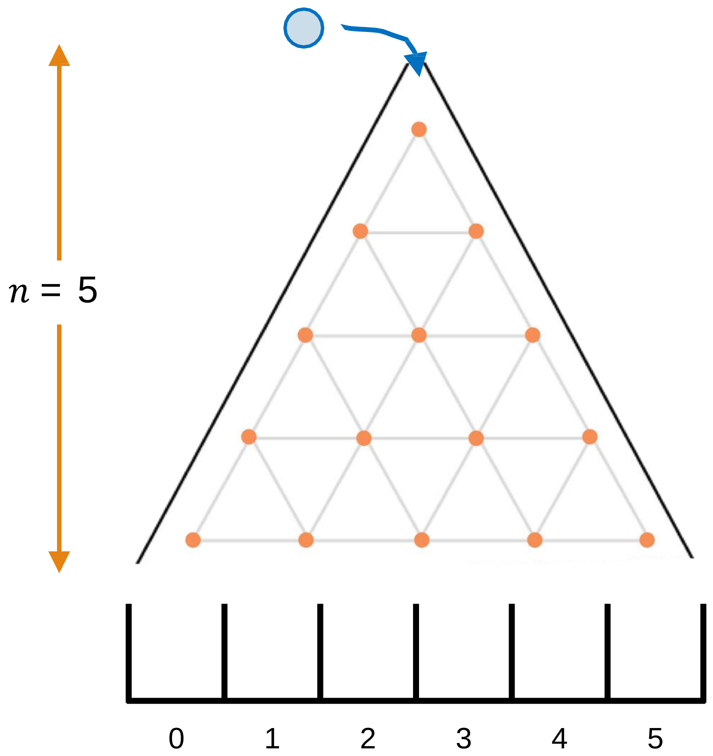




Galton board, a.k.a., [bean machine]

Galton Board

$X \sim b(n, p)$
$P(X = k) = C_n^k p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.
Let $B =$ the bucket index a ball drops into. What is the distribution of B?

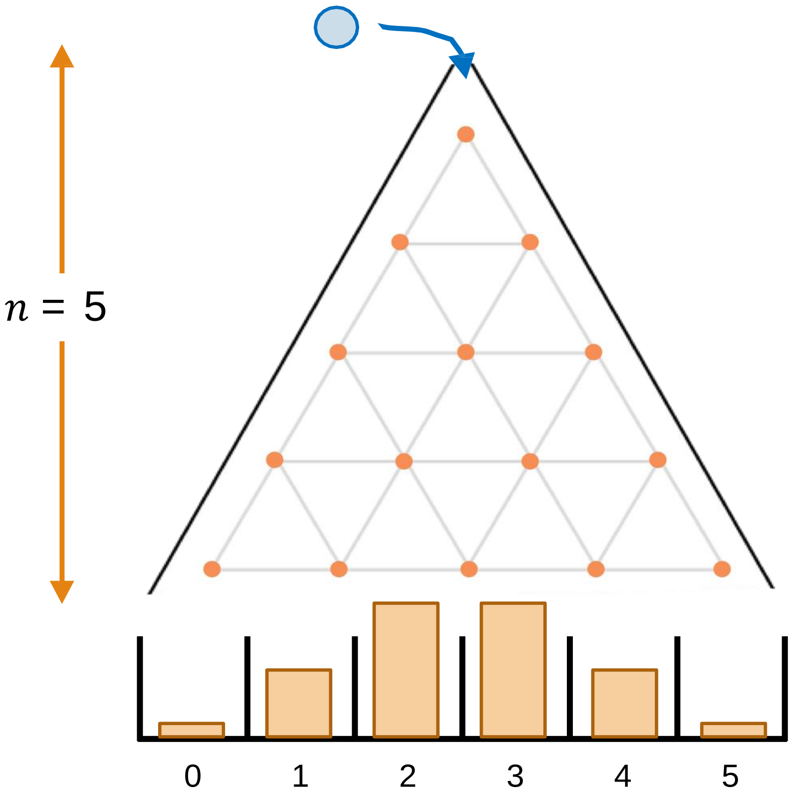
- Each pin is an independent trial
- One decision made for each level $i = 1, 2, \ldots, 5$
- Consider a Bernoulli R.V. with success if a ball went right on the i-th level
- Bucket index $B = \#$ times ball went right.

$$B \sim b(n = 5, p = 0.5)$$

Galton Board

Let B = the bucket index a ball drops into.
B is distributed as a Binomial R.V.

$$B \sim b(n = 5, p = 0.5)$$



Calculate the probability of a ball landing in the k-th bucket.

$$P(B = 0) = C_5^0 0.5^5 \approx 0.03$$
$$P(B = 1) = C_5^1 0.5^5 \approx 0.16$$
$$P(B = 2) = C_5^2 0.5^5 \approx 0.31$$

PMF of Binomial RV!

NBA Finals

$X \sim b(n, p)$
$P(X = k) = C_n^k p^k (1-p)^{n-k}$

- The Golden State Warriors are going to play the Toronto Raptors in a 7-game series during the 2019 NBA finals.
  - The Warriors have a probability of 58% of winning each game, independently.
  - A team wins the series if they win at least 4 games (we play all 7 games).

- What is P(Warriors winning)?

1. Define events / RVs & state goal

X: # games Warriors win
$X \sim \text{Bin}(7, 0.58)$

Desired probability? (select all that apply)

A. $P(X > 4)$
B. $P(X \geq 4)$
C. $P(X > 3)$
D. $1 - P(X \leq 3)$
E. $1 - P(X < 3)$

NBA Finals

$X \sim b(n, p)$
$P(X = k) = C_n^k p^k (1-p)^{n-k}$

- The Golden State Warriors are going to play the Toronto Raptors in a 7-game series during the 2019 NBA finals.
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- What is P(Warriors winning)?

1. Define events / RVs & state goal

X: # games Warriors win
$X \sim$Bin(7, 0.58)

wants: $P(X \geq 4)$

2. Solve

$$P(X \geq 4) = \sum_{k=4}^{7} C_7^k \cdot 0.58^k \cdot 0.42^{7-k}$$

Some common PMFs

Poisson distribution (泊松分布), $X \sim \pi(\lambda)$

$$P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0,1,2,\ldots$$

where $\lambda > 0$ is a constant.

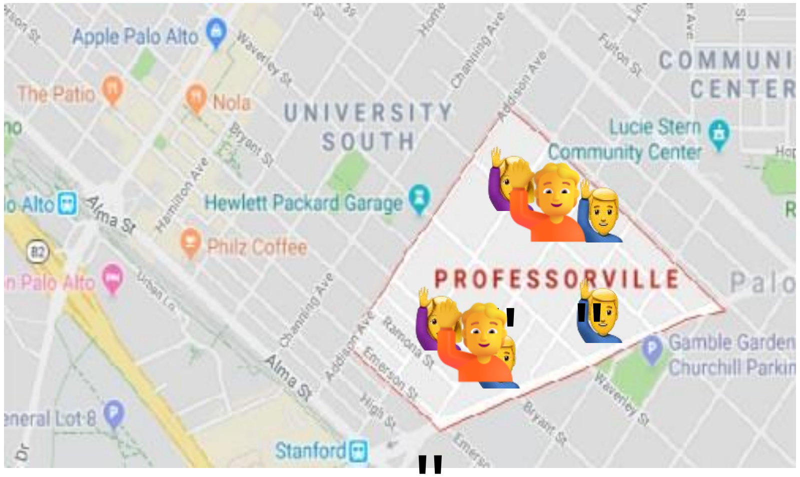
- can verify that $P\{X = k\} \geq 0$ and $\sum_{k=0}^{\infty} P\{X = k\} = 1$

- models the probability of a given number of events occurring in a fixed interval of time

Note: what is the meaning of parameter λ?

- events occur with a known constant mean rate λ.

Poisson ride sharing



Probability of k requests from this area in the next 1 minute?

Suppose we know:

On average,
$\lambda = 5$ (requests per minute)

By Poisson distribution:

$$P\{X = k\} = \frac{5^k e^{-5}}{k!}$$

We will link Poisson and Binomial distributions.

Poisson ride sharing

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60 seconds:

0	0	1	0	1	…	0	0	0	0	1
1	2	3	4	5						60

At each second:

- Independent trial
- a request (1) or no request (0).

$$P(X = k) = C_{60}^{k}\left(\frac{5}{60}\right)^{k}\left(1 - \frac{5}{60}\right)^{n-k}$$

Let $X = \#$ of requests in minute.

$X \sim b(60, 5/60)$

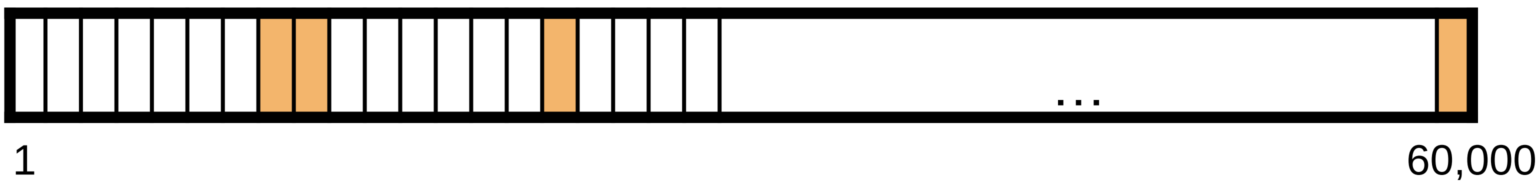
Wait! What if there are two requests in each second?

Poisson ride sharing

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60,000 milliseconds:



At each millisecond:

- Independent trial
- a request (1) or no request (0).

$$P(X = k)$$
$$= C_{60000}^{k}\left(\frac{5}{60000}\right)^{k}\left(1 - \frac{5}{60000}\right)^{n-k}$$

Let $X = \#$ of requests in minute.

$X \sim b(60000, 5/60000)$

Wait! What if there are two requests in each millisecond?

Poisson ride sharing

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into n infinitely small buckets:

OMG so small

1 ∞

At each time bucket:

- Independent trial
- a request (1) or no request (0).

$$P(X = k) = \lim_{n \to \infty} C_n^k \left(\frac{5}{n}\right)^k \left(1 - \frac{5}{n}\right)^{n-k}$$

Let $X = \#$ of requests in minute.

$X \sim b(n, 5/n)$

We will prove this can be approximated as a Poisson distribution.

Expand

$$P(X=k)=\lim_{n\to\infty} C_n^k \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} = \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\frac{\lambda^k}{n^k}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Rearrange

Def. of natural exponent

$$= \lim_{n\to\infty} \frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} = \lim_{n\to\infty} \frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty} \frac{n(n-1)\ldots(n-k+1)}{n^k}\frac{(n-k)!}{(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

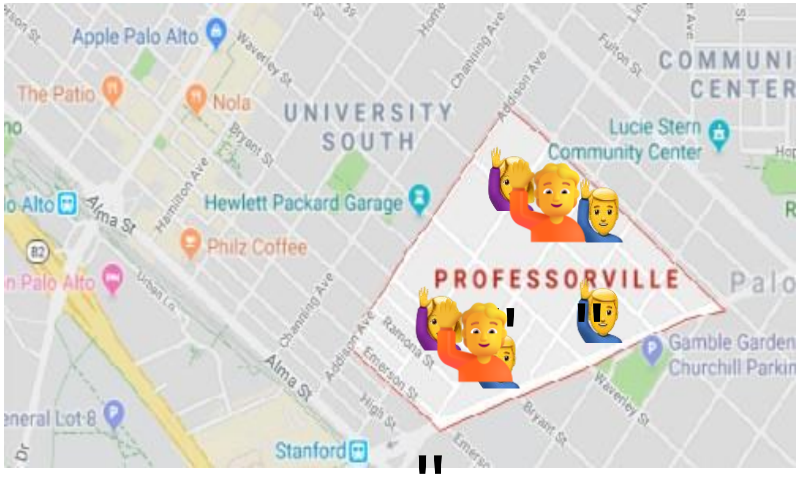
Cancel + Limit analysis

$$= \lim_{n\to\infty} \frac{n^k}{n^k}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{1} = \frac{\lambda^k}{k!}e^{-\lambda}$$

Cool! The Poisson distribution is a **limiting case** of the binomial distribution.
-------- **Poisson limit theorem**

$$\boldsymbol{\lambda = np}$$

Poisson ride sharing



Probability of k requests from this area in the next 1 minute?

Suppose we know:

On average,
$\lambda = 5$ (requests per minute)

By Poisson distribution:

$$P\{X = k\} = \frac{5^k e^{-5}}{k!}$$

By Binomial distribution:

$$P(X = k) = \lim_{n \to \infty} C_n^k \left(\frac{5}{n}\right)^k \left(1 - \frac{5}{n}\right)^{n-k}$$

Summary of common PMFs

Modelling the probability of success/occurrence in

One trial:

$$X \sim \mathrm{Ber}(p)$$

$$\Big\uparrow \; n = 1$$

Several trails:

$$X \sim b(n, p)$$

$$\Big\downarrow \; n \to \infty$$

An interval of time:

$$X \sim \pi(\lambda)$$

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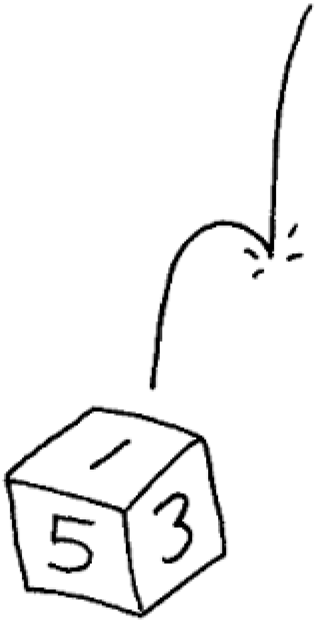
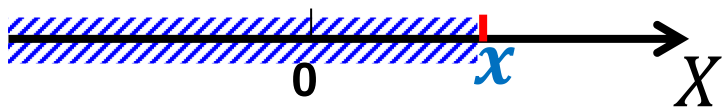


Cumulative Distribution Functions (概率分布函数)

For a random variable X, the cumulative distribution function (CDF) is defined as

$$F(x) = F_X(x) = P(X \leq x), \text{ where } -\infty < x < \infty$$



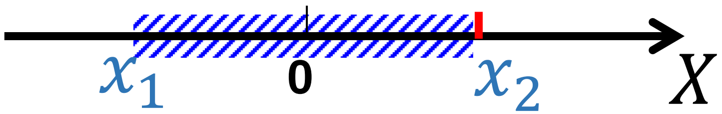
For a discrete R.V. X, the CDF is:

$$F(x) = P(X \leq x) = \sum_{x \leq a} p(x)$$

Cumulative Distribution Functions (概率分布函数)

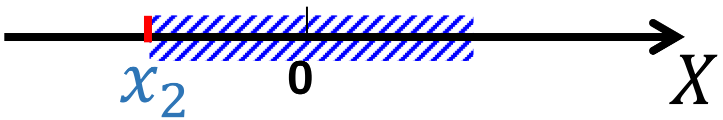
For any $x_1, x_2\ (x_1 < x_2)$, we have

$$P\{x_1 < X \leq x_2\} = P\{X \leq x_2\} - P\{X \leq x_1\} = F(x_2) - F(x_1)$$

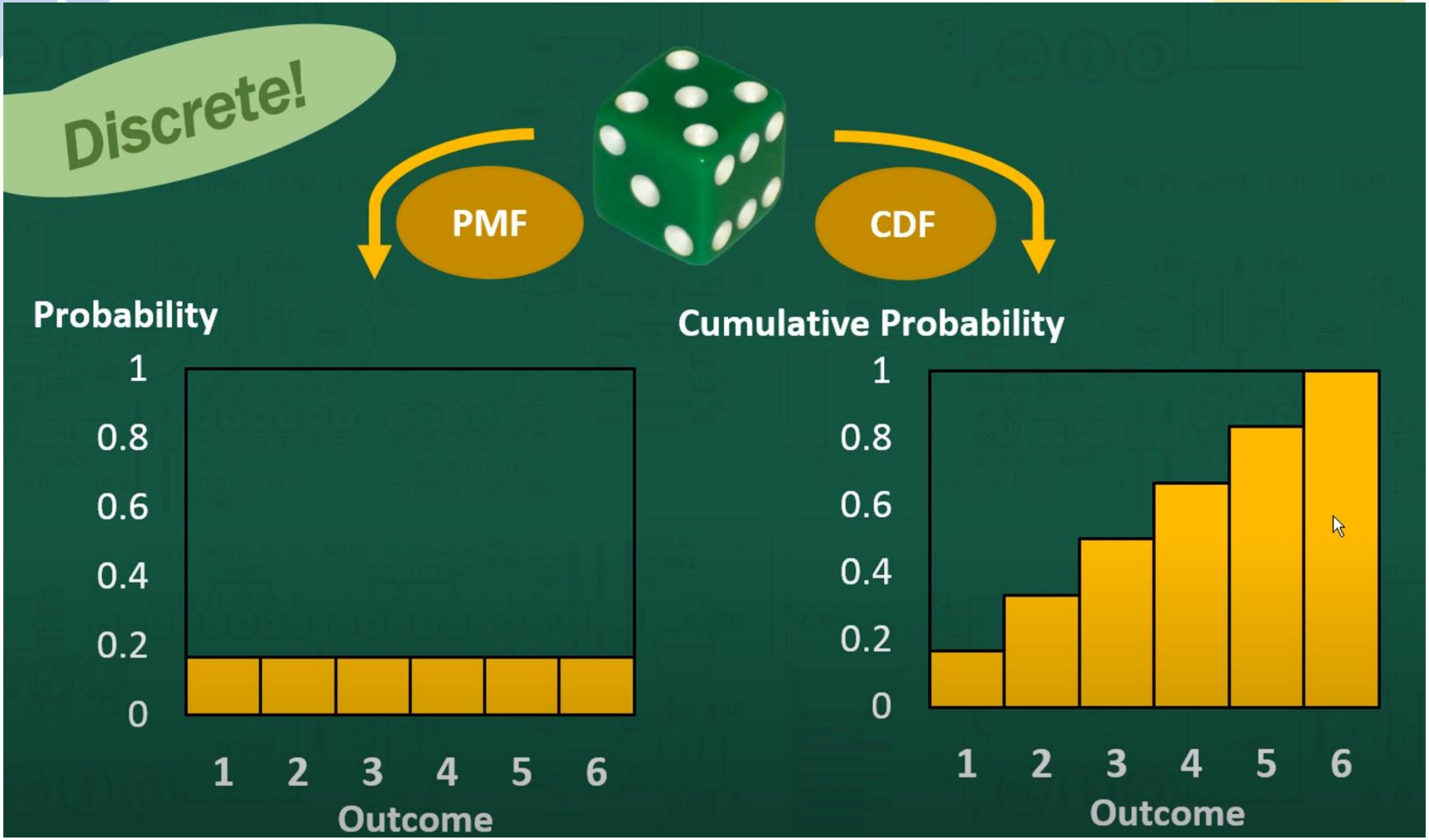


The interval is open and closed at the left and right endpoints, respectively.

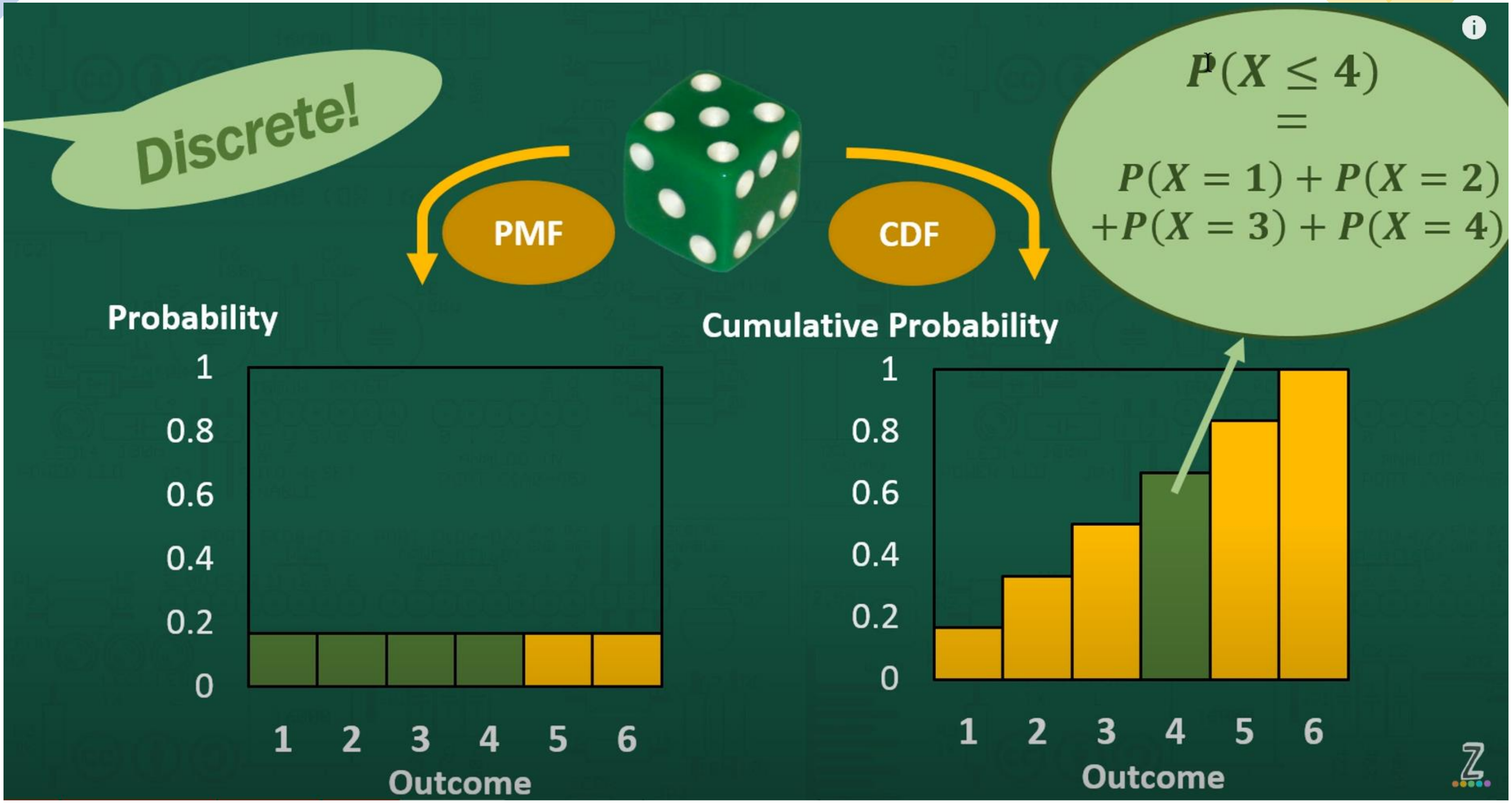
$$P\{X > x_2\} = 1 - P\{X \leq x_2\} = 1 - F(x_2)$$



$$F(x) = F_X(x) = P(X \leq x)$$

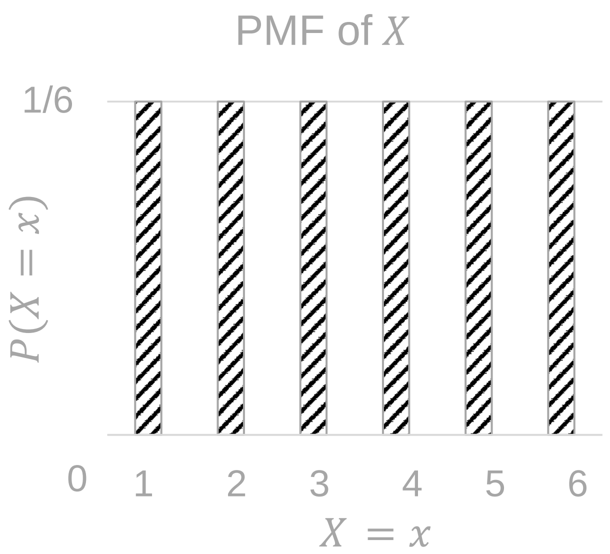


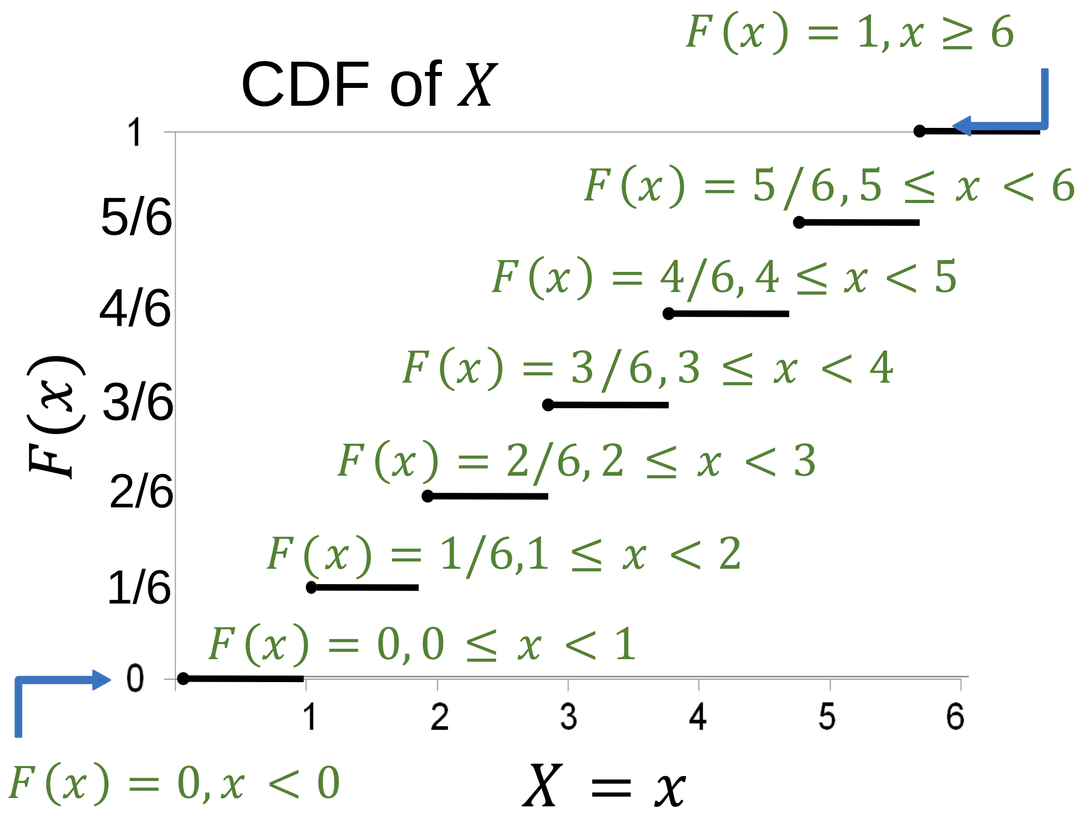
$$F(x) = F_X(x) = P(X \leq x)$$



CDFs as graphs

Let X be a R.V. that represents the result of a single dice roll.





Note: $F(x)$ is **right continuous**, i.e., $F(x+0) = F(x)$.

Ex. There are 6 balls in a bag, of which two are labeled ‘1’, three are labeled ‘2’, and one is labeled ‘3’. One ball is randomly drawn from the bag. Let X denotes the number of the ball.

(1) Find the CDF of X.
(2) Find the probability $P\{2 \leq X \leq 3\}$.

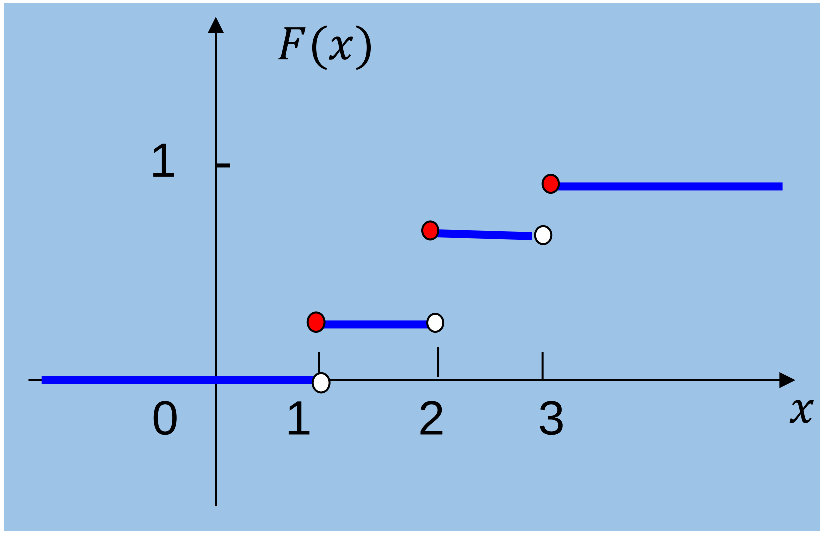
Ex. There are 6 balls in a bag, of which two are labeled ‘1’, three are labeled ‘2’, and one is labeled ‘3’. One ball is randomly drawn from the bag. Let X denotes the number of the ball.

(1) Find the CDF of X.
(2) Find the probability $P\{2 \le X \le 3\}$.

(1) The PMF of X is

X	1	2	3
p_k	1/3	1/2	1/6

$$F(x) = P\{X \le x\}$$
$$= \begin{cases} 0, & x < 1 \\ 1/3, & 1 \le x < 2 \\ 5/6, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

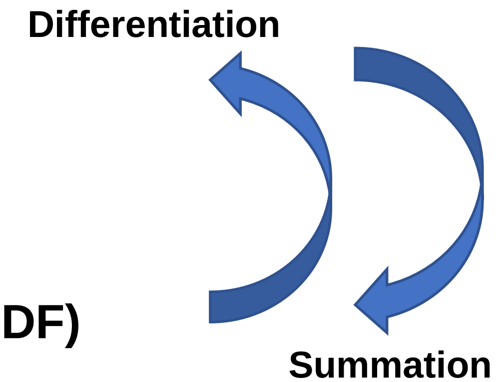


(2) $P\{2 \le X \le 3\}$
$= P\{X = 2\} + P\{X = 3\} = 4/6$

Discrete RV

（概率）分布律，
Probability Mass Function (PMF)

（累积）分布函数，
Cumulative Distribution Function (CDF)



Continuous RV

（概率）密度函数，
Probability Distribution Function (PDF)

（累积）分布函数，
Cumulative Distribution Function (CDF)