Chapter 5: Residues

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Outline

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1. Isolated Singularities
2. Residues

1 Concept of Isolated Singularities

Definition

If $f(z)$ is not analytic at z_0, but analytic everywhere in the deleted neighborhood $0 < |z - z_0| < \delta$, z_0 is called an isolated singularity of $f(z)$.

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Example 1.1

$z = 0$ *is an isolated singularity of* $e^{\frac{1}{z}}$ *and* $\dfrac{\sin z}{z}$.

$z = -1$ *is an isolated singularity of* $\dfrac{1}{z+1}$.

Note

Isolated singularities must be singularities,
but singularities are not necessarily isolated singularities.

Example 1.2

Determine the characteristics of the singularities at $z = 0$ *of* $f(z) = \dfrac{z^2}{\sin \frac{1}{z}}$.

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Solution.

The singularities are $z = 0, z = \dfrac{1}{k\pi}$ $(k = \pm 1, \pm 2, \cdots)$.
Because $\lim\limits_{k \to \infty} \dfrac{1}{k\pi} = 0$, there are always singularities in a deleted neighborhood of $z = 0$ no matter how small the deleted neighborhood is, $z = 0$ is not an isolated singularity.

Classification of isolated singularities

According to the Laurent series of $f(z)$ in the deleted neighborhood of its isolated singularity, isolated singularities can be divided into the following three categories:

1. Removable singularities 2. Poles 3. Essential singularities

1. Removable singularities
   1) Definition
   If there is no negative power term of $z - z_0$ in the Laurent series, the isolated singularity z_0 is called the removable singularity of $f(z)$.

Note

1. If z_0 is an isolated singularity of $f(z)$,
$f(z) = c_0 + c_1(z - z_0) + \cdots + c_n(z - z_0)^n + \cdots . \ (0 < |z - z_0| < \delta)$
The sum function of this power series must be an analytic function at z_0.
2. Whether $f(z)$ is defined in z_0 or not,
let $f(z_0) = c_0$, then $f(z)$ is analytic at z_0.
$$c_0 = f(z_0) = \lim_{z \to z_0} f(z), \qquad f(z) = \begin{cases} f(z) & z \neq z_0, \\ c_0 & z = z_0. \end{cases}$$

Classification of isolated singularities

2) Determine the removable singularity
   - i using definition
     If there is no negative power term of $z - z_0$ in the Laurent series, the isolated singularity z_0 is called the removable singularity of $f(z)$.
   - ii using limit $\lim\limits_{z \to z_0} f(z)$
     If the limit exists and is finite, z_0 is called the removable singularity of $f(z)$.

Example 1.3

$\dfrac{\sin z}{z} = 1 - \dfrac{1}{3!}z^2 + \dfrac{1}{5!}z^4 + \cdots$ *has no negative power term,*
$z = 0$ *is called the removable singularity of* $f(z)$.
If we add the definition: when $z = 0$, $\dfrac{\sin z}{z} = 1$,
then $\dfrac{\sin z}{z}$ *is analytic at* $z = 0$.

Example 1.4

Prove that $z = 0$ *is a removable singularity of* $\dfrac{e^z - 1}{z}$.

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Proof.

$$\frac{e^z - 1}{z} = \frac{1}{z}\left(1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n + \cdots - 1\right)$$

$$= \boxed{1 + \frac{1}{2!}z + \cdots + \frac{1}{n!}z^{n-1} + \cdots}, \quad (0 < |z| < +\infty)$$

no negative power term

so $z = 0$ is a removable singularity of $\dfrac{e^z - 1}{z}$.

Another proof: Because $\displaystyle\lim_{z \to z_0} \frac{e^z - 1}{z} = \lim_{z \to z_0} e^z = 1$,

$z = 0$ is a removable singularity of $\dfrac{e^z - 1}{z}$. □

Classification of isolated singularities

2 Poles

1) Definition

If there are only a finite number of negative power terms of $z-z_0$ in Laurent series, the highest power of $(z-z_0)^{-1}$ is $(z-z_0)^{-m}$,

$$\begin{aligned} f(z) =& c_{-m}(z-z_0)^{-m} + \cdots + c_{-2}(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} \\ &+ c_0 + c_1(z-z_0) + \cdots, \quad (m > 1, c_{-m} \neq 0) \end{aligned}$$

or $\boxed{f(z) = \dfrac{1}{(z-z_0)^m} g(z)}$.

Then the isolated singularity z_0 is called a pole of order m of $f(z)$.

Note

1. $g(z) = c_{-m} + c_{-m+1}(z-z_0) + c_{-m+2}(z-z_0)^2 + \cdots$
   1) $g(z)$ is an analytic function in $|z-z_0| < \delta$
   2) $g(z_0) \neq 0$
2. If z_0 is a pole of $f(z)$, $\lim\limits_{z \to z_0} f(z) = \infty$.

Example 1.5

For $f(z) = \dfrac{3z+2}{z^2(z+2)}$,
$z=0$ *is a pole of order* 2 *and* $z=-2$ *is a pole of order* 1.

Classification of isolated singularities

2) Determine the pole

i Determine by definition

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ii Determine by the equivalence form of the definition

$f(z) = \dfrac{g(z)}{(z - z_0)^m}$ holds in deleted neighborhood of z_0, where $g(z)$ is analytic in deleted neighborhood of z_0 and $g(z_0) \neq 0$.

iii Determine by computing limit $\lim\limits_{z \to z_0} f(z) = \infty$

Question

Find the singularities of $\dfrac{1}{z^3 - z^2 - z + 1}$. If they are poles, point out their orders.

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Solution.

Because $\dfrac{1}{z^3 - z^2 - z + 1} = \dfrac{1}{(z+1)(z-1)^2}$,
$z = -1$ is a pole of order 1 and $z = 1$ is a pole of order 2.

Classification of isolated singularities

3. Essential singularities

If there are infinite negative power terms of $z - z_0$ in Laurent series, the isolated singularity z_0 is called an essential singularity of $f(z)$.

For example,

$$e^{\frac{1}{z}} = \boxed{1 + z^{-1} + \frac{1}{2!}z^{-2} + \cdots + \frac{1}{n!}z^{-n} + \cdots}, \quad (0 < |z| < \infty)$$

infinite negative power terms

so $z = 0$ is an essential singularity and $\lim\limits_{z \to 0} e^{\frac{1}{z}}$ does not exist.

Note: $\lim\limits_{z \to z_0} f(z)$ does not exist and is not ∞ in the neighborhood of the essential singularities.

Isolated Singularities	Characteristics of Laurent series	$\lim_{z \to z_0} f(z)$
Removable Singularities	no negative power term	exist and finite
Poles of Order m	a finite number of negative power terms, the highest power of $(z-z_0)^{-1}$ is $(z-z_0)^{-m}$	∞
Essential Singularities	infinite negative power terms	not exist and infinite

2 Zeros and Poles

1 Definition of Zeros

Definition

Suppose that a function $f(z)$ that is not constantly equal to zero, if it can be expressed as $f(z) = (z - z_0)^m \varphi(z)$, where $\varphi(z)$ is analytic at z_0, $\varphi(z) \neq 0$ and m is a positive integer, then z_0 is called a zero of order m of $f(z)$.

Example 1.6

For $f(z) = z(z-1)^3$,
$z = 0$ is a zero of order 1 and $z = 1$ is a zero of order 3.

Note

The zeros of an analytic function that is not constantly equal to zero are isolated.

2. Determine the Zeros
   If $f(z)$ is analytic at z_0, then the necessary and sufficient condition for z_0 to be the zero of order m of $f(z)$ is
   $f^{(n)}(z_0) = 0, (n = 0, 1, 2, \cdots, m-1) \quad , f^{(m)}(z_0) \neq 0.$

Proof.

Necessity. If z_0 is a zero of order m of $f(z)$,
according to $f(z) = (z - z_0)^m \varphi(z)$,
expand $f(z)$ by Taylor's theorem:
$\varphi(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots,$
where $c_0 = \varphi(z_0) \neq 0$.

Proof (Cont.)

Taylor expansion of $f(z)$ at z_0:
$f(z) = c_0(z - z_0)^m + c_1(z - z_0)^{m+1} + c_2(z - z_0)^{m+2} + \cdots$.
The coefficients of the first m terms of the expansion are all zero, and the coefficient formula of the Taylor series shows that:
$f^{(n)}(z_0) = 0, \ (n = 0, 1, 2, \cdots, m - 1)$
and $\dfrac{f^{(m)}(z_0)}{m!} = c_0 \neq 0$.
Sufficiency. Omitted. □

Example 1.7

Find the order of zeros of the following functions.

1. $f(z) = z^3 - 1;$
2. $f(z) = \sin z.$

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Solution.

1. Because $f'(1) = 3z^2\big|_{z=1} = 3 \neq 0$,
   $z = 1$ is a zero of order 1 of $f(z)$.
2. Because $f'(0) = \cos z\big|_{z=0} = 1 \neq 0$,
   $z = 0$ is a zero of order 1 of $f(z)$.

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Question

Find the order of zeros of $f(z) = z^5\left(z^2+1\right)^2$.

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Question

Find the order of zeros of $f(z) = z^5 \left(z^2 + 1\right)^2$.

Solution.

$z = 0$ is a zero of order 5 and $z = \pm i$ are zeros of order 2.

3. Relationship between the zeros and poles
If z_0 is a pole of order m of $f(z)$, then z_0 is a zero of order m of $\dfrac{1}{f(z)}$. And vice versa.

Proof.

If z_0 is a pole of order m of $f(z)$,

$f(z) = \dfrac{1}{(z-z_0)^m} g(z) \quad (g(z_0) \neq 0)$.

When $z \neq z_0$, $\dfrac{1}{f(z)} = (z-z_0)^m \dfrac{1}{g(z)} = (z-z_0)^m h(z)$,

$h(z)$ is analytic at z_0 and $h(z_0) \neq 0$.

Proof (Cont.)

Because $\lim\limits_{z\to z_0} \dfrac{1}{f(z)} = 0$, let $\dfrac{1}{f(z_0)} = 0$,

then z_0 is a zero of order m of $\dfrac{1}{f(z)}$.

Conversely, if z_0 is a zero of order m of $\dfrac{1}{f(z)}$,

then $\dfrac{1}{f(z)} = (z - z_0)^m \varphi(z)$. analytic and $\varphi(z) \neq 0$

When $z \neq z_0$, $f(z) = \dfrac{1}{(z - z_0)^m}\Psi(z)$, $\boxed{\Psi(z) = \dfrac{1}{\varphi(z)}}$

So z_0 is a pole of order m of $f(z)$. □

Note

This theorem provides a simple method to determine the poles of functions.

Example 1.8

Find the singularities of $\dfrac{1}{\sin z}$.
If they are poles, indicate their orders.

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Solution.

The singularities of the function are the points at which $\sin z = 0$. These singularities are $z = k\pi \ (k = 0, \pm 1, \pm 2, \cdots)$, which are isolated singularities.
Because $(\sin z)'\big|_{z=k\pi} = \cos z|_{z=k\pi} = (-1)^k \neq 0$,
$z = k\pi$ are zeros of order 1 of $\sin z$,
that is $z = k\pi$ are singularities of order 1 of $\dfrac{1}{\sin z}$.

Example 1.9

Is $z = 0$ a pole of order 2 of $\dfrac{e^z - 1}{z^2}$?

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Solution.

$$\frac{e^z - 1}{z^2} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right)$$

analytic and $\varphi(z) \neq 0$

$$= \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots = \frac{1}{z}\boxed{\varphi(z)}$$

So $z = 0$ is not a pole of order 2, but a pole of order 1.

Example 1.9

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Solution.

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So $z = 0$ is not a pole of order 2, but a pole of order 1.

Question

Find the order of the pole at $z = 0$ of $\dfrac{\sinh z}{z^3}$.

Note: Conclusions cannot be drawn from the obvious form of the function.

Example 1.10

If point z_0 is a zero of order m_1 of function $f_1(z)$ and z_0 is a zero of order m_2 of function $f_2(z)$, please discuss characteristics of function $f(z) = \dfrac{f_1(z)}{f_2(z)}$ at point z_0.

Example 1.10

If point z_0 is a zero of order m_1 of function $f_1(z)$ and z_0 is a zero of order m_2 of function $f_2(z)$, please discuss characteristics of function $f(z) = \dfrac{f_1(z)}{f_2(z)}$ at point z_0.

Solution.

Since $f_1(z) = (z - z_0)^{m_1} g_1(z)$, and $f_2(z) = (z - z_0)^{m_2} g_2(z)$, we have

$$f(z) = \frac{(z - z_0)^{m_1} g_1(z)}{(z - z_0)^{m_2} g_2(z)} = (z - z_0)^{m_1 - m_2} g(z),$$

where $g(z) = \dfrac{g_1(z)}{g_2(z)}$ is analytic at z_0 and $g(z_0) \neq 0$.

Solution (Cont.)

1. 1) When $m_1 = m_2$, since $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = g(z_0)$, z_0 is a removable singularity of $f(z)$.
2. 2) When $m_1 > m_2$, since $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (z - z_0)^{m_1 - m_2} g(z) = 0$, z_0 is a removable singularity of $f(z)$.
3. 3) When $m_1 < m_2$, since $f(z) = \dfrac{1}{(z - z_0)^{m_2 - m_1}} g(z)$, z_0 is a pole of order $m_2 - m_1$ of $f(z)$

Example 1.11

Find the singularities of $f(z) = \dfrac{\left(z^2-1\right)(z-2)^3}{(\sin \pi z)^3}$.
If they are poles, indicate their orders.

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Solution.

$f(z)$ is analytic everywhere except at $z = 0, \pm 1, \pm 2, \cdots$.
Because $(\sin \pi z)' = \cos \pi z$ is not zero at $z = 0, \pm 1, \pm 2, \cdots$,
they are zeros of order 1 of $\sin \pi z$,
they are poles of order 3 of $f(z)$ except $1, -1, 2$.

Solution (Cont.)

Because $z^2 - 1 = (z-1)(z+1)$, 1 and -1 are zeros of order 1, 1 and -1 are poles of order 2 of $f(z)$.

When $z = 2$,
$$\lim_{z\to 2} f(z) = \lim_{z\to 2} \frac{\left(z^2-1\right)(z-2)^3}{(\sin \pi z)^3} = \frac{3}{\pi^3},$$
so $z = 2$ is a removable singularity of $f(z)$.

4. **Summary and Thinking**

   Understand the concept and classification of isolated singularities; Master the characteristics of removable singularities, poles and essential singularities; Be familiar with the relationship between zeros and poles.

Question

Find the types of the isolated singularities of $f(z) = \dfrac{1}{z^3\left(e^{z^3} - 1\right)}$.

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Find the types of the isolated singularities of $f(z) = \dfrac{1}{z^3\left(e^{z^3} - 1\right)}$.

Solution.

$z = 0$ is a zero of order 6 of $z^3\left(e^{z^3} - 1\right)$,
which means that $z = 0$ is a pole order 6 of $f(z)$.

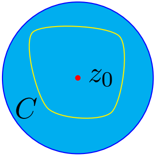
Outline

1. Isolated Singularities

2. Residues

1 Motivation

Suppose that z_0 is an isolated singularity of function $f(z)$, a deleted neighborhood z_0 is $0 < |z - z_0| < R$, and C is an any positive simple closed curve surrounding z_0 in this deleted neighborhood of z_0.



The Laurent series of $f(z)$ in $0 < |z - z_0| < R$ is written as

$$f(z) = \cdots + c_{-n}(z - z_0)^{-n} + \cdots + c_{-1}(z - z_0)^{-1} + \cdots + c_0 + c_1(z - z_0) + c_n(z - z_0)^n + \cdots$$

We want to find $\oint_C f(z) = ?$

$$\oint_C f(z)dz = \cdots + c_{-n} \underbrace{\oint_C (z-z_0)^{-n}\,\mathrm{d}z}_{=0} + \cdots + c_{-1} \underbrace{\oint_C (z-z_0)^{-1}\,\mathrm{d}z}_{=2\pi i} + \cdots$$

(Higher-order Derivatives) (Cauchy's integral formula)

$$+ c_0 \underbrace{\oint_C \mathrm{d}z}_{=0} + c_1 \underbrace{\oint_C (z-z_0)\,\mathrm{d}z}_{=0} + \cdots + c_n \underbrace{\oint_C (z-z_0)^n\,\mathrm{d}z}_{=0} + \cdots$$

(Cauchy's Theorem)

$$= 2\pi i\, \boxed{c_{-1}},$$

which is the coefficient of the negative power term $c_{-1}(z-z_0)^{-1}$ in the Laurent series.

That is, $c_{-1} = \dfrac{1}{2\pi i} \oint_C f(z)\,\mathrm{d}z = \boxed{\mathrm{Res}\left[f(z), z_0\right]}$

residue of $f(z)$ at z_0

Definition

If z_0 is an isolated singularity of $f(z)$, the integral of $f(z)$ along any simple closed curve C that is surrounding z_0 and in a deleted neighborhood of z_0 divided by $2\pi i$, i.e., $\dfrac{1}{2\pi i}\oint_C f(z)\,\mathrm{d}z$, is called the residue of $f(z)$ at z_0. We denote it by $\operatorname{Res}\left[f(z), z_0\right]$. (The residue of $f(z)$ at z_0 is the coefficient of the negative power term $c_{-1}(z-z_0)^{-1}$ in the Laurent series of $f(z)$ expanded in the annulus centered on z_0)

2 Compute Integral Using Residue

1) Residue Theorem

Theorem 2.1

$f(z)$ is analytic everywhere except for a finite number of singularities $z_1, z_2, \cdots, z_n$ in domain D, and C is a positive closed curve enclosing the singularities in D, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left[f(z), z_k\right].$$

Note:
1. $f(z)$ is analytic on C;
2. The residue theorem transforms the integral along a closed curve C into the residues of the integrand function at isolated singularities inside C.

Proof.

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z + \cdots + \oint_{C_n} f(z)\,\mathrm{d}z$$

Divide both sides by $2\pi i$,

$$\frac{1}{2\pi i}\oint_{C_1} f(z)\,\mathrm{d}z + \frac{1}{2\pi i}\oint_{C_2} f(z)\,\mathrm{d}z + \cdots + \frac{1}{2\pi i}\oint_{C_n} f(z)\,\mathrm{d}z$$

$$= \operatorname{Res}\left[f(z), z_1\right] + \operatorname{Res}\left[f(z), z_2\right] + \cdots + \operatorname{Res}\left[f(z), z_n\right]$$

$$= \sum_{k=1}^{n} \operatorname{Res}\left[f(z), z_k\right] \qquad \square$$

2) Calculation Method of Residues

Calculation Method of Residues

1. If z_0 is a removable singularity of $f(z)$, then $\text{Res}\left[f(z), z_0\right] = 0$.
2. If z_0 is an essential singularity of $f(z)$, then c_{-1} can be obtained from the Laurent series of $f(z)$.
3. If z_0 is a pole of $f(z)$, the calculation rules are given as follows:

1) If z_0 is a pole of order 1 of $f(z)$, then

$$\text{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} (z - z_0) f(z).$$

Calculation Method of Residues

2) If z_0 is a pole of order m of $f(z)$, then

$$\operatorname{Res}\left[f(z), z_0\right] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z-z_0)^m f(z)\right].$$

Proof.

$$\begin{aligned} f(z) =& c_{-m}(z-z_0)^{-m} + \cdots + c_{-2}(z-z_0)^{-2} \\ &+ c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \cdots \\ (z-z_0)^m f(z) =& c_{-m} + c_{-m+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{m-1} \\ &+ c_0(z-z_0)^m + c_1(z-z_0)^{m+1} + \cdots \end{aligned}$$

Calculation Method of Residues

Take the $m-1$ order derivative of both sides,

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$=(m-1)!c_{-1} + (\text{positive power terms of } z-z_0),$$

$$\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-z_0)^m f(z)\right] = (m-1)!c_{-1}.$$

So $\mathrm{Res}\left[f(z), z_0\right] = \dfrac{1}{(m-1)!}\lim_{z\to z_0}\dfrac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-z_0)^m f(z)\right].$ $\square$

Calculation Method of Residues

3) Let $f(z) = \dfrac{P(z)}{Q(z)}$, $P(z)$ and $Q(z)$ are analytic at z_0.
If $P(z_0) \neq 0$, $Q(z_0) = 0$ and $Q'(z_0) \neq 0$,
then z_0 is a pole of order 1 of $f(z)$ and $\operatorname{Res}\left[f(z), z_0\right] = \dfrac{P(z_0)}{Q'(z_0)}$.

Proof.
Because $Q(z_0) = 0$ and $Q'(z_0) \neq 0$,
z_0 is a zero of order 1 of $Q(z)$
and a pole of order 1 of $\dfrac{1}{Q(z)}$.

Calculation Method of Residues

Therefore $\dfrac{1}{Q(z)} = \dfrac{1}{z - z_0} \cdot \varphi(z)$, where $\varphi(z)$ is analytic at z_0 and $\varphi(z_0) \neq 0$.

We now have $f(z) = \dfrac{1}{z - z_0} \cdot \boxed{P(z)\varphi(z)}$.

Since $P(z_0)\varphi(z_0) \neq 0$ and it is analytic at z_0, z_0 is a pole of order 1 of $f(z)$.

$$\begin{aligned}\operatorname{Res}\left[f(z), z_0\right] &= \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \frac{P(z)}{\frac{Q(z) - Q(z_0)}{z - z_0}} \\ &= \frac{P(z_0)}{Q'(z_0)}.\end{aligned}$$

3 Example

Example 2.1

Find the residue of $f(z) = \dfrac{e^z}{z^n}$ *at* $z = 0$.

3 Example

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Find the residue of $f(z) = \dfrac{e^z}{z^n}$ *at* $z = 0$.

Solution.

Because $z = 0$ is a pole of order n of $f(z)$,

$$\begin{aligned}\operatorname{Res}\left[\frac{e^z}{z^n}, 0\right] &= \frac{1}{(n-1)!}\lim_{z\to 0}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(z^n \cdot \frac{e^z}{z^n}\right) \\ &= \frac{1}{(n-1)!}.\end{aligned}$$

Example 2.2

Find the residue of $f(z) = \dfrac{P(z)}{Q(z)} = \dfrac{z - \sin z}{z^6}$ *at* $z = 0$.

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Find the residue of $f(z) = \dfrac{P(z)}{Q(z)} = \dfrac{z - \sin z}{z^6}$ *at* $z = 0$.

Solution.

Because $P(0) = P'(0) = P''(0) = 0$ and $P'''(0) \neq 0$,
$z = 0$ is a zero of order 3 of $z - \sin z$,
$z = 0$ is a pole of order 3 of $f(z)$.
According to rule 3 of [Calculation Method of Residues],

$$\operatorname{Res}\left[f(z), 0\right] = \frac{1}{(3-1)!} \lim_{z \to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left[z^3 \cdot \frac{z - \sin z}{z^6} \right].$$

Solution (Cont.)

It is more convenient to use Laurent's expansion to find c_{-1}:

$$\frac{z-\sin z}{z^6} = \frac{1}{z^6}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right]$$

$$= \frac{z^{-3}}{3!} - \frac{z^{-1}}{5!} + \cdots ,$$

$$\operatorname{Res}\left[f(z), 0\right] = c_{-1} = -\frac{1}{5!}.$$

Note

1. Calculation rules should be applied flexibly in practice. If z_0 is a pole of order m and m is large, the derivative in rule 2) is difficult to calculate. For this case, the residues may can be calculated by directly expanding the Laurent series and finding c_{-1}.
2. When applying rule 2), it is generally not necessary to make m higher than the actual order of the pole. However, sometimes it makes the calculation easier when taking m higher than the actual order of the pole. For example, if we take $m = 6$ in Example 2.2, we have

$$\operatorname{Res}\left[f(z), 0\right] = \frac{1}{(6-1)!} \lim_{z \to 0} \frac{\mathrm{d}^5}{\mathrm{d}z^5} \left[z^6 \cdot \frac{z - \sin z}{z^6} \right] = -\frac{1}{5!}.$$

Example 2.3

Find the residue of $f(z) = \dfrac{e^z - 1}{z^5}$ *at* $z = 0$.

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Find the residue of $f(z) = \dfrac{e^z - 1}{z^5}$ *at* $z = 0$.

Solution.

Here, $z = 0$ is a pole of order 4 of $f(z)$.
The Laurent series of $f(z)$ in $0 < |z| < +\infty$ is

$$\frac{e^z - 1}{z^5} = \frac{1}{z^5}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots - 1\right)$$
$$= \frac{1}{z^4} + \frac{1}{2!z^3} + \frac{1}{3!z^2} + \frac{1}{4!z} + \frac{1}{5!} + \frac{z}{6!} + \cdots,$$

$$\operatorname{Res}\left[f(z), 0\right] = c_{-1} = \frac{1}{4!} = \frac{1}{24}.$$

Example 2.4

Compute the integral $\oint_C \frac{e^z}{z(z-1)^2}\,\mathrm{d}z$, *where* C *is circle* $|z| = 2$ *taken counterclockwise.*

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Compute the integral $\oint_C \frac{e^z}{z(z-1)^2}\,\mathrm{d}z$, *where* C *is circle* $|z| = 2$ *taken counterclockwise.*

Solution.

Because $z = 0$ is a pole of order 1, $z = 1$ is a pole of order 2,

$$\begin{aligned}
\operatorname{Res}\left[f(z), 0\right] &= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z} z \cdot \frac{e^z}{z(z-1)^2} \\
&= \lim_{z\to 0} \frac{e^z}{z(z-1)^2} = 1, \\
\operatorname{Res}\left[f(z), 1\right] &= \frac{1}{(2-1)!} \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 \frac{e^z}{z(z-1)^2}\right] \\
&= \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^z}{z}\right) = \lim_{z\to 1} \frac{e^z(z-1)}{z^2} = 0.
\end{aligned}$$

Solution (Cont.)

So $$\oint_C \frac{e^z}{z(z-1)^2}\,\mathrm{d}z$$
$$\begin{aligned}
&=2\pi i\Big\{\operatorname{Res}\big[f(z),0\big]+\operatorname{Res}\big[f(z),1\big]\Big\}\\
&=2\pi i(1+0)\\
&=2\pi i.
\end{aligned}$$

4. **Summary and Thinking**
   In this section, we learned the concept and calculation of residues and the residue theorem. We should focus on the general method of calculating residues, especially the method of calculating residues at poles, and using residue theorem to calculate complex integrals.

Question

Compute the integral $\oint_C \frac{\sin^2 z}{z^2(z-1)}\,\mathrm{d}z$, where C is circle $|z| = 2$ taken counterclockwise.

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Compute the integral $\oint_C \frac{\sin^2 z}{z^2(z-1)}\,\mathrm{d}z$, where C is circle $|z| = 2$ taken counterclockwise.

Answer

$\sin^2 1$.