Chapter 1 homework

1. Find the real part, imaginary part, module, argument and conjugate complex of the following complex numbers:

   (a) $2 + 2i$;

   Solution. $2 + 2i$.
   $\mathrm{Re}(2+2i) = 2$; $\mathrm{Im}(2+2i) = 2$;
   $|2+2i| = \sqrt{2^2+2^2} = 2\sqrt{2}$;
   $$\begin{aligned}\mathrm{Arg}(2+2i) &= \arg(2+2i) + 2k\pi \\ &= \arctan\frac{2}{2} + 2k\pi \\ &= \frac{\pi}{4} + 2k\pi, k = 0, \pm 1, \pm 2, \cdots;\end{aligned}$$
   $\overline{2+2i} = 2 - 2i$. ∎

   (b) $i + \dfrac{1-i}{1+i}$;

   Solution. $i + \dfrac{1-i}{1+i} = 1 - i$.
   $\mathrm{Re}(1-i) = 1$; $\mathrm{Im}(1-i) = -1$;
   $|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$;
   $$\begin{aligned}\mathrm{Arg}(1-i) &= \arg(1-i) + 2k\pi \\ &= \arctan -\frac{1}{1} + 2k\pi \\ &= -\frac{\pi}{4} + 2k\pi, k = 0, \pm 1, \pm 2, \cdots;\end{aligned}$$
   $\overline{1-i} = 1 + 1i$. ∎

   (c) $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^{100} + \left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{100}$;

Solution. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{100}+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{100}$

$$=\left(1e^{i\frac{\pi}{3}}\right)^{100}+\left(1e^{-i\frac{\pi}{3}}\right)^{100}$$
$$=\left(\cos\left(100\frac{\pi}{3}\right)+i\sin\left(100\frac{\pi}{3}\right)\right)+\left(\cos\left(-100\frac{\pi}{3}\right)+i\sin\left(-100\frac{\pi}{3}\right)\right)$$
$$=2\cos\left(\frac{100\pi}{3}\right)$$
$$=2\cos\left(32\pi+\frac{4\pi}{3}\right)$$
$$=2\cos\left(\frac{2\pi}{3}\right).$$

$\mathrm{Re}\left(2\cos\left(\frac{2\pi}{3}\right)\right)=2\cos\left(\frac{2\pi}{3}\right)$; $\mathrm{Im}\left(2\cos\left(\frac{2\pi}{3}\right)\right)=0$;

$\left|2\cos\left(\frac{2\pi}{3}\right)\right|=2\cos\left(\frac{2\pi}{3}\right)$;

$\mathrm{Arg}\left(2\cos\left(\frac{2\pi}{3}\right)\right)=2k\pi, k=0,\pm1,\pm2,\cdots$;

$\overline{2\cos\left(\frac{2\pi}{3}\right)}=2\cos\left(\frac{2\pi}{3}\right)$. ∎

(d) $i^{10}-4i^{15}+i$.

Solution. $i^{10}-4i^{15}+i$

$$=-1-4\times(-i)+i$$
$$=-1+5i.$$

$\mathrm{Re}(-1+5i)=-1$; $\mathrm{Im}(-1+5i)=5$;

$|-1+5i|=\sqrt{1+25}=\sqrt{26}$;

$\mathrm{Arg}(-1+5i)=\arg(-1+5i)+2k\pi=\arctan\left(-\frac{5}{1}\right)+\pi+2k\pi, k=0,\pm1,\pm2,\cdots$;

$\overline{-1+5i}=-1-5i$. ∎

2. Convert the following complex numbers into triangular expressions and exponential expressions:

(a) $-6-4i$;

Solution. $-6-4i$ ∎

$$=2\sqrt{13}\left(\cos\left(\arctan\frac{2}{3}-\pi\right)+i\sin\left(\arctan\frac{2}{3}-\pi\right)\right)$$
$$=2\sqrt{13}e^{i\left(\arctan\frac{2}{3}-\pi\right)}$$

(b) $1+i\tan\theta$;

Solution.

$$
\begin{aligned}
&1+i\tan\theta\\
=&\sqrt{1+\tan^2\theta}(\cos(\arctan\tan\theta)+i\sin(\arctan\tan\theta))\\
=&\sqrt{1+\tan^2\theta}\times\begin{cases}\cos\theta+i\sin\theta, & 0<\theta<\frac{\pi}{2}\cup-\frac{\pi}{2}<\theta<0\\ \cos(\theta+\pi)+i\sin(\theta+\pi), & \frac{\pi}{2}<\theta<\pi\\ \cos(\theta-\pi)+i\sin(\theta-\pi), & -\pi<\theta<-\frac{\pi}{2}\\ i, & \theta=\frac{\pi}{2}\\ -i, & \theta=-\frac{\pi}{2}\\ -1, & \theta=-\pi\\ 1, & \theta=\pi\end{cases}
\end{aligned}
$$

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(c) $1-\cos\varphi+i\sin\varphi, \quad 0\le\varphi\le\pi;$

Solution. $1-\cos\varphi+i\sin\varphi$ ■

$$
\begin{aligned}
=&2\sin^2\frac{\varphi}{2}+i2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\\
=&2\sin\frac{\varphi}{2}\left(\sin\frac{\varphi}{2}+i\cos\frac{\varphi}{2}\right)\\
=&2\sin\frac{\varphi}{2}\left(\cos\frac{\pi-\varphi}{2}+i\sin\frac{\pi-\varphi}{2}\right)\\
=&2\sin\frac{\varphi}{2}e^{i\left(\frac{\pi-\varphi}{2}\right)},0\le\varphi\le\pi
\end{aligned}
$$

(d) $\dfrac{(\cos 3\varphi+i\sin 3\varphi)^3}{(\cos 2\varphi-i\sin 2\varphi)^{10}}.$

Solution. $\dfrac{(\cos 3\varphi+i\sin 3\varphi)^3}{(\cos 2\varphi-i\sin 2\varphi)^{10}}$ ■

$$
\begin{aligned}
=&\frac{\left(e^{i3\varphi}\right)^3}{\left(e^{-i2\varphi}\right)^{10}}\\
=&\frac{e^{i9\varphi}}{e^{-i20\varphi}}\\
=&e^{i9\varphi}e^{i20\varphi}\\
=&e^{i29\varphi}
\end{aligned}
$$

3. Point out the relationship between complex z and complex iz.

   Solution. $|z|=|iz|$, $\operatorname{Arg} iz=\operatorname{Arg} z+\frac{\pi}{2}$. ■

4. Find the values of the following formulas:

(a) $\left(\dfrac{1+i}{1-i}\right)^8$;

Solution.

$$\begin{aligned}&\left(\frac{1+i}{1-i}\right)^8\\=&(i)^8\\=&1\end{aligned}$$

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(b) $(\sqrt{3}+i)^4$;

Solution.

$$\begin{aligned}&(\sqrt{3}+i)^4\\=&(2e^{i\frac{\pi}{6}})^4\\=&2^4e^{i\frac{\pi}{6}4}\\=&16e^{i\frac{2\pi}{3}}\\=&16\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\\=&16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\=&-8+8\sqrt{3}i\end{aligned}$$

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(c) $\sqrt[6]{-1}$;

Solution.

$$\begin{aligned}&\sqrt[6]{-1}\\=&\sqrt[6]{e^{i\pi}}\\=&\cos\left(\frac{\pi+2k\pi}{6}\right)+i\sin\left(\frac{\pi+2k\pi}{6}\right), k=0,1,2,3,4,5\end{aligned}$$

$k=0:\dfrac{\sqrt{3}}{2}+\dfrac{i}{2}$
$k=1:i$
$k=2:-\dfrac{\sqrt{3}}{2}+\dfrac{i}{2}$
$k=3:-\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}$
$k=4:-i$
$k=5:\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}$

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(d) $(1-i)^{\frac{1}{3}}$.

Solution. $(1-i)^{\frac{1}{3}}$

$$=\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{\frac{1}{3}}$$

$$=\sqrt[6]{2}e^{-i\left(\frac{\pi}{4}+2k\pi\right)\times\frac{1}{3}},k=0,1,2$$

$k=0:\sqrt[6]{2}e^{-i\frac{\pi}{12}}$
$k=1:\sqrt[6]{2}e^{-i\frac{7\pi}{12}}$
$k=2:\sqrt[6]{2}e^{-i\frac{5\pi}{4}}$ ■

5. Convert the following coordinate transformation formula into complex number form:

   (a) Translation formula $\begin{cases} x=x_1+a_1 \\ y=y_1+b_1 \end{cases}$;

   (b) Rotation formula $\begin{cases} x=x_1\cos\alpha-y_1\sin\alpha \\ y=x_1\cos\alpha+y_1\sin\alpha \end{cases}$.

   Solution. Let $A=a+ib$, $z=x+iy$, $z_1=x_1+iy_1$.

   (a) $z=z_1+A$

   (b) $z=z_1(\cos\alpha+i\sin\alpha)=z_1e^{i\alpha}$

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6. Let ω be the n-th power root of 1, but $\omega\neq1$, prove that ω satisfies equation $1+z+z^2+\cdots+z^{n-1}=0$.

   Proof. Since ω is the n-th root of 1, we have $\omega^n=1$.
   Then, we compute

   $$(1-\omega)\left(1+\omega+\omega^2+\cdots+\omega^{n-2}+\omega^{n-1}\right)$$
   $$=\left(1+\omega+\omega^2+\cdots+\omega^{n-2}+\omega^{n-1}\right)-\left(\omega+\omega^2+\cdots+\omega^{n-1}+\underbrace{\omega^n}_{=1}\right)$$
   $$=\left(1+\omega+\omega^2+\cdots+\omega^{n-1}\right)-\left(\omega+\omega^2+\cdots+\omega^{n-1}+1\right)$$
   $$=0$$

   $\because\omega\neq1$
   $\therefore 1-\omega\neq0$
   $\therefore 1+\omega+\omega^2+\cdots+\omega^{n-1}=0$ □

7. Find the curve represented by the following equation (where t is the real parameter):

   (a) $z=t+\dfrac{1}{t}i\quad(t\neq0)$;

   (b) $z=a+re^{it}\quad$ ($r>0$ is a real constant and a is a complex number).

Solution.

(a) Let $z = x + iy$
$\because z = t + \frac{1}{t}i$
$\therefore \begin{cases} x = t \\ y = \frac{1}{t} \end{cases}$
$\therefore xy = 1$ (Hyperbolic equation)

(b) Let $z = x + iy$
$\because z = a + r^{it}$
$= \mathrm{Re}(a) + i\,\mathrm{Im}(a) + r(\cos t + i \sin t)$
$= \mathrm{Re}(a) + r\cos t + i(\mathrm{Im}(a) + r \sin t)$
$\therefore \begin{cases} x = \mathrm{Re}(a) + r\cos t \\ y = \mathrm{Im}(a) + r\sin t \end{cases}$
$\therefore (x - \mathrm{Re}(a))^2 + (y - \mathrm{Im}(a))^2 = r^2$ (A circle with radius r centered at a)

$\blacksquare$

8. Find the curve represented by the following equation:

(a) $|z - 2| = 4$;

(b) $\arg(z - i) = \frac{\pi}{4}$;

(c) $z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} = b\overline{b}$ (a and b are complex constants).

Solution.

(a) Let $z = x + iy$
$\because |z - 2| = 4$
$\therefore |x + iy - 2| = 4$
$\therefore \sqrt{(x-2)^2 + y^2} = 4$
$\therefore (x-2)^2 + y^2 = 4^2$ (A circle with radius 4 centered at 2)

(b) Let $z = x + iy$
$\because \arg(z - i) = \frac{\pi}{4}$
$\therefore \frac{y-1}{x} = \tan\frac{\pi}{4} = 1$
$\therefore y = x + 1, x > 0$ (Ray with starting point i)

(c) Let $z = x + iy$
$\because z\overline{z} - \overline{a}z - a\overline{z} + a\overline{a} = b\overline{b}$
$\therefore z\overline{z} - (\overline{a}z + a\overline{z}) + a\overline{a} = b\overline{b}$
$\therefore z\overline{z} - (\overline{a}z + \overline{\overline{a}z}) + a\overline{a} = b\overline{b}$
$\therefore x^2 + y^2 - 2\,\mathrm{Re}(\overline{a}z) + \mathrm{Re}^2(a) + \mathrm{Im}^2(a) = \mathrm{Re}^2(b) + \mathrm{Im}^2(b)$
$\therefore x^2 + y^2 - 2(\mathrm{Re}(a)x + \mathrm{Im}(a)y) + \mathrm{Re}^2(a) + \mathrm{Im}^2(a) = \mathrm{Re}^2(b) + \mathrm{Im}^2(b)$

$\therefore \underbrace{[x^2 - 2\operatorname{Re}(a)x + \operatorname{Re}^2(a)]}_{(x-\operatorname{Re}(a))^2} + \underbrace{[y^2 - 2\operatorname{Im}(a)y + \operatorname{Im}^2(a)]}_{(y-\operatorname{Im}(a))^2} = \operatorname{Re}^2(b)+\operatorname{Im}^2(b)$

$\therefore (x - \operatorname{Re}(a))^2 + (y - \operatorname{Im}(a))^2 = \operatorname{Re}^2(b) + \operatorname{Im}^2(b)$

$\therefore |z - a|^2 = |b|^2$ (A circle with radius $|b|$ centered at a)

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9. Draw the trajectory graph of the point z that satisfies the following inequalities, and indicate whether it is bounded or unbounded, simply connected or multi-connected.

   (a) $|z - 3| > 4$;

   (b) $\left|\dfrac{z-3}{z-2}\right| \geq 1$;

   (c) $|z - 2| - |z + 2| > 3$;

   (d) $z\overline{z} - (2 + i)z - (2 - i)\overline{z} \leq 4$.

   Solution. Let $z = x + iy$.

   (a) The outside of circle $(x - 3)^2 + y^2 = 4$, unbounded, and multi-connected.

   (b) $\because \left|\dfrac{z-3}{z-2}\right| \geq 1$

   $\therefore |z - 3|^2 \geq |z - 2|^2$

   $\therefore (x - 3)^2 + y^2 \geq (x - 2)^2 + y^2$

   $\therefore (x - 3)^2 \geq (x - 2)^2$

   $\therefore x^2 - 6x + 9 \geq x^2 - 4x + 4$

   $\therefore x \leq \dfrac{5}{2}$

   Left half plane bounded by line $x = \dfrac{5}{2}$ and including the line.

   (c) $\because |z - 2| - |z + 2| > 3$

   $\therefore |z - 2| > 3 + |z + 2|$

   $\therefore (x - 2)^2 + y^2 > \left(3 + \sqrt{(x + 2)^2 + y^2}\right)^2$

   $\therefore (x - 2)^2 + y^2 > 9 + 6\sqrt{(x + 2)^2 + y^2} + (x + 2)^2 + y^2$

   $\therefore -4x > 9 + 6\sqrt{(x + 2)^2 + y^2} + 4x$

   $\therefore -8x - 9 > 6\sqrt{(x + 2)^2 + y^2}$

   $\therefore \dfrac{-8x - 9}{6} > \sqrt{(x + 2)^2 + y^2}$

   $$\therefore \begin{cases} -8x - 9 > 0 \\ \left(\dfrac{-8x - 9}{6}\right)^2 > (x + 2)^2 + y^2 \end{cases}$$

   $$\therefore \begin{cases} x < -\dfrac{9}{8} \\ \dfrac{4}{9}x^2 - \dfrac{4}{7}y^2 > 1 \end{cases}$$

Inside the left branch of hyperbola $\frac{4}{9}x^2 - \frac{4}{7}y^2 > 1$, unbounded, simply connected.

(d) According to homework 8(d), the inner area of a circle with a center of $2 - i$ and a radius of 3, bounded, simply connected.

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10. Function $w = \frac{1}{z}$ changes the following curve on the z-plane to what curve on the w-plane?

(a) $x^2 + y^2 = 3$;

(b) $y = -x$.

Solution. Let $z = x + iy$, $\omega = u + iv$

$\because \omega = \frac{1}{z} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$

$\therefore \begin{cases} u = \dfrac{x}{x^2 + y^2} \\ v = \dfrac{-y}{x^2 + y^2} \end{cases}$

(a) $u^2 + v^2 = \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{3}{9} = \frac{1}{3}$

Change the circle $x^2 + y^2 = 3$ in the z-plane to the circle $u^2 + v^2 = \frac{1}{3}$ in the ω-plane.

(b) $u = \frac{x}{x^2 + y^2} = \frac{-y}{x^2 + y^2} = v$

Change the line $y = -x$ in the z-plane to the line $u = v$ in the ω-plane.

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11. Let

$$f(z) = \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) \quad (z \neq 0)$$

and prove that when $z \to 0$, the limit of $f(z)$ does not exist.

Proof. Let $z = x + iy$,

$f(z) = \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) = \frac{2xy}{x^2 + y^2}$

$$\begin{aligned}
&\lim_{\substack{z\to 0\\ y=kx}} f(z)\\
=&\lim_{\substack{z\to 0\\ y=kx}} \frac{2xy}{x^2+y^2}\\
=&\lim_{\substack{z\to 0\\ y=kx}} \frac{2kx^2}{x^2+k^2x^2}\\
=&\lim_{\substack{z\to 0\\ y=kx}} \frac{2k}{1+k^2}
\end{aligned}$$

The limit value is related to the path approaching 0, so the limit does not exist. $\square$