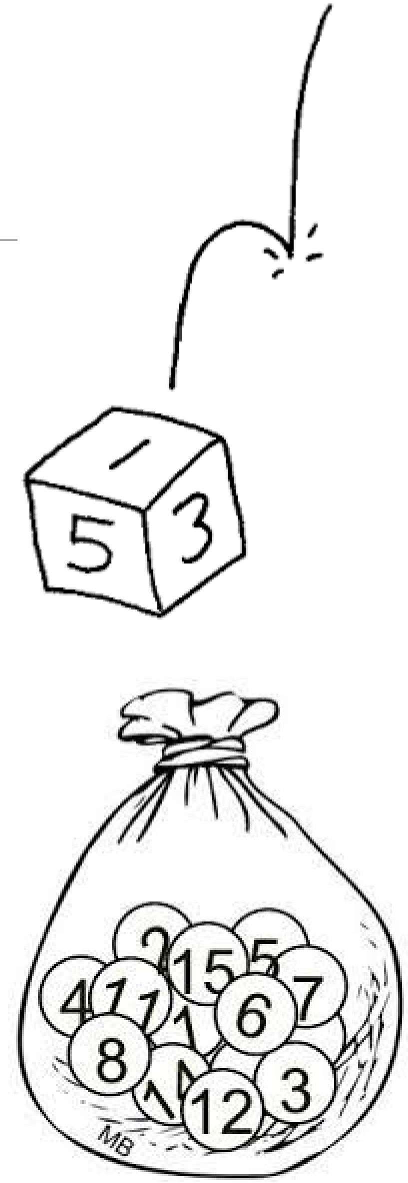
Lecture 11

- Expectation of R.V.s
- Variance of R.V.s



Expectation of discrete RVs	Expectation of continuous RVs
$E(X) = \sum_{k=1}^{\infty} x_k p_k$	$E(X) = \int_{-\infty}^{+\infty} x\, f(x) dx$
$E[g(X)] = \sum_{k=1}^{\infty} g(x_k)\, p_k$	$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) dx$
$E[g(X,Y)] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g\,(x_i, y_j)\, p_{ij}$	$E[g(X,Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g\,(x\,, y) f(x,y) dxdy$
$E(X) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i p_{ij}$	$E(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x\, f(x,y) dx\, dy$
$E(Y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y_j p_{ij}$	$E(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y\, f(x,y) dx\, dy$

Ex. Let the joint PMF of (X, Y) be

$Y \backslash X$	1	2
1	0.4	0.2
2	0.3	0.1

Find the expectation of $Z_1 = XY^2$, $Z_2 = X + Y$.

$Y \backslash X$	1	2
1	0.4	0.2
2	0.3	0.1

Sol. The PMF of $(X,Y), Z_1$ and Z_2 are

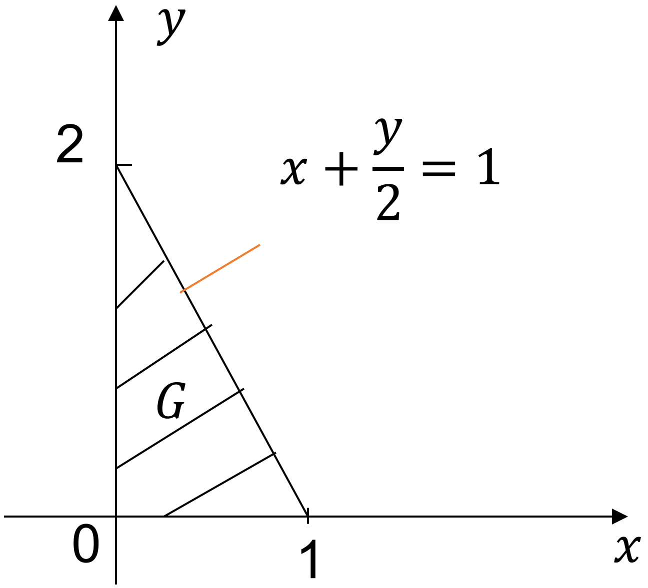
(X,Y)	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
XY^2	1	4	2	8
$X+Y$	2	3	3	4
p_k	0.4	0.3	0.2	0.1

$$E(Z_1) = E(XY^2) = 1 \times 0.4 + 4 \times 0.3 + 2 \times 0.2 + 8 \times 0.1 = 2.8$$

$$E(Z_2) = E(X+Y) = 2 \times 0.4 + 3 \times 0.3 + 3 \times 0.2 + 4 \times 0.1 = 2.7$$

$$\boxed{E(Z) = E[g(X,Y)] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(x_i, y_j)\, p_{ij}}$$

Ex. R.V.s X, Y follow uniform distribution in the defined region G (as shown in the figure). Find the expected values of X, Y and XY.

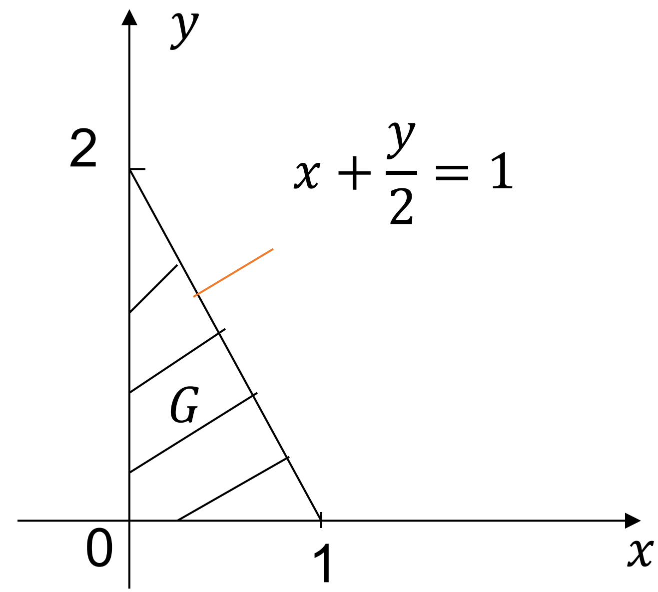


$$E(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x\, f(x, y) dx\, dy$$

$$E(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y\, f(x, y) dx\, dy$$

$$E[g(X, Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f(x, y) dx dy$$

Solution 1:



$$f(x, y) = \begin{cases} 1, & (x, y) \in \mathrm{G} \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy = \int_0^{2(1-x)} 1\, dy$$

$$= \begin{cases} 2(1-x) & , \quad 0 \le x \le 1 \\ 0, & \quad \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_0^1 2x(1-x) dx = \frac{1}{3}$$

Solution 1:

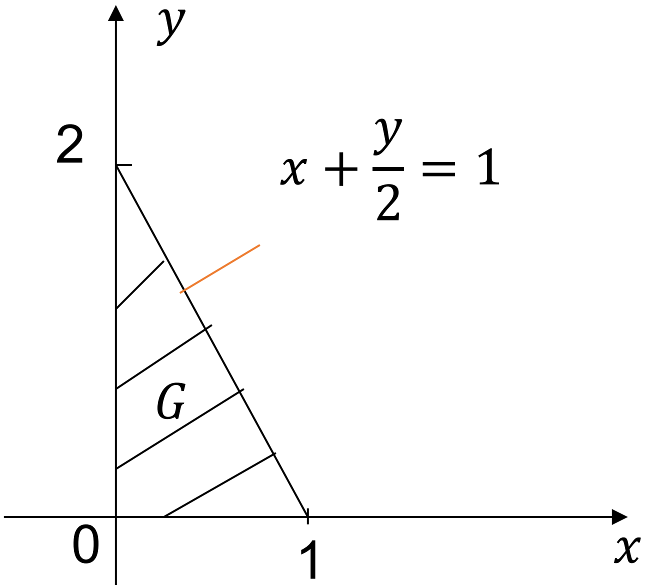
$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \begin{cases} 2(1-x) & , \ 0 \le x \le 1 \\ 0 & , \text{ otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \boxed{\int_0^1 2x(1-x)dx} = \frac{1}{3}$$

Solution 2:

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f(x,y) dx dy = \int_0^1 dx \int_0^{2(1-x)} x dy$$

$$= \boxed{\int_0^1 2x(1-x)dx} = \frac{1}{3}$$



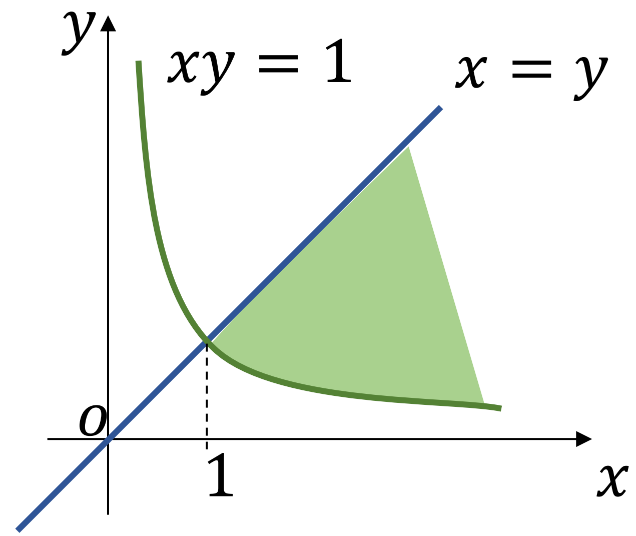
Similarly

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} yf(x,y)dxdy = \int_0^1 dx \int_0^{2(1-x)} ydy = \frac{2}{3}$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)dxdy = \int_0^1 dx \int_0^{2(1-x)} xydy = \frac{1}{6}$$

Ex. For R.V.s (X, Y), the joint PDF is given by

$$f(x,y) = \begin{cases} \dfrac{3}{2x^3y^2}, & \dfrac{1}{x} < y < x, x > 1 \\ 0, & \text{otherwise} \end{cases}$$

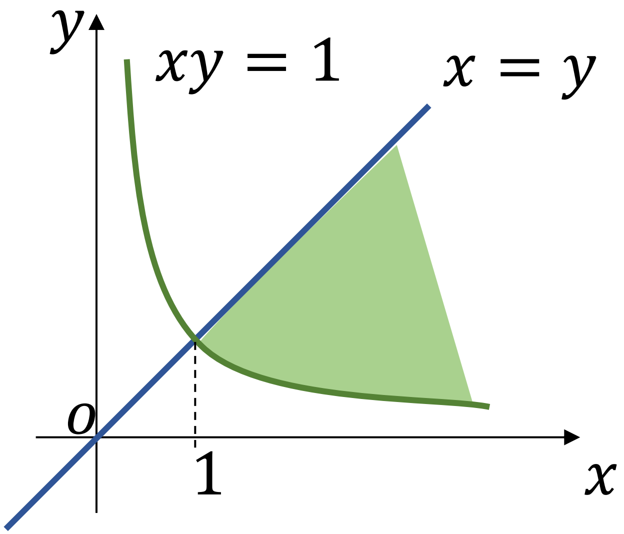


Find the following expectations $E(Y)$, $E(1/XY)$.

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, f(x,y)dx\, dy$$

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\,(x\,,y)f(x,y)dxdy$$

$$f(x,y)=\begin{cases}\dfrac{3}{2x^3y^2}, & \dfrac{1}{x}<y<x, x>1\\ 0, & \text{otherwise}\end{cases}$$



Sol.

$$E(Y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} yf(x,y)dxdy \quad =\int_{1}^{+\infty} dx\int_{1/x}^{x}\frac{3}{2x^3y}dy=\frac{3}{4}$$

$$E\left(\frac{1}{XY}\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{xy}f(x,y)dxdy \quad =\int_{1}^{+\infty} dx\int_{1/x}^{x}\frac{3}{2x^4y^3}dy=\frac{3}{5}$$

Important properties of expectation

1. Expectation of a constant.

$$E(C) = C$$

2. Linearity.

$$E(CX) = C \cdot E(X)$$

3. Expectation of a sum.

$$E(X \pm Y) = E(X) \pm E(Y)$$

4. Expectation of **independent** R.V.s.

$$E(XY) = E(X)E(Y)$$

Note:

- 3 and 4 can be extended to multiple R.V.s.
- Can be proved from definition of expectation. (Try it!)

Ex. If X, Y are independent, and follow exponential distribution.

$$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, & x > 0, \\ 0, & x \leq 0. \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{\beta} e^{-\frac{y}{\beta}}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

Find $E[e^{-(cX+dY)}]$, $(c > 0, d > 0)$. Hint: $\int e^{ax} dx = \frac{1}{a} e^{ax}$

$f_X(x) = \begin{cases} \frac{1}{\alpha} e^{-\frac{x}{\alpha}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$ $f_Y(y) = \begin{cases} \frac{1}{\beta} e^{-\frac{y}{\beta}}, & y > 0, \\ 0, & y \leq 0. \end{cases}$ Find $E[e^{-(cX+dY)}]$.

Sol. Since X and Y are independent,

$$E\left[e^{-(cX+dY)}\right] = \int\int e^{-(cX+dY)} f(x,y)dxdy$$

$$= \int e^{-cx} f(x)dx \cdot \int e^{-dy} f(y)dy = E(e^{-cX})E(e^{-dY})$$

$$= \int_0^{+\infty} e^{-cx} \frac{1}{\alpha} e^{-\frac{x}{\alpha}} dx \cdot \int_0^{+\infty} e^{-dy} \frac{1}{\beta} e^{-\frac{y}{\beta}} dy$$

$$= \frac{1}{(c\alpha + 1)(d\beta + 1)}$$

Alternative Sol. $E[g(X,Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f(x,y) dxdy$

Ex. A shuttle bus at an airport carries 20 passengers and departs from the airport. There are 10 stations where passengers can get off. If no passengers get off at a station, the bus will not stop. Assume that it is equally possible for each passenger to get off at each station, and the station at which the passengers get off is independent of each other. Let X denotes the number of stoppings, find $E(X)$.

Define variable

$$X_i = \begin{cases} 0, & \text{bus passes the } i-\text{th station}, \\ 1, & \text{bus stops at the } i-\text{th station}, \end{cases} \qquad i = 1, 2, \dots, 10$$

Then $X = X_1 + X_2 + \cdots + X_{10}$.

Sol. Define variable

$$X_i = \begin{cases} 0, & \text{bus passes the } i-\text{th station}, \\ 1, & \text{bus stops at the } i-\text{th station}, \end{cases} \quad i = 1,2,\ldots,10$$

Then $X = X_1 + X_2 + \cdots + X_{10}$.

(Decomposing one R.V. into multiple R.V.s)

Since the probability of getting off is independent, the probability of passing and stopping at the i-th station is

$$P(X_i = 0) = \left(\frac{9}{10}\right)^{20}, P(X_i = 1) = 1 - \left(\frac{9}{10}\right)^{20}$$

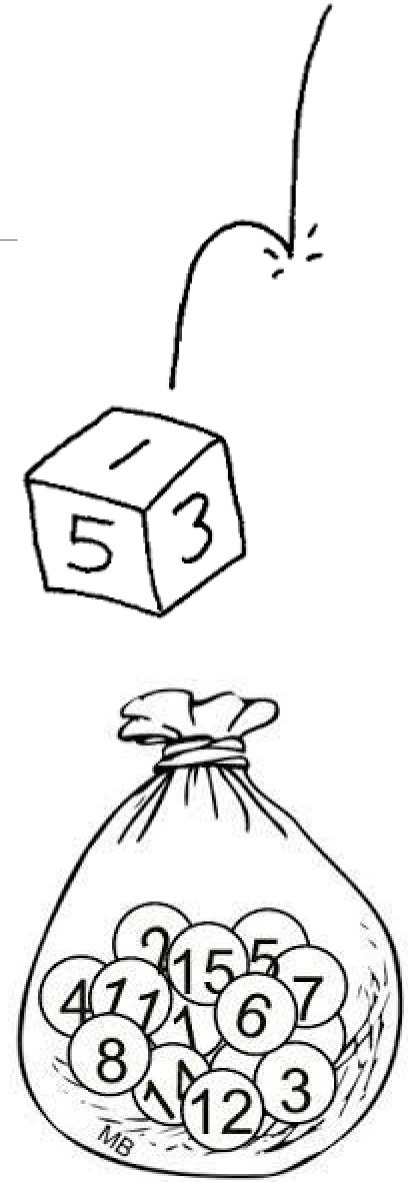
$$E(X_i) = 1 - 0.9^{20}, \qquad i = 1,2,\ldots,10$$

$$E(X) = E(X_1 + \cdots + X_{10}) = 10 \times (1 - 0.9^{20}) \approx 8.784$$

Lecture 11

- Expectation of R.V.s

- Variance of R.V.s



Expectation of Daily Temperature

Shanghai

$E(\text{Temperature}) = 17.1\ °\text{C}$

Kunming

$E(\text{Temperature}) = 15.5\ °\text{C}$

Climate data for Shanghai (normals 1981–2010, extremes 1951–present)													
Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Year
Record high °C (°F)	22.1 (71.8)	27.0 (80.6)	29.6 (85.3)	34.3 (93.7)	36.4 (97.5)	37.5 (99.5)	39.2 (102.6)	39.9 (103.8)	38.2 (100.8)	34.0 (93.2)	28.7 (83.7)	23.4 (74.1)	39.9 (103.8)
Average high °C (°F)	8.1 (46.6)	10.1 (50.2)	13.8 (56.8)	19.5 (67.1)	24.8 (76.6)	27.8 (82.0)	32.2 (90.0)	31.5 (88.7)	27.9 (82.2)	22.9 (73.2)	17.3 (63.1)	11.1 (52.0)	20.6 (69.0)
Daily mean °C (°F)	4.8 (40.6)	6.6 (43.9)	10.0 (50.0)	15.3 (59.5)	20.7 (69.3)	24.4 (75.9)	28.6 (83.5)	28.3 (82.9)	24.9 (76.8)	19.7 (67.5)	13.7 (56.7)	7.6 (45.7)	17.1 (62.7)
Average low °C (°F)	2.1 (35.8)	3.7 (38.7)	6.9 (44.4)	11.9 (53.4)	17.3 (63.1)	21.7 (71.1)	25.8 (78.4)	25.8 (78.4)	22.4 (72.3)	16.8 (62.2)	10.6 (51.1)	4.7 (40.5)	14.1 (57.5)
Record low °C (°F)	−10.1 (13.8)	−7.9 (17.8)	−5.4 (22.3)	−0.5 (31.1)	6.9 (44.4)	12.3 (54.1)	16.3 (61.3)	18.8 (65.8)	10.8 (51.4)	1.7 (35.1)	−4.2 (24.4)	−8.5 (16.7)	−10.1 (13.8)
Average precipitation mm (inches)	74.4 (2.93)	59.1 (2.33)	93.8 (3.69)	74.2 (2.92)	84.5 (3.33)	181.8 (7.16)	145.7 (5.74)	213.7 (8.41)	87.1 (3.43)	55.6 (2.19)	52.3 (2.06)	43.9 (1.73)	1,166.1 (45.91)
Average precipitation days (≥ 0.1 mm)	9.9	9.2	12.4	11.2	10.4	12.7	11.4	12.3	9.1	6.9	7.6	7.7	120.8
Average relative humidity (%)	74	73	73	72	72	79	77	78	75	72	72	71	74
Mean monthly sunshine hours	114.3	119.9	128.5	148.5	169.8	130.9	190.8	185.7	167.5	161.4	131.1	127.4	1,775.8

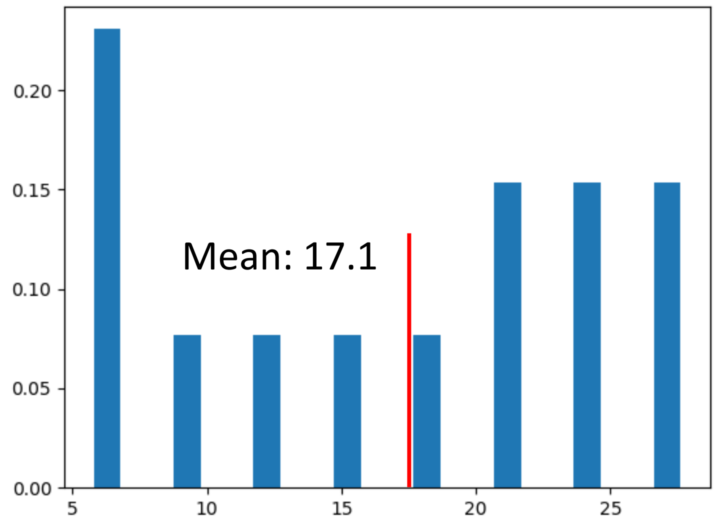
https://en.wikipedia.org/wiki/Shanghai

Climate data for Kunming (1981–2010 normals)													
Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Year
Record high °C (°F)	23.3 (73.9)	25.6 (78.1)	28.2 (82.8)	30.4 (86.7)	31.3 (88.3)	30.0 (86.0)	30.3 (86.5)	30.3 (86.5)	30.4 (86.7)	27.4 (81.3)	25.3 (77.5)	25.1 (77.2)	31.3 (88.3)
Average high °C (°F)	15.9 (60.6)	17.9 (64.2)	21.1 (70.0)	24.0 (75.2)	24.6 (76.3)	24.6 (76.3)	24.4 (75.9)	24.7 (76.5)	23.1 (73.6)	20.9 (69.6)	18.0 (64.4)	15.5 (59.9)	21.2 (70.2)
Daily mean °C (°F)	8.9 (48.0)	10.9 (51.6)	14.1 (57.4)	17.3 (63.1)	19.2 (66.6)	20.3 (68.5)	20.2 (68.4)	19.9 (67.8)	18.3 (64.9)	16.0 (60.8)	12.1 (53.8)	9.0 (48.2)	15.5 (59.9)
Average low °C (°F)	3.5 (38.3)	5.0 (41.0)	8.0 (46.4)	11.4 (52.5)	14.7 (58.5)	17.0 (62.6)	17.3 (63.1)	16.8 (62.2)	15.2 (59.4)	12.7 (54.9)	7.9 (46.2)	4.2 (39.6)	11.1 (52.1)
Record low °C (°F)	−2.8 (27.0)	−1.6 (29.1)	−5.2 (22.6)	2.0 (35.6)	5.5 (41.9)	10.8 (51.4)	11.6 (52.9)	11.5 (52.7)	6.2 (43.2)	4.0 (39.2)	−0.8 (30.6)	−7.8 (18.0)	−7.8 (18.0)
Average precipitation mm (inches)	15.8 (0.62)	14.6 (0.57)	17.6 (0.69)	25.2 (0.99)	85.5 (3.37)	170.4 (6.71)	200.2 (7.88)	203.9 (8.03)	113.9 (4.48)	81.7 (3.22)	36.7 (1.44)	13.6 (0.54)	979.1 (38.54)
Average precipitation days (≥ 0.1 mm)	4.4	4.6	5.5	6.8	12.2	17.4	20.3	19.3	15.8	13.0	7.3	3.8	130.4
Average relative humidity (%)	66	60	56	56	66	77	81	80	79	79	75	72	71
Mean monthly sunshine hours	224.5	219.6	255.4	244.8	212.2	135.0	124.3	144.9	123.5	143.7	169.8	200.0	2,197.7

https://en.wikipedia.org/wiki/Kunming

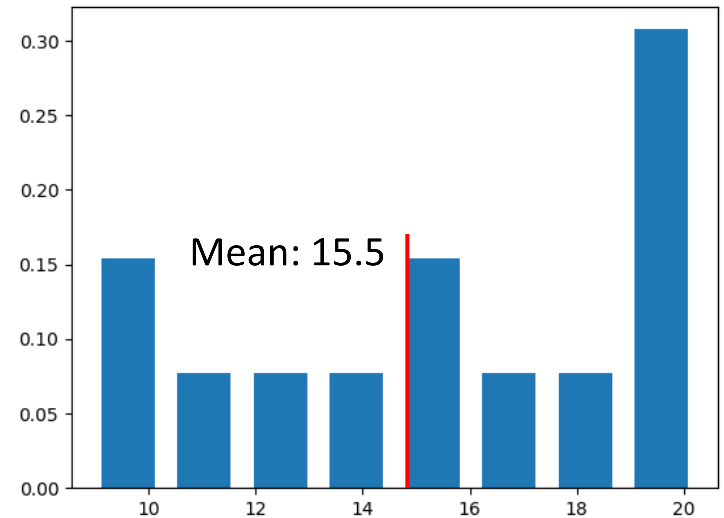
Monthly weather:
4.8, 6.6, 10, 15.3, 20.7, 24.4,
28.6, 28.3, 24.9, 19.7, 13.7, 7.6

Monthly weather:
8.9, 10.9, 14.1, 17.3, 19.2, 20.3,
20.2, 19.9, 18.3, 16.0, 12.1, 9.0



28.6 - 4.8 = 23.8

The PMF for monthly temperature in Shanghai.



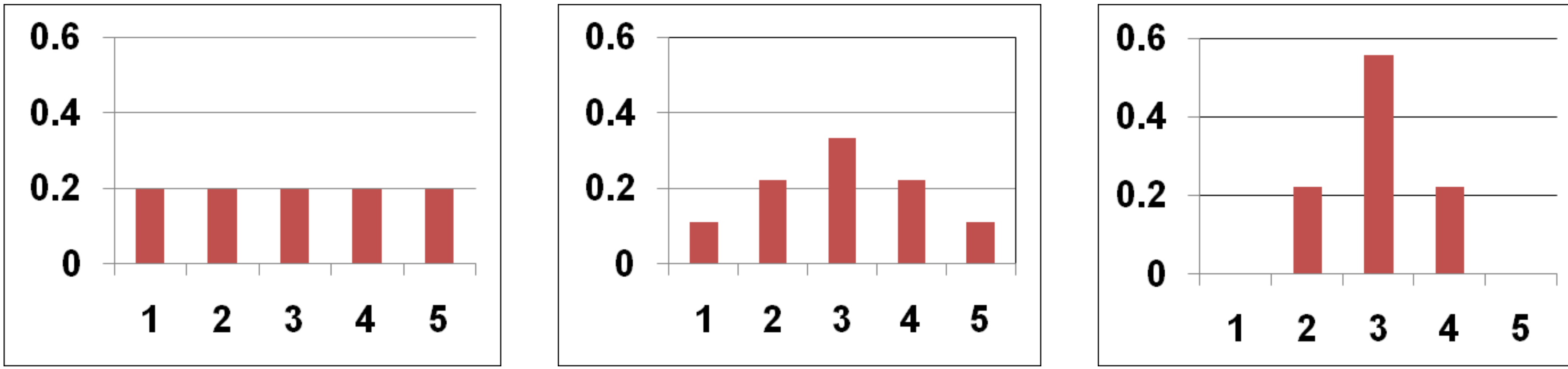
17.3 - 3.5 = 13.8

The PMF for monthly temperature in Kunming.

Expectation is not sufficient!

Variance = “spread”

Consider the following three distributions (PMFs):



- Expectation: $E[X] = 3$ for all distributions.
- But the “spread” in the distributions is different!
- Variance, $D[X]$: a formal quantification of “spread”.
- Variance measures the “stability” of a R.V. !

Variance

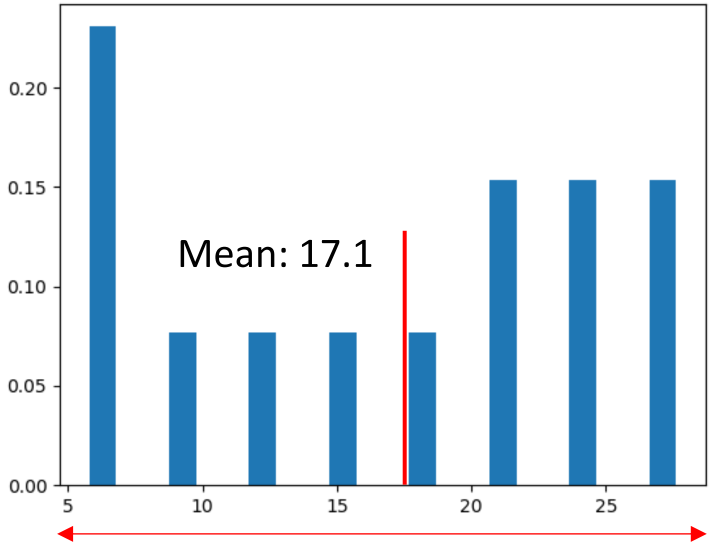
The variance of a random variable X with mean $E[X] = \mu$ is

$$D(X) = E[(X - \mu)^2]$$

- Also written as: VAR$[X]$, $\sigma^2(X)$, σ_X^2, $E[(X - E[X])^2]$.
- Also called: the 2nd order central **moment**.
- Note: $D(X) \geq 0$. When $D(X) = 0$?
- $\sigma(X) = \sqrt{D(X)}$ is called the standard deviation.

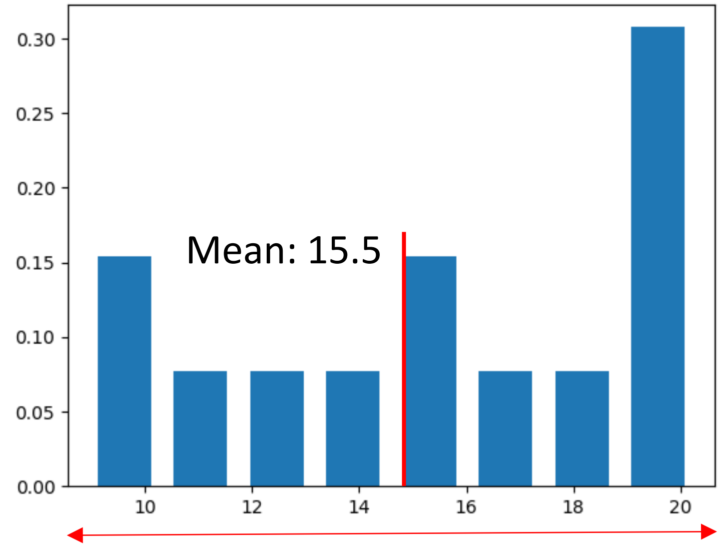
Monthly weather:
4.8, 6.6, 10, 15.3, 20.7, 24.4,
28.6, 28.3, 24.9, 19.7, 13.7, 7.6

Monthly weather:
8.9, 10.9, 14.1, 17.3, 19.2, 20.3,
20.2, 19.9, 18.3, 16.0, 12.1, 9.0



$\sigma \approx 8.57$

The PMF for monthly temperature in Shanghai.



$\sigma \approx 4.37$

The PMF for monthly temperature in Kunming.

Calculating Variance

From definition: $D(X) = E[(X-\mu)^2]$

- Discrete: $D(X) = \sum_{k=1}^{\infty}[x_k - E(X)]^2 p_k$
- Continuous: $D(X) = \int_{-\infty}^{+\infty}[x - E(X)]^2 f(x)dx$

Another way:

$$D(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$

Proof:

$$\begin{aligned} D(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2X \cdot \mu + \mu^2] \\ &= E(X^2) - 2E(X)\mu + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Variance of a 6-sided die

Let Y denote the outcome of a single die roll.
Recall $E[Y] = 7/2$. Calculate the variance of Y.

$$D(Y) = E[(Y-\mu)^2]$$

Approach #1: Definition

$$D(Y) = \frac{1}{6}\left(1-\frac{7}{2}\right)^2 + \frac{1}{6}\left(2-\frac{7}{2}\right)^2 + \frac{1}{6}\left(3-\frac{7}{2}\right)^2 + \frac{1}{6}\left(4-\frac{7}{2}\right)^2 + \frac{1}{6}\left(5-\frac{7}{2}\right)^2 + \frac{1}{6}\left(6-\frac{7}{2}\right)^2 = 35/12$$

$$D(Y) = E(Y^2) - \mu^2$$

Approach #2: Alternative Way

$$E(Y^2) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = 91/6$$

$$D(Y) = E(Y^2) - [E(Y)]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

1. Let R.V. X follows Bernoulli distribution $X \sim \mathrm{Ber}(p)$, find $D(X)$.

2. Let R.V. X follows Poisson distribution $X \sim \pi(\lambda)$, find $D(X)$.

3. Let R.V. X follows Uniform distribution $X \sim U(a, b)$, find $D(X)$.

4. Let R.V. X follows Exponential distribution $X \sim \exp(\theta)$, find $D(X)$.

5. Given a R.V. X with expectation $E(X) = \mu$, variance $D(X) = \sigma^2$, denote $X^* = \frac{X-\mu}{\sigma}$. Show that $E(X^*) = 0$, $D(X^*) = 1$.

Ex. Let R.V. X follows Bernoulli distribution $X \sim \mathrm{Ber}(p)$, find $D(X)$.

Sol.

$$E(X) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$E(X^2) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

$$D(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1-p)$$

Ex. Let R.V. X follows Poisson distribution $X \sim \pi(\lambda)$, find $D(X)$.

$$E(X) = \sum_{k=0}^{+\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$E(X^2) = E[X(X-1)+X] = E[X(X-1)] + E(X)$$
$$= \sum_{k=0}^{+\infty} k(k-1) \cdot \frac{\lambda^k e^{-\lambda}}{k!} + \lambda = \lambda^2 e^{-\lambda} \sum_{k=2}^{+\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda = \lambda^2 + \lambda$$

$$D(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Ex. Let R.V. X follows Uniform distribution $X \sim U(a, b)$, find $D(X)$.

Sol.

$$D(X) = E(X^2) - [E(X)]^2$$

$$= \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Ex. Let R.V. X follows Exponential distribution $X \sim \exp(\theta)$, find $D(X)$.

Sol.

$$E(X^2) = \int_0^{\infty} x^2 \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

$$= \left[-x^2 \cdot e^{-\frac{x}{\theta}}\right]_0^{\infty} + \int_0^{\infty} 2x \cdot e^{-\frac{x}{\theta}} dx \;= 2\theta^2$$

$$D(X) = E(X^2) - [E(X)]^2 = 2\theta^2 - \theta^2 = \theta^2$$

Ex. Given a R.V. X with expectation $E(X) = \mu$, variance $D(X) = \sigma^2$, denote $X^* = \frac{X-\mu}{\sigma}$.

Show that $E(X^*) = 0$, $D(X^*) = 1$.

Sol. $E(X^*) = \frac{1}{\sigma}E(X-\mu) = \frac{1}{\sigma}[E(X) - \mu] = 0$

$$D(X^*) = E\left({X^*}^2\right) - [E(X^*)]^2 = E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right]$$

$$= \frac{1}{\sigma^2}E[(X-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1$$

Note: the PDF of X is unknown.

Variances of common distributions

Distribution	Notation	Expected Value $E(X)$	Variance $D(X)$
Bernoulli	$X \sim \mathrm{Ber}(p)$	p	$p(1-p)$
Binomial	$X \sim b(n, p)$	np	$np(1-p)$
Poisson	$X \sim \pi(\lambda)$	λ	λ
Uniform	$X \sim U(a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$X \sim \exp(\theta)$	θ	θ^2
Normal	$X \sim \mathcal{N}(\mu, \sigma^2)$	μ	σ^2
Standard Normal	$X \sim \mathcal{N}(0,1)$	0	1

Verify these results by yourselves

Properties of variance

- $D(aX+b) \xRightarrow{Y=aX+b} E\left[\left(Y-E(Y)\right)^2\right] = a^2 \cdot D(X)$

1) Multiplication with a constant a, change the variance by a factor of a^2.
2) Adding a constant b does not change the variance.

Proof: $D(aX+b) = E\{[(aX+b)-(a\mu+b)]^2\}$
$= E[a^2(X-\mu)^2]$
$= a^2E[(X-\mu)^2] = a^2D(X)$

Wait! How about $D(aX+bY)$?

- $D(X \pm Y) = D(X) + D(Y) \pm 2E\{[X-E(X)][Y-E(Y)]\}$

If X and Y independent, $D(X \pm Y) = D(X) + D(Y)$

$$D(X \pm Y) = D(X) + D(Y) \pm 2E\{[X - E(X)][Y - E(Y)]\}$$

Proof:

$$\begin{aligned} D(X \pm Y) &= E\{[(X \pm Y) - E(X \pm Y)]^2\} \\ &= E\left\{\left[\left(X - E(X)\right) \pm \left(Y - E(Y)\right)\right]^2\right\} \\ &= E\left[\left(X - E(X)\right)^2\right] + E\left[\left(Y - E(Y)\right)^2\right] \\ &\quad \pm 2E\{[X - E(X)] \cdot [Y - E(Y)]\} \\ &= D(X) + D(Y) \pm 2E\{[X - E(X)] \cdot [Y - E(Y)]\} \end{aligned}$$

Moreover, if X and Y are independent

$$\Rightarrow \ 2E\{[X - E(X)] \cdot [Y - E(Y)]\} = 2[\boldsymbol{E(XY) - E(X)E(Y)}] = 0$$

$$\Rightarrow \ D(X \pm Y) = D(X) + D(Y), \ D(aX \pm bY) = a^2D(X) + b^2D(Y)$$

Properties of variance

- Given that $X_1, X_2, \dots X_n$ are i.i.d., let $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$

$$D(\bar{X}) = D\left[\frac{1}{n}(X_1 + \cdots + X_n)\right] = \frac{1}{n^2}D[X_1 + \cdots + X_n] = \frac{D(X_i)}{n}$$

Employing n copy of X_i reduces the variance to a factor of n!

The future of photography
Averaging from multiple images to reduce distortions.

Ex. Given R.V. X follows Poisson distribution $X \sim \pi(\lambda)$, and

$$3P\{X = 1\} + 2P\{X = 2\} = 4P\{X = 0\},$$

find $E(X)$ and $D(X)$.

Poisson distribution: $P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0,1,2, \ldots$

Ex. Given R.V.s $X \sim \mathcal{N}(1,2)$, $Y \sim \pi(3)$, and X, Y are independent, find $D(XY)$.

$$D(XY) = E(X^2Y^2) - E^2(XY)$$

Ex. Given R.V. X follows Poisson distribution $X \sim \pi(\lambda)$, and

$$3P\{X=1\}+2P\{X=2\}=4P\{X=0\},$$

find $E(X)$ and $D(X)$.

Sol.

Poisson distribution: $P\{X=k\}=\dfrac{\lambda^k e^{-\lambda}}{k!}, k=0,1,2,\ldots$

$$3\cdot\frac{\lambda^1 e^{-\lambda}}{1!}+2\cdot\frac{\lambda^2 e^{-\lambda}}{2!}=4\cdot\frac{\lambda^0 e^{-\lambda}}{0!}$$

$$\Rightarrow \lambda=1, \qquad \Rightarrow E(X)=D(X)=1$$

Ex. Given R.V.s $X \sim \mathcal{N}(1,2)$, $Y \sim \pi(3)$, and X, Y are independent, find $D(XY)$.

Sol.

$$
\begin{aligned}
D(XY) &= E(X^2Y^2) - E^2(XY) \\
&= E(X^2)E(Y^2) - [E(X)E(Y)]^2 \\
&= [E^2(X) + D(X)][E^2(Y) + D(Y)] - [E(X)E(Y)]^2 \\
&= (1+2)(9+3) - (1 \cdot 3)^2 = 27
\end{aligned}
$$