Lecture 03

- More on Bayes rule
- Independence
- Quiz

Probabilistic Deep Learning

Input:

Output:

Model: Tesla Model S

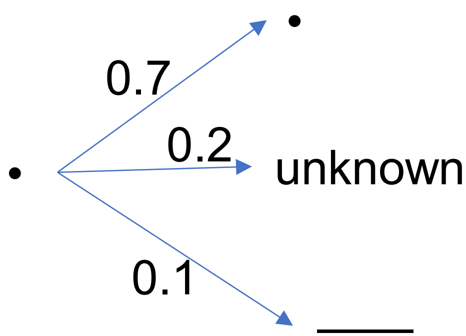
$$p(\text{hypothesis}|\text{data}) = \frac{p(\text{data}|\text{hypothesis}) \cdot p(\text{hypothesis})}{p(\text{data})}$$

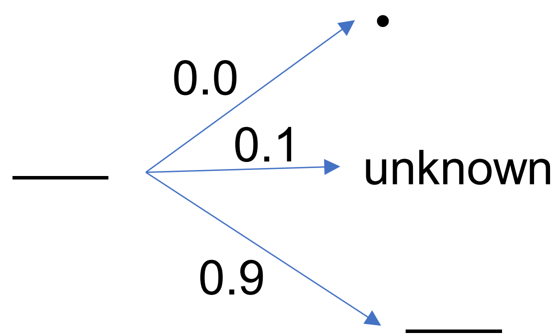
$$P(F_i|E) = \frac{P(E|F_i)P(F_i)}{P(E)}$$

$$P(F_i|E) = \frac{P(E|F_i)P(F_i)}{P(E)}$$

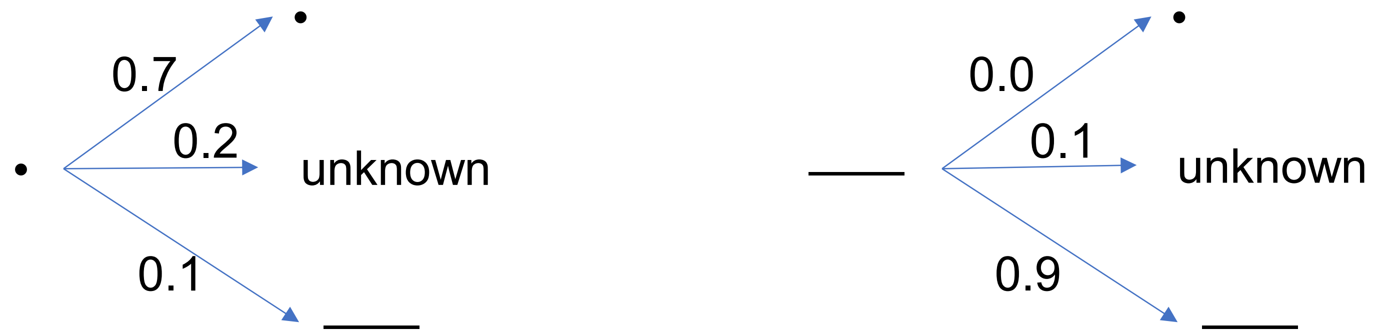
Ex. A signal "•" is sent out in the radio communication. Because of random noise interference, the received signals could be "•", “unknown", or "——" with probabilities of 0.7, 0.2, and 0.1, respectively. Similarly, if a signal "——" is sent out, the received signals could be "•", “unknown", or "——" with probabilities of 0.0, 0.1, and 0.9, respectively. The probabilities of "•" and "——" found in the signal are 0.6 and 0.4, respectively.

Question: If the received signal is “unknown", find the probabilities of the sent signal are "•" and "——", respectively?





$$P(F_i|E) = \frac{P(E|F_i)P(F_i)}{P(E)}$$



Sol. F_1: The sent signal is "•"; F_2: The sent signal is "——";
E: "unknown" is received.

Given: $P(F_1) = 0.6,\ P(F_2) = 0.4,\ P(E|F_1) = 0.2,\ P(E|F_2) = 0.1$

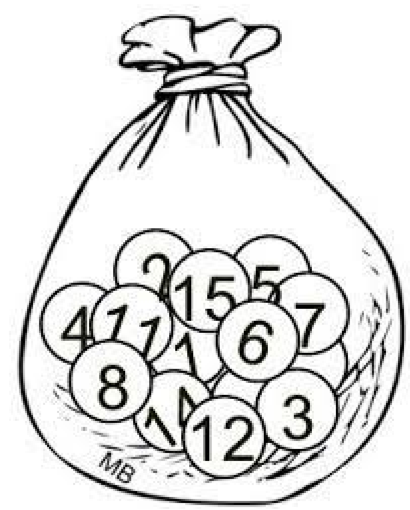
$$P(E) = P(F_1)P(E|F_1) + P(F_2)P(E|F_2) = 0.6 \cdot 0.2 + 0.4 \cdot 0.1 = 0.16$$

$$P(F_1|E) = \frac{P(F_1)P(E|F_1)}{P(E)} = \frac{3}{4}, \qquad P(F_2|E) = \frac{P(F_2)P(E|F_2)}{P(E)} = \frac{1}{4}$$

If the received signal is "unknown", the sent signal is more likely to be "•".

Lecture 03

- More on Bayes rule
- Independence
- Quiz



Independence of events

- Two events E and F are defined as independent if:
$$P(EF) = P(E)P(F)$$

Otherwise, E and F are called dependent events.

- If E and F are independent, then:
$$P(E|F) = P(E) = P(E|\bar{F})$$

- **Note**: How to interpret independence in Venn diagram?

Independence of events

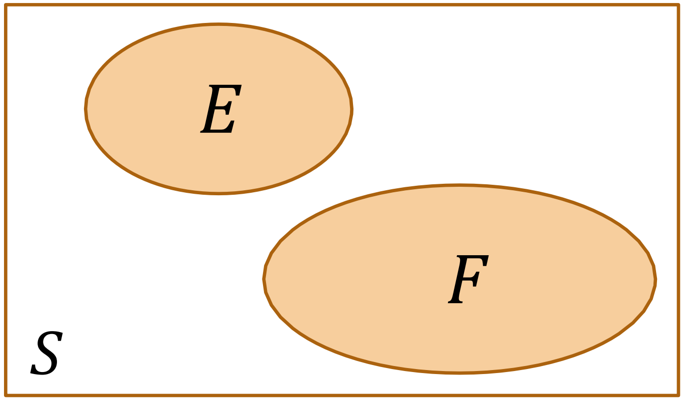
- **Note**: Mutually exclusive events vs. Independent events

$$E \cap F = \emptyset \quad \text{vs.} \quad P(E|F) = P(E)$$

Different!!

For events $E \neq \emptyset$ and $F \neq \emptyset$, it follows:

If $E \cap F = \emptyset$, then E and F are not independent.



Probability of events

E or F

$P(E \cup F)$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

De Morgan's Law

E and F

$P(EF)$

Mutually exclusive

Not mutually exclusive

Just add!

$$P(E \cup F)$$
$$= P(E) + P(F)$$

Inclusion-Exclusion Principle

$$P(E \cup F) =$$
$$P(E) + P(F) - P(E \cap F)$$

Probability of events

E and F

$P(EF)$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

De Morgan's Law

E or F

$P(E \cup F)$

Independent?

Dependent?

Just multiply!

$$P(EF) = P(E)P(F)$$

Chain Rule

$$P(EF) = P(E)P(F|E)$$

or

$$P(EF) = P(F)P(E|F)$$

A tricky example of independence

- There are 4 numbers: 3, 5, 4, and 60.
- Let **event A** be “the selected number is a multiple of 3", and **event B** is “the selected number is a multiple of 4". Then there are:

$$P(A) = 1/2, P(B) = 1/2, P(AB) = 1/4 = P(A)P(B)$$

- A and B are independent events.

- How about if we add another number to the original setting: 1, 3, 5, 4, and 60?

$$P(A) = 2/5, P(B) = 2/5, P(AB) = 1/5 \neq P(A)P(B) = 4/25$$

- A and B are no longer independent! But why?
- The sample spaces are different. Not the same anymore!

Generalizing independence

Three events E, F, and G are independent if:

$$\begin{cases} P(EFG) = P(E)P(F)P(G) \\ P(EF) = P(E)P(F) \\ P(FG) = P(F)P(G) \\ P(EG) = P(E)P(G) \end{cases}$$

n events $E_1, E_2, \dots, E_n$ are independent if:

for $r = 1, \dots n$:

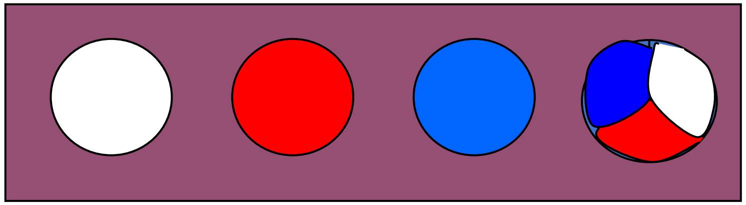
for every subset of $E_1, E_2, \dots, E_r$:

$$P(E_1 E_2 \dots E_r) = P(E_1)P(E_2) \dots P(E_r)$$

Note: Independent vs. Pairwise Independent (相互独立 vs.两两独立)

Independent and Pairwise Independent

Ex. There are 4 table tennis balls in a bag, one is painted white, one is painted red, one is painted blue, and the other is painted white, red and blue. Now a ball is randomly drawn from the bag. Suppose event A="the selected ball is painted with white", B="the selected ball is painted with red", C="the selected ball is painted with blue“. Verify that events A, B, and C are pairwise independent, but not independent of each other.



Pairwise independent: $P(AC) = P(A)P(C) = 1/4$, $P(BC) = P(B)P(C) = 1/4,\ P(A)P(B) = P(AB) = 1/4$

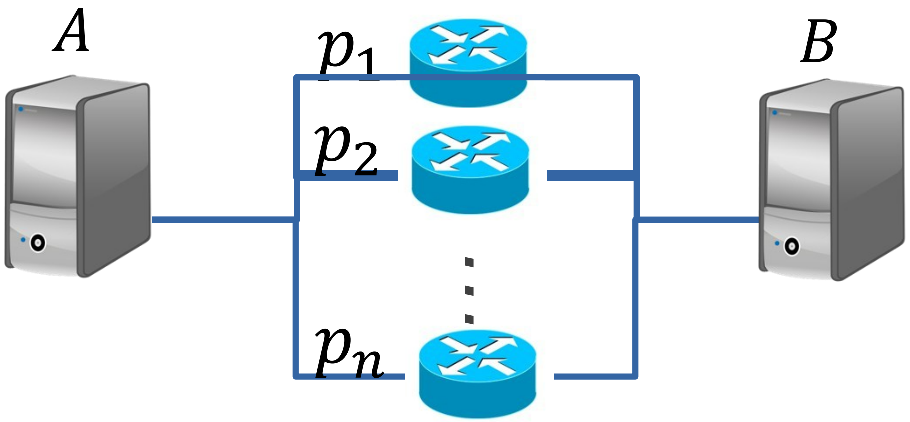
Independent: $P(ABC) = 1/4 \neq P(A)P(B)P(C) = 1/8$

System reliability

<u>Ex.</u> Consider the following parallel network:

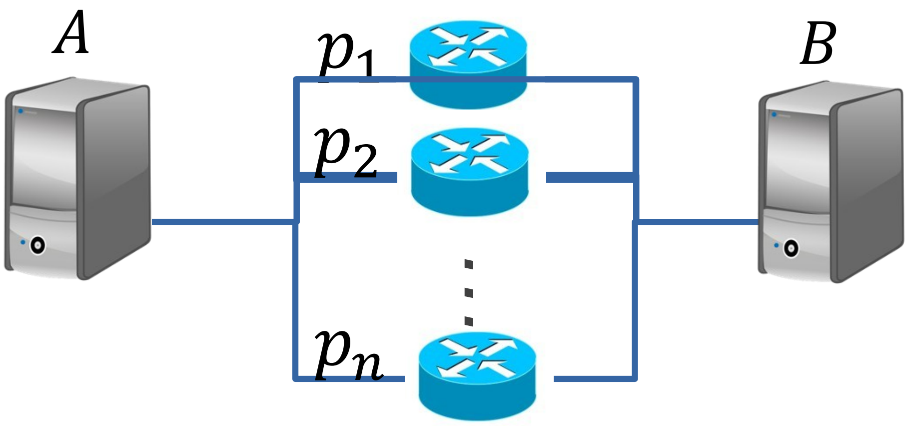
- n independent routers, each with
- probability p_i of functioning (where $1 \leq i \leq n$)
- E = a functional path from A to B exists.

- What is $P(E)$?



System reliability

What is $P(E)$?



$P(E) = P(\geq 1 \text{ router works})$

$= 1 - p(\text{all routers fail})$

$= 1 - (1 - p_1)(1 - p_2) \dots (1 - p_n)$

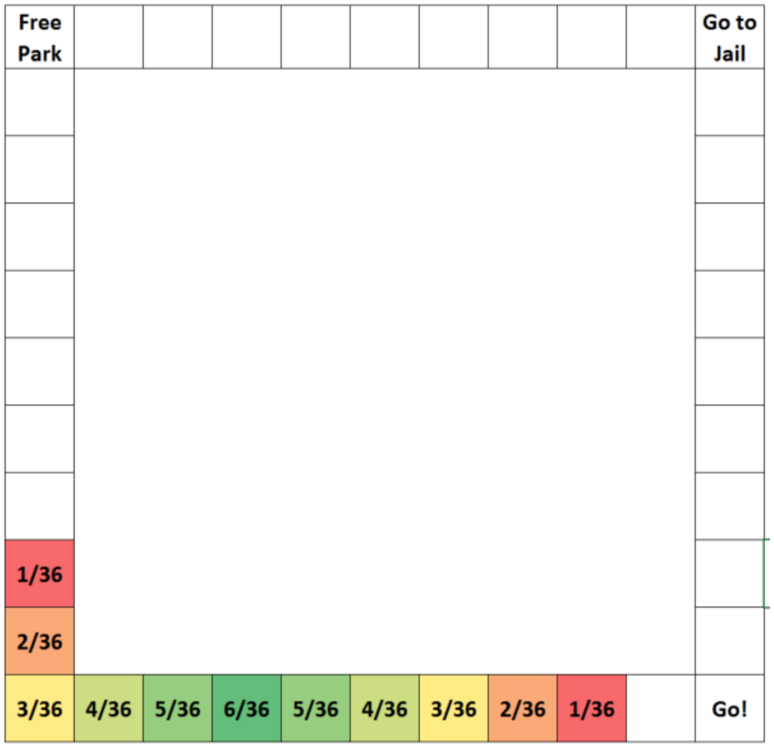
$= 1 - \prod_{i=1}^{n}(1 - p_i)$

Probability in Monopoly

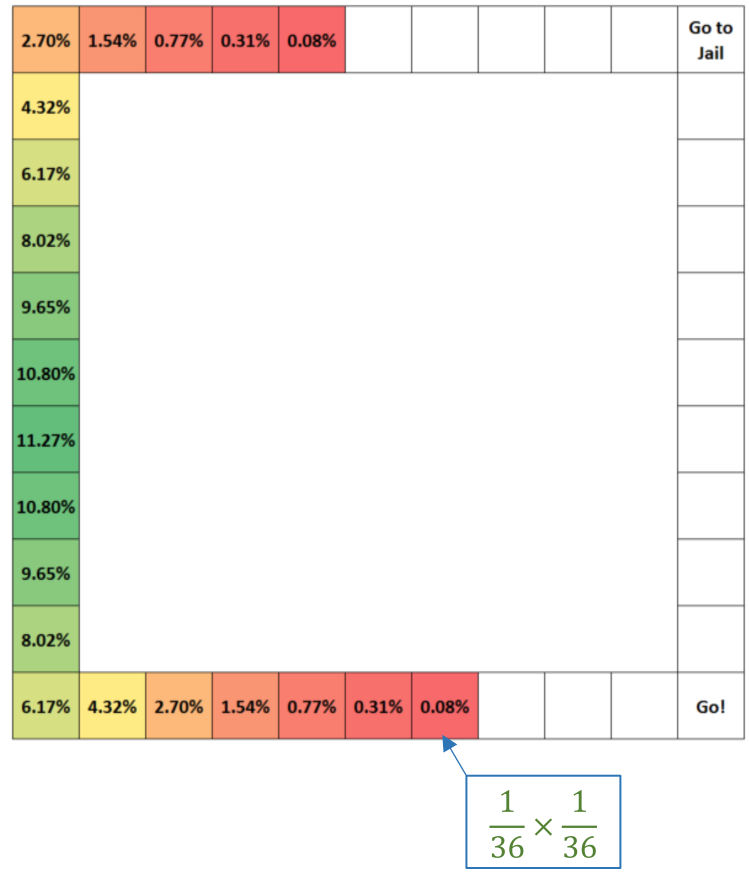
Monopoly is a multi-player economics-themed board game. In the game, players roll ***two dice*** to move around the game board, buying and trading properties, and developing them with houses and hotels.

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Probability in Monopoly



Probability in the first round.



Probability in the second round.

[Excel at Monopoly #2]

Chapter 1: Probability basics

- Experiment, sample space and event
- Relative frequency and probability
- Equally likely outcomes
- Laws of set operations
- Simple propositions of probability
- Conditional probability, law of total probability, Bayes rule
- Independence

Quiz 1

Date: 21-Sep-2023

Scope: Chapter 1-3 (basics of probability theory)

Open-book exam, with fill-in-the-blank and multiple-choice questions.

Time: ~45 mins