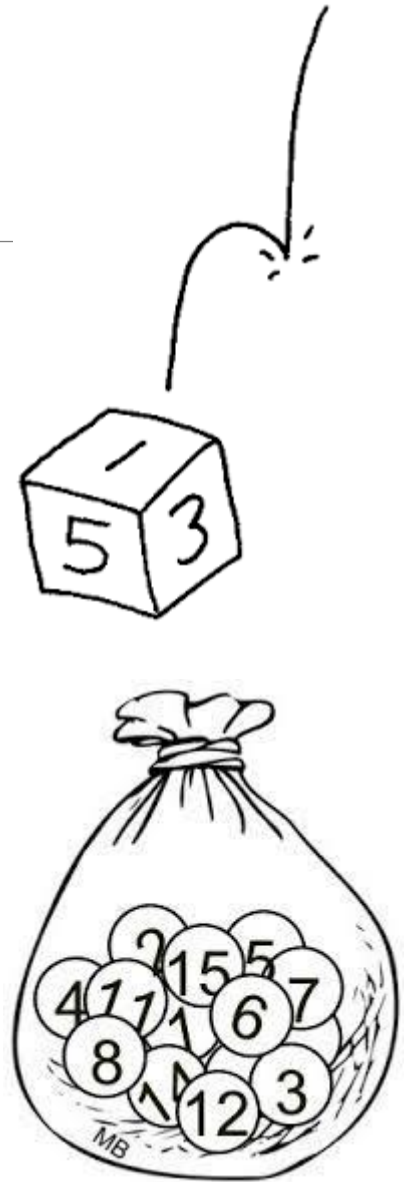


Lecture 08

- Marginal PMF, PDF, CDF
- Some common 2D PDF
- Conditional PMF, PDF, CDF
(条件分布律, 分布函数, 概率密度函数)



Marginal Cumulative Density Function (边缘概率分布函数)

Def. Given $F(X, Y)$ as the joint CDF of (X, Y) , the marginal CDF of (X, Y) are defined as

$$F_X(x) = P(X \leq x) = P(X \leq x, Y < +\infty) = F(x, +\infty)$$

Marginal CDF of X

$$F_Y(y) = P\{Y \leq y\} = P(X < +\infty, Y \leq y) = F(+\infty, y)$$

Marginal CDF of Y



Note:

Marginal CDF	vs.	CDF
$F_X(x)$		$F(x)$

Depend on how many variables
are in the context!

$$\begin{array}{ccc}
 F(x, y) & \begin{array}{c} \xrightarrow{f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}} \\ \xleftarrow{F(x, y) = \int_{-\infty}^y \left[\int_{-\infty}^x f(u, v) du \right] dv} \end{array} & f(x, y)
 \end{array}$$

???

Let $y \rightarrow +\infty$,

$$F(x, +\infty) = F_X(x)$$

Let $x \rightarrow +\infty$,

$$F(+\infty, y) = F_Y(y)$$

$$\begin{array}{ccc}
 F_X(x), F_Y(y) & \begin{array}{c} \xrightarrow{f(x) = \frac{dF(x)}{dx}} \\ \xleftarrow{F(x) = \int_{-\infty}^x f(t) dt} \end{array} & f_X(x), f_Y(y)
 \end{array}$$

Marginal Probability Density Function

Def. Given $f(X, Y)$ as the joint PDF of (X, Y) ,

$$F_X(x) = F(x, +\infty) = \int_{-\infty}^x \left[\int_{-\infty}^{+\infty} f(x, y) dy \right] dx, \quad -\infty < x < +\infty$$

Taking derivative of both sides, it yields the **marginal PDF of X**

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy, \quad -\infty < x < +\infty$$

Similarly, we obtain the marginal PDF of Y

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx, \quad -\infty < y < +\infty$$

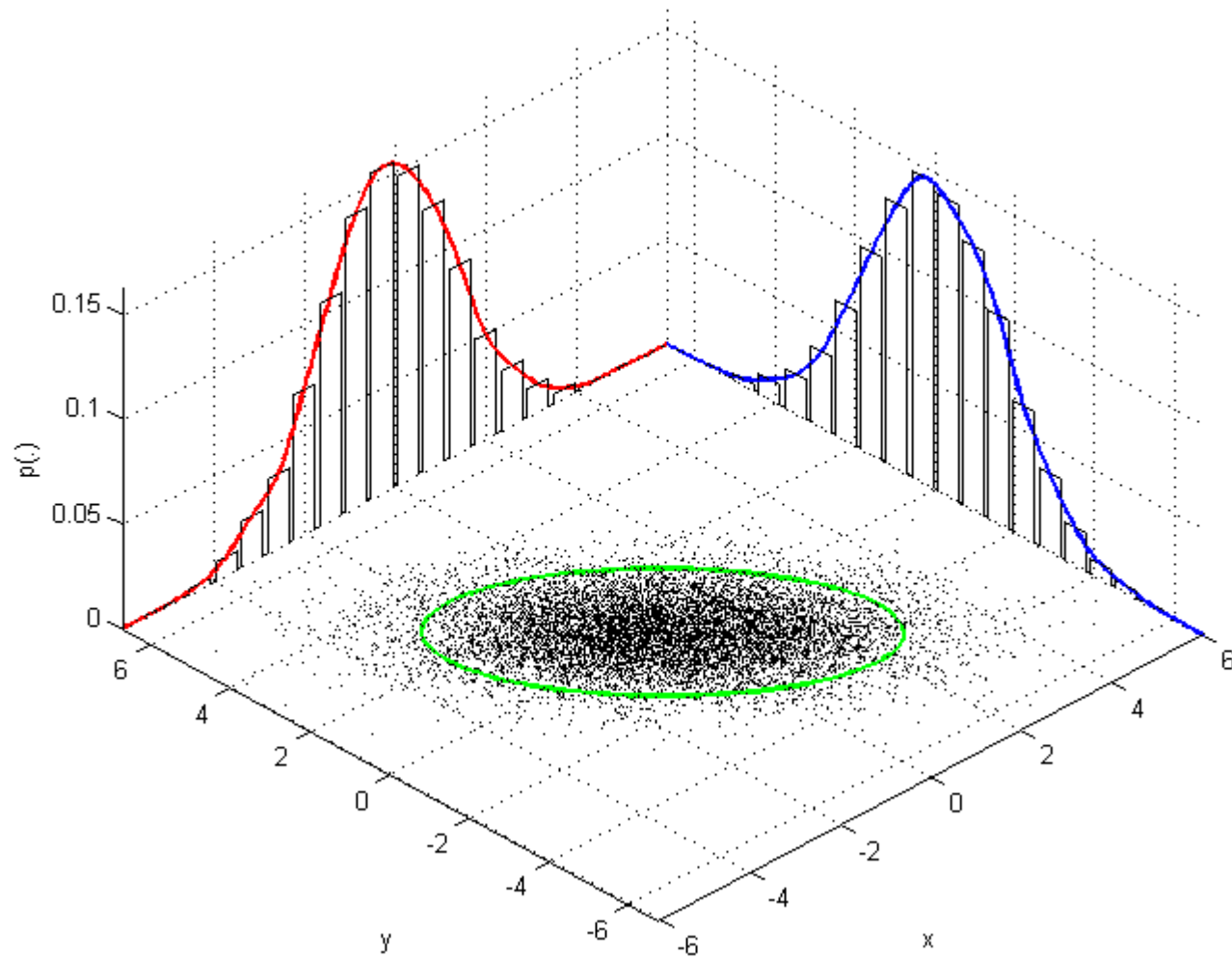


Illustration of marginal PDF or PMF

2-Dimensional

$$F(x, y)$$

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

$$f(x, y)$$

$$F(x, y) = \int_{-\infty}^y \left[\int_{-\infty}^x f(u, v) du \right] dv$$

???

Let $y \rightarrow +\infty$,
 $F(x, +\infty) = F_X(x)$
 Let $x \rightarrow +\infty$,
 $F(+\infty, y) = F_Y(y)$

???

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

$$F_X(x), F_Y(y)$$

$$f(x) = \frac{dF(x)}{dx}$$

$$f_X(x), f_Y(y)$$

$$F(x) = \int_{-\infty}^x f(t) dt$$

1-Dimensional

Probability concept explained: Marginalization



Probability Concepts Explained: Marginalization,
<https://towardsdatascience.com/probability-concepts-explained-marginalisation-2296846344fc>

- Marginalisation is a method that requires summing over the possible values of one variable to determine the marginal contribution of another.
- Suppose we're interested in people's happiness under different weather condition. We can write this mathematically as $P_{\text{😊}}(\text{weather})$.
- The problem is that people live in different region, and we can't just get rid of it in the measurement. What we're actually measuring is $P_{\text{😊}}(\text{weather}, \text{region})$.
- Marginalization tells us that we can calculate the quantity we want if we sum over all regions. $P_{\text{😊}}(\text{weather}) = P_{\text{😊}}(\text{weather}, \text{region}=R_1) + P_{\text{😊}}(\text{weather}, \text{region}=R_2) + \dots + P_{\text{😊}}(\text{weather}, \text{region}=R_n)$.
- **Note:** Compare with the total probability law.

Common 2D PDF

Uniform distribution (均匀分布),

$$f(x, y) = \begin{cases} \frac{1}{A}, & (x, y) \in G \\ 0 & \text{otherwise} \end{cases}$$

G is an area in arbitrary shape.

$$F(x, y) = \iint_R f(x, y) dx dy = \frac{1}{A} \cdot S$$

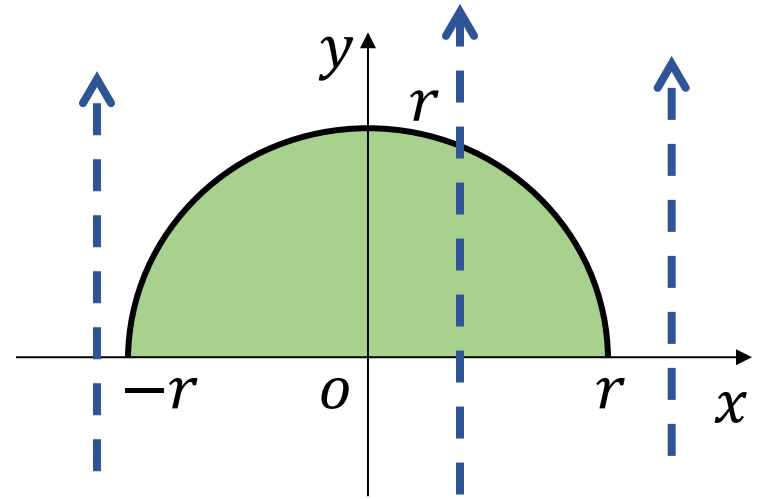
where R denotes the ROI, and S is the area of R .

Ex. Let G be the domain of $(X, Y) \in \{x^2 + y^2 \leq r^2 \cap y \geq 0\}$, (X, Y) follows uniform distribution, find the marginal PDF of X and Y .

Ex. Let G be the domain of $(X, Y) \in \{x^2 + y^2 \leq r^2 \cap y \geq 0\}$, (X, Y) follows uniform distribution, find the marginal PDF of X and Y .

Sol.
$$f(x, y) = \begin{cases} \frac{2}{\pi r^2}, & (x, y) \in G, \\ 0, & \text{otherwise.} \end{cases}$$

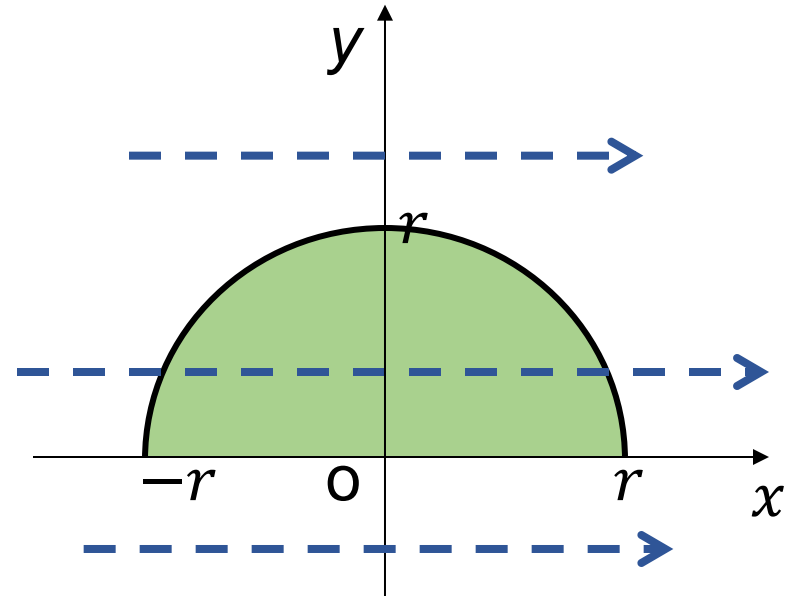
$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy$$



$$= \begin{cases} \int_0^{\sqrt{r^2 - x^2}} \frac{2}{\pi r^2} dy, & -r < x < r, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{2}{\pi r^2} \sqrt{r^2 - x^2}, & |x| < r \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

$$= \begin{cases} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2} dx, & 0 < y < r \\ 0, & \text{otherwise} \end{cases}$$



$$= \begin{cases} \frac{4}{\pi r^2} \sqrt{r^2 - y^2}, & 0 < y < r \\ 0, & \text{otherwise} \end{cases}$$

Common 2D PDF

Two-dimensional Normal Distribution,

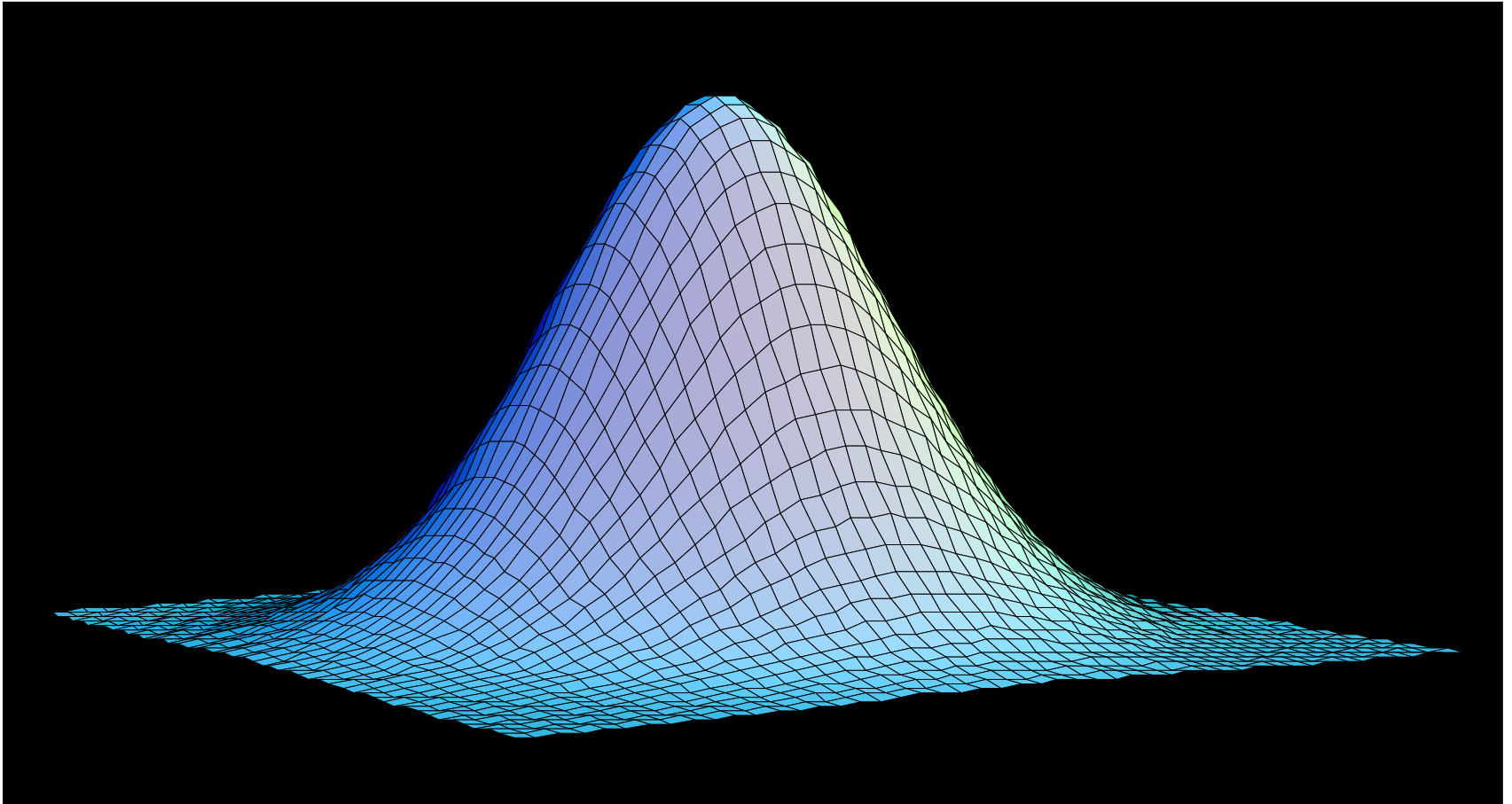
$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}$$

$$-\infty < x < \infty, \quad -\infty < y < \infty$$

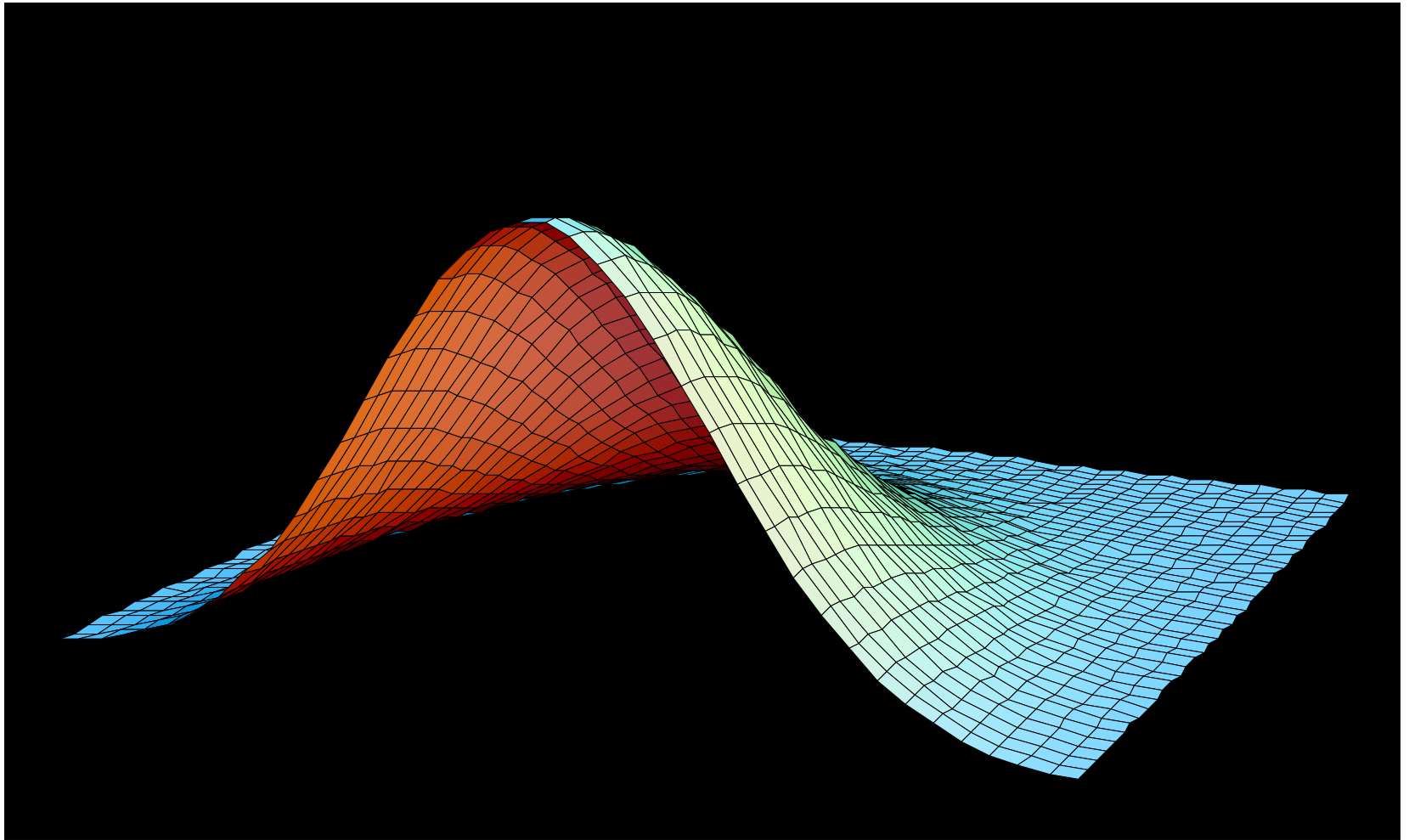
where $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho$ are constant, and $\sigma_1 > 0, \sigma_2 > 0, |\rho| \leq 1$. 2D normal distribution is denoted as

$$(X, Y) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

where ρ is the **correlation coefficient** between X and Y (to be covered in a few weeks).



2D PDF of normal distribution



Section view of 2D normal distribution

Ex. Let $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, find the marginal PDF of X and Y .

Sol.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, -\infty < x < +\infty$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}, -\infty < y < +\infty$$

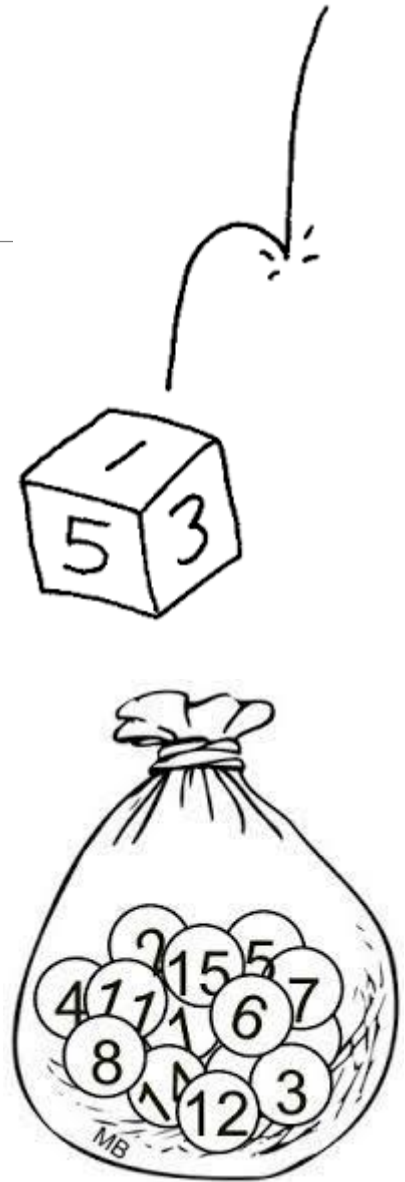
The marginal PDF of X and Y follows normal distribution.

$$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$$

Verify the result by yourselves.

Lecture 08

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(条件分布律, 分布函数, 概率密度函数)



Conditional PMF vs. Marginal PMF for discrete R.V.s

条件分布律 vs. 边缘分布率

	y_1	y_2	...	y_j	...	$p_{i\cdot}$
x_1	p_{11}	p_{12}	...	p_{1j}	...	$p_{1\cdot}$
x_2	p_{21}	p_{22}	...	p_{2j}	...	$p_{2\cdot}$
\vdots	\vdots	\vdots		\vdots		\vdots
x_i	p_{i1}	p_{i2}	...	p_{ij}	...	$p_{i\cdot}$
\vdots	\vdots	\vdots		\vdots		\vdots
$p_{\cdot j}$	$p_{\cdot 1}$	$p_{\cdot 2}$...	$p_{\cdot j}$...	1

The practical implication
of the Marginal PMF?

$$P(X = x_i | Y = y_j)$$

$$P(Y = y_j | X = x_i)$$

Conditional probability?

Discrete conditional distributions

Recall the definition of the **conditional probability** of event E given event F :

$$P(E|F) = \frac{P(EF)}{P(F)}$$

For discrete random variables X, Y , the **conditional probability mass function (PMF) of X given $Y = y_j$** is

$$P(X = x_i | Y = y_j) = \frac{P(X=x_i, Y=y_j)}{P(Y=y_j)}, \quad \textbf{Note: } P(Y = y_j) \neq 0, \text{ and } i = 1, 2, \dots$$

Different notation,
same meaning:

$$p_{X|Y}(x_i|y_j) = \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)} = \frac{p_{ij}}{p_{\cdot j}}$$

Discrete conditional distributions

For discrete random variables X, Y , the conditional probability mass function (PMF) of X given $Y = y_j$ is

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}, \quad i = 1, 2, \dots$$

The conditional cumulative distribution function (CDF) of X given $Y = y$ is

$$F_{X|Y}(x_i | y_j) = P(X \leq x_i | Y = y_j) = \sum_{x \leq x_i} P_{X|Y}(x | y_j), \quad i = 1, 2, \dots$$

Note: Compare with the unconditional case, a condition $P(Y = y_j)$ is considered in both PMF and CDF.

Conditional PMF vs. Marginal PMF for discrete R.V.s

条件分布律 vs. 边缘分布率

	y_1	y_2	...	y_j	...	$p_{i\cdot}$
x_1	p_{11}	p_{12}	...	p_{1j}	...	$p_{1\cdot}$
x_2	p_{21}	p_{22}	...	p_{2j}	...	$p_{2\cdot}$
\vdots	\vdots	\vdots		\vdots		\vdots
x_i	p_{i1}	p_{i2}	...	p_{ij}	...	$p_{i\cdot}$
\vdots	\vdots	\vdots		\vdots		\vdots
$p_{\cdot j}$	$p_{\cdot 1}$	$p_{\cdot 2}$...	$p_{\cdot j}$...	1

Marginal PMF: the PMF of a multi-dimensional variable along a dimension of interest;

Conditional PMF: the PMF of a multi-dimensional variable under a given condition.

Notations in Conditional PMF vs. Marginal PMF

$$P_X(x) = p_{i.} = \sum_{j=1}^{+\infty} p_{ij} \quad (i = 1, 2, \dots) \quad P_Y(y) = p_{.j} = \sum_{i=1}^{+\infty} p_{ij} \quad (j = 1, 2, \dots)$$

Marginal PMF of X

Marginal PMF of Y

$$P\{X = x_i | Y = y_j\} = \frac{p_{ij}}{p_{.j}},$$

$i = 1, 2, \dots$

Conditional PMF of X

$$P\{Y = y_j | X = x_i\} = \frac{p_{ij}}{p_{i.}},$$

$j = 1, 2, \dots$

Conditional PMF of Y

Note:

- 1) Conditional PMF = Joint PMF \div Marginal PMF;
- 2) Conditional PMF = Conditional Probability + Mass Function.

条件概率 + 分布律

Ex. A Safety Officer for an auto insurance company was interested in learning how **the extent of an individual's injury** in an automobile accident relates to **the type of safety restraint** the individual was wearing at the time of the accident. As a result, the Safety Officer used statewide ambulance and police records to compile the following **two-way table of joint probabilities**:

Extent of Injury (X)	Type of Restraint (Y)			$P_X(x)$
	None (0)	Belt only (1)	Belt and Harness (2)	
None (0)	0.065	0.075	0.06	0.20
Minor (1)	0.175	0.16	0.115	0.45
Major (2)	0.135	0.10	0.065	0.30
Death (3)	0.025	0.015	0.01	0.05
$P_Y(y)$	0.40	0.35	0.25	1.00

The Safety Officer was interested in answering the following questions:

- 1) What is the probability that a randomly selected person in an automobile accident was **wearing a seat belt and had only a minor injury**? **0.16**
- 2) If a randomly selected person wears **no restraint**, what is **the probability of death**? **0.025/0.40**
- 3) If a randomly selected person sustains **no injury**, what is the probability the person was **wearing a belt and harness**? **0.06/0.20**

Quick check

$$P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

Number or function?

1. $P(X = 2 | Y = 5)$

number

2. $P(X = x_i | Y = 5)$

1-D function

3. $P(X = 2 | Y = y_j)$

1-D function

4. $P(X = x_i | Y = y_j)$

2-D function

True or false?

1. $\sum_i P(X = x_i | Y = 5) = 1$

true

2. $\sum_j P(X = 2 | Y = y_j) = 1$

false


3. $\sum_i \sum_j P(X = x_i | Y = y_j) = 1$

false

4. $\sum_i (\sum_j P(X = x_i | Y = y_j) P(Y = y_j)) = 1$

true

Conditional probability in Amazon



Roll over image to zoom in

FINEDINE
Stainless Steel Mixing Bowls by Finedine (Set of 6) Polished Mirror Finish Nesting Bowl, 1/4 - 1.5-3 - 4-5 - 8 Quart - Cooking Supplies
 ★★★★★ 2,566 customer reviews | 75 answered questions
 Amazon's Choice for "stainless steel mixing bowls"

Price: **\$24.99** & **FREE Shipping** on orders over \$25 shipped by Amazon. Details
 Get \$40 off instantly: Pay \$0.00 upon approval for the Amazon.com Store Card.

✓prime | Try Fast, Free Shipping *

- With graduating sizes of 1/4, 1.5, 3, 4, 5 and 8 quart, the bowl set allows users to be well equipped for serving fruit salads, marinating for the grill, and adding last ingredients for dessert.
- Stainless steel bowls with commercial grade metal that can be used as both baking mixing bowls and serving bowls. These metal bowls won't stain or absorb odors and resist rust for years of durability.
- An easy to grip rounded-up on the stainless steel bowl set makes handling easier while a generous wide rim allows contents to flow evenly when pouring; flat base stabilizes the silver bowls making mixing all the easier.
- A space saving stackable design helps de-clutter kitchen cupboards while the attractive polished mirror finish on the large mixing bowls adds a luxurious aesthetic.
- This incredible stainless steel mixing bowl set is refrigerator, freezer, and dishwasher safe for quick and easy meal prep and clean up. They'd also make a great gift!

Compare with similar items

Used & new (7) from \$20.62 & FREE shipping on orders over \$25.00. Details









Report incorrect product information.

Packaging may reveal contents. Choose **Conceal Package** at checkout.

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$$P(\text{buy item } X \mid \text{bought item } Y)$$

The conditional relationship between purchasing different items.

Conditional PDF and CDF for continuous R.V.s

Conditional PDF of X given $Y = y$.

$$f_{X|Y}(x|y) \triangleq \frac{f(x, y)}{f_Y(y)}$$

Joint PDF

Marginal PDF

$$F_{X|Y}(x|y) = \int_{-\infty}^x \frac{f(u, y)}{f_Y(y)} du$$



Conditional CDFs

Conditional PDF of Y given $X = x$.

$$f_{Y|X}(y|x) \triangleq \frac{f(x, y)}{f_X(x)}$$

Joint PDF

Marginal PDF

$$F_{Y|X}(y|x) = \int_{-\infty}^y \frac{f(x, v)}{f_X(x)} dv$$

Conditional PDF and CDF for continuous R.V.s

Conditional PMF $p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$ Divide the histograms into small bins \longrightarrow $f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$ Conditional PDF

Conditional PDF $f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$ Multiple lhs by Δ_x and rhs by $\Delta_x \Delta_y / \Delta_x$.
 $\Delta_x, \Delta_y = \varepsilon$

$$f_{X|Y}(x|y)\Delta_x = \frac{f(x, y)\Delta_x \Delta_y}{f_Y(y)\Delta_y} \approx \frac{P\{x \leq X \leq x + \Delta_x, y \leq Y \leq y + \Delta_y\}}{P\{y \leq Y \leq y + \Delta_y\}}$$

$$= P\{x \leq X \leq x + \varepsilon | y \leq Y \leq y + \varepsilon\}$$

Consider a large range of X .

$$P\{X \leq x | Y = y\} = F_{X|Y}(x|y) = \lim_{\varepsilon \rightarrow 0^+} P\{X \leq x | y \leq Y \leq y + \varepsilon\} = \int_{-\infty}^x \frac{f(u, y)}{f_Y(y)} du$$

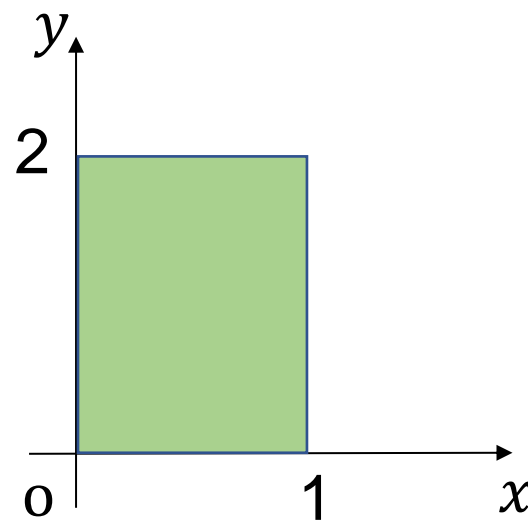
Conditional
CDF

Ex. Let the joint PDF of (X, Y) as

$$f(x, y) = \begin{cases} x^2 + \frac{1}{3}xy, & 0 < x < 1, \\ & 0 < y < 2, \\ 0, & \text{otherwise} \end{cases}$$

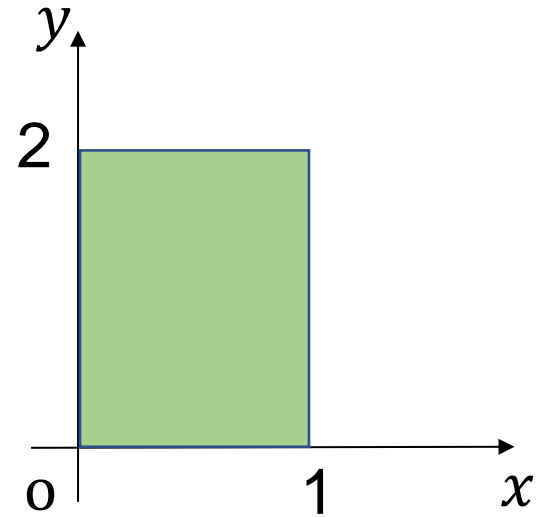
(1) $f_X(x)$, $f_Y(y)$;

(2) $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$.



Ex. Let the joint PDF of (X, Y) as

$$f(x, y) = \begin{cases} x^2 + \frac{1}{3}xy, & 0 < x < 1, \\ & 0 < y < 2, \\ 0, & \text{otherwise} \end{cases}$$



(1) $f_X(x)$, $f_Y(y)$;

(2) $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$.

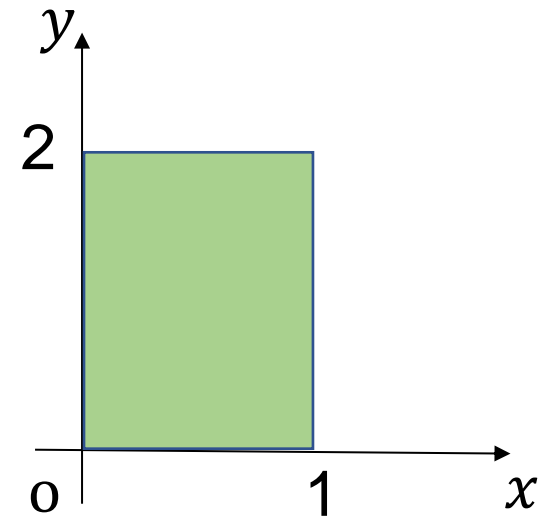
Sol. (1)
$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy = \int_0^2 \left(x^2 + \frac{1}{3}xy \right) dy$$
$$= 2x^2 + \frac{2}{3}x$$

$$f_X(x) = \begin{cases} 2x^2 + \frac{2}{3}x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly,
$$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{1}{6}y, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Ex. Let the joint PDF of (X, Y) as

$$f(x, y) = \begin{cases} x^2 + \frac{1}{3}xy, & 0 < x < 1, \\ & 0 < y < 2, \\ 0, & \text{otherwise} \end{cases}$$



(1) $f_X(x)$, $f_Y(y)$;

(2) $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$.

(2) $0 < y < 2$,

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(Y)} = \begin{cases} \frac{6x^2 + 2xy}{2 + y}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$0 < x < 1$,

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(X)} = \begin{cases} \frac{3x + y}{6x + 2}, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Ex. Assume the joint PDF of (X, Y) is

$$f(x, y) = \begin{cases} 24xy, & x^2 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional PDF of $Y|X$ and $X|Y$.

Ex. Assume the joint PDF of (X, Y) is

$$f(x, y) = \begin{cases} 24xy, & x^2 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

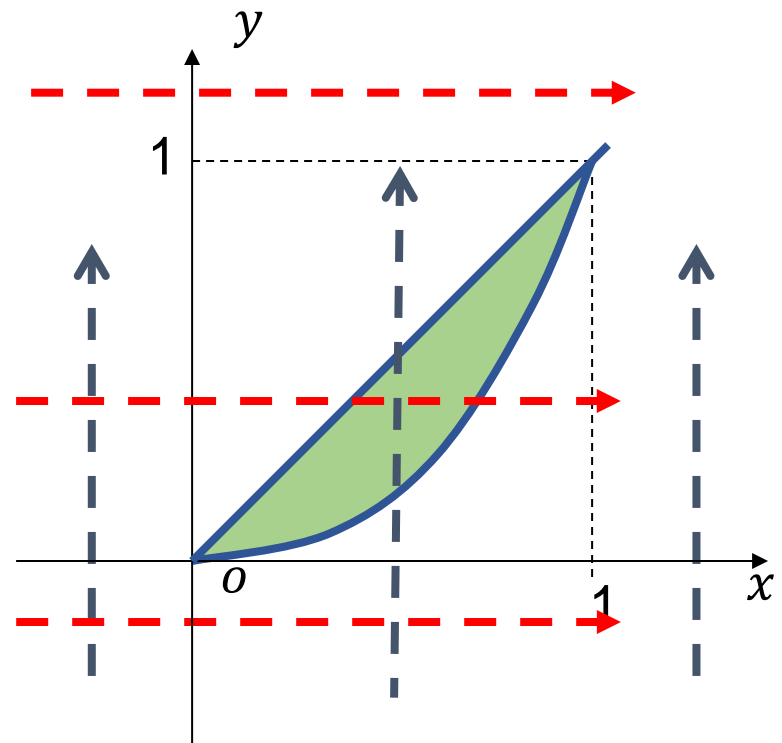
Find the conditional PDF of $Y|X$ and $X|Y$.

Sol.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f(x, y) dy \\ &= \begin{cases} \int_{x^2}^x 24xy dy, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 12x^3(1 - x^2), & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

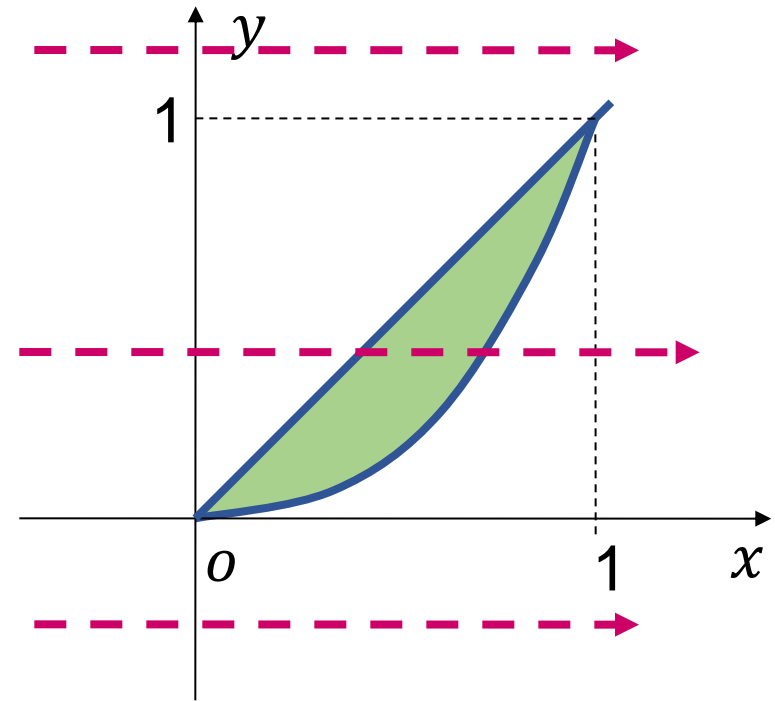
Given any x ($0 < x < 1$),

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{2y}{x^2(1 - x^2)}, & x^2 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

$$= \begin{cases} 12y(y - y^2), & 0 \leq y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$



Given any y ($0 < y < 1$),

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} \frac{2x}{y - y^2}, & y \leq x \leq \sqrt{y} \\ 0, & \text{otherwise} \end{cases}$$