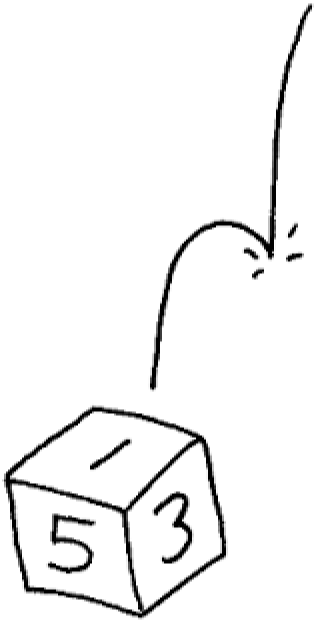
Lecture 07

- Multidimensional Variables

- Joint PMF, PDF, CDF

- Quiz 2



Why we need more variables

$$P\left\{\begin{matrix}\text{Score of Warriors}\\ > \text{Score of Lakers}\end{matrix}\right\}$$

An event defined by the relationship between two random variables!

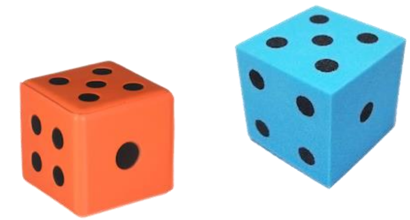
$$P\left\{\begin{matrix}20\ ℃ \le \text{Temperature} \le 25\ ℃\\ \text{and}\\ 0\ \text{mm} \le \text{Precipation} \le 5\ \text{mm}\\ \text{and}\ \ \dots\dots\end{matrix}\right\}$$

An event described by multiple dimensions.

a nice weather

Joint Probability Mass Function

Roll two 6-sided dice, yielding values X and Y.



X

random variable

$P(X = 1)$

probability of an event

$P(X = k)$

probability mass function

X, Y

random variables

$P(X = 1 \cap Y = 6)$

$P(X = 1, Y = 6)$

new notation: the comma

probability of the intersection of two events

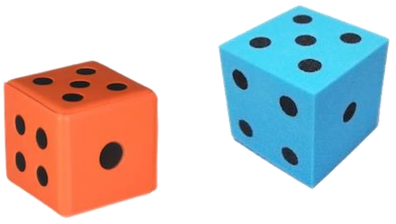
$P\left(X = x_i, Y = y_j\right)$

$P\left(x_i, y_j\right)$

Joint PMF

Two dice

Roll two 6-sided dice, yielding values X and Y.



1. What is the joint PMF of X and Y?

$$p(x_i, y_j) = 1/36, \qquad (x_i, y_j) \in \{(1,1), \ldots, (6,6)\}$$

		X					
		1	2	3	4	5	6
Y	1	1/36	...	...	...	...	1/36
	2	...	...	...	...		
	3	...	...	...	...	...	...
	4	...	...	...	...	...	...
	5	...	...	...	...	...	...
	6	1/36	...	...	...	...	1/36

$P(X = 4, Y = 2) = 1/36$

Probability table of discrete RVs.

Joint Probability Mass Function

The joint PMF of discrete R.V.s can be computed by taking on x_i and y_j.

$$p(x_i, y_j) = P\{X = x_i, Y = y_j\} \stackrel{\text{def}}{=} p_{ij}, (i, j = 1,2, \dots)$$

The PMF of X can be obtained from the joint PMF by

$$\begin{aligned} P_{i\cdot} = P_X(x) &= P(X = x) \\ &= P\left(\cup_j \{X = x, Y = y_j\}\right) \\ &= \textstyle\sum_j P\left(X = x, Y = y_j\right) \\ &= \textstyle\sum_j p\left(x, y_j\right) \end{aligned}$$

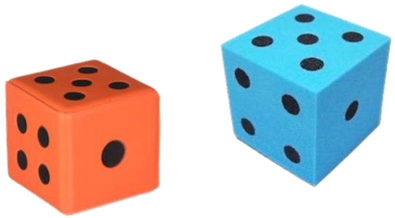
Similarly, $P_{\cdot j} = P_Y(y) = \sum_i p(x_i, y)$

Also called, Marginal PMF (边缘分布律).

	y_1	y_2	…	y_i	…	$p_{i\cdot}$
x_1	p_{11}	p_{12}	…	p_{1j}	…	$p_{1\cdot}$
x_2	p_{21}	p_{22}	…	$p_{2\mathrm{j}}$	…	$p_{2\cdot}$
⋮	⋮	⋮		⋮		⋮
x_i	p_{i1}	p_{i2}	…	p_{ij}	…	$p_{i\cdot}$
⋮	⋮	⋮		⋮		⋮
$p_{\cdot j}$	$p_{\cdot 1}$	$p_{\cdot 2}$	…	$p_{\cdot j}$	…	1

Two dice

Roll two 6-sided dice, yielding values X and Y.



1. What is the joint PMF of X and Y?

$p(x_i, y_j) = 1/36, \qquad (x_i, y_j) \in \{(1,1), \dots, (6,6)\}$

2. What is the marginal PMF of X?

$$P_{i\cdot} = p_X(x_i) = P(X = x_i)$$
$$= \sum_y p_{X,Y}(x_i, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6}, \qquad x_i \in \{1, \dots, 6\}$$

Ex. Assume that the random variable X is equally likely to take the values from $\{1, 2, 3, 4\}$, and the other random variable Y is equally likely to take an integer value between 1 and X. Try to find the joint PMF of (X, Y) and marginal PMF of X and Y.

Ex. Assume that the random variable X is equally likely to take the values from $\{1, 2, 3, 4\}$, and the other random variable Y is equally likely to take an integer value between 1 and X. Try to find the joint PMF of (X, Y) and marginal PMF of X and Y.

Sol. $X = i,\ i = 1, 2, 3, 4, \quad Y = j, j \leq i.$

$$P\{X = i, Y = j\} = P\{Y = j | X = i\} P\{X = i\} = \frac{1}{i}\frac{1}{4} (\ i = 1,2,3,4, \quad j \leq i)$$

X \ Y	1	2	3	4	
1	1/4	0	0	0	1/4
2	1/8	1/8	0	0	1/4
3	1/12	1/12	1/12	0	1/4
4	1/16	1/16	1/16	1/16	1/4
	25/48	13/48	7/48	1/16	1

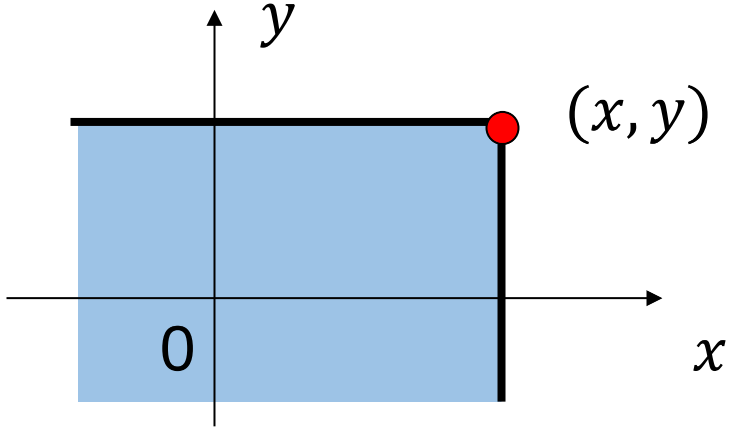
Joint Cumulative Distribution Function (CDF)

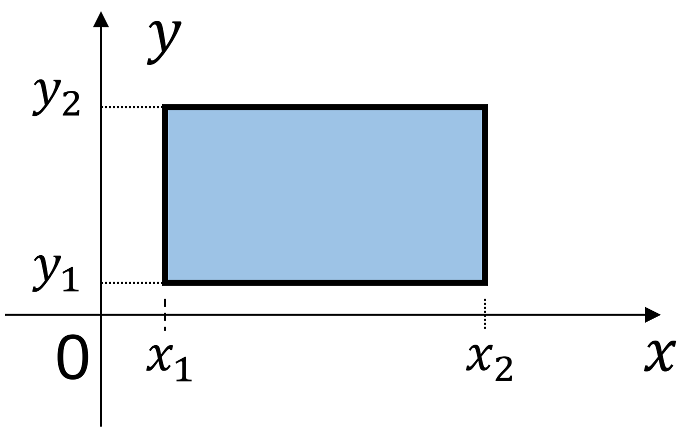
(联合分布函数)

For any two R.V.s X and Y, the joint CDF of X, Y is defined as

$$F(x, y) = P\{(X \leq x) \cap (Y \leq y)\} \stackrel{\text{def}}{=} P\{X \leq x, Y \leq y\}$$

The probability of X and Y can be computed from joint CDF.





$$P\{x_1 < X \leq x_2, y_1 < Y \leq y_2\} = F(x_2, y_2) - F(x_2, y_1) + F(x_1, y_1) - F(x_1, y_2)$$

Joint Probability Density Function

One-dimensional random variable:

A random variable X is continuous if there is a probability density function (PDF) $f(x) \geq 0$ such that for $-\infty < x < \infty$:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt.$$

<u>Def.</u> Given the distribution function of a two-dimensional random variable (X, Y) is $F(x, y)$, if there is a non-negative function $f(x, y)$, so that for any x, y

$$F(x, y) = P\{X \in (-\infty, x], Y \in (-\infty, y]\}$$
$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u, v)dudv$$

$f(x, y)$ is the joint probability density function of X and Y.

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$f(x, y)$ is the joint probability density function of X and Y.

It follows that

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

wherever the partial derivatives are defined, or $f(x, y)$ are continuous.

2-Dimensional

$$F(x,y) \underset{F(x,y)=\int_{-\infty}^{y}\left[\int_{-\infty}^{x} f(u,v)du\right]dv}{\overset{f(x,y)=\frac{\partial^2 F(x,y)}{\partial x \partial y}}{\rightleftarrows}} f(x,y)$$

$$F_X(x), F_Y(y) \underset{F(x)=\int_{-\infty}^{x} f(t)\,dt}{\overset{f(x)=\frac{dF(x)}{dx}}{\rightleftarrows}} f_X(x), f_Y(y)$$

1-Dimensional

Ex. Given the joint PDF of X and Y as

$$f(x,y)=\begin{cases}2e^{-(2x+y)}, & x>0, y>0\\ 0, & \text{otherwise.}\end{cases}$$

Find (1) the joint CDF $F(x,y)$; (2) $P\{Y \le X\}$.

hint: $\int e^{ax}dx=\frac{1}{a}e^{ax}$

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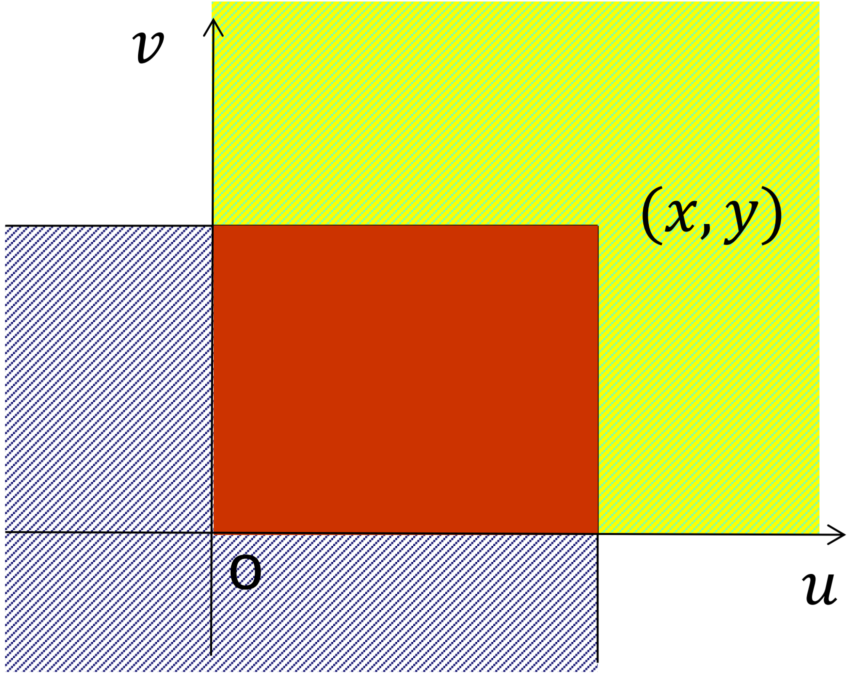
Sol.

hint: $\int e^{ax}dx=\frac{1}{a}e^{ax}$



$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x}f(u,v)dudv$$

$$=\begin{cases}\int_0^y\int_0^x 2e^{-(2u+v)}dudv, & x>0, y>0\\ 0, & \text{otherwise}\end{cases}$$

$$=\begin{cases}(1-e^{-2x})(1-e^{-y}), & x>0, y>0\\ 0, & \text{otherwise}\end{cases}$$

Some notes for integral of multi-dimensional variables

1) Why the integration variables are u, v instead of x, y ?

$\int_0^y \int_0^x 2e^{-(2u+v)} du\, dv$ (Mind the differences between u, v and x, y)

$= 2\int_0^y e^{-v}\left(\int_0^x e^{-2u} du\right) dv$

$= 2\int_0^y e^{-v}\left[-\frac{1}{2}e^{-2u}\right]_0^x dv$

Note: x, y are the upper limits of integration, while u, v are the variables for integration.

What if we do not distinguish u, v and x, y?

$\int_0^y \int_0^x 2e^{-(2x+y)} dx\, dy$

$= 2\int_0^y e^{-y}\left(\int_0^x e^{-2x} dx\right) dy$

$= 2\int_0^y e^{-y}\left[-\frac{1}{2}e^{-2x}\right]_0^x dy$ Get confused.

(2) Suppose $\{Y \le X\} = \{(X,Y) \in G\}$

$$P\{Y \le X\} = P\{(X,Y) \in G\}$$

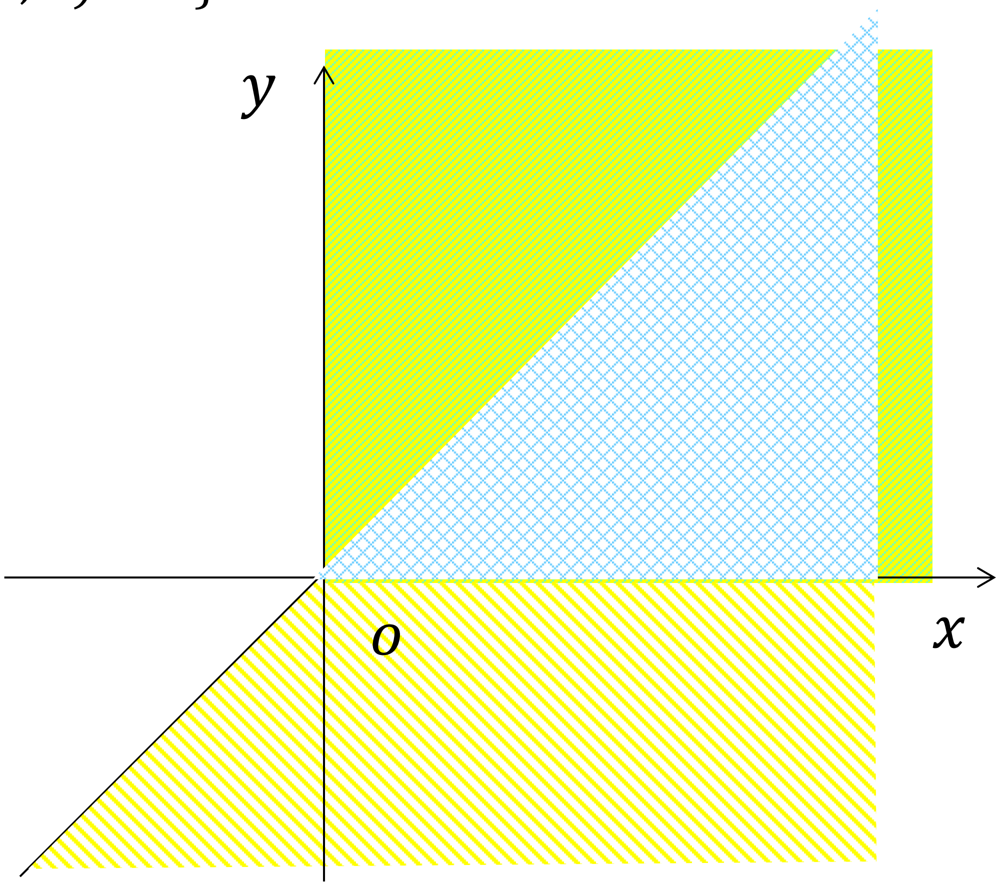
$$= \iint_G f(x,y)dxdy$$

$$\overset{(1)}{=} \int_0^{+\infty} \left[\int_y^{+\infty} 2e^{-(2x+y)}dx\right] dy$$

$$\overset{(2)}{=} \int_0^{+\infty} \left[\int_0^{x} 2e^{-(2x+y)}dy\right] dx$$

$$= 2\int_0^{+\infty} \left[-\frac{1}{2}e^{-2x}\right]_y^{+\infty} e^{-y}dy$$

$$= \int_0^{+\infty} [0 + e^{-2y}]e^{-y}dy = \left[-\frac{1}{3}e^{-3y}\right]_0^{+\infty} = \frac{1}{3}$$



Some notes for integral of multi-dimensional variables

2) How to determine the integration limit?

$$\iint_D f(x,y)d\sigma = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx, \quad x \in [a,b]$$

The limits of the inner integration are represented by the variable in the outer integration.

Ex. Let the joint PDF of (X, Y) as

$$f(x,y) = \begin{cases} Cxy, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the following:

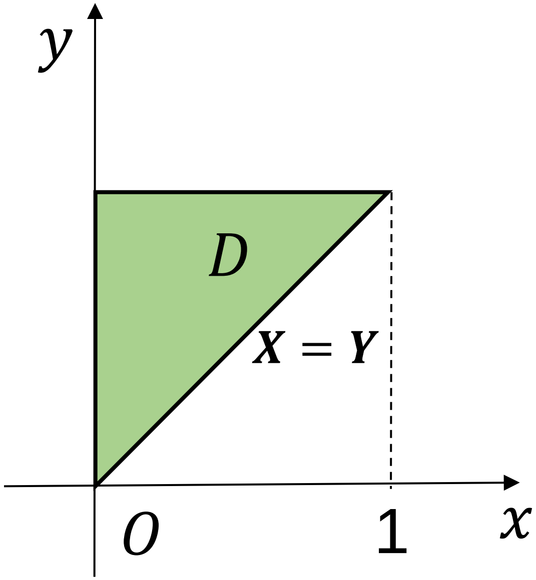
(1) value of C; (2) $P\{X + Y \le 1\}$; (3) $F(x, y)$.

Ex. Let the joint PDF of (X, Y) as

$$f(x,y) = \begin{cases} Cxy, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) determine the value of C.

Sol. (1) $D = \{(x,y) |\ 0 \le x \le y,\ 0 \le y \le 1\}$



$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)dxdy$$

$$= \iint_D f(x,y)dxdy$$

$$= \int_0^1 dx \int_x^1 Cxydy = \frac{C}{8} \Rightarrow C = 8$$

Ex. Let the joint PDF of (X, Y) as

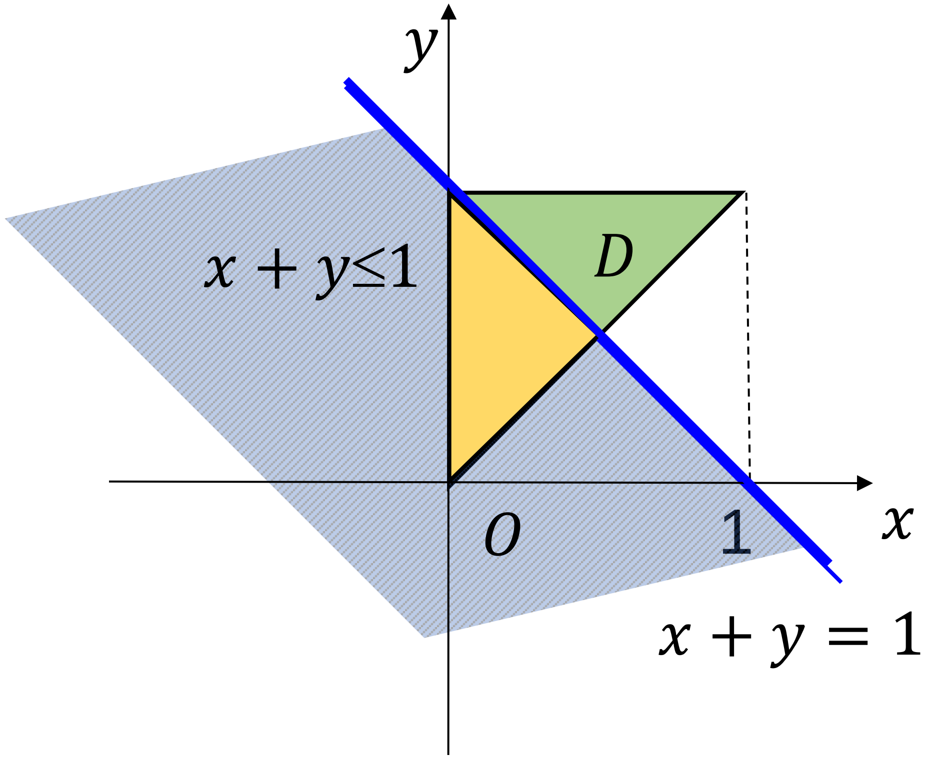
$$f(x,y)=\begin{cases} Cxy, & 0\le x\le y\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

(2) determine $P\{X+Y\le 1\}$.

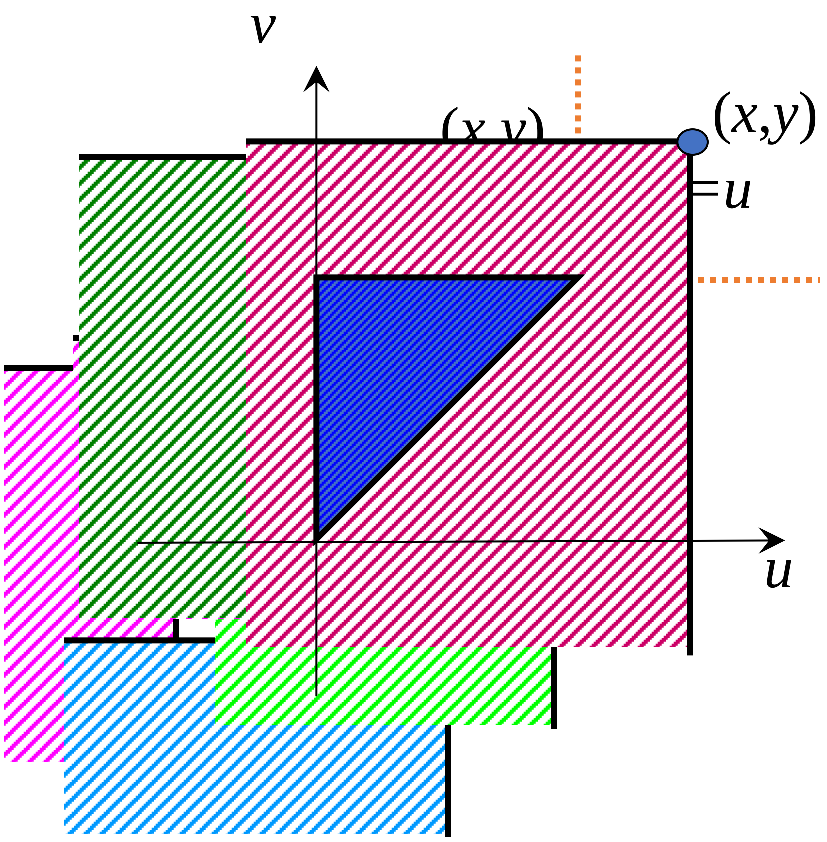
(2) $P\{X+Y\le 1\}$

$$=\iint\limits_{x+y\le 1} f(x,y)\,dxdy$$

$$=\int_0^{1/2}dx\int_x^{1-x}8xy\,dy=\frac{1}{6}$$



(3) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dvdu$



$x < 0$ or $y < 0$, $F(x,y) = 0$

$x \le y < 1, 0 \le x < 1,$

$$F(x,y) = \int_0^x du \int_u^y 8uvdv$$
$$= 2x^2y^2 - x^4$$

$x > y, 0 \le y < 1,$ $F(x,y) = \int_0^y dv \int_0^v 8uvdu = y^4$

$y \ge 1, 0 \le x < 1,$ $F(x,y) = \int_0^x du \int_u^1 8uvdv = 2x^2 - x^4$

$x \ge 1, y \ge 1,$ $F(x,y) = 1$

(3) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du$

$$F(x,y) = \begin{cases} 0, & x < 0 \text{ or } y < 0 \\ 2x^2y^2 - x^4 & x \le y < 1, 0 \le x < 1 \\ y^4 & x > y, 0 \le y < 1 \\ 2x^2 - x^4 & y \ge 1, 0 \le x < 1 \\ 1, & x \ge 1, y \ge 1 \end{cases}$$

Ex. Let the joint PDF of (X, Y) as

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(1) $P\{X > 1, Y < 1\}$; (2) $P\{X < Y\}$; (3) $P\{X < a\}$.

Hint: $\int e^{ax}dx = \frac{1}{a}e^{ax}$

Ex. Let the joint PDF of (X,Y) as

$$f(x,y)=\begin{cases}2e^{-x}e^{-2y}, & 0<x<\infty, 0<y<\infty\\ 0, & \text{otherwise.}\end{cases}$$

(1) $P\{X>1,Y<1\}$; (2) $P\{X<Y\}$; (3) $P\{X<a\}$.

Hint: $\int e^{ax}dx=\frac{1}{a}e^{ax}$

(1) $P\{X>1,Y<1\}=\int_0^1\int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$

$=\int_0^1 2e^{-2y}(-e^{-x}|_1^\infty)dy=e^{-1}\int_0^1 2e^{-2y}\,dy=e^{-1}(1-e^{-2})$

(2) $P(X<Y)=\iint_{x<y}2e^{-x}e^{-2y}dxdy=\int_0^\infty\int_0^y 2e^{-x}e^{-2y}dxdy$

$=\int_0^\infty 2e^{-2y}(1-e^{-y})dy=\int_0^\infty 2e^{-2y}dy-\int_0^\infty 2e^{-3y}dy=1-\frac{2}{3}=\frac{1}{3}$

(3) $P\{X<a\}=\int_0^a\int_0^\infty 2e^{-2y}e^{-x}\,dydx=\int_0^a e^{-x}\,dx=1-e^{-a}$

(Marginal PDF!)

Quiz 2

Date: 19-Oct-2023

Scope: discrete & continuous random variable

Open-book exam, with fill-in-the-blank and multiple-choice questions.

Time: ~45 mins