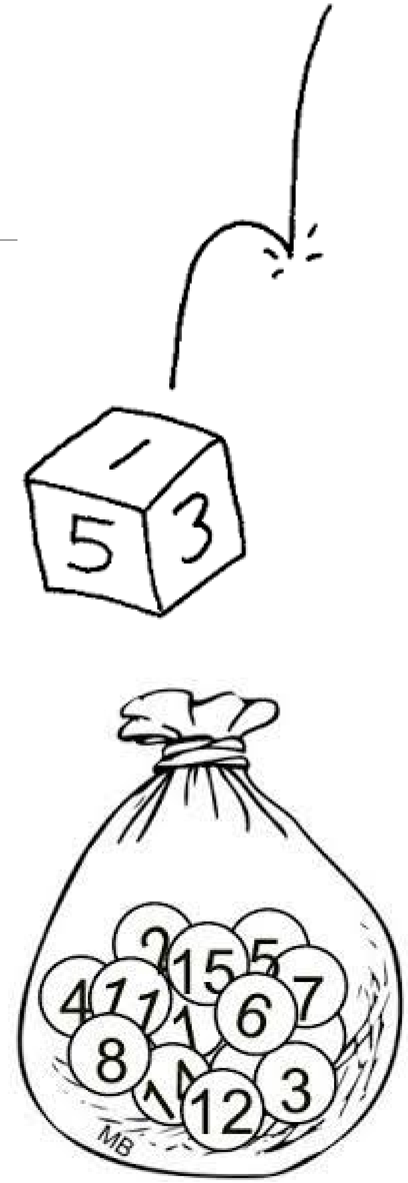
About our Final Exam

- Final exam ONLY accounts for 40% of the overall score.
- Including 7 **problem-solving** questions with sufficient hints.
- **Read and understand** each question carefully.
- Bring your calculator.

Lecture 10

- Joint PDF of Functions of R.V.s
- Expectation of R.V.s
- Quiz 3



What if we consider (Y_1, Y_2) instead of a single transformed variable Z?

$$Y_1 = g_1(X_1, X_2),\ Y_2 = g_2(X_1, X_2)$$

Joint PDF of functions of continuous R.V.s

(X_1, X_2) are continuous R.V.s with joint PDF $f(X_1, X_2)$,

Given $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$.

Check P224, single variable case: $f_Y(y) = f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right|$

Conditions:

1. y_1, y_2 can be uniquely solved for x_1, x_2, i.e., $x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$;
2. g_1, g_2 are continuous at all points (x_1, x_2), such that

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_1}{\partial x_2} \neq 0$$

Joint PDF of (y_1, y_2) is given by

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1},$$

$$x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2)$$

Joint PDF of functions of continuous R.V.s

Single variable case (from P225 of textbook):

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right|$$

$\Longrightarrow$

$$f_Y(y) = f_X[x]\left|\frac{dx}{dy}\right|, x = g^{-1}(y)$$

$\Longrightarrow$

$$\boxed{f_Y(y) = f_X[x]\left|\frac{dy}{dx}\right|^{-1}, x = g^{-1}(y)}$$

Terms to terms correspondence.

Joint PDF of (y_1, y_2) (from P280 of textbook):

$$\boxed{\begin{array}{l} f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}, \\ x_1 = h_1(y_1, y_2), x_2 = h_2(y_1, y_2) \end{array}}$$

$$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

Ex. 7a (P281) Let X_1 and X_2 be jointly continuous random variables with probability density function $f_{X_1,X_2}(X_1,X_2)$. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of $f_{X_1,X_2}(X_1,X_2)$.

Sol. Let $Y_1 = g_1(X_1,X_2) = X_1 + X_2$; $Y_2 = g_2(X_1,X_2) = X_1 - X_2$.

$$J(x_1,x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

and $x_1 = \frac{y_1+y_2}{2}$, $x_2 = \frac{y_1-y_2}{2}$

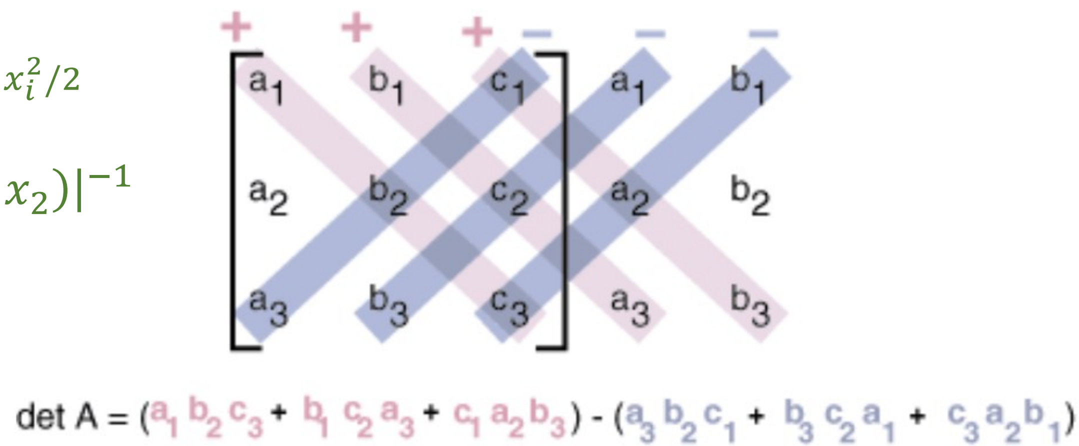
$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(x_1,x_2)|J(x_1,x_2)|^{-1} \\ &= \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) \end{aligned}$$

Further readings on n random variables in Chapter 6.7.

Ex. 7d (P285) Let X_1, X_2 and X_3 be independent standard normal variables. If $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, and $Y_3 = X_1 - X_3$, compute the joint density function of Y_1, Y_2, Y_3.

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} e^{-\sum_{i=1}^{3} x_i^2/2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}$$



Sol. Let $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2$, and $Y_3 = X_1 - X_3$, the Jacobian matrix is given as

$$J(x_1, x_2, x_3) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_3}{\partial x_1} & \frac{\partial g_3}{\partial x_2} & \frac{\partial g_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

and $x_1 = \frac{y_1+y_2+y_3}{3}, x_2 = \frac{y_1-2y_2+y_3}{3}, x_3 = \frac{y_1+y_2-2y_3}{3}$

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3} f_{X_1,X_2,X_3}\left(\frac{y_1+y_2+y_3}{3}, \frac{y_1-2y_2+y_3}{3}, \frac{y_1+y_2-2y_3}{3}\right)$$

We see that

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3(2\pi)^{3/2}} e^{-Q(y_1,y_2,y_3)/2}$$

where

$$Q(y_1, y_2, y_3) = \left(\frac{y_1 + y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - 2y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 + y_2 - 2y_3}{3}\right)^2$$

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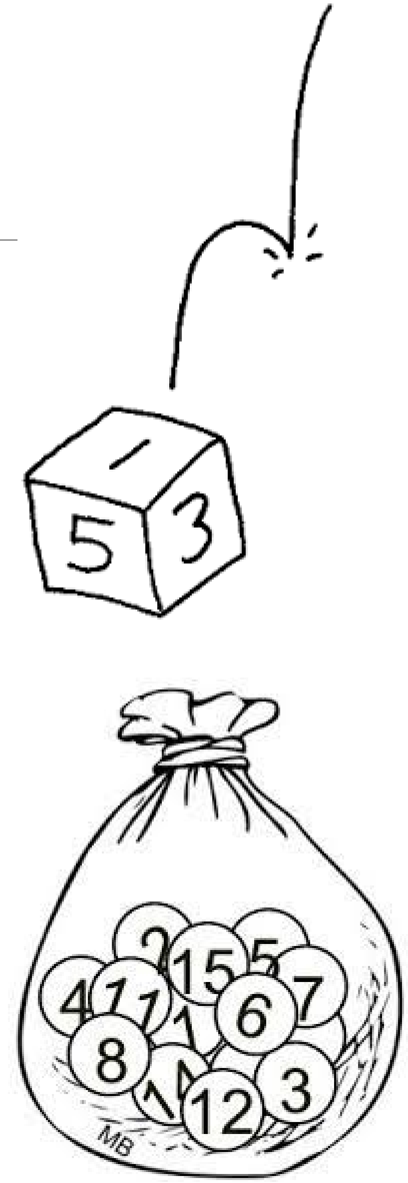
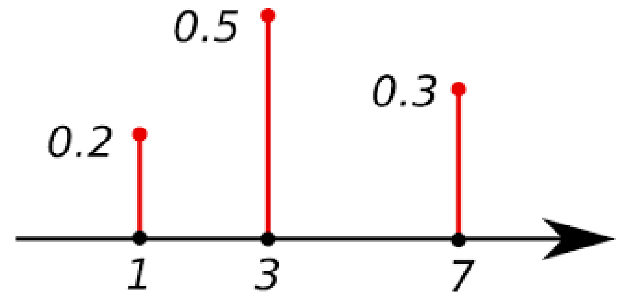


Discrete random variables

Experiment outcomes



PMF

$$P(X = x) = p(x)$$

Definition

Discrete Random Variable, X

Properties

Support

Without performing the experiment:

- The support is a set of possible values our R.V. will take on.
- Next up: How do we predict the next value?

The "expected value"!

Expectation of a discrete R.V.

The expectation of a discrete random variable X is defined as:

$$E(X) = \sum_{x:p(x)>0} x \cdot p(x)$$

- Note: weighted sum over all values of $X = x$ that have non-zero probability.

- Other names: mean, expected value, weighted average, center of mass, first moment.

Expectation of a die rolling

What is the expected value of a 6-sided die roll?

1. Define random variables

X = R.V. for value of roll

2. Solve

$$P(X = x) = \begin{cases} 1/6 & x \in \{1, \ldots, 6\} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5$$

Expectation of a roulette spinning

Game of Roulette: Bet 1 dollar

Win: get 35 dollars;
Lose: lose 1 dollar.

$$E(X) = 35 \cdot \frac{1}{38} - 1 \cdot \frac{37}{38} \approx -0.0526$$

What is the meaning of expected value?

It is a value that closest to the value of next occurrence.

Expectation of a continuous R.V.

Discrete

$$E(X) = \sum_x \boxed{x\,p(x)}$$

$$f(x)\Delta_X \approx P\{x \le X \le x + \Delta_x\}$$

$$\Sigma \Rightarrow \int$$

Continuous

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Quick check

Find $E[X]$ when the density function of X is

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Sol.

$$E[X] = \int xf(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

Ex. Assume R.V. $X \sim \pi(\lambda)$, find $E(X)$.

Ex. Assume R.V. $X \sim U(a, b)$, find $E(X)$.

Ex. Assume R.V. $X \sim \exp(\theta)$, find $E(X)$.

Hint: $\int e^{ax} dx = \frac{1}{a} e^{ax}$

Ex. Assume R.V. $X \sim \pi(\lambda)$, find $E(X)$.

Sol.

$$P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad k = 0,1,2,\cdots$$

$$E(X) = \sum_{k=0}^{+\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Power series expansion

Ex. Assume R.V. $X \sim U(a, b)$, find $E(X)$.

Sol.

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{a+b}{2}$$

Ex. R.V. X follows exponential distribution

$$f(x)=\begin{cases}\frac{1}{\theta}e^{-\frac{x}{\theta}}, & x>0 \\ 0, & x\le 0\end{cases} \qquad (\theta>0)$$

Find $E(X)$. Hint: $\int e^{ax}dx=\frac{1}{a}e^{ax}$

Sol. $E(X)=\int_0^{\infty}x\cdot\frac{1}{\theta}e^{-\frac{x}{\theta}}dx$

Integration by part: $\int_a^b u(x)v'(x)dx$

$$=[u(x)v(x)]_a^b-\int_a^b u'(x)v(x)dx$$

$$=\left[-x\cdot e^{-\frac{x}{\theta}}\right]_0^{\infty}+\int_0^{\infty}e^{-\frac{x}{\theta}}dx$$
$$=[0-0]+\left[-\theta e^{-\frac{x}{\theta}}\right]_0^{\infty}$$
$$=0+(0+\theta)=\theta$$

Expectations of common distributions

Distribution	Notation	Expected Value $E(X)$
Bernoulli	$X \sim \mathrm{Ber}(p)$	p
Binomial	$X \sim b(n, p)$	np
Poisson	$X \sim \pi(\lambda)$	λ
Uniform	$X \sim U(a, b)$	$\frac{a+b}{2}$
Exponential	$X \sim \exp(\theta)$	θ
Normal	$X \sim \mathcal{N}(\mu, \sigma^2)$	μ
Standard Normal	$X \sim \mathcal{N}(0,1)$	0

Verify these results by yourselves

Ex. If the system L consists of 5 independent sub-systems with lifetime following PDFs

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & x \leq 0 \end{cases} \qquad (\theta > 0)$$

Find the expected system lifetime under the following connections. (1) series. (2) parallel.

$$F_{\max}(x) = [F(x)]^5$$

$$F_{\min}(x) = 1 - [1 - F(x)]^5$$

Sol. The CDF of X is $F(x) = \begin{cases} 1 - e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & x \le 0 \end{cases}$

(1) Under series connection, the system lifetime N is related to $X_1, \ldots, X_5$ by $N = \min(X_1, X_2, \cdots, X_5)$, the CDF of system lifetime is

$$F_{\min}(x) = 1 - [1 - F(x)]^5 = \begin{cases} 1 - e^{-\frac{5x}{\theta}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

$$f_{\min}(x) = \begin{cases} \frac{5}{\theta} e^{-\frac{5x}{\theta}}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Exponential distribution

$$E(N) = \int_{-\infty}^{+\infty} x \, f_{\min}(x) dx = \int_{0}^{+\infty} x \frac{5}{\theta}^{-\frac{5x}{\theta}} dx = \frac{\theta}{5}$$

Or, identify that $N \sim \exp\left(\frac{\theta}{5}\right) \Rightarrow E(N) = \frac{\theta}{5}$

(2) Under parallel connection, the system lifetime M is related to $X_1, \dots, X_5$ by $M = \max(X_1, X_2, \dots, X_5)$, the CDF of system lifetime is

$$F_{\max}(x) = [F(x)]^5 = \begin{cases} (1 - e^{-\frac{x}{\theta}})^5, & x > 0 \\ 0, & x \le 0 \end{cases}$$

Therefore, the PDF is

$$f_{\max}(x) = \begin{cases} \frac{5}{\theta}(1 - e^{-\frac{x}{\theta}})^4 e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$E(M) = \int_{-\infty}^{+\infty} x\, f_{\max}(x) dx = \int_0^{+\infty} x \frac{5}{\theta}\, e^{-\frac{x}{\theta}} \left(1 - e^{-\frac{x}{\theta}}\right)^4 dx = \frac{137}{60}\theta$$

Expectations of $g(X)$

From the PMF of discrete R.V. X

$Y = g(X)$	$g(x_1)$	$g(x_2)$	...	$g(x_k)$	...
p_k	p_1	p_2	...	p_k	...

$$E(Y) = E[g(X)] = \sum_{k=1}^{\infty} g(x_k)\, p_k$$

Note: We do not need $f(y)$ to compute $E(Y)$!

Expectations of $g(X)$

Discrete R.V. X with values x_i and PMF $p(x_i)$, then for any real function $Y = g(X)$,

$$E[g(X)] = \sum_i g(x_i) \cdot p(x_i)$$

Continuous R.V. X with PDF $f(x)$, then for any real function $Y = g(X)$,

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) \cdot f(x) dx$$

Note:

- We do not need $f(y)$ to compute $E(Y)$.
- By exploiting the relationship between X and Y, only $f(x)$ is needed.

Proof

$$f_X(x) \Rightarrow f_Y(y) \Rightarrow E(Y)$$

The PDF of $Y = g(X)$ is

$$f_Y(y) = \begin{cases} f_X[h(y)]|h'(y)|, & \alpha < y < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

$$E(Y) = \int_{-\infty}^{+\infty} y\, f_Y(y)dy = \int_{\alpha}^{\beta} y f_X[h(y)]|h'(y)|dy$$

$h'(y) > 0$:

$$E(Y) = \int_{\alpha}^{\beta} y f_X[h(y)]h'(y)dy \qquad h'(y) = dx/dy$$

$$\xRightarrow{y=g(x)} \int_{h(\alpha)}^{h(\beta)} g(x)\, f_X\big[h\big(g(x)\big)\big] \frac{dx}{dy}\, dy = \int_{-\infty}^{+\infty} g(x) f_X(x)dx$$

$h'(y) < 0$:

$$E(Y) = -\int_{\alpha}^{\beta} y f_X[h(y)]h'(y)dy$$

$$= -\int_{+\infty}^{-\infty} g(x) f_X(x)dx = \int_{-\infty}^{+\infty} g(x) f_X(x)dx$$

Ex. The annual demand of rare earth in the international market is denoted by a random variable X (ton). X follows a uniform distribution $X \sim U(a, b)$. Each ton of rare earth earns s thousands yuan. However, if it is not sold and is stored in the warehouse, it costs l thousands yuan per ton. How many tons of rare earth should be prepared to maximize the expected value of the revenue?

Let t denote the amount of rare earth prepared, and X denotes the amount of rare earth sold, the revenue is:

$$Y = g(X) = \begin{cases} sX - (t - X)l, & a < X \le t \\ st, & t < X \le b \end{cases}$$

Sol. Let t and X denote the amount of rare earth prepared and sold, respectively, and $t \in [a, b]$. The revenue can be written as

$$Y = g(X) = \begin{cases} sX - (t - X)l, & a < X \leq t \\ st, & t < X \leq b \end{cases}$$

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

$$E(Y) = E[g(X)] = \int_a^b g(x) f(x) dx$$

$$= \int_a^t [sx - (t - x)l] \frac{1}{b-a} dx + \int_t^b st \frac{1}{b-a} dx$$

$$= \frac{1}{2(b-a)} [-(l+s)t^2 + 2(la + sb)t - (l+s)a^2]$$

Let $\frac{d}{dt} E(Y) = 0$, which yields

$-(l+s)t + (la + sb) = 0$, that is $t = \frac{la+sb}{l+s}$

Expectation of discrete RVs

$$E(X) = \sum_{k=1}^{\infty} x_k p_k$$

$$E[g(X)] = \sum_{k=1}^{\infty} g(x_k)\, p_k$$

$$E[g(X,Y)] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g\,(x_i, y_j)\, p_{ij}$$

$$E(X) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i p_{ij}$$

$$E(Y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} y_j p_{ij}$$

Expectation of continuous RVs

$$E(X) = \int_{-\infty}^{+\infty} x\, f(x) dx$$

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) dx$$

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\,(x\,,y) f(x,y) dxdy$$

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, f(x,y) dx\, dy$$

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, f(x,y) dx\, dy$$