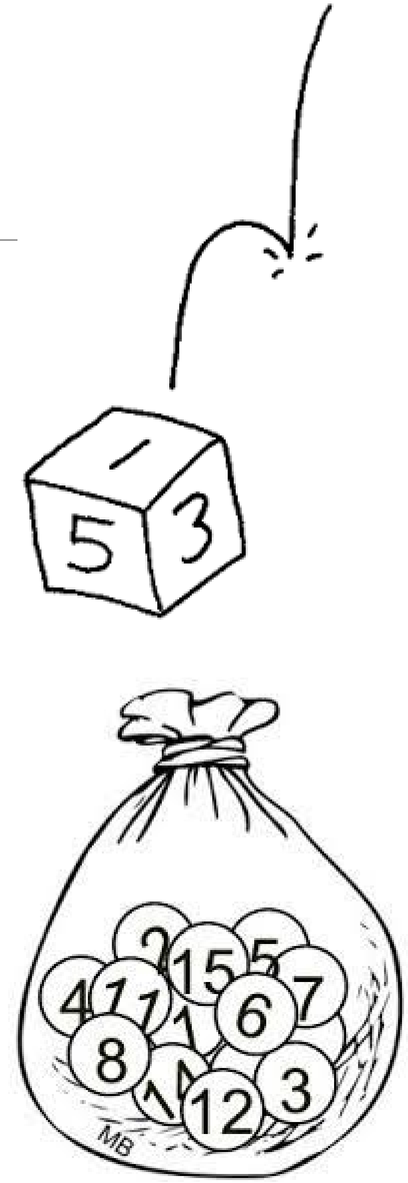
Lecture 06

- Normal distribution
- Probability function of a function of a R.V.
- Quiz 2



Campus travel time

Example: You spend some minutes X traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X follows normal distribution.

What is the probability you spend ≥ 6 minutes in travelling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^{\infty} f(x)dx = \int_6^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

OMG??! Analytical intractable.

Computing probabilities for Normal R.V.s

For a Normal R.V. $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

Difficult to solve analytically.

However, we can solve for probabilities numerically:

$$F(x) = ?(x)$$

To get there, we'll first need to know some properties of Normal RVs.

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved for numerically

Computing probabilities for Normal R.V.s

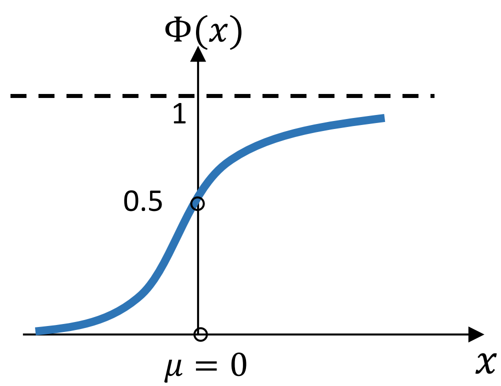
Let $X \sim \mathcal{N}(\mu, \sigma^2)$, where $\mu \neq 0, \sigma^2 \neq 1$.

To compute the CDF, $P\{X \leq x\} = F(x)$:

- We cannot analytically solve the integral (it has no closed form)
- …but we *can* solve numerically using a function Φ:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

CDF of a standard Normal variable



Probabilities for a general Normal R.V.

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P\{X \leq x\} = F(x)$, we use $\Phi(x)$, the CDF for the standard normal distribution $Z \sim \mathcal{N}(0, 1)$:

$$\boxed{F(X) = \Phi\left(\frac{x-\mu}{\sigma}\right)}$$

Proof:

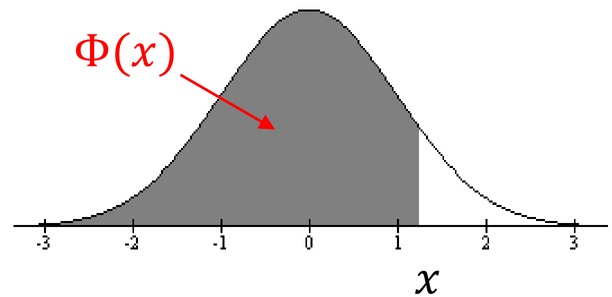
$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= P(X - \mu \leq x - \mu) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{x-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{x-\mu}{\sigma}\right)
\end{aligned}$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of X.
- $\frac{X-\mu}{\sigma} \sim \mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0,1)$.
- $\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$ with CDF Φ.

Φ has been numerically computed

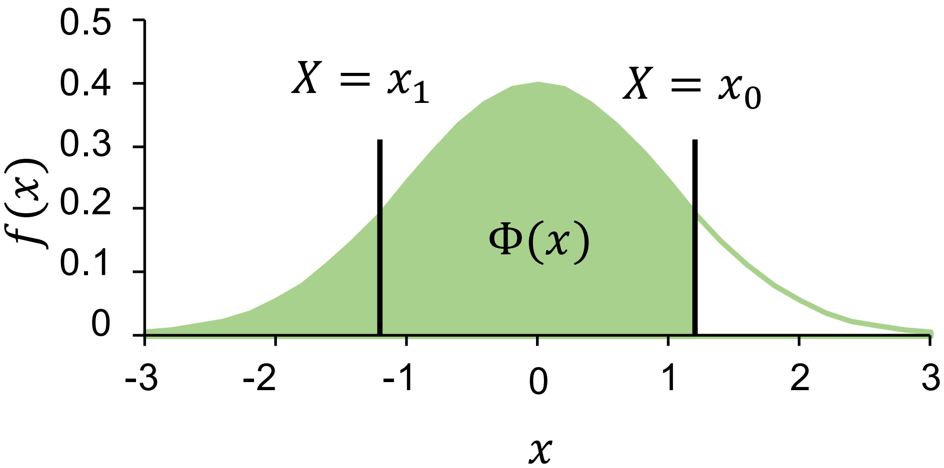
Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X.

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998



Given x, obtain $\Phi(x)$ numerically from the table.

$$P(X \leq x_0) = \Phi(x_0)$$



Input x_0, obtain $P(X < x_0)$.

Standard Normal Table only has probabilities $\Phi(x_0)$ for $x_0 \geq 0$.

What if $x_1 = -x_0 < 0$?

By symmetric of normal distribution,

$$\Phi(x_1) = 1 - \Phi(-x_1)$$
$$= 1 - \Phi(x_0)$$

Campus travel time

Example: You spend some minutes X traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes^2

Suppose X follows normal distribution.

What is the probability you spend ≥ 6 minutes in travelling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = 1 - F_X(6)$$

$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right) = 1 - \Phi(1.41) \approx 1 - 0.9207 = 0.0793$$

Ex. Let $X \sim \mathcal{N}(0,1)$, find $P\{|X| \leq 1.5\}$

Sol. $P\{|X| \leq 1.5\}$

$$= P\{-1.5 \leq X \leq 1.5\} \quad = \Phi(1.5) - \Phi(-1.5)$$

$$= \Phi(1.5) - [1 - \Phi(1.5)] \quad = 2 \cdot \Phi(1.5) - 1$$

$$= 2 \times 0.9332 - 1 = 0.8664$$

Ex. Let $X \sim \mathcal{N}(1.5, 4)$, find $P\{1 < X < 3.5\}$.

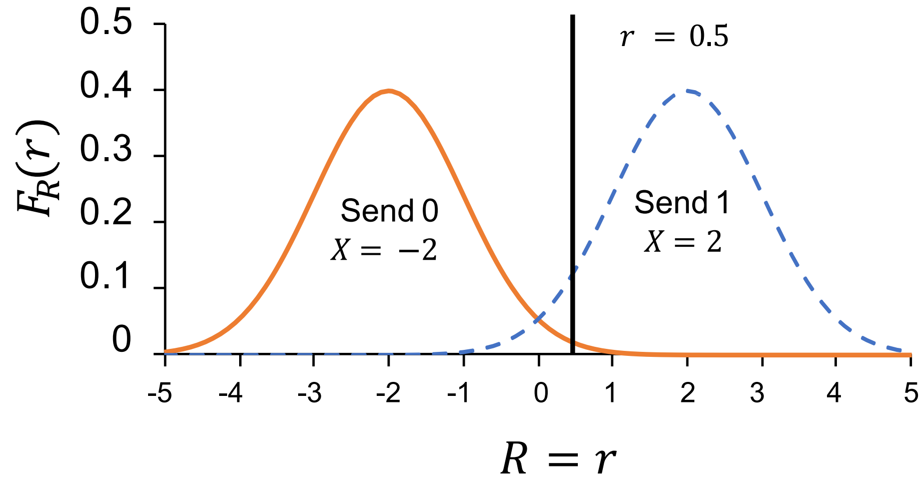
Sol. $P\{1 < X < 3.5\}$

$$= P\left\{\frac{1 - 1.5}{2} < \frac{X - 1.5}{2} < \frac{3.5 - 1.5}{2}\right\}$$

$$= \Phi\left(\frac{3.5 - 1.5}{2}\right) - \Phi\left(\frac{1 - 1.5}{2}\right) = \Phi(1) - \Phi(-0.25)$$

$$= \Phi(1) - [1 - \Phi(0.25)] \quad = 0.8413 - 1 + 0.5987 \quad = 0.44$$

Noisy wires



Send a voltage of 2 V or −2 V on wire to denote 1 and 0, respectively.

- X = voltage sent (2 or −2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$

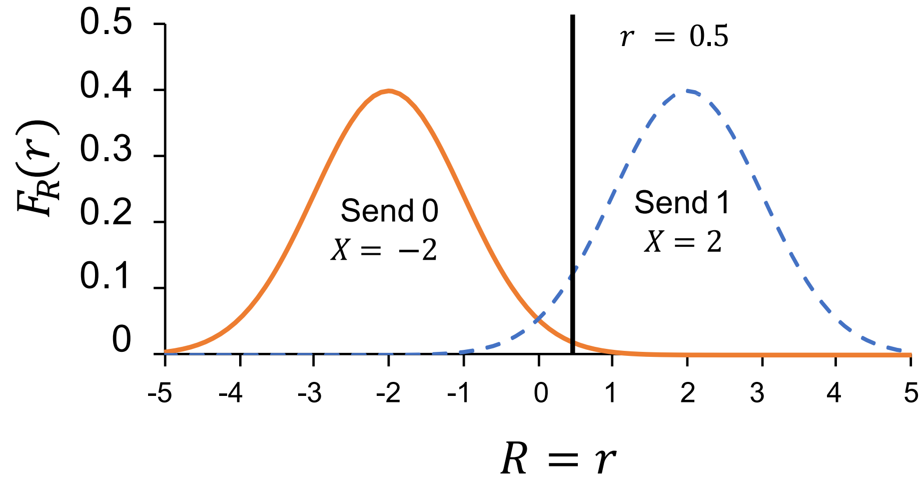
0 otherwise.

1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?
2. What is P(decoding error | original bit is 0)?

Noisy wires

Send a voltage of 2 V or −2 V on wire to denote 1 and 0, respectively.

- X = voltage sent (2 or −2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.



Decode: 1 if $R \geq 0.5$

0 otherwise.

1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

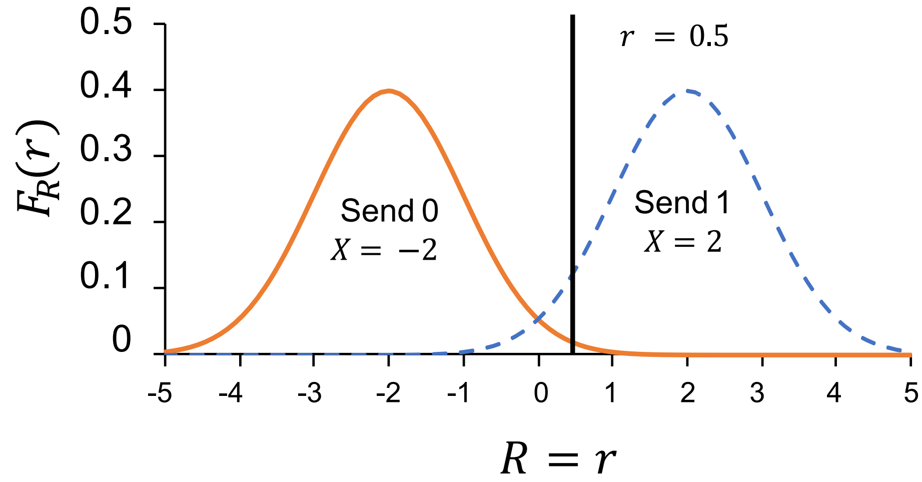
$$P\{R < 0.5 | X = 2\} = P\{2 + Y < 0.5\} = P\{Y < -1.5\}$$

$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

Noisy wires

Send a voltage of 2 V or −2 V on wire to denote 1 and 0, respectively.

- X = voltage sent (2 or −2)
- Y = noise, $Y \sim \mathcal{N}(0,1)$
- $R = X + Y$ voltage received.



Decode: 1 if $R \geq 0.5$

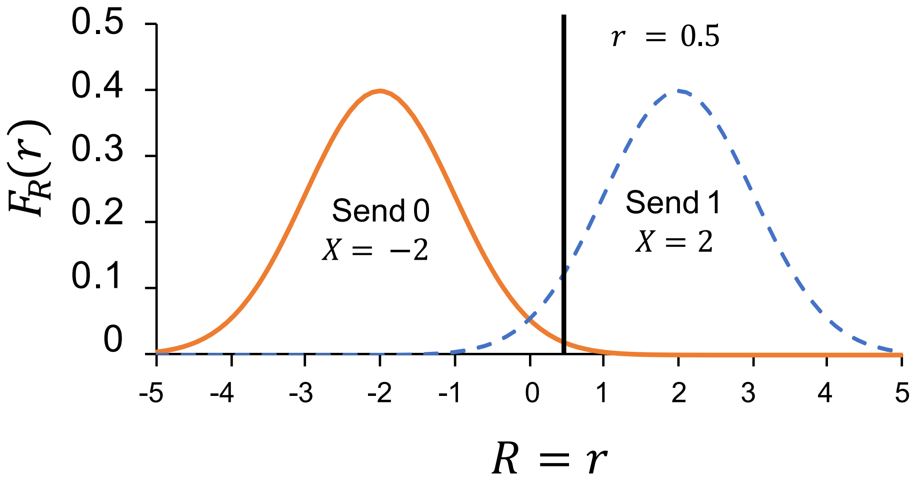
0 otherwise.

2. What is P(decoding error | original bit is 0)?

$$P\{R \geq 0.5 | X = -2\} = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

3. What is the probability of decoding error?

Noisy wires



Send a voltage of 2 V or −2 V on wire to denote 1 and 0, respectively.

- X = voltage sent (2 or −2)
- Y = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$

0 otherwise.

3. What is the probability of decoding error?

$$P\{\text{error}\} = P\{R \geq 0.5 | X = -2\}\, P\{X = -2\} + P\{R < 0.5 | X = 2\}\, P\{X = 2\}$$

Law of total probability

Ex. An expert witness in a paternity suit testifies that the length (in days) of human gestation is approximately normally distributed with parameters $\mu = 270$ and $\sigma^2 = 100$. The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before the birth.

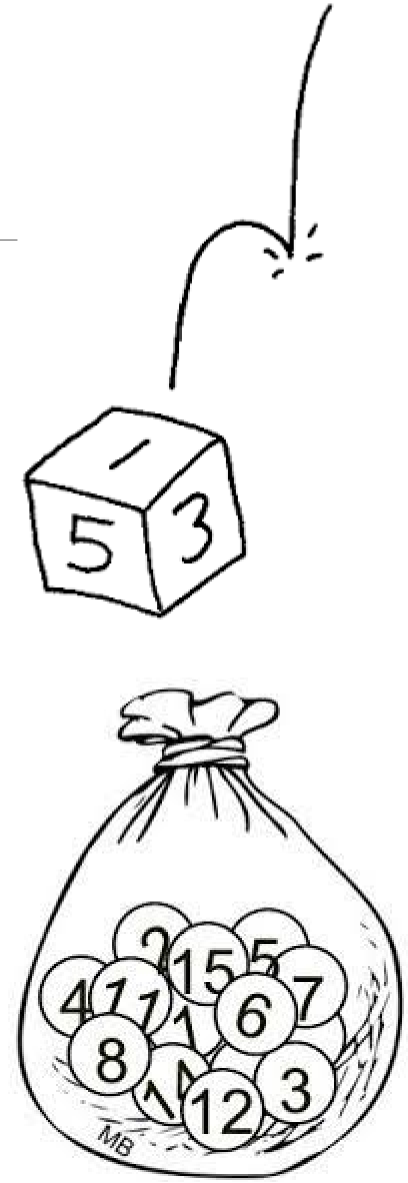
If the defendant was, in fact, the father of the child, what is the probability that the mother could have had the very long or very short gestation indicated by the testimony?

Sol. Let X denote the length of the gestation and assume that the defendant is the father. Then the probability that the birth could occur within the indicated period is

$$
\begin{aligned}
P\{X > 290 \text{ or } X < 240\} &= P\{X > 290\} + P\{X < 240\} \\
&= P\left\{\frac{X-270}{10} > \frac{290-270}{10}\right\} + P\left\{\frac{X-270}{10} < \frac{240-270}{10}\right\} \\
&= P\left\{\frac{X-270}{10} > 2\right\} + P\left\{\frac{X-270}{10} < -3\right\} \\
&= 1 - \Phi(2) + 1 - \Phi(3) \approx 0.0241
\end{aligned}
$$

Lecture 05

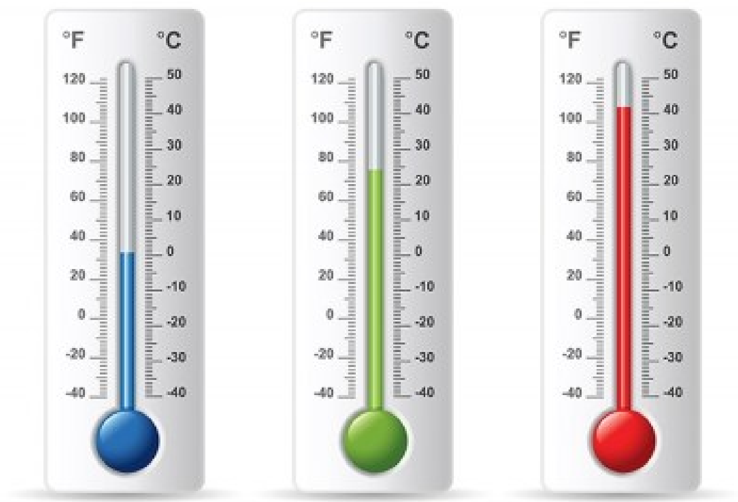
- Normal distribution
- Probability Function of a function of a R.V.
- Quiz 2



PDF of a function of a R.V.

Known: PDF of X; Want: PDF of $Y = g(x)$.

How do we measure temperature?



Cold Comfort Hot

Height of the mercury bar (x)
⇒Temperature (y)

$$Y = g(X) = a \cdot X + b$$

Known: PDF of X; Want: PDF of Y.

Since $g(X)$ is a linear function
⇒
$f_X(x)$ and $f_Y(y)$ are related by linear relationship (e.g., the uniform distribution).

What if $g(X)$ is a nonlinear function?

Two ways of getting PMF/PDF of $Y = g(X)$

Path 1:

$$X, f(x) \longrightarrow Y = g(X) \longrightarrow F(y), f(y)$$

Path 2:

$$F(x) \longrightarrow Y = g(X)$$

- ❑ For discrete R.V.s, both paths are feasible, and Path 1 is simpler.

- ❑ For continuous R.V.s, only Path 2 is feasible.

1. The probability of a single point is zero, $f(x)$ and $f(y)$ cannot be related directly.
2. $F(X \leq x)$ and $F(Y \leq y)$ can be related by $Y = g(x)$ (as shown in later slides).
3. Taking derivative of $F(y)$ yields $f(y)$.

PMF of a function of a R.V.

Ex. Given the PMF of discrete R.V. X as

X	-1	0	1	2	3
p_k	1/5	1/10	1/10	3/10	3/10

Find the PMFs of (1) $Y = X - 1$, (2) $Y = -2X$, (3) $Y = X^2$.

Sol：

X	- 1	0	1	2	3
$X - 1$	- 2	- 1	0	1	2
$-2X$	2	0	- 2	- 4	- 6
X^2		0	1	4	9
p_k		1/10	3/10	3/10	3/10

Review: Linear transformation of a Normal R.V.

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$. Linear transformations of X are also Normal.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

$$F_Y(y) = P\{Y \leq y\}$$
$$= P\{aX + b \leq y\}$$
$$= P\left\{X \leq \frac{y-b}{a}\right\}$$
$$= F_X\left(\frac{y-b}{a}\right)$$

Differentiation

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$
$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 /2\sigma^2\right\}$$
$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2(a\sigma)^2}\right\}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

PDF of a function of a R.V.

Let X be a continuous random variable having probability density function $f(X)$. Suppose that $g(x)$ is a strictly monotonic (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a probability density function given by

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right| \qquad \text{(in the domain of } X\text{)}$$

Note:

- $g(X)$ is strictly monotonic and differentiable.
- $X = g^{-1}(Y)$ is the inverse function of $Y = g(X)$.

(Partial) Proof:

For an increasing function $Y = g(X)$.

What if $Y = g(X)$ is a decreasing function?

$$F_Y(y) = P\{g(X) \leq y\} = P\{X \leq g^{-1}(y)\} = F_X\left(g^{-1}(y)\right)$$

Differentiating both sides, yields $\qquad f_Y(y) = f_X[g^{-1}(y)] \frac{d}{dy} g^{-1}(y)$

Review: Linear transformation of a Normal R.V.

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$. Linear transformations of X are also Normal.

If $Y = aX + b$, then $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$X = g^{-1}(y) = \frac{Y - b}{a}$$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}a\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2(a\sigma)^2}\right\}$$

Ex. Assume a R.V. X with PDF $f_X(x) = \begin{cases} \frac{x}{8}, & 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

Find the PDF of $Y = 2X + 8$.

Sol.

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right| \qquad X = g^{-1}(y) = \frac{Y-8}{2}$$

$$f_Y(y) = f_X\left(\frac{y-8}{2}\right)\left(\frac{y-8}{2}\right)'$$

$$= \begin{cases} \frac{1}{8}\left(\frac{y-8}{2}\right) \cdot \frac{1}{2}, & 0 < \frac{y-8}{2} < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{y-8}{32}, & 8 < y < 16 \\ 0, & \text{otherwise} \end{cases}$$

Let $f_X(x)$ defines over $x \in [a, b]$, the domain of y is computed by $[\min\{g(a), g(b)\}, \max\{g(a), g(b)\}]$.

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right|$$

Ex. 7d (P226) Let X be a continuous positive random variable with density function $f_X(X)$, and let $Y = X^n$, $n > 0$. Find $f_Y(Y)$, the probability density function of Y.

Sol.

If $Y = g(x) = x^n$, with $x > 0$, $\frac{d}{dx}g(x) = n \cdot x^{n-1} > 0$

then

$$g^{-1}(y) = y^{1/n}$$

and

$$\frac{d}{dy}[g^{-1}(y)] = \frac{1}{n}y^{\frac{1}{n}-1}$$

Hence, we obtain, for $y > 0$

$$f_Y(y) = \frac{1}{n}y^{\frac{1}{n}-1}f\left(y^{1/n}\right)$$

Ex. Assume the voltage in a system is related with phase by $V = A\sin\theta$. A is a positive constant. θ is a continuous R.V., $\theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find the PDF of V.

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right|$$

Sol.

The PDF of θ is $f(\theta) = \begin{cases} \frac{1}{\pi}, & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$

Define $v = g(\theta) = A\sin\theta$, $\frac{d}{d\theta}g(\theta) = A\cos\theta > 0$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\theta = g^{-1}(v) = \arcsin\frac{v}{A}, \qquad \frac{d}{dv}g^{-1}(v) = \frac{1}{\sqrt{A^2 - v^2}}$$

$$\psi(v) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{\sqrt{A^2 - v^2}}, & -A < v < A \\ 0, & \text{otherwise} \end{cases}$$

Quiz 2

Date: 14-Oct-2024

Scope: discrete & continuous random variable

Open-book exam, with fill-in-the-blank and multiple-choice questions.

Time: ~45 mins