Chapter 1: Complex Number and Complex Function

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Outline

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1. **Concept of Complex Numbers**

   We already know that in the range of real numbers, the equation $x^2 = -1$ has no solution. Due to the need of solving the equation, people introduced an imaginary unit i (or j) and specified $i^2 = -1$, so it is concluded that i is a root of equation $x^2 = -1$.

   For any two real numbers x and y, we call $z = x + iy$ (or written as $z = x + yi$) as a complex number, where x and y are called the real part and imaginary part of z respectively, written as $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

When the imaginary part is a number, it is customary to put the imaginary number i after the number, such as $2+3i$; When the imaginary part is a letter, it is customary to put the imaginary number i before the letter, such as $a+ib$.
When $x=0, y \neq 0, z=0+iy=iy$, it is called pure imaginary number; when $y=0, z=x+0i=x$, we think of it as a real number x.

2 **Expressions of Complex Numbers**

Since a complex number $z = x + iy$ is uniquely determined by a pair of ordered real numbers (x, y), if the complex number is described in the rectangular coordinate system on the two-dimensional (2-D) plane, the set of all complex numbers and the set of points on the plane form a one-to-one corresponding relationship. Therefore, we can represent the complex number $z = x + iy$ using the point with coordinate (x, y) on the plane, which is a commonly used geometric expression of the complex number. Also, we call this 2-D plane as the complex plane (also the z plane), the x-axis as the real axis and the y-axis as the imaginary axis .

It is easy to see that the complex number corresponds to the point on the complex plane in a one-by-one manner, that is, each complex number $z = x + iy$ determines a point with coordinate (x, y) on the complex plane, and vice versa. On one hand, we can study the problem of complex function with the help of geometric language and methods. On the other hand, it also gives a foundation for the application of complex function in practice.

On the complex plane, the complex number z is not only one-to-one corresponding to the point (x, y), but also one-to-one corresponding to the vector pointing from the origin O to the point $z = x + iy$. Therefore, the complex z can also be represented by the vector $\overrightarrow{OP}$ (see Figure 1.1). This is the vector representation of the complex number. The length of the vector is called the modulus or absolute value of the complex number z and is denoted as

$$|z| = r = \sqrt{x^2 + y^2} \tag{1.1.1}$$

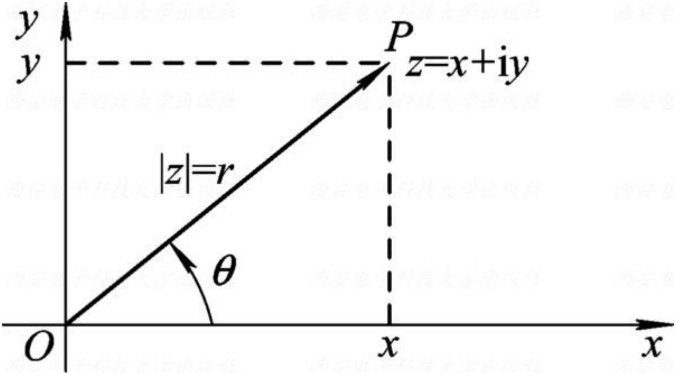


Figure: 1.1

Obviously, the following relationships are valid:

$$|x| \leq |z|, |y| \leq |z|, |z| \leq |x| + |y|$$

When $z \neq 0$, the angle θ determined by taking the positive real axis as the starting edge and the vector $\overrightarrow{OP}$ representing z as the ending edge is called as the argument of z and denoted as $\operatorname{Arg} z = \theta$.
And we have

$$\tan(\operatorname{Arg} z) = \frac{y}{x} \tag{1.1.2}$$

We know that for any complex number that is not zero $z \neq 0$, there are infinite arguments (see Figure 1.2). That is, $\operatorname{Arg} z$ is a multi-valued function, and the arguments of z satisfy: if θ_1 is one of the arguments, then

$$\operatorname{Arg} z = \theta_1 + 2k\pi, (k \in \mathbb{Z}) \tag{1.1.3}$$

The above formula gives the relationship between all the arguments of z.

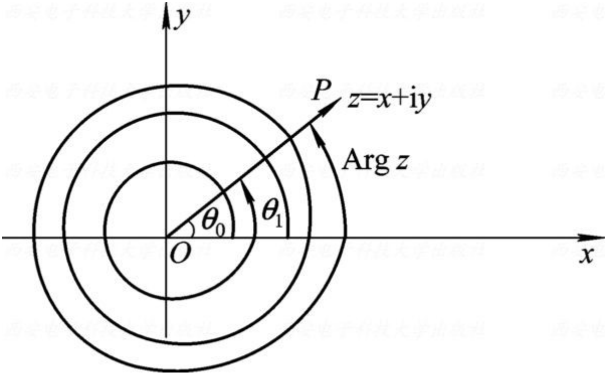


Figure: 1.2

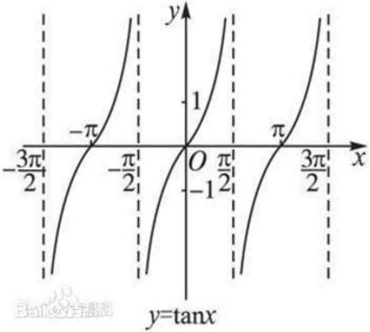
For the all possible arguments of z, we call the argument θ_0 that satisfies $-\pi < \theta_0 \leq \pi$ as the principal value of the arguments $\operatorname{Arg} z$ and it is written as $\arg z = \theta_0$, i.e.

$$\operatorname{Arg} z = \arg z + 2k\pi \quad (k \in \mathbb{Z}) \tag{1.1.4}$$

When $z = 0$, i.e. $|z| = 0$, its argument is uncertain and thus it is not defined.

The principal value of the arguments $\arg z$ ($z = x + iy$ and $z \neq 0$) can be determined by the arctangent function $\arctan \dfrac{y}{x}$ and considering z in different quadrants. Specifically, it is determined according to the following relationship:

$$\arg z(z \neq 0) = \begin{cases} \arctan \dfrac{y}{x} & (z \text{ is in the 1st and 4th quadrants}) \\ \arctan \dfrac{y}{x} + \pi & (z \text{ is in the 2nd quadrant}) \\ \arctan \dfrac{y}{x} - \pi & (z \text{ is in the 3rd quadrant}) \\ \dfrac{\pi}{2} & (z \text{ is on the positive imaginary axis}) \\ -\dfrac{\pi}{2} & (z \text{ is on the negative imaginary axis}) \\ \pi & (z \text{ is on the negative real axis}) \\ 0 & (z \text{ is on the positive real axis}) \end{cases} \tag{1.1.5}$$


y
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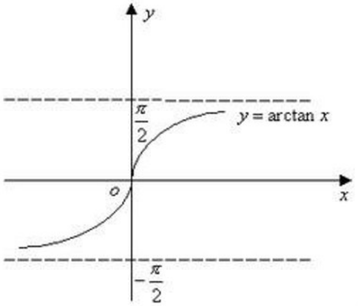
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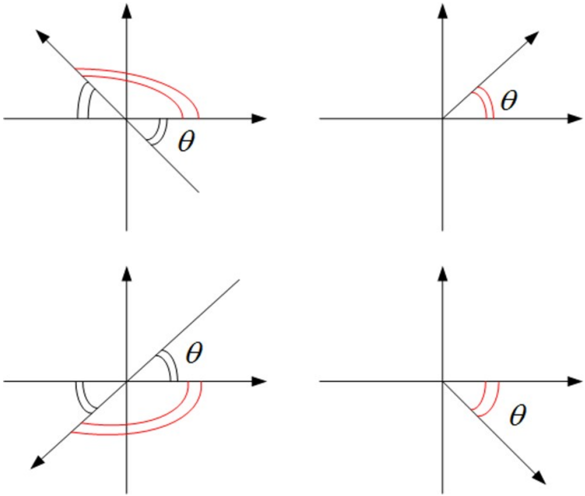
θ
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Example 1.1

Calculate the arguments and the principal argument of complex numbers $1 + i$ *and* $-3\sqrt{3} - 3i$.

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Solution.

For $z=1+i$, z is in the first quadrant, the principal argument is

$$\arg z = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4}$$

The arguments are given by

$$\operatorname{Arg} z = \arg z + 2k\pi = \frac{\pi}{4} + 2k\pi \quad (k \in \mathbb{Z}).$$

Solution (Cont.)

For $z = -3\sqrt{3} - 3i$, z is in the third quadrant, the principal argument is

$$\arg z = \arctan \frac{y}{x} - \pi = \arctan \frac{-3}{-3\sqrt{3}} - \pi = -\frac{5}{6}\pi.$$

Its any argument is given by

$$\operatorname{Arg} z = \arg z + 2k\pi = -\frac{5}{6}\pi + 2k\pi \quad (k \in \mathbb{Z}).$$

According to the conversion relationship between rectangular coordinates and polar coordinates: $x = r\cos\theta, y = r\sin\theta$, z can also be expressed in the following form:

$$z = r(\cos\theta + i\sin\theta) \tag{1.1.6}$$

where r is the modulus of z, θ is the argument of z ($\mathrm{Arg}\, z$). The above formula is called the triangular (polar) representation of the complex number z.

By substituting Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ into the triangular representation of the complex number, we can get:

$$z = re^{i\theta} \tag{1.1.7}$$

This representation is called the exponential representation of the complex number. Due to the multivalued nature of arguments, the triangular representation and exponential representation of complex z are not unique. Various representations of complex numbers can be converted to each other to meet the needs of studying different problems.

Example 1.2

Transform complex number $z = 1 + \sin 1 + i \cos 1$ *into triangular expression and exponential expression.*

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Transform complex number $z = 1 + \sin 1 + i\cos 1$ into triangular expression and exponential expression.

Solution.

First, find the modulus r of z and the principal value of arguments $\operatorname{Arg} z$:

$$
\begin{aligned}
r^2 &= (1+\sin 1)^2 + \cos^2 1 = 2(1+\sin 1) \\
&= 2\left[1 + \cos\left(\frac{\pi}{2} - 1\right)\right] = 4\cos^2\left(\frac{\pi}{4} - \frac{1}{2}\right) \\
r &= 2\cos\left(\frac{\pi}{4} - \frac{1}{2}\right) > 0
\end{aligned}
$$

Solution (Cont.)

Because $1+\sin 1>0, \cos 1>0$, so z is in the first quadrant.

$$\begin{aligned}
\arg z &= \arctan\frac{y}{x} = \arctan\left(\frac{\cos 1}{1+\sin 1}\right) \\
&= \arctan\left[\frac{\sin\left(\frac{\pi}{2}-1\right)}{1+\cos\left(\frac{\pi}{2}-1\right)}\right] \\
&= \arctan\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{1}{2}\right)\cos\left(\frac{\pi}{4}-\frac{1}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{1}{2}\right)}\right] \\
&= \left(\frac{\pi}{4}-\frac{1}{2}\right)
\end{aligned}$$

Solution (Cont.)

So the triangular expression of z is

$$z = 2\cos\left(\frac{\pi}{4} - \frac{1}{2}\right)\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

The exponential expression of z is

$$z = 2\cos\left(\frac{\pi}{4} - \frac{1}{2}\right)e^{i\left(\frac{\pi}{4} - \frac{1}{2}\right)}$$

Usually, when we convert z into its triangular or exponential forms, θ can be replaced by the principal value of the arguments.

On the multivalued nature of complex number's arguments :

1. The argument of a complex number $1+i$ is $\pi/4+2k\pi$;
2. The complex number with modulus $\sqrt{2}$ and argument $\pi/4+2k\pi$ is expressed in exponential form $\sqrt{2}e^{i\pi/4}$;
3. The complex number with modulus $\sqrt{2}$ and principal argument $\pi/4$ of is expressed in exponential form, strictly speaking, it is $\sqrt{2}e^{i\pi/4+2k\pi}$, which is usually expressed as $\sqrt{2}e^{i\pi/4}$.

Outline

1. **Four Arithmetic Operations of Complex Numbers**
The addition and subtraction between two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined as follows:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \quad (1.2.1)$$

The above formula shows that the addition and subtraction of two complex numbers is the addition and subtraction of real part and real part, and the addition and subtraction of imaginary part and imaginary part. Obviously, when z_1 and z_2 are real numbers (that is, when $y_1 = y_2 = 0$), the above formula is consistent with the operation rules for real numbers.

Let the complex numbers z_1 and z_2 be represented by the corresponding vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ respectively, then the addition and subtraction of the complex numbers are consistent with the addition and subtraction of the vectors. Thus, the diagonal $\overrightarrow{OP}$ of the parallelogram with $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ as adjacent edges on the plane represents the complex number $z_1 + z_2$ (as shown in Figure 1.3), diagonal $\overrightarrow{P_2P_1}$ represents the complex $z_1 - z_2$.

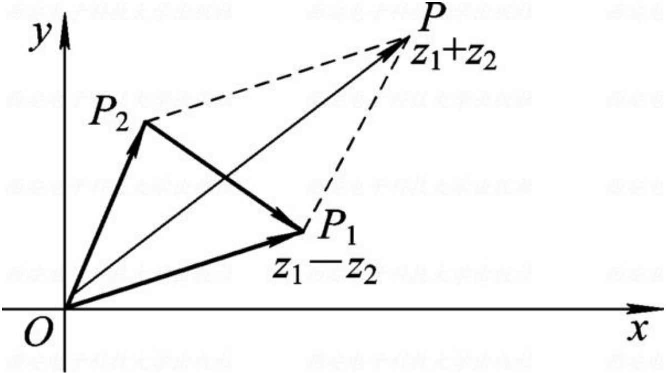

Figure: 1.3

According to the vector parallel rule, edge $|z_1|$, edge $|z_2|$ and edge $|z_1 - z_2|$ form a triangle, where $|z_1 - z_2|$ represents the length of vector $\overrightarrow{P_2P_1}$, that is, the distance between points z_1 and z_2 on the complex plane. In fact,

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

this formula just gives the distance between two points on the plane.

Edge $|z_1|$, edge $|z_2|$, and edge $|z_1 + z_2|$, also form a triangle. According to the geometric knowledge, the following two inequalities are true:

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad |z_1 - z_2| \geq ||z_1| - |z_2||$$

The multiplication and division of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined as follows:

$$z = z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \quad (z_2 \neq 0) \tag{1.2.2}$$

Similarly, the multiplication and division of complex $|z_1|$ and $|z_2|$ expressed in triangular representation or exponential representation can be derived.
If there are two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\
&= r_1 r_2 [(\cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2)] \qquad (1.2.3)
\end{aligned}
$$

Also have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \tag{1.2.4}$$

Suppose $z_1 = r_1 e^{i\theta_1}$,$z_2 = r_2 e^{i\theta_2}$. The exponential expression of multiplication and division is

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \quad (r_2 \neq 0) \end{aligned} \tag{1.2.5}$$

We can observe to get the following formulas:

$$|z_1 z_2| = |z_1|\,|z_2|\,, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{1.2.6}$$

$$\begin{cases} \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \\ \operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 \end{cases} \tag{1.2.7}$$

For example, if $z_1 = -1$ and $z_2 = i$, then $z_1 z_2 = -i$, so

$$\operatorname{Arg} z_1 = \pi + 2n\pi \quad (n \in \mathbb{Z})$$

$$\operatorname{Arg} z_2 = \frac{\pi}{2} + 2m\pi \quad (m \in \mathbb{Z})$$

$$\operatorname{Arg} z_1 z_2 = -\frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z})$$

Substituting these arguments into equation (1.2.7), we have

$$\frac{3\pi}{2} + 2(m+n)\pi = -\frac{\pi}{2} + 2k\pi$$

To let the above formula valid, $k = m + n + 1$ must be hold. As long as m and n take a certain value respectively, the value of k can always be taken to make $k = m + n + 1$, and vice versa. For example, if $m = n = 0$, then $k = 1$; if $k = -1$, then $m = 0, n = -2$ or $m = -2, n = 0$.

We now can use the triangular expression or exponential expression of complex number to understand the geometric meaning of multiplication and division of complex numbers. When we represent complex numbers as vectors, it can be said that the vector representing product $z_1 z_2$ is obtained by rotating the vector representing z_1 by an angle $\operatorname{Arg} z_2$, and stretching the amplitude of z_1 by $r_2 = |z_2|$ times, as shown in Figure 1.4.

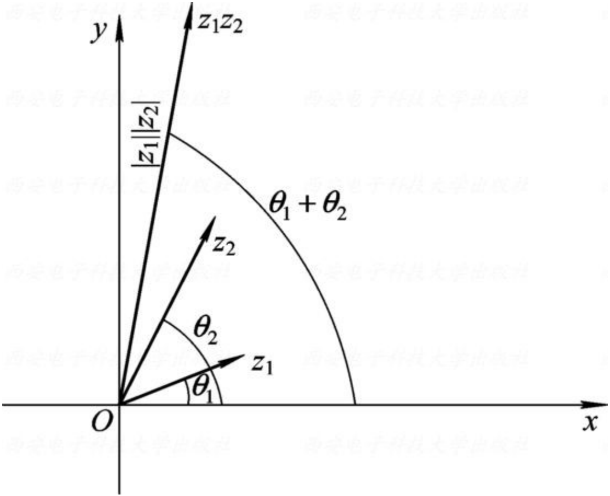

Figure: 1.4

- To extend the amplitude of the vector representing the complex z by $\sqrt{2}$ times and rotate the angle counterclockwise by $\frac{\pi}{4}$ on the complex plane, which of the following options is right:

$$\mathrm{A}.z\sqrt{2}e^{-j\frac{\pi}{4}} \qquad \mathrm{B}.\frac{z}{\sqrt{2}}e^{-j\frac{\pi}{4}} \qquad \mathrm{C}.\frac{z}{\sqrt{2}}e^{j\frac{\pi}{4}} \qquad \mathrm{D}.z\sqrt{2}e^{j\frac{\pi}{4}}$$

- Expression $\dfrac{z}{1-\sqrt{3}j}$ means to ________ on modulus and ________ on argument of z.
- There are three complex numbers z_1, z_2, z_3, we know that $z_3 = \dfrac{z_1}{z_2}$, the modulus of z_3 is $\dfrac{1}{5}$ times that of z_1, the argument of z_3 is the argument of z_1 turn counterclockwise by $\dfrac{\pi}{6}$. $z_2 =?$

Example 2.1

Let z_1 *and* z_2 *be any two points on the complex plane, to prove the inequality:* $|z_1 - z_2| \geq ||z_1| - |z_2||$.

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Proof.

The geometric meaning of this inequality is a triangle with z_1, z_2 and $(z_1 - z_2)$ as edges, and the length $|z_1 - z_2|$ of one edge is not less than the absolute value of the difference between the lengths of both edges ($||z_1| - |z_2||$). We now can prove this inequality using the triangular inequality $|z_1 + z_2| \leq |z_1| + |z_2|$.
Because

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

Proof (Cont.)

$$|z_1| - |z_2| \leq |z_1 - z_2| \tag{1}$$

Because

$$|z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1|$$

$$|z_2| - |z_1| \leq |z_2 - z_1| \tag{2}$$

Using inequalities (1) and (2), we have

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

□

Example 2.2

Given that the two vertices of an equilateral triangle are $z_1 = 1 + 2i$ and $z_2 = 3 + 2i$, find the third vertex.

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Solution.

As shown in Figure 1.5, we fix the point of z_1 and rotate the vector $z_2 - z_1$ by angle of $\frac{\pi}{3}$ (or $-\frac{\pi}{3}$) to obtain another vector, and its end point is the third vertex z_3.

Solution (Cont.)

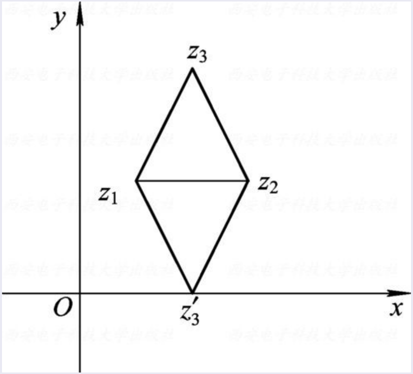


Figure: 1.5

As in the case of real numbers, the four arithmetic operations of complex numbers also satisfy the commutative law, the associative law and the distributive law:

$$z_1 + z_2 = z_2 + z_1; \; z_1 z_2 = z_2 z_1 \qquad \text{(commutative law)}$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3; \; z_1(z_2 z_3) = (z_1 z_2) z_3 \qquad \text{(associative law)}$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \text{(distributive law)}$$

After introducing the above four operations, all complex numbers are called complex field, which is usually represented by symbol ($\mathbb{C}$).

2 Powers and Roots of Complex Numbers

The product of n identical non-zero finite complex numbers, z, is called the n-th power of z and is written as z^n, i.e.

$$z^n = \underbrace{z \cdot z \cdots z}_{n}$$

If $z = re^{i\theta}$, then

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) \quad (n \in \mathbb{N})$$

In particular, when $r = 1$, that is, $z = \cos\theta + i\sin\theta$, then de Moivre formula is obtained:

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) \qquad (1.2.8)$$

For $n \in \mathbb{N}$, we have $z^n = r^n(\cos n\theta + i \sin n\theta)$. Now, we want to ask $z^{-n} =?$ for $n \in \mathbb{Z}$?
When $n \in \mathbb{N}$, we compute:

$$
\begin{aligned}
z^{-n} = z^{-n} = (z^{-1})^n &= \frac{1}{z^n} \\
&= \frac{1}{\underbrace{z \cdot z \cdots z}_{n}} \\
&= \frac{1}{re^{j\theta} re^{j\theta} \cdots re^{j\theta}} \\
&= \frac{1}{r^n e^{j\theta n}} \\
&= r^{-n} e^{-j\theta n} \\
\Rightarrow z^{-n} &= r^{-n}[\cos(-n\theta) + j \sin(-n\theta)]
\end{aligned}
$$

Therefore, we finally have $z^n = r^n(\cos n\theta + i \sin n\theta)$ for $n \in \mathbb{Z}$.

If $z = w^n$, w is called the n-th root of z, i.e., $w = \sqrt[n]{z}$. Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and $w = \rho e^{i\varphi} = \rho^n(\cos\varphi + i\sin\varphi)$, according to de Moivre formula (1.2.8)

$$\rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta) \qquad (1.2.9)$$

which gives $\rho^n = r$, $\cos n\varphi = \cos\theta$, $\sin n\varphi = \sin\theta$. Obviously, the necessary and sufficient conditions for the corrections of the equation (1.2.9) are

$$\rho^n = r, \quad n\varphi = \theta + 2k\pi \quad (k \in \mathbb{Z})$$

Therefore, we have $\rho = r^{\frac{1}{n}}$ and $\varphi = \dfrac{\theta + 2k\pi}{n}$, where $r^{\frac{1}{n}}$ is the arithmetic root of real number r. To combine the above results, the n-th root of z is given by

$$\omega = \sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \qquad (1.2.10)$$

When $k = 0, 1, 2, \cdots, n-1$, n different roots are obtained:

$$\omega_0 = r^{\frac{1}{n}} \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)$$

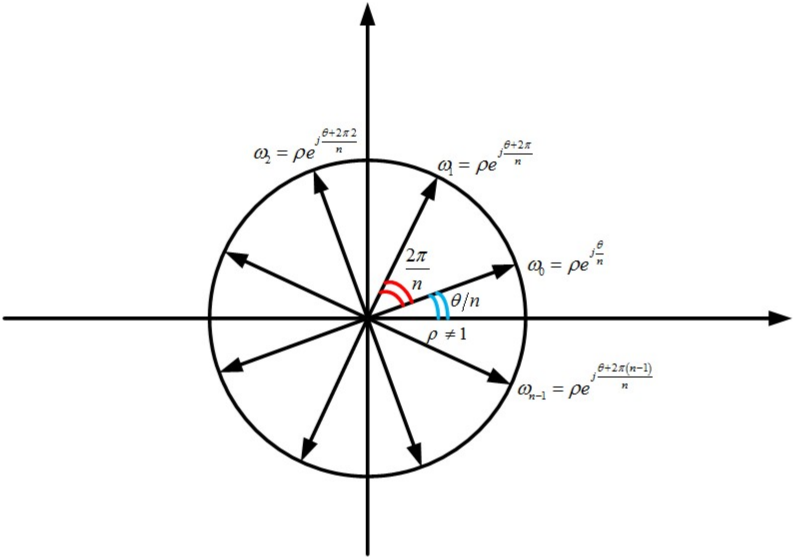
$$\omega_1 = r^{\frac{1}{n}} \left(\cos \frac{\theta + 2\pi}{n} + i \sin \frac{\theta + 2\pi}{n} \right)$$

$$\vdots$$

$$\omega_{n-1} = r^{\frac{1}{n}} \left(\cos \frac{\theta + 2(n-1)\pi}{n} + i \sin \frac{\theta + 2(n-1)\pi}{n} \right)$$

When k is substituted with other integer values, these roots appear again. For example, when $k = n$, there is

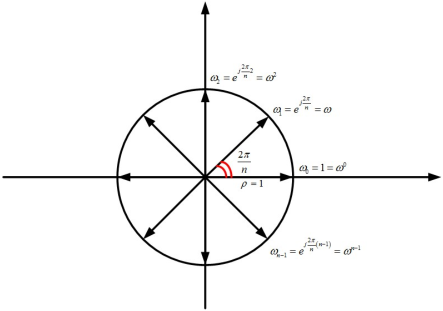
$$\omega_n = r^{\frac{1}{n}} \left(\cos \frac{\theta + 2n\pi}{n} + i \sin \frac{\theta + 2n\pi}{n} \right) = \omega_0$$


$\omega_2 = \rho e^{j\frac{\theta+2\pi 2}{n}}$
$\omega_1 = \rho e^{j\frac{\theta+2\pi}{n}}$
$\frac{2\pi}{n}$
$\omega_0 = \rho e^{j\frac{\theta}{n}}$
θ/n
$\rho \neq 1$
$\omega_{n-1} = \rho e^{j\frac{\theta+2\pi(n-1)}{n}}$

In particular, when $z = 1$, let

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

then the n-th roots of 1 is $1, \omega, \omega^2, \cdots, \omega^{n-1}$ respectively.



Example 2.3

Find the value of $\left(\dfrac{\sqrt{3}+i}{1-i}\right)^6$

Example 2.3

Find the value of $\left(\dfrac{\sqrt{3}+i}{1-i}\right)^6$

Solution.

Because

$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2e^{\frac{\pi}{6}i}$$

$$1-i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = \sqrt{2}e^{-\frac{\pi}{4}i}$$

Solution (Cont.)

$$\frac{\sqrt{3}+i}{1-i} = \frac{2e^{\frac{\pi}{6}i}}{\sqrt{2}e^{-\frac{\pi}{4}i}} = \sqrt{2}e^{\frac{5\pi}{12}i}$$

then

$$\left(\frac{\sqrt{3}+i}{1-i}\right)^6 = \left(\sqrt{2}e^{\frac{5\pi}{12}i}\right)^6 = 8e^{\frac{5\pi}{2}i} = 8\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = 8i$$

Example 2.4

Solve equation $(1+z)^6 = (1-z)^6$.

Example 2.4

Solve equation $(1+z)^6 = (1-z)^6$.

Solution.

Obviously, the root of the equation $z \neq 1$, so the original equation can be written as

$$\left(\frac{1+z}{1-z}\right)^6 = 1$$

Let $\omega = \dfrac{1+z}{1-z}$, then $\omega^6 = 1$.

Solution (Cont.)

Because $1 = \cos 0 + i \sin 0$, then

$$\sqrt[6]{1} = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6} \quad (k = 0, 1, 2, 3, 4, 5)$$

then

$$\omega_0 = (\cos 0 + i \sin 0)$$
$$\omega_1 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Solution (Cont.)

$$\omega_2 = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\omega_3 = (\cos\pi + i\sin\pi)$$

$$\omega_4 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$\omega_5 = \left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

These six roots are the six vertices of a regular hexagon inscribed on a circle with radius 1 centered at the origin (see Figure 1.6).

Solution (Cont.)

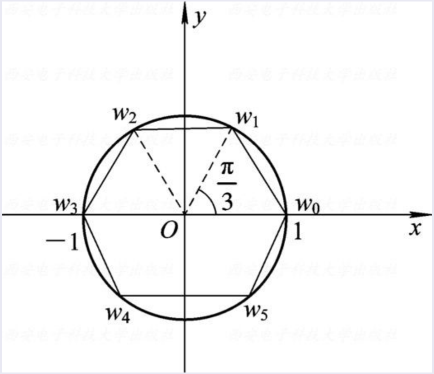


Figure: 1.6

Solution (Cont.)

From

$$\omega = \frac{1+z}{1-z} \Rightarrow z = \frac{\omega - 1}{\omega + 1}$$

$$z = \frac{e^{i\alpha} - 1}{e^{i\alpha} + 1} = \frac{\cos\alpha + i\sin\alpha - 1}{\cos\alpha + i\sin\alpha + 1}$$

$$z = \frac{2\sin\frac{\alpha}{2}\left(-\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\right)}{2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)} = \tan\frac{\alpha}{2}\frac{e^{i\left(\frac{\pi}{2}+\frac{\alpha}{2}\right)}}{e^{i\frac{\alpha}{2}}} = i\tan\frac{\alpha}{2}$$

Therefore, the root of the original equation is $i\tan\dfrac{\alpha}{2}$, where $\alpha = 0, \dfrac{2\pi}{6}, \dfrac{4\pi}{6}, \dfrac{6\pi}{6}, \dfrac{8\pi}{6}, \dfrac{10\pi}{6}$

3 Complex Conjugate

We say $\overline{z}$ is z's complex conjugate, if the positions of a pair of the complex numbers z and $\overline{z}$ in the complex plane are symmetrical about the real axis x (see Figure 1.7). According to this definition, there is $|z| = |\overline{z}|$; and if z is not on the negative real axis and origin, there is $\arg z = -\arg \overline{z}$.

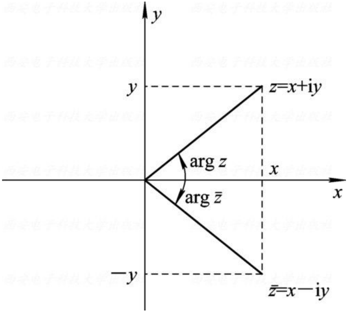


Figure: 1.7

Some computation properties about complex conjugates

1. $|z| = |\overline{z}|$, $\arg z = -\arg \overline{z}$, $\overline{\overline{z}} = z$
2. $z\overline{z} = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = |z|^2$
3. $z + \overline{z} = 2\mathrm{Re}(z), z - \overline{z} = 2i\mathrm{Im}(z)$
4. $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2, \overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2, \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$
5. If $P(z)$, and $Q(z)$ are real coefficient polynomial of z and $Q(z) \neq 0$, we have $\overline{\left(\dfrac{P(z)}{Q(z)}\right)} = \dfrac{\overline{P(z)}}{\overline{Q(z)}} = \dfrac{P(\overline{z})}{Q(\overline{z})}$.

Example 2.5

Let $z_1 = 1 + 2i, z_2 = -3 - 4i$, *find* $\dfrac{z_1}{z_2}, \overline{\left(\dfrac{z_1}{z_2}\right)}, \mathrm{Re}\left(\dfrac{z_1}{z_2}\right), \mathrm{Im}\left(\dfrac{z_1}{z_2}\right)$.

Example 2.5

Let $z_1 = 1 + 2i, z_2 = -3 - 4i$, *find* $\frac{z_1}{z_2}, \overline{\left(\frac{z_1}{z_2}\right)}, \mathrm{Re}\left(\frac{z_1}{z_2}\right), \mathrm{Im}\left(\frac{z_1}{z_2}\right)$.

Solution.

$$\frac{z_1}{z_2} = \frac{1+2i}{-3-4i} = \frac{(1+2i)(-3+4i)}{(-3-4i)(-3+4i)}$$
$$= \frac{-3-8+(4-6)i}{25} = -\frac{11}{25} - \frac{2}{25}i$$

Solution (Cont.)

So

$$\overline{\left(\frac{z_1}{z_2}\right)} = -\frac{11}{25} + \frac{2}{25}i$$

that is

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = -\frac{11}{25}, \quad \mathrm{Im}\left(\frac{z_1}{z_2}\right) = -\frac{2}{25}$$

Example 2.6

Let a *and* b *be real numbers,* $b \neq 0, |a+ib| = 1$*, then prove that there must be a real number* c *so that* $a + ib = \dfrac{c+i}{c-i}$.

Proof.

$$
\begin{aligned}
& z = a + ib \\
\Rightarrow\ & z = \frac{c+i}{c-i} \\
\Rightarrow\ & c = \frac{z+1}{z-1} i \\
\Rightarrow\ & c = \frac{a+ib+1}{a+ib-1} i = \frac{(a+1)+ib}{(a-1)+ib} i = \frac{[(a+1)+ib][(a-1)-ib]}{[(a-1)+ib][(a-1)-ib]} i \\
& = \frac{(a+1)(a-1)+b^2+ib[(a-1)-(a+1)]}{(a-1)^2+b^2} i \\
& = \frac{a^2+b^2-1-2bi}{a^2-2a+1+b^2} i \\
& = \frac{b}{(1-a)} \quad (\because a^2+b^2=1)
\end{aligned}
$$

$\square$

Example 2.7

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ *be two arbitrary complex numbers, and prove*

1. $z_1\overline{z_2} + \overline{z_1}z_2 = 2\,\mathrm{Re}(z_1\overline{z_2})$;
2. $|z_1 + z_2| \leq |z_1| + |z_2|$.

Proof.

1. $$\begin{aligned} z_1\overline{z_2} + \overline{z_1}z_2 &= (x_1 + iy_1)(x_2 - iy_2) + (x_1 - iy_1)(x_2 + iy_2) \\ &= (x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2) \\ &\quad + (x_1x_2 + y_1y_2) + i(-x_2y_1 + x_1y_2) \\ &= 2(x_1x_2 + y_1y_2) \\ &= 2\,\mathrm{Re}(z_1\overline{z_2}) \end{aligned}$$

   or $z_1\overline{z_2} + \overline{z_1}z_2 = z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\,\mathrm{Re}(z_1\overline{z_2})$

Proof (Cont.)

2. We can use the geometric method to get the triangle inequality; however, we now use the complex number operation to prove it.

   Because

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_2\overline{z_2} + \overline{z_1}z_2 + z_1\overline{z_2} \\
&= |z_1|^2 + |z_2|^2 + \overline{z_1}z_2 + z_1\overline{z_2}
\end{aligned}$$

Proof (Cont.)

From $z_1\overline{z_2} + \overline{z_1}z_2 = 2\,\mathrm{Re}(z_1\overline{z_2})$ (1), there is

$$\begin{aligned}
|z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \\
&\le |z_1|^2 + |z_2|^2 + 2\,|z_1\overline{z_2}| \\
&= |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Taking square root on both sides can finish the proof of the trigonometric inequality. $\square$

Example 2.8

Proof: The necessary and sufficient condition for the three complex numbers z_1, z_2, z_3 to become the vertices of an equilateral triangle is that they satisfy the equation $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

Proof.

Method 1.

The necessary and sufficient condition for $\triangle z_1 z_2 z_3$ to be an equilateral triangle is: vector $\overrightarrow{z_1 z_2}$ rotates $\dfrac{\pi}{3}$ or $-\dfrac{\pi}{3}$ around z_1 to get the vector $\overrightarrow{z_1 z_3}$, which is $z_3 - z_1 = (z_2 - z_1)e^{\pm \frac{\pi}{3} i}$ or

Proof (Cont.)

$$\frac{z_3 - z_1}{z_2 - z_1} = e^{\pm\frac{\pi}{3}i} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}, \quad \frac{z_3 - z_1}{z_2 - z_1} - \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i$$

Simplify the square on both sides, we can get

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

□

Proof.

Method 2.

The equation $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ can be written as

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3} \tag{1}$$

If the complex numbers z_1, z_2, z_3 satisfy equation (1), then

$$|z_3 - z_1|^2 = |z_1 - z_2|\,|z_2 - z_3| \tag{2}$$

Proof (Cont.)

$$\frac{z_2 - z_1}{z_3 - z_1} - 1 = \frac{z_1 - z_3}{z_2 - z_3} - 1$$

which is

$$\frac{z_2 - z_3}{z_3 - z_1} = \frac{z_1 - z_2}{z_2 - z_3} \tag{3}$$

Taking the absolute values of both sides can give

$$|z_2 - z_3|^2 = |z_1 - z_2|\,|z_3 - z_1| \tag{4}$$

Proof (Cont.)

Conversely, if $\triangle z_1 z_2 z_3$ is an equilateral triangle, then

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1$$

$$\begin{cases} \arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_1) = \pm\dfrac{\pi}{3} \\ \arg\left(\dfrac{z_1 - z_3}{z_2 - z_3}\right) = \arg(z_1 - z_3) - \arg(z_2 - z_3) = \pm\dfrac{\pi}{3} \end{cases}$$

Proof (Cont.)

And the signs are the same, so

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}$$

$\square$

Finally, we briefly illustrate the application of complex numbers with examples. We discuss the following two questions in detail: i) how to use a complex number equation to express a plane curve $F(x, y) = 0$; and ii) how to determine the plane curve represented by a complex number equation.

Example 2.9

The straight line passing through the two points $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ *is represented by a complex equation.*

Example 2.9

The straight line passing through the two points $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ *is represented by a complex equation.*

Solution.

In plane geometry, we know that a straight line passing through the points (x_1, y_1) and (x_2, y_2) can be expressed by a parametric equation as

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \end{cases} \qquad (-\infty < t < +\infty)$$

Let $z = x + iy$. Then, we have

$$\begin{aligned} z &= x_1 + t(x_2 - x_1) + i[y_1 + t(y_2 - y_1)] \\ &= x_1 + iy_1 + t[x_2 + iy_2 - (x_1 + iy_1)] \\ &= z_1 + t(z_2 - z_1) \end{aligned}$$

Solution (Cont.)

In fact, if $z = x + iy$ is set, then by the nature of conjugate complex numbers,

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}$$

Substituting them into the given linear equation $ax + by = c$, we have

$$a\left(\frac{z + \overline{z}}{2}\right) + b\left(\frac{z - \overline{z}}{2i}\right) = c$$

which can be simplified as

$$(a - ib)z + (a + ib)\overline{z} = 2c$$

Example 2.10

Find the graphics represented by the following equation:

1. $|z + 2i| = 2$
2. $|z - i| + |z + i| \leq 4$
3. $0 \leq \arg(z - 1) \leq \dfrac{\pi}{4}, 2 \leq \operatorname{Re} z \leq 3$

Example 2.10

Find the graphics represented by the following equation:

1. $|z+2i|=2$
2. $|z-i|+|z+i|\leq 4$
3. $0\leq \arg(z-1)\leq \dfrac{\pi}{4}, 2\leq \operatorname{Re} z\leq 3$

Solution.

1. It is not difficult to see geometrically that the equation $|z+2i|=2$ represents the trajectory of all points with a distance of 2 from the point $-2i$, that is, a circle with a center of $-2i$ and a radius of 2 (see Figure 1.8 (a)).

Solution (Cont.)

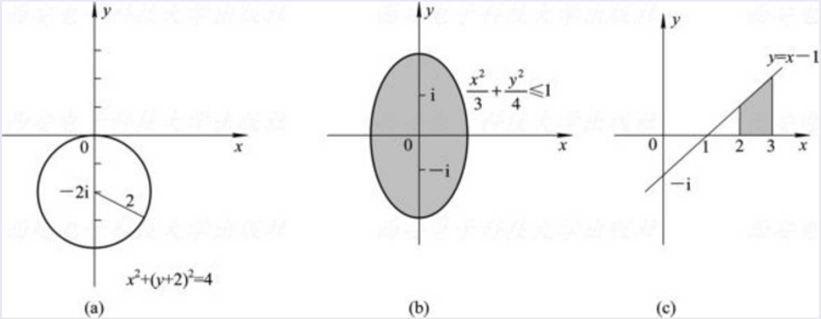


Figure: 1.8

4 **Complex Spheres and the Point at Infinity**

We take a sphere that is tangent to the complex plane at the coordinate origin O. A point S on the sphere coincides with the origin, and a straight line perpendicular to the complex plane is made through O. The line intersects the sphere at another point N. S and N are called the south and north poles of the sphere (see Figure 1.9). Take any point z on the complex plane, then the line connecting zN must intersect the only point P on the sphere; conversely, if P is any point on the sphere that is different from N, then the line connecting NP must intersect the only point on the complex plane z.

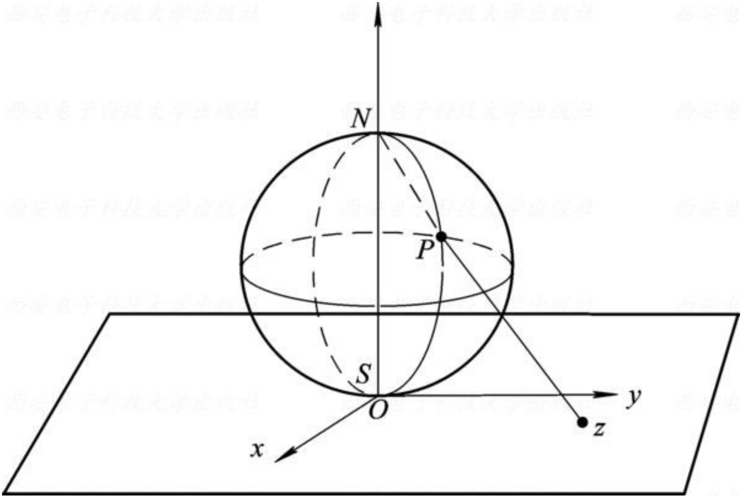


Figure: 1.9

The complex plane plus the point ∞ is called the extended complex plane, which corresponds to the entire sphere, called the complex sphere. The complex sphere can clearly express the infinity point of the extended complex plane. This is why it is superior to the complex plane. In other words, a geometric model of the extended complex plane is the complex sphere.

Finally, it should be pointed out that there is only one point at infinity on the extended complex plane, which is different from the concepts of $+\infty$ and $-\infty$ in calculus. ∞ is not a symbol, but a definite complex number. For ∞, the real part, imaginary part, and argument are meaningless, and only the modulus $|\infty| = +\infty$ is specified.

Regarding its operation, we also specify that if a is a finite complex number, the four arithmetic operations between a and ∞ are

- Addition: $a + \infty = \infty + a = \infty$
- Subtraction: $a - \infty = \infty - a = \infty$
- Multiplication: $a \cdot \infty = \infty \cdot a = \infty \quad (a \neq 0)$
- Division: $\dfrac{a}{\infty} = 0, \dfrac{\infty}{a} = \infty, \dfrac{a}{0} = \infty \quad (a \neq 0$, but can be $\infty)$

Outline

1. Concepts of Point Sets on Complex Plane
   Let us first take a point z_0 on the complex plane.

Definition

A circle centered at z_0 and with a radius of positive number δ in the complex plane is expressed as

$$|z - z_0| < \delta$$

The set of all points inside this circle but not on the circle is called the δ neighborhood of point z_0 (see Figure 1.10). And the point set determined by the inequality $0 < |z - z_0| < \delta$ is called the deleted δ neighborhood of z_0.

Definition (Cont.)

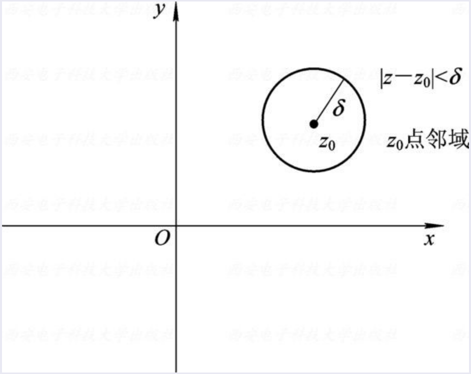


Figure: *1.10*

Definition

Let G be a set of points on the plane, z_0 be any point that belongs to G, $z_0 \in G$.

- *If there is a neighborhood of z_0 that only contains the points belonging to G, then z_0 is called an interior point of G.*
- *If every point in G is its interior point, then G is called an open set.*
- *If there are points belonging to G and some points not belonging to G in any neighborhood of point z_0, then point z_0 is called a boundary point of G.*
- *The set P consisting of all the boundary points of G is called the boundary of G. The boundary of a point set may be composed of several curves and some isolated points (see Figure 1.11).*

Definition (Cont.)

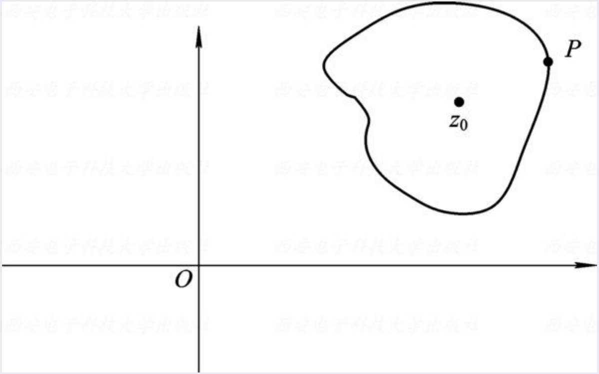


Figure: *1.11*

Definition

Suppose G is a set of points on the plane, z_0 is any point belonging to G. If all points in the neighborhood of z_0 do not belong to G, then z_0 is called an isolated point of G.

Definition

If the point set G can be completely contained in the circle with the origin as the center and a certain positive number R as the radius, then G is called a bounded set, otherwise, G is called an unbounded set.

Definition

In the complex plane, the outside region of a circle centered at the origin and with a positive number R *as its radius is called a neighborhood of the infinity point, denoted as* $|z| > R$*; excluding the infinity point itself, only the set of all points satisfying* $|z| > R$ *is called the deleted neighborhood of the infinity point, which can be expressed as* $R < |z| < \infty$*.*

2 The Concept of Domains

Definition

The non-empty set of points on the plane D is called a domain, if it meet the following two conditions:

- *D is an open set;*
- *D is connected, which means that any two points in D can be connected by a polygonal line that belongs to D.*

From the definition of domain, it is known that the domain does not include boundary points. Thus the domain is generally called an open domain. The point set formed by the domain D and the boundary P is called the closed domain, denoted as $\overline{D}$.

For example, the following regions are all domains:

(1) A circle with the origin as the center and R as the radius (i,e, a circular region) on the z-plane is represented as $|z| < R$;
(2) The concentric rings (i.e., a ring region) shown in Figure 1.12(a) are represented as $r < |z| < R$;
(3) The band-shaped region shown in Figure 1.12(b) is expressed as $y_1 < \mathrm{Im}(z) < y_2$;
(4) The two unbounded regions separated by the real axis $\mathrm{Im}(z) = 0$ on the z plane are the upper half plane and the lower half plane, which are expressed as $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$;
(5) The two unbounded regions separated by the imaginary axis $\mathrm{Re}(z) = 0$ on the z plane are the left half plane and the right half plane, which are respectively represented as $\mathrm{Re}(z) < 0$ and $\mathrm{Re}(z) > 0$;
(6) The angular region shown in Figure 1.12(c) is expressed as $0 < \arg z < \varphi$.

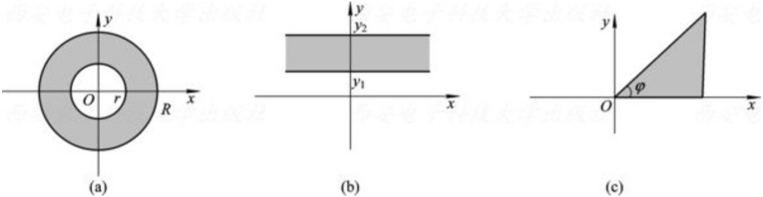


Figure: 1.12

3 Jordan Curve

Definition

If $x(t)$ and $y(t)$ are two continuous real-variable functions, then the system of equations

$$x = x(t), y = y(t) \quad (a \leq t \leq b)$$

represents a curve on the plane and it is called a continuous curve. If you make

$$z(t) = x(t) + iy(t)$$

Then the curve can use the following complex equation to represent:

$$z = z(t) \quad (a \leq t \leq b)$$

which is the complex expression of the curve, and $z(a)$ and $z(b)$ are called the start and end points of the continuous curve C, respectively.

Definition (Cont.)

- *For t_1 and t_2 satisfying $a < t_1 < b$ and $a \leq t_2 \leq b$, when $t_1 \neq t_2$ and $z(t_1) = z(t_2)$, the point $z(t_1)$ is called a coincidence point of curve C.*
- *The continuous curve C that has no coincidence point is called the simple curve or the Jordan curve.*
- *If the start point and the end point of the simple curve C is the same point, that is, $z(a) = z(b)$, then the curve C is called a simple closed curve.*

For example, the circle $|z| = R$ is a simple closed curve, which divides the complex plane into two regions with no common point: $|z| < R$ and $|z| > R$, the former is bounded, and the latter is unbounded. And they are all bounded by $|z| = R$.

Definition

Let $C : z = z(t) = x(t) + iy(t)(\alpha \leq t \leq \beta)$ *is a simple (or simple closed) curve. If* $x'(t)$ *and* $y'(t)$ *are continuous, and for each value of* t*, there is* $[x'(t)]^2 + [y'(t)]^2 \neq 0$*, then the curve is called a smooth curve. The curve composed of several smooth curves is called a piece-wise smooth curve.*

4 Simply Connected Domain and Multiply Connected Domain

A simple closed curve can be used to distinguish between a simply connected domain and a multiply connected domain.

Definition

Definition: If the part enclosed by any simple closed curve in the domain D is still in the domain D, then the domain D is called a single connected domain; otherwise, it is called a multiply connected domain.

For example, the circular region $|z| < R$ is a simply connected domain; the ring region $0 < |z| < R$ is a multiply connected domain, because $0 < |z| < R$ is $|z| < R$ with the center of the circle $z = 0$ deleted and thus the so called "holes" are formed. We can conclude that all domains with "holes" are multi-connected.

Example 3.1

Find the set of points $P = \{z = r(\cos\theta + i\sin\theta)\}$ *where* z *satisfying the relational expression:*

$$2\cos\theta < r < 4\cos\theta\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$

If the set is a domain, is it bounded or unbounded? Is it a simply connected domain or a multiply connected domain?

Solution.

Suppose

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

then

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

Solution. (Cont.)

Because $2\cos\theta < r < 4\cos\theta$,

$$2r\cos\theta < r^2 < 4r\cos\theta, \quad 2x < x^2 + y^2 < 4x$$

From $x^2 + y^2 > 2x$, we get $(x-1)^2 + y^2 > 1$;and from $x^2 + y^2 < 4x$, we get $(x-2)^2 + y^2 < 4$. Thus the point set is the shaded part in Figure 1.13:

$$\begin{cases} (x-1)^2 + y^2 > 1 \\ (x-2)^2 + y^2 < 4 \end{cases}$$

This is a domain, which is bounded, and it is simply connected.

Solution. (Cont.)

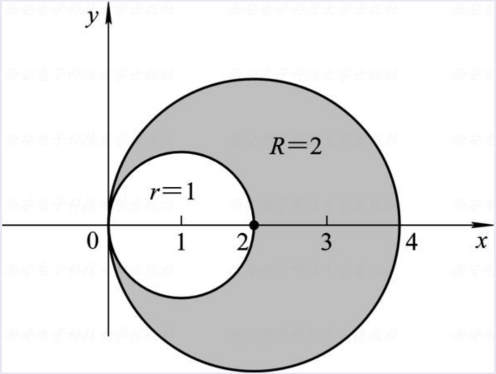


Figure: 1.13

Example 3.2

Does the set composed of points z satisfying the condition $\mathrm{Im}(z) > 1$ and $|z| < 2$ constitute a domain? If the set is a domain, is it simply connected and is it a bounded?

Example 3.2

Does the set composed of points z satisfying the condition $\mathrm{Im}(z) > 1$ and $|z| < 2$ constitute a domain? If the set is a domain, is it simply connected and is it a bounded?

Solution.

Let $z = x + iy, \mathrm{Im}(z) > 1$ and $|z| < 2$, which gives $y > 1, x^2 + y^2 < 4$. As shown by the shaded part in Figure 1.14, it is bounded and simply connected domain.

Solution. (Cont.)

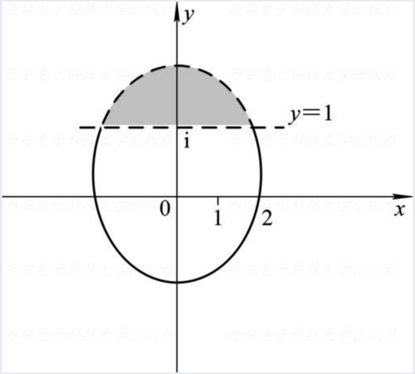


Figure: 1.14

Outline

1. Complex Numbers and Their Expressions
2. Operations of Complex Numbers
3. Regions on Complex Plane
4. Complex Function
5. Limit and Continuity of Complex Function

1 Definition of Complex Function

Definition

Let G be a set of complex numbers. Suppose there is a certain mapping rule f. If for every complex number z in the set G, the rule maps z to a certain complex number w, then it is said that a complex function f is determined on G, denoted as $w = f(z)(z \in G)$. G is called the domain of function $f(z)$.

Definition (Cont.)

If a complex number z is mapped to one w, we call the function $f(z)$ as a single valued function; if a complex number z is mapped to two or more complex numbers w, then we call the function $f(z)$ as a multi-valued function. The set E of all w mapped from all z in G is called the set of function values or the range of f.

Definition (Cont.)

Giving a complex number $z = x + iy$ is equivalent to giving two real numbers x and y, and the complex number $w = u + iv$ also corresponds to a pair of real numbers u and v. Thus, the complex function is equivalent to two real functions:

$$u = u(x, y), v = v(x, y)$$

which are two real variable functions with two independent variables x and y.

Definition (Cont.)

Letting $w = f(z), w = u + iv, z = x + iy$, *then we have*

$$\begin{cases} f(z) = w = u + iv \\ f(x + iy) = u(x, y) + iv(x, y) \end{cases} \Rightarrow u = u(x, y), v = v(x, y)$$

For example, let us consider the function $w = z^2$, and let $z = x + iy, w = u + iv$. Then, we have

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

thus, the function $w = z^2$ corresponds to two real functions with two real variables:

$$u = x^2 - y^2, v = 2xy$$

2 The Concept of Mapping

If the point z in set G is mapped $w = f(z)$ to the point w in set E, then the point $w = f(z)$ is called the image point (or image) of z, and z is called the original image of $w = f(z)$; the complex plane of z is called the z plane and the complex plane of w is called the w plane (see Figure 1.15).

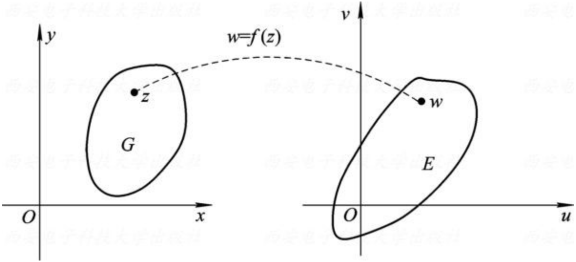


Figure: 1.15

For example, the mapping of the function $w = \overline{z}$ maps the point $z = x + iy$ on the z plane to the point $w = x - iy$ on the w plane; $z_1 = 1 + 4i$ is mapped into $w_1 = 1 - 4i$; $z_2 = 1 - 3i$ is mapped to $w_2 = 1 + 3i$, $\triangle ABC$ is mapped to $\triangle A'B'C'$ (see Figure 1.16).

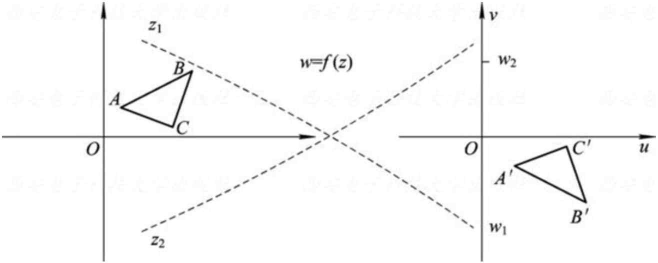


Figure: 1.16

Example 4.1

Investigate the mapping properties of the function $w = z^2$.

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Investigate the mapping properties of the function $w = z^2$.

Solution.

1. If $z = re^{i\theta}$ and $z = \rho e^{i\varphi}$, the function $w = z^2$ can be expressed as

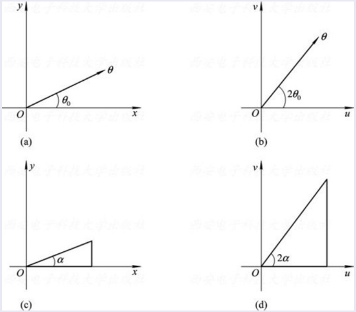
$$\begin{cases} \rho = r^2 \\ \varphi = 2\theta \end{cases}$$

Then the following mapping properties of the function $w = z^2$ can be obtained:

Solution. (Cont.)

(1) The ray $\theta = \theta_0(-\pi < \theta_0 \le \pi)$ from the origin on the z plane is mapped to the ray $\varphi = 2\theta_0(-\pi < \theta_0 \le \pi)$ on the w plane, as shown in Figure 1.17(a), (b).

(2) The angular area $D = \{z : 0 \le \arg z \le \alpha\}$ on the z plane is mapped to the angular area $G = \{w : 0 \le \arg w \le 2\alpha\}$ on the w plane, as shown in Figure 1.17(c), (d).



Solution. (Cont.)

2. If $z = x + iy, w = u + iv$, the function $w = z^2$ can be expressed as

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

And the following properties can be obtained:

(1) The two families of isometric hyperbolas whose asymptotes are the lines $y = \pm x$ and the x-axis/y-axis on the z-plane, i.e., $x^2 - y^2 = c_1, 2xy = c_2$, are mapped to two parallel lines $u = c_1, v = c_2$ on the w-plane (as shown in Figure 1.18).

Solution. (Cont.)

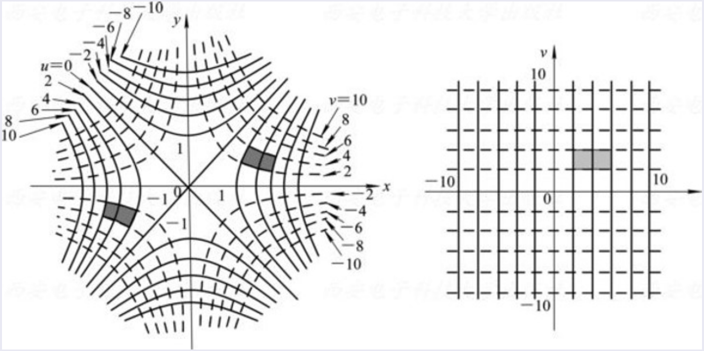


Figure: 1.18

Solution. (Cont.)

2. Next, we will further study the image of the two family of straight lines $x = c_1, y = c_2$ on the z-plane under the mapping of function $w = z^2$.
   When $x = c_1$, it is expressed as a straight line parallel to the imaginary axis on the z-plane, which is represented by a solid line in Figure 1.19(a). Under the mapping $w = z^2$, this straight line becomes

$$\begin{cases} u = c_1^2 - y^2 \\ v = 2c_1 y \end{cases} \quad (-\infty < y < \infty).$$

   We eliminate y to get

$$v^2 = 4c_1^2(c_1^2 - u)$$

Solution. (Cont.)

This is a family of parabolas on the w plane, represented by a dashed line in Figure 1.19(b).
In particular, when $c_1 = 0$, the equation of the straight line is $x = 0$. Under the mapping, this straight line becomes

$$\begin{cases} u = -y^2 \\ v = 0 \end{cases} \Rightarrow \begin{cases} u \leq 0 \\ v = 0 \end{cases}$$

Therefore, we know that under the mapping $w = z^2$, the y-axis $(x = 0)$ on the z plane is mapped to negative part of the x-axis on the w plane.

Solution. (Cont.)

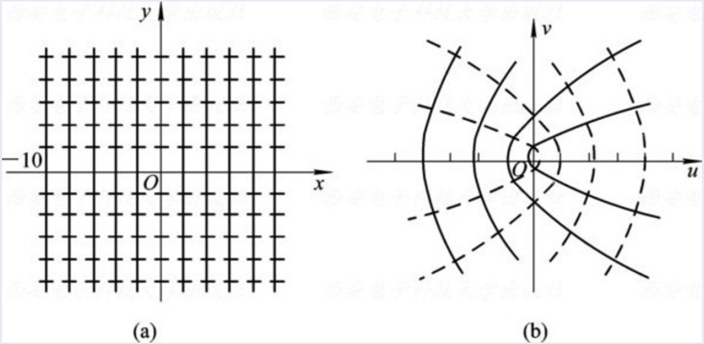


Figure: 1.19

Solution. (Cont.)

Like the discussion about $x = c_1$, we consider the straight line $y = c_2$ (the dotted line in Figure 1.19(a)) under the mapping of $w = z^2$. The image of the straight lines $y = c_2$ on the z plane is a family of parabolas on the w plane

$$v^2 = 4c_2^2 \left(u + c_2^2\right)$$

That is, the solid lines in Figure 1.19(b). In particular, when $c_2 = 0$, the equation of the line becomes $y = 0$ and the image $w = z^2$ is given by

$$\begin{cases} u = x^2 \\ v = 0 \end{cases} \text{that is} \begin{cases} u \geq 0 \\ v = 0 \end{cases}$$

which is the origin and the positive real axis on the w plane.

Example 4.2

For the mapping $w = z + \dfrac{1}{z}$, *find the image of the circle* $|z| = 2$.

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For the mapping $w = z + \dfrac{1}{z}$, *find the image of the circle* $|z| = 2$.

Solution.

Set $z = x + iy, w = u + iv$, then the mapping $w = z + \dfrac{1}{z}$ is equivalent to

$$u + iv = x + iy + \frac{x - iy}{x^2 + y^2},$$

that is

$$\begin{cases} u = x + \dfrac{x}{x^2 + y^2} \\ v = y - \dfrac{y}{x^2 + y^2} \end{cases} \tag{1}$$

Solution. (Cont.)

For the circle $|z| = 2$, the parametric equation is

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases} \quad (0 \le \theta \le 2\pi)$$

Substituting into equation (1), the parametric equation changes to

$$\begin{cases} u = \dfrac{5}{2}\cos\theta \\ v = \dfrac{3}{2}\sin\theta \end{cases} \quad (0 \le \theta \le 2\pi)$$

which actually represents an ellipse on the w plane
$$\frac{u^2}{\left(\frac{5}{2}\right)^2} + \frac{v^2}{\left(\frac{3}{2}\right)^2} = 1$$

Outline

1 Limit of Function

Definition

Suppose that the function $w = f(z)$ *is defined in a deleted neighborhood of* z_0, $0 < |z - z_0| < \rho$. *If there is a complex number* A, *for each positive number* ,$\forall \varepsilon > 0$, *there is a positive number,* $\exists \delta > 0$, *such that* $|f(z) - A| < \varepsilon$ *whenever* $0 < |z - z_0| < \delta$, *where* A *is called the limit of* $f(z)$ *as* z *approaches* z_0 *and denoted as*

$$\lim_{z \to z_0} f(z) = A$$

or denoted as $f(z) \to A$ *when* $z \to z_0$ *(see Figure 1.20).*
Note: the result must hold no matter how z *approaches* z_0.

Definition (Cont.)

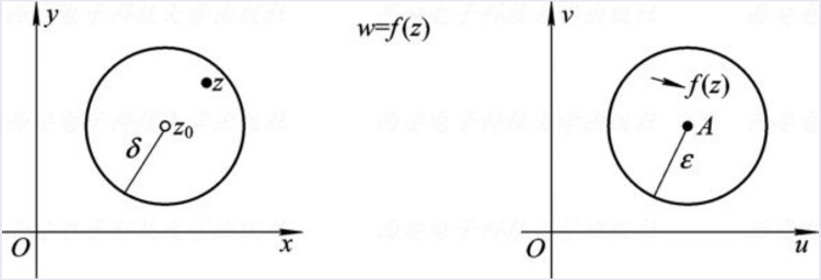

Figure: *1.20*

Theorem 5.1

Suppose $f(z) = u(x,y) + iv(x,y), A = a + ib, z_0 = x_0 + iy_0$.
Then, the necessary and sufficient condition of

$$\lim_{z \to z_0} f(z) = A$$

is

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = a, \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = b.$$

Proof.

If $\lim\limits_{z\to z_0} f(z) = A$, then according to the definition of limit, we get that when $0 < |(x+iy)-(x_0+iy_0)| < \delta$, there is

$$|(u+iv)-(a+ib)| < \varepsilon$$

or when $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$, there is

$$|(u-a)-i(v-b)| < \varepsilon$$

Proof (Cont.)

So, when $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, there is

$$|u - a| < \varepsilon, \quad |v - b| < \varepsilon$$

This means that

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = a, \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = b$$

Proof (Cont.)

On the contrary, if the above two formulas are true, then when

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

there is

$$|u - a| < \frac{\varepsilon}{2}, \quad |v - b| < \frac{\varepsilon}{2}$$

First, we have

$$|f(z) - A| = |(u - a) + i(v - b)| \le |u - a| + |v - b|$$

Then when $0 < |z - z_0| < \delta$, we get

$$|f(z) - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which means

$$\lim_{z \to z_0} f(z) = A$$

Property 5.1

If the limit of $f(z)$ *exists when* z *approaches* z_0*, then the limit is unique.*

Property 5.2

If the limit of $f(z)$ *exists when* z *approaches* z_0*, then* $f(z)$ *is bounded in some deleted neighborhood of* z_0*.*

Theorem 5.2

Suppose there are complex functions $f(z)$ and $g(z)$. If

$$\lim_{z\to z_0} f(z) = A, \quad \lim_{z\to z_0} g(z) = B$$

, then

1. $\lim_{z\to z_0} [f(z) \pm g(z)] = A \pm B$;
2. $\lim_{z\to z_0} [f(z)g(z)] = AB$;
3. $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B} \quad (B \neq 0)$.

Example 5.1

Prove that the limit of function $f(z) = \dfrac{\mathrm{Re}(z)}{|z|}$ *does not exist when* $z \to 0$.

Proof.

Let $z = x + iy$, then $f(z) = \dfrac{x}{\sqrt{x^2 + y^2}}$, thus

$$u(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad v(x, y) = 0$$

Proof (Cont.)

Obviously, we have $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} v(x,y)=0$. Then, we investigate the limit of $u(x,y)$ as $z=x+iy$ approach zero.
Let z approach zero along the straight line $y=kx$, then

$$\lim_{\substack{x\to x_0\\ y=kx}} u(x,y)=\lim_{\substack{x\to 0\\ y=kx}}\frac{x}{\sqrt{x^2+y^2}}=\lim_{x\to 0}\frac{x}{\sqrt{x^2(1+k^2)}}=\pm\frac{1}{\sqrt{1+k^2}}$$

Obviously, it varies with k. Thus, $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} u(x,y)$ does not exist.
According to Theorem 1, $\lim\limits_{z\to 0} f(z)$ does not exist. □

Example 5.2

Find the following limits:

1. $$\lim_{z\to i} \frac{z-i}{z\left(1+z^2\right)}$$
2. $$\lim_{z\to 1} \frac{z\overline{z}+2z-\overline{z}-2}{z^2-1}$$

Solution.

1

$$\lim_{z\to i}\frac{z-i}{z\left(1+z^2\right)}=\lim_{z\to i}\frac{z-i}{z(z-i)(z+i)}=\lim_{z\to i}\frac{1}{z(z+i)}=-\frac{1}{2}$$

2

$$\lim_{z\to 1}\frac{z\overline{z}+2z-\overline{z}-2}{z^2-1}=\lim_{z\to 1}\frac{(\overline{z}+2)(z-1)}{(z+1)(z-1)}=\lim_{z\to 1}\frac{(\overline{z}+2)}{(z+1)}=\frac{3}{2}$$

2 Function Continuity

Definition

Function $w = f(z)$ is defined in a certain neighborhood of point z_0. If $\lim\limits_{z \to z_0} f(z) = f(z_0)$, we say that $f(z)$ is continuous at point z_0. Similarly, the continuity of the function $w = f(z)$ at point ∞ can be defined. If $f(z)$ is continuous everywhere in a domain D, we say that $f(z)$ is continuous in the domain D.

Theorem 5.3

The necessary and sufficient conditions for that the complex function $f(z) = u(x, y) + iv(x, y)$ is continuous at the point $z_0 = x_0 + iy_0$ are the real functions $u = u(x, y)$ and $v = v(x, y)$ are continuous at (x_0, y_0), that is

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u(x_0, y_0)$$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v(x_0, y_0)$$

Theorem 5.4

1. *The sum, difference, product and quotient of the two continuous functions $f(z)$ and $g(z)$ at z_0 (the denominator at z_0 is not zero) are still continuous at z_0;*
2. *If the function $h = g(z)$ is continuous at z_0, and the function $w = f(h)$ is continuous at $h_0 = g(z_0)$, then the composite function $w = f[g(z)]$ is continuous at z_0.*

Example 5.3

Prove that f(z) is not continuous at the origin.

$$f(z) = \begin{cases} \dfrac{xy}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Proof.

If $f(z)$ is continuous at the origin, according to the definition, it should have

$$\lim_{z \to 0} f(z) = f(z) = 0$$

Proof (Cont.)

Let's observe the limit of $\lim_{z\to 0} f(z)$.

When $z \to 0$ we choose $z = x + ix(x \to 0)$, then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

When $z \to 0$ we choose $z = iy(y \to 0)$, then

$$\lim_{z\to 0} f(z) = \lim_{y\to 0} \frac{0 \cdot y}{0 + y^2} = 0$$

Thus, the limit dose not exist. □

Finally, we point out that the meaning of the function $f(z)$ being continuous at a point z_0 on a curve C is:

$$\lim_{z \to z_0} f(z) = f(z_0) \quad (z \in C)$$

According to the definition, we can prove that the continuous function $f(z)$ on the closed curve or the curve including the end points of the curve is bounded on the curve, that is, i.e., there is a positive number M, the result of $|f(z)| \leq M$ is always hold on the curve.