BLENDING SIR AND PREDATOR-PREY MODELS TO PREDICT THE LABOR MARKET

JASON VASQUEZ, DYLAN SKINNER, BENJAMIN MCMULLIN, AND ETHAN CRAWFORD

Abstract. The labor market, including the unemployment rate and the amount of workers looking for jobs, can have a large impact on the economy. The more people employed means more money being spent, which in turn means more money being made. Furthermore, rise in unemployment can lead to a recession. Being able to predict the labor market can help us prepare for a recession and help us understand the economy better. In this paper, we adapt an SIR model to characterize the dynamics of employed, unemployed, and retired individuals in the labor market. Additionally, we employ a quasi predator-prey model to illustrate the oscillatory behavior observed in the white-collar and blue-collar industries. By comparing the SIR model to the predator-prey model, we aim to enhance our understanding of the complex interactions within the labor market, providing potential insights for recession prediction and economic analysis.

1. Background/Motivation

One thing that is certain in life is that people will always need jobs. Not only this, but people will often lose their jobs. Furthermore, people will (eventually) retire from their jobs. The focus of our project is modeling this situation.

In recent studies exploring the complexities of employment trajectories and occupational sectors, researchers have employed various modeling approaches such as Agent-Based Modeling [9], and Markov Chains [14]. However, in a departure from conventional methodologies, our investigation takes an innovative turn by adapting the SIR (Susceptible-Infectious-Recovered) model [5], typically utilized for studying disease dynamics, to the realm of employment dynamics. This unique application aims to unravel the intricate propagation of employment statuses, specifically delving into the transitions between being employed, unemployed, and retired.

A discernible trend has emerged in recent times, notably influenced by the technological revolution. The surge in interest and demand for tech-oriented careers has prompted a significant shift away from traditional blue-collar professions. This migration has led to a dual challenge: a scarcity of skilled workers in the blue-collar sector and an oversaturation of the tech industry [3]. To capture this relationship between white and blue collar jobs, we also create a quasi-predator-prey framework inspired by ecological models, which offers insights into the cyclical dynamics between these sectors.

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Motivated by the imperative to comprehend and address the consequences of this evolving employment landscape, our research aims to contribute valuable insights for informing strategic policies and industry interventions. By analyzing the strengths of both the predator-prey framework and the SIR model, we aspire to provide a comprehensive understanding of the intricate dynamics shaping the contemporary employment scenario.

2. Modeling

2.1. **Theoretical Framework.** The Susceptible-Infectious-Recovered (SIR) model, developed by Kermack and McKendrick in 1927 [5], is a foundational mathematical framework for understanding the spread of infectious diseases in populations. It divides individuals into susceptible, infectious, and recovered compartments, capturing the dynamics of disease transmission. In our model, we adapt the SIR model to represent the dynamics of the employment market through the labor force, unemployed, and retired populations.

2.2. **Previous Work.** We begin by building off the work of ElFadily et. al. [2]. In their work, ElFadily et. al. proposed a model representing the labor force and unemployed populations. They begin by defining their equations as

$$\frac{dL}{dt} = \gamma U - (\sigma + \mu)L, \quad \frac{dU}{dt} = \rho \left(1 - \frac{L_\tau + U_\tau}{N_c}\right) L_\tau + \sigma L - (\gamma + \mu)U, \tag{1}$$

where L is the labor force, U is the unemployed population, with initial conditions for (1) defined as:

$$\begin{gathered} L(0) > 0, \quad U(0) > 0, \\ (L(\theta), U(\theta)) = (\varphi_1(\theta), \varphi_2(\theta)), \quad \theta \in [-\tau, 0], \end{gathered} \tag{2}$$

where $\varphi_i \in C([-\tau, 0], \mathbb{R}^+), \ \ i = 1, 2$.

The parameters are defined as follows:

- γ: employment rate
- σ: rate of job loss
- μ: mortality rate
- ρ: maximum population growth rate
- N_c: population carrying capacity
- τ: time lag needed to contribute in the reproductive process of a new individual looking for a job

2.3. **Modifications: The Retirement Group.** With this information in mind, we can begin to adapt this model to fit our desired model structure. We begin by adding a third population, the retired population, R. We can then define our new equations as

$$
\begin{aligned}
\frac{dL}{dt} &= \gamma U - (\sigma+\mu)L - \left(\frac{\Sigma}{L+U}\right)L + \omega\left(\frac{\Sigma}{L+U}\right)R + \rho L,\\
\frac{dU}{dt} &= \rho\left(1-\frac{L+U}{N_c}\right)L + \sigma L - (\mu+\gamma)U,\\
\frac{dR}{dt} &= \left(\frac{\Sigma}{L+U}\right)L - \omega\left(\frac{\Sigma}{L+U}\right)R - \mu R,
\end{aligned} \tag{3}
$$

which simplify to

$$
\begin{aligned}
\frac{dL}{dt} &= \gamma U - (\sigma+\mu)L + (\omega R - L)\left(\frac{\Sigma}{L+U}\right) + \rho L,\\
\frac{dU}{dt} &= \rho\left(1-\frac{L+U}{N_c}\right)L + \sigma L - (\mu+\gamma)U\\
\frac{dR}{dt} &= (L-\omega R)\left(\frac{\Sigma}{L+U}\right) - \mu R.
\end{aligned} \tag{4}
$$

One of the first things to note from our equations is the removal of the time lag τ. This is because, instead of factoring in people when they are born, we are instead factoring them in when they turn of working age (for simplification, age 16). This reduces unnecessary complexity in our model. Additionally, we make two assumptions about unemployed people, being they will neither retire directly from unemployment, nor contribute to the growth of the population. A final thing to note is that we have added two new parameters, ω and Σ. We define Σ to be the number of people who retire each year in the United States, and ω to be the rate at which retired people enter back into the full-time workforce (which is a dimensionless constant). We can then define our new initial conditions as:

$$
\begin{gathered}
L(0)>0, \quad U(0)>0, \quad R(0)>0,\\
(L(\theta),U(\theta),R(\theta)) = (\varphi_1(\theta),\varphi_2(\theta),\varphi_3(\theta)), \quad \theta\in[-\tau,0],
\end{gathered} \tag{5}
$$

where $\varphi_i \in C([-\tau,0],\mathbb{R}^+)$, $i=1,2,3$. Incorporating nuanced dynamics into our model, we introduce the following terms and elucidate their significance within the equations:

- $\pm\left(\frac{\Sigma}{L+U}\right)L$: Captures retirements relative to the total workforce, considering the annual number of retirees (Σ) as a percentage of the employed population (L).
- $\pm\omega\left(\frac{\Sigma}{L+U}\right)R$: Models retired individuals re-entering the workforce, with ω representing the transition rate.
- ρL: Represents natural job growth, proportional to the employed population.
- $-\mu R$: Represents the natural attrition of retired individuals due to mortality at each time step.

2.4. **Modifications: The White-Collar and Blue-Collar Groups.** We can further model the labor market by examining the relationship between two industries commonly referred to as the white-collar and blue-collar industries. In recent years, we have seen an explosion of jobs and interest in the white-collar field, specifically in the tech industry, while the blue-collar industry has seen a decline in interest. This has led to an oversaturation of the tech industry and a scarcity of labor in the blue-collar industry, which in turn has slowed growth in the former and led to increased growth in the latter. This cyclical relationship mirrors that of a predator-prey relationship, and we can model it as such.

Choosing a predator-prey model to represent blue-collar and white-collar jobs is justified by its ability to capture dynamic interactions and cyclic behavior between the two job categories. This modeling approach incorporates feedback loops, reflecting the influence each job type has on the other, and naturally represents population dynamics in response to economic, educational, or technological factors. Additionally, the model's adaptability allows for the inclusion of additional factors, offering flexibility in addressing the multifaceted dynamics of the labor market. This choice opens avenues for research and exploration of hypothetical scenarios, providing a structured framework to analyze and understand the interplay between different job categories over time.

The classical predator-prey model is given by the following equations:

$$\frac{dx}{dt} = \rho x - axy, \quad \frac{dy}{dt} = -\mu y + \varepsilon axy. \tag{6}$$

where x is the prey population, y is the predator population, and ρ, a, μ, and ε are parameters [6] [13]. We can adapt this model by defining the following:

$$\frac{dx}{dt} = \rho x\left(1 - \frac{x}{k}\right) - axy, \quad \frac{dy}{dt} = -\mu y + \varepsilon axy + \beta y\left(1 - \frac{y}{C}\right). \tag{7}$$

where

- x: Blue-collar jobs
- y: White-collar jobs
- ρ: Growth rate of blue-collar jobs
- a: Rate at which people switch from blue collar to white collar jobs
- μ: Decay rate of white-collar jobs
- ε: The rate at which job transitioning affects the labor market
- k: Carrying capacity for blue-collar jobs
- C: Carrying capacity for white-collar jobs.
- β: Growth rate of white-collar jobs

Thus, we can interpret the new additions to our model as:

- $\rho x(1 - \frac{x}{k})$: Carrying capacity term for the blue-collar (prey) population, reflecting growth proportional to the number of blue-collar workers.
- $\beta y(1 - \frac{y}{C})$: Carrying capacity term for the white-collar (predator) population, constrained by the workforce size and capable of independent growth.

The reason that only white-collar jobs have a decay rate is meant to model the current market situation. The white-collar market is becoming oversaturated so as it increases in population it decays, whereas the blue-collar market has not reached this point and never does with the initial conditions given. We also made the assumption that the white-collar market converts whereas the blue-collar market does not. This was a choice to model the specific circumstance we attempt to analyze. In the real world, this could go both ways with varying degrees, but for simplification we made it only one way.

2.5. **Labor Force, Unemployement, and Retirement Simulations.** We now give the specific values for hyperparameters for our model and how we came to those values. We define the following:

- $\sigma = 0.013905$: Derived from comprehensive data on total layoffs and discharges in the United States (2000-2023) [4].
- $\rho = 0.014577$: Maximum growth rate calculated from MacroTrends Excel data (2000-2022) [7].
- $\gamma = 0.6062$: Employment rate average (2000-2022) from the Bureau of Labor Statistics [11].
- $\mu = 0.008498$: Mortality rate derived from 2000-2022 mortality data [12].
- $N_c = 260,000,000$: Population of individuals aged 16 and above in the United States in 2022 [1].
- $\Sigma = 775,045$: Annual retirees in the U.S. (2000-2021) using Social Security Administration data [10], calculated by

$$\Sigma = \frac{1}{2021 - 2001} \sum_{i=2001}^{2021} (x_i - x_{i-1})$$

- $\omega = 0.063$: Rate at which retirees re-enter the workforce based on research by Maestas [8].

We began testing our model by running it for 60 years with the current numbers for the United States (see figure 1).

To test the robustness of our model, we ran it with different initial conditions that do not represent the current situation in the United States. We first decreased the number in the labor force, increased the number of unemployed, and decreased the number of retired. We made these changes rather conservative by only slightly perturbing the real numbers. We then ran the model for 60 years (see figure 2). Parallel to figure 1, we can see that the model still reaches an equilibrium despite the initial conditions being skewed from their true values.

We ran our model, once again, against a different set of initial conditions. This time, we significantly decreased the number of people in the occupied labor force, significantly increased the number of unemployed (ensuring that the number of unemployed was much greater than the number of people in the labor force), and moderately decreased the number of retired. We then ran the model for 60 years (see figure 3). As you can see, the model

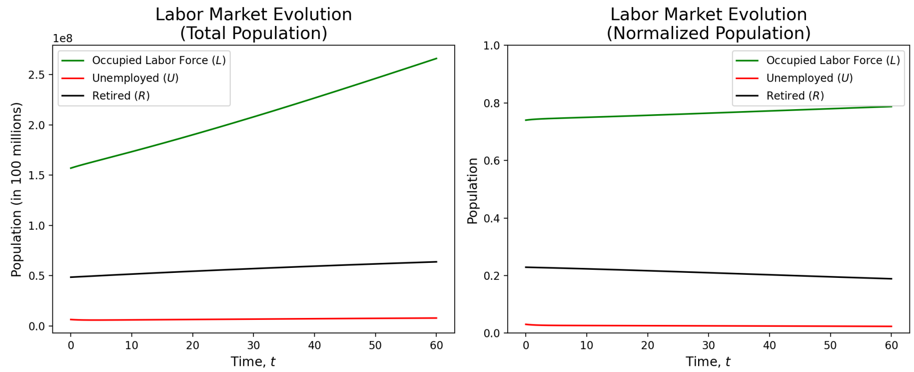

FIGURE 1. Initial conditions: $L(0) = 157,000,000$, $U(0) = 6,500,000$, $R(0) = 48,590,000$.

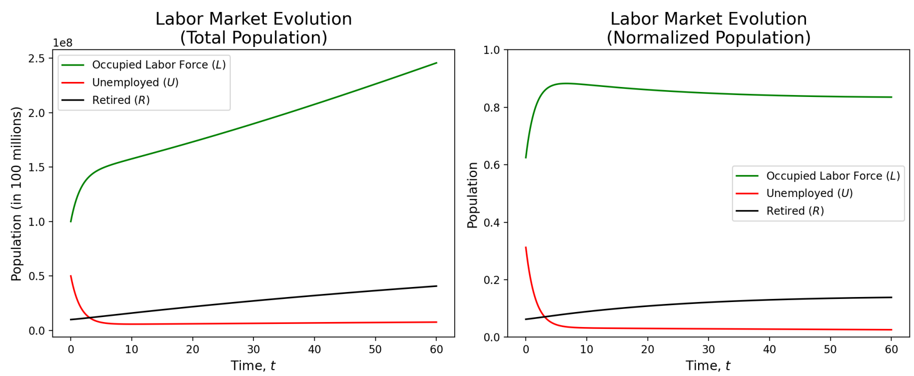

FIGURE 2. Initial conditions: $L(0) = 100,000,000$, $U(0) = 50,000,000$, $R(0) = 10,000,000$.

still reaches an equilibrium, despite the initial conditions being heavily skewed from their true values.

We ran a final test, this time having the number of retired people set as greater than the number of people in the labor force and unemployed combined. We then ran the model for 60 years (see figure 4).

Unlike the previous graphs, we can see that the model does not reach an equilibrium in 60 years. While the number of people in the labor force rises and the number of retired people falls, this change does not appear to be significant enough to reach an equilibrium. However, when ran again for $T = 100$ years, we can see that the model gets closer to an equilibrium, but still does not reach one (see figure 5).

2.6. **White-Collar and Blue-Collar Simulations.** For our white- and blue-collar model, we experimented with different hyperparameters to see how they would affect the model.

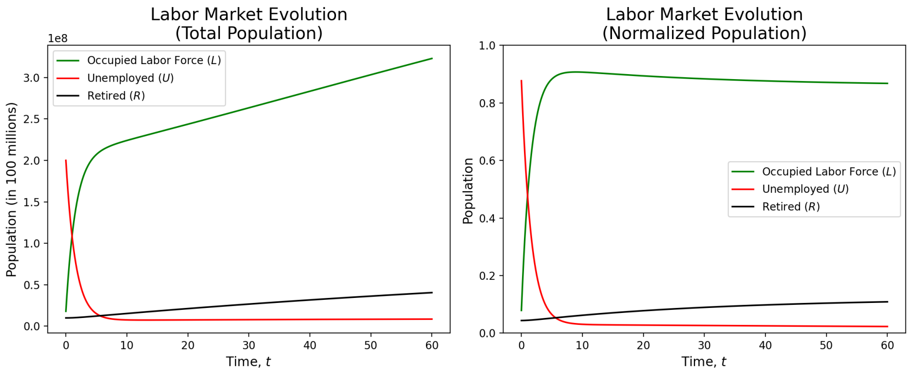

FIGURE 3. Initial conditions: $L(0) = 18,000,000$, $U(0) = 200,000,000$, $R(0) = 10,000,000$.

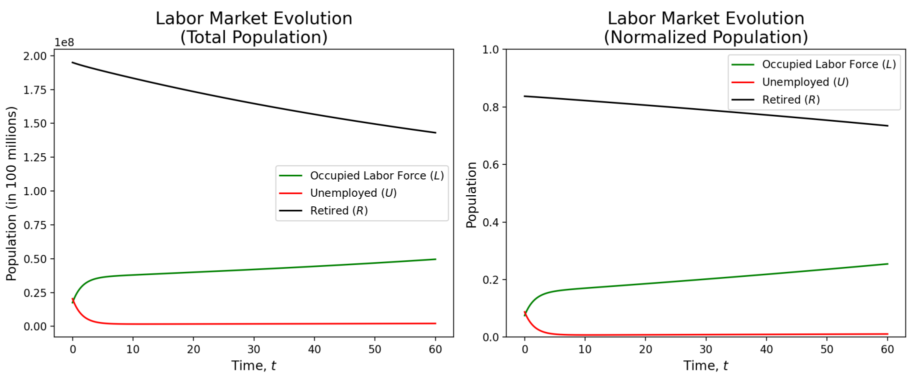

FIGURE 4. Initial conditions: $L(0) = 17,500,000$, $U(0) = 20,400,000$, $R(0) = 195,000,000$.

In our first run of the model, we used parameters $\rho = 7$, $a = 5$, $\mu = 1$, $\varepsilon = .2$, $k = 3$, $\beta = 1$, $C = 1.5$. As we see, the model oscilates slightly in the beginning, and then settles into a stable equilibrium (see figure 6). The initial conditions come from data on the US Labor market and percentage of workers in white-collar or blue-collar jobs [11].

Consider now a new set of initial conditions, namely $\rho = 7$, $a = 5$, $\mu = 2$ $\varepsilon = .2$, $k = 3$, $\beta = 1$, $C = 1.5$, and the same set with $\mu = 1$. Despite only the slight change of μ by 1, our model predicts completely different results (see figure 7).

3. Results

The SIR and predator-prey models in the context of employment dynamics offers a comprehensive framework for understanding the complex interactions within the labor market. Here are some key observations and conclusions drawn from the presented models:

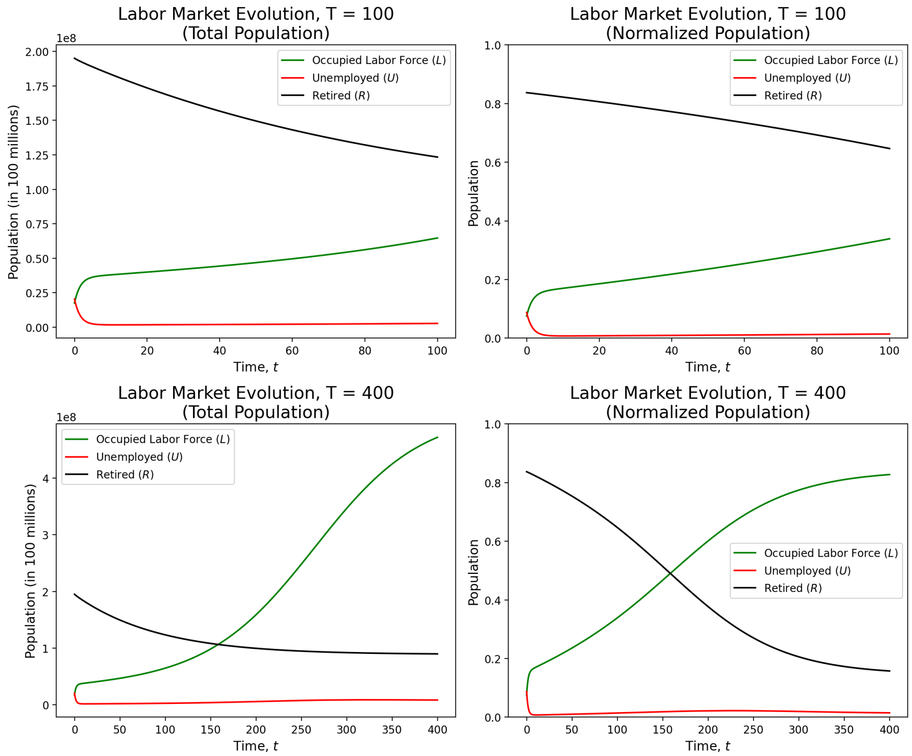

FIGURE 5. Initial conditions: $L(0) = 17,500,000$, $U(0) = 20,400,000$, $R(0) = 195,000,000$.

Overall, our SIR model of the labor market shows remarkable stability. We can see that, regardless of the initial conditions, the model reaches an equilibrium, with the number of employed, unemployed, and retired individuals remaining relatively constant. When we used initial conditions that reflected the current numbers for the United States, the model saw relatively little change as time went on (see figure 1). With initial conditions that represented a larger than average unemployed population, the model corrected itself and reached a similar equilibrium as the previous model (see figure 2). Finally, when presented with initial populations that were flipped, the model still stabilized to the same equilibrium (see figure 3).

The predator-prey model is very sensitive to changes in the hyperparameters as even a small change can cause the model to behave very differently. We see in figure 7 that a change of 1 in μ causes the populations to entirely flip. This model, while not as robust as the SIR labor force model, still shows some interesting results. It is interesting to see how the relationships in the model caused oscillations in the different populations. The oscillations are small enough that they are not visible on the graph, but they are still present, and can mimic the overall labor force where a swing of thousands of jobs is noticed by the economy as a whole (see figure 6). A strength of this second model is exactly that, being able to see

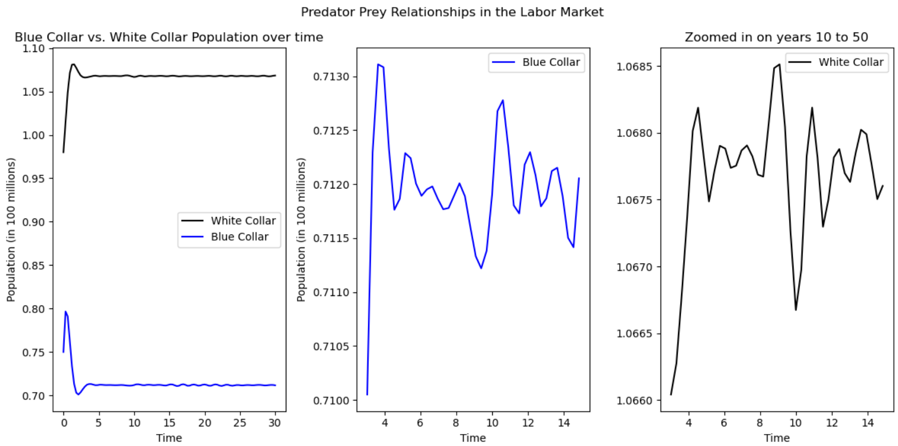

FIGURE 6. Parameters $\rho = 7$, $a = 5$, $\mu = 1$, $\varepsilon = .2$, $k = 3$, $\beta = 1$, $C = 1.5$. Zooming in between years $10 - 50$, we can see the oscillations more clearly.

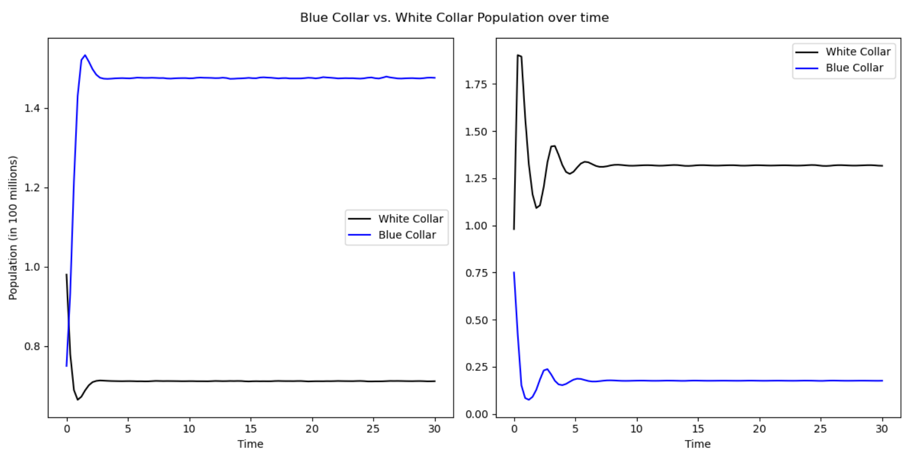

FIGURE 7. The left graph corresponds to $\mu = 2$, while the right graph corresponds to $\mu = 1$, others parameters are kept the same.

the oscillations while keeping the oscillations to a scale that would be realistic in the real world.

4. Analysis/Conclusions

In our model, modified from ElFadily et. al. [2], we have extended their adapted SIR framework to capture more interesting dynamics of the labor market. In our adaptation, we introduced an additional compartment for Retired (R) individuals, reflecting the life cycle of employment. The key modifications involve incorporating terms that represent the natural attrition of the retired population, their potential re-entry into the workforce, and the growth of job opportunities proportional to the number of employed individuals. These adjustments provide a nuanced representation of the labor market's temporal evolution, accounting for retirement dynamics, mortality, and the cyclical nature of job creation and re-entry. This enhanced model allows for a more comprehensive understanding of the complex interactions within the labor market over time.

Despite the strengths of our SIR model, there are weaknesses present. One weakness is that changing the initial conditions can cause the results to differ significantly between each other during the first few years. While it is true that the solutions end up reaching similar values as T grows, those first few years of difference can pose a problem. Another more significant weakness is that this model only considers how the labor markets interact with each other. One major factor in the labor market is the current state of the economy, and our model does not take that into account. Thus, one improvement that can be made is finding a way to include present economic conditions.

The predator-prey model, while not as robust as the SIR model, still shows some exciting results. It is interesting to see how the relationships in the model cause oscillations between the different populations. The oscillations are small enough that they are not visible on the graph, but they are still present, and can mimic the overall labor force. However, this second model is very unstable and requires several large simplifications. This model fails to be a accurate representation of the labor market, and does little else other than show small oscillations. Given more time, we would have loved to expand on this idea and come up with a stable robust model that can illustrate the oscillations between the white-collar and blue-collar industries.

Despite its weaknesses, our models provide insights into the long-term stability and equilibrium of the workforce. The inclusion of retirement-related terms allows policymakers and economists to analyze the impact of demographic shifts on employment trends and anticipate workforce fluctuations. Moreover, the explicit consideration of job creation and re-entry mechanisms offers a more realistic representation of economic dynamics, enabling better predictions of labor market behavior. Understanding the cyclical nature of job opportunities and retiree contributions provides valuable insights for economic planning, workforce management, and policy development. This modified SIR model, by bridging epidemiological principles with labor market dynamics, contributes to a holistic framework for studying the interplay between demographic factors and economic trends, supporting informed decision-making in the real world.

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In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
import pandas as pd
```

Equation 1

$$\frac{dL}{dt} = \gamma U - (\sigma + \mu)L,$$
$$\frac{dU}{dt} = \rho\left(1 - \frac{L_\tau + U_\tau}{N_c}\right)L_\tau + \sigma L - (\mu + \gamma)U$$

where the variable L and U can be interpreted as the number of occupied labor force and the number of unemployed, respectively.

The initial conditions for the above system are:

$$L(0) > 0,$$
$$U(0) > 0,$$
$$(L(\theta), U(\theta)) = (\varphi_1(\theta), \varphi_2(\theta)), \quad \forall\theta \in [-\tau, 0]$$

where $\varphi_i \in C([-\tau, 0], \mathbb{R}^+), \ \ i = 1, 2$

Explanation of Variables

First equation

- γ: employment rate
- γU: the number of new employees among people looking for a job
- σL: the number of employees who have lost thier job
- μL: the number of employees who have died

Second equation

In general, the second equation models the evolution of the unemployed population.

- $\rho(1 - (L + U/N_c))L$: Indicates the number of new people looking for a new job.
  - This reflects maximum growth of job seekers when the unemployement rate is very low and very low labor force growth when the unemployment rate is very high.
  - In other words, population growth is limited by the sharp increase in the number of unemployed.
- N_c: The carrying capacity.
- τ: The time lag needed to contribute in the reproductive process of a new individual lookin for a job.

Values Used in Paper

Top figure 1

- $\sigma = 0.1$
- $\rho = 0.01054$
- $\tau = 25$ years
- $\gamma = 0.7$
- $\mu = 0.000481$
- $N_c = 11000000$

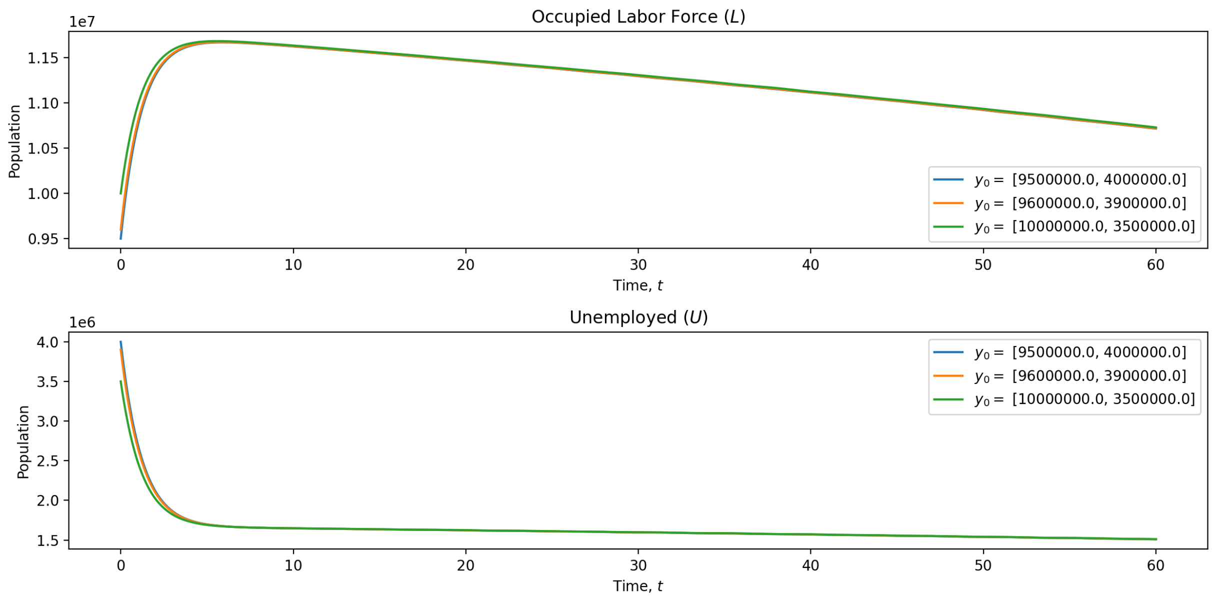
Bottom figure 1

- $\sigma = 0.1$
- $\rho = 0.01054$
- $\tau = 25$ years
- $\gamma = 0.2$
- $\mu = 0.00481$
- $N_c = 51000000$

Trying to Reproduce the Graphs

In [ ]:
```
# Figure 1
sigma, rho, tau, gamma, mu = 0.1, 0.01054, 25, 0.7, 0.000481
N_c = 11000000

def f(t, y, sigma, rho, tau, gamma, mu, N_c):
    # Build and return the model
    # L' = gamma * U(t) - (sigma + mu) * L(t)
    # U' = rho * (1 - (L(t) - U(t)) / N_c) * L(t) + sigma L(t) - (mu + gamma
    return (gamma * y[1] - (sigma + mu) * y[0],
            rho * (-1 + (y[0] - y[1]) / N_c) * y[0] + sigma * y[0] - (mu + g

# plt.figure(figsize=(12,6),dpi=100)
fig, ax = plt.subplots(2, 1, figsize=(12,6), dpi=200)
# Initialize variables used for problem.
y0 = np.array([0.1, 0.2])
t = np.linspace(0, 60, 10000)

# Plug these values into solve_ivp
for y0 in [[9.5e6, 4e6], [9.6e6, 3.9e6], [10e6, 3.5e6]]:
    solution = solve_ivp(f, (0, 60), y0, t_eval=t, args=(sigma, rho, tau, ga

    # Get the number of rabbits and wolves
    labor_force = solution.y[0, :]
    ax[0].plot(solution.t, labor_force, label=f'$y_0 = $ {y0}')
    ax[0].set_title(f'Occupied Labor Force ($L$)')
    ax[0].set_xlabel('Time, $t$')
    ax[0].set_ylabel('Population')
```

```
    ax[0].legend()

    unemployed = solution.y[1, :]
    ax[1].plot(solution.t, unemployed, label=f'$y_0 = $ {y0}')
    ax[1].set_title(f'Unemployed ($U$)')
    ax[1].set_xlabel('Time, $t$')
    ax[1].set_ylabel('Population')
    ax[1].legend()

plt.tight_layout()
plt.show()
```



Me and Ethan's Next Task!

Modify the paper to include a retirement group (and other parameters that are hithertoo, unknown).

Original Equations:

$$\begin{aligned}\frac{dL}{dt} &= \gamma U - (\sigma + \mu)L, \\ \frac{dU}{dt} &= \rho\left(1 - \frac{L_\tau + U_\tau}{N_c}\right)L_\tau + \sigma L - (\mu + \gamma)U\end{aligned}$$

We want to modify these equations to include retirement. We can call the retirement group R. Our new equations will look something like this:

```
\begin{align*}
    \frac{dL}{dt} &= \gamma U - (\sigma + \mu)L {\color{red}\;- \left(\frac{\Sigma}{L_{\ta
    \frac{dU}{dt} &= \rho\left(1 - \frac{L_{\tau} + U_{\tau}}{N_c} \right)L_{\tau} + \sigma L
    \frac{dR}{dt} &= {\color{red}\left(\frac{\Sigma}{L_{\tau} + U_{\tau}}\right) L - \omega\
\end{align*}
```

Simplifying, we get:

$$\begin{aligned}
\frac{dL}{dt} &= \gamma U - (\sigma + \mu)L + (\omega R - L)\left(\frac{\Sigma}{L_\tau + U_\tau}\right),\\
\frac{dU}{dt} &= \rho\left(1 - \frac{L_\tau + U_\tau}{N_c}\right)L_\tau + \sigma L - (\mu + \gamma)U\\
\frac{dR}{dt} &= (L - \omega R)\left(\frac{\Sigma}{L_\tau + U_\tau}\right) - \mu R
\end{aligned}$$

Of course, we will need to modify $\frac{dL}{dt}$ and $\frac{dU}{dt}$ to account for the retirement group. We will also need to modify the initial conditions to account for the retirement group.

Looking at this link, we can find the number of people who retire each year in the United States by looking at the increase in social security use each year.

Social Security provides more than just retirement benefits, however. *source

- Retired workers and their dependents accounted for 76.9% of total benefits paid in 2022.
- Disabled workers and their dependents accounted for 11.6% of total benefits paid in 2022.
  - About 90 percent of workers aged 21-64 in covered employment in 2022 and their families have protection in the event of a severe and prolonged disability.
  - About 1 in 4 of today's 20 year-olds will become disabled and entitled to Social Security disabled worker benefits before reaching age 67.
  - 65% of the private sector workforce has no long-term disability insurance.
- Survivors of deceased workers accounted for 11.5% of total benefits paid in 2022.
  - More than one in eight of today's 20-year-olds will die before reaching age 67.

For our model, we will consider all people on social security as retired from the work force.

It is also important to consider however, that many people who retire from the work force, return to the work force. This is especially true for people who retire early. According to BankRate, 2.1 percent left retirement to go back to work in June 2020, and 3.2 percent returned back to work in March 2022. This increase makes sense given the pandemic.

More helpful, however, is this article from the NIH's National Library of Medicine that states 12.9% of people who fully retire return to full-time work, and 6.3% of people who fully retire return to part-time work. This is a much more reasonable number to use in our model. Additionally, while our model is considering the labor force, unemployed, and retired, we will group these statistics together (i.e., we are considering the labor force to be people who are both full- and part-time). Thus, on average, 19.2% of people who retire return to the labor force. This is ω. (It is important to note that the above source breaks retired people into full retirement and partial retirement. People who are fully retired had a clear final day of work. Partially retired people do not and instead phase out. Since partially retired people are still *technically* working, we consider them in the labor force.)

Documenting our Equation Changes

- Σ: The number of people who retire each year in the United States. We calculate this number from data from this website from the Social Security Administration, looking at the years 2000 - present. The formula for this is:

$$\Sigma = \frac{1}{2021-2001}\sum_{i=2001}^{2021} x_i - x_{i-1}$$

  where x_i is the number of people on social security for a given year i. This number is $775,045$.

  - In our equation, we have $\left(\frac{\Sigma}{L_\tau + U_\tau}\right)$. This is the number of people who retire each year in the United States divided by the number of people who are in the labor force and are unemployed.

- ω: This is the rate at which retired people enter back into the full-time work force. According to sources, this number is 6.3% or 0.063.

- $\omega\left(\frac{\Sigma}{L_\tau + U_\tau}\right)L$: This is the number of retired people who enter back into the work force. We calculated it this way because we believe that people who recently retired are the ones most likely to go back to work. We subtract it from dR/dt because those are people no longer classified as retired, and add it to dL/dt because those are people who are now classified as part of the labor force.

- μR: This is the number of people in the retired group who die. We subtract it from dR/dt because those are people no longer classified as retired. We add it no where because we are not keeping track of the dead.

In [ ]:
```
# Calculate the retirement rate
social_security_data = pd.read_html('https://www.ssa.gov/oact/STATS/OASDIber
total_by_year = social_security_data.Total.iloc[3:55].astype(float)

total_by_year['diff'] = total_by_year.diff()
total_by_year.index = social_security_data.Year.iloc[3:55].astype(int)['Year
retirement_rate = total_by_year['diff'].iloc[-20:].mean()
#total_by_year['diff'].iloc[1:].plot.bar(figsize=(12,6), title='Change in So
print('Retirement Rate per year (Σ):', retirement_rate)
print('Average of the last 20 years')
print(total_by_year['diff'].mean())
```

Changing Previous Hyperparameters to Represent United States

For our model, we will use the following hyperparameters: $\sigma, \rho, \tau, \gamma, \mu, N_c, \Sigma, \omega$.

For our values, we will consider the following:

- $\sigma = 0.013905$
- $\rho = 0.014577$
- $\tau = 25$ years
- $\gamma = 0.6062$ *source
- $\mu = 0.008498$
- $N_c = 260000000$
- $\Sigma = 775045$
- $\omega = 0.192$

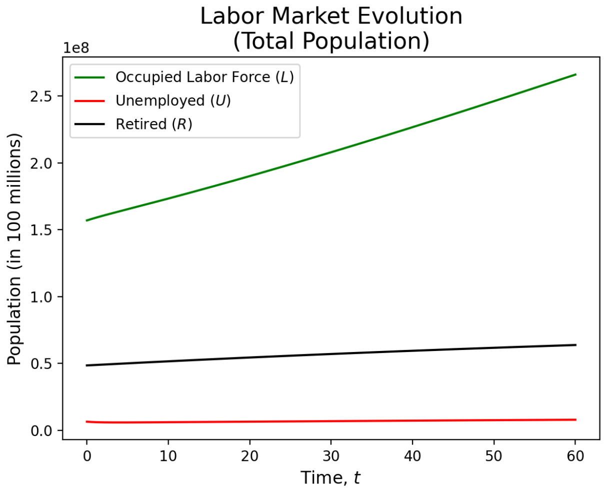
How we calculated each

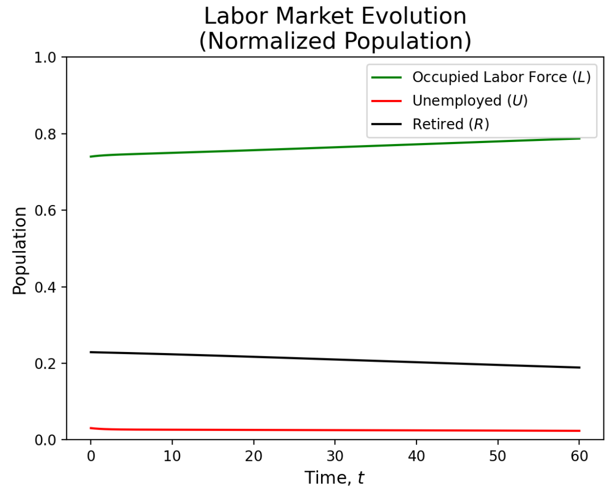
- σ: We found this website which outlined the total layoffs and discharges in the United States, by month, between 2000-2023 (present month). We calculated the average and got 0.013905.
- ρ: We looked at an Excel spreadsheet from this website. We got the growth rate from 2000 - 2022 and averaged those numbers. However, since this is the **maximum** growth rate and not the average, we included three standard deviations. This number is 0.014577.
- γ: We looked at this pdf from the Bureau of Labor Statistics. We simply calculated the average of the employment rate from 2000 - 2022. This number is 0.6062.
- μ: We looked at this website. We simply calculated the average deaths per 100,000 people from 2000 - 2022, then divided by 100k to get the mortality rate. This number is 0.008498.
- N_c: We calculate this to be the population of people 18+ in the United States in 2022. This number is 260000000. (Got this number here.)
- Σ: (see above work.)
- ω: (see above work.)

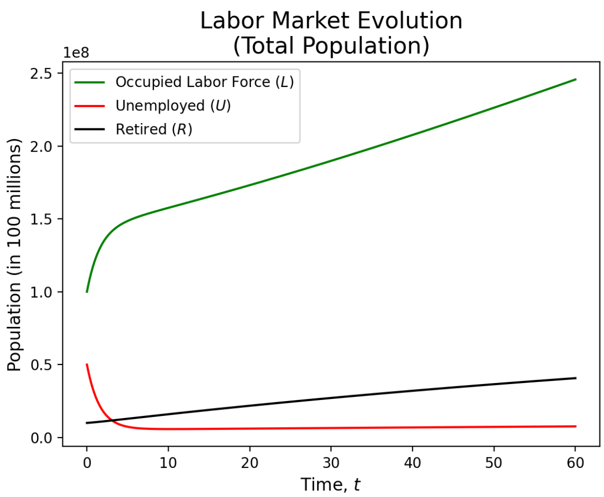
In [ ]:

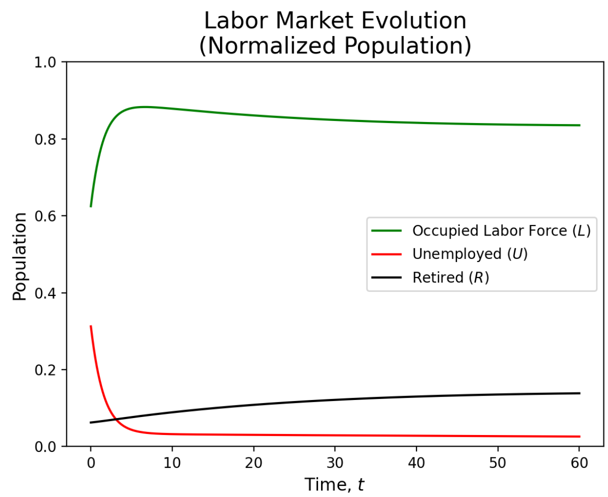
```
# Trying to graph our above stuff.
sigma, rho, tau, gamma, mu, Sigma, omega = 0.013905, 0.014577, 25, 0.6062, 0
# sigma, rho, tau, gamma, mu, Sigma, omega = 0.013905, 0.014577, 25, 0.6062,
# N_c = 2.6e8
N_c = 4e8

def f(t, y, sigma, rho, tau, gamma, mu, N_c, Sigma, omega):
    # Build and return the model
```

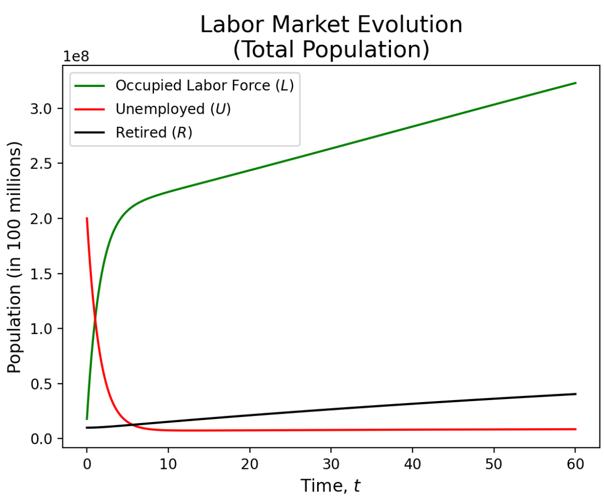
```
    # L' = gamma * U(t) - (sigma + mu) * L(t) - (Sigma/(L(t)+U(t))) * L(t) +
    # U' = rho * (1 - (L(t) - U(t)) / N_c) * L(t) + sigma L(t) - (mu + gamma
    # R' = (Sigma/(L(t)+U(t))) * L(t) - omega * R(t) - mu * R(t)
    # return (gamma * y[1] - (sigma + mu) * y[0] - (Sigma / (y[0] + y[1])) *
    return (gamma * y[1] - (sigma + mu) * y[0] - (Sigma / (y[0] + y[1])) * y
            rho * (1 - (y[0] + y[1]) / N_c) * y[0] + sigma * y[0] - (mu + ga
            # (Sigma / (y[0] + y[1])) * y[0] - omega * y[2] - mu * y[2])
            (Sigma / (y[0] + y[1])) * y[0] - omega * (Sigma / (y[0] + y[1]))

# plt.figure(figsize=(12,6),dpi=100)
# fig, ax = plt.subplots(2, 1, figsize=(12,10), dpi=200)

# Initialize variables used for problem.
T = 60
t = np.linspace(0, T, 1000)

# Plug these values into solve_ivp
# for i, y0 in enumerate([[157e6, 6.5e6, 48.59e6], [157.1e6, 1e6, 1e6], [156
for i, y0 in enumerate([[157e6, 6.5e6, 48.59e6], [100e6, 50e6, 10e6], [18e6,
    # fig, ax = plt.subplots(2, 1, figsize=(8,10), dpi=200)
    fig, ax = plt.subplots(1, 2, figsize=(12,5), dpi=200)

    if i != 4:
        solution = solve_ivp(f, (0, T), y0, t_eval=t, args=(sigma, rho, tau,
    elif i == 4:
        T = 200
        t = np.linspace(0, T, 1000)
        solution = solve_ivp(f, (0, T), y0, t_eval=t, args=(sigma, rho, tau,

    sum_people = np.sum(solution.y, axis=0)
    fontsize = 16

    ax[0].plot(solution.t, solution.y[0, :], label=f'Occupied Labor Force ($
    ax[0].plot(solution.t, solution.y[1, :], label=f'Unemployed ($U$)', colo
    ax[0].plot(solution.t, solution.y[2, :], label=f'Retired ($R$)', color='
    ax[0].set_title(f'Labor Market Evolution\n(Total Population)', fontsize=
    ax[0].set_xlabel('Time, $t$', fontsize=fontsize - 4)
    ax[0].set_ylabel('Population (in 100 millions)', fontsize=fontsize - 4)
    # ax[i].set_ylim(0, 1)
    ax[0].legend(fontsize=fontsize - 6)

    ax[1].plot(solution.t, solution.y[0, :]/sum_people, label=f'Occupied Lab
    ax[1].plot(solution.t, solution.y[1, :]/sum_people, label=f'Unemployed (
    ax[1].plot(solution.t, solution.y[2, :]/sum_people, label=f'Retired ($R$
    # ax[1].set_title(f'Population Model, Initial Conditions: {numerize(y0[0
    ax[1].set_title(f'Labor Market Evolution\n(Normalized Population)', font
    ax[1].set_xlabel('Time, $t$', fontsize=fontsize - 4)
    ax[1].set_ylabel('Population', fontsize=fontsize - 4)
    ax[1].set_ylim(0, 1)
    ax[1].legend(fontsize=fontsize - 6)
    plt.tight_layout()
    plt.show()


# plt.tight_layout()
# plt.show()
```

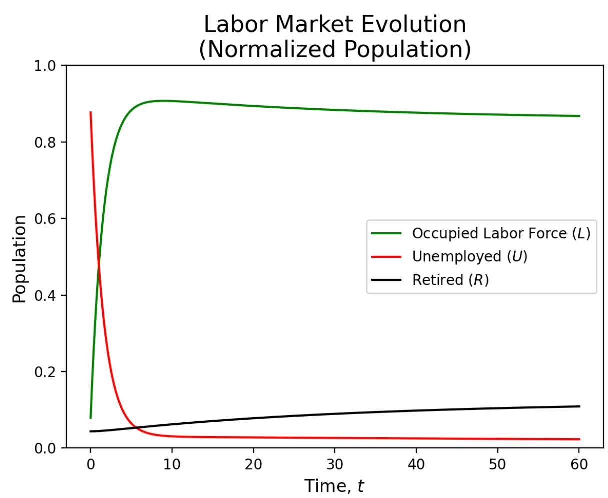

Labor Market Evolution
(Total Population)
1e8
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population (in 100 millions)
Time, t

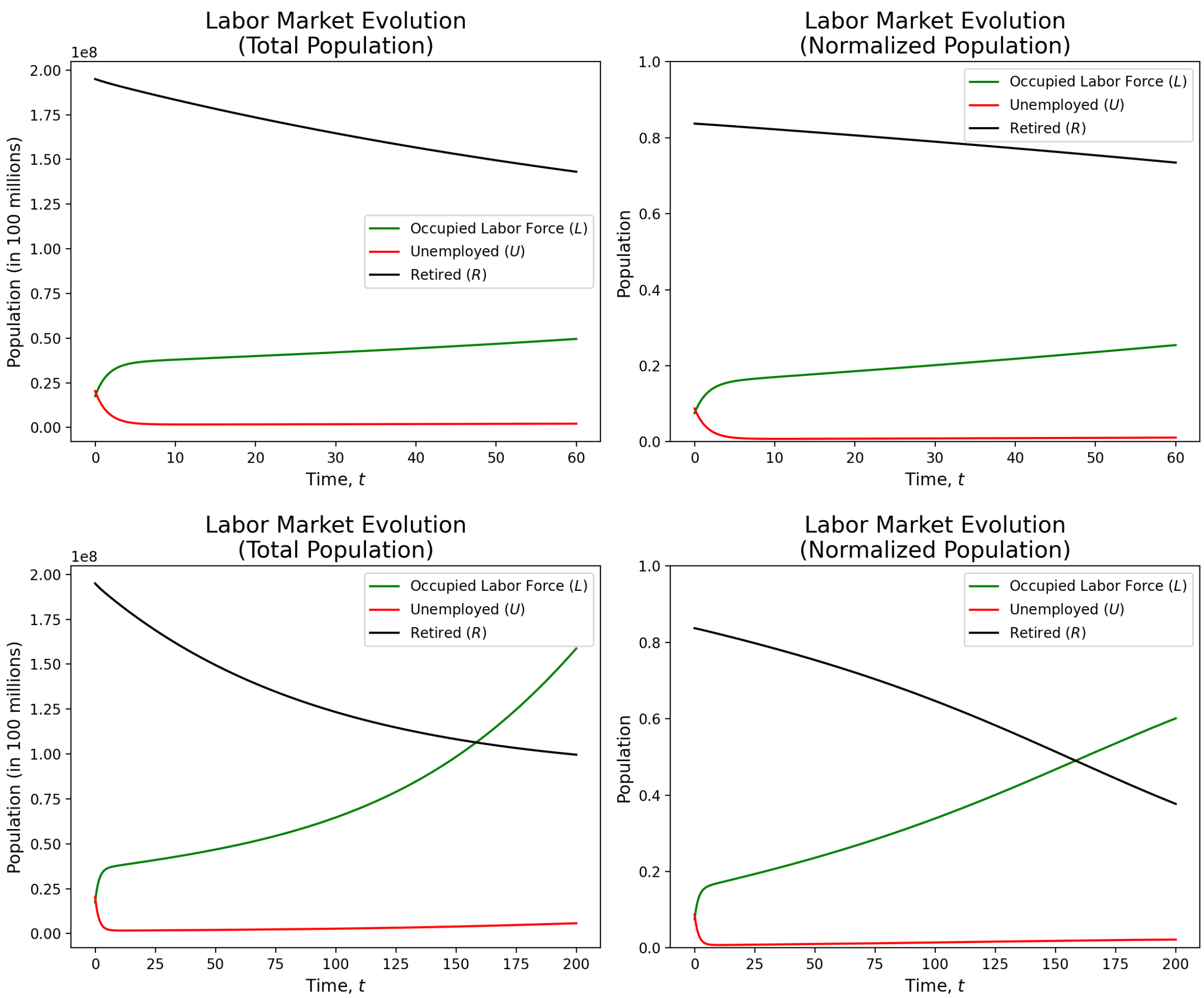

Labor Market Evolution
(Normalized Population)
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population
Time, t


Labor Market Evolution
(Total Population)
1e8
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population (in 100 millions)
Time, t


Labor Market Evolution
(Normalized Population)
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population
Time, t


Labor Market Evolution
(Total Population)
1e8
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population (in 100 millions)
Time, t


Labor Market Evolution
(Normalized Population)
Occupied Labor Force (L)
Unemployed (U)
Retired (R)
Population
Time, t



In [ ]:
```
# Initialize variables used for problem.
T = 100
t = np.linspace(0, T, 1000)

# Plug these values into solve_ivp
# for i, y0 in enumerate([[157e6, 6.5e6, 48.59e6], [157.1e6, 1e6, 1e6], [156
fig, ax = plt.subplots(2, 2, figsize=(12,10), dpi=200)
for i, y0 in enumerate([[17.5e6, 20.4e6, 195e6], [17.5e6, 20.4e6, 195e6]]):
    if i != 1:
        solution = solve_ivp(f, (0, T), y0, t_eval=t, args=(sigma, rho, tau,
    elif i == 1:
        T = 400
        t = np.linspace(0, T, 1000)
        solution = solve_ivp(f, (0, T), y0, t_eval=t, args=(sigma, rho, tau,

    sum_people = np.sum(solution.y, axis=0)
    fontsize = 16

    ax[i, 0].plot(solution.t, solution.y[0, :], label=f'Occupied Labor Force
    ax[i, 0].plot(solution.t, solution.y[1, :], label=f'Unemployed ($U$)', c
    ax[i, 0].plot(solution.t, solution.y[2, :], label=f'Retired ($R$)', colo
    ax[i, 0].set_title(f'Labor Market Evolution, T = {T}\n(Total Population)
    ax[i, 0].set_xlabel('Time, $t$', fontsize=fontsize - 4)
    ax[i, 0].set_ylabel('Population (in 100 millions)', fontsize=fontsize -
    # ax[i].set_ylim(0, 1)
```

```
    ax[i, 0].legend(fontsize=fontsize - 6)

    ax[i, 1].plot(solution.t, solution.y[0, :]/sum_people, label=f'Occupied
    ax[i, 1].plot(solution.t, solution.y[1, :]/sum_people, label=f'Unemploye
    ax[i, 1].plot(solution.t, solution.y[2, :]/sum_people, label=f'Retired (
    # ax[1].set_title(f'Population Model, Initial Conditions: {numerize(y0[0
    ax[i, 1].set_title(f'Labor Market Evolution, T = {T}\n(Normalized Popula
    ax[i, 1].set_xlabel('Time, $t$', fontsize=fontsize - 4)
    ax[i, 1].set_ylabel('Population', fontsize=fontsize - 4)
    ax[i, 1].set_ylim(0, 1)
    ax[i, 1].legend(fontsize=fontsize - 6)
plt.tight_layout()
plt.show()
```

Predator Prey Models

In [ ]:
```
# Define the differential equations
def f(t, x, rho, a, mu, eps, k, e, C):
    return np.array([
        rho * x[0] * (1 - x[0] / k) - a * x[0] * x[1],
        -mu * x[1] + eps * a * x[0] * x[1] + e * x[1] * (1 - x[1] / C)
    ])

# Set up initial conditions and parameters
t0, tf = 0, 30
t_span = (t0, tf)
steps = 100
t = np.linspace(*t_span, steps)
x0 = [.75, .98]
rho, a, mu, eps, k, e, C = 7, 5, 1, .2, 3, 1, 1.5

# Solve the differential equations
solution = solve_ivp(f, t_span, x0, t_eval=t, args=(rho, a, mu, eps, k, e, C

# Create plots
plt.figure(figsize=(12, 6), dpi=100)
plt.suptitle('Predator-Prey Relationships in the Labor Market')

# Subplot 1: Blue Collar vs. White Collar Population over time
plt.subplot(1, 3, 1)
plt.plot(t, solution.y[1], 'k-', label='White Collar')
plt.plot(t, solution.y[0], 'b-', label='Blue Collar')
plt.xlabel('Time')
plt.ylabel('Population (in 100 millions)')
plt.title('Blue Collar vs. White Collar Population over time')
plt.legend()

# Subplot 2: Blue Collar Population from year 10 to 50
plt.subplot(1, 3, 2)
t0, tf = 10, 50
plt.plot(t[t0:tf], solution.y[0, t0:tf], 'b-', label='Blue Collar')
plt.legend()
plt.xlabel('Time')
plt.ylabel('Population (in 100 millions)')
```

```
# Subplot 3: White Collar Population from year 10 to 50
plt.subplot(1, 3, 3)
plt.title('Zoomed in on years 10 to 50 to see the oscillations')
plt.plot(t[t0:tf], solution.y[1, t0:tf], 'k-', label='White Collar')
plt.legend()
plt.xlabel('Time')

# Adjust layout and save the plot
plt.tight_layout()
plt.savefig('blue_vs_white.png')
plt.show()
```

In [ ]:
```
plt.figure(figsize=(12,6),dpi=100)

plt.suptitle('Zoomed in on years 10 to 50 to see the oscillations')
plt.subplot(1, 2, 1)
t0, tf = 10, 50
plt.plot(t[t0:tf], solution.y[0, t0:tf], 'b-', label='Blue Collar')
plt.legend()
plt.xlabel('Time')
plt.ylabel('Population (in 100 millions)')

plt.subplot(1, 2, 2)
plt.plot(t[t0:tf], solution.y[1, t0:tf], 'k-', label='White Collar')
plt.legend()
plt.tight_layout()
plt.xlabel('Time')
plt.savefig('blue_vs_white_zoomed.png')
plt.show()
```

In [ ]:
```
plt.figure(figsize=(12,6),dpi=100)

def f(t, x, rho, a, mu, eps, k, e, C):
    return np.array(
                    [rho * x[0] * (1 - x[0] / k) - a*x[0]*x[1],
                     -mu*x[1] + eps*a*x[0]*x[1] + e * x[1] * (1 - x[1] / C)]
                     )

t0, tf = 0, 30
t_span = (t0, tf)
steps = 100
t = np.linspace(*t_span, steps)

x0 = [.75, .98]

rho, a, mu, eps, k, e, C = 7, 5, 2, .2, 3, 1, 1.5
solution = solve_ivp(f, t_span, x0, t_eval=t, args=(rho, a, mu, eps, k, e, C

plt.subplot(1, 2, 1)
plt.plot(t, solution.y[1], 'k-', label='White Collar')
plt.plot(t, solution.y[0], 'b-', label='Blue Collar')
plt.xlabel('Time')
plt.ylabel('Population (in 100 millions)')
plt.suptitle('Blue Collar vs. White Collar Population over time')
```

```
plt.legend()

rho, a, mu, eps, k, e, C = 7, 5, 1, 1, 3, 1, 1.5
solution = solve_ivp(f, t_span, x0, t_eval=t, args=(rho, a, mu, eps, k, e, C
plt.subplot(1, 2,2)
plt.plot(t, solution.y[1], 'k-', label='White Collar')
plt.plot(t, solution.y[0], 'b-', label='Blue Collar')
plt.xlabel('Time')
plt.legend()
plt.tight_layout()

plt.savefig('figures/bad_paramenters.png')

plt.show()
```

To introduce a carrying capacity parameter, I've modified the equations to include a new parameter C as the carrying capacity for white-collar jobs. The modified equations would then be:

$$\frac{dx_0}{dt} = \rho x_0 \left(1 - \frac{x_0}{k}\right) - a x_0 x_1$$

$$\frac{dx_1}{dt} = -\mu x_1 + \epsilon a x_0 x_1 \left(1 - \frac{x_0}{C}\right)$$

where

- ρ: Growth rate of blue-collar jobs.
- a: Rate at which people switch from blue collar to white collar jobs.
- μ: Decay rate of white-collar jobs.
- ϵ: Efficiency of white-collar jobs in utilizing blue-collar jobs.
- k: Carrying capacity for blue-collar jobs.
- C: Carrying capacity for white-collar jobs.

In [ ]:
```
def pred_prey_blue_white_collar(t, x, rho, a, mu, eps, k, C):
    def job_model(t, x, rho, a, mu, eps, k, C):
        return np.array([
            rho * x[0] * (1 - x[0] / k) - a * x[0] * x[1],
            -mu * x[1] + eps * a * x[0] * x[1] * (1 - x[1] / C)
        ])

    t0, tf = 0, 30
    t_span = (t0, tf)
    steps = 100
    t = np.linspace(*t_span, steps)

    # Solve the system of equations
    solution = solve_ivp(job_model, t_span, x0, t_eval=t, args=(rho, a, mu,

    # Plot the results
    plt.plot(t, solution.y[0], label='Blue Collar Jobs')
    plt.plot(t, solution.y[1], label='White Collar Jobs')
    plt.legend()
```

```
    plt.xlabel('Time')
    plt.ylabel('Population')
    plt.title('Blue Collar and White Collar Job Dynamics with Carrying Capac
    plt.show()
```