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Systemic Risk Illustrated

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Abstract We study the behavior of diffusions coupled through their drifts in a way that each component mean-reverts to the mean of the ensemble. In particular, we are interested in the number of components reaching a "default" level in a given time. This coupling creates stability of the system in the sense that there is a large probability of "nearly no default" as opposed to the case of independent Brownian motions for which the distribution of number of defaults is of binomial type. However, we show that this "swarming" behavior also creates a small probability that a large number of components default corresponding to a "systemic risk event". The goal of this work is to illustrate systemic risk with a toy model of lending and borrowing banks, using mean-field limit and large deviation estimates for a simple linear model.

17.1 Introduction

In the toy model discussed below, the diffusion processes $Y_t^{(i)}$, $i=1,\dots,N$ represent the log-monetary reserves of N banks possibly lending and borrowing to each other. The system is driven by N independent standard Brownian motions $W_t^{(i)}$, $i=1,\dots,N$ and starts at time $t=0$ from $Y_0^{(i)}=y_0^{(i)}$, $i=1,\dots,N$. For simplicity and without loss of generality for the purpose of this chapter, we assume that the diffusion coefficients are constant and identical, denoted by $\sigma>0$. In the case of no lending or borrowing, $Y_t^{(i)}$, $i=1,\dots,N$ are independent and simply given by driftless Brownian motions:

$$dY_t^{(i)}=\sigma dW_t^{(i)} \qquad i=1,\dots,N. \tag{17.1}$$

Our toy model of lending and borrowing consists in introducing an interaction through drift terms of the form $(Y_t^{(j)}-Y_t^{(i)})$ representing the rate at which bank i borrows from or lends to bank j. In this case, the rates are proportional to the difference in log-monetary reserves. Our model is:

$$dY_t^{(i)} = \frac{\alpha}{N}\sum_{j=1}^{N}(Y_t^{(j)} - Y_t^{(i)})\,dt + \sigma dW_t^{(i)}, \quad i = 1,\ldots,N, \tag{17.2}$$

where the overall rate of mean-reversion α/N has been normalized by the number of banks and we assume $\alpha > 0$. Note that in the case $\alpha = 0$, the system (17.2) reduces to the independent system (17.1).

In the spirit of structural models of default, we introduce a default level $\eta < 0$ and say that bank i defaults by time T if its log-monetary reserve reached the level η before time T (note that in this simplified model, bank i stays in the system until time T).

Here, we want to comment on the difference between "systemic risk" which we will discuss below and "credit risk". In the latter case, $Y^{(i)}$ denotes the log-value of a firm (or its stock price as a proxy) for instance, and dependency between firms can be created by introducing a correlation structure between the Brownian motions' $W^{(i)}$s (dependency can also be created through volatilities, see Fouque et al. (2008), but for the sake of this comment we assume that volatilities remain constant and identical). In pricing credit derivatives, the drifts are imposed by risk-neutrality and do not play a role in the correlation of defaults. In the independent case, as in system (17.1), and assuming symmetry (same initial value), the loss distribution (distribution of the number of defaults) is simply binomial. In the correlated cases, for reasonable level of correlation, the shape of the loss distribution is roughly preserved with some skewness and fatter-tail effects. We will show that the shape of the loss distribution generated by the coupled system (17.2) is very different, with mainly a large mass near zero (stability of the system) and a small (but present) mass in the tail near N (systemic risk).

In the next section, we illustrate the stability of system (17.2) by simulations for various values of the mean-reversion rate α and we compare with the independent case $\alpha = 0$ as in (17.1). As expected, the possibility for a bank to borrow money from other banks with larger monetary reserves creates this stability of the system.

In Section 17.3, we derive the mean-field limit of system (17.2) as the number of banks becomes large. In this limit, banks become independent and their log-monetary reserves follow OU processes. Interestingly, before taking this limit, we observe that each component mean-reverts to a common Brownian motion with a small diffusion of order $1/\sqrt{N}$. We exploit this fact in Section 17.4, to explain systemic risk as the small-probability event where this mean level reaches the default barrier, with a typically large number of components "following" the mean and defaulting. Moreover, this small probability of systemic risk is independent of

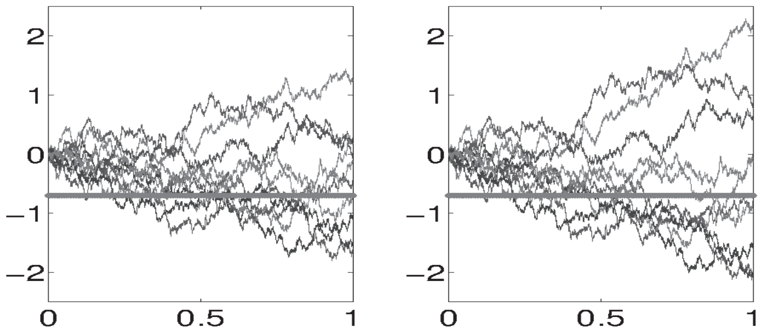

Figure 17.1 One realization of the trajectories of the coupled diffusions (17.2) with $\alpha = 1$ (left plot) and trajectories of the independent Brownian motions (17.1) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $\eta = -0.7$.

the mean-reversion rate α so that a large α corresponds to more stability but at the same time to (or "at the price of") a larger systemic event.

17.2 Stability illustrated by simulations

We first compare the coupled diffusions (17.2) to the independent case (17.1) by looking at typical trajectories. For simplicity of our simulation, we assume $y_0^{(i)} = 0$, $i = 1,\ldots,N$. Also, we choose the common parameters $\sigma = 1$, $\eta = -0.7$, and $N = 10$, and we used the Euler scheme with a time-step $\Delta = 10^{-4}$, up to time $T = 1$. In Figures 17.1, 17.2 and 17.3, we show a typical realization of the N trajectories with $\alpha = 1$, $\alpha = 10$, and $\alpha = 100$ respectively. We see that the trajectories generated by (17.2) are more grouped than the ones generated by (17.1). This is the "swarming" or "flocking" effect more pronounced for a larger α. Consequently, less (or almost no) trajectories will reach the default level η, creating stability of the system.

Next, we compare the loss distributions for the coupled and independent cases. We compute these loss distributions by Monte Carlo method using 10^4 simulations, and with the same parameters as previously.

In the independent case, the loss distribution is Binomial(N, p) with parameter p given by

$$p = \mathbb{P}\left(\min_{0\le t\le T}(\sigma W_t) \le \eta\right)$$
$$= 2\Phi\left(\frac{\eta}{\sigma\sqrt{T}}\right),$$

where Φ denotes the $\mathcal{N}(0,1)$-cdf, and we used the distribution of the minimum of

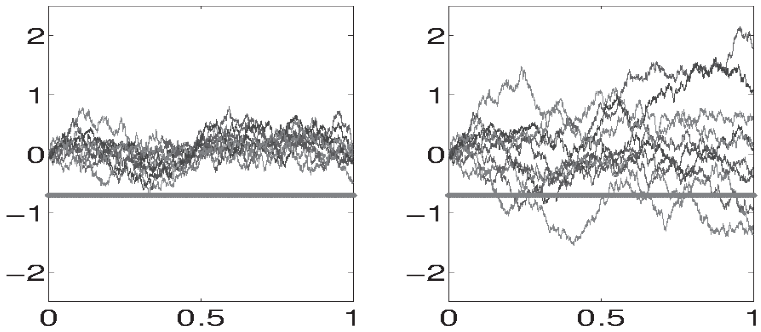

Figure 17.2 One realization of the trajectories of the coupled diffusions (17.2) (left plot) with $\alpha = 10$ and trajectories of the independent Brownian motions (17.1) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $\eta = -0.7$.

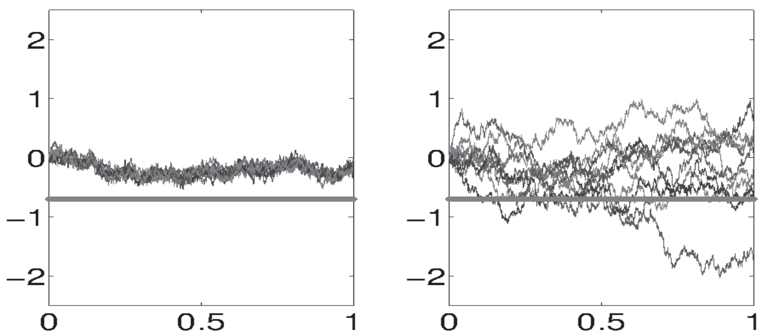

Figure 17.3 One realization of the trajectories of the coupled diffusions (17.2) (left plot) with $\alpha = 100$ and trajectories of the independent Brownian motions (17.1) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $\eta = -0.7$.

a Brownian motion (see Karatzas and Shreve (2000) for instance). With our choice of parameters, we have $p \approx 0.5$ and therefore the corresponding loss distribution is almost symmetric as can be seen on the left panels (dashed lines) in Figures 17.4, 17.5, and 17.6. Observe that in the independent case, the loss distribution does not depend on α, and therefore is the same on these three figures (up to the Monte Carlo error estimate).

Next, we compare the loss distribution generated by our coupled system (17.2) for increasing values of α (solid lines, $\alpha = 1$, $\alpha = 10$, and $\alpha = 100$ in Figures 17.4, 17.5, and 17.6, respectively). We see that increasing α, that is the rate of borrowing and lending, pushes most of the mass to zero default, in other words, it improves the stability of the system by keeping the diffusions near zero (away

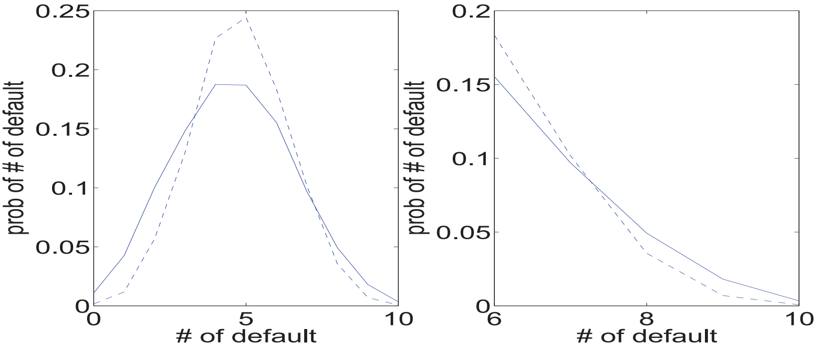

Figure 17.4 On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = 1$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

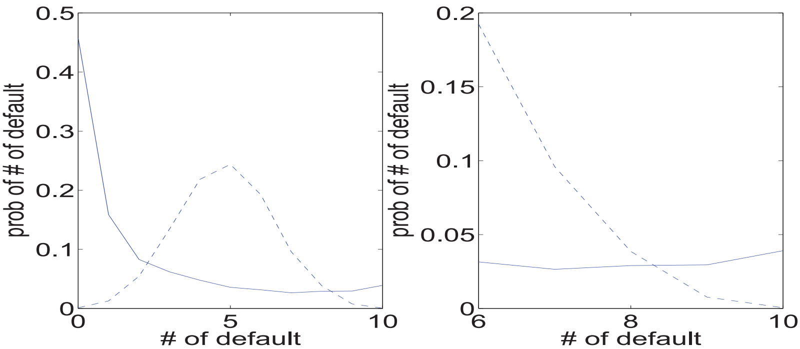

Figure 17.5 On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = 10$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

from default) most of the time. However, we also see that there is small but non-negligible probability, that almost all diffusions reach the default level. On the right panels of Figures 17.4, 17.5, and 17.6 we zoom on this tail probability. In fact, we will see in the next section that this tail corresponds to the small probability of the ensemble average reaching the default level, and to almost all diffusions following this average due to "flocking" for large α.

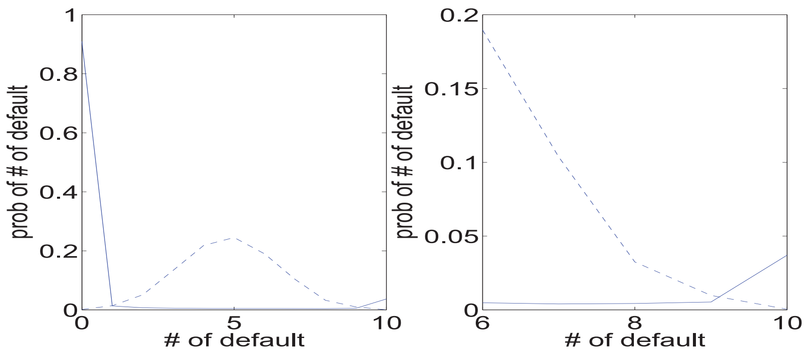

Figure 17.6 On the left, we show plots of the loss distribution for the coupled diffusions with $\alpha = 100$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

17.3 Mean-field limit

In order to understand the behavior of the coupled system (17.2), we rewrite its dynamics as:

$$dY_t^{(i)} = \frac{\alpha}{N}\sum_{j=1}^{N}(Y_t^{(j)} - Y_t^{(i)})\,dt + \sigma dW_t^{(i)}$$
$$= \alpha\left[\left(\frac{1}{N}\sum_{j=1}^{N} Y_t^{(j)}\right) - Y_t^{(i)}\right] dt + \sigma dW_t^{(i)}. \tag{17.3}$$

In other words, the $Y^{(i)}$ are Ornstein–Uhlenbeck (OU) processes mean-reverting to the ensemble average. Next, we observe that this ensemble average satisfies

$$d\left(\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)}\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right),$$

and assuming for instance that $y_0^{(i)} = 0$, $i = 1,\ldots,N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} Y_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \tag{17.4}$$

and consequently

$$dY_t^{(i)} = \alpha\left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)}\right) - Y_t^{(i)}\right] dt + \sigma dW_t^{(i)}. \tag{17.5}$$

Note that in fact the ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^{(j)} \to 0 \quad a.s.,$$

and therefore, the processes $Y^{(i)}$'s converge to independent OU processes with long-run mean zero. In order to make this result precise, one can solve (17.5)

$$Y_t^{(i)} = \frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(i)} - \frac{\sigma}{N}\sum_{j=1}^{N}\left(e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(j)}\right),$$

and derive that $Y_t^{(i)}$ converges to $\sigma e^{-\alpha t}\int_0^t e^{\alpha s} dW_s^{(i)}$ which are independent OU processes. This is in fact a simple example of a mean-field limit and propagation of chaos studied in general in Sznitman (1991).

Note that the distributions of hitting times for OU processes have been studied in Alili et al. (2005). Let us denote

$$p' = \mathbb{P}(\tau \leq T),$$

τ being the hitting time of the default level for an OU process with long-run mean zero, given by

$$dY_t = -\alpha Y_t dt + \sigma dW_t.$$

In the interesting regime where $p'N \to \lambda > 0$, obtained as $N \to \infty$ and $\eta \to -\infty$ appropriately, the loss distribution converges to a Poisson distribution with parameter λ. In this stable regime, the mass is mainly concentrated on a small number of defaults.

In the next section, we investigate the small probability of a large number of defaults when the default level η is fixed.

17.4 Large deviations and systemic risk

In this section, we focus on the event where the ensemble average given by (17.4) reaches the default level. The probability of this event is small (when N becomes large), and is given by the theory of large deviations. In our simple example, this probability can be computed explicitly as follows:

$$\begin{aligned}\mathbb{P}\left(\min_{0\leq t\leq T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right) \leq \eta\right) &= \mathbb{P}\left(\min_{0\leq t\leq T}\widetilde{W}_t \leq \frac{\eta\sqrt{N}}{\sigma}\right)\\ &= 2\Phi\left(\frac{\eta\sqrt{N}}{\sigma\sqrt{T}}\right), \qquad (17.6)\end{aligned}$$

where $\widetilde{W}$ is a standard Brownian motion. Therefore, using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N}\log\mathbb{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N}W_t^{(i)}\right)\le\eta\right)=\frac{\eta^2}{2\sigma^2 T}. \tag{17.7}$$

In other words, for a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp(-\eta^2 N/(2\sigma^2 T))$. Recalling (17.4), we identify

$$\left\{\min_{0\le t\le T}\left(\frac{1}{N}\sum_{i=1}^{N}Y_t^{(i)}\right)\le\eta\right\}$$

as a *systemic event*. Observe that this event does not depend on $\alpha > 0$, in other words, increasing stability by increasing α (that is increasing the rate of borrowing and lending) does not prevent a systemic event where a large number of banks default. In fact, once in this event, increasing α creates even more defaults by "flocking to default". This is illustrated in the Figure 17.6, where $\alpha = 100$ and the probability of systemic risk is roughly 3% (obtained using formula (17.6)). One could object that with this definition of a systemic event, in fact, only one bank could default (far below the barrier) and all the others be above the default barrier since only the average counts. But, this type of event is easily seen to be of probability of smaller order. What we try to capture here, is the fact that for large α, the $Y^{(i)}$s are close to each other and once in the default event they will all be at (or near) the default level.

17.5 Conclusion

We proposed a simple toy model of coupled diffusions to represent lending and borrowing between banks. We show that, as expected, this activity stabilizes the system in the sense that it decreases the number of defaults. Indeed, and naively, banks in difficulty can be "saved" by borrowing from others. In fact, the model illustrates the fact that stability increases as the rate of borrowing and lending increases. It shows also that this coupling through the drifts is very different from correlation through the driving Brownian motions or volatilities as it is the case in the structural approach for credit risk (see for instance Fouque et al. (2008)). This can be seen by comparing loss distributions as we did in Section 17.2. In the latter case, the loss distribution is shaped as a binomial while in the former case, it is bimodal with a large mass on the left on small numbers of defaults and a small mass on the right on very large numbers of defaults. This last observation is explained through the mean-field limit of the system (for large number of banks) combined with a large deviation argument. The model is rich enough to exhibit this property and simple enough to be tractable. In particular, the mean-field limit is easy to derive. The diffusions mean-revert to the average of the ensemble, and this average converges, as the number of banks becomes large, to a level away from the default level. That explains the stabilization of the system. However, there is a small probability, computed explicitly in our model, that the average of the ensemble reaches the default level. Combined with the “flocking” behavior (“everybody follows everybody”), this leads to a *systemic event* where almost all default, in particular when the rate of borrowing and lending is large.

To summarize, our simple model shows that “lending and borrowing improves stability but also contributes to systemic risk”. We have quantified this behavior and identified the crucial role played by the rate of borrowing and lending.

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