Title

Team 674

Problem A

Abstract

In order to design the optimal trajectory of a solar sail spacecraft launched from Earth to Mars, in this paper, we discuss the influence of the surface area of solar sail and its attitude angle about the largest-payload and shortest-time problem. We treat this influencial problem as an optimal control problem.
will affect the sail area and the attitude angle as endpoint time equation of state optimal constrained time and relative distance inequality the constraint control problem, designed by Matlab iterative algorithm, linear search the optimal attitude angle and the best turn sail area, shorten the time of flight tasks for the engineering application of solar sail, it has practical reference value.

Contents

1 Background

With the increase of flight distance and the extension of working time on deep space exploration mission, the tranditional spacecraft which relys on the reaction propulsion will need to carry more fuels and energy, which will largely increase the payload of the spacecraft and thus limit its maximum speed.

In recent years, solar sail as a new type of spacecraft has attracted much attention. It can be powered by solar light pressure without having to carry a large amount of fuel. Therefore, it has been widely used in the interplanetary navigation and deep space exploration[1].

Solar sail deep space trajectory optimization research began in the last century in 70s, there have been many research achievements, of which the most famous is the task of the United States in 1978 NASA proposed in 1986 plans to rendezvous with Halley's comet[2]. After that, several classical literatures have studied the optimization and application of the indirect trajectory of solar sail, which has promoted the development of the solar sail optimization theory[3].

2 Introduction

A spacecraft equipped with solar sails is powered by sunlight instead of rockets or fuels. It can sustain a thrust from the inexhaustible sunlight by sailing out a giant ultra thin sail whose thickness is about only 100 atoms. It flies like a sailing boat in the ocean, and adjusts the direction of the sail by changing the angle of the sail(also named attitude angle). As long as its geometry and the attitude angle are appropriate, it can fly in any direction, including the direction of light source. With the thrust of sunlight, the spacecraft can fly to the edge of the solar system and enter into the interstellar space to complete deep space exploration missions. The advantages of solar sail spacecraft is that it does not need to carry a large amount of propellant. Although the solar radiation is very small, but the continuous acceleration from the large, ultrathin mirrors can hardness the faint pressure of the Sun's reflected light to move throught the vacuum sapce and reach a considerable speed which is about 5-10 times faster than traditional spacecraft, and suitable for deep space exploration.

In this paper, we firstly establish the structure model of the solar sail, and analysis its forces. By comparing the magnitude, we ignore the other forces apart from the Sun and the Sun's gravitational pressure. Then lists the differential equation of the motion by Newton Second Law, and give the expression of acceleration which relate to its position. Finally, we obatained the optimal flight plan of solar sail with the realisation of the shortest transit-time and the maximum payload.

3 Problem statement

The spacecraft can reflect the sunlight by carrying a large, lightweight reflective surface that exerts a pressure equal to twice its energy density, thus gaining impetus. This impetus is related to the acceleration of the spacecraft, and its acceleration is influenced by the attitude of the spacecraft. When the attitude angle α of the spacecraft changes, its trajectory will change accordingly.

When the distance between Earth and Mars is the closest, a rocket will launch a total mass of 2000 kg (sail plus payload) to escape velocity from Earth to Mars, and the spacecraft must have a relative velocity of no more than 9 km/s to ensure a safe landing close to Mars. Assume that the sail is made of material of mass 7 g/m^2, we need to design a flight plan for the solar sail spacecraft in order to find the optimal size of the sail so that to maximum the payload and to reduce transit time.

amsmath graphicx subfigure float

4 List of Symbols

Symbol	Description	Value
$S.$	Solar constant/W$\cdot s^{-2}$	1367
G	Universal gravitational constant/$m^3kg^{-1}s^{-2}$	6.67259×10^{-11}
R_E	The distance from the sun to the earth/m	1.496×10^{11}
R_M	The distance from the sun to the Mars/m	2.2794×10^{11}
M_S	The masse of the sun/kg	1.9891×10^{30}
M_E	The masse of the earth/kg	5.965×10^{24}
M_M	The masse of the Mars/kg	6.4219×10^{23}
m	Total mass (sail plus payload)/kg	2000
T_E	The period of revolution of earth/s	3.1536×10^7 (365days)
T_M	The period of revolution of Mars/s	5.93568×10^7 (687days)
ω_E	The angular velocity of revolution of earth/rad$\cdot s^{-1}$	1.9924×10^{-7}
ω_M	The angular velocity of revolution of Mars/rad$\cdot s^{-1}$	1.0585×10^{-7}
α	The attitude angle of solar sail	$-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$
A	The area of solar sail	

Table 1: List of symbols

5 Assumption and Theory

5.1 Method

We can analyse the force state of a solar sail spacecraft as usual to get its kinematical equation, and then the trajectory under the influence of spacecraft in the solar photovoltaic can be solved with several differential equations. Then we design some programs with the aid of the software Matlab in order to find out the optimal solution which satisfies the constraints linearly by iterative method. And then we can make many analysis on the parameters that affect the orbit.

5.2 Basic model of the solar sail

In this paper, the rigid body hypothesis is used in the modeling of attitude dynamics of solar sail spacecraft (Fighre 1). The solar sail surface is a square geometry, with one control blades at each end of the four support rods. The payload is connected with the surface center of solar sail through a control rod with universal joint.

The simplified solar sail configuration is shown in Figure 2. The simplified configuration regards the solar sail surface as a square plane with an area A, and the center of mass coincides with its geometric center, which is located at the position of the gimbal. The attitude angle α of the solar sail is defined as the angle between the normal direction of the sail and the radial direction of the Sun, with a value of $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

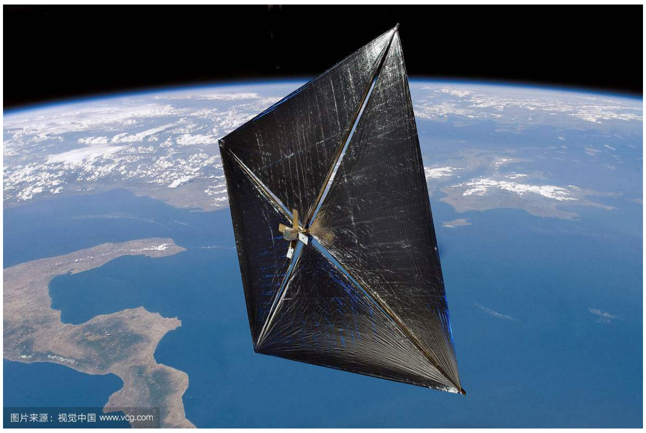

Figure 1: Solar sail

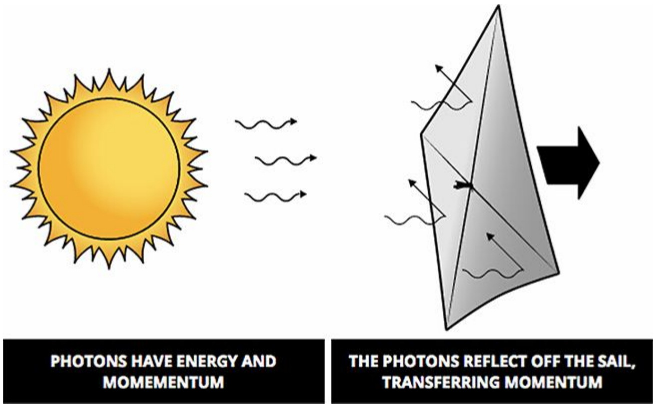

Figure 2: Basic model of the solar sail

5.3 Force of light pressure

5.3.1 Assumptions

The following calculation of the force of light pressure is based on these three hypotheses:

(1) The energy emitted by Sun during unit time is constant.

(2) The radiant energy absorbed by Mercury, Venus, and Earth is negligible;.

(3) Only considered in the solar system.

5.3.2 Calculation

The solar constant (S.) refers to the solar radiation energy that received per second by per unit area of the top of the atmosphere circle perpendicular to the sunlight at the average distance from Sun to Earth ($D = 1.496 \times 10^8 km$). Similarly we denote S(r) as the energy flow density at the distance r from the Sun, and we denote A as the area of solar sail.

According to the conversation of energy:

$$S.\frac{4}{3}\pi {R_E}^2 = S(r)\frac{4}{3}\pi r^2$$

$$S(r) = \frac{S.R_E^2}{r^2} \tag{5.1}$$

The energy density is:

$$\rho_E = \frac{E}{V} = \frac{S.tAcos\alpha}{ctAcos\alpha} = \frac{S.}{c} \tag{5.2}$$

The total light pressure is equal to twice of its energy density:

$$P(r) = 2\rho_E = \frac{2S(r)}{c} = \frac{2S.{R_E}^2}{cr^2} \tag{5.3}$$

The ideal plane solar sail model assumed that the impact of solar photon on the sail surface can be regarded as an ideal reflection, and the sail surface is an ideal plane. Therefore, the forces exert by incident photons and reflected photons are equal, shown as follows:

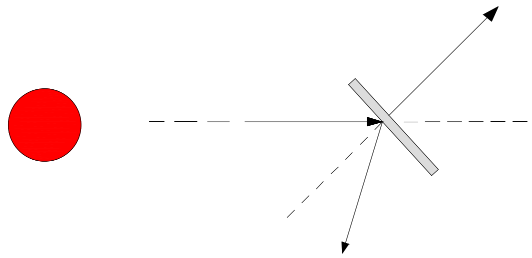

Figure 3: incident and reflected photon

The light pressure produced by the incident photon is:

$$P_i = \frac{P(r)}{2} A_z (cos\alpha\, \vec{n} - sin\alpha\, \vec{t}) \tag{5.4}$$

In this formula, A_z is the effective sail area, α is the angle between the incident ray and the solar sail unit normal direction $\vec{n}$, the unit vector of solar sail normal direction $\vec{n}$ is perpendicular to the sail plane and pointing out away from the sun. $\vec{t}$ is the tangent unit vector perpendicular to the normal $\vec{n}$, the anticlockwise direction is defined positive. Then the light pressure produced by the reflected light is as bellow:

$$P_r = \frac{P(r)}{2} A_z (cos\alpha\, \vec{n} + sin\alpha\, \vec{t}) \tag{5.5}$$

The effective area refers to the area on which the sail is projected on the plane perpendicular to the sunlight, $A_z = Acos\alpha$. Therefore, the total light pressure acting on the ideal plane solar sail is:

$$\vec{F_p} = \vec{F_i} + \vec{F_r} = P(r) Acos^2\alpha\, \vec{n} = \frac{2S.{R_E}^2 Acos^2\alpha}{cr^2}\, \vec{n} \tag{5.6}$$

Therefore, the force of light pressure F_p is always along the normal direction of solar sail surface, $\vec{f} = \vec{n}$, $\vec{f}$ is the unit vector of thrust, it is always in the direction along the pressure direction and deviates from the sun.

5.4 Analysis of the gravitation

We calculated the gravitation from the Sun, Earth, Mars when the spacecraft has not escaped the Earth and has not entered the orbit of Mars. It can be seen accorfing to the order of magnitude in figures bellow that during the process of flight, the gravitation exerted by the Earth and Mars both can be negligible in comparing to the Sun's gravity. Thus we can get a simplified model.

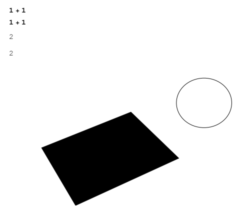

Figure 4: ...

5.5 Force analysis and expression of acceleration

In this problem, the solar sail is mainly affected by the gravity of Sun and the pressure of sunlight, because the magnitude of solar sail due to the gravitational action of Mercury, Venus and Earth is very small which can be neglected in comparing to the relative solar gravitation.

The solar sail is subjected to the small thrust from the continuous light pressure, and thrust itself is inversely proportional to the square of distance from Sun to the solar sail, so it can be realized with spiral trajectory.

Light pressure is always deviated from the sun, by controling the attitude of solar sail, when the solar sail attitude angle α is positive, i.e. $F_p \cdot \alpha > 0$, solar sail get orbital angular momentum, solar sail and spiral outward away from the sun; when the sail angle is negative, i.e. $F_p \cdot \alpha < 0$, solar sail lose orbital angular momentum, and the solar sail moves toward the sun.

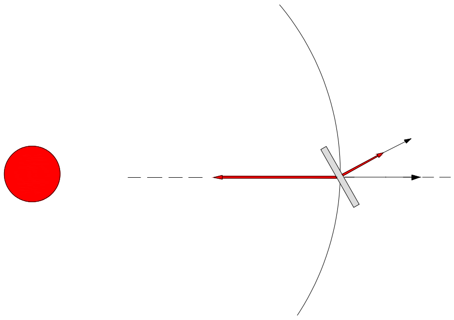

Figure 5: spiraling outward($F_P \cdot \alpha > 0$)

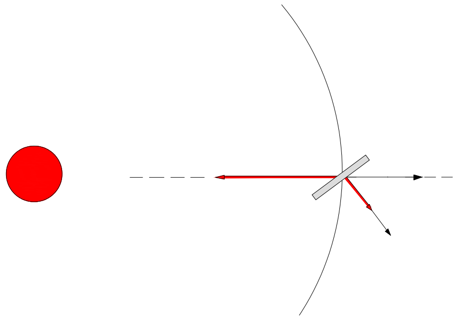

Figure 6: spiraling inward($F_P \cdot \alpha < 0$)

Denoted F_g as the gravity of Sun received by solar sail and $\vec{F_g} = G\frac{M_S m}{r^2}\vec{u}_r$.
According to the Newton Second Law:

$$\vec{F_g} + \vec{F_p} = m\vec{a} \tag{5.7}$$

projected to the radial direction $\vec{u}_r$:

$$G\frac{M_S m}{r^2} + \frac{2S.{R_E}^2 Acos^2\alpha}{cr^2} \cdot cos\alpha = ma \tag{5.8}$$

so we obtain the normal acceleration as:

$$\vec{a} = \left(\frac{GM_S}{r^2} + \frac{2S.{R_E}^2 Acos^3\alpha}{mcr^2}\right)\vec{n} \tag{5.9}$$

5.6 Kinematical equation

With the equation (2.6) when we suppose the surface of the solar sail is A, the attitude angle is α (the attitude angle is defined by the included angle between the vector $\vec{n}$ and the vector $\vec{u}_r$ which have presented in the Figure.3)

$$\vec{F_p} = P(r)Acos^2(\alpha)\vec{n} \tag{5.10}$$

and the gravitation from the Sun:

$$\vec{F_g} = -\frac{GM_S m}{r^2}\vec{u}_r \tag{5.11}$$

We have the conversion relation between these vectors :

$$\vec{u_r} = cos\theta\vec{u_x} + sin\theta\vec{u_y} \tag{5.12}$$

and

$$\vec{u_\theta} = -sin\theta\vec{u_x} + cos\theta\vec{u_y} \tag{5.13}$$

so the normal should be:

$$\begin{aligned}\vec{n} &= cos\alpha\vec{u_r} + sin\alpha\vec{u_\theta} \\ &= cos\alpha(cos\theta\vec{u_x} + sin\theta\vec{u_y}) + sin\alpha(-sin\theta\vec{u_x} + cos\theta\vec{u_y}) \\ &= (cos\alpha cos\theta - sin\alpha sin\theta)\vec{u_x} + (cos\alpha sin\theta + sin\alpha cos\theta)\vec{u_y}\end{aligned}$$

Then we project this two force to the axe x and axe y:

$$\vec{F_x} = (-\frac{GMmx}{(\sqrt{x^2+y^2})^3} + P(r)Acos^2\alpha(cos\alpha\frac{x}{\sqrt{x^2+y^2}} - sin\alpha\frac{y}{\sqrt{x^2+y^2}}))\vec{u_x} \tag{5.14}$$

$$\vec{F_y} = (-\frac{GMmy}{(\sqrt{x^2+y^2})^3} + P(r)Acos^2\alpha(cos\alpha\frac{y}{\sqrt{x^2+y^2}} + sin\alpha\frac{x}{\sqrt{x^2+y^2}}))\vec{u_y} \tag{5.15}$$

According to the basic principle of dynamics, we can get these kinematical equation:

$$\frac{d^2x}{dt^2} = \frac{F_x}{m} \tag{5.16}$$

$$\frac{d^2y}{dt^2} = \frac{F_y}{m} \tag{5.17}$$

After we finish the basic kinematical equation, we can use MATLAB to solve with initial conditions to get its trajectory.

5.7 Trajectory

5.7.1 Assumption

Here, we make some simplificaition in considering the orbits:
(1) these revolution orbits of Earth and Mars are periphery,
(2) these revolution plane of Earth and Mars is the same plane.

5.7.2 The initial state of the solar sail spacecraft

First, considering that the solar sail spacecraft will be launched from Earth to Mars, and it needs to use the rocket to accelerate the spacecraft to Earth's escape velocity. The best and most fuel-efficient situation here is that the energy that solar sails spacecraft is given by the rocket is just enough to get the spacecraft out of the Earth's gravitational field, so, we are treating the speed of the solar sail with the speed of the Earth's revolution speed and direction.
Next we consider the mass of the spacecraft is 2000kg.

5.7.3 Trajectory without power

According to the initial condition, without the power and the influence of the gravitation from Earth, it is simple to understand that the trajectory of spacecraft is the revolutionary orbit.

5.7.4 Trajectory with power

In this section we have got some trajectories of the spacecraft by setting the value of surface area A of the solar sail and the attitude angle α. We can figure out that if we want to send the solar sail to Mars or outer planet, we have to make the attitude angle positive ($\alpha > 0$), i.e. the situation: spiraling outward. And the other situation is spiraling inward ($\alpha < 0$) which is corresponding to sending the solar sail to inner planet e.g. Mercury. We want to change the surface area A and the attitude angle α to make it to the revolutionary orbit of Mars. The test verifies that the spacecraft can indeeds get to the revolutionary orbit of Mars, i.e. sending the equipment to Mars by using the solar sail is absolutely viable in some conditions.

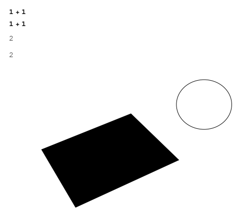

Figure 7: ...

6 Realization

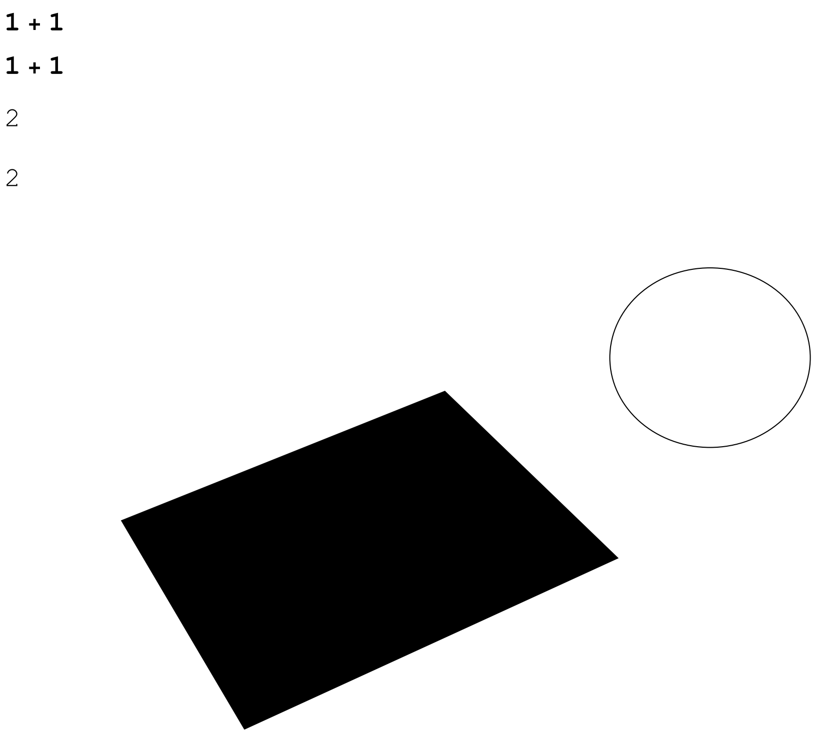

Figure 8: ...

7 Conclusion

8 Trajectory Optimization Model

8.1 Establishment of optimal model

8.1.1 Description

In the previous section we have demonstrated the feasibility of launching solar sail spacecraft from earth to Mars orbit. In comparing the previous figures, we can find the attitude angle α and the surface of the solar sail A have certain influence on the trajectory of spacecraft. Thought understanding the differential equation (5.14) (5.14) the transit time t is determined by the attitude angle α and the surface of the solar sail A. So this section is in order to find the optimal solution, to decrease the transit time and insure the payload, to maximize the advantage of solar sails.

8.1.2 Conditions of Capture by Mars

According to actual condition, when the spacecraft near Mars, the influence of the Martian gravity is bigger and bigger, so we set up a Martian gravity range. When the spacecraft into the scope, we ignore the role of the sun's gravity, and the Martian gravity plays a leading role. In other words, the spacecraft was captured by Mars. At this time, if the speed constraint is under certain conditions, it can land safely, so we can get the spacecraft to be captured by Mars as one of the constraints.
We suppose that when $\frac{G_M}{G} = \varepsilon$ the spacecraft will be captured by Mars, according to the gravitational equation:

$$\frac{GM_M m}{(r_{range})^2} = \frac{GM_S m}{(R_M - r_{range})^2} \cdot \varepsilon$$

So here we get the distance constrain:

$$\varepsilon \cdot r_{range}^2 + 2 \cdot R_E \cdot \frac{M_M}{M_S} r_{range} - R_E^2 \cdot \frac{M_M}{M_S} = 0$$

E.g. we suppose that $\varepsilon = 10$ after calculating we get the range of gravitational influence of Mars is 2.68×10^7. So in the constraint condition we set this order of magnitude (10^7) is reasonable.

8.1.3 Mathematical formulation

As discussion above, we can take the below mathematical formulation of optimization model

❶ **Objective function:** $\min t = t(\alpha, A)$

❷ **Decision variable:** α, A

❸ $t(\alpha, A)$ **satisfy system of differential equations** (5.14) (5.14)

❹ **Constraint conditions:**

- 4 Initial conditions:

$$\begin{cases} x(t=0) = R_E \\ y(t=0) = 0 \\ v_x(t=0) = 0 \\ v_y(t=0) = V_E(Earth\ revolution\ speed) \end{cases}$$

- 4 End conditions:

$$\begin{cases} x(t=t_{arrive})^2 + y(t=t_{arrive})^2 = R_M^2 \\ v_x(t=t_{arrive})^2 + v_y(t=t_{arrive})^2 \le 9000 m \cdot s^{-1} \\ x(t=t_{arrive}) = R_M \cos\left[\dfrac{t_{arrive}}{T_M} \cdot 2\pi\right] \\ y(t=t_{arrive}) = R_M \sin\left[\dfrac{t_{arrive}}{T_M} \cdot 2\pi\right] \end{cases}$$

Where the last 2 equations mean: when the solar sail get to Mars, time spent is t_{arrive}, at the end of this time, the position of Mars are $\left(R_M \cos\left[\frac{t_{arrive}}{T_M} \cdot 2\pi\right], R_M \sin\left[\frac{t_{arrive}}{T_M} \cdot 2\pi\right]\right)$, where T_M is the period of revolution of Mars, the position of solar sail $(x(t = t_{arrive}), y(t = t_{arrive}))$ should meet this point.

- 2 conditions of range of decision conditions:

$$0 < A < \frac{m}{\sigma} = 285714m^2$$

$$0 < \alpha < \frac{\pi}{2}$$

8.2 Realization by program

STEP0 : Start;

STEP1 : Input kinematical differential equations (5.14) (5.14) and initial conditions;

STEP2 : Give the range of the surface A and the attitude angle α;

STEP3 : Set the step length of A and α, start to loop;

STEP4 : Judge the solution : the relative speed between Mars and the spacecraft is less than $9km/s$, the distance between Mars and the spacecraft is less than r_{range};

STEP5 : Output the (A,α) that meet the judgement;

The detailed program code will be pasted to Appendix.

8.3 Result

After the iterative process by Matlab , luckily we seek 7 optimal values that satisfy all constraint conditions. By the program, we also obtain the transit time(time when the solar sailing arrive the destination) t_{arrive} and the effective load(the total mass removes the quality of the solar sail) m_e, and

$$m_e = m - \sigma \cdot A$$

where A is the area of solar sail, $\sigma = 7g \cdot m^{-2}$ is the density of solar sail and $m = 2000kg$ is the total mass of solar sail and effective load m_e.

Next is the table of 7 optimal values found by program and we list the corrosponding values of A, t_{arrive} and m_e.

	1	2	3	4	5	6	7
$\alpha(rad)$	0.21195	0.2198	0.14915	0.1099	0.1256	0.0658	0.0942
$\sqrt{A}(m)$	420	420	440	460	460	480	480
$A(m^2)$	176400	176400	193600	211600	211600	230400	230400
$t_{arrive}(day)$	765	770	548	525	564	483	840
$m_e(kg)$	765.2	765.2	644.8	518.8	518.8	387.2	387.2

Table 2: List of 7 optimal values

By this table, we can find the minimum transit time t_{arrive} is 483 days, the corrosponding effective load is 387.2 kg account for 19.36% of the total mass. If we want to get the maximum effective load m_e, we need about 770 days.

Next are specific trajectories of the 7 cases above.

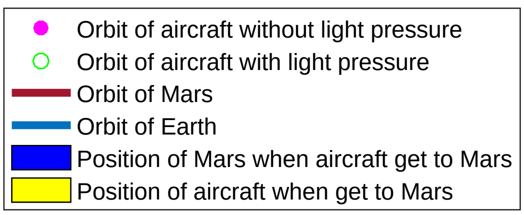

Figure 9: Legend of lines and points in figures below

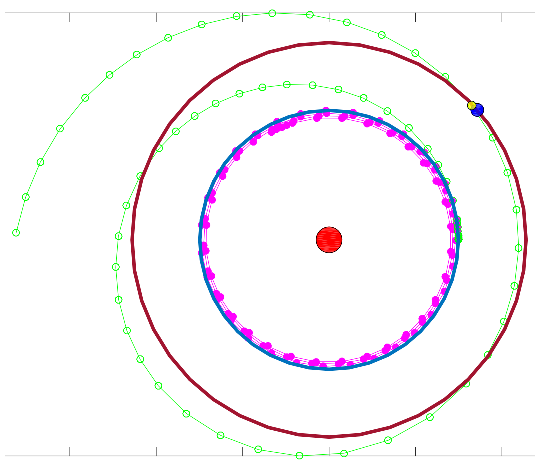

Figure 10: $alpha = 0.21195, A = 176400$

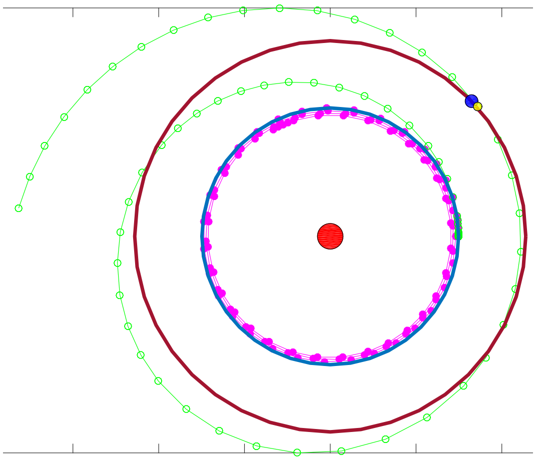

Figure 11: $alpha = 0.2198, A = 176400$

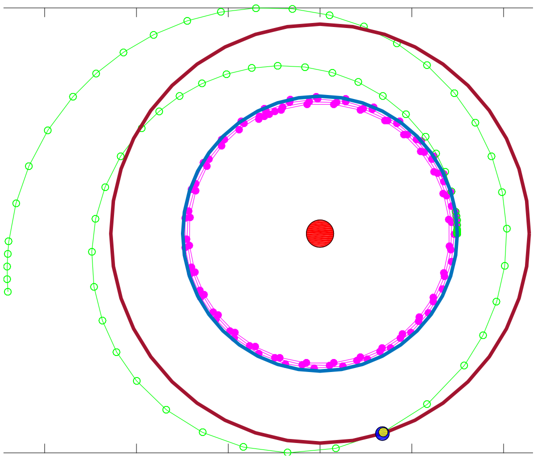

Figure 12: $alpha = 0.14915, A = 193600$

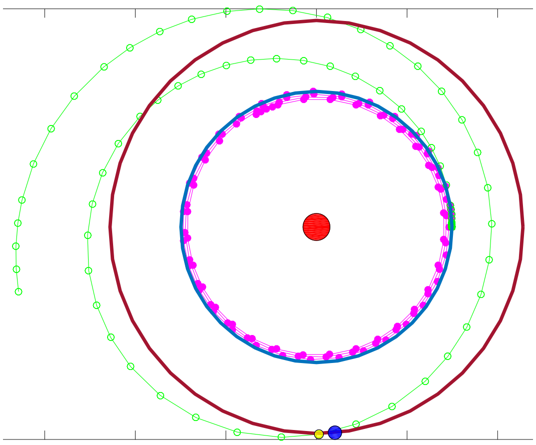

Figure 13: $alpha = 0.1099, A = 211600$

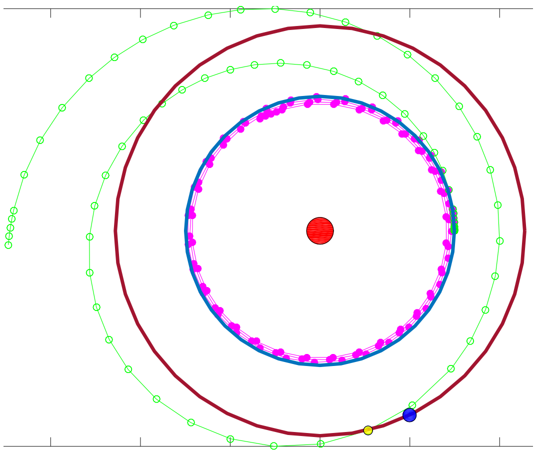

Figure 14: $alpha = 0.1256, A = 211600$

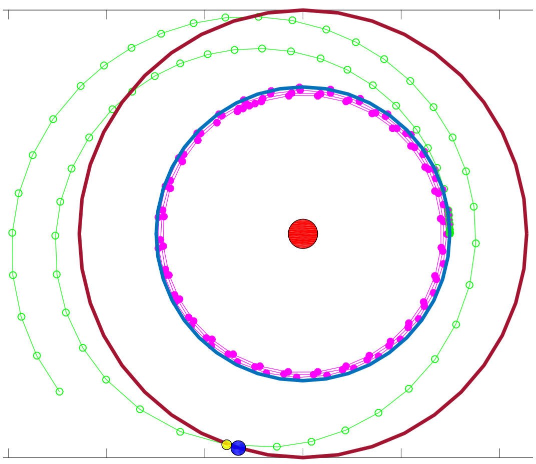

Figure 15: $alpha = 0.0658, A = 230400$

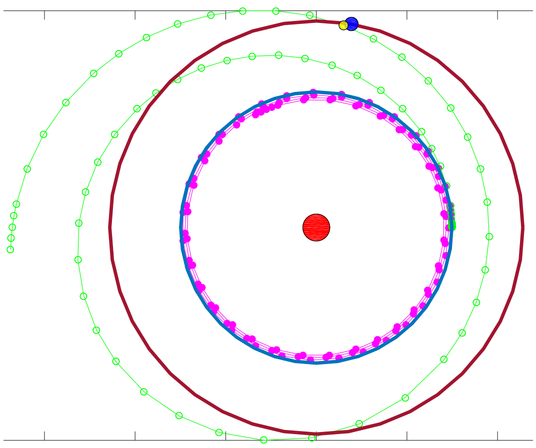

Figure 16: $alpha = 0.0942, A = 230400$

CJK

References

[1] 贺晶,龚胜平,李俊峰. 利用逃逸能量的太阳帆最快交会轨迹优化[J]. 深空探测学报,2014,1(01):60-66.

[2] Friedman L，Carroll W，Goldstein R，et al．Solar sailing—the concept made realistic[J]．AIAA Paper78—82，Jan．1978．

[3] 贾灵伟. 太阳帆航天器轨道转移优化及可视化仿真平台的设计与实现[D].中国科学技术大学,2010.

[4] 张辉. 太阳帆航天器轨道动力学与控制研究[D].中国科学技术大学,2014.

[5] 崔乃刚,刘家夫,荣思远. 太阳帆航天器动力学建模与求解[J]. 航空学报,2010,31(08):1565-1571.

[6]