# Finding $\pi$ in Unexpected Places

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This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

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# Chapter 1

# Introduction

$\pi$ is found in several areas of mathematics such as in geometry, physics, and even certain integrals. We know that $\pi$ is the ratio of a circle's circumference to its diameter, so we would expect it to appear in geometry, particularly with angles due to radians. However it is less obvious how $\pi$ appears in probability, complex analysis, and number theory. For example, why are the following equations true:

- $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$,
- $e^{i\pi} = -1$,
- $\prod_p^{\infty}(1 - \frac{1}{p^2}) = \frac{6}{\pi^2}$.

Most of the time you will end up finding a hidden equation for a circle or a trigonometric function [1] [2] [3], and from there it is obvious that $\pi$ will show up. So in a way it is the circle or trigonometric function that is unexpected.

In this report I will be exploring two particular examples of $\pi$ turning up in an unexpected way, and why it turns up. These two places are: firstly in collisions between two balls and a wall in a straight line, and secondly in the Mandelbrot set fractal. In later chapters we will be going deeper into aspects of the Mandelbrot set and also seeing if we can find any more occurrences of $\pi$ hiding in there. Specifically we will look at the Buddhabrot (a different way of seeing the Mandelbrot set), Julia and Fatou sets (which are both very similar to the Mandelbrot set).

# Chapter 2

# Playing pool with $\pi$

## 2.1 The problem

Back in 2003 Gregory Galperin found a new deterministic way to calculate $\pi$ [4]. He found $\pi$ by counting the number of collisions between two balls, one with a mass $100^N$ ($N \in \mathbb{N}_0$) that of the other, like in a game of pool or billiards.

In figure 2.1 we have two balls and a wall. Say that the ball on the right has a mass equal to that of the other ball (hence $M = m$), and the wall is assumed to have infinite mass. We push the right ball into the one on the left, with some velocity $-\alpha$ ($\alpha \in \mathbb{R}_{>0}$) to the right, and count the number of collisions (assuming that the collisions are purely elastic). First the right ball hits the other into the wall, and then this ball bounces back into the right ball again, causing the right ball to roll off to infinity and the left ball is sat stationary. So in total there are three collisions, this can be shown mathematically using the conservation of momentum and energy (just kinetic energy in this case):

$$momentum = mass \times velocity, \tag{2.1}$$
$$kinetic\ energy = \frac{1}{2} \times mass \times velocity^2. \tag{2.2}$$

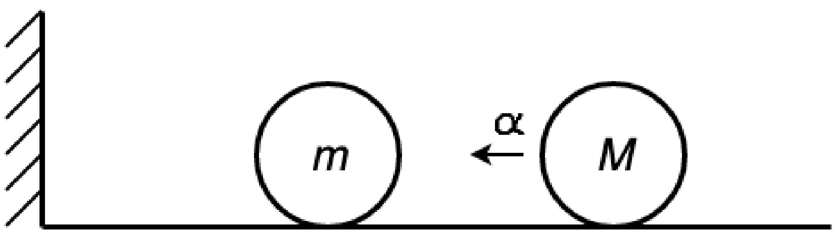


Figure 2.1: The set-up of the problem in question.

The collision between the left ball and the wall is simple, since the wall has infinite mass the ball if simply reflected off of it and travels away with the same speed. This means that the total momentum between the balls will change when the left ball hits the wall, but energy is still conserved due to squaring of the velocities (the minus sign has no effect). For the balls colliding we look at the total momentum and energy before and after each collision. For the first collision, if we say the velocities after the collision for the left and right ball are $u_1$ and $v_1$ respectively, the equations of momentum and energy show:

$$\begin{aligned} -M\alpha &= Mv_1 + mu_1, \\ \frac{1}{2}M\alpha^2 &= \frac{1}{2}Mv_1^2 + \frac{1}{2}mu_1^2. \end{aligned} \tag{2.3}$$

Then we simply solve this to find $u_1$ and $v_1$ in terms of $\alpha$ and repeat this technique for the other collisions. Once there can be no more collisions we will have one of three cases:

- both balls are moving to the right and the right ball faster than the left,
- both balls are moving to the right at the same speed,
- the left ball is stationary and the right is traveling to the right.

Note that both balls can not be stationary, although this would not cause any more collisions, it is is impossible due to the conservation of energy.

These cases can be translated into a simple inequality:

$$0 \leq u \leq v. \tag{2.4}$$

Note that the number of collisions does not depend on $\alpha$ or $m$, we just need the right ball to travel towards the left ball.

Now let us increase the mass of the right ball to 100 times the mass of the left ball, and again count the number of collisions when we push the right ball to the left. This time there turn out to be 31 collisions. Now increase the right ball's mass by 100 again, so that now it is $10,000$ times the mass of the left ball. Pushing the right ball now results in 314 collisions. And if $M = 100^3 m$ then there are 3141 collisions. A pattern has emerged (see the table below for more values of $M$): the number of collisions are the first $N+1$ digits of $\pi$, where $N$ is such that $M = 100^N m$.

In other words

$$number\ of\ collisions \times 10^{-N} \approx \pi. \tag{2.5}$$

| $N$ | $M$ | *No. of Collisions* |
|---|---|---|
| 0 | $m$ | 3 |
| 1 | $10^2m$ | 31 |
| 2 | $10^4m$ | 314 |
| 3 | $10^6m$ | 3141 |
| 4 | $10^8m$ | 31415 |
| 5 | $10^{10}m$ | 314159 |
| 6 | $10^{12}m$ | 3141592 |
| 7 | $10^{14}m$ | 31415926 |
| 8 | $10^{16}m$ | 314159265 |
| 9 | $10^{18}m$ | 3141592653 |
| 10 | $10^{20}m$ | 31415926535. |

## 2.2 Solving the problem

The reason why (2.5) holds true is due to features of the conversations of energy and momentum, and can be proven using a phase diagram. We use $\sqrt{m}u$ for the $y$-axis and $\sqrt{M}v$ for the $x$-axis. The reason why we do not simply use the velocities for the axes is because this way the conservation of energy is represented by a circle (rather than just an ellipse), and it is useful to find circles where we can since they are fundamentally related to $\pi$. Using the equations (2.3) we can find the equation for the conservation of momentum and energy in this new coordinate system, as follows:

$$\begin{aligned} -M\alpha &= Mv + mu, \\ &= \sqrt{M}x + \sqrt{m}y, \\ \Leftrightarrow y &= -\frac{\sqrt{M}}{\sqrt{m}}x - \frac{M}{\sqrt{m}}\alpha. \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{1}{2}M\alpha^2 &= \frac{1}{2}Mv^2 + \frac{1}{2}Mu^2, \\ &= \frac{1}{2}x^2 + \frac{1}{2}y^2, \\ \Leftrightarrow x^2 + y^2 &= (\sqrt{M}\alpha)^2. \end{aligned} \tag{2.7}$$

But since the wall collision effects the momentum between the balls, the $y$-intercept of equation (2.6) can vary, so instead lets simple call it $c$. Hence we have:

$$y = -\frac{\sqrt{M}}{\sqrt{m}}x + c. \tag{2.8}$$

We see this in action in figure 2.2. The different velocities of the balls are plotted throughout the interaction (as red dots), as is the order in which the system moves between those velocity pairs (the pink lines). Notice how all points are on the circle, which

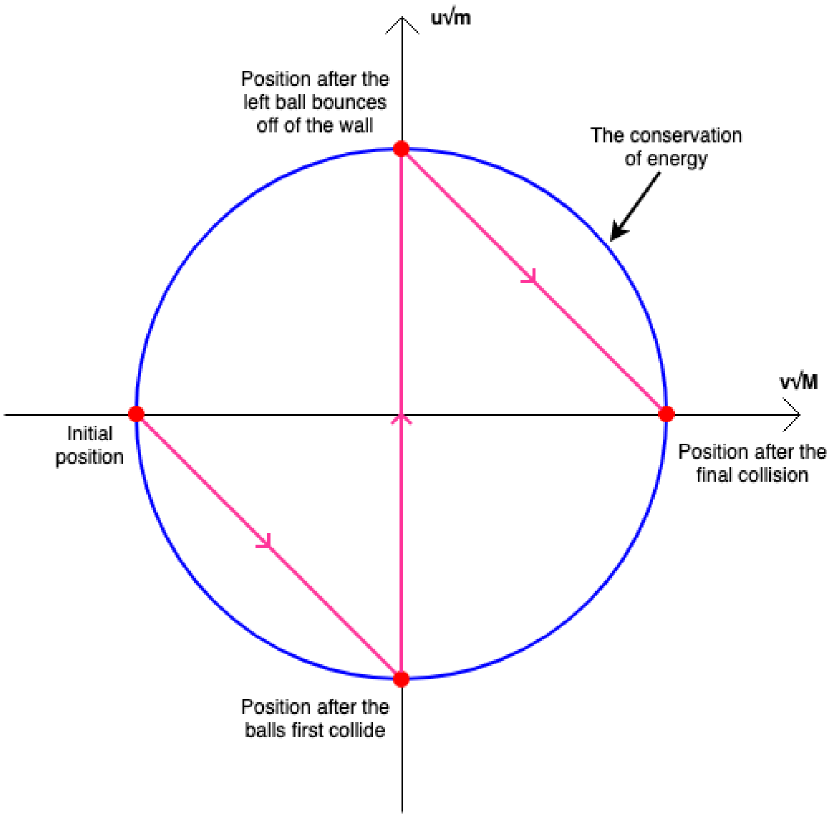


Figure 2.2: A phase diagram for the case where M=m.

represents the conservation of energy. The conservation of momentum is represented by the **diagonal** pink lines, the gradient of those lines are both $-1$, which is equal to

$$\frac{-\sqrt{M}}{\sqrt{m}} = \frac{-\sqrt{100^N m}}{\sqrt{m}} = -10^N.$$

The vertical pink lines represent the left ball bouncing off of the wall, which just changes the sign of the left ball's velocity. This means that we can find the velocity coordinates simply by putting the slope with gradient $-10^N$ (the conservation of momentum) through the initial point and see where it intersect the circle with radius $\sqrt{M}\alpha$ (the conservation of energy) to find the next point. Then the coordinate after that is directly above from the previous one on the circle, since the wall "reflects" the left ball. We then repeat these steps until the end condition (2.4) is satisfied. In this coordinate system this means that:

$$\begin{aligned} & 0 \leq \frac{y}{\sqrt{m}} \leq \frac{x}{\sqrt{M}}, \\ \Leftrightarrow \qquad & 0 \leq y \leq \frac{\sqrt{m}}{\sqrt{M}} x. \end{aligned} \tag{2.9}$$

This condition is shown by the green line in figure 2.3; so once a velocity pair is between the line and the $x$-axis we know there will be no more collisions. We can apply this method for other values of $N$ and plot their phase diagrams counting how many collisions occur (the number of pink lines). This can easily be programmed (see appendix $A$.1 for the complete programme) to produce diagrams for any $N \in \mathbb{R}$, see figure 2.3.

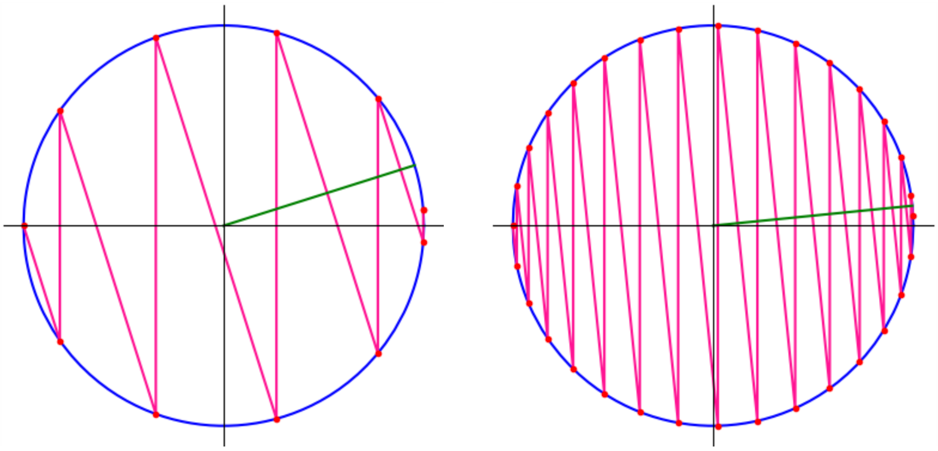

Figure 2.3: Phase diagrams for N=0.5 (left) and N=1 (right).

Note that the arc lengths between each velocity pair are equal (excluding the last arc). This is due to the well-known inscribed angle theorem, as shown in figure 2.4. This also means that the arc length is $2\theta$ where $\theta$ is the acute angle between the conservation of momentum line (2.8) and the vertical. Equation (2.8) says that the lines all have the same gradient and therefore all the same angle $\theta$, and hence corresponding arc length $2\theta\alpha$ $\square$.

The number of arcs separated by the velocity pairs is equal to the number of collisions plus one, so we just need to count the number of arcs. Looking at the diagrams this is asking how many whole $2\theta\alpha$ arc lengths it takes to add up to the whole circumference $2\pi\alpha$, or just above. The answer to this is obviously $\lceil \frac{\pi}{\theta} \rceil$, and so the number of collisions is equal to $\lceil \frac{\pi}{\theta} \rceil - 1$. We can simply then calculate $\theta$ using trigonometry:

$$\begin{aligned} & \tan\theta = \frac{opposite}{adjacent} = |\frac{1}{gradient}| = |\frac{1}{-10^N}| = 10^{-N}, \\ \Rightarrow \quad & \theta = \arctan 10^{-N}, \\ \Rightarrow \quad & number\ of\ collisions = \lceil \frac{\pi}{\arctan 10^{-N}} \rceil - 1. \end{aligned} \tag{2.10}$$

$\tan\theta$ also has the property that as $\theta \to 0$, $\tan\theta \to \theta$, so we can put this into equation

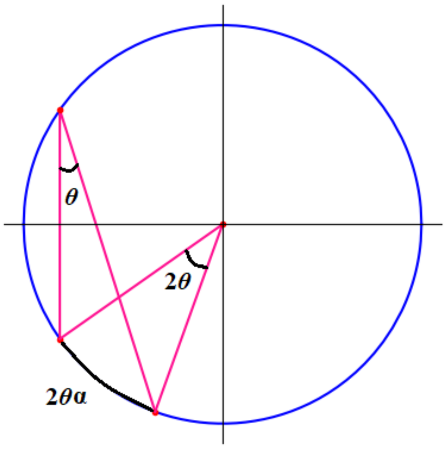


Figure 2.4: The inscribed angle theorem, shown on part of the diagram for N=0.5.

(2.10) and we get:

$$number\ of\ collisions = \lceil \frac{\pi}{10^{-N}} \rceil - 1 = \lceil 10^N \pi \rceil - 1, \tag{2.11}$$

which is equal to the the first $N + 1$ digits of $\pi$ $\square$.

Note that since $10^{-N}\pi$ is never an integer (due to $\pi$ being irrational) $\lceil 10^N \pi \rceil - 1 = \lfloor 10^N \pi \rfloor$ and this is more obviously the first $N + 1$ digits of $\pi$.

# Chapter 3

# $\pi$ in the Mandelbrot Set

## 3.1 Fractals

The Mandelbrot set is one of the most famous, if not the most famous, fractal in mathematics. A fractal is a shape which shows self-similarity. A fern, for example, has a fractal nature since each frond resembles the whole plant, this way the same bit of genetic code can be reused. More precisely a fractal is a "subset of Euclidean space whose "fractal dimension" exceeds its topological dimension" [5], although the exact definition is still debated. In Euclidean space the topological dimension is simply the dimension of the space. The fractal, or Hausdorff, dimension essentially generalises vector space and most fractals will have non-integer fractal dimension. The Sierpinski triangle has fractal dimension equal to $\frac{log(3)}{log(2)} \approx 1.58$ [6], and the coastline of Britain is estimated to have a fractal dimension of around 1.24 [5].

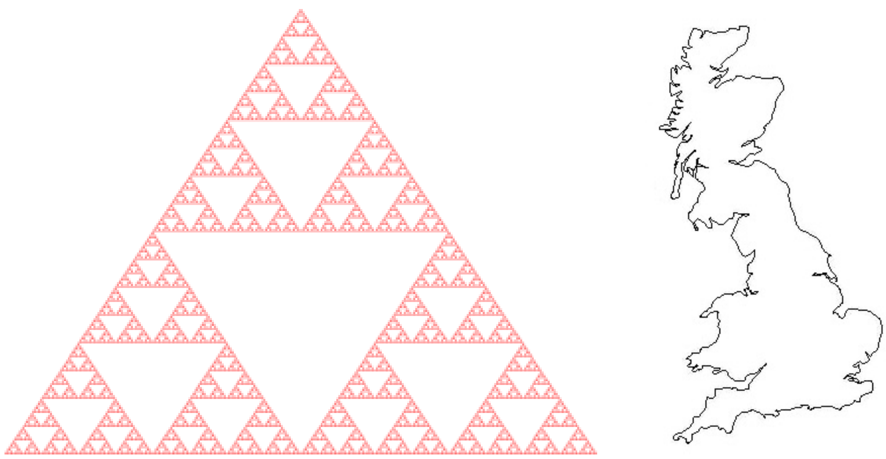

Figure 3.1: The Sierpinski triangle [7] (left), and the coast of Britain (right).

## 3.2 Introduction to the Mandelbrot set

The Mandelbrot set is in the complex plane and consists of all points $c \in \mathbb{C}$ such that the sequence

$$z_{n+1} = z_n^2 + c \tag{3.1}$$

does not diverge for $z_0 = 0$. In other words the $c$ values for which the corresponding orbit of 0 under $z^2 + c$ is bounded.

For example if we take $c = -\frac{1}{2}$ then the sequence is $0, -\frac{1}{2}, -\frac{1}{4}, -\frac{7}{16}, -\frac{79}{256}, \ldots$, which is bounded and therefore in the Mandelbrot set. Whereas if we take $c = 1$ the sequence is $0, 1, 2, 5, 26, \ldots$, this grows to infinity and so is not in the set.

Doing this for the whole complex plane gives figure 3.2.

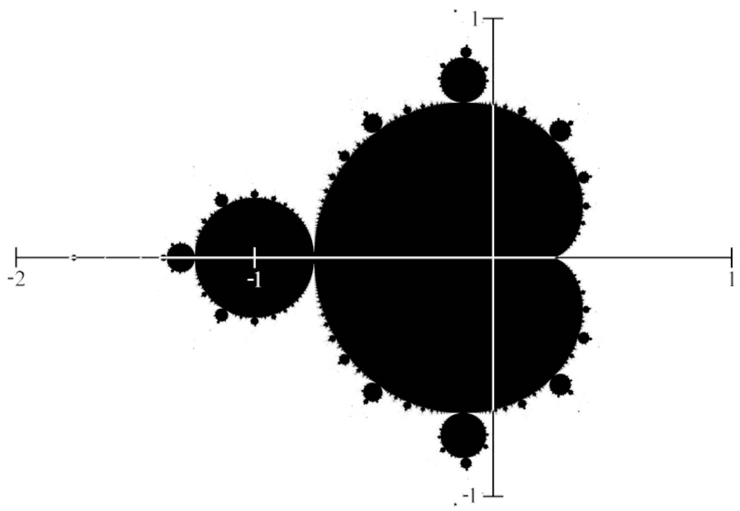


Figure 3.2: The Mandelbrot set [9].

## 3.3 The Mandelbrot set's boundary circle

In fact if the sequence grows beyond 2 in modulus then it is certain that the sequence is unbounded, and thus that $c$ is not in the Mandelbrot set. We can prove this by showing that if $|z_n| > 2$ and $|z_n| \geq |c|$ then $|z_{n+1}| > |z_n| > |c|$, and therefore the sequence keeps growing and is hence unbounded.

Let $|w_0| > 2$ and $|w_0| \geq |c|$. Then define

$$|w_1| \quad := \quad |w_0^2 + c|$$

$$\begin{aligned}
&\geq \quad |w_0|^2 - |c| \text{ (by the triangle inequality)} \\
&\geq \quad |w_0|^2 - |w_0| \\
&= \quad |w_0|(|w_0| - 1). \qquad (3.2)
\end{aligned}$$

Since $|w_0| > 2$ $|w_0| - 1 > 1$, and so $|w_1| > |w_0| > 2$ by (3.2). Repeating this tells us that

$$|w_2| \geq |w_1|(|w_1| - 1) \geq |w_0|(|w_0| - 1)(|w_1| - 1) > |w_0|(|w_0| - 1)(|w_0| - 1).$$

Then repeatedly applying this method we see that $|z_n| > |w_0|(|w_0| - 1)^n$ and as $n \to \infty$ we show that the sequence goes off to infinity, and hence is unbounded $\square$.

## 3.4 Programming the Mandelbrot set

Instead of simply seeing whether a value of $c$ is in the Mandelbrot set or not, we can count how many iterations it takes until the sequence gets beyond the circle of radius 2. To program this we simply define a function which iterates equation (3.1) until $|z_n| > 2$ or a maximum number of iterations has been reached, call this integer 'imax'. The function will then return the number of iterations it has done, if this is equal to 'imax' then we say that this point is in the Mandelbrot set, otherwise the result will show how quickly the sequence diverged for that point. To plot this we create a large array of evenly spaced coordinates in a $3 \times 3$ grid centred on (-0.5,0), this way it will include the whole Mandelbrot set. Then for each coordinate in the array we put it through the previously defined function, as a complex number, and plot the returned iteration number using a heat map. We can also change the function to return the square-root of the iterations to make the image more visually appealing (note that this is equivalent to simply changing the colour map), resulting in figure 3.3. For the actual Python code I wrote see appendix $A$.2.

## 3.5 An unexpected $\pi$

Surprisingly $\pi$ can be found in the Mandelbrot set fractal, as David Boll did when he was trying to verify if the connection between the bulbs at -0.75 is a single point [8]. If we travel along the real axis (from right to left) towards $\frac{1}{4}$ using $\frac{1}{4} + \varepsilon$, taking $\varepsilon = 1$ initially, then dividing by 100 each time, we find that the number of iterations before the sequence grows beyond 2 are:
As in the previous chapter we have a relation to the digits of $\pi$, except this time the last digit is not always correct. But we do have that $\sqrt{\varepsilon}N \approx \pi$. Note that $\frac{1}{4}$ is in the Mandelbrot set, but $\frac{1}{4} + \varepsilon$ is not for any $\varepsilon > 0$.

$$\begin{aligned}
&0^2 + \frac{1}{4} = \frac{1}{4} = 0.25, \\
&\frac{1}{4}^2 + \frac{1}{4} = \frac{1}{16} + \frac{1}{4} = \frac{5}{16} = 0.3125,
\end{aligned}$$

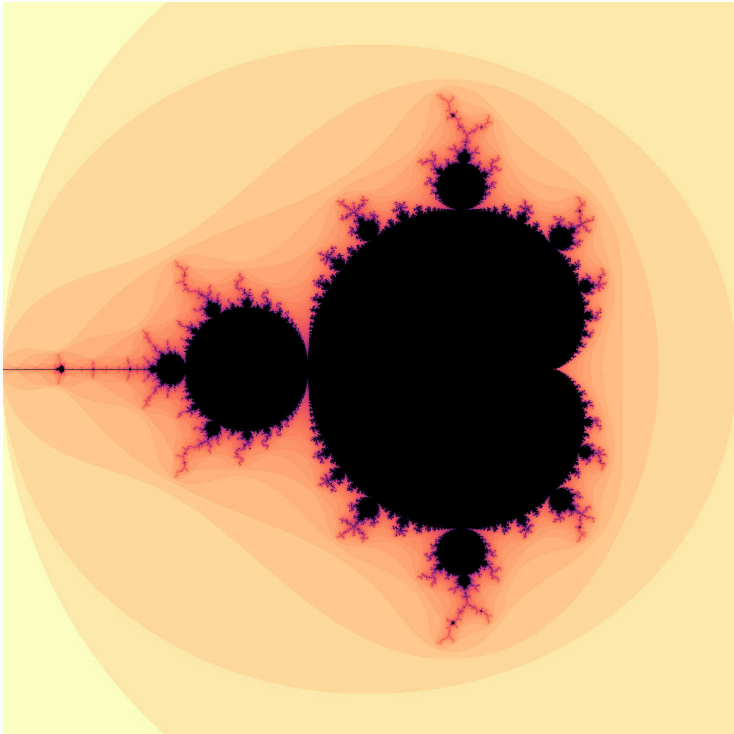

Figure 3.3: The Mandelbrot set.

| $\varepsilon$ | *No. of Iterations* |
|---|---|
| 1 | 2 |
| $10^{-2}$ | 30 |
| $10^{-4}$ | 312 |
| $10^{-6}$ | 3140 |
| $10^{-8}$ | 31414 |
| $10^{-10}$ | 314157. |

$$\left(\frac{5}{16}\right)^2 + \frac{1}{4} = \frac{25}{256} + \frac{1}{4} = \frac{89}{256} \approx 0.34766,$$
$$\left(\frac{89}{256}\right)^2 + \frac{1}{4} = \frac{7921}{65536} + \frac{1}{4} = \frac{24305}{65536} \approx 0.37086,$$

and the differences between each term are decreasing:

$$\frac{5}{16} - \frac{1}{4} = \frac{1}{16} = 0.0625,$$
$$\frac{89}{256} - \frac{5}{16} = \frac{9}{256} = 0.0351,$$

$$\frac{24305}{65536} - \frac{89}{256} = \frac{1521}{65536} = 0.0232.$$

## 3.6 Proving the $\pi$ result near $\frac{1}{4}$

It is not immediately clear how we should go about proving this $\pi$ result, but it is a good idea to try to find a continuous function to approximate the sequence [11]. Firstly we need to certify that it is fairly reasonable to have such an approximation (the following is not entirely rigorous but a completely thorough proof is beyond the scope of this report). For $c = \frac{1}{4} + \varepsilon$ in the Mandelbrot sequence most of the terms are around $\frac{1}{2}$; this can be shown using a web diagram, figure 3.4.

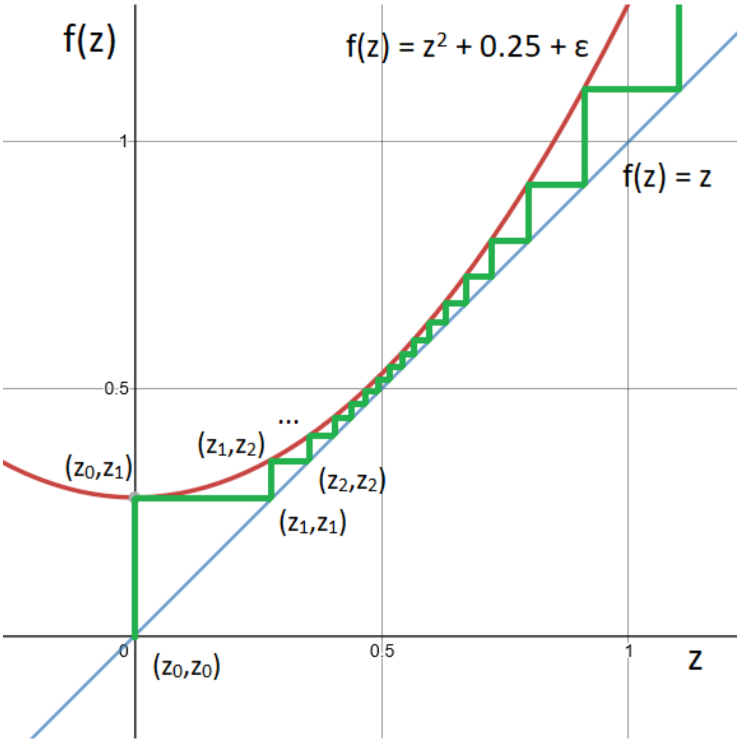


Figure 3.4: A web diagram for the sequence.

The web diagram shows how the sequence changes and importantly that the points in the sequence follow a “continuous” pattern (by that we mean that the terms increase smoothly and slowly) particularly around $z_n = \frac{1}{2}$ and as $\varepsilon \to 0$. Now we must shift the sequence by $-\frac{1}{2}$, making most of the terms occur around 0; call this new sequence $x_n$.

$$x_n := z_n - \frac{1}{2}. \tag{3.3}$$

Putting (3.3) into the Mandelbrot sequence equation (3.1) with $c = \frac{1}{4} + \varepsilon$ we see:

$$\begin{aligned}
x_{n+1} + \frac{1}{2} &= (x_n + \frac{1}{2})^2 + \frac{1}{4} + \varepsilon, \\
\Leftrightarrow x_{n+1} &= x_n^2 + x_n + \frac{1}{4} + \frac{1}{4} - \frac{1}{2} + \varepsilon, \\
x_{n+1} &= x_n^2 + x_n + \varepsilon. \qquad (3.4)
\end{aligned}$$

With this shifted sequence, which can be assumed to be a function of $n$ too, we can say that:

$$\frac{dx}{dn} = x_{n+1} - x_n, \qquad (3.5)$$

which is the difference between two terms in the sequence, and by (3.4):

$$\frac{dx}{dn} = x_n^2 + \varepsilon.$$

The above is a separable ordinary differential equation, and we can thus solve it as follows:

$$\begin{aligned}
\frac{1}{x^2 + \varepsilon} \frac{dx}{dn} &= 1, \\
\int \frac{dx}{x^2 + \varepsilon} &= \int dn, \\
\int \frac{dx}{x^2 + \varepsilon} &= n + \alpha,
\end{aligned}$$

for $\alpha$ a constant from the integration. Note that since $\varepsilon \neq 0$ we will need some kind of substitution to calculate the integral. Let $x = \sqrt{\varepsilon} \tan(\theta(n))$, therefore $\frac{dx}{d\theta} = \sqrt{\varepsilon} \sec^2 \theta$, then substitute it into the integral above giving:

$$\begin{aligned}
\int \frac{\sqrt{\varepsilon} \sec^2 \theta}{\varepsilon \tan^2 \theta + \varepsilon} d\theta &= n + \alpha, \\
\int \frac{\sqrt{\varepsilon}}{\varepsilon \sin^2 \theta + \varepsilon \cos^2 \theta} d\theta &= n + \alpha, \\
\int \frac{d\theta}{\sqrt{\varepsilon}} &= n + \alpha, \\
\frac{\theta}{\sqrt{\varepsilon}} &= n + \alpha, \\
\theta &= \sqrt{\varepsilon}(n + \alpha).
\end{aligned}$$

Since $\sqrt{\varepsilon}\alpha$ is constant we can merge that $\sqrt{\varepsilon}$ into $\alpha$. By doing that and putting $\theta$ back in terms of $x$ we get:

$$\begin{aligned}
\arctan \frac{x}{\sqrt{\varepsilon}} &= \sqrt{\varepsilon} n + \alpha, \\
\frac{x}{\sqrt{\varepsilon}} &= \tan(\sqrt{\varepsilon} n + \alpha), \\
x &= \sqrt{\varepsilon} \tan(\sqrt{\varepsilon} n + \alpha). \qquad (3.6)
\end{aligned}$$

Here we have a trigonometric function, tan, and it is now much more obvious how $\pi$ will occur.

We can compare this new equation (3.6) to the sequence and see how little they differ for small $\varepsilon$, as in figure 3.5.

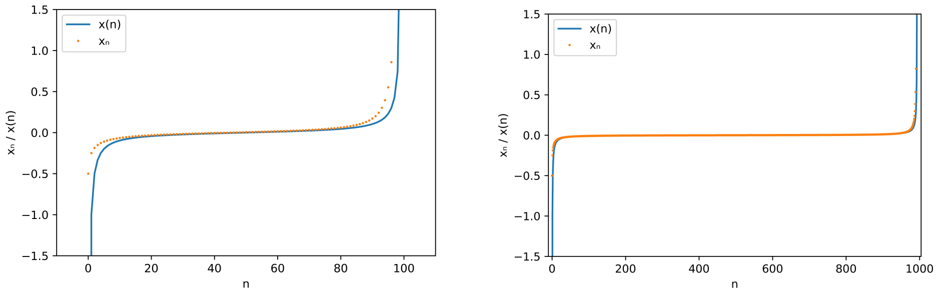


Figure 3.5: A plot of equation (3.6) (blue) and the sequence (3.4) (orange) for $\varepsilon = 0.001$ and 0.00001 respectively.

We substitute the first and last term of the sequence into (3.6) and solve the resulting simultaneous equation. Since $z_0 = 0$ therefore $x_0 = -\frac{1}{2}$. The last term of the original Mandelbrot sequence, say with index $N$, is defined to be the term where $z_N^2 + c > 2$. Hence $(x_N + \frac{1}{2})^2 + \frac{1}{4} + \varepsilon > 2 \Leftrightarrow x_N^2 + x_N + \varepsilon > \frac{3}{2}$ (since the sequences are of the same length), and $0 < x_N \leq \frac{3}{2}$.

Putting $x_0 = -\frac{1}{2}$ into (3.6) we have:

$$\begin{aligned} -\frac{1}{2} &= \sqrt{\varepsilon}\tan(\alpha), \\ -\frac{1}{2\sqrt{\varepsilon}} &= \tan(\alpha), \end{aligned} \tag{3.7}$$

and as $\varepsilon \to 0$ the left-hand side of (3.7) will go to $-\infty$ and thus the right-hand side, $\tan(\alpha)$, will also go to $-\infty$. This means that $\alpha$ is one of the poles of the tangent function, so let us say that $\alpha = -\frac{\pi}{2}$ (this is an arbitrary choice and does not effect the result). We now have that

$$x = \sqrt{\varepsilon}\tan(\sqrt{\varepsilon}n - \frac{\pi}{2}). \tag{3.8}$$

Similarly we put $x_N$ into (3.8) and see that:

$$\begin{aligned} x_N &= \sqrt{\varepsilon}\tan(\sqrt{\varepsilon}N - \frac{\pi}{2}), \\ \frac{x_N}{\sqrt{\varepsilon}} &= \tan(\sqrt{\varepsilon}N - \frac{\pi}{2}), \end{aligned} \tag{3.9}$$

and as $\varepsilon \to 0$ the left-hand side of (3.9) will go to $+\infty$ and thus the right-hand side, $\tan\left(\sqrt{\varepsilon}N - \frac{\pi}{2}\right)$ will also go to $+\infty$. Since at $x_0$ we had (3.7) going to $-\infty$ from the right (obviously it can not come in from the left anyway) we want $\sqrt{\varepsilon}N - \frac{\pi}{2}$ to be the next pole of the tangent function along from $\alpha$ $(= -\frac{\pi}{2})$. This way there are no discontinuities in the tangent function for the values of $\sqrt{\varepsilon}n - \frac{\pi}{2}$. In other words:

$$\begin{aligned} \sqrt{\varepsilon}N - \frac{\pi}{2} &\to \frac{\pi}{2}, \\ \sqrt{\varepsilon}N &\to \pi, \end{aligned} \tag{3.10}$$

and we have our result since $N$ is the last term of the sequence, and thus the number of terms in the sequence, which in turn is the number of iterations of (3.1) before it grows beyond 2 $\square$.

## 3.7 Other $\pi$ results

Taking a line down into the point-bridge at $c = -0.75$ gives a $\pi$ result, but slightly different to before. For the path $-0.75 + \varepsilon i$ we start with $\varepsilon = 1$ and divide it by 10 each time (rather than by 100 like in section 3.5) and see that the number of iterations before growing past 2 is as follows:

| $\varepsilon$ | *No. of Iterations* |
|---|---|
| 1 | 3 |
| $10^{-1}$ | 33 |
| $10^{-2}$ | 315 |
| $10^{-3}$ | 3143 |
| $10^{-4}$ | 31417 |
| $10^{-5}$ | 314160. |

We can also take a path into the point-bridge at -1.25, but this time we can not simply take a straight line into the point since that would bump into the Mandelbrot set, so instead we take a parabola $-1.25 - \varepsilon^2 + \varepsilon i$:

| $\varepsilon$ | *No. of Iterations* | $2N$ |
|---|---|---|
| 1 | 1 | 2 |
| $10^{-1}$ | 18 | 36 |
| $10^{-2}$ | 159 | 318 |
| $10^{-3}$ | 1586 | 3172 |
| $10^{-4}$ | 15731 | 31462 |
| $10^{-5}$ | 157085 | 314170. |

Here we have even worse accuracy than before, the last **two** digits are often wrong. Instead we could use a parabola not so close to the Mandelbrot set, such as $-1.25 - 2\varepsilon^2 + \varepsilon i$, and we get a better result: 2, 30, 314, 3144, 31416, 314160, 3141592,....

In fact any path into a point-bridge (or cusp like at $\frac{1}{4}$) will be related to $\pi$.

# Chapter 4

# Julia and Fatou Sets

## 4.1 Introduction to Julia and Fatou sets

Julia sets are very similar to the Mandelbrot set, they also use the sequence:

$$z_{n+1} = z_n^2 + c, \tag{4.1}$$

except $c \in \mathbb{C}$ is fixed and we see if the sequence is bounded for different values of $z_0$, the values of $z_0$ for which that sequence is bounded are in that particular Julia set. In other words the set consists of the seeds of orbits which are bounded under $z^2 + c$ for a particular fixed $c$ value. This means that there are multiple Julia sets (not just one like the Mandelbrot set) but they are only connected if $c$ is in the Mandelbrot set [14]. In fact if $c$ is not in the Mandelbrot set the set is completely disconnected and made up of infinitely many separate single points, we call these Fatou sets.

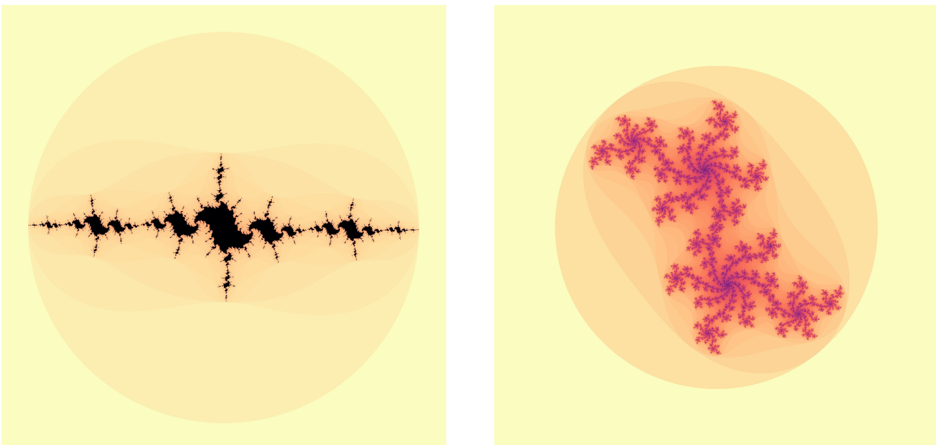

Figure 4.1: The Julia set with $c = -1.3 + 0.05i$ (left), and the Fatou set with $c = 0.156 + 0.618i$ (right).

In fact Julia sets resemble the Mandelbrot set in a small neighbourhood of that $c$. This means that Julia sets can be arranged in a grid dependent on their $c$ values and it will resemble the Mandelbrot set, see figure 4.2. We can think of the Mandelbrot set as a map of Julia sets, and the rest of the complex plane as a map of Fatou sets (this will come in useful later in section 7.1).

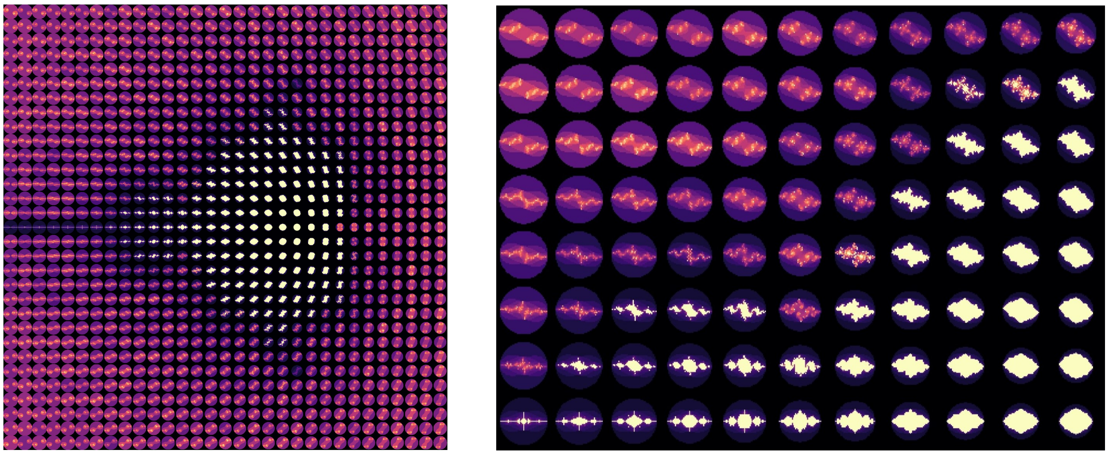

Figure 4.2: 961 Julia sets arranged by their $c$ values with a close-up on part of the upper-left quadrant.

Additionally all Julia sets have rotational symmetry of order 2, and sets where the imaginary part of $c$ is 0 (such as the bottom row in the above figure on the right) have horizontal and vertical symmetry too. Julia sets with $Im(c) > 0$ are mainly in the upper-left and bottom-right quadrants, and Julia sets with $Im(c) < 0$ the opposite is true.

## 4.2 Programming Julia and Fatou sets

Recall how for the Mandelbrot set that if, for some $n \in \mathbb{N}$, $|z_n| > 2$ then the sequence is unbounded and thus that $c$ is not in the Mandelbrot set. Julia sets have a similar property [15] but instead of 2 the "escape radius" is:

$$R \in \mathbb{R}_{>0} \text{ such that } R^2 - R \geq |c|.$$

Using the quadratic formula we see that:

$$R = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-|c|)}}{2(1)},$$

and since $|c| \geq 0$ (by definition of modulus) $R \not> 0$ whenever we take the negative root above. Therefore the escape radius $R$ is:

$$R = \frac{1 + \sqrt{1 + 4|c|}}{2}. \tag{4.2}$$

We code this very similarly to how we programmed the Mandelbrot set in section 3.4, see appendix A.3 for the programme in its entirety.

# Chapter 5

# Periodicity of Orbits in the Mandelbrot Set

## 5.1 Examples of the periods of orbits

For orbits of 0 under $z^2 + c$ with $c$ in the Mandelbrot set there is a periodic behaviour of the sequence, or it approaches one, the vast majority of the time. For example with $c = -1$ we have:

$$z_0 = 0, \; z_1 = 0^2 - 1 = -1, \; z_2 = (-1)^2 - 1 = 0, \; z_3 = 0^2 - 1 = -1, \ldots,$$

which has period 2.
Similarly for $c = -0.122561 + 0.744862i$ (6 d.p.):

$$\begin{aligned}
&z_0 = 0, \; z_1 = 0^2 - 0.122561 + 0.744862i = -0.122561 + 0.744862i, \\
&z_2 = (-0.122561 + 0.744862i)^2 - 0.122561 + 0.744862i = -0.662359 + 0.562280i, \\
&z_3 = (-0.662359 + 0.562280i)^2 - 0.122561 + 0.744862i = 0, \ldots,
\end{aligned}$$

thus this orbit has period 3.
For $c = -1.1$ we get:

$$\begin{aligned}
&z_0 = 0, \; z_1 = 0^2 - 1.1 = -1.1, \; z_2 = (-1.1)^2 - 1.1 = 0.11, \\
&z_3 = 0.11^2 - 1.1 = -1.0879, \; z_4 = (-1.0879)^2 - 1.1 = 0.08352641, \ldots,
\end{aligned}$$

here we can see that this sequence approaches a 2-cycle.
Finally for $c = -1.9$ (which is still in the Mandelbrot set) we see chaotic behaviour so no periodicity, see figure 5.1.

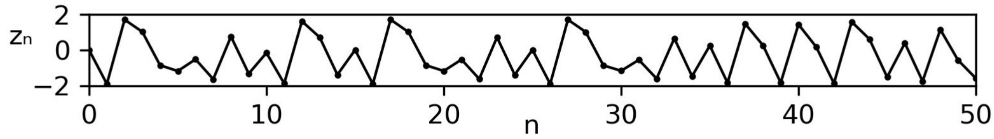


Figure 5.1: The orbit of 0 under $z^2 + c$ for $c = -1.9$.

## 5.2 Structure of the Mandelbrot set

### 5.2.1 The location of primary bulbs

The Mandelbrot set is made up of the main cardioid, circular bulbs coming off of the cardioid and each other, mini Mandelbrots, and hairs with no width which connect up the whole set [16] [17].

It turns out that the pattern in the orbits depends on which shape it is in. In the previous examples the orbit of 0 with:

- $c = -1$ is in the centre of the main bulb (which is the large disc to the left of the main cardioid),
- $c = -0.122561 + 0.744862i$ is in the large bulb above the main cardioid,
- $c = -1.1$ is in the main bulb with $-1$,
- $c = -1.9$ is on an infinitely thin hair of the Mandelbrot set on the negative real axis.

For $c$ in the main cardioid the orbit of 0 tends towards a single point, so we say the main cardioid has period 1. The main bulb of the Mandelbrot set has period 2, and the large bulbs above and below the main cardioid both have period 3. In fact there is a primary bulb (a bulb attached to the main cardioid) for every $\frac{p}{q}$ between 0 and 1, where $p$ and $q$ are coprime natural numbers, and these primary bulbs are attached to the main cardioid at:

$$c_{\frac{p}{q}} = \frac{1}{2}e^{2\pi i\frac{p}{q}}(1 - \frac{1}{2}e^{2\pi i\frac{p}{q}}), \tag{5.1}$$

and $q$ is the period of the bulb. Note that $e^{2\pi i\frac{p}{q}}$ is on the unit circle and of an angle $\frac{p}{q}\pi$ around the unit circle. Another property is that the total number of primary bulbs with periodicity $q$ is equal to Euler totient function of $q$, $\varphi(q)$. The Euler totient function for a natural number $n \in \mathbb{N}$ is the number of natural numbers up to $n$ that are coprime to $n$:

$$\varphi(n) := |\{m \in \{1, 2, ..., n\} \text{ s.t. } gcd(m, n) = 1\}|.$$

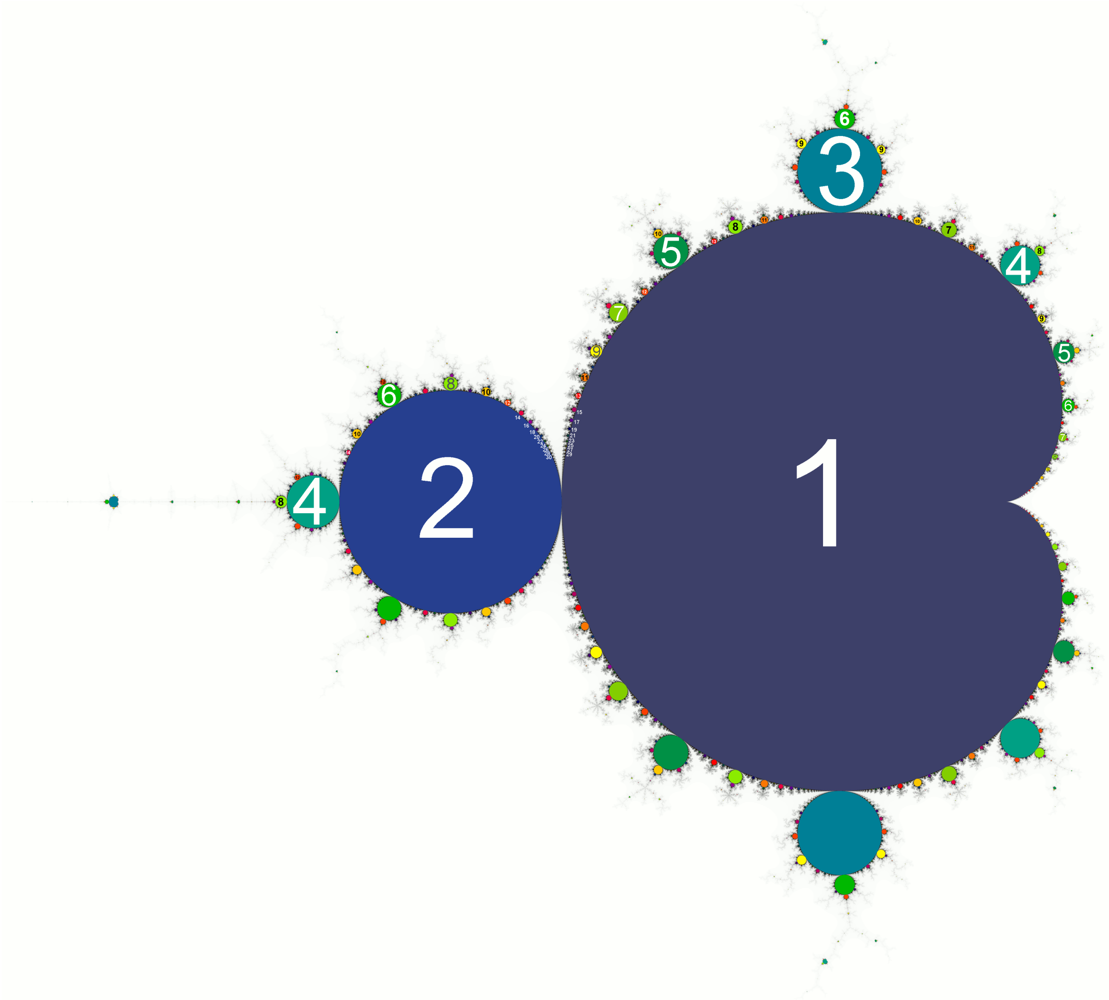


Figure 5.2: The periods of regions in the Mandelbrot set [18].

This is because we are counting the rational numbers between 0 and 1; we can do so by starting with the lowest denominator 2 and working upwards skipping numerators which would simplify the fraction to one we have counted before; this is the same as only picking numerators coprime to the denominator hence we have the aforementioned property. We count these rational numbers like so:

$$\frac{1}{2}, \quad \frac{1}{3}, \frac{2}{3}, \quad \frac{1}{4}, \frac{3}{4}, \quad \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \quad \frac{1}{6}, \frac{5}{6}, \quad \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \ldots.$$

Interestingly we can use this way of counting to show how there are multiple types of infinity which are larger than one another. The number of rational numbers between 0 and 1 is countably infinite (as are the number of natural numbers) but we can not count the number of real numbers between 0 and 1 and we know that these contain all the rational numbers between 0 and 1 (as well as the irrational numbers) hence this infinity is larger [19].

There is a conformal transform that we can do on the Mandelbrot set so that the main cardioid is mapped to a disc. Under this mapping the $\frac{1}{3}$ bulb goes to the point $\frac{1}{3}$ the way around the circle (anti-clockwise as is the convention), the $\frac{2}{5}$ bulb $\frac{2}{5}$ around the circle, and so on. We can see in figure 5.2 that the $\frac{2}{5}$ bulb is between the $\frac{1}{3}$ bulb and the $\frac{1}{2}$ bulb, just as $\frac{2}{5}$ is between $\frac{1}{3}$ and $\frac{1}{2}$. This property applies to every primary bulb.

### 5.2.2 Non-primary bulbs

We see in figure 5.2 that bulb's periods are multiples of the bulb they come off of. For example the secondary bulbs coming off of the primary bulb with period 2 all have even periods, the largest of these secondary bulbs has period 4 ($= 2 \times 2$), the second largest bulbs have period 6 ($= 3 \times 2$), the third largest bulbs have period 8 ($= 4 \times 2$), and so on. Additionally they are located $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$ around the main bulb; this is the case for all other bulbs too (we say 0 of the way around a bulb is the point where it is attached to the larger bulb with smaller period than it, and go around anti-clockwise as normal), and the cardioid when it is transformed. See figure 5.3 for close-ups of some non-primary bulbs. Also notice that non-primary bulbs have periods directly related to their size.

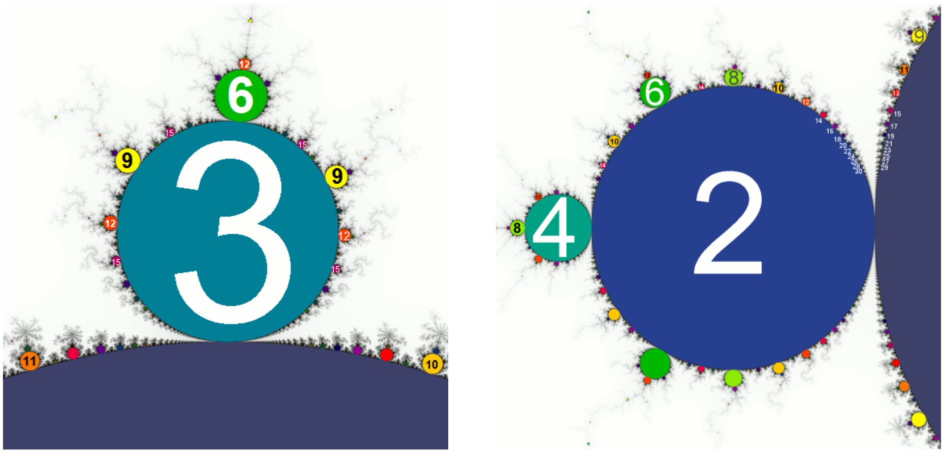


Figure 5.3: The periods of regions in the Mandelbrot set zoomed in on the primary $\frac{1}{3}$ bulb (left) and the main disc (right).

We can call these non-primary bulbs by the sequence of the $\frac{p}{q}$ bulbs they come off of, starting with the primary bulb. For example the bulb in figure 5.4 is the $\frac{1}{3} - \frac{1}{3} - \frac{2}{3}$ bulb, representing the upper primary bulb with period 3 it comes off of then the right period

9 ($= 3 \times 3$) bulb (a third of the way around the primary bulb from its attachment point) and finally itself which is the bulb two thirds of the way around the secondary bulb.


Figure 5.4: The $\frac{1}{3} - \frac{1}{3} - \frac{2}{3}$ bulb.

### 5.2.3 Strands

Each primary bulb has a set of strands coming off of the top of them (technically they come off the bulb at the top of the stack of infinitely many bulbs above the primary bulb) and the number of branches is equal to the period of the bulb. You can see this in figure 5.2, and note that the strand coming off of the primary $\frac{1}{2}$ bulb can be thought of as two branches on top of each other.

What about the strands coming off of non-primary bulbs? These have branches equal to their relative period (which is their period divided by the period of the bulb they come off of, for example the bulb in figure 5.4 has relative period 3) near the bottom of the strand, but a longer strand comes off of this with branches equal in number to the period of the previous bulb and so on. In addition the branch which the next set of branches come off of is dependent on the location of the bulb around the previous bulb. For the ... $- \frac{p}{q}$ bulb the second set of branches comes off of the first set at the $(q - p)^{th}$ branch (anti-clockwise) and so forth. You can see this on the strands from the period 15 bulbs in figure 5.4 and the bulbs in figure 5.5.

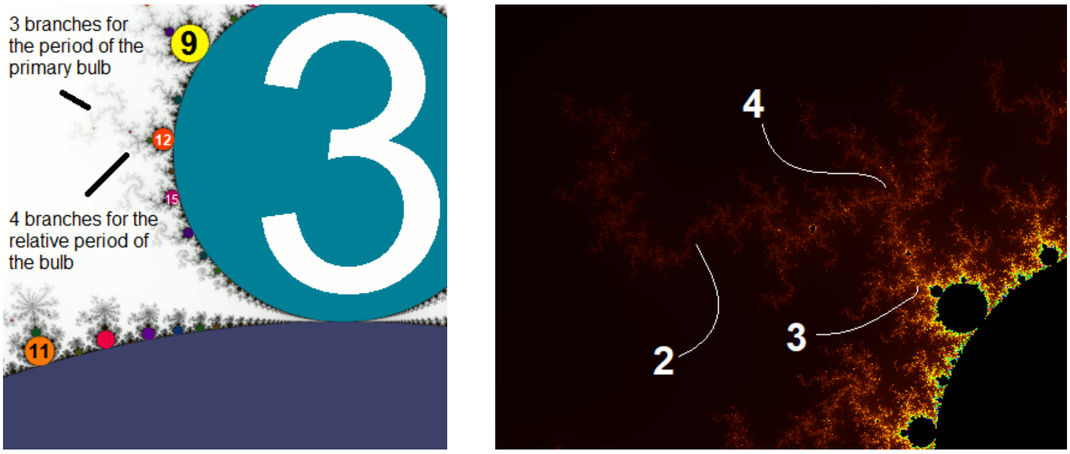


Figure 5.5: The strands for the $\frac{1}{3}-\frac{3}{4}$ bulb (left), and the $\frac{1}{2}-\frac{1}{4}-\frac{2}{3}$ bulb [20].

## 5.3 Periodicity of point bridges between bulbs

What about the points between two bulbs, such as $c = -0.75$? Does this orbit approach a cycle with a period of 1 or with a period of 2?

Well after $50,000$ iterations the sequence is still converging:

$$z_{50001} = -0.50315670, \ z_{50002} = -0.49683333,$$
$$z_{50003} = -0.50315664, \ z_{50004} = -0.49683340 \text{ (8 d.p.)},$$

whereas for other orbits, such as $c = -0.751$, have converged to 2 distinct points (to sixteen decimal places) by this stage. Even after 10 million iterations it is still going:

$$z_{10000001} = -0.500223581508, \ z_{10000002} = -0.499776368503,$$
$$z_{10000003} = -0.500223581486, \ z_{10000004} = -0.499776368526 \text{ (12 d.p.)},$$

and appearing as if it will eventually get to $-0.5$ (since that is a fixed point of $z_{n+1} = z_n^2 - 0.75$) which seemingly would take an infinite number of iterations. A very similar thing happens for the orbit with $c = -1.25$ and it seems to approach a 2-cycle between 1.207 and 0.207 (3 d.p.) (similarly solutions to $z = (z^2 - 1.25)^2 - 1.25$).

## 5.4 The relationship between bulb period and Julia sets

If we look at the Julia set with $c$ in a bulb you may notice a link between the Julia set's appearance and the period of the bulb that $c$ is in. See in figure 5.6 how there are five

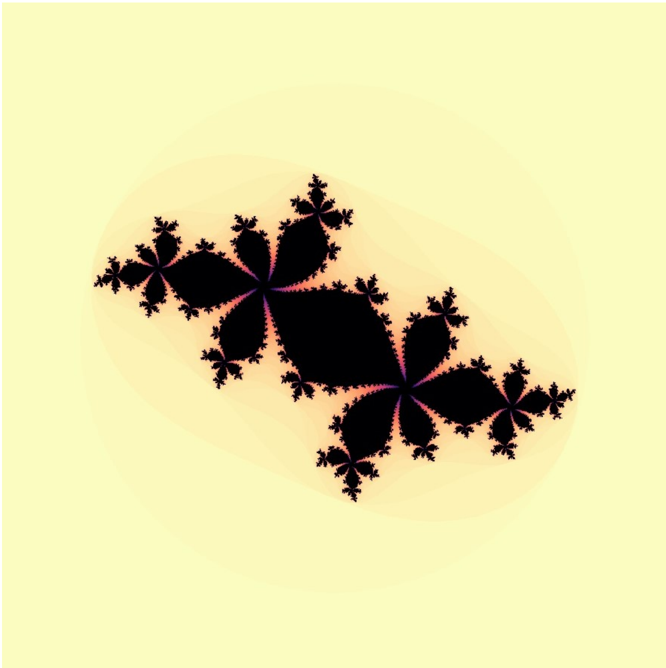

Figure 5.6: The Julia set with $c = -0.5 + 0.55i$ which is in the primary $\frac{2}{5}$ bulb.

limbs coming out of points all over the set.

The point-bridge surrounded by the largest limbs in the upper left quadrant is a fixed point and the first solution to:

$$\begin{aligned} z &= z^2 - 0.5 + 0.55i \\ \Leftrightarrow z &= -0.41653 + 0.300045i,\ 1.41653 - 0.300045i. \end{aligned} \tag{5.2}$$

The second solution is the point at the far-right tip of the Julia set.

Points in the set will have a sequence which eventually enters a loop between five points in the centres of the five largest limbs to the left, as in figure 5.7.

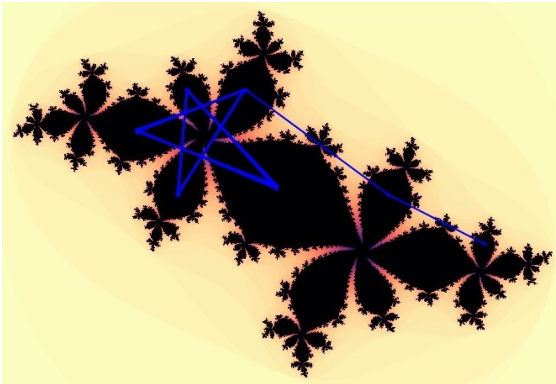

Figure 5.7: The sequence (4.1) for $z_0 = -1.06 - 0.26i$ for the Julia set with $c = -0.5 + 0.55i$.

Julia sets with $c$ in a non-primary bulb with a relative period of $q$ will have a similar appearance to the Julia set with $c$ in a primary bulb with period $q$. We can see this in figure 5.8 where Julia sets with $c$ in a secondary ... $-\ \frac{2}{5}$ bulb have what looks like the Julia set with $c$ in the $\frac{2}{5}$ bulb (figure 5.6) all over the set rotated and shrunk. Notice how the central copy of figure 5.6 is rotated in the second and third sets in figure 5.8, this directly corresponds to how far around the main cardioid the primary bulb is. Each set also has secondary branches with the number of branches equal to the period of the primary bulb it is attached to, similar to the properties of strands in subsection 5.2.3.

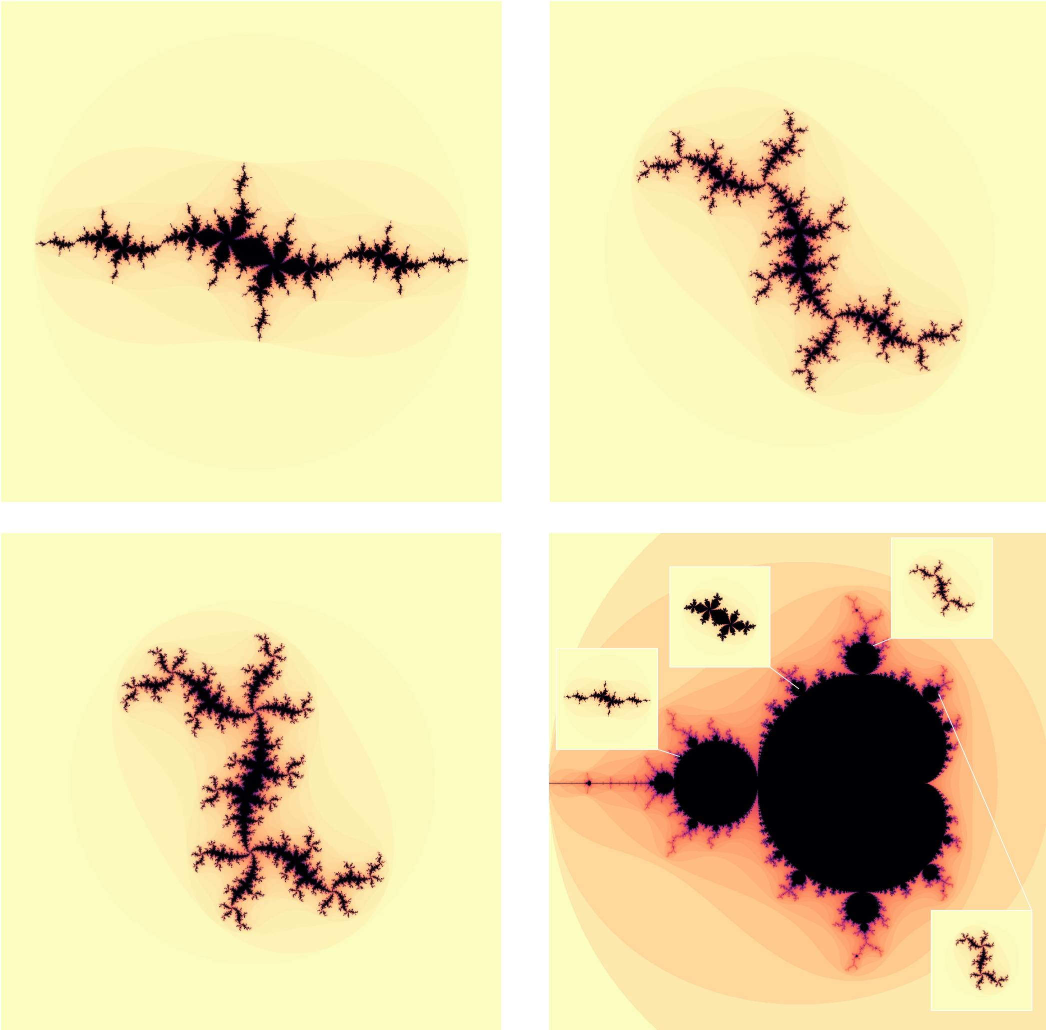

Figure 5.8: From left to right and top to bottom: The Julia set with $c = -1.21 + 0.15i$ in the $\frac{1}{2} - \frac{2}{5}$ bulb, the Julia set with $c = -0.06 + 0.82i$ in the $\frac{1}{3} - \frac{2}{5}$ bulb, the Julia set with $c = 0.327 + 0.533i$ in the $\frac{1}{4} - \frac{2}{5}$ bulb, and their locations in relation to the Mandelbrot set.

For 'less primary' bulbs the set will get smaller and again like the strands it will have sets of branches corresponding to the bulbs location. In figure 5.9 with $c$ in the $\frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{2}{5}$ bulb we see that again a copy of figure 5.6 is in the centre (and elsewhere around the set too), then traveling away from this we reach junctions with four branches, then a junction with three branches, and finally a junction with two branches. All very similar to the strands in figure 5.5.

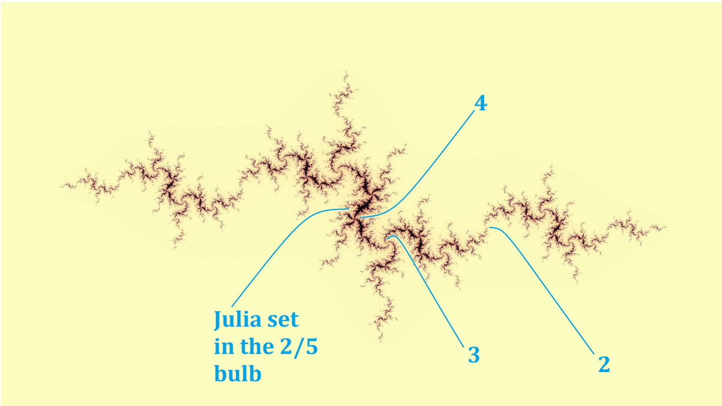


Figure 5.9: The Julia set with $c = -1.11235 + 0.25315i$ in the $\frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{2}{5}$ bulb.