Programming & Problem Solving in Engineering

ENGR220

Matrix Algebra

The following slides examine some basic ideas from Matrix Algebra. The intention here is not use a lot of sophisticated concepts; the idea is to be able to form and solve sets of simultaneous equations in our engineering analyses. The thrust here is on simultaneous equations, so this brief review/introduction of the concept of matrices is helpful.

The following topics are included in the slides:

Objectives
Introduction
Definition
Nomenclature
Operations
Multiplication
Determinate
Inverse
Simultaneous Equations
Gaussian Elimination
Matrix Partitioning

Objectives

All of this may be review. You have probably seen this material at one time or another.

In general, the *solution* to a complex circuit will involve the formation and solution to a set of *simultaneous equations*.

In order to efficiently work with these equations, we need to write them in what is called *matrix notation*.

Here we will investigate a few of the *principles of matrix algebra* that are useful when working with sets of simultaneous equations.

Introduction

In this class, we need a *few basic ideas from matrix algebra*.

It will not be necessary to know complicated or esoteric concepts of linear algebra, just a level of knowledge that will allow us to manipulate simultaneous equations.

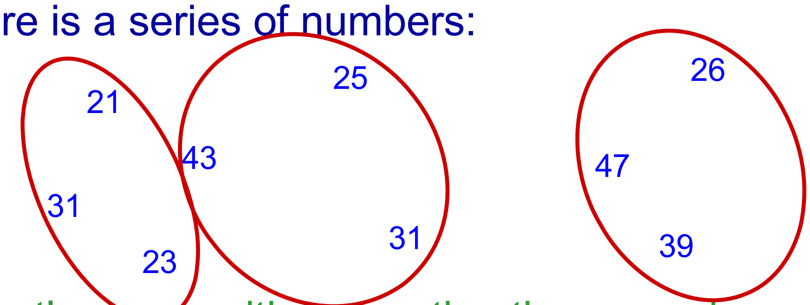
Matrix Definition

- So, what is a matrix?
- Simply stated, a matrix consists of a series of numbers that all describe the same or related things
- We wish to keep the numbers together because they are all related
- For example, we could have a matrix of numbers that would give the ages of all students in this class

Nothing complicated about the idea of a matrix. Just a set of related numbers.

Just Numbers

- Here is a series of numbers:



- What's wrong with presenting these numbers this way?
- It is very difficult to talk about the individual numbers since they are so scattered around
- Let's arrange them into columns:

We do this purely for convenience.

Organized Numbers

- It is much better to arrange them in rows and columns

$$\begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{A}$$

- This matrix might represent the ages of the students in each line of desks in our classroom
- We give the matrix a name, $\underline{A}$ in this case
- The line under it, or **bold type**, shows that it is more than a scalar variable, i.e. there are many numbers associated with $\underline{A}$, nine in this case

In general, a matrix contains a set of numbers that describes the same or a similar thing. When you write a matrix, do not forget the line under the name to signify that it is a matrix.

Rows and Columns

$$\begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{A}$$

- We divide a matrix into columns and rows
- The columns are vertical, since columns hold up bridges and buildings

It is easy to remember which are the columns. They go up and down. The columns in a building are vertical.

Rows and Columns

$$\begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{\mathbf{A}}$$

- We divide a matrix into columns and rows
- The columns are vertical, since columns hold up bridges and buildings
- The rows are horizontal, they are the ones that aren't columns

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The rows are the ones that aren't the columns. So rows go sideways.

Subscripts

$$\begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{\mathbf{A}}$$

- We can refer to an individual number by using the matrix variable without the line and specifying subscripts

$$\mathbf{A}_{21} = \mathbf{31}$$

$$\mathbf{A}_{13} = \mathbf{26}$$

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The subscripts specify the row number (side-to-side) and the column number (up and down), in that order.

Subscript Meaning

$$\begin{array}{cc} & \text{col 1} \\ \text{row 2} & \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{A} \end{array}$$

- We can refer to an individual number by using the matrix variable without the line and specifying subscripts

$A_{21}=31$ row 2, column 1

- The first subscript is the row number
- The second subscript is the column number

This example shows the meaning of the subscripts.

Subscript Meaning-Example 2

$$\begin{array}{cc} & \text{col 3} \\ \text{row 1} & \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \underline{A} \end{array}$$

- We can refer to an individual number by using the matrix variable without the line and specifying subscripts

$A_{13}=26$ row 1, column 3

- The first subscript is the row number
- The second subscript is the column number

Here is another example. Don't confuse the meaning of the subscripts with the convention used to designate a particular cell in a spreadsheet. **Spreadsheets do it backwards.** They indicate the column first and then the row.

Order-Square-Diagonal

- **Order** (size, dimension, rank): $\underline{A}_{mxn}$
  - mxn means m rows and n columns where the columns go up and down
- **Square Matrix**: m = n, same number of rows and columns - It looks square

$$\underline{A} = \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix}$$

This is a 3x3 matrix

- **Diagonal Matrix**: $B_{ij} = 0$ if $i \neq j$
  Here is an example of a 3x3 diagonal matrix

$$\underline{B} = \begin{vmatrix} 27 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

Can a diagonal term=0?

Just some definitions. The principal diagonal, or usually we just say diagonal, consists in this case of the numbers 27, -2, 5 in the line going from upper left to lower right. The definition of a diagonal matrix only says that the off diagonal terms are zero. Therefore, a diagonal term can equal zero.

Identity-Row-Column Matrices

- **Identity Matrix**: $\underline{I}$ is a diagonal matrix with only 1's on the diagonal
  $I_{ij} = 0$ if $i \neq j$ and $I_{ij} = 1$ if $i = j$

$$\underline{I}_{3x3} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

- **Row Matrix**: 1 x n

$$\underline{C} = \begin{vmatrix} 4 & 9 & -21 \end{vmatrix}$$

- **Column Matrix**: m x 1
  (Also called a vector)

$$\underline{D} = \begin{vmatrix} 7 \\ 9 \\ 0 \end{vmatrix}$$

More definitions. The identity matrix is the matrix equivalent of the scalar number 1. You used column matrices (vectors) in Statics a lot.

Nomenclature

- **Symmetric Matrix**: **A** is a symmetric matrix when:

  $A_{ij} = A_{ji}$ $(A_{12}=2; A_{21}=2)$

$$\mathbf{A}_{3x3} = \begin{vmatrix} 8 & 2 & 1 \\ 2 & 9 & -2 \\ 1 & -2 & 10 \end{vmatrix}$$

- **This means that if you interchange the rows and columns, you have the same matrix**
- **Another way of saying this:**
  - **row 1 = column 1**

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We see that a symmetric matrix is kinda mirrored about the diagonal. We will primarily be using symmetric matrices in this class. Look at the first row and first column. If the matrix is symmetric, the identical numbers will be in both row 1 and column 1.

Symmetric Matrix-Definition

- **Symmetric Matrix**: **A** is a symmetric matrix when:

  $A_{ij} = A_{ji}$ $(A_{12}=2; A_{21}=2)$

$$\mathbf{A}_{3x3} = \begin{vmatrix} 8 & 2 & 1 \\ 2 & 9 & -2 \\ 1 & -2 & 10 \end{vmatrix}$$

- **This means that if you interchange the rows and columns, you have the same matrix**
- **Another way of saying this:**
  - **row 1 = column 1**
  - **row 2 = column 2**

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Likewise, the same numbers are in the second row and in the second column.

Symmetric Matrix

- **Symmetric Matrix**: **A** is a symmetric matrix when:

  $A_{ij} = A_{ji}$ $(A_{12}=2; A_{21}=2)$

$$\underline{\mathbf{A}}_{3x3} = \begin{bmatrix} 8 & 2 & 1 \\ 2 & 9 & -2 \\ 1 & -2 & 10 \end{bmatrix}$$

- **This means that if you interchange the rows and columns, you have the same matrix**
- **Another way of saying this:**
  - **row 1 = column 1**
  - **row 2 = column 2**
  - **row 3 = column 3**

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The third row and the third column also contain the same numbers. Symmetric matrices are very important in this class, since the matrix that describes the response of a structure to applied loads will always be a symmetric matrix.

Triangular-Equal Matrices

- **Upper Triangular Matrix**:

  $U_{ij} = 0$ if $i > j$ Must be zeros!

Non zero values can be in this triangular region

$$\underline{\mathbf{U}} = \begin{bmatrix} 7 & 67 & -5 \\ 0 & 43 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- **Lower Triangular Matrix**:

  $L_{ij} = 0$ if $i < j$ Must be zeros!

$$\underline{\mathbf{L}} = \begin{bmatrix} 7 & 0 & 0 \\ 6 & 43 & 0 \\ 5 & 22 & 4 \end{bmatrix}$$

Nonzero values can be in this triangular region

- **Equal Matrices**: **A** = **B** means the same size and $A_{ij} = B_{ij}$

  They are exactly the same

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Triangular matrices are used in the solution of simultaneous equations. If two matrices are equal, then you can interchange the two and you cannot tell any difference.

Matrix Operations

- **Addition/Subtraction**: $\underline{C} = \underline{A} + \underline{B}$ means $\underline{A}$ and $\underline{B}$ are the same size and $C_{ij} = A_{ij} + B_{ij}$

$$\boxed{\underline{C} = \begin{vmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \\ A_{31}+B_{31} & A_{32}+B_{32} & A_{33}+B_{33} \end{vmatrix}}$$

- **Multiplication by a scalar**: $\underline{A} = 13\,\underline{B}$ means $A_{ij} = 13\,B_{ij}$, i.e. multiply each element of $\underline{B}$ by 13

$$\boxed{\underline{A} = \begin{vmatrix} 13B_{11} & 13B_{12} & 13B_{13} \\ 13B_{21} & 13B_{22} & 13B_{23} \\ 13B_{31} & 13B_{32} & 13B_{33} \end{vmatrix}}$$

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These operations on matrices make sense. They are very similar to the equivalent operations involving scalars.

Matrix Operations-Transpose

- **Transpose of a matrix**: $\underline{C} = \underline{A}^T$ means $\underline{C}$ is obtained by interchanging the rows and columns of $\underline{A}$ $\quad C_{ij} = A_{ji}$

if :

$$\underline{A} = \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix}$$

then :

$$\underline{C} = \underline{A}^T = \begin{vmatrix} 21 & 31 & 23 \\ 25 & 43 & 31 \\ 26 & 47 & 39 \end{vmatrix}$$

Note: If a matrix is symmetric, then it equals its transpose

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To find the transpose of a matrix we switch the rows and the columns. Row 1 becomes column 1. Column 1 becomes row 1, etc. Since in a symmetric matrix, the rows are the same as the columns, switching them does not change anything.

Structural Analysis

Matrix Multiplication

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Now we'll examine some basic ideas from Matrix Algebra. We will not need a lot of sophisticated concepts here, but we do need to be able to form and solve sets of simultaneous equations.

Matrix Multiplication

- **Multiplication**: C = A B means A and B are to be multiplied and we do the following:

If we have: $\underset{(2x3)}{\underline{A}} = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 5 & -3 \end{vmatrix}$; $\underset{(3x2)}{\underline{B}} = \begin{vmatrix} -3 & 4 \\ 3 & 0 \\ 1 & -1 \end{vmatrix}$

- To get C_{11} (row 1, col 1), we take row 1 of A, stand it on end beside col 1 of B, multiply the adjacent numbers and add them up

$$\begin{matrix} 4 \\ -1 \\ 0 \end{matrix} * \begin{matrix} -3 \\ 3 \\ 1 \end{matrix} = \begin{matrix} -12 \\ -3 \\ 0 \end{matrix}$$

$C_{11} = -15$

A is 2x3
B is 3x2
C will be 2x2

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Matrix multiplication is a different story. This definition is different from any such operation dealing with scalars. You should remember the process. To get the number in position row 1, col 1, we take row 1 of the first matrix, stand it on end next to column 1 of the second matrix, multiply the numbers that are side-by-side, and add up these products. This -15 is then the number that goes into row1 and column 1 in the product matrix.

Matrix Multiplication

- **Multiplication**: $\underline{C} = \underline{A}\ \underline{B}$ means $\underline{A}$ and $\underline{B}$ are to be multiplied and we do the following:

$$\text{If we have: } \underline{\mathbf{A}} = \begin{vmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{2} & \mathbf{5} & \mathbf{-3} \end{vmatrix} \;;\; \underline{\mathbf{B}} = \begin{vmatrix} \mathbf{-3} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} \end{vmatrix}$$

- To get C_{11} (row 1, col 1), we take row 1 of A, stand it on end beside col 1 of B, multiply the adjacent numbers and add them up

$$\begin{matrix} 4 \\ -1 \\ 0 \end{matrix} \; * \; \begin{matrix} -3 \\ 3 \\ 1 \end{matrix} \; = \; \begin{matrix} -12 \\ -3 \\ 0 \end{matrix}$$

$$C_{11} = -15$$

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Matrix multiplication is a different story. This definition is different from any such operation dealing with scalars. You should remember the process. To get the number in position row 1, col 1, we take row 1 of the first matrix, stand it on end next to column 1 of the second matrix, multiply the numbers that are side-by-side, and add up these products. This –15 is then the number that goes into row1 and column 1 in the product matrix.

Matrix Multiplication

- **Multiplication**: $\underline{C} = \underline{A}\,\underline{B}$ means $\underline{A}$ and $\underline{B}$ are to be multiplied and we do the following:

$$\text{If we have: } \underline{\mathbf{A}} = \begin{vmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{2} & \mathbf{5} & \mathbf{-3} \end{vmatrix} \; ; \; \underline{\mathbf{B}} = \begin{vmatrix} \mathbf{-3} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} \\ \mathbf{1} & \mathbf{-1} \end{vmatrix}$$

- To get C_{12} (row 1, col 2), we take row 1 of A, stand it on end beside col 2 of B, multiply the adjacent numbers and add them up

$$\begin{matrix} 4 \\ -1 \\ 0 \end{matrix} \; * \; \begin{matrix} 4 \\ 0 \\ -1 \end{matrix} \; = \; \begin{matrix} 16 \\ 0 \\ 0 \end{matrix}$$

$$C_{12} = \; 16$$

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To get the number in position row 1, col 2, we take row 1 of the first matrix, stand it on end, stand it on end next to column 2 of the second matrix, multiply the numbers that are side-by-side, and add up these products. This 16 is then the number that goes into row1 and column 2 in the product matrix.

Matrix Multiplication

- **Multiplication**: $\underline{C} = \underline{A}\,\underline{B}$ means $\underline{A}$ and $\underline{B}$ are to be multiplied and we do the following:

$$\text{If we have: } \underline{\mathbf{A}} = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 5 & -3 \end{vmatrix} \;;\; \underline{\mathbf{B}} = \begin{vmatrix} -3 & 4 \\ 3 & 0 \\ 1 & -1 \end{vmatrix}$$

- To get C_{21} (row 2, col 1), we take row 2 of A, stand it on end beside col 1 of B, multiply the adjacent numbers and add them up

$$\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} * \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 15 \\ -3 \end{pmatrix}$$

$C_{21} = \; 6$

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To get the number in position row 2, col 1, we take row 2 of the first matrix, stand it on end, stand it on end next to column 1 of the second matrix, multiply the numbers that are side-by-side, and add up these products. This 6 is then the number that goes into row 2 and column 1 in the product matrix.

Matrix Multiplication

- **Multiplication**: $\underline{C} = \underline{A}\ \underline{B}$ means $\underline{A}$ and $\underline{B}$ are to be multiplied and we do the following:

$$\text{If we have: } \underline{A} = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 5 & -3 \end{vmatrix} \quad ; \quad \underline{B} = \begin{vmatrix} -3 & 4 \\ 3 & 0 \\ 1 & -1 \end{vmatrix}$$

- To get C_{22} (row 2, col 2), we take row 2 of A, stand it on end beside col 2 of B, multiply the adjacent numbers and add them up

$$\begin{matrix} 2 \\ 5 \\ -3 \end{matrix} * \begin{matrix} 4 \\ 0 \\ -1 \end{matrix} = \begin{matrix} 8 \\ 0 \\ 3 \end{matrix}$$

$C_{22}=$ 11

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To get the number in position row 2, col 2, we take row 2 of the first matrix, stand it on end, stand it on end next to column 2 of the second matrix, multiply the numbers that are side-by-side, and add up these products. This 11 is then the number that goes into row 2 and column 2 in the product matrix.

Matrix Multiplication

- **Multiplication**: $\underline{C} = \underline{A}\ \underline{B}$ means $\underline{A}$ and $\underline{B}$ are to be multiplied and we do the following:

$$\text{If we have: } \underline{A} = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 5 & -3 \end{vmatrix} \quad ; \quad \underline{B} = \begin{vmatrix} -3 & 4 \\ 3 & 0 \\ 1 & -1 \end{vmatrix}$$

- All together now:

$$\underline{AB} = \begin{vmatrix} 4(-3)-1(3)+0(1) & 4(4)-1(0)+0(-1) \\ 2(-3)+5(3)-3(1) & 2(4)+5(0)-3(-1) \end{vmatrix} = \begin{vmatrix} -15 & 16 \\ 6 & 11 \end{vmatrix}$$

- In general, $\underline{AB} \neq \underline{BA}$
- Why?

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Study each of these calculations and make sure you understand the process and can get each of the terms. Since you take a row from the first matrix and a column from the second matrix to get each term, you cannot in general switch the order of the mult.

Activity-Multiplication

- OK, now try it!
- For the different matrices shown below, find the coefficient that goes into the product matrix **C** in position 21

$$\underline{A} = \begin{vmatrix} 4 & -1 \\ 2 & 5 \end{vmatrix} \;;\; \underline{B} = \begin{vmatrix} -3 & 4 & 1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$\underline{C} = \underline{A}\,\underline{B}$$

$$\boxed{C_{21} = \quad}$$

- Take out a piece of paper and write down this result

Here is one for you to try. Do the multiplication to get this term. Then go to the next slide to check your answer.

Activity-Multiplication Solution

- OK, now try it!
- For the matrices shown below, find the coefficient that goes into the product matrix **C** in position 21

$$\underline{A} = \begin{vmatrix} 4 & -1 \\ 2 & 5 \end{vmatrix} \;;\; \underline{B} = \begin{vmatrix} -3 & 4 & 1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$\underline{C} = \underline{A}\,\underline{B}$$

$$\boxed{C_{21} = 2(-3)+5(3) = -6+15 = 9}$$

- Here is the answer

How did you do? If not great, go back and review.

Pre and Post Multiplication

- Note: if **A** is (mxn) and **B** is (nxp) then **C**=**AB** will be (mxp) - The inner dimensions MUST be the same, and the product will have a size equal to the outer dimensions

$$\underline{\mathbf{A}}_{\mathbf{mxn}}\underline{\mathbf{B}}_{\mathbf{nxp}} = \underline{\mathbf{C}}_{\mathbf{mxp}}$$

same

Product size

- We say that A is post multiplied by B
- or:
- We say B is pre multiplied by A

So the order of the multiplication is important. It determines the size of the product matrix. In general the product matrix A B will not even be the same size as the product matrix B A.

So there are two products of matrix A and matrix B. You must specify the order. You cannot just say multiply A and B, you have to say which matrix comes first.

Matrix Multiplication by $\underline{\mathbf{I}}$

- The Identity matrix is a special case:

$$\underline{\mathbf{AI}} = \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix}$$

and

$$\underline{\mathbf{IA}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix} = \begin{vmatrix} 21 & 25 & 26 \\ 31 & 43 & 47 \\ 23 & 31 & 39 \end{vmatrix}$$

so for this special case:

$$\boxed{\underline{\mathbf{AI}} = \underline{\mathbf{IA}} = \underline{\mathbf{A}}}$$

Remember, this is an exception to the rule - Usually $\underline{\mathbf{AB}} \neq \underline{\mathbf{BA}}$

Recall that the Identity matrix, $\underline{\mathbf{I}}$, is the matrix equivalent of the scalar number 1. If you multiply the number 1 by any other number, you get that same number. Likewise, if you multiply any matrix by the identity matrix $\underline{\mathbf{I}}$, you get that same matrix.

Structural Analysis

Inverse of A Matrix

Now we'll examine some basic ideas from Matrix Algebra. We will not need a lot of sophisticated concepts here, but we do need to be able to form and solve sets of simultaneous equations.

Determinant of a Matrix

- Determinant of a Matrix: The sum of all possible products of elements of the matrix where each product does not contain more than one element from each row and column and each product must be multiplied by (-1) raised to the proper power - It is usually indicated by:

det **A** = |**A**|

- **That should be perfectly clear?**
- **No?** OK, we'll illustrate it by examples

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You are all familiar with the concept of a determinant. Here is the real definition of a determinant. It is pretty obscure and hard to use to actually find the value of a determinant. The important idea in this definition is that when you multiply numbers from the matrix, you cannot use two numbers from the same row or from the same column.

Determinant of a Matrix-Example

- Look at the following problem:

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

Recall the process:You take the **positive** of all products from upper left to lower right

$$\underline{A} = \left|\begin{array}{ccc:cc} 2 & -1 & 3 & 2 & -1 \\ 0 & 4 & -2 & 0 & 4 \\ 1 & -3 & 5 & 1 & -3 \end{array}\right|$$

Rewrite the first 2 columns:

+[2 (4) 5]= 40

+[-1 (-2)1]= 2

+[3 (0) (-3)]= 0

It is easier to visualize if you rewrite the first 2 columns to the right of the matrix. For these products you just multiply the matrices and use whatever sign you get, either plus or minus.

Determinant of a Matrix-Example

- Look at the following problem:

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

Recall the process:You take the **positive** of all products from upper left to lower right and;

The **negative** of all products from upper right to lower left

$$\underline{A} = \left|\begin{array}{ccc:cc} 2 & -1 & 3 & 2 & -1 \\ 0 & 4 & -2 & 0 & 4 \\ 1 & -3 & 5 & 1 & -3 \end{array}\right|$$

+[2 (4) 5]= 40 **-[3 (4)1]= -12**

+[-1 (-2)1]= 2 **-[2 (-2) (-3)]= -12**

+[3 (0) (-3)]= 0 **-[-1 (0) 5]= 0**

Here you change the sign of whatever product you get. Note that each product contains numbers from every row and every column.

Determinant of a Matrix-Example

- Look at the following problem:

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\underline{A} = \left|\begin{array}{ccc:cc} 2 & -1 & 3 & 2 & -1 \\ 0 & 4 & -2 & 0 & 4 \\ 1 & -3 & 5 & 1 & -3 \end{array}\right|$$

Recall the process:You take the **positive** of all products from upper left to lower right and;

The **negative** of all products from upper right to lower left

+[2 (4) 5]= 40

+[-1 (-2)1]= 2

+[3 (0) (-3)]= 0

-[3 (4)1]= -12

-[2 (-2) (-3)]= -12

-[-1 (0) 5]= 0

Now add them up:

|A|=40 + 2 + 0 -12 -12 + 0

$\boxed{\textbf{|A|=18}}$

This is the determinate of the matrix. Note that the determinate of a matrix is a scalar number.

Inverse of a Matrix-Definition

- **Inverse of a Matrix**: If $\underline{A}$ is a square matrix then define $\underline{A}^{-1}$ as the Inverse of $\underline{A}$
  - Such that: $\underline{A}\,\underline{A}^{-1} = \underline{A}^{-1}\,\underline{A} = \underline{I}$
- Here is another exception to the rule that says you cannot switch the order of multiplication
- The inverse of a matrix is as close as we come to defining matrix division
- It is the matrix equivalent of multiplying a number by its inverse and getting 1 - $\underline{I}$ is the matrix version of the scalar 1

Note that the inverse of a matrix is a matrix also, and it is the same size as $\underline{A}$.

Inverse of a Matrix-Why Important

- Why is this idea of an inverse important?

Look at the following sumultaneous equations:

$$2x_1 - 1x_2 + 3x_3 = 1$$
$$0x_1 + 4x_2 - 2x_3 = 2$$
$$1x_1 - 3x_2 + 5x_3 = 1$$

← **Scalar form of the equations**

3 ways to write the same equations

This can be written in Matrix form using our definition of matrix multiplication as follows:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

← **Matrix form**

or in Matrix Notation **as:** $\underline{A}\underline{x} = \underline{b}$ ← **Matrix notation**

We will be solving simultaneous equations when we solve Statically Indeterminate problems, so they are important.

Inverse of a Matrix-Equations

- We will usually write our description of a problem solution in the form of simultaneous equations - Then we can do the following:

$$\underline{A}\underline{x} = \underline{b}$$

Multiply both sides by $\underline{A}^{-1}$

$$\underbrace{(\underline{A}^{-1}\underline{A})}\underline{x} = \underline{A}^{-1}\underline{b}$$

$$\underline{I}\underline{x} = \underline{x} = \underline{A}^{-1}\underline{b}$$

So the solution to the equations can be expressed by

$$\underline{x} = \underline{A}^{-1}\underline{b}$$

- We can express the solution to the equations in terms of the inverse of the matrix $\underline{A}$ - So we need to find the matrix $\underline{A}^{-1}$

We perform some simple matrix algebra to solve for the unknowns in the matrix equation, $\underline{x}$. It is written as the product of the inverse of the coefficient matrix $\underline{A}$, post-multiplies by the vector $\underline{x}$. Let's now tern our attention toward finding the inverse of the matrix $\underline{A}$.

Inverse of a Matrix-Cofactor

- To begin, we need to find a matrix called the Cofactor Matrix, $\underline{A}^C$ - It is defined as follows:

$$\underline{\mathbf{A}}^C = \begin{vmatrix} \mathbf{A}_{11}^C & \mathbf{A}_{12}^C & \mathbf{A}_{13}^C \\ \mathbf{A}_{21}^C & \mathbf{A}_{22}^C & \mathbf{A}_{23}^C \\ \mathbf{A}_{31}^C & \mathbf{A}_{32}^C & \mathbf{A}_{33}^C \end{vmatrix} = \text{Cofactor Matrix}$$

$$A_{11}^C = (-1)^{1+1} \begin{vmatrix} \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{32} & \mathbf{A}_{33} \end{vmatrix}$$

$$\mathbf{A}_{12}^C = (-1)^{1+2} \begin{vmatrix} \mathbf{A}_{21} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{33} \end{vmatrix}$$

The cofactor matrix $\underline{A}^C$ comes from a known matrix $\underline{A}$ and is the same size

- Let's look at an example:

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The cofactor matrix is the first step. Each term in the cofactor matrix comes from the determinate of a submatrix of A that was formed by deleting 1 row and 1 column of $\underline{A}$.

Cofactor Matrix Example

- Using this definition:

For Example:

$$\underline{\mathbf{A}} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}; \text{ then: } \underline{\mathbf{A}}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

$$\text{where: } \mathbf{A}_{12}^C = (-1)^{1+2} \begin{vmatrix} 0 & -2 \\ 1 & 5 \end{vmatrix} = (-1)[(0)(5) - (1)(-2)] = -2$$

- Note that each cofactor term comes from the determinant of a smaller matrix, a **submatrix**, created by deleting one row and one column, times -1 raised to the (row + column) power
- Let's examine the process in detail:

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To find the term in the cofactor matrix in row 1 and column 1, we delete row 1 and column from A and find the determinate of this submatrix. We then multiply this determinate by (-1) raised to the (1+1)=2 power.

Cofactor Matrix Example (1,1)

- The process:
- Cross out a row and a column, in this case (1,1)
- Take the determinant of what is left
- Place this number into the position of the row and column of A^C that were crossed out, (1,1)
- **We'll ignore the $(-1)^{i+j}$ for now**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 4 & -2 \\ -3 & 5 \end{vmatrix} = +[(4)(5)] - [(-2)(-3)] = 14$$

$$\begin{vmatrix} 14 & & \\ & & \\ & & \end{vmatrix} \Rightarrow A^C$$

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Again, looking at the term in position row 1 and column 1. This is the determinate of the submatrix and this number goes into the cofactor matrix in row 1 and column 1.

Cofactor Matrix Example (1,2)

- Next term:
- Cross out a row and a column, in this case (1,2)
- Take the determinant of what is left
- Place this number into the position of the row and column that were crossed out, (1,2)
- **Again, ignore the $(-1)^{i+j}$ for now**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 0 & -2 \\ 1 & 5 \end{vmatrix} = +[(0)(5)] - [(-2)(1)] = 2$$

$$\begin{vmatrix} 14 & 2 & \\ & & \\ & & \end{vmatrix}$$

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Do the same thing for row 1 and column 2. The submatrix consists of the terms not crossed out. The determinate of the submatrix goes into the row 1 and column 2 position.

Cofactor Matrix Example (1,3)

- Next term :
- Cross out a row and a column, in this case (1,3)
- Take the determinant of what is left
- Place this number into the position of the row and column that were crossed out, (1,3)
- **Again, ignore the $(-1)^{i+j}$ for now**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 4 \\ 1 & -3 \end{vmatrix} = +[(0)(-3)] - [(4)(1)] = -4$$

$$\begin{vmatrix} 14 & 2 & -4 \\ & & \\ & & \end{vmatrix}$$

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Now we do row 1 and column 3. The determinate of the submatrix goes into the row 1 and column 3 position.

Cofactor Matrix Example (2,1)

- Same thing:
- Cross out a row and a column, in this case (2,1)
- Take the determinant of what is left
- Place this number into the position of the row and column that were crossed out, (2,1)
- **Again, ignore the $(-1)^{i+j}$ for now**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 3 \\ -3 & 5 \end{vmatrix} = +[(-1)(5)] - [(3)(-3)] = 4$$

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & & \\ & & \end{vmatrix}$$

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Now we do row 2 and column 1. The determinate of the submatrix goes into the row 2 and column 1 position.

Cofactor Matrix Example (2,2)

- Let's show one more:
- Cross out a row and a column, in this case (2,2)
- Take the determinant of what is left
- Place this number into the position of the row and column that were crossed out, (2,2)
- **We ignored the $(-1)^{i+j}$ throughout**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ \cancel{0} & \cancel{4} & \cancel{-2} \\ 1 & -3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = +[(2)(5)] - [(3)(1)] = 7$$

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & \\ & & \end{vmatrix}$$

Now row 2 and column 2.

Cofactor Matrix Example-The Rest

- Do the rest the same way

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & -5 \\ -10 & -4 & 8 \end{vmatrix}$$

The remaining terms are found the same way. Make certain you can find them correctly. If not, go back and review.

Cofactor Matrix Example-Signs

- Do the rest the same way

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & -5 \\ -10 & -4 & 8 \end{vmatrix} \quad (+1)$$

i is the row number; j is the column number

- Look at the sign that multiplies each term in this matrix, $(-1)^{i+j}$
- This multiplier will be +1 when (i+j) is an even number, and -1 when it is an odd number
- Along the diagonal, i=j, so (i+j) is even and the sign multiplier is +1

Now the signs. They can usually be determined by inspection. Along the diagonal the signs will not change. This does not mean they are all positive. It merely means that you do not multiply the number you obtained by -1.

Cofactor Matrix Example-Signs

- Do the rest the same way

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & -5 \\ -10 & -4 & 8 \end{vmatrix} \quad (-1)$$

(-1)

i is the row number; j is the column number

- Look at the sign that multiplies each term in this matrix, $(-1)^{i+j}$
- This multiplier will be +1 when (i+j) is an even number, and -1 when it is an odd number
- Along the diagonal, i=j, so (i+j) is even and the sign multiplier is +1
- One position from the diagonal, it will be -1

One position below, above, left, or right of the diagonal the term $(-1)^{i+j}$ will be -1 so you will need to change the sign of these terms. Whatever number was calculated in these positions, whether positive or negative, will be multiplied by -1, resulting in a sign change.

Cofactor Matrix Example-Signs

- Do the rest the same way

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & -5 \\ -10 & -4 & 8 \end{vmatrix}$$

(+1)

(+1)

i is the row number; j is the column number

- Look at the sign that multiplies each term in this matrix, $(-1)^{i+j}$
- This multiplier will be +1 when (i+j) is an even number, and -1 when it is an odd number
- Along the diagonal, i=j, so (i+j) is even and the sign multiplier is +1
- One position from the diagonal, it will be -1
- Two positions from the diagonal, it will be +1
- etc.

Two positions away from the diagonal will have (-1) raised to an even power, which is 1. These terms will not undergo a sign change.

Cofactor Matrix Example-Signs

- Do the rest the same way

$$\begin{vmatrix} 14 & 2 & -4 \\ 4 & 7 & -5 \\ -10 & -4 & 8 \end{vmatrix}$$

(+1) (-1) (+1)

(+1) (-1)

- Look at the sign that multiplies each term in this matrix, $(-1)^{i+j}$
- This multiplier will be +1 when (i+j) is an even number, and -1 when it is an odd number
- Along the diagonal, i=j, so (i+j) is even and the sign multiplier is +1
- One position from the diagonal, it will be -1
- Two positions from the diagonal, it will be +1
- etc.

So here it is all together. The diagonal terms and all terms 2 positions away from the diagonal will not undergo a change of sign. All those terms 1 position, 3 positions, 5 positions away will have their signs changed.

Cofactor Matrix Example-Signs

- We now step through the matrix, changing the sign of every other term, remembering that the diagonal terms are multiplied by +1
- These terms are multiplied by (+1) and **do not change sign**
- These other terms are multiplied by (-1) and so their **signs change**
- We change the signs of the circled terms and we get the cofactor matrix, **$\underline{A}^C$**:

$$\begin{vmatrix} 14 & \boxed{2} & -4 \\ \boxed{4} & 7 & \boxed{-5} \\ -10 & \boxed{-4} & 8 \end{vmatrix}$$

$$\underline{A}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Move through the matrix changing the signs of every other term. The result is the cofactor matrix, $\underline{A}^C$.

Activity-Cofactors

- OK, it's time to try it
- For the different matrix shown below, find the **cofactor element** that goes into position 21

$$\underline{A} = \begin{vmatrix} 4 & -2 & 3 \\ 3 & 2 & -2 \\ 0 & -2 & 4 \end{vmatrix}$$

- Find the numerical value of this term (Don't forget the sign!)

$A^C_{21} =$

Now you try it. After you have found the value, go to the next slide to see if you did it correctly. Don't cheat!

Activity-Cofactors Solution

- OK, it's time to try it
- For the different matrix shown below, find the **cofactor element** that goes into position 21

$$\underline{A} = \begin{vmatrix} 4 & -2 & 3 \\ 3 & 2 & -2 \\ 0 & -2 & 4 \end{vmatrix}$$

- Find the numerical value of this term (Don't forget the sign!)

$$\boxed{A^C_{21}=(-1)^{2+1}[-2(4)-3(-2)]=-1[-8+6]=2}$$

How did you do? Great, I hope. If not, go back and review.

Determinant Using Cofactors-Row 1

- We can easily find the **determinant** of ***A*** **using** any row or column of $\underline{A}^C$

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \qquad \underline{A}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Row 1: [2(14)+(-1)(-2)+3(-4)]=18

Once we have found the cofactor matrix, it is easy to get the determinate of the original matrix, A. For example, look at the first rows of $\underline{A}$ and $\underline{A}^C$. Multiply the terms in corresponding positions and add the products. We get 18, the determinate of the matrix $\underline{A}$.

Determinant Using Cofactors-Row 2

- We can easily find the **determinant** of ***A*** **using** any row or column of **$\underline{A}^C$**

$$\underline{\mathbf{A}} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \qquad \underline{\mathbf{A}}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Row 1: [2(14)+(-1)(-2)+3(-4)]=18

Row 2: [0(-4)+(4)(7)+(-2)(5)]=18

Now do the same thing for row 2 of $\underline{A}$ and $\underline{A}^C$. Multiply the terms in corresponding positions and add the products. We get 18, the determinate of the matrix $\underline{A}$.

Determinant Using Cofactors-Column 1

- We can easily find the **determinant** of ***A*** **using** any row or column of **$\underline{A}^C$**

$$\underline{\mathbf{A}} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \qquad \underline{\mathbf{A}}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Row 1: [2(14)+(-1)(-2)+3(-4)]=18

Row 2: [0(-4)+(4)(7)+(-2)(5)]=18

Col 1: [2(14)+(0)(-4)+(1)(-10)]=18

Now do the same thing for column 1 of $\underline{A}$ and $\underline{A}^C$. Multiply the terms in corresponding positions and add the products. We get 18, the determinate of the matrix $\underline{A}$.

Determinant Using Cofactors-Column 3

- We can easily find the **determinant** of ***A*** **using** any row or column of **$\underline{A}^C$**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \qquad \underline{A}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Row 1: [2(14)+(-1)(-2)+3(-4)]=18

Row 2: [0(-4)+(4)(7)+(-2)(5)]=18

Col 1: [2(14)+(0)(-4)+(1)(-10)]=18

Col 3: [3(-4)+(-2)(5)+(5)(8)]=18

Now do the same thing for column 3 of $\underline{A}$ and $\underline{A}^C$. Multiply the terms in corresponding positions and add the products. We get 18, the determinate of the matrix $\underline{A}$.

Determinant Using Cofactors-Any One

- We can easily find the **determinant** of ***A*** **using** any row or column of **$\underline{A}^C$**

$$\underline{A} = \begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \qquad \underline{A}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

Row 1: [2(14)+(-1)(-2)+3(-4)]=18

Row 2: [0(-4)+(4)(7)+(-2)(5)]=18

Col 1: [2(14)+(0)(-4)+(1)(-10)]=18

Col 3: [3(-4)+(-2)(5)+(5)(8)]=18

- Any other row or column will also give you the determinant

We can use any row or column, just make sure you use the same one in each matrix. We get the same answer no matter which we use.

Adjoint Matrix

- We now need to take the transpose of this matrix - We call this matrix the **adjoint matrix** and designate it by $\underline{A}^A$
  - Just interchange the rows and columns
- **Remember what we want to find?** The inverse matrix such that: $\underline{A}^{-1}\,\underline{A} = \underline{I}$
- Ok, we almost have the inverse - Let's look at this next expression:

$$\underline{A}^C = \begin{vmatrix} 14 & -2 & -4 \\ -4 & 7 & 5 \\ -10 & 4 & 8 \end{vmatrix}$$

$$\boxed{\underline{A}^A = \begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}}$$

Remember what we are trying to find? The inverse of the matrix $\underline{A}$. We need to define just one more matrix. To get the **Adjoint matrix** we take the **transpose of the Cofactor matrix**. Many of our matrices will be symmetric. If $\underline{A}$ is symmetric, then $\underline{A}^C$ is also symmetric, so the Adjoint matrix will be the same as the Cofactor matrix. This is true only when $\underline{A}$ is a symmetric matrix.

Inverse of a Matrix-We Have It

Let's find:

$$\left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}\underline{A} = \overbrace{\frac{1}{18}}^{1/|\underline{A}|}\overbrace{\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}}^{\underline{A}^{A}}\overbrace{\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}}^{\underline{A}}$$

Multiply the matrices and simplify:

$$\boxed{\left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}}\underline{A} = \frac{1}{18}\begin{vmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{I}$$

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To find the inverse, let's look at this product. Take the scalar determinate and multiply it by the product of the Adjoint matrix post multiplied by the matrix $\underline{A}$. This yields the identity matrix $\underline{I}$. This satisfies our definition of the inverse matrix. The equation in the box then defines the inverse matrix.

Inverse of a Matrix-Our Example

Let's find:

$$\left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}\underline{A} = \overbrace{\frac{1}{18}}^{1/|\underline{A}|}\overbrace{\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}}^{\underline{A}^{A}}\overbrace{\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix}}^{\underline{A}}$$

Multiply the matrices and simplify:

$$\boxed{\left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}}\underline{A} = \frac{1}{18}\begin{vmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{I}$$

Therefore:

$$\boxed{\underline{A}^{-1} = \left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}} = \boxed{\frac{1}{18}\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}}$$

Here is our inverse matrix!

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One divided by the determinant times the Adjoint matrix is the expression for the inverse matrix. For our example, here is the inverse matrix.

Singular Matrix

$$\boxed{\underline{A}^{-1} = \left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}} = \frac{1}{18}\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}$$

Here is our inverse matrix!

- When will this inverse not exist?
- When the determinant equals zero
- This brings up another definition
- A matrix is singular if: $\boxed{\det \underline{A} = |\underline{A}| = 0}$

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There will not always be an inverse matrix. When the determinant equals zero, we cannot find the inverse. This defines a type of matrix called a singular matrix.

Singular Matrix-No Solution

$$\boxed{\underline{A}^{-1} = \left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}} = \frac{1}{18}\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}$$

Here is our inverse matrix!

- When will this inverse not exist?
- When the determinant equals zero
- This brings up another definition
- A matrix is singular if: $\boxed{\det \underline{A} = |\underline{A}| = 0}$
- If the matrix **A** is singular, the inverse does not exist, and we cannot solve the simultaneous equations

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Since the solution to a set of simultaneous equations, **A** **x** = **b**, can be written as **x** = $\underline{A}^{-1}$ **b**, we can see that there will not be a solution if we cannot find the inverse matrix.

Simultaneous Equations-Our Solution

- Now back to the solution to our simultaneous equations: $\underline{x}=\underline{A}^{-1}\,\underline{b}$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} = \underline{A}\,\underline{x} = \underline{b}$$

- Plug in our inverse matrix

$$\underline{x} = \left(\underline{A}^{-1}\right)\underline{b} = \left(\frac{1}{18}\begin{vmatrix} 14 & -4 & -10 \\ -2 & 7 & 4 \\ -4 & 5 & 8 \end{vmatrix}\right)\begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

$$\boxed{\underline{A}^{-1} = \left[\frac{1}{|\underline{A}|}\right]\underline{A}^{A}}$$

- Multiply through and simplify

$$\underline{x} = \frac{1}{18}\begin{vmatrix} -4 \\ 16 \\ 14 \end{vmatrix} = \begin{vmatrix} -\frac{4}{18} \\ \frac{16}{18} \\ \frac{14}{18} \end{vmatrix} = \boxed{\begin{vmatrix} -\frac{2}{9} \\ \frac{8}{9} \\ \frac{7}{9} \end{vmatrix} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}}$$

- We now have our solution:

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Use our inverse matrix to find $\underline{x}$, the solution to the simultaneous equations.

Simultaneous Equations-Check

- To check our solution, we substitute our answer back into the original equations and multiply

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} -\frac{2}{9} \\ \frac{8}{9} \\ \frac{7}{9} \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} \quad \text{so the solution checks}$$

Yippee!
It checks

Our solution appears correct

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We always plug back into the original equations to check our work. Post-multiply ***A*** by the calculated $\underline{x}$ and see if it equals $\underline{b}$. Ours does so we can feel assured that our solution is correct.

Simultaneous Equations-Much Work

- But what is the problem with finding the solution in this way?
- Hey, it's a lot of work!
- The process of finding all the cofactors involves many calculations
- We need a more efficient method for solving systems of equations containing more than than three equations illustrated in this example
- There are many such procedures available
- One of the easiest of most efficient methods is Gaussian Elimination

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For the three simultaneous equations in our example, each of the cofactor terms was the determinant of a 2x2 matrix. This is almost trivial to find. If we have more equations, then we must find determinants of larger matrices to evaluate the cofactor matrix. The amount of work increases by the square of the number of equations and quickly gets out of control. We need a more efficient method.

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \underline{A}\underline{x} = \underline{b}$$

Structural Analysis

Gaussian Elimination

Now we'll examine some basic ideas from Matrix Algebra. We will not need a lot of sophisticated concepts here, but we do need to be able to form and solve sets of simultaneous equations.

Gaussian Elimination Method

- This technique involves what mathematicians refer to as "elementary row transformations "
  - Each row of the square matrix A, and the corresponding term on the right hand side, represents one of the scalar equations
  - If we add, or subtract the same term to both sides of the equal sign, we have not changed the solution to the equations
  - Likewise if we multiply, or divide, both sides by the same value, we do not change the solution
  - The rows, and therefore the equations, can also be rearranged if so desired

Let's demonstrate the Gaussian Elimination method on our equations in our example problem. We will use these "elementary row transformations" to simplify our equations and get them into a form that will be far easier to solve.

Gaussian Elimination-Explanation

•Let's look at our set of simultaneous equations:

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

•The first row of matrix **A** and matrix **b** represent the first equation: $2\,x_1 - 1\,x_2 + 3\,x_3 = 1$

- This equation basically says: 1 = 1

•If we multiply both sides of this equation by (-½), the solution is the same: $-x_1 + \frac{1}{2}\,x_2 - \frac{1}{2}\,(3)\,x_3 = -\frac{1}{2}$

- This equation now really says: - ½ = - ½

•The solution to the equations has not changed

All we have done here is multiply both sides of an equality by $-\frac{1}{2}$. The solution is still the same.

Gaussian Elimination-Explanation

•If we take the resulting equation and add it to equation 3, again we have the same solution because we would be adding (- ½) to each side

•The resulting equation reads:

$$0\,x_1 - \frac{1}{2}\,(5)\,x_2 + \frac{1}{2}\,(7)\,x_3 = \frac{1}{2}$$

•This equation will have the same solution as the original equation, but is obviously simpler since it no longer contains the variable x_1

•We perform this process on the equations written in matrix form

We are just adding the same values to each side of an equality, we are just doing it with the equations. Let's now perform the computations on the numbers written in matrix form.

Gaussian Elimination-The Goal

$$\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

- In the **Gaussian elimination** method we want to get an **upper or a lower triangular matrix** by performing these row transformations
- This means we take one row (i.e. one equation), multiply it by a number and add the resulting equation to one of the other equations to make one of the numbers in **A** go to 0
- Here we will strive to obtain 0's for these numbers to achieve an upper triangular matrix

We want to "uncouple" the equations, by eliminating some of the unknown variables, the x's, from some of the equations. Why did we decide to find an upper triangular matrix for the problem instead of a lower triangular matrix?

Gaussian Elimination-Triangular Matrix

- Here are our equations:

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

- We want to get our triangular matrix
- To make the numbers in column 1 go to 0, we use row 1, i.e. equation 1

In general, to make the numbers in column i go to zero, we use equation (row) i.

Gaussian Elimination-Triangular Matrix

- Here are our equations:

$$\begin{vmatrix} 2 & -1 & 3 \\ \boxed{0} & 4 & -2 \\ 1 & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

- We want to get our triangular matrix
- To make the numbers in column 1 go to 0, we use row 1, i.e. equation 1
- The number in column 1 of equation 2 is already 0, so no work is required here

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This term being equal to zero is why we decided to obtain an upper triangular matrix. Had it not already been equal to zero, we would have had to make it zero.

Gaussian Elimination-Triangular Matrix

- Here are our equations:

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ \boxed{1} & -3 & 5 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

- We want to get our triangular matrix
- To make the numbers in column 1 go to 0, we use row 1, i.e. equation 1
- The number in column 1 of equation 2 is already 0, so no work is required here
- To make the 1 go to 0, we take row 1, multiply it by - (½) and add it to row 3, as we illustrated before

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When we multiply equation 1 by $-\frac{1}{2}$, the first term in that equation becomes -1. When equation 1 is added to equation 3, the number 1 in the first position in equation 3 becomes zero. Remember that this is our goal, to make all terms below the line equal to zero.

Gaussian Elimination-Position 3.1

- Let's operate on the matrices:
- Multiply row 1 by (- ½)
- Add this to row 3

$$\left|\begin{array}{ccc} 2 & -1 & 3 \\ \{-1 & \frac{1}{2} & -3(\frac{1}{2})\} \\ 0 & 4 & -2 \\ 1 & -3 & 5 \\ \{-1 & \frac{1}{2} & -3(\frac{1}{2})\} \end{array}\right| \left|\begin{array}{c} x_1 \\ \\ x_2 \\ x_3 \\ \\ \end{array}\right| = \left|\begin{array}{c} 1 \\ \{-\frac{1}{2}\} \\ 2 \\ 1 \\ \{-\frac{1}{2}\} \end{array}\right|$$

The numbers within the braces are the revised numbers. We add the modified first equation to the third equation. Remember to perform the same operations on the right hand side of the equations.

Gaussian Elimination-Position 3.1

- Let's operate on the matrices:
- Multiply row 1 by (- ½)
- Add this to row 3

$$\left|\begin{array}{ccc} 2 & -1 & 3 \\ \{-1 & \frac{1}{2} & -3(\frac{1}{2})\} \\ 0 & 4 & -2 \\ 1 & -3 & 5 \\ \{-1 & \frac{1}{2} & -3(\frac{1}{2})\} \end{array}\right| \left|\begin{array}{c} x_1 \\ \\ x_2 \\ x_3 \\ \\ \end{array}\right| = \left|\begin{array}{c} 1 \\ \{-\frac{1}{2}\} \\ 2 \\ 1 \\ \{-\frac{1}{2}\} \end{array}\right|$$

- Resulting in:
- We have now created a 0 in position (3,1)
- This is what we wanted to do

$$\left|\begin{array}{ccc} 2 & -1 & 3 \\ 0 & 4 & -2 \\ \boxed{0} & -(\frac{1}{2})5 & (\frac{1}{2})7 \end{array}\right| \left|\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right| = \left|\begin{array}{c} 1 \\ 2 \\ \frac{1}{2} \end{array}\right|$$

Add this modified first equation to equation three and we obtain a 0 in the first column of equation 3. This was our goal.

Gaussian Elimination-Position 3.2

- Now we look at column 2 - We need the -(½)5 to go to 0
- Multiply row 2 by (5/8)
- Add this to row 3

$$\begin{array}{ccc|c|c|c}
2 & -1 & 3 & x_1 & & 1 \\
0 & 4 & -2 & x_2 & = & 2 \\
\{0 & (\tfrac{1}{2})5 & -(\tfrac{1}{4})5\} & & & \{\tfrac{1}{4}\}5 \\
0 & -(\tfrac{1}{2})5 & (\tfrac{1}{2})7 & x_3 & & \tfrac{1}{2} \\
\{0 & (\tfrac{1}{2})5 & -(\tfrac{1}{4})5\} & & & \{\tfrac{1}{4}\}5
\end{array}$$

- Resulting in:

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & \boxed{0} & (\tfrac{1}{4})9 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ \tfrac{1}{2}(7) \end{vmatrix}$$

- We have now created a 0 in position (3,2)
- This is what we wanted to do

The multiplier is equal to the number that we want, $(\frac{1}{2})5$, divided by the number that we have, 4. Which equals 5/8. We now have the zero we sought.

Activity-Gaussian Elimination

- **New Matrix: Try it**
- **Try to get a 0 in position 31**
- We use equation 1 to make 0's in column 1

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ 4 & -3 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$$

- **What number do we use to multiply the first equation so that when we add it to equation 3 we will get a zero?**
- Perform the calculations on the matrix and **find the revised third equation**
- **Revised matrix with a 0 in position 31:**

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -1 \\ ? & ? & ? \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ ? \end{vmatrix}$$

Now you try it. Multiply equation 1 by some number so that when you add the resulting equation to equation 3, you will get a zero in column 1. When you get the answer, go to the next slide and check your answer.

Activity-Gaussian Elimination Solution

- Let's operate on the matrices:
- Multiply row 1 by (-2)
- Add this to row 3

$$\begin{vmatrix} 2 & -1 & 2 \\ \{-4 & 2 & -4\} \\ 1 & 2 & -1 \\ 4 & -3 & 1 \\ \{-4 & 2 & -4\} \end{vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{vmatrix} 1 \\ \{-2\} \\ 2 \\ 1 \\ \{-2\} \end{vmatrix}$$

- Resulting in:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ \boxed{0} & -1 & -3 \end{vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$$

- We have now created a 0 in position (3,1)
- This is what we wanted to do

How'd you do? If you did not get it, go back and review the material to find out what you did wrong. Note that we do not permanently change row 1 in this step. Multiplying row 1 by -2 is just for changing row 3.

Product of the Diagonal Terms

- We now have the desired upper triangular matrix
- What do we get if we multiply the diagonal terms?

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 9/4 \end{vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 7/4 \end{vmatrix}$$

- $2(4)(9/4)=\boxed{18}$
- This is again the determinate of the matrix!
- **Why?**

The last equation has only one unknown, x_3. We can solve for it directly.

Uncoupled Equations-Find x_3

- We now have the desired upper triangular matrix
- The equations are easily uncoupled
- We can solve for x_3 from the last equation

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 9/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7/4 \end{bmatrix}$$

$$\frac{9}{4}x_3 = \frac{7}{4}$$

$$\boxed{x_3 = \frac{7}{9}}$$

The last equation has only one unknown, x_3. We can solve for it directly.

Uncoupled Equations-Find x_2

- We now have the desired upper triangular matrix
- The equations are easily uncoupled
- We can solve for x_3 from the last equation
- Now from the second equation we find x_2

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 9/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7/4 \end{bmatrix}$$

$$\frac{9}{4}x_3 = \frac{7}{4}$$

$$\boxed{x_3 = \frac{7}{9}}$$

$$4x_2 - 2x_3 = 4x_2 - 2\left(\frac{7}{9}\right) = 2$$

$$\boxed{x_2 = \frac{8}{9}}$$

The second equation only contains 2 unknowns, x_3 and x2, and we just evaluated x_3. We can now solve directly for x_2.

Back Substitution-Find x_1

- From the upper triangular matrix we have solved for x_2 and x_3
- We now plug these values into the first equation
- We can solve for the last unknown, x_1
- The entire solution to the equations has been found

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & \frac{9}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{7}{4} \end{bmatrix}$$

$$\boxed{x_2 = \frac{8}{9}} \qquad \boxed{x_3 = \frac{7}{9}}$$

$$2x_1 - 1x_2 + 3x_3 = 1$$

$$2x_1 - 1\left(\frac{8}{9}\right) + 3\left(\frac{7}{9}\right) = 1$$

$$\boxed{x_1 = -\frac{2}{9}}$$

We now have two of the unknowns evaluated. We take these values and substitute them into the first equation to find x_1. We now have all of $\underline{\mathbf{x}}$, the vector of unknowns. This process is called forward elimination, because we start with equation 1 and eliminate numbers in that column, followed by backward substitution, because we solve for the last unknown first, and then move back up to find the other unknowns.

Gaussian Elimination-The Solution

- Our original equations

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

- The solution to the equations

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{9} \\ \frac{8}{9} \\ \frac{7}{9} \end{Bmatrix}$$

- This is, of course, the same solution we obtained using the inverse method
- We check the solution as we did before by substituting back into the equations and multiplying

$\boxed{\text{It checks!}}$

Same solution, of course. Always check to verify that your solution is correct by substituting back into the original equations and multiplying **A** **x** to make sure you get **b**.

Matrix Partitioning

- Matrix partitioning is a useful tool in structural analysis
- In matrix partitioning, a matrix is subdivided, or partitioned, into smaller matrices and then the individual matrices operated on as individual matrices
- It allows different aspects of the structure to be examined, since the submatrices can be chosen to describe particular characteristics of the structure
- Here is a brief example of matrix partitioning

Matrix partitioning is quite important in many areas of structural analysis.

Matrix Partitioning-Example

- Let's say that the first 2 rows and columns of this example symmetric matrix describe an important aspect of a structure

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\left[\begin{array}{cc|ccc} 5 & -3 & 5 & 13 & 7 \\ -3 & 18 & 2 & 8 & 0 \\ \hline 5 & 2 & 10 & 9 & 1 \\ 13 & 8 & 9 & 20 & 3 \\ 7 & 0 & 1 & 3 & 14 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{c} 2 \\ 5 \\ \hline -2 \\ 0 \\ 6 \end{array}\right]$$

- We can partition the matrix to separate the first 2 rows and columns
- Each of these submatrices follows all the rules we have discussed governing matrix operations
- We can even name each submatrix as a new matrix

We have divided, or partitioned, the matrix into 4 smaller matrices.

Matrix Partitioning-Example

- Let's identify the partitioned matrices by the subscripts L and R for left and right

$$\underline{\mathbf{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\left[\begin{array}{cc|ccc} 5 & -3 & 5 & 13 & 7 \\ -3 & 18 & 2 & 8 & 0 \\ \hline 5 & 2 & 10 & 9 & 1 \\ 13 & 8 & 9 & 20 & 3 \\ 7 & 0 & 1 & 3 & 14 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \\ x_5 \end{array}\right] = \left[\begin{array}{c} 2 \\ 5 \\ \hline -2 \\ 0 \\ 6 \end{array}\right]$$

- We can now perform the matrix multiplication using the partitioned matrices

$$\left[\begin{array}{c|c} \underline{\mathbf{A}}_{LL} & \underline{\mathbf{A}}_{LR} \\ \hline \underline{\mathbf{A}}_{RL} & \underline{\mathbf{A}}_{RR} \end{array}\right] \left[\begin{array}{c} \underline{\mathbf{x}}_{L} \\ \hline \underline{\mathbf{x}}_{R} \end{array}\right] = \left[\begin{array}{c} \underline{\mathbf{b}}_{L} \\ \hline \underline{\mathbf{b}}_{R} \end{array}\right]$$

$$\boxed{\begin{aligned} \underline{\mathbf{A}}_{LL}\underline{\mathbf{x}}_{L} + \underline{\mathbf{A}}_{LR}\underline{\mathbf{x}}_{R} &= \underline{\mathbf{b}}_{L} \\ \underline{\mathbf{A}}_{RL}\underline{\mathbf{x}}_{L} + \underline{\mathbf{A}}_{RR}\underline{\mathbf{x}}_{R} &= \underline{\mathbf{b}}_{R} \end{aligned}}$$

- Each partition is a separate matrix itself

Let's name the matrices using subscripts that have basically the same meaning as the subscripts defining the individual elements of the matrix. The partitioned matrices follow the same rules of multiplication as do regular matrices.

These partitioned matrices can now be manipulated to help describe particular aspects of a structure.

The End

Matrix Algebra Section

There is, of course, much more to be learned in matrix algebra, but this is enough to get us started in our analysis of structures.