Time is precious; waste it wisely.

18.01 Single Variable Calculus

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Contents

1 Differentiation

1.1 Definition and Basic Rules

Definition 1. *The derivative $f'(x_0)$ of f at x_0 is the slope of the tangent line to $y = f(x)$ at the point $P = (x_0, f(x_0))$.*

Formula for the derivative:

$$\underbrace{f'(x_0)}_{\text{derivative of } f \text{ at } x_0} = \lim_{x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to 0} \underbrace{\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}_{\text{difference quotient}}$$

Problem 1
Does $f(x) = \lfloor x \rfloor$ have a derivative? If so, what is it? If not, why not?

Solution. The "limit as Δx approaches 0" isn't well defined, so $f(x)$ is not differentiable at $x = 0$. (The left-hand limit and right hand limit are not equal)

Notations

Just as there are many ways to express the same thing, there are many notations for the derivative.

a) $\Delta y = \Delta f$

b) Taking the limit as $\Delta x \to 0$, we get

   (a) $\frac{\Delta y}{\Delta x} \to \frac{\mathrm{d}y}{\mathrm{d}x}$ (Leibniz' notation)

   (b) $\frac{\Delta f}{\Delta x} \to f'(x_0)$ (Newton's notation)

Example 1. *Find the derivative of $f(x) = x^n$ where $n = 1, 2, 3 \ldots$*
Here we have:

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

*The **binomial theorem** tells us that:*

$$x^n + n(\Delta x)x^{n-1} + O\left((\Delta x)^2\right)$$

where $O(\Delta x)^2$ is shorthand for "all of the terms with $(\Delta x)^2$,$(\Delta x)^3$,and so on up to $(\Delta x)^n$ "
Now we have:

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{(x^n + n(\Delta x)(x^{n-1}) + O(\Delta x)^2) - x^n}{\Delta x} = nx^{n-1} + O(\Delta x)$$

and therefore,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Since we think about $\frac{\Delta y}{\Delta x}$ as the average change in y over an interval of size Δx The derivatives $\frac{dy}{dx}$ can also be taken as the instantaneous rate of change.

Definition 2. *The* ***right(left)-hand limit*** *of a function $f(x)$ as x approaches a ,denoted as $\lim_{x\to a^+} f(x)$, represents the value that $f(x)$ approaches as x gets arbitrarily close to a from the right (left) side (i.e., from values greater than a).*

Definition 3. *A function f is* ***continuous*** *at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. Which means:*

* $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$*; both of these one sided limits exist.*
* $f(x_0)$ *is defined.*
* $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.

Definition 4. *Discontinuity*

a) A ***jump discontinuity*** *occurs when the right-hand and left-hand limits exist but are not equal.*

b) At a ***removable discontinuity****, the left-hand and right-hand limits are equal but either the function is not defined or not equal to these limits:*

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \neq f(x_0)$$

c) In an ***infinite discontinuity****, the left- and right-hand limits are infinite.(e.g hyperbola)*

Theorem 1. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. To show that:

$$\lim_{x\to x_0} f(x) - f(x_0) = 0$$

$$\begin{aligned}
\lim_{x\to x_0} f(x) - f(x_0) &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\
&= f'(x_0) \cdot 0 \\
&= 0
\end{aligned}$$

(we used the assumption that f was differentiable when we wrote down $f'(x)$.) □

Derivative of $\sin x$ and $\cos x$, Algebraic Proof

Begin with the definition of the derivative:

$$\frac{d}{dx}\sin x = \lim_{\Delta x\to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

By using $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ we can get:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0} \cos x \left(\frac{\sin \Delta x}{\Delta x}\right)\end{aligned}$$

Here we introduce two important facts: **a)**$\lim_{x\to 0} \frac{\sin x}{x} = 1$ **b)**$\lim_{x\to 0} \frac{\cos x - 1}{x}$. Hence, we conclude:

$$\frac{d}{dx}\sin x = \cos x$$

The calculation of the derivative of $\cos x$ is similar to that of the derivative of $\sin x$. The proof of the two properties above are ommitted here.

Theorem 2. ***Product Rule***

$$(uv)' = u'v + uv'$$

Proof.

$$\begin{aligned}(uv)' &= \lim_{\Delta x \to 0} \frac{(uv)(x+\Delta x) - (uv)(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\left(\frac{u(x+\Delta x) - u(x)}{\Delta x}\right) v(x) + u(x+\Delta x)\left(\frac{v(x+\Delta x) - v(x)}{\Delta x}\right)\right] \\ &= u'(x)v(x) + u(x)v'(x)\end{aligned}$$

□

Theorem 3. ***Quotient Rule***

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Theorem 4. ***Chain Rule:****The derivative of a composition of functions is a product.*

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Notations

Higher derivatives are derivatives of derivatives.

$f'(x)$	Df	$\frac{df}{dx}$	$\frac{d}{dx}f$
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$	$\left(\frac{d}{dx}\right)^2 f$
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$	$\left(\frac{d}{dx}\right)^3 f$
$f^{(n)}(x)$	$D^n f$	$\frac{d^n f}{dx^n}$	$\left(\frac{d}{dx}\right)^n f$

The symbol $\frac{d}{dx}$ represent "operators" which can be applied to a function. This explains why the two powers are in different locations.

1.2 Implicit Differentiation and Inverse Functions

Implicit Differentiation (Rational Exponent Rule)

$$(x^a)' = ax^{a-1}, \forall x \in \mathbb{Q}$$

Example 2. ***Slope of a line tangent to a circle - Direct version***
The graph of $x^2 + y^2 = 1$ is a circle of ridius 1 centered at the origin. This equation can't be written in a form of $y = f(x)$ since every x has two corresponding y values.

$$x^2 + y^2 = 1$$

$$y = \pm\sqrt{1 - x^2}$$

Now we just focus on the top half of the unit circle. By using the chain rule, we can have:

$$\frac{dy}{dx} = \frac{1}{2}u^{-1/2} \cdot (-2x) = -x \cdot (1 - x^2)^{-1/2} = \frac{-x}{\sqrt{1 - x^2}}.$$

Slope of a line tangent to a circle - Implicit version
Instead of solving for y, we could just imply the operator $\frac{d}{dx}$ to both side of the original equation:

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

We get the same answer and it works for both sides of the unit circle. Implicit differentiation simplified this calculation.

Example 3. ***Derivative of the Inverse of a Function***
If $f(x) = y$ and $g(y) = x$, then g is the inverse of f ($g = f^{-1}$) and f is the inverse of g. The graph of f^{-1} is the reflection of the graph of f across the line $y = x$. So we have:

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}.$$

An example of this is the derivative of $y = \arctan(x)$. We can start from its inverse:

$\tan y = x$

$$\begin{aligned}
\tan y &= x \\
\frac{d}{dx}\tan y &= \frac{d}{dx}x \\
\left(\frac{1}{(\cos y)^2}\right)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2(y) \\
\frac{d}{dx}\arctan(x) &= \frac{1}{1+x^2}.
\end{aligned}$$

Derivative of a^x

Proof.

$$\begin{aligned}
a^x = \left(e^{\ln(a)}\right)^x &= e^{x\ln(a)} \\
\frac{d}{dx}e^{(\ln a)x} &= (\ln a)e^{(\ln a)x} \\
\frac{d}{dx}a^x &= (\ln a)a^x
\end{aligned}$$

□

Example 4. ***Derivative of*** x^x *First, let x denote x^x, then we take the natural log of both sides:*

$$\ln v = x\ln x$$

Next, we differentiate both sides of the equation,:

$$\begin{aligned}
(\ln v)' &= \ln x + x\cdot\frac{1}{x} \\
\frac{v'}{v} &= \frac{1}{x}
\end{aligned}$$

Plugging in x^x for v and solving for v', we get:

$$\frac{d}{dx}x^x = x^x(1+\ln x)$$

2 Applications of Differentiation

2.1 Approximation and Curve Sketching

Linear Approximation

Example 5. ***Linear Approximation to lnx at*** $x = 1$
For a given curve $y = f(x)$*, it is approximately the same as its tangent line:*

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

Let $f(x) = lnx$*. Then the formula for linear approximation tells us that:*

$$\begin{aligned} f(x) \approx& \quad f(x_0) + f'(x_0)(x - x_0) \\ ln\ x \approx& \quad \ln(1) + 1(x - 1) \\ ln\ x \approx& \quad 0 + (x - 1) \\ \ln x \approx& \quad (x - 1) \end{aligned}$$

When x *is close to the base point* x_0*, the point of linear approximation is that the curve is approximately the same as the tangent line.*

Example 6. ***Approximations at 0 for Sine, Cosine and Exponential Functions***
Based on the formula $f(x) \approx f(0) + f'(0)x$*, we have:*

a) $\sin x \approx x$ *(if* $x \approx 0$*)*

b) $\cos x \approx 1$ *(if* $x \approx 0$*)*

c) $e^x \approx 1 + x$ *(if* $x \approx 0$*)*

Example 7. ***Approximations at 0 for*** $ln(1 + x)$ ***and*** $(1 + x)^r$
1. $ln(1 + x) \approx x$ *(if* $x \approx 0$*)*
2. $(1 + x)^r \approx 1 + rx$ *(if* $x \approx 0$*)*

Quadratic Approximation

Quadratic approximation is an extension of linear approximation by adding one more term:

$$\boxed{f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Linear Part}} + \underbrace{\frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Quadratic Part}} \quad (x \approx x_0)}$$

According to the equation above, we can calculate the following approximations:

- $\sin x \approx x \quad (\text{if } x \approx 0)$
- $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
- $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
- $\ln(1 + x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$

- $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad$ (if $x \approx 0$)

Problem 2

The linear approximation of $\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}$.

Solution.

$$e^{-3x}(1+x)^{-1/2} \approx \left(1 + (-3x) + \frac{1}{2}(-3x)^2\right)\left(1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\right)$$

$$e^{-3x}(1+x)^{-1/2} \approx 1 - \frac{7}{2}x + \frac{51}{8}x^2$$

Definition 5. *If $f'(x_0) = 0$, we call x_0 a critical point and $y_0 = f(x_0)$ is a critical value of f.*

General Strategy for Graph Sketching

a) **Special points:** discontinuities of f, end points, easy points...

b) **Check $f'(x)$:** critical points

c) **Check $f''(x)$:** concave up or down?

2.2 Optimization, Related Rates and Newton's Method

Problem 3

Find the box (without a top) with least surface area for a fixed volume.[square bottom]

Solution. [**Direct Solution**] Let x denote width and length, y denote height. We have the *constraint* that the box must have a certain volume:

$$y = \frac{V}{x^2}$$

The surface of the box can be written as:

$$A(x) = x^2 + \frac{4V}{x}$$

To find the critical points we take the derivative of $A(x)$ and set it equal to zero.

$$A'(x) = 2x - \frac{4V}{x^2} = 0$$

$$x = (2V)^{\frac{1}{3}}$$

Then we can check end points and obtain the final answer (Here we use dimensionless variables):

$$\frac{x}{y} = 2$$

[Implicit Solution]

$$\frac{d}{dx}V = 2xy + x^2\frac{dy}{dx} \Longrightarrow 0 = 2xy + x^2y'$$

Problem 4

Related Rates, A Conical Tank

Consider a conical tank whose radius at the top is 4 feet and whose depth is 10 feet. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is at depth 5 feet?

Solution. The volume of a cone is $\frac{1}{3}\pi r^2 h$. We have:

$$V = \frac{1}{3} \cdot \underbrace{\pi r^2}_{\text{base}} \cdot \underbrace{h}_{\text{height}}$$

We can use the Chain Rule to find the rate of change of height with respect to time:

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dh}\frac{dh}{dt} \\ &= \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2\frac{dh}{dt} \\ &= \frac{4}{25}\pi h^2 h' \end{aligned}$$

We know that $V' = 2$ and $h = 5$ when we want to find h', so we can plug these values in:

$$\begin{aligned} 2 &= \frac{4}{25}\pi \cdot 5^2 \cdot h' \\ h' &= \frac{1}{2\pi} \end{aligned}$$

Theorem 5. ***Newton's Method***

Newton's method is a way to approximate the roots of a function. It is based on the idea that if x is close to a root of f, then $f(x)$ is close to 0. So we can approximate the root by finding the x-intercept of the tangent line to the graph of f at $(x, f(x))$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The size of the error is proportional to the square of the size of the previous error. Newton's method works well if the initial guess is close to the root.

2.3 Mean Value Theorem, Antiderivatives and Differential Equations

Theorem 6. ***Mean Value Theorem***
If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point c in (a,b) such that:

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

The Mean Value Theorem and Linear Approximation

The linear approximation of f at $x = a$ has the formula:

$$f(x) \approx f(a) + f'(a)(x-a)$$

If we let $\Delta x = x - a$, then we can rewrite this as:

$$\frac{\Delta Y}{\Delta X} \approx f'(a)$$

Similarly, the Mean Value Theorem says that:

$$\exists c \in (a,b) \quad s.t. \quad f(b) = f(a) + f'(c)(b-a)$$

which can be rewritten as:

$$\exists c \in (a,b) \quad s.t. \quad \frac{\Delta Y}{\Delta X} = f'(c)$$

The average change in y over an interval is between the maximum and minimum values of $f'(x)$. During a trip, the average speed of a car is between the maximum and minimum speeds:

$$\min_{a \leq x \leq b} f'(x) \leq \frac{f(b)-f(a)}{b-a} = f'(c) \leq \max_{a \leq x \leq b} f'(x)$$

Definition 6. ***Differential***
The differential of a function $y = f(x)$ is defined as:

$$dy = f'(x)dx$$

$*$Recall the relation between differentials and linear approximation

Definition 7. ***Antiderivative***
$G(x) = \int g(x)dx$ is an antiderivative of $g(x)$. Other ways of writing antiderivatives are:

$$G'(x) = g(x) \quad or \quad dG = g(x)dx$$

Example 8. ***Antiderivative of*** $\frac{1}{x}$

$$\int \frac{1}{x}dx = \int x^{-1}dx$$
$$= \ln|x| + c$$

Antiderivatives are Unique up to a Constanttitle

Theorem 7. *If $F'(x) = f(x)$ and $G'(x) = f(x)$, then* $\boxed{F(x) = G(x) + c}$ *for some constant c.*

Proof. If $F'(x) = G'(x)$, then $(F - G)'(x) = 0$. By the **Mean Value Theorem**, $\exists c$ such that $G(x) - F(x) = c$. So $G(x) = F(x) + c$. □

"This is a very important fact. It's the basis for calculus; the reason why it makes sense to do calculus at all."

Introduction to Ordinary Differential Equations

Example 9. $\frac{dy}{dx} + xy = 0$
The first step to solve it is to separate $\frac{dy}{dx}$*:*

$$\frac{dy}{y} = -xdx$$

Then we integrate both sides:

$$\int \frac{dy}{y} = \int -xdx$$

$$\ln y = -\frac{x^2}{2} + c \quad \textit{assume } y > 0$$
$$y \quad = \quad Ae^{-x^2/2} \quad (A = e^c)$$

This function is known as the normal distribution.

3 The Definite Integral and its Applications

3.1 Definition of the Definite Integral and First Fundamental Theorem

Definition 8. ***Riemann Sum***
Let f be a function defined on $[a, b]$. The general procedure for computing the definite integral $\int_a^b f(x)dx$ is to approximate the area under the curve $y = f(x)$ by the sum of the areas of rectangles:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$.
In the limit as n goes to infinity, this sum approaches the value of the definite integral:

$$\lim_{n\to\infty} S_n = \int_a^b f(x)dx$$

The Fundamental Theorem of Calculus

Theorem 8. ***First Fundamental Theorem of Calculus***
If f is continuous on $[a,b]$ and $F'(x) = f(x)$ for all x in $[a,b]$, then:

$$\boxed{\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b}$$

Problem 5
Area under one "hump" of sin(x).

Solution.

$$\begin{aligned}\int_0^\pi \sin x dx &= -\cos x\Big|_0^\pi \\ &= -\cos\pi + \cos 0 \\ &= 2\end{aligned}$$

An Interpretation of the Fundamental Theorem of Calculus

Speed and distance are very helpful in understanding the definite integral!!!
Let t denote time, $v(t)$ denote velocity, and $s(t)$ denote distance. Then:

$$\int_a^b v(t)dt = s(b) - s(a)$$

It's very reasonable to think that the distance traveled is the area under the velocity curve.

$$\sum_{i=1}^n v(t_i)\Delta t \approx \int_a^b v(t)dt$$

The Riemann sum, on the left, is approximately how far the object travels, and the definite integral, on the right, is the exact distance traveled.

Properties of Integrals

a)
$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

b)
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

c)
$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

d)
$$\int_a^a f(x)dx = 0$$

e)
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

f) (Estimation) If $f(x) \leq g(x)$ and $a < b$, then:
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

g) (Change of Variables or "Substitution") If $u = u(x)$ then $du = u'(x)$ and $\int g(u)du = \int g(u(x))u'(x)dx$.
$$\int_{u_1}^{u_2} g(u)du = \int_{x_1}^{x_2} g(u(x))u'(x)dx$$
where u' does not change sign. (If it does, we need to split the integral into pieces.)

The Fundamental Theorem and the Mean Value Theorem

The **first fundamental theorem** can be weitten as:
$$\Delta F = \int_a^b f(x)dx$$

If we divide both sides by Δx, we get:
$$\frac{\Delta F}{\Delta x} = \underbrace{\frac{1}{b-a}\int_a^b f(x)dx}_{\text{average value of } f(x)}$$

We can rewrite **FTC1** as:
$$\Delta F = Average(F')\Delta x$$

And we rewrite **MVT** as:
$$\Delta F = F'(c)\Delta x \quad \text{(Here } c \text{ is not specific)}$$

Even if we don't know exactly what c is, we can still get:

$$\left(\min_{a<x<b} F'(x)\right) \Delta x \leq \Delta F = F'(c)\Delta x \leq \left(\max_{a<x<b} F'(x)\right) \Delta x.$$

The **FTC1** gives us a little more information:

$$\left(\min_{a<x<b} F'(x)\right) \Delta x \leq \Delta F = \text{Average}F'\Delta x \leq \left(\max_{a<x<b} F'(x)\right) \Delta x.$$

Problem 6

The Mean Value Theorem and Estimation

Given that $F'(x) = \frac{1}{x+1}$ and $F(0) = 1$, the mean value theorem implies that $A < F(4) < B$ for which A and B?

1. **MVT**:

$$F(4) - F(0) = F'(c)(4-0)$$
$$\frac{1}{5} < \frac{F(4) - F(0)}{4} < 1$$
$$\frac{9}{5} < F(4) < 5$$

2. **FTC1**: Based on the **FTC1**, we can get:

$$F(4) - F(0) = \int_0^4 \frac{1}{x+1} dx$$

Based on the graph of $\frac{1}{x+1}$, we can get:

$$F(4) - F(0) = \int_0^4 \frac{dx}{1+x} < \int_0^4 1dx = 4 \text{ and } f(4) - f(0) = \int_0^4 \frac{dx}{1+x} > \int_0^4 \frac{1}{5}dx = \frac{4}{5}$$

Again we can get:

$$\frac{9}{5}(\text{lower Riemann sum}) < F(4) < 5(\text{upper Riemann sum})$$