# 18.01 Single Variable Calculus

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# 1  Differentiation

## 1.1  Definition and Basic Rules

**Definition 1.** *The derivative $f'(x_0)$ of $f$ at $x_0$ is the slope of the tangent line to $y = f(x)$ at the point $P = (x_0, f(x_0))$.*

Formula for the derivative:

$$\underbrace{f'(x_0)}_{\text{derivative of } f \text{ at } x_0} = \lim_{x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to 0} \underbrace{\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}_{\text{difference quotient}}$$

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### Problem 1

Does $f(x) = \lfloor x \rfloor$ have a derivative? If so, what is it? If not, why not?

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**Solution.** The "limit as $\Delta x$ approaches 0" isn't well defined, so $f(x)$ is not differentiable at $x = 0$. (The left-hand limit and right hand limit are not equal)

**Notations**

Just as there are many ways to express the same thing, there are many notations for the derivative.

a) $\Delta y = \Delta f$

b) Taking the limit as $\Delta x \to 0$, we get

    (a) $\frac{\Delta y}{\Delta x} \to \frac{\mathrm{d} y}{\mathrm{d} x}$ (Leibniz' notation)

    (b) $\frac{\Delta f}{\Delta x} \to f'(x_0)$ (Newton's notation)

**Example 1.** *Find the derivative of $f(x) = x^n$ where $n = 1, 2, 3 \ldots$*
*Here we have:*
$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

*The **binomial theorem** tells us that:*

$$x^n + n(\Delta x)x^{n-1} + O\left((\Delta x)^2\right)$$

*where $O(\Delta x)^2$ is shorthand for "all of the terms with $(\Delta x)^2$, $(\Delta x)^3$ ,and so on up to $(\Delta x)^n$"*
*Now we have:*

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{(x^n + n(\Delta x)(x^{n-1}) + O(\Delta x)^2) - x^n}{\Delta x} = nx^{n-1} + O(\Delta x)$$

*and therefore,*

$$\frac{d}{dx}x^n = nx^{n-1}$$

Since we think about $\frac{\Delta y}{\Delta x}$ as the average change in $y$ over an interval of size $\Delta x$ The derivatives $\frac{dy}{dx}$ can also be taken as the instantaneous rate of change.

**Definition 2.** *The **right(left)-hand limit** of a function $f(x)$ as $x$ approaches $a$ ,denoted as $\lim_{x \to a^+} f(x)$, represents the value that $f(x)$ approaches as $x$ gets arbitrarily close to $a$ from the right (left) side (i.e., from values greater than $a$).*

**Definition 3.** *A function $f$ is **continuous** at $x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Which means:*

* $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$; *both of these one sided limits exist.*

* $f(x_0)$ *is defined.*

* $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0)$.

**Definition 4.** *Discontinuity*

a) *A **jump discontinuity** occurs when the right-hand and left-hand limits exist but are not equal.*

b) *At a **removable discontinuity**, the left-hand and right-hand limits are equal but either the function is not defined or not equal to these limits:*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$$

c) *In an **infinite discontinuity**, the left- and right-hand limits are infinite.(e.g hyperbola)*

**Theorem 1.** *If $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$.*

*Proof.* To show that:
$$\lim_{x \to x_0} f(x) - f(x_0) = 0$$

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \\ &= f'(x_0) \cdot 0 \\ &= 0 \end{aligned}$$

（we used the assumption that $f$ was differentiable when we wrote down $f'(x)$.） $\qquad \square$

**Derivative of $\sin x$ and $\cos x$, Algebraic Proof**

Begin with the definition of the derivative:

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

By using $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ we can get:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \sin x \left( \frac{\cos \Delta x - 1}{\Delta x} \right) + \lim_{\Delta x \to 0} \cos x \left( \frac{\sin \Delta x}{\Delta x} \right)$$

Here we introduce two important facts: **a)**$\lim_{x\to 0} \frac{\sin x}{x} = 1$ **b)**$\lim_{x\to 0} \frac{\cos x - 1}{x}$. Hence, we conclude:

$$\frac{d}{dx}\sin x = \cos x$$

The calculation of the derivative of $\cos x$ is similar to that of the derivative of $\sin x$. The proof of the two properties above are ommitted here.

**Theorem 2.** ***Product Rule***

$$(uv)' = u'v + uv'$$

*Proof.*

$$(uv)' = \lim_{\Delta x \to 0} \frac{(uv)(x + \Delta x) - (uv)(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x)v(x) - u(x)v(x) + u(x + \Delta x)v(x + \Delta x) - u(x + \Delta x)v(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[ \left( \frac{u(x + \Delta x) - u(x)}{\Delta x} \right) v(x) + u(x + \Delta x) \left( \frac{v(x + \Delta x) - v(x)}{\Delta x} \right) \right]$$
$$= u'(x)v(x) + u(x)v'(x)$$

$\square$

**Theorem 3.** ***Quotient Rule***

$$\left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}$$

**Theorem 4.** ***Chain Rule:****The derivative of a composition of functions is a product.*

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

**Notations**

Higher derivatives are derivatives of derivatives.

| | | | |
|---|---|---|---|
| $f'(x)$ | $Df$ | $\frac{df}{dx}$ | $\frac{d}{dx}f$ |
| $f''(x)$ | $D^2 f$ | $\frac{d^2 f}{dx^2}$ | $\left(\frac{d}{dx}\right)^2 f$ |
| $f'''(x)$ | $D^3 f$ | $\frac{d^3 f}{dx^3}$ | $\left(\frac{d}{dx}\right)^3 f$ |
| $f^{(n)}(x)$ | $D^n f$ | $\frac{d^n f}{dx^n}$ | $\left(\frac{d}{dx}\right)^n f$ |

The symbol $\frac{d}{dx}$ represent "operators" which can be applied to a function. This explains why the two powers are in different locations.

## 1.2   Implicit Differentiation and Inverse Functions

**Implicit Differentiation (Rational Exponent Rule)**

$$(x^a)' = ax^{a-1}, \forall x \in \mathbb{Q}$$

**Example 2.** *Slope of a line tangent to a circle - Direct version*
*The graph of $x^2 + y^2 = 1$ is a circle of ridius 1 centered at the origin. This equation can't be written in a form of $y = f(x)$ since every x has two corresponding y values.*

$$x^2 + y^2 = 1$$

$$y = \pm\sqrt{1 - x^2}$$

*Now we just focus on the top half of the unit circle. By using the chain rule, we can have:*

$$\frac{dy}{dx} = \frac{1}{2}u^{-1/2} \cdot (-2x) = -x \cdot (1 - x^2)^{-1/2} = \frac{-x}{\sqrt{1 - x^2}}.$$

*Slope of a line tangent to a circle - Implicit version*
*Instead of solving for y, we could just imply the operator $\frac{d}{dx}$ to both side of the original equation:*

$$x^2 + y^2 = 1$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

*We get the same answer and it works for both sides of the unit circle. Implicit differentiation simplified this calculation.*

**Example 3.** *Derivative of the Inverse of a Function*
*If $f(x) = y$ and $g(y) = x$, then g is the inverse of f ($g = f^{-1}$) and f is the inverse of g. The graph of $f^{-1}$ is the reflection of the graph of f across the line $y = x$. So we have:*

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}.$$

*An example of this is the derivative of $y = \arctan(x)$. We can start from its inverse:*

$\tan y = x$

$$\tan y = x$$

$$\frac{d}{dx} \tan y = \frac{d}{dx} x$$

$$\left( \frac{1}{(\cos y)^2} \right) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2(y)$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}.$$

**Derivative of $a^x$**

*Proof.*

$$a^x = \left( e^{\ln(a)} \right)^x = e^{x \ln(a)}$$

$$\frac{d}{dx} e^{(\ln a)x} = (\ln a) e^{(\ln a)x}$$

$$\frac{d}{dx} a^x = (\ln a) a^x$$

$\square$

**Example 4.** ***Derivative of*** $x^x$ *First, let x denote $x^x$, then we take the natural log of both sides:*

$$\ln v = x \ln x$$

*Next, we differentiate both sides of the equation,:*

$$(\ln v)' = \ln x + x \cdot \frac{1}{x}$$

$$\frac{v'}{v} = \frac{1}{x}$$

*Plugging in $x^x$ for v and solving for $v'$, we get:*

$$\frac{d}{dx} x^x = x^x (1 + \ln x)$$

# 2  Applications of Differentiation

## 2.1  Approximation and Curve Sketching

**Linear Approximation**

**Example 5.** *Linear Approximation to lnx at* $x = 1$

*For a given curve $y = f(x)$, it is approximately the same as its tangent line:*

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

*Let $f(x) = lnx$. Then the formula for linear approximation tells us that:*

$$\begin{aligned}
f(x) &\approx & f(x_0) + f'(x_0)(x - x_0) \\
ln\ x &\approx & \ln(1) + 1(x - 1) \\
ln\ x &\approx & 0 + (x - 1) \\
\ln x &\approx & (x - 1)
\end{aligned}$$

*When $x$ is close to the base point $x_0$, the point of linear approximation is that the curve is approximately the same as the tangent line.*

**Example 6.** *Approximations at 0 for Sine, Cosine and Exponential Functions*

*Based on the formula $f(x) \approx f(0) + f'(0)x$, we have:*

a) $\sin x \approx x$ *(if $x \approx 0$)*

b) $\cos x \approx 1$ *(if $x \approx 0$)*

c) $e^x \approx 1 + x$ *(if $x \approx 0$)*

**Example 7.** *Approximations at 0 for* $ln(1 + x)$ *and* $(1 + x)^r$

*1. $ln(1 + x) \approx x$ (if $x \approx 0$)*

*2. $(1 + x)^r \approx 1 + rx$ (if $x \approx 0$)*

**Quadratic Approximation**

Quadratic approximation is an extension of linear approximation by adding one more term:

$$\boxed{f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Linear Part}} + \underbrace{\frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Quadratic Part}} \quad (x \approx x_0)}$$

According to the equation above, we can calculate the following approximations:

- $\sin x \approx x$    (if $x \approx 0$)

- $\cos x \approx 1 - \frac{x^2}{2}$   (if $x \approx 0$)

- $e^x \approx 1 + x + \frac{1}{2}x^2$   (if $x \approx 0$)

- $\ln(1 + x) \approx x - \frac{1}{2}x^2$   (if $x \approx 0$)

- $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2$    (if $x \approx 0$)

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**Problem 2**

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The linear approximation of $\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}$.

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**Solution.**

$$e^{-3x}(1+x)^{-1/2} \approx \left(1 + (-3x) + \frac{1}{2}(-3x)^2\right)\left(1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\right)$$

$$e^{-3x}(1+x)^{-1/2} \approx 1 - \frac{7}{2}x + \frac{51}{8}x^2$$

**Definition 5.** *If $f'(x_0) = 0$, we call $x_0$ a critical point and $y_0 = f(x_0)$ is a critical value of f.*

**General Strategy for Graph Sketching**

a) **Special points:** discontinuities of $f$, end points, easy points...

b) **Check $f'(x)$:** critical points

c) **Check $f''(x)$:** concave up or down?

## 2.2 Optimization, Related Rates and Newton's Method

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**Problem 3**

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Find the box (without a top) with least surface area for a fixed volume.[square bottom]

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**Solution. [Direct Solution]** Let x denote width and length, y denote height. We have the *constraint* that the box must have a certain volume:

$$y = \frac{V}{x^2}$$

The surface of the box can be written as:

$$A(x) \quad = \quad x^2 + \frac{4V}{x}$$

To find the critical points we take the derivative of $A(x)$ and set it equal to zero.

$$A'(x) \quad = \quad 2x - \frac{4V}{x^2} = 0$$
$$x = (2V)^{\frac{1}{3}}$$

Then we can check end points and obtain the final answer (Here we use dimensionless variables):

$$\frac{x}{y} = 2$$

**[Implicit Solution]**

$$\frac{d}{dx}V = 2xy + x^2\frac{dy}{dx} \implies 0 = 2xy + x^2y'$$

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**Problem 4**

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**Related Rates, A Conical Tank**

Consider a conical tank whose radius at the top is 4 feet and whose depth is 10 feet. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is at depth 5 feet?

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**Solution.** The volume of a cone is $\frac{1}{3}\pi r^2 h$. We have:

$$V = \frac{1}{3} \cdot \underbrace{\pi r^2}_{\text{base}} \cdot \underbrace{h}_{\text{height}}$$

We can use the Chain Rule to find the rate of change of height with respect to time:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV}{dh}\frac{dh}{dt} \\
&= \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2\frac{dh}{dt} \\
&= \frac{4}{25}\pi h^2 h'
\end{aligned}$$

We know that $V' = 2$ and $h = 5$ when we want to find $h'$, so we can plug these values in:

$$\begin{aligned}
2 &= \frac{4}{25}\pi \cdot 5^2 \cdot h' \\
h' &= \frac{1}{2\pi}
\end{aligned}$$

**Theorem 5.** *Newton's Method*

*Newton's method is a way to approximate the roots of a function. It is based on the idea that if $x$ is close to a root of $f$, then $f(x)$ is close to 0. So we can approximate the root by finding the x-intercept of the tangent line to the graph of $f$ at $(x, f(x))$.*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

*The size of the error is proportional to the square of the size of the previous error. Newton's method works well if the initial guess is close to the root.*

## 2.3 Mean Value Theorem, Antiderivatives and Differential Equations

**Theorem 6.** *Mean Value Theorem*
*If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a point $c$ in $(a, b)$ such that:*

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

**The Mean Value Theorem and Linear Approximation**

The linear approximation of $f$ at $x = a$ has the formula:

$$f(x) \approx f(a) + f'(a)(x - a)$$

If we let $\Delta x = x - a$, then we can rewrite this as:

$$\frac{\Delta Y}{\Delta X} \approx f'(a)$$

Similarly, the Mean Value Theorem says that:

$$\exists c \in (a, b) \quad s.t. \quad f(b) = f(a) + f'(c)(b - a)$$

which can be rewritten as:

$$\exists c \in (a, b) \quad s.t. \quad \frac{\Delta Y}{\Delta X} = f'(c)$$

The average change in $y$ over an interval is between the maximum and minimum values of $f'(x)$. During a trip, the average speed of a car is between the maximum and minimum speeds:

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} = f'(c) \le \max_{a \le x \le b} f'(x)$$

**Definition 6.** *Differential*
*The differential of a function $y = f(x)$ is defined as:*

$$dy = f'(x)dx$$

$*$Recall the relation between differentials and linear approximation

**Definition 7.** *Antiderivative*
*$G(x) = \int g(x)dx$ is an antiderivative of $g(x)$. Other ways of writing antiderivatives are:*

$$G'(x) = g(x) \quad or \quad dG = g(x)dx$$

**Example 8. *Antiderivative of $\frac{1}{x}$***

$$\int \frac{1}{x} dx = \int x^{-1} dx$$

$$= \ln|x| + c$$

**Antiderivatives are Unique up to a Constanttitle**

**Theorem 7.** *If $F'(x) = f(x)$ and $G'(x) = f(x)$, then $\boxed{F(x) = G(x) + c}$ for some constant $c$.*

*Proof.* If $F'(x) = G'(x)$, then $(F - G)'(x) = 0$. By the **Mean Value Theorem**, $\exists c$ such that $G(x) - F(x) = c$. So $G(x) = F(x) + c$. $\qquad\square$

"This is a very important fact. It's the basis for calculus; the reason why it makes sense to do calculus at all."

**Introduction to Ordinary Differential Equations**

**Example 9.** $\frac{dy}{dx} + xy = 0$
*The first step to solve it is to separate $\frac{dy}{dx}$:*

$$\frac{dy}{y} = -x dx$$

*Then we integrate both sides:*

$$\int \frac{dy}{y} = \int -x dx$$

$$\ln y = -\frac{x^2}{2} + c \quad \text{assume } y > 0$$

$$y \quad = \quad A e^{-x^2/2} \quad (A = e^c)$$

*This function is known as the normal distribution.*

# 3 The Definite Integral and its Applications

## 3.1 Definition of the Definite Integral and First Fundamental Theorem

**Definition 8. *Riemann Sum***
*Let f be a function defined on $[a, b]$. The general procedure for computing the definite integral $\int_a^b f(x) dx$ is to approximate the area under the curve $y = f(x)$ by the sum of the areas of rectangles:*

$$S_n = \sum_{i=1}^{n} f(x_i^*) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^*$ is any point in the ith subinterval $[x_{i-1}, x_i]$.

In the limit as n goes to infinity, this sum approaches the value of the definite integral:

$$\lim_{n \to \infty} S_n = \int_a^b f(x)dx$$

## The Fundamental Theorem of Calculus

**Theorem 8.** *First Fundamental Theorem of Calculus*

*If $f$ is continuous on $[a, b]$ and $F'(x) = f(x)$ for all $x$ in $[a, b]$, then:*

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

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### Problem 5

Area under one "hump" of sin(x).

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**Solution.**

$$\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi$$
$$= -\cos \pi + \cos 0$$
$$= 2$$

## An Interpretation of the Fundamental Theorem of Calculus

**Speed and distance are very helpful in understanding the definite integral!!!**

Let t denote time, $v(t)$ denote velocity, and $s(t)$ denote distance. Then:

$$\int_a^b v(t)dt = s(b) - s(a)$$

It's very reasonable to think that the distance traveled is the area under the velocity curve.

$$\sum_{i=1}^n v(t_i)\Delta t \approx \int_a^b v(t)dt$$

The Riemann sum, on the left, is approximately how far the object travels, and the definite integral, on the right, is the exact distance traveled.

## Properties of Integrals

a)

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

b)
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

c)
$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

d)
$$\int_a^a f(x)dx = 0$$

e)
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

f) (Estimation) If $f(x) \le g(x)$ and $a < b$, then:

$$\int_a^b f(x)dx \le \int_a^b g(x)dx$$

g) (Change of Variables or "Substitution") If $u = u(x)$ then $du = u'(x)$ and $\int g(u)du = \int g(u(x))u'(x)dx$.

$$\int_{u_1}^{u_2} g(u)du = \int_{x_1}^{x_2} g(u(x))u'(x)dx$$

where $u'$ does not change sign. (If it does, we need to split the integral into pieces.)