Time is precious; waste it wisely.

18.01 Single Variable Calculus

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Unit 1: Differentiation

Part A: Definition and Basic Rules

Definition 1. *The derivative $f'(x_0)$ of f at x_0 is the slope of the tangent line to $y = f(x)$ at the point $P = (x_0, f(x_0))$.*

Formula for the derivative:

$$\underbrace{f'(x_0)}_{\text{derivative of } f \text{ at } x_0} = \lim_{x\to 0} \frac{\Delta f}{\Delta x} = \lim_{x\to 0} \underbrace{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}_{\text{difference quotient}}$$

Problem 1
Does $f(x) = \lfloor x \rfloor$ have a derivative? If so, what is it? If not, why not?

Solution. The "limit as Δx approaches 0" isn't well defined, so $f(x)$ is not differentiable at $x = 0$. (The left-hand limit and right hand limit are not equal)

Notations

Just as there are many ways to express the same thing, there are many notations for the derivative.

a) $\Delta y = \Delta f$

b) Taking the limit as $\Delta x \to 0$, we get

   (a) $\frac{\Delta y}{\Delta x} \to \frac{\mathrm{d}y}{\mathrm{d}x}$ (Leibniz' notation)

   (b) $\frac{\Delta f}{\Delta x} \to f'(x_0)$ (Newton's notation)

Example 1. *Find the derivative of $f(x) = x^n$ where $n = 1, 2, 3 \ldots$*
Here we have:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

*The **binomial theorem** tells us that:*

$$x^n + n(\Delta x)x^{n-1} + O\left((\Delta x)^2\right)$$

where $O(\Delta x)^2$ is shorthand for "all of the terms with $(\Delta x)^2$,$(\Delta x)^3$,and so on up to $(\Delta x)^n$"

Now we have:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x} = \frac{(x^n + n(\Delta x)(x^{n-1}) + O(\Delta x)^2) - x^n}{\Delta x} = nx^{n-1} + O(\Delta x)$$

and therefore,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Since we think about $\frac{\Delta y}{\Delta x}$ as the average change in y over an interval of size Δx The derivatives $\frac{dy}{dx}$ can also be taken as the instantaneous rate of change.

Definition 2. *The* ***right(left)-hand limit*** *of a function $f(x)$ as x approaches a ,denoted as $\lim_{x\to a^+} f(x)$, represents the value that $f(x)$ approaches as x gets arbitrarily close to a from the right (left) side (i.e., from values greater than a).*

Definition 3. *A function f is* ***continuous*** *at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. Which means:*

* $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$*; both of these one sided limits exist.*
* $f(x_0)$ *is defined.*
* $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x_0)$.

Definition 4. *Discontinuity*

a) *A* ***jump discontinuity*** *occurs when the right-hand and left-hand limits exist but are not equal.*

b) *At a* ***removable discontinuity****, the left-hand and right-hand limits are equal but either the function is not defined or not equal to these limits:*

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \neq f(x_0)$$

c) *In an* ***infinite discontinuity****, the left- and right-hand limits are infinite.(e.g hyperbola)*

Theorem 1. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. To show that:

$$\lim_{x\to x_0} f(x) - f(x_0) = 0$$

$$\begin{aligned}\lim_{x\to x_0} f(x) - f(x_0) &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)\\ &= f'(x_0)\cdot 0\\ &= 0\end{aligned}$$

(we used the assumption that f was differentiable when we wrote down $f'(x)$.) □

Derivative of $\sin x$ and $\cos x$, Algebraic Proof

Begin with the definition of the derivative:

$$\frac{d}{dx}\sin x = \lim_{\Delta x\to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

By using $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ we can get:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0} \frac{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0} \sin x\left(\frac{\cos\Delta x - 1}{\Delta x}\right) + \lim_{\Delta x\to 0}\cos x\left(\frac{\sin\Delta x}{\Delta x}\right)\end{aligned}$$

Here we introduce two important facts: **a)**$\lim_{x\to 0}\frac{\sin x}{x} = 1$ **b)**$\lim_{x\to 0}\frac{\cos x - 1}{x}$. Hence, we conclude:

$$\frac{d}{dx}\sin x = \cos x$$

The calculation of the derivative of $\cos x$ is similar to that of the derivative of $\sin x$. The proof of the two properties above are ommitted here.

Theorem 2. ***Product Rule***

$$(uv)' = u'v + uv'$$

Proof.

$$\begin{aligned}(uv)' &= \lim_{\Delta x\to 0}\frac{(uv)(x+\Delta x)-(uv)(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x+\Delta x)-u(x)v(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{u(x+\Delta x)v(x)-u(x)v(x)+u(x+\Delta x)v(x+\Delta x)-u(x+\Delta x)v(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left[\left(\frac{u(x+\Delta x)-u(x)}{\Delta x}\right)v(x)+u(x+\Delta x)\left(\frac{v(x+\Delta x)-v(x)}{\Delta x}\right)\right]\\ &= u'(x)v(x)+u(x)v'(x)\end{aligned}$$

□

Theorem 3. ***Quotient Rule***

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Theorem 4. ***Chain Rule***:*The derivative of a composition of functions is a product.*

$$\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Notations

Higher derivatives are derivatives of derivatives.

$f'(x)$	Df	$\frac{df}{dx}$	$\frac{d}{dx}f$
$f''(x)$	D^2f	$\frac{d^2f}{dx^2}$	$\left(\frac{d}{dx}\right)^2 f$
$f'''(x)$	D^3f	$\frac{d^3f}{dx^3}$	$\left(\frac{d}{dx}\right)^3 f$
$f^{(n)}(x)$	D^nf	$\frac{d^nf}{dx^n}$	$\left(\frac{d}{dx}\right)^n f$

The symbol $\frac{d}{dx}$ represent “operators” which can be applied to a function. This explains why the two powers are in different locations.

Part B: Implicit Differentiation and Inverse Functions

Implicit Differentiation (Rational Exponent Rule)

$$(x^a)' = ax^{a-1}, \forall x \in \mathbb{Q}$$

Example 2. ***Slope of a line tangent to a circle - Direct version***
The graph of $x^2 + y^2 = 1$ *is a circle of ridius 1 centered at the origin. This equation can't be written in a form of* $y = f(x)$ *since every* x *has two corresponding* y *values.*

$$x^2 + y^2 = 1$$

$$y = \pm\sqrt{1 - x^2}$$

Now we just focus on the top half of the unit circle. By using the chain rule, we can have:

$$\frac{dy}{dx} = \frac{1}{2}u^{-1/2} \cdot (-2x) = -x \cdot (1 - x^2)^{-1/2} = \frac{-x}{\sqrt{1 - x^2}}.$$

Slope of a line tangent to a circle - Implicit version
Instead of solving for y*, we could just imply the operator* $\frac{d}{dx}$ *to both side of the original equation:*

$$\begin{aligned} x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

We get the same answer and it works for both sides of the unit circle. Implicit differentiation simplified this calculation.

Example 3. ***Derivative of the Inverse of a Function***
If $f(x) = y$ *and* $g(y) = x$*, then* g *is the inverse of* f $(g = f^{-1})$ *and* f *is the inverse of* g*. The graph of* f^{-1} *is the reflection of the graph of* f *across the line* $y = x$*. So we have:*

$$\frac{d}{dy}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}.$$

An example of this is the derivative of $y = \arctan(x)$*. We can start from its inverse:* $\tan y = x$

$$\begin{aligned} \tan y &= x \\ \frac{d}{dx}\tan y &= \frac{d}{dx}x \\ \left(\frac{1}{(\cos y)^2}\right)\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2(y) \\ \frac{d}{dx}\arctan(x) &= \frac{1}{1 + x^2}. \end{aligned}$$

Derivative of a^x

Proof.

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}$$
$$\frac{d}{dx}e^{(\ln a)x} = (\ln a)e^{(\ln a)x}$$
$$\frac{d}{dx}a^x = (\ln a)a^x$$

□

Example 4. ***Derivative of*** x^x *First, let* x *denote* x^x*, then we take the natural log of both sides:*

$$\ln v = x\ln x$$

Next, we differentiate both sides of the equation,:

$$(\ln v)' = \ln x + x\cdot\frac{1}{x}$$
$$\frac{v'}{v} = \frac{1}{x}$$

Plugging in x^x *for* v *and solving for* v'*, we get:*

$$\frac{d}{dx}x^x = x^x(1+\ln x)$$

Unit 2: Applications of Differentiation

Part A: Approximation and Curve Sketching

Linear Approximation

Example 5. ***Linear Approximation to lnx at*** $x = 1$
For a given curve $y = f(x)$*, it is approximately the same as its tangent line:*

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$$

Let $f(x) = lnx$*. Then the formula for linear approximation tells us that:*

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$
$$ln\ x \approx \ln(1) + 1(x-1)$$
$$ln\ x \approx 0 + (x-1)$$
$$\ln x \approx (x-1)$$

When x *is close to the base point* x_0*, the point of linear approximation is that the curve is approximately the same as the tangent line.*

Example 6. ***Approximations at 0 for Sine, Cosine and Exponential Functions*** *Based on the formula $f(x) \approx f(0) + f'(0)x$, we have:*

a) $\sin x \approx x$ *(if $x \approx 0$)*

b) $\cos x \approx 1$ *(if $x \approx 0$)*

c) $e^x \approx 1 + x$ *(if $x \approx 0$)*

Example 7. ***Approximations at 0 for*** $ln(1+x)$ ***and*** $(1+x)^r$
1. $ln(1+x) \approx x$ *(if $x \approx 0$)*
2. $(1+x)^r \approx 1 + rx$ *(if $x \approx 0$)*

Quadratic Approximation

Quadratic approximation is an extension of linear approximation by adding one more term:

$$\boxed{f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Linear Part}} + \underbrace{\frac{f''(x_0)}{2}(x - x_0)^2}_{\text{Quadratic Part}} \quad (x \approx x_0)}$$

According to the equation above, we can calculate the following approximations:

- $\sin x \approx x \quad (\text{if } x \approx 0)$
- $\cos x \approx 1 - \frac{x^2}{2} \quad (\text{if } x \approx 0)$
- $e^x \approx 1 + x + \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
- $\ln(1+x) \approx x - \frac{1}{2}x^2 \quad (\text{if } x \approx 0)$
- $(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2 \quad (\text{if } x \approx 0)$

Problem 2

The linear approximation of $\frac{e^{-3x}}{\sqrt{1+x}} = e^{-3x}(1+x)^{-1/2}$.

Solution.

$$e^{-3x}(1+x)^{-1/2} \approx \left(1 + (-3x) + \frac{1}{2}(-3x)^2\right)\left(1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2\right)$$

$$e^{-3x}(1+x)^{-1/2} \approx 1 - \frac{7}{2}x + \frac{51}{8}x^2$$

Definition 5. *If $f'(x_0) = 0$, we call x_0 a critical point and $y_0 = f(x_0)$ is a critical value of f.*

General Strategy for Graph Sketching

a) **Special points:** discontinuities of f, end points, easy points...

b) **Check** $f'(x)$**:** critical points

c) **Check** $f''(x)$**:** concave up or down?