Delta Functors

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1 Abelian Categories and Exact Functors

Def: A category $\mathfrak{U}$ is called an ***Ab**-category* if every $\mathrm{Hom}(A, B)$ has the structure of an abelian group such that composition is bilinear $f \circ (g + h) \circ e = f \circ g \circ e + f \circ h \circ e$.

Def: An object $A \in \mathfrak{U}$ is initial, respectively terminal, if for any object $B \in \mathfrak{U}$, there is exactly one morphism $A \to B$, respectively $B \to A$.

Prop 1.1

$A \in \mathfrak{U}$ is initial iff $A \in \mathfrak{U}^{op}$ is terminal.

Proof. If A is initial, then for any object $B \in \mathfrak{U}^{op}$, there is a unique morphism $f \in \mathrm{Hom}_{\mathfrak{U}}(A, B)$ and $\mathrm{Hom}_{\mathfrak{U}^{op}}(B, A) = \mathrm{Hom}_{\mathfrak{U}}(A, B)$, therefore there is a unique $f \in \mathrm{Hom}_{\mathfrak{U}^{op}}(B, A)$, so A is terminal in $\mathfrak{U}^{op}$. Conversely, if $A \in \mathfrak{U}^{op}$ is terminal, then for any $B \in \mathfrak{U}$, there is a unique $f \in \mathrm{Hom}_{\mathfrak{U}^{op}}(B, A) = \mathrm{Hom}_{\mathfrak{U}}(A, B)$, so A is initial in $\mathfrak{U}$. □

Prop 1.2

Initial and terminal objects are unique up to unique isomorphism.

Proof. Suppose the statement is true for initial objects, then let $A, B \in \mathfrak{U}$ be terminal objects. Since $(\mathfrak{U}^{op})^{op} = \mathfrak{U}$ by definition, then by 1.1, A, B are initial objects in $\mathfrak{U}^{op}$, therefore there exists a unique isomorphism $f \in \mathrm{Hom}_{\mathfrak{U}^{op}}(A, B) = \mathrm{Hom}_{\mathfrak{U}}(B, A)$. Therefore f is a unique isomorphism $B \to A$, so the statement holds for terminal objects.

Let $A, B \in \mathfrak{U}$ be initial objects. If there is some $f : A \to B$ an isomorphism, then it is necessarily unique since A is initial, so $\mathrm{Hom}(A, B)$ has only one element, which would be f. Therefore we need only show that there is such an isomorphism. We know that $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(B, A)$ each contain exactly one element since A and B are initial, let them be f and g respectively. Then $f \circ g in \mathrm{Hom}(A, A)$. Since A is initial, then there is exactly one element in $\mathrm{Hom}(A, A)$, but we know that $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, so $\mathrm{Hom}(A, A) = \{\mathrm{id}_A\}$. Since $f \circ g \in \mathrm{Hom}(A, A)$, then $f \circ g = \mathrm{id}_A$. Similarly, $g \circ f \in \mathrm{Hom}(B, B)$ and since B is initial, then we also have that $\mathrm{Hom}(B, B) = \{\mathrm{id}_B\}$, therefore $g \circ f = \mathrm{id}_B$, so f is an isomorphism as desired. □

> **Def:** An object $A \in \mathfrak{U}$ is called a *zero object* if A is both an initial and terminal object.

Example: The trivial group 0 in **Ab** is a zero object. For any abelian group G, we must send the identity to the identity. The only element in the 0 group is the identity, so there is at most one morphism from 0 to any abelian group. The morphism $f : 0 \to G$ given by $f(0) = 0$ is a homomorphism since $f(0+0) = f(0) = 0 = 0 + 0 = f(0) + f(0)$. Therefore 0 is initial. To see that 0 is terminal, since 0 has only one element, then there is at most one map from any group into 0, namely the map $f : G \to 0$ where $f(g) = 0$ for all $g \in G$. This map is a homomorphism since for any $a, b \in G$, $f(a+b) = 0 = 0 + 0 = f(a) + f(b)$. Therefore 0 is the zero object in **Ab**. Since both initial and terminal objects are unique up to unique isomorphism, then so too are zero objects.

> **Def:** For any $A, B \in \mathfrak{U}$, we call C the product of A and B if there are morphisms $\mathrm{pr}_A, \mathrm{pr}_B$ from C to A and B respectively such that for any object D with morphisms f, g from D to A and B respectively, then there exists a unique morphism $h : D \to C$ such that $\mathrm{pr}_A \circ h = f$ and $\mathrm{pr}_B \circ h = g$.

We can understand the above definition by the following diagram:

$$\begin{tikzcd} & D \arrow[dl, "f"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "g"] & \\ A & C \arrow[l, "\mathrm{pr}_A"] \arrow[r, "\mathrm{pr}_B"'] & B \end{tikzcd}$$

The commutativity of this diagram says that all paths from one vertex to another compose to give the same morphism. In this case, it is exactly the statement that $\mathrm{pr}_A \circ h = f$ and $\mathrm{pr}_B \circ h = g$. Therefore this diagram encodes the notion that C is the product of A and B. Alternatively, this says that a product is the limit of a diagram with exactly two points (I have not defined a limit, and I will continue to not do so).

If C is the product of A and B, we write $C = A \times B$. This makes sense assuming that the product is unique up to isomorphism, otherwise $A \times B$ would not say which product it is referring to.

> **Prop 1.3**
>
> Products, if they exist, are unique up to isomorphism.

Proof. Let $A, B \in \mathfrak{U}$ and suppose that $C, D \in \mathfrak{U}$ are both products of A and B. Let $\mathrm{pr}_A^C, \mathrm{pr}_B^C$ be the projection morphisms of C and $\mathrm{pr}_A^D, \mathrm{pr}_B^D$ be those for D. Since D has morphisms to A and B, then by definition of a product, there exists a unique morphism $f : D \to C$ such that $\mathrm{pr}_A^C \circ f = \mathrm{pr}_A^D$ and $\mathrm{pr}_B^C \circ f = \mathrm{pr}_B^D$. Similarly, since C has morphisms to A and B, then since D is a product, there is a unique morphism $g : C \to D$ such that $\mathrm{pr}_A^D \circ g = \mathrm{pr}_A^C$ and $\mathrm{pr}_B^D \circ g = \mathrm{pr}_B^C$.

We want to show that $f \circ g = \mathrm{id}_C$ and that $g \circ f = \mathrm{id}_D$. By relabelling C as D and D as C, we in fact need only show that $f \circ g = \mathrm{id}_C$. Since C has morphisms to A and B, then by definition of a product, there is a unique morphism $h : C \to C$ such that $\mathrm{pr}_A^C \circ h = \mathrm{pr}_A^C$ and $\mathrm{pr}_B^C \circ h = \mathrm{pr}_B^C$, but notice that the identity id_C satisfies both these conditions. Therefore the only morphism $h : C \to C$

preserving the projection maps is the identity. Now notice that $f \circ g$ is a morphism from $C \to C$ and:

$$\mathrm{pr}_A^C \circ f \circ g = (\mathrm{pr}_A^C \circ f) \circ g = \mathrm{pr}_A^D \circ g = \mathrm{pr}_A^C$$
$$\mathrm{pr}_B^C \circ f \circ g = (\mathrm{pr}_B^C \circ f) \circ g = \mathrm{pr}_B^D \circ g = \mathrm{pr}_B^C$$

Above, we said that the only such morphism preserving the projections is the identity, therefore $f \circ g = \mathrm{id}_C$. It follows that $g : C \to D$ is an isomorphism. □

Note that products are not unique up to unique isomorphism. For instance, in the category **Set**, two sets are isomorphic iff they have the same cardinality. Now notice that for the sets $A = \{0, 1\}, B = \{2\}$, there is the product $A \times B = \{(0, 2), (1, 2)\}$ where the projection maps send (a, b) to a and to b respectively. Notice however that we can take $A \times B = \{a, b\}$ with projections pr_A sending a to 0 and b to 1 and pr_B sending both a and b to 2. The sets $\{a, b\}$ and $\{(0, 2), (1, 2)\}$ are not the same sets, however there is an isomorphism given by sending $a \mapsto (0, 2)$ and $b \mapsto (1, 2)$. However, there is another isomorphism sending $a \mapsto (1, 2)$ and $b \mapsto (0, 2)$. We see that there is not a unique isomorphism between two products.

If we ask that the isomorphism also respect the projection maps, then there is a unique isomorphism. That is to say that the product $A \times B$ is unique up to unique isomorphism over the category of objects in $\mathfrak{U}$ over A and B (here I have not and will continue not to define comma categories).

Prop 1.4

Let $A, B \in \mathfrak{U}$. Suppose $A \times B$ exists, then $B \times A$ exists and $A \times B \cong B \times A$.

Proof. Let $\mathrm{pr}_A : A \times B \to A$ and $\mathrm{pr}_B : A \times B \to B$ be the projections. For any object C with morphisms $f : C \to B$ and $g : C \to A$, then g, f are morphisms C to A, B respectively, so there is a unique morphism $h : C \to A \times B$ such that $\mathrm{pr}_A \circ h = g$ and $\mathrm{pr}_B \circ h = f$. This is exactly the statement that $A \times B$ is the product of B and A. Therefore $B \times A$ exists and since products are unique up to isomorphism, then regardless of which product we choose for $B \times A$, since $A \times B$ is a product of B and A, then $A \times B \cong B \times A$. □

Prop 1.5

Let $A, B, C \in \mathfrak{U}$ and suppose that $A \times B$, $B \times C$, and $(A \times B) \times C$ exist, then $A \times (B \times C)$ exists and $(A \times B) \times C \cong A \times (B \times C)$.

Proof. As in the previous proof, the isomorphism is automatic as long as we can show that $A \times (B \times C)$ is the product of A and $B \times C$. Let $\mathrm{pr}_{A \times B}$ and pr_C be the projection maps from $(A \times B) \times C$. Let pr_A and pr_B be the projection maps from $A \times B$. Let $\mathrm{pr}'_B, \mathrm{pr}'_C$ be the projections from $B \times C$. Let $D \in \mathfrak{U}$ and $f : D \to A$ and $g : D \to B \times C$. We then get morphisms $\mathrm{pr}'_B \circ g : D \to B$ and $\mathrm{pr}'_C \circ g : D \to C$. By definition of the product $A \times B$, there exists a unique morphism $h : D \to A \times B$ such that $\mathrm{pr}_A \circ h = f$ and $\mathrm{pr}_B \circ h = \mathrm{pr}'_B \circ g$. It follows by the definition of $(A \times B) \times C$ that there exists a unique morphism $k : D \to (A \times B) \times C$ such that $\mathrm{pr}_{A \times B} \circ k = h$ and that $\mathrm{pr}_C \circ k = \mathrm{pr}'_C \circ g$.

We now need to say what the projection morphisms from $(A \times B) \times C$ to A and $B \times C$ are, then check that they compose with k to give back f and g. We have the morphism $\mathrm{pr}''_A = \mathrm{pr}_A \circ \mathrm{pr}_{A \times B}$ which maps to A, then we have two morphisms $\mathrm{pr}_B \circ \mathrm{pr}_{A \times B}$ and pr_C, then by definition of the product, we get a morphism $\mathrm{pr}''_{B \times C} : (A \times B) \times C \to B \times C$. Now $\mathrm{pr}''_A \circ k = \mathrm{pr}_A \circ (\mathrm{pr}_{A \times B} \circ k) = \mathrm{pr}_A \circ h = f$ as desired.

We want to show that $\mathrm{pr}''_{B\times C} \circ k = g$. We don't really know that $\mathrm{pr}''_{B\times C}$ is, however we know that $\mathrm{pr}''_{B\times C} \circ k$ maps from D to $B \times C$, therefore by the uniqueness of morphisms to a product, if we can show that $\mathrm{pr}'_B \circ \mathrm{pr}''_{B\times C} \circ k = \mathrm{pr}'_B \circ g$ and $\mathrm{pr}'_C \circ \mathrm{pr}''_{B\times C} \circ k = \mathrm{pr}'_C \circ g$. Now by definition of $\mathrm{pr}''_{B\times C}$, we know that $\mathrm{pr}'_B \circ \mathrm{pr}''_{B\times C} = \mathrm{pr}_B \circ \mathrm{pr}_{A\times B}$ and $\mathrm{pr}'_C \circ \mathrm{pr}''_{B\times C} = \mathrm{pr}_C$. Therefore we need to check that $\mathrm{pr}_B \circ \mathrm{pr}_{A\times B} \circ k = \mathrm{pr}'_B \circ g$ and that $\mathrm{pr}_C \circ k = \mathrm{pr}'_C \circ g$. The latter of the two follows from the definition of k. For the prior, we know that $\mathrm{pr}_{A\times B} \circ k = h$ and that $\mathrm{pr}_B \circ h = \mathrm{pr}'_B \circ g$ as desired. It follows that $(A \times B) \times C$ is the product of A and $B \times C$ as desired. $\square$

From the above two propositions, we see that the order and bracketing of products does not matter.

Prop 1.6

Let $\{A_i\}_{i=1}^n$ be a finite collection of objects. Suppose that $A_1 \times \cdots \times A_n$ exists. Then there exists morphisms $\mathrm{pr}_i : A_1 \times \cdots A_n \to A_i$ for each i. Furthermore, given morphisms $f_i : D \to A_i$ for each i, there exists a unique morphism $f : D \to A_1 \times \cdots \times A_n$ such that $\mathrm{pr}_i \circ f = f_i$.

Proof. We proceed by induction on n. For $n = 2$, this is the definition of a product of two objects. Suppose that it is true for $n-1$, then we know that bracketing and order of products is irrelevant, so we may write out product as $(A_1 \times \cdots A_{n-1}) \times A_n$. By definition of a product, there are two projection maps pr and pr_n to $A_1 \times \cdots \times A_{n-1}$ and A_n respectively. By induction, there are projection maps $\mathrm{pr}'_i : A_1 \times \cdots \times A_{n-1} \to A_i$. Let $\mathrm{pr}_i = \mathrm{pr}'_i \circ \mathrm{pr}$. Let $f_i : D \to A_i$ be morphisms, then by induction, there is a unique morphism $f' : D \to A_1 \times \cdots \times A_{n-1}$ such that $\mathrm{pr}'_i \circ f' = f_i$. By definition of a product, there is a unique morphism $f : D \to (A_1 \times \cdots \times A_{n-1}) \times A_n$ such that $\mathrm{pr}' \circ f = f'$ and $\mathrm{pr}_n \circ f = f_n$. Then we see that $\mathrm{pr}_i \circ f = \mathrm{pr}'_i \circ \mathrm{pr}' \circ f = \mathrm{pr}'_i \circ f' = f_i$ for all $i < n$ and for $i = n$, we already have that $\mathrm{pr}_n \circ f = f_n$. Therefore f is exactly the unique morphism which we wanted. $\square$

The above proposition states that a product of finitely many objects is the same as many products of two objects.

Def: A category $\mathfrak{U}$ is called an *additive category* if it is an **Ab**-category with a zero object and for all $A, B \in \mathfrak{U}$, the product $A \times B$ exists in $\mathfrak{U}$.

Def: Let $\mathfrak{U}$ be a category with a zero object 0, then for any $A, B \in \mathfrak{U}$, the morphism given by the composition $A \to 0$ and $0 \to B$ is the morphism $0_{AB} : A \to B$ called the *zero morphism*.

There is a more general definition of a collection of zero morphisms in a category. However, here we will really only care about abelian categories where these morphisms defined above gives a canonical collection of zero morphisms.

Prop 1.7

Let $A, B \in \mathfrak{U}$, then for any $f : C \to A$ we have that $0_{AB} \circ f = 0_{CB}$ and for any $g : B \to C$ we have that $g \circ 0_{AB} = 0_{AC}$.

Proof. We have unique morphisms $\varphi_A : A \to 0$ and $\psi_B : 0 \to B$, then $0_{AB} = \psi_B \circ \varphi_A$. Now there is a unique morphism $\varphi_C : C \to 0$, but we have that $\varphi_A \circ f$ is a morphism from $C \to 0$, therefore $\varphi_A \circ f = \varphi_C$. It follows that $0_{AB} \circ f = \psi_B \circ \varphi_A \circ f = \psi_B \circ \varphi_C = 0_{CB}$. Similarly, we have a unique morphism $\psi_C : 0 \to C$, then $g \circ \psi_B$ is a morphism $0 \to C$, therefore $g \circ \psi_B = \psi_C$. It follows that $g \circ 0_{AB} = g \circ \psi_B \circ \varphi_A = \psi_C \circ \varphi_A = 0_{AC}$ as desired. $\square$

Def: Let $\mathfrak{U}$ be a category with a zero object, then for any morphisms $f : A \to B$, the *kernel* of f is an object K with a morphism $k : K \to A$ such that $f \circ k = 0_{AB}$ and for any other object K' with a morphism $k' : K \to A$ such that $f \circ k' = 0_{AB}$, then there is a unique morphism $u : K' \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
 & K \arrow[u, "k"] \arrow[r, "0_{KB}"'] & B \\
K' \arrow[uur, "k'"] \arrow[ur, dashed, "\exists! u"] \arrow[urr, "0_{K'B}"'] & &
\end{tikzcd}$$

Def: Let $\mathfrak{U}$ be a category with a zero object, then for any morphisms $f : A \to B$, the *cokernel* of f is an object Q with a morphism $q : B \to Q$ such that $q \circ f = 0_{AQ}$ and for any other object Q' with a morphism $q' : B \to Q'$ such that $q' \circ f = 0_{AQ'}$, then there is a unique morphism $u : Q \to Q'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & B \arrow[d, "q"'] \arrow[ddr, "q'"] & \\
A \arrow[ur, "f"] \arrow[r, "0_{AQ}"'] \arrow[drr, "0_{AQ'}"'] & Q \arrow[dr, dashed, "\exists! u"] & \\
 & & Q'
\end{tikzcd}$$

Note that the compositions with a zero morphism in them will always commute since zero morphisms compose with other morphisms according to 1.7. Notice that if $\mathfrak{U}$ is a category with a zero object, then the zero object of $\mathfrak{U}$ is also a zero object in $\mathfrak{U}^{op}$ since it is still initial and terminal by 1.1. Reversing all of the arrows in the definition of the cokernel, we see that the cokernel is in fact just the kernel of f in $\mathfrak{U}^{op}$. This is usually true of objects whose names start with "co".

Prop 1.8

Let $f : A \to B$ be a morphism, then the kernel, if it exists, is unique up to isomorphism

Proof. Let K, K' be two kernels with morphisms k, k', then by definition of kernel, there exist morphisms $u_1 : K \to K'$ and $u_2 : K' \to K$ such that $k' \circ u_1 = k$ and $k \circ u_2 = k'$. We know that there is a unique morphism $h : K \to K$ such that $k \circ h = k$. This morphism h is exactly id_K. We want to show that $u_2 \circ u_1 = \mathrm{id}_K$. The other composition follows by swapping the roles of K and K'. To see that $u_2 \circ u_1 = \mathrm{id}_K$, we just need to check that $k \circ u_2 \circ u_1 = k$, since then we know by the above uniqueness that this will force $u_2 \circ u_1 = \mathrm{id}_K$. We know that $(k \circ u_2) \circ u_1 = k' \circ u_1 = k$ as desired. It follows that the composition is the identity, so $K \cong K'$. ☐

Prop 1.9

Let $f : A \to B$ be a morphism, then the cokernel, if it exists, is unique up to isomorphism

Proof. Let Q, Q' be two cokernels with morphisms q, q'. By definition of the cokernel, there exists morphisms $u_1 : Q' \to Q$ and $u_2 : Q \to Q'$ such that $u_1 \circ q' = q$ and $u_2 \circ q = q'$. There is a unique morphism $h : Q \to Q$ such that $u \circ q = q$. This morphism is just id_Q, therefore to show that $u_1 \circ u_2 = \mathrm{id}_Q$ we just need to show that $u_1 \circ u_2 \circ q = q$, but we know that $u_1 \circ (u_2 \circ q) = u_1 \circ q' = q$, so indeed $u_1 \circ u_2 = \mathrm{id}_Q$. The other composition $u_2 \circ u_1 = \mathrm{id}_{Q'}$ follows by relabelling Q as Q' and vice versa. Therefore $Q \cong Q'$ ☐

Def: A morphism $f : A \to B$ is a *monomorphism* if for $g, h : C \to A$ such that $f \circ g = f \circ h$, then $g = h$.

Def: A morphism $f : A \to B$ is a *epimorphism* if for $g, h : B \to C$ such that $g \circ f = h \circ f$, then $g = h$.

Def: Let $A, B \in \mathfrak{U}$. An object C is called the *coproduct* of A and B if C is the product of A and B in $\mathfrak{U}^{op}$.

Since we know that products are unique up to isomorphism, then an two coproducts of A and B are isomorphic in $\mathfrak{U}^{op}$. Two objects are isomorphic in $\mathfrak{U}$ iff they are isomorphic in $\mathfrak{U}^{op}$, so coproducts are unique up to isomorphic in $\mathfrak{U}$. The coproduct of A and B is denoted $A \coprod B$.

Prop 1.10

Let $\mathfrak{U}$ be an additive category, then for any $A, B \in \mathfrak{U}$, 0_{AB} is the identity element in $\mathrm{Hom}(A, B)$.

Proof. In $\mathrm{Hom}(A, 0)$, we have both the identity element and 0_{A0}, but since 0 is a terminal object, then $\mathrm{Hom}(A, 0)$ has a single morphism, so 0_{A0} is the identity element in $\mathrm{Hom}(A, 0)$. Similarly, 0_{0B} is the identity element in $\mathrm{Hom}(0, B)$. Now $0_{AB} = 0_{0B} \circ 0_{A0}$. Since composition is bilinear, then we have that $0_{AB} = (0_{0B} + 0_{0B}) \circ 0_{A0} = 0_{AB} + 0_{AB}$, so $0_{AB} = 0$ is the identity element as desired. ☐

Prop 1.11

Let $\mathfrak{U}$ be an additive category, then for any $A, B \in \mathfrak{U}$, the coproduct $A \coprod B$ exists and is isomorphic to $A \times B$.

Proof. We have morphisms $\mathrm{id}_A : A \to A$ and $0_{AB} : A \to B$ which gives a morphism $i_A : A \to A \times B$ such that $\mathrm{pr}_A \circ i_A = \mathrm{id}_A$ and $\mathrm{pr}_B \circ i_A = 0_{AB}$. Similarly, the morphisms $0_{BA} : B \to A$ and $\mathrm{id}_B : B \to A \times B$ give a unique morphism i_B such that $\mathrm{pr}_A \circ i_B = 0_{BA}$ and $\mathrm{pr}_B \circ i_B = \mathrm{id}_B$. We now have morphisms $i_A : A \to A \times B$ and $i_B : B \to A \times B$, which are the projection morphisms in $\mathfrak{U}^{op}$.

To see that $A \times B$ is the coproduct, let C be any object in $\mathfrak{U}$ with morphisms $i_A^C : A \to C$ and $i_B : B \to C$, then we want to show that there is a unique morphism $i : A \times B \to C$ such that $i_A^C = i \circ i_A$ and $i_B^C = i \circ i_B$. Consider the morphism $i : A \times B \to C$ given by $i_A^C \circ \mathrm{pr}_A + i_B^C \circ \mathrm{pr}_B$. Here the addition makes sense since $\mathfrak{U}$ is an **Ab**-category. Now when we compose we get:

$$i \circ i_A = i_A^C \circ \mathrm{pr}_A \circ i_A + i_B^C \circ \mathrm{pr}_B \circ i_A = i_A^C \circ \mathrm{id}_A + i_B^C \circ 0_{AB} = i_A^C + 0_{AC} = i_A^C$$

Composing $i \circ i_B$ gives i_B^C by the same computation with A replaced by B. To see that this morphism i is unique, we note that if $i_A^C = i \circ i_A$, then composing with pr_A we get that $i_A^C \circ \mathrm{pr}_A = i \circ (i_A \circ \mathrm{pr}_A)$. Similarly, composing $i_B^C = i \circ i_B$ with pr_B we get that $i_B^C \circ \mathrm{pr}_B = i \circ (i_B \circ \mathrm{pr}_B)$. It follows that $i \circ (i_A \circ \mathrm{pr}_A + i_B \circ \mathrm{pr}_B) = i_A^C \circ \mathrm{pr}_A + i_B^C \circ \mathrm{pr}_B$. Let $h = i_A \circ \mathrm{pr}_A + i_B \circ \mathrm{pr}_B$, then notice that:

$$\begin{aligned}
\mathrm{pr}_A \circ h &= (\mathrm{pr}_A \circ i_A) \circ \mathrm{pr}_A + (\mathrm{pr}_A \circ i_B) \circ \mathrm{pr}_B \\
&= \mathrm{id}_A \circ \mathrm{pr}_A + 0_{BA} \circ \mathrm{pr}_B \\
&= \mathrm{pr}_A + 0_{A \times B, A} \\
&= \mathrm{pr}_A
\end{aligned}$$

Replacing A with B and vice versa, we get that $\mathrm{pr}_B \circ h = \mathrm{pr}_B$. The identity map $\mathrm{id}_{A\times B}$ is the unique morphism $A \times B \to A \times B$ which preserve the projections, thus $h = \mathrm{id}_{A\times B}$. It follows that $i \circ h = i = i_A \circ \mathrm{pr}_A + i_B \circ \mathrm{pr}_B$, thus i is unique. □

The above proposition is a rather remarkable fact about additive categories that is very untrue in general. For example, in **Top**, set theoretically, the coproduct of two topological spaces is the disjoint union, whereas the product is the set theoretic product.

Prop 1.12

If $\mathfrak{U}$ is an additive category, then $\mathfrak{U}^{op}$ is also an additive category.

Proof. We first need to show that if $\mathfrak{U}$ is an **Ab**-category, then $\mathfrak{U}^{op}$ is also an **Ab**-category. For any $A, B \in \mathfrak{U}^{op}$, we make $\mathrm{Hom}_{\mathfrak{U}^{op}}(A, B)$ into an abelian group by giving it the same structure a $\mathrm{Hom}_{\mathfrak{U}}(B, A)$. To see that this is still bilinear, let $f : B \to A$, $g, g' : C \to B$, and $h : D \to C$ in $\mathfrak{U}^{op}$, then these correspond to morphisms $\widetilde{f} : A \to B, \widetilde{g}, \widetilde{g}' : B \to C$, and $\widetilde{h} : C \to D$. We know that $\widetilde{h} \circ (\widetilde{g} + \widetilde{g}') \circ \widetilde{f} = \widetilde{h} \circ \widetilde{g} \circ \widetilde{f} + \widetilde{h} \circ \widetilde{g}' \circ \widetilde{f}$. In $\mathfrak{U}^{op}$, all compositions are reversed, so we have that $f \circ (g + g') \circ h = f \circ g \circ h + f \circ g' \circ h$. This is exactly what we need for $\mathfrak{U}^{op}$ to be an **Ab**-category.

To see that $\mathfrak{U}^{op}$ is an additive category, we note that by 1.1, a zero object in $\mathfrak{U}$ is also a zero object in $\mathfrak{U}^{op}$ since initial objects in $\mathfrak{U}$ are terminal in $\mathfrak{U}^{op}$ and terminal objects in $\mathfrak{U}$ are initial in $\mathfrak{U}^{op}$, so zero objects remain zero objects. To see that $\mathfrak{U}^{op}$ has products, we notice that a product in $\mathfrak{U}^{op}$ is a coproduct in $\mathfrak{U}$ by definition and we showed in 1.11 that $\mathfrak{U}$ has coproducts. It follows that $\mathfrak{U}^{op}$ is an additive category. □

Prop 1.13

Let $\mathfrak{U}$ be an additive category. Then $A \in \mathfrak{U}$ is initial iff it is terminal iff it is zero.

Proof. If A is zero, then it is initial and terminal. Since initial and terminal objects are unique up to isomorphism, then if A is initial or terminal, then it is isomorphic to zero. $\square$

Prop 1.14

Let $f : A \to B$ be a morphism in an additive category. If $\ker(f)$ exists, with $k : \ker(f) \to A$, then k is a monomorphism. If $\operatorname{coker}(f)$ exists with $q : B \to \operatorname{coker}(f)$, then q is an epimorphism.

Proof. Suppose that $\ker(f)$ exists with $k : \ker(f) \to A$. Let $g, h : C \to \ker(f)$ such that $k \circ g = k \circ h$, then $k \circ (g - h) = 0_{CA}$. Now the morphism $0_{CA} : C \to A$ is such that $f \circ 0_{CA} = 0_{CB}$, therefore there is a unique morphism $u : C \to \ker(f)$ such that $k \circ u = 0_{CA}$. We know that $u = g - h$ makes this true, but we also know that $k \circ 0_{C,\ker(f)} = 0_{CA}$, therefore $u = 0_{C,\ker(f)}$ as well, so $g - h = 0$, therefore $g = h$. It follows that k is a monomorphism.

Suppose that $\operatorname{coker}(f)$ exists with $q : B \to \operatorname{coker}(f)$. Let $g, h : \operatorname{coker}(f) \to C$ be two morphisms such that $g \circ q = h \circ q$, then $(g - h) \circ q = 0_{BC}$. The morphism 0_{BC} is a morphism such that $0_{BC} \circ f = 0_{AC}$, therefore there is a unique morphism $u : \operatorname{coker}(f) \to C$ such that $u \circ q = 0_{BC}$. We know that $(g - h) \circ q = 0_{BC}$, therefore $u = g - h$. However, we also know that $0_{\operatorname{coker}(f),C} \circ q = 0_B$, therefore $u = 0$, so $g - h = 0$ hence $g = h$ as desired. It follows that q is an epimorphism. $\square$

Def: Given a morphism $f : A \to B$

1. If f has cokernel $\operatorname{coker}(f)$ with morphism $q : B \to \operatorname{coker}(f)$. The *image* of f is the kernel of q, denoted $\operatorname{im}(f)$.

2. If f has kernel $\ker(f)$ with morphism $k : \ker(f) \to A$. The *coimage* of f is the cokernel of k, denoted $\operatorname{coim}(f)$.

Suppose that $f : A \to B$ is a morphism such that $\operatorname{im}(f)$ and $\operatorname{coim}(f)$ both exist, then there is a morphism $q' : A \to \operatorname{coim}(f)$ and a morphism $k' : \operatorname{im}(f) \to B$ by definition of cokernel and kernel. Since $\operatorname{coim}(f)$ is the cokernel of the morphism $k : \ker(f) \to A$, then for any morphism $g : A \to C$ with $g \circ k = 0$, then we get a morphism $u : \operatorname{coim}(f) \to C$ such that $u \circ q' = g$. In particular, we have a unique morphism $f : A \to B$ such that $f \circ k = 0$ by definition of the kernel. It follows that we get a morphism $u : \operatorname{coim}(f) \to B$. Now by definition of the kernel of q, if we can show that $q \circ u = 0$, then we will get a unique morphism $u' : \operatorname{coim}(f) \to \operatorname{im}(f)$ such that $k' \circ u' = u$. To see that $q \circ u = 0$ we notice that since q' is the morphism to a cokernel, then it is an epimorphism, thus $q \circ u$ is 0 iff $q \circ u \circ q' = q \circ f = 0$ but this is obvious. It follows that there is a unique morphism $u' : \operatorname{coim}(f) \to \operatorname{im}(f)$ such that $k' \circ u' = u$.

Def: A category $\mathfrak{U}$ is called an *abelian category* if $\mathfrak{U}$ is an additive category such that:

1. Every morphism in $\mathfrak{U}$ has a kernel and a cokernel.

2. For any $f : A \to B$, the canonical morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

Let us check that **Ab**, the category of abelian groups is in fact an abelian category. We first need to show that it is an **Ab**-category. Let A, B be abelian groups, then we can define a group structure on $\operatorname{Hom}(A, B)$ by saying that for $f, g : A \to B$, $f + g : A \to B$ is given set theoretically by $(f+g)(x) = f(x) + g(x)$. We check that $f + g$ is a group homomorphism: $(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y)$ and since B is abelian, then this is the same as $f(x) + g(x) + f(y) + g(y) = (f+g)(x) + (f+g)(y)$. We now check that this makes $\mathrm{Hom}(A, B)$ into a group. The identity element is given by the zero homomorphism $0 \in \mathrm{Hom}(A, B)$ which sends $0(x) = 0$ for all $x \in A$. We can see that this is the identity element easily since $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$. Inverses are given by $(-f)(x) = -f(x)$. To see that it is abelian, we notice that $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$, therefore $f + g = g + f$. It follows that **Ab** is an **Ab**-category.

To see that **Ab** is an additive category, we already saw that the trivial group 0 is the zero object. For products, let A, B be abelian groups, then we have the direct sum $A \oplus B$ whose elements are of the form (a, b) with $a \in A, b \in B$. Set theoretically $A \oplus B$ is the product of A and B, so the uniqueness statement of the product definition is satisfied. That is to say that if C is an abelian group with homomorphisms $f : C \to A$ and $g : C \to B$, then we get $(f, g)(x) = (f(x), g(x))$ as a set map $(f, g) : C \to A \oplus B$. To see that it is a group homomorphism, we see $(f, g)(x + y) = (f(x+y), g(x+y)) = (f(x) + f(y), g(x) + g(y)) = (f(x), g(x)) + (f(y), g(y)) = (f, g)(x) + (f, g)(y)$. It follows that $A \oplus B$ is the product of A and B. Therefore **Ab** is an additive category.

To show that **Ab** is an abelian category, we need to check a few things. We first show that for any homomorphism $f : A \to B$, $\ker(f)$ is the kernel and $\mathrm{coker}(f)$ is the cokernel.

To see that $\ker(f)$ is the kernel of f, we need a morphism $\ker(f) \to A$. This morphism is the inclusion map $i : \ker(f) \to A$. By definition, $\ker(f) = \{x \in A \big| f(x) = 0\}$, so indeed $f \circ i = 0_{\ker(f),B}$. Now let C be any abelian group and $k : C \to A$ such that $f \circ k = 0$. It follows that for all $b \in C$, $k(b) \in \ker(f)$, so in fact the map $k : C \to A$ has image lying in $\ker(f)$, so as a set map k maps $C \to \ker(f)$. Since the group structure on $\ker(f)$ is inherited from that of A, then $u = k : C \to \ker(f)$ is a group homomorphism. Since we want $i \circ u = k$ and i is injective, then we must have that $u = k$, so it is unique. It follows that $\ker(f)$ with the inclusion map is indeed the categorical kernel.

To see that $\mathrm{coker}(f)$ is the cokernel of f, we have the quotient map $q : B \to B/\mathrm{im}(f) = \mathrm{coker}(f)$. Let Q be an abelian group and let $q' : B \to Q$ be a morphism such that $q' \circ f = 0_{AQ'}$. We want to show that there is a unique morphism $u : \mathrm{coker}(f) \to Q$ such that $u \circ q = q'$. Since $q' \circ f = 0$, then we know that $\mathrm{im}(f) \subseteq \ker(q')$. Since q is surjective, then for any $\overline{x} \in \mathrm{coker}(f)$, we must have that $u(\overline{x}) = (u \circ q)(x) = q'(x)$. This uniquely define u set theoretically, so uniqueness is satisfied. To see that this is well-defined, note that if $\overline{x} = \overline{y}$, then $x - y \in \mathrm{im}(f) \subseteq \ker(q')$, so $q'(x) = q'(y)$. To see that this is a homomorphism, for $\overline{x}$ and $\overline{y}$ in $\mathrm{coker}(f)$ we have that $u(\overline{x} + \overline{y}) = u(\overline{x+y}) = q'(x+y) = q'(x) + q'(y) = u(\overline{x}) + u(\overline{y})$. Therefore cokernels in **Ab** are indeed $B/\mathrm{im}(f)$.

From the above, we see that **Ab** has the kernel and cokernel of every morphism. We now need to check that every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel. We first show that monomorphisms are injective homomorphisms and that epimorphisms are surjective homomorphisms. To do so, we want to show that f is a monomorphism iff $\ker(f)$ is the zero object and an epimorphisms iff $\mathrm{coker}(f)$ is the zero object, at which point these equivalent conditions to being monomorphisms/epimorphisms can be obtained.

Prop 1.15

Let $f : A \to B$ be a morphism in an additive category $\mathfrak{U}$ such that $\ker(f)$ and $\operatorname{coker}(f)$ exist, then:

1. f is a monomorphism iff $\ker(f) \cong 0$.

2. f is an epimorphism iff $\operatorname{coker}(f) \cong 0$.

Proof. 1) Suppose that f is a monomorphism but that $\ker(f) \ncong 0$, then $\ker(f)$ is not terminal. For any objects X, Y we have that $\operatorname{Hom}(X, Y)$ has the structure of a group, so in particular it is non-empty. It follows that for $\ker(f)$ to not be terminal, there must be some C such that $\operatorname{Hom}(C, \ker(f))$ has at least two elements, say $g \neq h$. Let $k : \ker(f) \to A$ be the associated morphism, then we have that $f \circ k = 0$. Therefore $f \circ k \circ g = f \circ k \circ h = 0$. Since f is a monomorphism, then we have that $k \circ g = k \circ h$. By 1.15, k is a monomorphism so $g = h$ which is a contradiction. Conversely, suppose that $\ker(f) = 0$, then let $g, h : C \to A$ such that $f \circ g = f \circ h$, then $f \circ (g - h) = 0$, so by definition of $\ker(f)$, there exists a unique morphism $u : C \to \ker(f)$ such that $k \circ u = g - h$. Since $\ker(f) = 0$, then the only morphism $C \to \ker(f)$ is 0, thus $u = 0$, so $g - h = k \circ 0 = 0$, thus $g = h$, so f is a monomorphism.

2) f is an epimorphism iff f is a monomorphism in $\mathfrak{U}^{op}$ iff $\ker_{\mathfrak{U}^{op}}(f) = 0$ by 1), but by the remark made after the definition of a cokernel, we know that $\operatorname{coker}_{\mathfrak{U}}(f) = \ker_{\mathfrak{U}^{op}}(f)$, therefore f is an epimorphism iff $\operatorname{coker}(f) = 0$. □

Note that the way we proved 2) in 1.15, we could have used the same argument to prove the second part of 1.14.

We can now finish showing that **Ab** is an abelian category. From 1.15, we know that $f : A \to B$ is a monomorphism iff $\ker(f) = 0$. The zero object in **Ab** is the trivial group, so $\ker(f) = 0$ iff f is injective, therefore monomorphisms in **Ab** are just injective maps. Similarly, f is an epimorphism iff $\operatorname{coker}(f) = 0$ which is equivalent to saying that $\operatorname{im}(f) = B$, i.e. that f is surjective.

We now need to show that the canonical morphism $\operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism for all morphisms f. We know that there is a unique morphism $u : \operatorname{coim}(f) \to B$ such that $u \circ q' = f$. We know that $\operatorname{coim}(f) = \operatorname{coker}(k)$ where k is the inclusion $\ker(f) \to A$, therefore $\operatorname{coim}(f) = A/\ker(f)$ and the map q' is the quotient map $A \to A/\ker(f)$. It follows that $u(\overline{x}) = f(x)$. Now once we quotient B by the image of f we have a quotient map $q : B \to \operatorname{coker}(f) = B/\operatorname{im}(f)$ and we get that $q \circ u = 0$, so in fact u maps into the kernel of q which is just $\operatorname{im}(f)$. It follows that the categorical image is also the standard image. Therefore our canonical morphism is the map $u : A/\ker(f) \to \operatorname{im}(f)$ given by $u(\overline{x}) = f(x)$. The statement that this is an isomorphism is exactly the first isomorphism theorem. Therefore **Ab** is an abelian category.

Def: Let A, B be object with morphisms $f : A \to C$ and $g : B \to C$. An object D with morphisms $\operatorname{pr}_A : D \to A, \operatorname{pr}_B : D \to B$ is called the *fiber product* of A and B over C if for any E with morphisms f', g' from E to A and B such that $f \circ f' = g \circ g'$, then there exists a unique morphism $u : E \to D$ such that $f' = \operatorname{pr}_A \circ u$ and $g' = \operatorname{pr}_B \circ u$.

The above definition should look very similar to the definition of a product. This is because fiber products are products, not in the category $\mathfrak{U}$, but in the comma category $\mathfrak{U}/C$. It follows from this description that fiber products are unique up to isomorphism. Since they are unique up to isomorphism, then the fiber product of A and B over C is denoted $A \times_C B$.

Prop 1.16

Let $\mathfrak{U}$ be an abelian category, then for all A, B, C and $f : A \to C, g : B \to C$, the fiber product $A \times_C B$ exists.

Proof. Consider the product $A \times B$, then we have a morphism $h : A \times B \to C$ given by $h = f \circ \mathrm{pr}_A - g \circ \mathrm{pr}_B$. Let $D = \ker(h)$, then D comes with a morphism $k : D \to A \times B$. Let $\mathrm{pr}_A^D = \mathrm{pr}_A \circ k$ and $\mathrm{pr}_B^D = \mathrm{pr}_B \circ k$. We want to show that $D = A \times_C B$. Let E be an object with morphisms $f' : E \to A$ and $g' : E \to B$ such that $f \circ f' = g \circ g'$. Since $E \to A$ and $E \to B$, then there is a unique morphism $u : E \to A \times B$ such that $f' = \mathrm{pr}_A \circ u, g' = \mathrm{pr}_B \circ u$. We compute $h \circ u = f \circ (\mathrm{pr}_A \circ u) - g \circ (\mathrm{pr}_B \circ u) = f \circ f' - g \circ g' = 0$. Since $h \circ u = 0$, then by definition of the kernel, there is a unique $u' : E \to D$ such that $u = k \circ u'$. If $u = k \circ u'$, then $\mathrm{pr}_A^D \circ u' = \mathrm{pr}_A \circ u = f'$ and similarly for g'. Conversely if u' is such that $f' = \mathrm{pr}_A^D \circ u', g' = \mathrm{pr}_B^D \circ u'$, then we want to show that $u = k \circ u'$ so that u' is uniquely determined. Since u and $k \circ u'$ are both morphisms into a product, then by the definition of the product, we just need to show that the projections are the same. We compute $\mathrm{pr}_A \circ k \circ u' = \mathrm{pr}_A^D \circ u' = f'$ by assumption, but $\mathrm{pr}_A \circ u = f'$ as well. The same holds for the projection to B. If follows that $u = k \circ u'$, so u' is unique. Therefore D is the fiber product of A and B over C. $\square$

The above proof can be understood to say that the fiber product is the object which makes two morphisms equal. This is true in abelian categories as the above proof shows. However, in more general categories, fiber products can look more peculiar. It is always true that fiber products represent the fiber product of two functors (which really does just make the two functors equal since they are to **Set**). However, the object representing the fiber product may not be so simple. For a prime example of this, one can look at the fiber product in **Sch**, which is set theoretically a subset of a quadruple product, not just a product.

Prop 1.17

Let $f : A \to C$ be an epimorphism. Let $g : B \to C$ be any morphism. Then $\mathrm{pr}_B : A \times_C B \to B$ is an epimorphism.

Proof. We first show that C is a fiber product of A and B over $A \times_C B$ in $\mathfrak{U}^{op}$. D is a fiber product of A and B over C iff $D = \ker(f \circ \mathrm{pr}_A - g \circ \mathrm{pr}_B)$, i.e. the sequence $0 \to D \to A \times B \to C$ is exact where $A \times B \to C$ is the map $f \circ \mathrm{pr}_A - g \circ \mathrm{pr}_B$. It follows that C is a fiber product of A and B over $A \times_C B$ iff $h = f \circ \mathrm{pr}_A - g \circ \mathrm{pr}_B$ is an epimorphism. Since f is an epimorphism, then composing with i_A we get f which is an epimorphism so $h \circ i_A$ is an epimorphism. It follows that h is also an epimorphism. Therefore C is the fiber product of A and B over $A \times_C B$.

We get a morphism $\mathrm{coker}(f) \to \mathrm{coker}(\mathrm{pr}_B)$ by considering $q_1 \circ g$. Now notice that $q_1 \circ g \circ \mathrm{pr}_B = q_1 \circ f \circ \mathrm{pr}_A = 0$ since $q_1 \circ f = 0$. By the universal property of the cokernel, it follows that there is a unique morphism $\tilde{g} : \mathrm{coker}(\mathrm{pr}_2) \to \mathrm{coker}(f)$ such that $\tilde{g} \circ q_2 = q_1 \circ g$.

We now need to get a morphism $\mathrm{coker}(f) \to \mathrm{coker}(\mathrm{pr}_B)$. A morphism out of $\mathrm{coker}(f)$ is equivalent to giving a morphism from C whose composition with f is 0. Since C is the fiber product in the opposite category, then giving a morphism out of C is equivalent to giving maps out of A and B such that the composition with $\mathrm{pr}_A, \mathrm{pr}_B$ from the fiber product are the same. We have $q_2 : B \to \mathrm{coker}(\mathrm{pr}_B)$ and $q_2 \circ \mathrm{pr}_B = 0$ and similarly we have the zero morphism $0 : A \to \mathrm{coker}(\mathrm{pr}_B)$ then $0 \circ \mathrm{pr}_A = 0$, therefore $q_2 \circ \mathrm{pr}_A = 0 = q_2 \circ \mathrm{pr}_B$. It follows that we obtain a morphism $s : C \to \mathrm{coker}(\mathrm{pr}_B)$ such that $s \circ g = q_2$ and $s \circ f = 0$. Since $s \circ f = 0$, then in fact s yields a unique morphism $\widetilde{s} : \text{coker}(f) \to \text{coker}(\text{pr}_B)$ such that $s = \widetilde{s} \circ q_1$. We can list out the compositions that we have:

$$\begin{aligned} s &= \widetilde{s} \circ q_1 \\ q_2 \circ \text{pr}_A &= q_2 \circ \text{pr}_B = 0 \\ \widetilde{g} \circ q_2 &= q_1 \circ g \\ s \circ g &= q_2 \\ s \circ f &= 0 \\ q_1 \circ f &= 0 \end{aligned}$$

We know that q_1 is an epimorphism, so to show that $\widetilde{g} \circ \widetilde{s}$ is the identity, we need only show that $\widetilde{g} \circ \widetilde{s} \circ q_1 = q_1$. We know that $\widetilde{s} \circ q_1 = s$, so we want to show that $\widetilde{g} \circ s = q_1$. q_1 is a morphism out of C which is a fiber product in $\mathfrak{U}^{op}$, so $q_1 = \widetilde{g} \circ s$ iff $q_1 \circ g = \widetilde{g} \circ s \circ g$ and $q_1 \circ f = \widetilde{g} \circ s \circ f$. We know that $s \circ g = q_2$, so $\widetilde{g} \circ s \circ g = \widetilde{g} \circ q_2 = q_1 \circ g$ and $s \circ f = 0$, so $\widetilde{g} \circ s \circ f = 0 = q_1 \circ f$. It follows that $\widetilde{g} \circ \widetilde{s} = \text{id}_{\text{coker}(f)}$. For the other direction, we have that q_2 is an epimorphism, therefore we need only check $\widetilde{s} \circ \widetilde{g} \circ q_2 = q_2$. We know that $\widetilde{g} \circ q_2 = q_1 \circ g$, therefore $\widetilde{s} \circ q_1 \circ g = s \circ g = q_2$. It follows that $\widetilde{s} \circ \widetilde{g} = \text{id}_{\text{coker}(\text{pr}_B)}$. Therefore $\text{coker}(\text{pr}_B) = \text{coker}(f)$. Since f is an epimorphism, then $\text{coker}(f) = 0$, so $\text{coker}(\text{pr}_B) = 0$, thus pr_B is an epimorphism. $\square$

The above says that epimorphisms are stable under base change. Many properties of morphism are stable under base change.

Def: Let $\mathfrak{U}$ be an abelian category. A sequence $\cdots \to A_{i-1} \xrightarrow{f} A_i \xrightarrow{g} A_{i+1} \to \cdots$ is said to be *exact* at A_i if $\text{im}(f) = \ker(g)$. The sequence is exact if it is exact at all A_i.

Note that in the above definition, equality of the image and kernel is more than just an isomorphism. The image and kernel are not just objects, they also have morphisms $k : \ker(g) \to A_i$ and $k' : \text{im}(f) \to A_i$. When we say that $\text{im}(f) = \ker(g)$, we mean that there is an isomorphism $q : \text{im}(f) \to \ker(g)$ such that $k' = k \circ q$. This is the same as saying that $\text{im}(f)$ is a kernel of g.

Def: Let $\mathfrak{U}, \mathfrak{B}$ be abelian categories. A covariant functor $F : \mathfrak{U} \to \mathfrak{B}$ is *exact* if for all exact sequences $A \xrightarrow{f} B \xrightarrow{g} C$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact.

Def: Let $\mathfrak{U}$ be an abelian category. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence.*

Def: Let $\mathfrak{U}, \mathfrak{B}$ be abelian categories. A covariant functor $F : \mathfrak{U} \to \mathfrak{B}$ is called *left exact* if for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ we have that $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact. F is called *right exact* if $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

Example: Let X, Y be topological spaces and $f : X \to Y$ a continuous map. Let $\mathscr{F}$ be a sheaf of abelian groups on X, then we have a sheaf of abelian groups $f_*\mathscr{F}$ on Y given by $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$. We check that $f_*\mathscr{F}$ is a sheaf. The restriction maps are $\rho_{UV}^{f_*\mathscr{F}} : f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$

given by $\rho_{UV}^{f_*\mathscr{F}} = \rho^{\mathscr{F}}_{f^{-1}(U),f^{-1}(V)}$. Clearly $(f_*\mathscr{F})(\varnothing) = \mathscr{F}(\varnothing) = 0$. If $U = \bigcup_i U_i$ and $a_i \in (f_*\mathscr{F})(U_i)$ such that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ for all i, j, then this means that $a_i \in \mathscr{F}(f^{-1}(U_i))$ and $f^{-1}(U) = \bigcup_i f^{-1}(U_i)$ with $a_i|_{f^{-1}(U_i) \cap f^{-1}(U_j)} = a_j|_{f^{-1}(U_i)\cap f^{-1}(U_j)}$ for all i, j. It follows that there is a unique $a \in \mathscr{F}(f^{-1}(U))$ such that $a|_{f^{-1}(U_i)} = a_i$. This means that there is a unique $a \in f_*\mathscr{F}(U)$ such that $a|_{U_i} = a_i$. Therefore $f_*\mathscr{F}$ is a sheaf. We need to make sure that $f_*\varphi$ is indeed a morphism of sheaves, for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ we obtain a morphism $f_*\varphi : f_*\mathscr{F} \to f_*\mathscr{G}$ given by $(f_*\varphi)(U) = \varphi(f^{-1}(U))$. To see that it is a morphism, for any $U \subseteq V$ we have that

$$\begin{aligned}
\rho_{UV}^{f_*\mathscr{G}}(f_*\varphi)(U)(x) &= \rho^{\mathscr{G}}_{f^{-1}(U),f^{-1}(V)}\varphi(f^{-1}(U))(x) \\
&= \varphi(f^{-1}(V))(\rho^{\mathscr{F}}_{f^{-1}(U),f^{-1}(V)}(x)) \\
&= (f_*\varphi)(V)(\rho_{UV}^{f_*\mathscr{F}}(x))
\end{aligned}$$

It follows that $f_*\varphi$ is a morphism. To show that f_* is a functor, we need to show that $f_*\mathrm{id}_{\mathscr{F}} = \mathrm{id}_{f_*\mathscr{F}}$ and that $f_*(\varphi \circ \psi) = f_*\varphi \circ f_*\psi$. For the prior, we have that for any $U \subseteq Y$, $(f_*\mathrm{id}_{\mathscr{F}})(U) = \mathrm{id}_{\mathscr{F}}(f^{-1}(U)) = \mathrm{id}_{\mathscr{F}(f^{-1}(U))} = \mathrm{id}_{(f_*\mathscr{F})(U)}$, therefore $f_*\mathrm{id}_{\mathscr{F}} = \mathrm{id}_{f_*\mathscr{F}}$. For the latter, we have that $(f_*(\varphi \circ \psi))(U) = (\varphi \circ \psi)(f^{-1}(U)) = \varphi(f^{-1}(U)) \circ \psi(f^{-1}(U)) = (f_*\varphi)(U) \circ (f_*\psi)(U)$, therefore $f_*(\varphi \circ \psi) = f_*\varphi \circ f_*\psi$. It follows that f_* is a functor.

We now show that f_* is a left exact functor. Let $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{R} \to 0$ be a short exact sequence of sheaves of abelian groups. We want to show that $0 \to f_*\mathscr{F} \xrightarrow{f_*\varphi} f_*\mathscr{G} \xrightarrow{f_*\psi} f_*\mathscr{R}$ is exact. We first show that $f_*\varphi$ is injective, i.e. $\ker(f_*\varphi) = 0 = \mathrm{im}(0)$. The kernel of a morphism of sheaves is a sheaf given by $\ker(f_*\varphi)(U) = \ker((f_*\varphi)(U))$. We have that $(f_*\varphi)(U) = \varphi(f^{-1}(U))$. Since φ is injective, then the kernel of $\varphi(f^{-1}(U))$ is 0, so $f_*\varphi$ is also injective. The image of $f_*\varphi$ is a sheaf $\mathrm{im}(f_*\varphi)$ where an element $a \in \mathrm{im}(f_*\varphi)(U)$ is given by a cover $U = \bigcup_i U_i$ and a tuple $\{a_i\}$ of $a_i \in \mathrm{im}((f_*\varphi)(U_i))$ such that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$. Since $a_i \in \mathrm{im}((f_*\varphi)(U_i))$, then $a_i = (f_*\varphi)(U_i)(b_i)$ for some b_i. From the compatibility conditions on the a_i we get that $(f_*\varphi)(U_i \cap U_j)(b_i|_{U_i \cap U_j}) = (f_*\varphi)(U_i \cap U_j)(b_j|_{U_i \cap U_j})$. Since $f_*\varphi$ is injective, then we have that $b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$ for all i, j. It follows that the b_i glue to some $b \in (f_*\mathscr{F})(U)$ such that $(f_*\varphi)(U_i)(b|_{U_i}) = a_i$, therefore $(f_*\varphi)(U)(b) = a$ since they agree on a cover of U. It follows that the naive image sheaf is the whole image sheaf. We can now check exactness at $f_*\mathscr{G}$. Since f_* is a functor, then $0 = f_*(\psi \circ \varphi) = f_*\psi \circ f_*\varphi$ so $\mathrm{im}(f_*\varphi) \subseteq \ker(f_*\psi)$. To show the other containment, let $a \in \ker((f_*\psi)(U)) = \ker(\psi(f^{-1}(U)))$, then by exactness of $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{R} \to 0$ we know that $a \in \mathrm{im}(\varphi)(f^{-1}(U))$. Since φ is injective, then as we saw, the naive image presheaf is a sheaf, so $a = \varphi(f^{-1}(U))(b)$ for some $b \in \mathscr{F}(f^{-1}(U))$, therefore $a = (f_*\varphi)(U)$ so $a \in \mathrm{im}(f_*\varphi)(U)$ as desired. It follows that f_* is a left exact functor.

We will see later that although f_* is a left exact functor, it is in general not an exact functor.

2 Derived Functors

Def: An object $P \in \mathfrak{U}$ is called *projective* if for any morphism $f : P \to M$ and any epimorphism $g : N \to M$, there exists a morphism $f' : P \to N$ such that $f = g \circ f'$.

Def: An object $I \in \mathfrak{U}$ is called *injective* if for any morphism $f : M \to I$ and any monomorphism $i : M \to N$, then there is a morphism $f' : N \to I$ such that $f = f' \circ i$.

Example: In the category of modules over a ring R, any free module is projective. Let R^I be a free module for some index set I, then any morphism $f : R^I \to M$ is determined by where it sends

$\{e_i\}_{i \in I}$. If $g : N \to M$ is an epimorphism, then it is a surjective morphism since the cokernel is 0, so for each i, we may choose some $n_i \in N$ such that $g(n_i) = f(e_i)$, then let $f' : R^I \to N$ be given by sending e_i to n_i. It follows that $g(f'(e_i)) = f(e_i)$, so $g \circ f' = f$. Therefore R^I is projective. In fact, if R is a local ring, then all finitely generated projective modules are free.

Def: A category $\mathfrak{U}$ has *enough projectives* if for any object $M \in \mathfrak{U}$, there is an epimorphism $P \to M$ with P a projective object.

Def: A category $\mathfrak{U}$ has *enough injectives* if for any object $M \in \mathfrak{U}$, there is a monomorphism $M \to I$ with I an injective object.

Def: Let $M \in \mathfrak{U}$. A *projective resolution* of M is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with all the P_i projective.

Def: Let $M \in \mathfrak{U}$. An *injective resolution* of M is an exact sequence $0 \to M \to I^0 \to I^1 \to \cdots$ with all the I^i injective.

We will often abbreviate a resolution. Instead of writing $\cdots \to P_1 \to P_0 \to M \to 0$, one may instead write just $P_\bullet \to M \to 0$ and likewise $0 \to M \to I^\bullet$. If F is a functor, then by $F(P_\bullet)$ or $F(I^\bullet)$, we mean the sequence obtained by applying F to the original sequence. Note that the indices being in superscripts for injective resolutions and subscripts for projective is not a mistake and rather has to do with the notation for homology and cohomology.

Def: A sequence $\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$ is called a chain complex if $d_{n+1} \circ d_n = 0$.

If $\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$ is an exact sequence, then we have that $\mathrm{im}(d_{n+1}) = \ker(d_n)$, so we get that $d_{n+1} \circ d_n = 0$. Therefore all exact sequences are chain complexes.

Def: Let $\mathfrak{U}$ be an abelian category and let $\cdots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0$ be a chain complex. We define $H_i(A_\bullet)$ to be $\ker(d_i)/\mathrm{im}(d_{i+1})$. H_i is the i^{th} homology group of the sequence $A_\bullet$.

Def: Let $\mathfrak{U}$ be an abelian category and let $A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$ be a cochain complex. We define $H^i(A^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$. H^i is the i^{th} cohomology group of the sequence $A^\bullet$.

A cochain complex really just means that the indices are increasing rather than decreasing. Notice that a sequence $A_\bullet$ or $A^\bullet$ is exact iff $H_i(A_\bullet) = 0$ for all i or $H^i(A^\bullet) = 0$ for all i respectively.

Prop 2.1

Let $f : A \to B$ be an epimorphism and $g : B \to C$ a morphism, then $\operatorname{coker}(g) = \operatorname{coker}(g \circ f)$.

Proof. We want to show that $\operatorname{coker}(g \circ f)$ is the cokernel of g. Let $q : C \to \operatorname{coker}(g \circ f)$ be the morphism associated to the cokernel. We need to check that $q \circ g = 0$. By definition of the cokernel, we know that $q \circ g \circ f = 0$. Since f is an epimorphism, then by definition this means that $q \circ g = 0$ as required. Let Q be any object with a morphism $q' : C \to Q$ such that $q' \circ g = 0$. It follows that $q' \circ g \circ f = 0$, so there is a unique morphism $u : \operatorname{coker}(g \circ f) \to Q$ such that $q' = u \circ q$. It follows that $\operatorname{coker}(g \circ f)$ is the cokernel of g. $\square$

Note that working in $\mathfrak{U}^{op}$, we get that if $f : A \to B$ is a morphism and $g : B \to C$ is a monomorphism, then $\ker(f) = \ker(g \circ f)$.

Prop 2.2

Let $\mathfrak{U}$ be a category with enough projectives (injectives), then for any $M \in \mathfrak{U}$, we may find a projective (injective) resolution of M.

Proof. By definition of having enough projectives, we may find a projective object P_0 and an epimorphism $d_0 : P_0 \to M$. Let $K_0 = \ker(d_0)$ with morphism $k_0 : K_0 \to P_0$. We may find a projective object P_1 with an epimorphism $d_1' : P_1 \to K_0$. Let $d_1 = k_0 \circ d_1'$. By definition, $\operatorname{im}(d_1)$ is the kernel of the morphism $q : P_0 \to \operatorname{coker}(d_1)$. Since d_1' is an epimorphism, then we know that $\operatorname{coker}(d_1) = \operatorname{coker}(k_0)$. It follows by definition that $\operatorname{im}(d_1) = \operatorname{im}(k_0)$. Since k_0 is a monomorphism, then $\operatorname{im}(k_0)$ is the pair (K_0, k_0), therefore $\operatorname{im}(d_1) = \ker(d_0)$. We may now continue inductively to obtain a projective resolution of M. $\square$

Since the definition of projective and injective are dual ,then a projective resolution in $\mathfrak{U}^{op}$ is an injective resolution in $\mathfrak{U}^{op}$. Therefore if $\mathfrak{U}$ has enough injectives, then injective resolutions exist.

Def: Let $\mathfrak{U}, \mathfrak{B}$ be **Ab**-categories. A functor $F : \mathfrak{U} \to \mathfrak{B}$ is *additive* if the map $\varphi \mapsto F(\varphi)$ from $\operatorname{Hom}(A, B) \to \operatorname{Hom}(F(A), F(B))$ is a group homomorphism for all A, B.

Def: Let $\mathfrak{U}, \mathfrak{B}$ be abelian categories where $\mathfrak{U}$ has enough projectives. Let $F : \mathfrak{U} \to \mathfrak{B}$ be an additive right exact functor. Let $M \in \mathfrak{U}$, then choose a projective resolution $P_\bullet \to M \to 0$. We then get a sequence $F(P_\bullet)$ and the i^{th} *left derived functor* of F is given by $L_i F(M) = H_i(F(P_\bullet))$.

Def: Let $\mathfrak{U}, \mathfrak{B}$ be abelian categories where $\mathfrak{U}$ has enough injectives. Let $F : \mathfrak{U} \to \mathfrak{B}$ be an additive left exact functor. Let $M \in \mathfrak{U}$, then choose an injective resolution $0 \to M \to I^\bullet$. We then get a sequence $F(I^\bullet)$ and the i^{th} *right derived functor* of F is given by $R^i F(M) = H^i(F(I^\bullet))$.

It is not at all obvious that these objects are even well-defined. To show that these are well-defined, it will become clear why we chose projective and injective resolutions. I will only prove well-definedness for right derived functors. The proof for left derived functors then follows since it is just the right derived functor of $F : \mathfrak{U}^{op} \to \mathfrak{B}^{op}$.

We can check that $F(P_\bullet)$ is indeed a chain complex. This is because for all n we have that $d_{n+1} \circ d_n = 0$, therefore $F(d_{n+1}) \circ F(d_n) = F(0)$. Since F is an additive functor, then $F(\mathrm{id}_0) = \mathrm{id}_{F(0)}$ and id_0 is the identity in $\mathrm{Hom}(0, 0)$, so it is also sent to the identity, thus $\mathrm{id}_{F(0)}$ is the identity in $\mathrm{Hom}(F(0), F(0))$. We know that the identities are the zeros morphisms, so $\mathrm{id}_{F(0)} = 0_{F(0)}$. It follows that for any object A, for $f \in \mathrm{Hom}(F(A), F(0))$, we have that $f = \mathrm{id}_{F(0)} \circ f = 0_{F(0)} \circ f = 0_{F(A),F(0)}$. Therefore $F(0)$ is a terminal object. We similarly get that $F(0)$ is initial, so $F(0) = 0$. It follows that $F(d_{n+1}) \circ F(d_n) = F(0) = 0$ so $F(P_\bullet)$ is indeed a chain complex.

Def: Let $C_\bullet$ and $D_\bullet$ be chain complexes with morphisms $c_i : C_i \to C_{i-1}$ and $d_i : D_i \to D_{i-1}$. A collection of morphisms $\{f_i\}$ is a *morphism of chain complexes* if $f_i : C_i \to D_i$ and $d_i \circ f_i = f_{i-1} \circ c_i$.

Note that with the above notion of morphism, this makes the class of chain complexes into a category in and of itself. In fact, if $\mathfrak{U}$ is an abelian category, then the category of chain complexes in $\mathfrak{U}$ is also an abelian category.

Furthermore, since functors respect composition, then if F is an additive functor and $\{f_i\}$ is a morphism from $C_\bullet$ to $D_\bullet$, then $\{F(f_i)\}$ is a morphism from $F(C_\bullet)$ to $F(D_\bullet)$.

Given a morphism $\{f_i\} : C_\bullet \to D_\bullet$, we obtain morphisms $f_i : H_i(C_\bullet) \to H_i(D_\bullet)$. I will explain how to do this in **Ab**, and I leave the abstraction to general abelian categories to you. I think there is some embedding theorem that says that if you can make a morphism in **Ab**, then you can do in any abelian category under some suitable conditions, but I don't know the statement. For any $\overline{x} \in H_i(C_\bullet)$, we have that $x \in \ker(c_i)$, so we can send it to $\overline{f_i(c_i)}$. To check that this is well-defined, we just have to check that it is 0 on all of $\mathrm{im}(c_{i+1})$. If $x = c_{i+1}(y)$, then $f_i(x) = f_i(c_{i+1}(y)) = d_{i+1}(f_{i+1}(y))$, therefore $f_i(x) \in \mathrm{im}(d_{i+1})$, so it is 0 in $H_i(D_\bullet)$. It follows that f_i yield well-defined maps and these are obviously homomorphisms.

Prop 2.3

Let $\{f_i\}$ be a morphism from $C_\bullet$ to $D_\bullet$ such that there exist morphisms $s_i : C_i \to D_{i+1}$ with $f_i = s_{i-1}c_i + d_{i+1}s_i$, then f_i induce the zero morphism from $H_i(C_\bullet) \to H_i(D_\bullet)$ for all i.

Proof. Let $\overline{x} \in H_i(C_\bullet)$, then $f_i(\overline{x})$ is the image of $f_i(x)$ in $H_i(D_\bullet)$. We know that $f_i = s_{i-1}c_i + d_{i+1}s_i$. Now $d_{i+1}s_i(x)$ is 0 in $H_i(D_\bullet)$ since it is in the image of d_{i+1}. Since $x \in \ker(c_i)$, then $c_i(x) = 0$, so $s_{i-1}c_i(x) = 0$, therefore $f_i(x) = 0$ in $H_i(D_\bullet)$. Therefore f_i induce the 0 map $H_i(C_\bullet) \to H_i(D_\bullet)$ for all i. □

Prop 2.4

Let $\{g_i\} : P_\bullet \to D_\bullet$ be a morphism of chain complexes such that $P_i = D_i = 0$ for all $i < -1$ and P_i projective for all $i \geqslant 0$. Suppose that $g_{-1} = 0$, then g_i satisfy the assumptions of 2.3.

Proof. Let d_i and d'_i be the morphisms of $P_\bullet$ and $D_\bullet$ respectively. We have that $d'_0 \circ g_0 = 0$, therefore g_0 factors through the kernel of d'_0 which is exactly the image of $d'_1 : D_1 \to D_0$. Since d'_1 is an epimorphism from D_1 to $\mathrm{im}(D_1)$, then since P_0 is projective, then g_0 lifts to a morphism $s_0 : P_0 \to D_1$ such that $d'_1 \circ s_0 = g_0$. Now suppose that we have constructed $s_i : P_i \to D_{i+1}$

such that $g_i = s_{i-1} \circ d_i + d'_{i+1} \circ s_i$. Now consider $h = g_{i+1} - s_i d_{i+1}$. We then have that $d'_{i+1} h = g_i d_{i+1} - (d'_{i+1} s_i) d_{i+1}$. We know that $d'_{i+1} s_i = g_i - s_{i-1} d_i$, therefore

$$\begin{aligned} d'_{i+1} h &= g_i d_{i+1} - (g_i - s_{i-1} d_i) d_{i+1} \\ &= g_i d_{i+1} - g_i d_{i+1} + s_{i-1} d_i d_{i+1} \\ &= 0 + 0 \end{aligned}$$

It follows that h factors through the kernel of d'_{i+1}, but this is exactly the image of d'_{i+2}. We know that d'_{i+2} is an epimorphism from $D_{i+2} \to \operatorname{im}(d'_{i+2})$, therefore since P_{i+1} is projective, then in fact h factors through D_{i+2}, so $h = d'_{i+2} s_{i+1}$ for some morphism $s_{i+1} : P_{i+1} \to D_{i+2}$. Now writing out h, we have that $g_{i+1} = d'_{i+2} s_{i+1} + s_i d_{i+1}$ completing the induction. □

Thm 2.5

Let $P_\bullet \xrightarrow{\epsilon} M \to 0$ be a projective resolution of M, then given a morphism $\{f_i\}$ from $P_\bullet \xrightarrow{\epsilon} M \to 0$ to itself such that $f_{-1} : M \to M$ is the identity, then for any additive functor F, we have that $\{F(f_i)\}_{i=0}^{\infty}$ induce the identity morphism on $H_i(F(P_\bullet))$.

Proof. We have $\{\mathrm{id}_i\}$ which is the morphism $P_\bullet \to P_\bullet$ given by $\mathrm{id}_i : P_i \to P_i$ is the identity map. Let $g_i = f_i - \mathrm{id}_i$. We know that $F(g_i) = F(f_i) - F(\mathrm{id}_i)$ is 0 for $i = -1$, so by 2.4 and 2.3, $F(g_i)$ induce the 0 map on $H_i(P_\bullet)$. Since F is a functor, then $F(\mathrm{id}_i) = \mathrm{id}_{F(P_i)}$. It is fairly clear that $\mathrm{id}_{F(P_i)}$ induce the identity map on $H_i(F(P_\bullet))$, therefore we $0 = F(f_i) - \mathrm{id}_{H_i(F(P_\bullet))}$, so indeed $F(f_i)$ induce the identity map on $H_i(F(P_\bullet))$. □

Prop 2.6

Let F be an additive functor. Let $P_\bullet \to M \xrightarrow{\epsilon} 0$ be a projective resolution. Let $D_\bullet \xrightarrow{\epsilon'} N \to 0$ be an exact sequence. Let $f : M \to N$ be a morphism, then there is a morphism of chain complexes $\{f_i\}$ from $P_\bullet \xrightarrow{\epsilon} M \to 0$ to $D_\bullet \xrightarrow{\epsilon'} N \to 0$ such that $f_{-1} = f$. Furthermore for any other such morphism $\{f'_i\}$ with $f'_{-1} = f$, f_i and f'_i induce the same morphism $H_i(P_\bullet) \to H_i(D_\bullet)$.

Proof. We construct a morphism $\{f_i\}$ from $P_\bullet \xrightarrow{\epsilon} M \to 0$ to $D_\bullet \xrightarrow{\epsilon'} N \to 0$. We have the morphism $f \circ \epsilon$ from P_0 to N. Since ϵ' is an epimorphism, then we obtain a morphism $f_0 : P_0 \to D_0$ such that $\epsilon' \circ f_0 = f \circ \epsilon$. Suppose we have constructed $f_i : P_i \to D_i$, then we have that $f_i : \ker(d_i) \to D_i$ composes with d'_i to 0, so it factors through $\ker(d_i)$, therefore $f_i : \ker(d_i) \to \ker(d'_i)$. Now since $\ker(d'_i) = \operatorname{im}(d'_{i+1})$ and $d'_{i+1} : D_{i+1} \to \operatorname{im}(d'_{i+1})$ is an epimorphism, then since P_{i+1} is surjective, then we obtain a morphism $f_{i+1} : P_{i+1} \to D_{i+1}$ such that $d'_{i+1} \circ f_{i+1} = f_i \circ d_{i+1}$.

Now if f'_i is another lift of f, then $f_i - f'_i$ is 0 at $i = -1$, so the difference induces 0 on homology, thus f_i and f'_i induce the same morphism on homology as desired. □

We can now show that derived functors are well-defined.

Prop 2.7

Let F be an additive functor. Let $P_\bullet \xrightarrow{\epsilon} M \to 0$ and $P'_\bullet \xrightarrow{\epsilon'} M \to 0$ be two projective resolutions of M, then $H_i(F(P_\bullet)) \cong H_i(F(P'_\bullet))$ for all i. Furthermore, any morphism $\{f_i\}$ from $P_\bullet \xrightarrow{\epsilon} M \to 0$ to $P'_\bullet \xrightarrow{\epsilon'} M \to 0$ such that $f_{-1} = \mathrm{id}_M$ induces this isomorphism.

Proof. By swapping the labels of P' and P, we can construct a morphism $\{g_i\}$ from $P' \xrightarrow{\epsilon'} M \to 0$ to $P \xrightarrow{\epsilon} M \to 0$ such that $g_{-1} = \mathrm{id}_M$. It follows that $\{f_i \circ g_i\}$ and $\{g_i \circ f_i\}$ are morphisms from $P' \xrightarrow{\epsilon'} M \to 0$ to itself and $P \xrightarrow{\epsilon} M \to 0$ to itself respectively which are id_M on M. It follows by 2.3, that $f_i \circ g_i$ and $g_i \circ f_i$ induce the identity morphisms on $H_i(F(P_\bullet))$ and $H_i(F(P'_\bullet))$.

To now check that this means that $\{f_i\}$ are isomorphisms, we need only make the simple check that given two morphisms $\{f_i\} : B_\bullet \to C_\bullet$ and $\{g_i\} : A_\bullet \to B_\bullet$, then the morphism of $\{f_i \circ g_i\}$ induced on the homology is the composition of the induced morphisms of f_i and g_i. For any $\overline{x} \in H_i(A_\bullet)$, we have that $(f_i \circ g_i)(\overline{x}) = \overline{(f_i \circ g_i)(x)} = \overline{f_i(g_i(x))} = f_i(\overline{g_i(x)}) = f_i(g_i(\overline{x}))$. Note that I have again abused notation by denoting the morphism on homology by the same symbol as the morphism on the chain complexes. It follows from the above that the morphisms on homology do indeed compose to the identity, so our morphisms give isomorphism $H_i(F(P)) \to H_i(F(P'))$

To see the final claim, that this isomorphism is canonical, suppose that we chose another lift of id_M denoted $\{f'_i\}$ with $f'_{-1} = \mathrm{id}_M$, then $\{f_i - f'_i\}$ is a morphism such that $f_{-1} - f'_{-1} = 0$, therefore it induces the zero morphism on homology by 2.3, thus f_i and f'_i induce the same morphism on homology. Therefore this isomorphism is canonical. □

It is perhaps now clear why we ask for projective and injective objects. These are exactly the kinds of objects that allow us to propagate morphisms back along complexes.

Now that we know that $L_iF(M)$ are well-defined, then the next question that we can ask if whether or not it is a functor.

Prop 2.8

Let F be an additive functor, then L_iF is an additive functor for all i.

Proof. For any morphism $f : M \to N$, given projective resolutions, $P_\bullet \to M \to 0$ and $P'_\bullet \to N \to 0$, we obtain a morphism $\{f_i\}$ between the chain complexes by 2.6 such that $f_{-1} = f$. Furthermore, any such such morphism of chain complexes induce the same morphism $H_i(F(P_\bullet)) \to H_i(F(P'_\bullet))$. It follows that f yields a well-defined morphism $L_iF(M) \to L_iF(N)$ for all i. If $f = \mathrm{id}$, then by 2.7, it induces the identity on $L_iF(M)$, so $L_iF(\mathrm{id}) = \mathrm{id}$. Furthermore, if $f : M \to N$ and $g : N \to K$, then choosing $\{f_i\}$ and $\{g_i\}$ morphisms from the projective resolution of M to N and N to K respectively, then since $\{g_i \circ f_i\}$ lifts $g \circ f$, then it follows that $L_iF(g \circ f) = g_i \circ f_i = L_iF(g) \circ L_iF(f)$.

The fact that L_iF is additive is immediate since if we have $f, g : M \to N$, then letting $\{f_i\}, \{g_i\}$ lift f, g to a projective resolution, then $\{f_i + g_i\}$ is a lift of $f + g$ and therefore $L_iF(f + g) = f_i + g_i = L_iF(f) + L_iF(g)$ as desired. □

Prop 2.9

Let F be an additive right exact functor, then L_0F is naturally isomorphic to F.

Proof. Let $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$ be a projective resolution, then since F is right exact, it follows that $F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(\epsilon)} F(M) \to 0$ is exact. Therefore $\mathrm{im}(F(d_1)) = \ker(F(\epsilon))$, so $L_0F(M) = F(P_0)/\mathrm{im}(F(d_1)) = F(P_0)/\ker(F(\epsilon)) = \mathrm{coim}(F(\epsilon)) = \mathrm{im}(F(\epsilon)) = F(M)$. Denote this isomorphism by $\eta(M) : L_0F(M) \to F(M)$. We now need to show that this isomorphism on objects commutes. More precisely, if $f : M \to N$, then we get $\eta(M)$ and $\eta(N)$. We want that $F(f) \circ \eta(M) = \eta(N) \circ L_0F(f)$. If we can show this, then the isomorphism is a natural transformation and a natural transformation all of whose morphisms are isomorphisms automatically has an inverse which is also a natural transformation, thus F is naturally isomorphic to L_0F. We can do this by considering the following daigram:

$$\begin{tikzcd}
\ker(F(d_0)) \arrow[r] & F(P_0) \arrow[r, "d_0"] \arrow[d, "F(f_0)"'] & \mathrm{coim}(F(d_0)) \arrow[r, "\sim"] \arrow[d, "L_0F(f)"'] & \mathrm{im}(F(d_0)) \arrow[r, "\sim"] & F(M) \arrow[d, "F(f)"'] \\
\ker(F(d_0')) \arrow[r] & F(P_0') \arrow[r] & \mathrm{coim}(F(d_0')) \arrow[r, "\sim"] & \mathrm{im}(F(d_0')) \arrow[r, "\sim"] & F(N)
\end{tikzcd}$$

The composition of the maps from the coimage to $F(M)$ or $F(N)$ are $\eta(M)$ and $\eta(N)$ respectively. We know that the square with vertices $F(P_0), F(M), F(P_0'), F(N)$ commutes and by definition of $L_0F(f)$, we know the square with vertices $F(P_0), \mathrm{coim}(F(d_0)), F(P_0'), \mathrm{coim}(F(d_0'))$ commutes. Furthermore we know that d_0 is an epimorphism onto $\mathrm{coim}(F(d_0))$. Therefore the equation $F(f) \circ \eta(M) = \eta(N) \circ L_0F(f)$ follows from the following fact: □

Prop 2.10

Consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "a"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
A' \arrow[r, "g"] & B' \arrow[r, "b"] & C'
\end{tikzcd}$$

Suppose that the left square commutes, that the square A, C, A', C' commutes and that f is an epimorphism, then the right square commutes.

Proof. We need to check that $\gamma \circ a = b \circ \beta$. Since f is an epimorphism, then in fact we can check this after composition with f. It follows that:

$$\begin{aligned}
\gamma \circ a \circ f &= b \circ g \circ \alpha \\
&= b \circ \beta \circ f
\end{aligned}$$

Therefore since f is an epimorphism, then $\gamma \circ a = b \circ \beta$ and thus the right square commutes. □

I want to finish the discussion of derived functors by proving the long exact sequence. To do so, we will need simultaneous projective resolutions.

Prop 2.11

Consider the following commutative diagram with exact rows:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

Then there is an exact sequence $\ker a \to \ker b \to \ker c \to \mathrm{coker} a \to \mathrm{coker} b \to \mathrm{coker} c$

Proof. I will prove this for $\mathbf{Mod}(R)$. The general proof is left as an exercise that I don't want to do. For $x \in \ker(a)$, we have that $b(f(x)) = f'(a(x)) = 0$, therefore $f(x) \in \ker(b)$, so f yields a morphism $f : \ker a \to \ker b$. Similarly we get $g : \ker b \to \ker c$. Suppose that $g(x) = 0$, then $x \in \ker(g) = \mathrm{im}(f)$, so $x = f(y)$. Since f' is injective, then $a(y) = 0$ iff $f'(a(y)) = b(f(y)) = b(x) = 0$, thus $a(y) = 0$, so $y \in \ker a$. It follows that we have exactness at $\ker b$. Now given any $x \in \ker c$, e have that $x = g(y)$ for some $y \in B$ since g is surjective. Then $g'(b(y)) = c(x) = 0$ so $b(y) \in \ker(g') = \mathrm{im}(f')$, thus $b(y) = f'(z)$ for some z. We then let $\delta(x) = z$. If we choose y' such that $g(y') = g(y)$, then $g(y' - y) = 0$ so $y' - y = f(w)$ for some $w \in A$. Let $b(y') = f'(z')$. We know that $y' - y = f(w)$, so $b(y' - y) = b(f(w)) = f'(a(w))$. Since f' is injective, then $a(w) = z' - z$, so $z' = z$ in $\mathrm{coker} a$. It follows that δ is well-defined. It is fairly clearly a morphism. Now suppose that $\delta(x) = 0$, then $b(y) = f'(z)$ with z in $\mathrm{im}(a)$, so $z = a(w)$, thus $b(y) = f'(a(w)) = b(f(w))$. Then $y - f(w) \in \ker b$ and $g(y - f(w)) = g(y) - 0 = x$, so $x \in \mathrm{im}(g)$, thus we have exactness at $\ker c$. To get a morphism $\mathrm{coker} a \to \mathrm{coker} b$, we take $\overline{x} \in \mathrm{coker} a$, then map it to $\overline{f'(x)}$. This is well-defined since if $x = a(w)$, then $f'(x) = f'(a(w)) = b(f(w)) \in \mathrm{im}(b)$, so it is 0 in $\mathrm{coker} b$. We similarly get the map $\mathrm{coker} b \to \mathrm{coker} c$. We now check exactness. Suppose that $\overline{x} \in \mathrm{coker} a$ and $f'(x) \in \mathrm{im}(b)$, so $f'(x) = b(y)$, then $g'(b(y)) = f'(g'(y)) = 0$. Furthermore $g'(b(y)) = c(g(y)) = 0$, thus $g(y) \in \ker c$. It follows that $\overline{x} = \delta(g(y))$, so we have exactness at $\mathrm{coker} a$. Finally, if $g'(x) \in \mathrm{im}(c)$, then $g'(x) = c(y)$, then $y = g(z)$. We then see that $g'(x - b(z)) = g'(x) - c(g(z)) = g'(x) - g'(x) = 0$. therefore $x - b(z) \in \ker(g') = \mathrm{im}(f')$, so $x - b(z) = f'(w)$. Now modding out by $\mathrm{im}(b)$, we see that in $\mathrm{coker}(b)$, $x = f'(w)$, so $x \in \mathrm{im}(f')$ as desired.

Now note furthermore that if f is injective, then if $x \in \ker a$ with $f(x) = 0$, then $x = 0$ so $f : \ker a \to \ker b$ is injective. And similarly if g' is surjective, then for any $y \in \mathrm{coker} c$ we have that $y = g'(x)$ for some $x \in \mathrm{coker} b$, so $g' : \mathrm{coker} b \to \mathrm{coker} c$ is surjective. $\square$

Def: A short exact sequence $0 \to A \to B \xrightarrow{f} C \to 0$ is called split if there is a morphism $g : C \to B$ such that $f \circ g = \mathrm{id}_C$.

The most common example of a split short exact sequence is $0 \to A \to A \oplus B \to B \to 0$ where the morphism $A \oplus B \to B$ is pr_B and the map back is $i_B : B \to A \oplus B$. In fact, this is the only split short exact sequence in the following sense:

Def: Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be short exact sequences. A morphism of short exact sequences is a triple $f : A \to A', g : B \to B', h : C \to C'$ such that the following commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Prop 2.12

Let $0 \to A \xrightarrow{f'} B \xrightarrow{f} C \to 0$ be a split short exact sequence with $g : C \to B$ such that $f \circ g = \mathrm{id}_C$. Then there exists an isomorphism of short exact sequences $(\mathrm{id}_A, \alpha, \mathrm{id}_C)$ to the sequence $0 \to A \to A \oplus C \to C \to 0$.

Proof. We first show that we may find $g' : B \to A$ such that $g'f' = \mathrm{id}_A$. To do so, consider the morphism $h : B \to B$ given by $\mathrm{id}_B - gf$. We have that $f(\mathrm{id}_B - gf) = f - fgf = f - f = 0$, therefore h factors through $\ker(f) \cong \mathrm{im}(f') \cong A$, i.e. we may find $g' : B \to A$ such that $f'g' = h$. Since $(A, f') = \ker(f)$, then f' is a monomorphism, but $f'g'f' = hf' = (\mathrm{id}_B - gf)f' = f' - gff'$ and $ff' = 0$, so $f'g'f' = f' = f'\mathrm{id}_A$, then since f' is a monomorphism, we get that $g'f' = \mathrm{id}_A$ as desired.

We now have morphisms $f : B \to C$ and $g' : B \to A$, thus we obtain a morphism $\alpha : B \to A \oplus C$ such that $\mathrm{pr}_A \alpha = g'$ and $\mathrm{pr}_B \alpha = f$. Since pr_A is an epimorphism, then from $\mathrm{pr}_A \alpha = g'$ we get $\mathrm{pr}_A \alpha f' = g'f' = \mathrm{id}_A = \mathrm{pr}_A i_A$, therefore $\alpha f' = i_A$. It follows that the diagram commutes, so $(\mathrm{id}_A, \alpha, \mathrm{id}_C)$ is indeed a morphism of short exact sequences. To see that it is an isomorphism, we need only show that α is an isomorphism. To find an inverse, since products and coproducts are the same in abelian categories, then there exists a map $\beta : A \oplus C \to B$ such that $\beta i_A = f'$ and $\beta i_C = g$. We use the same idea to show that $\alpha\beta = \mathrm{id}_{A \oplus C}$ and $\beta\alpha = \mathrm{id}_B$.

We know that $\alpha\beta = \mathrm{id}$ iff $\alpha\beta i_A = i_A$ and $\alpha\beta i_C = i_C$ by the universal property of the coproduct. We have that $\alpha\beta i_A = \alpha f' = i_A$ as shown before. On the other hand, $\alpha\beta i_C = \alpha g$ and $\alpha g = i_C$ iff $\mathrm{pr}_A \alpha g = \mathrm{pr}_A i_C = 0$ and $\mathrm{pr}_C \alpha g = \mathrm{pr}_C \mathrm{id}_C = \mathrm{id}$. We know that $\mathrm{pr}_C \alpha = f$, so $\mathrm{pr}_C \alpha g = fg = \mathrm{id}_C$ as desired and $\mathrm{pr}_A \alpha g = g'g$. Since f' is a monomorphism, then $g'g = 0$ iff $f'g'g = 0$ and $f'g' = h$, then $hg = g - gfg = g - g = 0$ as desired. It follows that $\alpha\beta = \mathrm{id}_{A \oplus C}$.

Finally, we show that $\beta\alpha = \mathrm{id}_B$. We first show that β is an epimorphism. To see this, let $k : B \to D$ such that $k\beta = 0$, then $k\beta i_A = kf' = 0$, so k factors through $\mathrm{coker}(f') = \mathrm{coim}(f) = \mathrm{im}(f)$, thus $k = \widetilde{k}f$ for some $\widetilde{k} : C \to D$, but then $0 = k\beta i_C = kg = \widetilde{k}fg$ but $fg = \mathrm{id}_C$, so $\widetilde{k} = 0$, thus $k = 0$. It follows that β is an epimorphism, so we may check that $\beta\alpha = \mathrm{id}_B$ by showing that $\beta\alpha\beta = \beta$ but we know that $\alpha\beta = \mathrm{id}_{A \oplus C}$, so this follows immediately. Therefore α is an isomorphism, so the short exact sequences are isomorphic. □

In the above, the maps on A and C are the identity, so in fact we have better than an isomorphism of short exact sequences, this is what is an isomorphism of extensions. The above proposition essentially says that the identity element in $\mathrm{Ext}^1(C, A)$ is the isomorphism class of split short exact sequences which is a priori stronger than the isomorphism class of trivial extensions of C by A.

Prop 2.13

Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence, then we may find projective resolutions $P'_\bullet, P_\bullet, P''_\bullet$ of A', A, A'' respectively such that the induced morphisms give a split exact sequence of morphisms $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$.

Proof. I will again work in $\mathbf{Mod}(R)$. We first construct P_0 as $P''_0 \oplus P'_0$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P'_0 \arrow[r, "d'_0"] \arrow[d] \arrow[dr, dashed, "\widetilde{f}"] & A' \arrow[r] \arrow[d, "f"] & 0 \\
P''_0 \oplus P'_0 \arrow[r] \arrow[d] & A \arrow[d, "g"] & \\
P''_0 \arrow[r, "d''_0"] \arrow[d] \arrow[ur, dashed, "\widetilde{g}"] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

The vertical morphisms from $P'_0 \to P''_0 \oplus P'_0$ is the inclusion into the second coordinate and then from $P''_0 \oplus P'_0$ to P''_0 is the projection to the first coordinate. It follows that the left column is exact. Since P''_0 is projective and g is an epimorphism, then we obtain $\widetilde{g} : P''_0 \to A$ such that $g \circ \widetilde{g} = d''_0$. We simply let $\widetilde{f} = f \circ d'_0$. Then the map from $P''_0 \oplus P'_0 \to A$ is $(\widetilde{g}, \widetilde{f}) = \widetilde{g} \circ \mathrm{pr}_{P''_0} + \widetilde{f} \circ \mathrm{pr}_{P'_0}$. To see that it is surjective, let $x \in A$, then $g(x) = d''_0(y)$ for some $y \in P''_0$, so then $x - (\widetilde{g}, \widetilde{f})(y, 0)$ maps to $g(x) - g(x) = 0$ under g since $g(\widetilde{g}(y)) = d''_0(y) = g(x)$. It follows that $x - \widetilde{g}(y) \in \ker g = \operatorname{im} f$, so $x - \widetilde{g}(y) = f(z)$. Since d'_0 is surjective, then $z = d'_0(w)$, then we have that $(\widetilde{g}, \widetilde{f})(y, w) = \widetilde{g}(y) + \widetilde{f}(w) = \widetilde{g}(y) + x - \widetilde{g}(y) = x$, therefore $(\widetilde{g}, \widetilde{f})$ is surjective. Notice that the sequence $0 \to P'_0 \to P''_0 \oplus P'_0 \to P''_0 \to 0$ is split exact since we have the morphism $i_{P''_0} : P''_0 \to P''_0 \oplus P'_0$.

Now we want to continue the resolution. We consider the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
P'_{n+1} \arrow[r] & \operatorname{im}(d'_{n+1}) \arrow[r, "\sim"'] \arrow[d] & \ker(d'_n) \arrow[r] \arrow[d] & P'_n \arrow[r, "d'_n"'] \arrow[d] & P'_{n-1} \arrow[d] \\
& \operatorname{im}(d_{n+1}) \arrow[r, "\sim"'] \arrow[d] & \ker(d_n) \arrow[r] \arrow[d] & P_n \arrow[r, "d_n"'] \arrow[d] & P_{n-1} \arrow[d] \\
P''_{n+1} \arrow[r] & \operatorname{im}(d''_{n+1}) \arrow[r, "\sim"'] \arrow[d] & \ker(d''_n) \arrow[r] \arrow[d] & P''_n \arrow[r, "d''_n"'] \arrow[d] & P''_{n-1} \arrow[d] \\
& 0 & 0 & 0 & 0
\end{tikzcd}$$

Note that we obtain the morphisms of kernels by the snake lemma (2.11). Then the images of the d_{n+1}'s are the kernels, so obtain morphisms of the images. But since P'_{n+1} and P''_{n+1} surject onto their images, then we are back in the first case, so we know that we can extend the resolution one more step, i.e. we may find a morphism out of $P_{n+1} = P''_{n+1} \oplus P'_{n+1}$ to the image of d_{n+1} such that the diagram commutes. This map, which we call d_{n+1}, then composes with d_n to be 0, i.e. $d_n \circ d_{n+1} = 0$ since d_{n+1} factors through $\ker(d_n)$ by construction of d_{n+1}. It follows that this is a chain complex. □

We can now prove two of the most important fact about derived functors, that being the long exact sequence. This will then generalize derived functors to the notion of a delta functor.

Prop 2.14

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence and let $F : \mathfrak{U} \to \mathfrak{B}$ be an additive right exact functor. Then there exists a long exact sequence:

$$\cdots \xrightarrow{L_1F(f)} L_1F(B) \xrightarrow{L_1F(g)} L_1F(C) \xrightarrow{\delta_1} F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

where $\delta_i : L_iF(C) \to L_{i-1}F(A)$.

Proof. By 2.13, there exists a split exact simultaneous projective resolution $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ of $0 \to A \to B \to C \to 0$. For any n, the sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact, i.e. we have maps $P_n \to P'_n$ and $P''_n \to P_n$ which compose correctly to the identity. This must still be true after applying F, so we have $0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P'_\bullet) \to 0$ remains a short exact sequence of chain complexes. We now show that for any short exact sequence of chain complexes, we obtain a long exact sequence on their homology, which is exactly the long exact sequence that we require. □

Prop 2.15

Let $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ be a short exact sequence of chain complexes, then we obtain a long exact sequence of their homology:

$$\cdots \to H_{i+1}(C_\bullet) \xrightarrow{\delta_{i+1}} H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \xrightarrow{\delta_i} H_{i-1}(A_\bullet) \to \cdots$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & B_{i+2} \arrow[r, "g"] \arrow[d, "d^B_{i+2}"] & C_{i+2} \arrow[r] \arrow[d, "d^C_{i+2}"] & 0 \\
0 \arrow[r] & \ker(d^A_{i+1}) \arrow[r] \arrow[d] & \ker(d^B_{i+1}) \arrow[r, "g'"] \arrow[d] & \ker(d^C_{i+1}) \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d, "d^A_{i+1}"] & B_{i+1} \arrow[r] \arrow[d, "d^B_{i+1}"] & C_{i+1} \arrow[r] \arrow[d, "d^C_{i+1}"] & 0 \\
0 \arrow[r] & \ker(d^A_i) \arrow[r] \arrow[d] & \ker(d^B_i) \arrow[r] \arrow[d] & \ker(d^C_i) \arrow[d] & \\
 & H_i(A_\bullet) \arrow[r] & H_i(B_\bullet) \arrow[r] & H_i(C_\bullet) &
\end{tikzcd}$$

We want to show that $\delta d^C_{i+2} = 0$. Since g is an epimorphism, then it suffices to show that $\delta d^C_{i+2} g = 0$, but $d^C_{i+2} g = g' d^B_{i+2}$, so by exactness at $\ker(d^C_{i+1})$ in the snake lemma, 2.11, we get that $\delta g' = 0$, thus $\delta d^C_{i+2} = 0$. It follows that we obtain a map $\delta : H_{i+1}(C_\bullet) \to H_i(A_\bullet)$ since δ factors through the cokernel of d^C_{i+2} which is $H_{i+1}(C_\bullet)$. Since the image of $g' : \ker(d^B_{i+1}) \to H_{i+1}(C_\bullet)$ and $g' : H_{i+1}(B_\bullet) \to H_{i+1}(C_\bullet)$ are the same, then after quotienting the kernels, we retain exactness. Therefore, we have an exact sequence:

$$H_{i+1}(B_\bullet) \xrightarrow{g'} H_{i+1}(C_\bullet) \xrightarrow{\delta} H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet)$$

Joining these exact sequences together for all i yields the long exact sequence. □

Prop 2.16

Let F be an additive right exact functor, then let (f, g, h) be a morphism of short exact sequences, $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$. Then the following diagram commutes for all n:

$$\begin{tikzcd} L_nF(C) \arrow[r, "\delta_n"] \arrow[d, "L_nF(h)"] & L_{n-1}F(A) \arrow[d, "L_{n-1}F(f)"] \\ L_nF(C') \arrow[r, "\delta_n"] & L_{n-1}F(A') \end{tikzcd}$$

Proof. Omitted, mostly because it is annoying. The idea is to take the simultaneous projective resolutions and the maps from 2.6 and make them commute inductively. □

3 δ-Functors

Def: A covariant homological *δ-functor* from $\mathfrak{U}$ to $\mathfrak{B}$ is a collection of additive functors $T_n : \mathfrak{U} \to \mathfrak{B}$ for $n \geqslant 0$ ($T_n = 0$ for $n < 0$) with morphisms $\delta_n : T_n(C) \to T_{n-1}(A)$ for each short exact sequence $0 \to A \to B \to C \to 0$ such that:

1. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then

$$\cdots \to T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \to \cdots$$

2. For any morphism (f, g, h) of short exact sequences $0 \to A \to B \to C \to 0$ to $0 \to A' \to B' \to C' \to 0$, the δ's commute:

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"] & T_{n-1}(A) \arrow[d, "T_{n-1}(f)"] \\ T_n(C') \arrow[r, "\delta_n"] & T_{n-1}(A') \end{tikzcd}$$

We showed at the end of the last section that derived functors satisfy the conditions to be a δ-functor. In this sense, δ-functors are generalizations of derived functors. However, we will see that upon addition of another condition to δ-functors, we once again obtain derived functors. Derived functors are in a sense the δ-functors which are most appropriately determined by T_0, however general δ-functors are not determined by T_0, e.g. let F be a right exact functor, then $\{L_iF\}$ is a δ-functor, but so is the collection $\{T_i\}$ where $T_0 = F$ and $T_i = 0$ for all $i > 0$.

Def: A *morphism of δ-functors* $S \to T$ is a collection of natural transformations $\eta_n : S_n \to T_n$ which commute with δ_n.

Def: A δ-functor T is *universal* if for any other δ-functor S and a natural transformation $f_0 : S_0 \to T_0$, there exists a unique morphism of δ-functors $\eta : S \to T$ with $\eta_0 = f_0$.

Prop 3.1

Let S, T be universal δ-functors such that $f_0 : S_0 \to T_0$ is a natural isomorphism. Then S and T are isomorphic.

Proof. There exist unique morphisms $\eta : S \to T$ and $\zeta : T \to S$ such that $\eta_0 = f_0$ and $\zeta_0 = f_0^{-1}$. Then $\eta\zeta : T \to T$ is such that $(\eta\zeta)_0 = \eta_0\zeta_0 = \mathrm{id}_0 : T_0 \to T_0$. We know that $\mathrm{id} : T \to T$ extends $\eta_0\zeta_0$, then by uniqueness coming from the fact that T is universal, we get that $\eta\zeta = \mathrm{id}$. Exchanging the names of S and T, we see that $\zeta\eta = \mathrm{id}$, thus S and T are isomorphic δ-functors. $\square$

We see from the above that universal δ-functors are determined by their first functor. It follows that if we can show that derived functors are universal, then the universal δ-functors are exactly just the derived functors.

Def: An additive functor $F : \mathfrak{U} \to \mathfrak{B}$ is called *coeffaceable* if for every A, there is an epimorphism $u : P \to A$ with P projective such that $F(u) = 0$.

Prop 3.2

Let T be a δ-functor such that T_n is coeffaceable for all $n \geqslant 1$, then T is universal.

Proof. Let S be a δ-functor and $f_0 : S_0 \to F$ a natural transformation. Let $A \in \mathfrak{U}$, then let P be projective with $0 \to K \to P \xrightarrow{u} A \to 0$ exact with $T_n(u) = 0$ for $n \geqslant 1$. Then from the long exact sequences we get:

$$\begin{tikzcd}
 & S_{n+1}(A) \arrow[r, "\delta^S_{n+1}"] & S_n(K) \arrow[r] \arrow[d, "f_n(K)"] & S_n(P) \arrow[d, "f_n(P)"] \\
T_{n+1}(P) \arrow[r, "0"] & T_{n+1}(A) \arrow[r, "\delta^T_{n+1}"] & T_n(K) \arrow[r, "g"] & T_n(P)
\end{tikzcd}$$

Note that $T_{n+1}(u) = 0$ for all n since $n \geqslant 0$. Since f_n is a natural transformation, then the diagram commutes. It follows that $gf_n(K)\delta^S_{n+1} = 0$, so $f_n(K)\delta^S_{n+1}$ factors through $\ker(g) = \mathrm{im}(\delta^T_{n+1})$. Since $\ker(\delta^T_{n+1}) = 0$, then $\mathrm{im}(\delta^T_{n+1}) = T_{n+1}(A)$. It follows that there is a unique morphism $f\_n + 1(A) : S_{n+1}(A) \to T_{n+1}(A)$ such that $\delta^T_{n+1}f_{n+1}(A) = f_n(K)\delta^S_{n+1}$. This relation must hold if f_* is a morphism of δ-functors. Therefore if f_{n+1} is indeed a natural transformation and commutes with the δ's, then it is unique, which proves that T is a universal δ-functor.

To see that f_{n+1} is a natural transformation, let $g : A \to B$ be any morphism. Let $0 \to K_A \to P_A \xrightarrow{u_A} A \to 0$ and $0 \to K_B \to P_B \xrightarrow{u_B} B \to 0$ with P_A, P_B projective and u_A, u_B as in the definition of f_{n+1}. Now consider the following diagram:

$$\begin{tikzcd}
S_n(K_A) \arrow[rrr, "S_n(g)"] \arrow[ddd, "f_n(K_A)"] & & & S_n(K_B) \arrow[ddd, "f_n(K_B)"] \\
& S_{n+1}(A) \arrow[ul, "\delta^S_{n+1}"'] \arrow[r, "S_{n+1}(g)"] \arrow[d, "f_{n+1}(A)"] & S_{n+1}(B) \arrow[ur, "\delta^S_{n+1}"] \arrow[d, "f_{n+1}(B)"] & \\
& T_{n+1}(A) \arrow[dl, "\delta^T_{n+1}"'] \arrow[r, "T_{n+1}(g)"] & T_{n+1}(B) \arrow[dr, "\delta^T_{n+1}"] & \\
T_n(K_A) \arrow[rrr, "T_n(g)"] & & & T_n(K_B)
\end{tikzcd}$$

We want to show that the inside square commutes. We know that $\delta^T_{n+1} : T_{n+1}(B) \to T_n(K_B)$ is a monomorphism since we have $T_{n+1}(P_B) \xrightarrow{0} T_{n+1}(B) \xrightarrow{\delta^T_{n+1}} T_n(K_B)$. Therefore it suffices to show that the inside square commutes after composition with δ^T_{n+1}. By definition of f_*, S_*, and T_* and induction on n, we know that the five outside squares commute. We now compute:

$$\begin{aligned}
\delta^T_{n+1} f_{n+1}(B) S_{n+1}(g) &= f_n(K_b)\delta^S_{n+1} S_{n+1}(g) \\
&= f_n(K_B) S_n(g) \delta^S_{n+1} \\
&= T_n(g) f_n(K_A) \delta^S_{n+1} \\
&= T_n(g) \delta^T_{n+1} f_{n+1}(A) \\
&= \delta^T_{n+1} T_{n+1}(g) f_{n+1}(A)
\end{aligned}$$

Therefore the inside square commutes, so f_{n+1} is a natural transformation. Finally we need to check that $\delta^T_{n+1} f_{n+1} = f_n \delta^S_{n+1}$. Let $0 \to A \to B \to C \to 0$ be a short exact sequence and let $0 \to K \to P \xrightarrow{u} C \to 0$ with P projective and $T_n(u) = 0$. Then since $B \to C$ is an epimorphism, we get a morphism $P \to B$ and since $K \to P \to B$ composes with $B \to C$ to give 0, then we get a morphism $K \to A$ as below:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, dashed, "g"] & P \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

From the definition of a δ-functor, we get that $S_n(g)\delta^S_{n+1} = \delta^S_{n+1}$. Now consider the following diagram:

$$\begin{tikzcd}
S_n(A) \arrow[ddd, "f_n(A)"'] & & S_n(K) \arrow[ll, "S_n(g)"'] \arrow[ddd, "f_n(K)"] \\
& S_{n+1}(C) \arrow[ul, "\delta^S_{n+1}"] \arrow[ur, "\delta^S_{n+1}"'] \arrow[d, "f_{n+1}(C)"] & \\
& T_{n+1}(C) \arrow[dl, "\delta^T_{n+1}"'] \arrow[dr, "\delta^T_{n+1}"] & \\
T_n(A) & & T_n(K) \arrow[ll, "T_n(g)"]
\end{tikzcd}$$

We need to show that the left square commutes. From what was just said, we know that the top and bottom triangles commute. Furthermore, since f_n is a natural transformation, we know that the outside square commutes. By definition of $f_{n+1}(C)$, we know that the right square commutes. We now compute:

$$\begin{aligned}
\delta^T_{n+1} f_{n+1}(C) &= T_n(g)\delta^T_{n+1} f_{n+1}(C) \\
&= T_n(g) f_n(K) \delta^S_{n+1} \\
&= f_n(A) S_n(g) \delta^S_{n+1} \\
&= f_n(A) \delta^S_{n+1}
\end{aligned}$$

as desired. It follows that T is a universal δ-functor. □

Prop 3.3

Let F be an additive right exact functor, then $L_* F$ is a universal δ-functor.

Proof. Let $A \in \mathfrak{U}$, then for any projective $P \in \mathfrak{U}$ with an epimorphism $u : P \to A$ (which exists since $\mathfrak{U}$ has enough projectives), then we have that $L_i F(u) : L_i F(P) \to L_i F(A)$ but $L_i F(P) = 0$ for all $i > 0$, therefore $L_i F$ are coeffaceable for $i > 0$ and thus $L_* F$ is a universal δ-functor by 3.2. □

We can now see an application of this theory. Note that all of the above works for right derived functors and effaceable functors (e.g. for all A, there is an injective I and monomorphism $u : A \to I$ such that $F(u) = 0$).

Prop 3.4

Let X be a topological space and let $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{R} \to 0$ be a short exact sequence of sheaves on X. Let Γ be the global sections functor, then $0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \to \Gamma(\mathscr{R})$ is exact, i.e. Γ is left exact.

Proof. Let $x \in \mathscr{F}(X)$ and suppose that $\varphi(X)(x) = 0$, then since φ is injective, we have that $\varphi(X)$ is injective, so $x = 0$. Note that for sheaves, a monomorphism is a morphisms which is injective on all sections (the analogous statement about surjective morphisms is not true). To see exactness in the middle, suppose that $x \in \mathscr{G}(X)$ and $\psi(X)(x) = 0$, then $x \in \ker(\psi) = \operatorname{im}(\varphi)$. Recall that we showed that if φ is injective, then the naive image presheaf is in fact a sheaf, so $x \in \operatorname{im}(\varphi)(X)$ implies that $x = \varphi(X)(y)$ for some y, thus we have exactness in the middle. It follows that Γ is left exact. □

Note that Γ is not an exact functor. The category $\mathbf{Ab}(X)$ of sheaves of abelian groups has enough injectives. This allows us to give the following definition:

Def: Let X be a topological space and let $\mathscr{F}$ be a sheaf of abelian groups, then the i^{th} cohomology group of $\mathscr{F}$ is $R^i\Gamma(\mathscr{F})$, denoted $H^i(X, \mathscr{F})$.

Prop 3.5

Let $f : X \to Y$ be continuous. Let $\mathscr{F}$ be a sheaf of abelian groups on X, then the sheaf $R^i f_*(\mathscr{F})$ is the sheaf associated to the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$.

Proof. Let $\mathscr{H}^i(\mathscr{F})$ be the sheafification of the given presheaf. We first want to show that $\mathscr{H}^i$ is a δ-functor. For any short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$, since restriction to an open subset of X is exact, then $0 \to \mathscr{F}|_{f^{-1}(U)} \to \mathscr{G}|_{f^{-1}(U)} \to \mathscr{R}|_{f^{-1}(U)} \to 0$ is a short exact sequence. Then we obtain a long exact sequence of cohomology groups,

$$\cdots \to H^i(f^{-1}(U), \mathscr{G}|_{f^{-1}(U)}) \to H^i(f^{-1}(U), \mathscr{R}|_{f^{-1}(U)}) \to H^{i+1}(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)}) \to \cdots$$

For $V \subseteq U$, we have the restriction map $f_0 = \rho_{f^{-1}(U), f^{-1}(V)}$. Since H^i are universal δ-functors, then the definition of a morphism of sheaves is equivalent to saying that f_0 is a natural transformation, thus we obtain the restriction morphisms $f_n^{U,V}$ on the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$. We then have that $f_n^{U,W} = f_n^{U,V} f_n^{V,W}$ since they agree when $n = 0$. It follows that $\mathscr{H}^i(\mathscr{F})$ with restriction maps $f_n^{U,V}$ is a sheaf and since the $f_n^{U,V}$ are natural transformations, then this implies that in fact $\mathscr{H}^i$ is a functor, since $f_n^{U,V}$ being natural transformations says that it preserves morphisms of sheaves.

Since we obtain a long exact sequence for all sections, then we have a long exact sequence:

$$\cdots \to \mathscr{H}^i(\mathscr{G}) \to \mathscr{H}^i(\mathscr{R}) \xrightarrow{\delta^i} \mathscr{H}^{i+1}(\mathscr{F}) \to \cdots$$

We now need only show that the δ maps commute. Let (f, g, h) be a morphism short exact sequences of sheaves from $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$ to $0 \to \mathscr{F}' \to \mathscr{G}' \to \mathscr{R}' \to 0$. Applying the universal δ-functor $H^i(f^{-1}(U), -|_{f^{-1}(U)})$ we get:

$$\begin{tikzcd}
H^i(f^{-1}(U), \mathscr{R}|_{f^{-1}(U)}) \arrow[r, "\delta^i"'] \arrow[d] & H^{i+1}(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)}) \arrow[d] \\
H^i(f^{-1}(U), \mathscr{R}'|_{f^{-1}(U)}) \arrow[r, "\delta^i"'] & H^{i+1}(f^{-1}(U), \mathscr{F}'|_{f^{-1}(U)})
\end{tikzcd}$$

Furthermore, since the restrictions are natural transformations, then the commutativity of this diagram for each U gives commutativity for $\mathscr{H}^i$. It follows that $\mathscr{H}^i$ are δ-functors.

To see that $\mathscr{H}^i$ are universal, we can show that they are effaceable. Let $\mathscr{F}$ be any sheaf and let $u : \mathscr{F} \to \mathscr{I}$ be an injective map into an injective sheaf $\mathscr{I}$. Then $\mathscr{I}|_{f^{-1}(U)}$ is an injective sheaf since restriction of injective is injective. Therefore $\mathscr{H}^i(\mathscr{F}) = 0$ for all $i \geqslant 1$. It follows that $\mathscr{H}^i(u) = 0$ for all $i \geqslant 1$, thus $\mathscr{H}^*$ is a universal δ-functor.

Since $R^i f_*$ is a derived functor, then it is a universal δ-functor. Furthermore we have that $R^0 f_* = f_*$ and

$$(f_* \mathscr{F})(U) = \mathscr{F}(f^{-1}(U)) = \Gamma(f|_{f^{-1}(U)}) = H^0(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)}) = (\mathscr{H}^0 \mathscr{F})(U)$$

therefore $R^0 f_* = \mathscr{H}^0$, so $R^* f_* = \mathscr{H}^*$. It follows that $R^i f_* \mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto H^i(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)})$ as desired. $\square$