MAT327 Homework problems

Ethan Bottomley-Mason

October 2022

Question 1

Constant functions are continuous:

Proof:

Let $f : X \to Y$ be a constant function c, then for any open set U of Y, either $c \in Y$ or $c \notin Y$. If $c \in Y$, then $f^{-1}(Y) = X$ which is open and if $c \notin Y$, then $f^{-1}(Y) = \varnothing$ which is open. Therefore f is continuous.

Question 18.4

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by $f(x) = (x, y_0), g(y) = (x_0, y)$ are imbeddings.

Proof:

We want to show that these are continuous with continuous inverse from the image. I will only show one of them as the other is analogous. Given any basis element $U \times V \subseteq X \times Y$, we have that $f^{-1}(U \times V)$ is either U if $y_0 \in V$ or $\varnothing$ if $y_0 \notin V$ which are both open in X, thus f is continuous. The inverse of f on the image is $f^{-1} : X \times \{y_0\} \to X$ by $(x, y_0) \mapsto x$. Given any basis element $U \times \varnothing$ or $U \times \{y_0\}$, then the preimage is either U or $\varnothing$ both of which are open. We may also check that these are in fact inverses of each other:

$$f(f^{-1}(x, y_0)) = f(x) = (x, y_0) \qquad f^{-1}(f(x)) = f^{-1}(x, y_0) = x$$

Question 18.11

Let $F : X \times Y \to Z$. We say that F is continuous in each variable separately if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous and similarly for $k : Y \to Z$. Show that if F is continuous, then F is continuous in each variable separately.

Proof:

Let $R_{y_0} : x \mapsto (x, y_0)$ and $R_{x_0} : y \mapsto (x_0, y)$ which we have seen to be continuous, then $h = F \circ R_{y_0}$ is a composition of continuous functions and hence continuous and similarly for k.

Question 18.12

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation:

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & x, y \neq 0, 0 \\ 0 & x, y = 0, 0 \end{cases}$$

a) Show that F is continuous in each variable separately.
b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $F(x, x)$.
c) Show that F is not continuous.

Proof:

a) When we fix some y_0, then $h(x) = F(x, y_0)$ is continuous for all $x \neq 0$ since it is a rational function with non-zero denominator. If $y_0 \neq 0$, then the denominator is always positive and so the function is continuous everywhere. If y_0 is 0, then $h(x) = 0$ is the constant function and thus continuous. Analogous logic holds when we fix x.

b) $g(x) = xx/(x^2 + x^2) = \frac{1}{2}$ for $x \neq 0$ and $g(0) = 0$.

c) The preimage of $(-\frac{1}{2}, \frac{1}{2})$ under g is just 0 which is not open, thus g is not continuous and since $x \mapsto (x, x)$ is continuous, then if f were continuous g would be continuous, but it is not, thus f cannot be continuous. Note that $x \mapsto (x, x)$ is continuous since if we take any basis element $U \times V$, the preimage is just $U \cap V$ which is open.

Question 18.13

Let $A \subseteq X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f. Proof:

Suppose that there are two distinct extensions g_1 and g_2 of f which are continuous, then since they are distinct, they must disagree at some point $x \in \bar{A}$, that is $g_1(x) \neq g_2(x)$. Since they are distinct points in Y, then they may be separated by disjoint open sets U, V, $g_1(x) \in U, g_2(x) \in V, U \cap V = \varnothing$. Since g_1, g_2 are continuous, then $g_1^{-1}(U), g_2^{-1}(V)$ are open sets containing x and thus so is $O = g_1^{-1}(U) \cap g_2^{-1}(V)$. Since $x \in \bar{A}$, then $O \cap A \neq \varnothing$, and so there is some $y \in O$ which is in A, then since $y \in g_1^{-1}(U) \cap g_2^{-1}(V)$, which extend f, then $g_1(y) = f(y) = g_2(y) \in U \cap V = \varnothing$ is a contradiction. Therefore the extension is unique.

Question 18.9

Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|_{A\alpha}$ is continuous for each α.
a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.
c) An indexed family of sets A_α is said to be locally finite if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Proof:

a) For any U closed in Y, we have that:

$$f^{-1}(U) = f^{-1}(U) \cap X = \bigcup_\alpha \left(f^{-1}(U) \cap A_\alpha\right) = \bigcup_\alpha f|_{A\alpha}^{-1}(U)$$

Since each $f|_{A\alpha}$ is continuous, then $f|_{A\alpha}^{-1}(U)$ is closed and since A_α is closed, then $f^{-1}(U) \cap A_\alpha$ is closed in X and thus $f^{-1}(U)$ is a finite union of closed sets and is thus closed. Therefore f is continuous.

b) Let $f : [0, 1] \to \mathbb{R}$ be constant 1 on $(0, 1]$ and $f(0) = 0$, then consider the countable collection of closed sets $\{[\frac{1}{n}, 1]\} \cup \{\{0\}\}$, on each set the function is constant and hence continuous, but f is not continuous since the preimage of $(-\frac{1}{2}, \frac{1}{2})$ which is $\{0\}$ is not open.

c) Let U_x be the given open neighborhood of x which intersects only finitely many A_α, then $U_x = \bigcup_{i=1}^n A_{\alpha_i} \cap U_x$, so U_x is the union of finitely many closed sets $A_{\alpha_i} \cap U_x$ and since $f|_{A_{\alpha_i}}$ is continuous, then so is the restriction $f|_{A_{\alpha_i} \cap U_x}$, thus by part a, $f|_{U_x}$ is continuous. Since $X = \bigcup_{x\in X} U_x$, and each U_x is open with $f|_{U_x}$ continuous, then for any open set V of Y,

$$f^{-1}(V) = f^{-1}(V) \cap X = \bigcup_{x\in X} \left(f^{-1}(V) \cap U_x\right) = \bigcup_{x\in X} f|_{U_x}^{-1}(V)$$

Since U_x is open and $f|_{U_x}^{-1}(V)$ is open in U_x, then it is open in X and thus $f^{-1}(V)$ is a union of open sets and thus open.

Question 19.6

Let $x_1, x_2, \cdots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that the sequence converges to the point x if and only if the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \cdots$ converges to $\pi_\alpha(x)$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Proof:

$\Rightarrow$: If $x_1, x_2, \cdots$ converges to x, then for any open set U containing $\pi_\alpha(x)$, we have that $\pi^{-1}(U)$ is open and contains x and thus $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n \in \pi^{-1}(U) \Rightarrow \pi(x_n) \in U$, thus $\{\pi_\alpha(x_i)\}$ converges to $\pi_\alpha(x)$.

$\Leftarrow$: If $\{\pi_\alpha(x_n)\}$ converges to $\pi_\alpha(x)$ for every α, then given any open set U containing x, it contains a basic element; that is to say that there is some finite subset S of the α's such that $x \in \prod_{\beta\in S} U_\beta \times \prod_{\beta\notin S} X_\beta$. For each $\beta \in S$, there is some N_β such that for $n > N_\beta$ we have that $\pi_\beta(x_n) \in U_\beta$. Let $N = \max_{\beta\in S} N_\beta$

which is well-defined since S is finite, then for all $n > N$ we have that $\pi_\beta(x_n) \in U_\beta$ for all $\beta \in S$ and so

$$x_n \in \prod_{\beta \in S} U_\beta \times \prod_{\beta \notin S} X_\beta \subseteq U$$

It follows that $\{x_n\}$ converges to x.

This does not hold in the box topology. Consider the space $\prod_{i \in \mathbb{N}} \mathbb{R}$ with the box topology and the sequence $x_i = (\frac{1}{i}, \frac{1}{i}, \cdots)$. The projection of each of these sequences converge to 0 since for any open set U of 0 it contains some ball $(-\frac{1}{N}, \frac{1}{N})$ and so for $n > N$, $\pi_k(x_n) = \frac{1}{n} < \frac{1}{N}$ and so $\pi_k(x_n) \in U$. However in the box topology, for the open set $U = \prod_{i \in \mathbb{N}} (-\frac{1}{i}, \frac{1}{i})$, for any x_i, we have that $\frac{1}{i} \notin (-\frac{1}{i+1}, \frac{1}{i+1})$ and so $x_i \notin U$ and thus x_i does not converge to 0.

Question 6

Show that the product $(X \prod Y) \prod Z$ is also a product $X \prod Y \prod Z$.

Proof:

We want to show that $(X \prod Y) \prod Z$ has the universal property of the product. Let W be an object with morphisms f, g, h into X, Y, Z respectively. Consider the following diagram:

$$\begin{tikzcd}
& & (X \prod Y) \prod Z \arrow[dl, "\pi_{XY}"'] \arrow[ddll, bend right, "\pi_X \circ \pi_{XY}"'] \arrow[dd, "\pi_Y \circ \pi_{XY}"] \arrow[ddrr, "\pi_Z"] & & \\
& X \prod Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & & & \\
X & & Y & & Z \\
& & W \arrow[ull, "f"] \arrow[u, "g"] \arrow[urr, "h"'] \arrow[uul, "\exists ! p"] \arrow[uuu, bend right, "\exists ! q"'] & &
\end{tikzcd}$$

The existence and uniqueness of p is guaranteed by the universal property of the product of X and Y, then the existence and uniqueness of q is guaranteed by the universal property of $(X \prod Y) \prod Z$. Notice then that $(X \prod Y) \prod Z$ has morphisms $\pi_X \circ \pi_{XY}, \pi_Y \circ \pi_{XY}, \pi_Z$ into X, Y, Z respectively. We want to check that q and the projections from $(X \prod Y) \prod Z$ commute. We have that $\pi_X \circ \pi_{XY} \circ q = \pi_X \circ p = f$, then we have that $\pi_Y \circ \pi_{XY} \circ q = \pi_Y \circ p = g$ and finally $\pi_Z \circ q = h$. Therefore, the diagram commutes. It follows that $(X \prod Y) \prod Z$ satisfies the universal property of the product and is hence a product $X \prod Y \prod Z$.

Question 20.3

Let X be a metric space with metric d.
a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
b) Let X' denote the space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

Proof:

Let $d'((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$ be a metric on $X \times X$, then we want to show that $d : X \times X \to \mathbb{R}$ is continuous. Fix some point (x_1, x_2), then given any $\epsilon > 0$, if $d((x_1, x_2), (y_1, y_2)) < \frac{\epsilon}{2}$, we have that $d(x_1, y_1) < \frac{\epsilon}{2}$ and $d(x_2, y_2) < \frac{\epsilon}{2}$, so:

$$d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, x_2) \leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2)$$

$$\Rightarrow d(x_1, x_2) - d(y_1, y_2) \leq d(y_1, x_1) + d(y_2, x_2)$$

And additionally,

$$d(y_1, y_2) \leq d(y_1, x_1) + d(x_1, y_2) \leq d(y_1, x_1) + d(x_1, x_2) + d(x_2, y_2)$$

$$\Rightarrow d(y_1, y_2) - d(x_1, x_2) \leq d(x_1, y_1) + d(x_2, y_2)$$

It follows that $|d(y_1, y_2) - d(x_1, x_2)| \leq d(x_1, y_1) + d(x_2, y_2) < \epsilon$.

b) Let $B(x, \epsilon)$ be an epsilon ball centered at x in X'. If we take $d_{x_0}(y) = d(x_0, y)$, then d_{x_0} is continuous since it is a restriction to $\{x_0\} \times X'$. $B(x_0, \epsilon)$ is the preimage of $(-\epsilon, \epsilon)$ under d_{x_0} and is thus open. It follows that the basis of X is open in X', then X is finer than X'.

Question 8

For any metric space, show that $\bar{B}(x, r)$ is closed. Give an example when $\bar{B}(x, r) \neq \overline{B(x, r)}$.

Proof:

$\overline{B}(x, r) = \{y \in X \big| d(x, y) \leq r\}$. We can show that the complement is open. Let $y \in \overline{B}(x, r)^C$, then $d(x, y) > r$, so $d(x, y) = r + \epsilon$ for some $\epsilon > 0$, then take the ball $B(y, \epsilon)$. If $z \in B(y, \epsilon)$, then $d(y, z) < \epsilon$, so $d(y, x) \leq d(y, z) + d(x, z)$, thus $d(x, z) \geq d(y, x) - d(y, z) > r + \epsilon - \epsilon = r$. It follows that $z \notin \overline{B}(x, r)$ and thus $y \in B(y, \epsilon) \subseteq \overline{B}(x, r)^C$ and so the complement is open.

Give $\mathbb{R}$ the discrete metric, then $\overline{B}(0, 1) = \mathbb{R}$, but $\overline{B(0, 1)} = \{0\}$ since $\{0\}$ is closed and the closure is the intersection of all closed sets containing $B(0, 1) = \{0\}$.

Question 9

Suppose $X = A_1 \cup \cdots \cup A_n$ where each A_i is closed. Let $\sim$ be the equivalence relation on $\coprod A_i$ such that $x \sim y$ if x and y come from the same point on X. Show that X is homeomorphic to $\coprod A_i/\sim$. Draw the commutative diagram for $n = 2$.

Proof:

Consider the following diagram:

$$\begin{tikzcd}
\coprod A_i \arrow[r, "q"'] \arrow[d, "f"'] & \coprod A_i/\sim \arrow[dl, shift right, "g"'] \\
X \arrow[ur, shift right, "m"'] & A_i \arrow[l, "\iota"] \arrow[u, "h = q \circ i"'] \arrow[ul, "i"]
\end{tikzcd}$$

We have that ι is the inclusion of A_i into X, i is the inclusion of A_i into $\coprod A_i$, q is the quotient map. f is given to us by the universal property of the coproduct. Similarly, g is given to us by the universal property of the quotient since f is constant on the fibres of the quotient (if $(x, i) \sim (y, j)$, then $x = y$ so $f(x, i) = x = y = f(y, j)$). m is the map which makes $m \circ \iota = h$ for each A_i. This m is well-defined since for any $x \in X$, $x \in A_i$ for some i, thus $m(x) = m(\iota(x)) = h(x)$ and for any $x \in A_i$ and $x \in A_j$, we have that $m(x) = q(x, i) = \{(x, i) \big| x \in A_i\} = q(x, j) = m(x)$. Since $m|_{A_i}$ is just h which is the composition of continuous functions and each A_i is closed, then by the pasting lemma, m is continuous. Notice that $g(\{(x, i) \big| x \in A_i\}) = x$, so:

$$\begin{aligned}
g(m(x)) &= g(\{(x, i) \big| x \in A_i\}) = x \\
m(g(\{(x, i) \big| x \in A_i\})) &= m(x) = \{(x, i) \big| x \in A_i\}
\end{aligned}$$

It follows that $g \circ m = \mathrm{Id}_X$ and $m \circ g = \mathrm{Id}_{\coprod A_i/\sim}$ and so they are homeomorphic.

Question 10

Let $K = \{\frac{1}{n} \big| n \in \mathbb{N}\} \cup \{0\}$ with the subspace topology from $\mathbb{R}$. Define $X_1 = \coprod K^{(i)} \coprod K/\sim$ where the equivalence is $0^{(i)} \sim \frac{1}{i}$. Let $X = X_1 \setminus \{\{0^{(i)}, \frac{1}{i}\} \big| i \in \mathbb{N}\}$. Show that 0 is not an isolated point in X (i.e. $\overline{X \setminus \{0\}} = X$).

Proof:

Consider any open set U in X containing 0, then in X_1, U contained some $\frac{1}{n}$ and hence contains some $0^{(n)}$, therefore U contains some $(\frac{1}{m})^{(n)}$ and thus it also contains $(\frac{1}{m})^{(n)}$ in X. It follows that 0 is a limit point of $X \setminus \{0\}$.

Question 11

Show that $S^n/x \sim -x$ is homeomorphic to $\mathbb{RP}^n$. First show that $\mathbb{R}^{n+1} \setminus \{0\}/\sim$ is homeomorphic to S^n, where $x \sim kx, k > 0$, then use the face that the composition of quotient maps is quotient.

Proof:

We want to show that $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ is homeomorphic to S^n. We have the map $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by $x \mapsto \frac{x}{||x||}$ which is continuous since scaling is continuous and $||x||$ is continuous and nonzero on $\mathbb{R}^{n+1}$. We also have the inclusion $i : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$. Consider the following diagram:

$$\begin{tikzcd}
(\mathbb{R}^{n+1} \setminus \{0\})/\sim \arrow[dr, shift right, "h"'] & \\
\mathbb{R}^{n+1} \setminus \{0\} \arrow[u, "q"] \arrow[r, shift right, "r"'] & S^n \arrow[l, shift right, "i"'] \arrow[ul, shift right, "p = q \circ i"']
\end{tikzcd}$$

Notice that for any $x \sim y$ in $\mathbb{R}^{n+1} \setminus \{0\}$, we have that $y = kx$, so $r(y) = \frac{kx}{||kx||} = \frac{kx}{k||x||} = \frac{x}{||x||} = r(x)$, so we get a continuous map h from the quotient to S^n by the universal property of the quotient. Notice that p is continuous since it is the composition of continuous functions. We want to show that h and p are inverses of each other. For any element $x \in \mathbb{R}^{n+1}$ we have that $h(q(x)) = r(x)$, thus $h(\{kx \big| k > 0\}) = \frac{x}{||x||}$. We can show h and p are inverses as follows:

$$p(h(\{kx \big| k > 0\})) = p\left(\frac{x}{||x||}\right) = q\left(\frac{x}{||x||}\right) = \left\{\frac{k}{||x||}x \Big| k > 0\right\} = \{kx \big| k > 0\}$$

$$h(p(x)) = h(\{kx \big| k > 0\}) = \frac{x}{||x||} = x$$

Therefore $(\mathbb{R}^{n+1} \setminus \{0\})/\sim \cong S^n$.

We now show that $\mathbb{RP}^n \cong S^n/(x \sim -x)$. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{R}^{n+1} \arrow[r, "q"] \arrow[d, "p"'] & S^n \arrow[dl, "\exists h"] \\
\mathbb{RP}^n &
\end{tikzcd}$$

We consider the quotient maps $q : \mathbb{R}^{n+1} \to S^n$ and $p : \mathbb{R}^{n+1} \to \mathbb{RP}^n$. For any $y = kx$ with $k > 0$, we have that $p(y) = \{k'kx \big| k' \in \mathbb{R}\} = \{k'x \big| k' \in \mathbb{R}\} = p(x)$, so there exists a continuous function h from S^n to $\mathbb{RP}^n$. We have that $h^{-1}(\{kx \big| k \neq 0\}) = \{kx \big| k > 0\} \cup \{k(-x) \big| k > 0\} = q(x) \cup q(-x)$. Our homeomorphism from $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$ to S^n preserves these fibres when considering the quotient map $S^n/(x \sim -x)$ since $q(x) \mapsto x, q(-x) \mapsto -x$ and so $S^n/(x \sim -x) \cong \mathbb{RP}^n$.

Question 12

Show that $\mathbb{R}^\omega$ is disconnected.

Proof:

Let B be the set of all bounded sequences and U be the set of all unbounded sequences in $\mathbb{R}^\omega$. For any point $x \in B$, it is contained in the open set $V = \prod(x_i - 1, x_i + 1)$ and for any $y \in V$, since x is bounded, there is some M such that $|x_i| < M$ for all i, then $|y_i| \leq |x_i| + 1 < M + 1$, so y is bounded. It follows that $x \in V \subseteq B$, so B is open. For any unbounded sequence $x \in U$ it is contained in the open set $V = \prod(x_i - 1, x_i + 1$. For any M, there is some n such that $|x_n| > M + 1$ and thus $|y_n| > M$, so y is unbounded. It follows that $x \in V \subseteq U$ and thus U is open. Any sequence is either bounded or unbounded, so $\mathbb{R}^\omega = B \cup U$ which is a separation.

Question 13

Let U be an open neighborhood of S^{n-1} in $\overline{B}^n$. Show that there exists $\epsilon > 0$ such that $(\overline{B}(0, 1 - \epsilon))^c \subseteq U$.

Proof:

Consider the function $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \times \mathbb{R}^+$ given by $f(x) = (\frac{x}{||x||}, ||x||)$, each component of this function is continuous and its inverse is given by $f^{-1}(x, r) = rx$ is continuous. Note that these are inverses because $f(f^{-1}(x, r)) = (\frac{rx}{r||x||}, r||x||) = (\frac{x}{||x||}, r) = (x, r)$, this uses the fact that $x \in S^{n-1}$, so $||x|| = 1$. Conversely, $f^{-1}(f(x)) = f^{-1}(\frac{x}{||x||}, ||x||) = x$. Since U is an open neighborhood of S^{n-1} in $\overline{B}^n$, then $U \supseteq U' \cap \overline{B}^n$ with $S^{n-1} \subseteq U'$ and U' open in $\mathbb{R}^{n+1}$. For any $x \in S^{n-1} \times \{1\}$, since $S^{n-1} \subseteq U'$, then there is some $y \in S^{n-1} \subseteq U'$ with $f(y) = (y, 1)$, thus $f(U')$ is an open set containing $S^{n-1} \times \{1\}$. By the tube lemma, there is some open set W in $\mathbb{R}^+$ such that $S^{n-1} \times W$ is contained in $f(U')$. W contains an open ball $B(1, \epsilon)$ for some $\epsilon << 1$, thus $S^{n-1} \times B(1, \epsilon) \subseteq f(U')$ and so for $V = f^{-1}(S^{n-1} \times B(1, \epsilon))$, we have that $V \subseteq U'$. Notice that for any point $x \in V$, we have that $x = f(y, r) = ry$ with $r > 1 - \epsilon$ and $||y|| = 1$, thus $||x|| > 1 - \epsilon$. It follows that $V' = \overline{B}^n \cap V \subseteq U$ and so $(\overline{B}(0, 1 - \epsilon))^c \subseteq V'U$.

16.2

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X.

Proof:

We can say only that the subspace topology on Y will be finer with $(X, \mathcal{T}')$ than with $(X, \mathcal{T})$, but not strictly finer, for instance if $Y = \varnothing$, then the topology on Y will be the same regardless of the topology on X.

16.3

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

Proof:

$A = \{x \big| \frac{1}{2} < |x| < 1\}$, this set is open in $\mathbb{R}$ (it is an open interval) and thus is contained in Y and thus open in Y.

$B = \{x \big| \frac{1}{2} < |x| \leq 1\}$. This set is open in Y as it is $[-1, 1] \cap (\frac{1}{2}, 2)$, but it is not open in $\mathbb{R}$, since the point 1 has no open set containing it that is contained in B.

$C = \{x \big| \frac{1}{2} \leq |x| < 1$. This set is open in neither $\mathbb{R}$ not Y since any open set containing the point $\frac{1}{2}$ in Y or in $\mathbb{R}$ contains a point with value $< \frac{1}{2}$, which is not contained in C.

$D = \{x \big| \frac{1}{2} \leq |x| \leq 1\}$. This set is open in neither for the same reason as C.

$E = \{x \big| 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+\}$. This set can be written as $\bigcup_{i \geq 1} (\frac{1}{i+1}, \frac{1}{i})$ which is open in $\mathbb{R}$ and contained in Y and hence open in Y.

16.4

A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof:

We know that $f(\bigcup U_\alpha) = \bigcup f(U_\alpha)$ and $f(\bigcap U_\alpha) = \bigcap f(U_\alpha)$, so all we need to show is that π_1 and π_2 are open maps on the subbasis elements of the product. For any subbasis element $U \times Y$, we have that $\pi_1(U \times Y) = U$ is open and $\pi_2(U \times Y) = Y$ is open. Similarly, for any subbasis element $X \times V$, $\pi_1(X \times V) = X$ is open and $\pi_2(X \times V) = V$ is open. Note that if any of U, V, X, Y are empty, then the output of the respective functions is empty which is still open. It follows that π_1 and π_2 are open maps.

16.5

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively.Assume these sets are nonemtpy.
(a) Show that if $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.
(b) Does the converse of a hold? Justify your answer.

Proof:

a) We want to show that any basis element in $X \times Y$ is open in $X' \times Y'$, but this is trivially true since any basis element looks like $U \times V$ with U open in X and V open in Y, but since these are coarser than X, Y respectively, then U open in X' and V open in Y', thus $U \times V$ is open in $X' \times Y'$, so $X' \times Y'$ is finer than $X \times Y$.

b) Yes the converse does also hold. Suppose that $X' \times Y'$ is finer than $X \times Y$, then for any open set U in X, we have that $U \times Y$ is open in $X \times Y$ and hence in $X' \times Y'$. Since $\pi_1 : X' \times Y' \to X'$ is an open map, then $\pi_1(U \times X) = U$ is open in X', note that this is cannot be empty since X is nonempty by assumption. Thus X' is finer than X, and analogously, Y' is finer than Y.

16.6

Show that the countable collection $\{(a, b) \times (c, d) \big| a < b, c < d, a, b, c, d \in \mathbb{Q}\}$ is a basis for $\mathbb{R}^2$.

Proof:

We want to show that $\{(a, b) \big| a < b, a, b \in \mathbb{Q}\}$ is a basis of $\mathbb{R}$ and then the above statement follows trivially from the definition of the basis of the product topology. Notice that for any open set U in $\mathbb{R}$, we have that for any point $x \in U$, there is an open interval $(x - \epsilon_1, x + \epsilon_2)U$, then we may take $a \in (x - \epsilon_1, x)$ and $b \in (x, x + \epsilon_2)$ rational, then $a < b$ and $x \in (a, b)U$, so U is open in with this basis. Furthermore, this basis is clearly open in the standard basis of $\mathbb{R}$ as it is just a subset of the standard basis.

17.8

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subseteq$ or $\supseteq$ holds.
(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$.
(c) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

Proof:

a) $\overline{A}\cap\overline{B}$ is an intersection of closed sets and thus closed and contains $A\cap B$, so $\overline{A\cap B}\subseteq\overline{A}\cap\overline{B}$. The other inclusion is not true, for instance consider $\mathbb{R}$ with the standard topology, then $\overline{(0,1)\cap(1,2)}=\overline{\varnothing}=\varnothing$, but $\overline{(0,1)}\cap\overline{(1,2)}=[0,1]\cap[1,2]=\{1\}$.

b) $\bigcap\overline{A_\alpha}$ is an intersection of closed sets and thus closed and contains $\bigcap A_\alpha$, thus $\overline{\bigcap A_\alpha}\subseteq\bigcap\overline{A_\alpha}$. The other inclusion fails for the same reason as in a).

c) For any $x\in\overline{A}\setminus\overline{B}$, we have that for any open U containing x, that $U\cap A\neq\varnothing$ and that there is some open set V containing x such that $V\cap B=\varnothing$. Then for any U containing x, we have that $U\cap(A\setminus B)\supseteq U\cap V\cap(A\setminus B)$ and we get that this is equal to $U\cap(A\cap V\setminus B\cap V)$. Since $B\cap V$ is empty, then this is just $U\cap A\cap V=(U\cap V)\cap A$. Since $U\cap V$ is open, then it intersects non-trivially with A, thus $U\cap V\cap(A\setminus B)\subseteq U\cap(A\setminus B)$ must be nonempty. It follows that $\overline{A}\setminus\overline{B}\subseteq\overline{A\setminus B}$. The other inclusion does not hold. Consider the standard topology on $\mathbb{R}$, then take A to be the rationals and B to be the irrationals, then $A\setminus B=A$ and $\overline{A}=\mathbb{R}$, but $\overline{A}\setminus\overline{B}=\mathbb{R}\setminus\mathbb{R}=\varnothing$.

17.12

Show that a subspace of a Hausdorff space is Hausdorff.

Proof:

Let $Y\subseteq X$ with X Hausdorff. Consider any $x,y\in Y$, then $x,y\in X$, so there exists disjoint open sets U,V with $x\in U$ and $y\in V$ since X is Hausdorff, then $x\in U\cap Y$ and $y\in V\cap Y$ and $(U\cap Y)\cap(V\cap Y)=(U\cap V)\cap Y=\varnothing\cap Y=\varnothing$. Furthermore, $U\cap Y$ and $V\cap Y$ are open in Y. It follows that Y is Hausdorff with the subspace topology.

17.19

If $A\subseteq X$, we define the boundary of A by the equation: $\mathrm{Bd}A=\overline{A}\cap\overline{(X\setminus A)}$.
(a) Show that $\mathrm{Int}A$ and $\mathrm{Bd}A$ are disjoint and $\overline{A}=\mathrm{Int}A\cup\mathrm{Bd}A$.
(b) Show that $\mathrm{Bd}A=\varnothing$ iff A is clopen.
(c) Show that A is open iff $\mathrm{Bd}A=\overline{A}\setminus A$.
(d) If A is open, is it true that $A=\mathrm{Int}(\overline{A})$? Justify your answer.

Proof:

a) For any $x\in\mathrm{Bd}A$, we have that for any open set U containing x, $U\cap A\neq\varnothing$ and $U\cap(X\setminus A)\neq\varnothing$. Notice that $\mathrm{Int}A$ is an open set in X contained in A, then $\mathrm{Int}A\subseteq A\Rightarrow\mathrm{Int}A\cap X\setminus A=\varnothing$, thus no point in $\mathrm{Int}A$ belongs to A, so they are disjoint. If $x\in\int(A)$, then $x\in A$, so $x\in\overline{A}$. If $x\in\mathrm{Bd}A$, then if $x\in A$ then $x\in\overline{A}$ and if $x\notin A$, then x is a limit point of A, so $x\in\overline{A}$. If $x\in\overline{A}$, then if there is some U open containing x such that $U\subseteq A$, then $x\in\mathrm{Int}A$. If there is no U such that $x\in U\subseteq A$, then we must have that for all U open containing x, $U\cap(X\setminus A)\neq\varnothing$, but since $x\in\overline{A}$, then we have that $U\cap A\neq\varnothing$, so $x\in\mathrm{Bd}A$. Therefore, $\overline{A}=\mathrm{Int}A\cup\mathrm{Bd}A$.

b) If $\mathrm{Bd}A = \varnothing$, then $\overline{A} = \mathrm{Int}A$, but $\mathrm{Int}A \subseteq A \subseteq \overline{A}$, so we must have that $\mathrm{Int}A = A = \overline{A}$, thus A is clopen. If A is clopen, then $\mathrm{Int}A = A = \overline{A}$, so $\mathrm{Bd}A \subseteq \overline{A} = A$, but at the same time, $\mathrm{Bd}A$ and $\mathrm{Int}A = A$ are disjoint, so $\mathrm{Bd}A$ must be empty.

c) If A is open, then $A = \mathrm{Int}A$ and, then $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$, so $\mathrm{Bd}A = \overline{A} \setminus \mathrm{Int}A = \overline{A} \setminus A$ (note that this requires that $\mathrm{Bd}A$ and $\mathrm{Int}A$ are disjoint). If $\mathrm{Bd}A = \overline{A} \setminus A$, then since $A \subseteq \overline{A}$, then we may take the complement in $\overline{A}$ to get that $\mathrm{Int}A = A$, thus A is open.

d) No, this is not true. Consider $A = (0,1) \cup (1,2)$, then $\overline{A} = [0,2]$ and $\mathrm{Int}(\overline{A}) = (0,2) \neq A$.

Suggested Exercise 3

Let A, B be subsets of X. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$ and $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Give an example when there is no equality in the latter relation.

Proof:

Let A° and B° are open and thus $A^\circ \cap B^\circ$ is open and contained in $A \cap B$, thus $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. For any $x \in (A \cap B)^\circ$, there is an open set U s.t. $x \in U \subseteq A \cap B$, then $x \in U \subseteq A$ and $x \in U \subseteq B$, so $x \in A^\circ \cap B^\circ$. Therefore, $(A \cap B)^\circ = A^\circ \cap B^\circ$. $A^\circ \cup B^\circ$ is open and contained in $A \cup B$, so $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$. The other inclusion does not hold however, since for $A = (0,1)$ and $B = [1,2)$, we have that $(A \cup B)^\circ = (0,2)^\circ = (0,2)$, but $A^\circ \cup B^\circ = (0,1) \cup (1,2)$.

Suggested Exercise 4

Decide (and prove) where in the equivalent definitions of continuity once can use basis/subbasis elements instead of general open sets.

Proof:

Let B be a basis, then if f is continuous, then f is continuous on the elements of the basis since they are open. If f is continuous on the elements of the basis, then for any open set U, for any $x \in U$, there is a basis element B_x such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$ and $f^{-1}(U) = f^{-1}(\bigcup_{x \in U} B_x) = \bigcup_{x \in U} f^{-1}(B_x)$ is a union of open sets, so f is continuous.

Let S be a subbasis, then we may extend S to a basis by taking all intersections all elements in the subbasis. If f is continuous, then it is clearly continuous on all of the subbasis elements. If f is continuous on the subbasis elements, then for any basis element B, we may write it as $\bigcap_{i=1}^n S_i$, then $f^{-1}(B) = f^{-1}(\bigcap_{i=1}^n S_i) = \bigcap_{i=1}^n f^{-1}(S_i)$ which is a finite intersection of open sets, so f is continuous on the basis generated by S and thus continuous.

18.2

Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Proof:
No, $f(x)$ need not be a limit point of $f(A)$. Consider any constant function, $f(x) = c$, then $f(A) = \{c\}$ and so $\overline{f(A) \setminus f(x)} = \overline{\varnothing} = \varnothing \neq \{c\} = f(A)$.

18.5

Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Proof:
Consider the function $f : (a, b) \to (0, 1)$ given by $f(x) = \frac{x-a}{b-a}$, then f is bijective and continuous and $f^{-1} : (0, 1) \to (a, b)$ is given by $f(x) = a + (b - a)x$ is also bijective and continuous. The case for $[a, b] \to [0, 1]$ is analogous.

18.6

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Proof:
Consider the dirichlet function $d(x)$ and then the function $f(x) = |x|d(x)$, then for any $x \neq 0$ and for $\epsilon = |x|$, we see that for any $\delta > 0$, that there is an irrational number i in $B(x, \delta)$ with $|i| \geq |x|$ and a rational number $q \in B(x, \delta)$, then $f(i) = |i|d(i) = |i| \geq |x|$ and $f(q) = 0$. If x is irrational, then $|f(x) - f(q)| = |x|d(x) = |x| \geq |x|$ and if x is rational then $|f(x) - f(i)| = |f(i)| = |i| \geq |x|$. It follows that f is not continuous at any $x \neq 0$. At $x = 0$, notice that $0 \leq f(x) \leq |x|$, so by the squeeze theorem, f is continuous at 0.

18.10

Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $fg : A \times C \to B \times D$ by the equation: $(f \times g)(a, c) = (f(a), g(c))$. Show that $f \times g$ is continuous.

Proof:
We need only show that $f \times g$ is continuous on the subbasis elements. Consider any subbasis element $U \times D$ with U open in B, then $(f \times g)^{-1}(U \times D) = f^{-1}(U) \cap g^{-1}(D) = f^{-1}(U)$ is open and similarly, $(f \times g)^{-1}(B \times V) = f^{-1}(B) \cap g^{-1}(V) = g^{-1}(V)$ is open. It follows that $f \times g$ is continuous.

19.1

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices and $B_\alpha = X_\alpha$ for all remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$

Proof:

We want to show that this is a basis. Note that each such element is clearly open in the product topology. We now want to show that given U an open set in the product topology, for any point $x \in U$, we can find a basis element containing x contained in U. Since U is open, then there is a basis element $U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \notin \{\alpha_1, \cdots, \alpha_n\}} X_\alpha$ containing x which is contained in U. For each U_{α_i}, there is a basis element B_{α_i} containing x_{α_i} contained in U_{α_i}, thus $x \in B_{\alpha_1} \times \cdots \times B_{\alpha_n} \times \prod_{\alpha \notin \{\alpha_1, \cdots, \alpha_n\}} X_\alpha U$. It follows that this is indeed a basis of the product topology.

19.2

Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Proof:

In $\prod A_\alpha$, any basis element looks like $\prod (U_\alpha \cap A_\alpha)$ with only finitely many $U_\alpha \cap A_\alpha \neq A_\alpha$ in the product topology and arbitrary in the box topology. Notice that $\prod (U_\alpha \cap A_\alpha) = (\prod U_\alpha) \cap (\prod A_\alpha)$. This shows that it is a subspace in the box topology. In the product topology, however, we may have that infinitely many $U_\alpha \neq X_\alpha$, but we may have only finitely many such that $U_\alpha \cap A_\alpha \neq A_\alpha$. Notice that if ${}_\alpha \cap A_\alpha = A_\alpha$, then we may replace U_α with X_α and retain the same product. In this way, we now only have finitely many U_α such that $U_\alpha \neq X_\alpha$.

19.3

If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.

Proof:

Consider any two distinct points $x, y \in \prod X_\alpha$, then for some α, x_α and y_α are distinct, so there exists U_α, V_α disjoint and open in X_α such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$. Consider the open sets $U' = U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$ and $V' = V_\alpha \times \prod_{\beta \neq \alpha} X_\beta$. These are open in both the product and box topologies and $x \in U', y \in V'$ and $U' \cap V' = (U_\alpha \cap V_\alpha) \times \prod_{\beta \neq \alpha} (X_\beta \cap X_\beta) = \varnothing \times \prod_{\beta \neq \alpha} X_\beta = \varnothing$. Therefore, both the product topology and the box topology are Hausdorff.

19.5

One of the implications stated in Theorem 19.6 holds for the box topology. Which one?

Proof:

Theorem 19.6 states that if $f : A \to \prod_{\alpha \in J} X_\alpha$ is given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each α. Given the product topology we have that f is continuous iff each f_α is continuous.

Given the box topology, we still have that if f is continuous, then each f_α is continuous. We may take the projection $\pi_\alpha : \prod_{\beta \in J} X_\beta \to X_\alpha$ which is continuous from the box topology. It follows that $f_\alpha = \pi_\alpha \circ f$ is a composition of continuous functions and hence continuous. The other direction does not hold however. For instance, let $X_i = \mathbb{R}$ for each $i \in \mathbb{N}$, then consider the map $f(x) = (x, x, \cdots)$, then each $f_i(x) = x$ which is the identity function and hence continuous. However, we have that $\prod_{i \in \mathbb{N}}(-\frac{1}{i}, \frac{1}{i})$ is open in the box topology, and its preimage under f is $\bigcap_{i \in \mathbb{N}}(-\frac{1}{i}, \frac{1}{i}) = \{0\}$ which is not open in $\mathbb{R}$.

19.7

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero", that is, all sequences $(x_1, x_2, \cdots)$ such that $x_i \neq 0$ for only finitely many i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies?

Proof:

In the box topology, we have that for any $x \notin \mathbb{R}^\infty$, we have infinitely many nonzero values, so consider the open set $U_x = \prod_{i=1}^\infty U_i$ where $U_i = B(x_i, |x_i|)$ if $x_i \neq 0$ and $U_i = \mathbb{R}$ if $x_i = 0$. For any $x \notin \mathbb{R}^\infty$, we have that for any $y \in U_x$, y must have as many non-zero values as x, since for each i with $x_i \neq 0$, we have that $y_i \in B(x_i, |x_i|)$, so $|y_i - x_i| < |x_i|$, hence $|y_i| = |y_i - x_i + x_i| = |x_i - (x_i - y_i)| \geq |x_i| - |y_i - x_i| > 0$, so $y - i \neq 0$. It follows that $y \notin \mathbb{R}^\infty$ and so $U_x \in (\mathbb{R}^\infty)^c$. Therefore, $x \notin \overline{\mathbb{R}^\infty}$.

In the product topology, for any element $x \in \mathbb{R}^\omega$, any open set U containing x is of the form $U_1 \times U_2 \times \cdots \times U_n \times \prod_{i=n+1}^\infty \mathbb{R}$ since we can just take the largest index n. We have that $x_i \in U_i$ for each $i = 1, 2, \cdots, n$, then notice that $(x_1, x_2, \cdots, x_n, 0, 0, \cdots) \in U$ and that it is also in $\mathbb{R}^\infty$, thus $U \cap \mathbb{R}^\infty \neq \varnothing$, so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

19.8

Given sequences $(a_1, a_2, \cdots)$ and $b_1, b_2, \cdots)$ of all real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation $h((x_1, x_2, \cdots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \cdots)$. Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Proof:

Notice that for any $x \in \mathbb{R}^\omega$, we have that $h(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \cdots) = (x_1, x_2, \cdots) = x$. Furthermore, if $h(x) = h(x')$, then $\pi_i(h(x)) = \pi_i(h(x'))$ for each i, so $a_i x_i - b_i = a_i x_i' - b_i x_i = x_i'$ since $a_i \neq 0$. It follows that h is bijective. Furthermore, notice that each $\pi_i \circ h$ is linear and hence continuous so h is continuous. The inverse of h is given by $h^{-1}(x) = (\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \cdots)$ which is also continuous since it is linear in each component. Therefore h is a homeomorphism.

If we are working in the box topology, then h is still a homeomorphism. It is bijective for the same reason. We also have that for any basis element $\prod_{i=1}^{\infty}(x_i, y_i)$ we have that the preimage is $\prod_{i=1}^{\infty}(\frac{x_i-b_i}{a_i}, \frac{y_i-b_i}{a_i}$ which is still open in the box topology. Similarly, the inverse of h is a function of the same form an hence also continuous.

Suggested Exercise 7

Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces. Define $C = X_1 \times \{1\} \cup X_2 \times \{2\}$ with the topology $\mathcal{T}$ generated by $\{U \times \{1\} \big| U \in \mathcal{T}_1\} \cup \{U \times \{2\} \big| U \in \mathcal{T}_2\}$.
a) Show that $(C, \mathcal{T})$ is a coproduct $X_1 \coprod X_2$.
b) If $X_1 \cap X_2 = \varnothing$, then $X_1 \cup X_2$ with the topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$ is also a coproduct $X_1 \coprod X_2$ (show that by exhibiting the unique homeomorphism respecting the inclusions).

Proof:

a) Let Z be a topological space with continuous maps $f : X_1 \to Z$ and $g : X_2 \to Z$. There are maps $p_1 : X_1 \to C$ and $p_2 : X_2 \to C$ given by $p_1(x) = (x, 1)$ and $p_2(x) = (x, 2)$. For any basis element $U \times \{1\}$ we have that $p_1^{-1}(U \times \{1\}) = U$ is open and $p_2^{-1}(U \times \{1\}) = \varnothing$ is open. Similarly, if we have a basis element $U \times \{2\}$, $p_1^{-1}(U \times \{2\}) = \varnothing$ is open and $p_2^{-1}(U \times \{2\}) = U$ is open. Suppose we have a map $h : C \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
Z & X_1 \arrow[l, "f"'] \arrow[d, "p_1"] \\
X_2 \arrow[u, "g"] \arrow[r, "p_2"'] & C \arrow[ul, "h"]
\end{tikzcd}$$

We want to show that h is well-defined, unique and continuous. To make this diagram commute, we must have that $h \circ p_1 = f$ and $h \circ p_2 = g$. For any $(x, 1) \in C$, we have that $(x, 1) = p_1(x)$, then $h(x, 1) = h(p(x)) = f(x)$ and similarly, for any $(x, 2) \in C$, we have that $h(x, 2) = h(p_2(x)) = g(x)$. This definition makes h well-defined and makes the diagram commute. For the diagram to commute, we must have this be the definition of h and so h is unique. We now need only show that h is continuous. Notice that $X_1 \times \{1\}$ and $X_2 \times \{2\}$ are closed in C and union to C, so we need only show that $h|_{X_1\times\{1\}}$ and $h|_{X_2\times\{2\}}$ are continuous. Notice that $p_1^{X_1\times\{1\}} : X_1 \to X_1 \times \{1\}$ is a homeomorphism since $p_1(X_1) = X_1 \times \{1\}$ and is in fact a bijection, so $p_1^{X_1\times\{1\}}$ is continuous and it is an open map since if U is open in X_1, then $U \times \{1\}$ is open in C and hence in $X_1 \times \{1\}$. Similarly, $p_2^{X_2\times\{2\}}$ is a homeomorphism., For any open set U in Z, we have that $h|_{X_1\times\{1\}}^{-1}(U)$ is open iff $p_1^{-1}(h|_{X_1\times\{1\}}^{-1}(U))$ is open, but notice that this is just $(h|_{X_1\times\{1\}} \circ p_1)^{-1} = (h \circ p_1)^{-1} = f^{-1}$ and $f^{-1}(U)$ is open since f is continuous. Therefore $h|_{X_1\times\{1\}}^{-1}(U)$ is open and so $h|_{X_1\times\{1\}}$ is continuous. Similarly, $h|_{X_2\times\{2\}}$ is continuous and thus h is continuous.

b) We will consider the function $k : X_1 \cup X_2 \to X_1 \coprod X_2$ given by $k(x) = (x, 1)$ if $x \in X_1$ and $k(x) = (x, 2)$ if $x \in X_2$. Consider any open basis element in the coproduct then $k^{-1}(U \times \{1\}) = U$ and $k^{-1}(U \times \{2\}) = U$ are both open in $X_1 \cup X_2$ since they are basis elements, thus k is continuous. Since we have maps $f : X_1 \to X_1 \cup X_2$ given by inclusion and similarly $g : X_2 \to X_1 \cup X_2$, then as long as these are continuous, we will have that there is a unique map from the coproduct to $X_1 \cup X_2$. f is continuous since for any basis element $U \subseteq X_1$, we have that $f^{-1}(U) = U$ is open and similarly, for any basis element $V \subseteq X_2$, we have that $f^{-1}(V) = \varnothing$ since $X_1 \cap X_2 = \varnothing$. There is an analogous argument to show that g is continuous. We thus receive a continuous function $h : X_1 \coprod X_2 \to X_1 \cup X_2$ with $h((x, 1)) = f(x) = x$ and $h((x, 2)) = g(x) = x$. We want to show that $k \circ h$ and $h \circ k$ are the identity. For any $x \in X_1 \cup X_2$, if $x \in X_1$, then $h(f(x)) = h((x, 1)) = x$ and if $x \in X_2$, then $h(f(x)) = h((x, 2)) = x$. If $(x, 1) \in X_1 \coprod X_2$, then $f(h((x, 1))) = f(x) = (x, 1)$ and $f(h((x, 2))) = f(x) = (x, 2)$. It follows that $X_1 \cup X_2$ is homeomorphic to $X_1 \coprod X_2$.

20.1.a

In $\mathbb{R}^n$, define $d'(x, y) = \sum_{i=1}^n |x_i - y_i|$. Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

Proof:

We want to show that d' and the euclidean metric d are comparable. Notice that $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \leq \sum_{i=1}^n \sqrt{(x_i - y_i)^2}$ since the square root is convex, then $d(x, y) \leq d'(x, y)$. We can similarly get that $|\langle x - y, 1\rangle|^2 \leq ||x - y||^2 \cdot ||1||^2$ under the euclidean inner product with the Cauchy-Schwarz inequality, thus $d'(x, y) \leq d(x, y)\sqrt{n}$. It follows that for any open ball $B_{d'}(x, r)$ under d', we have that for any $y \in B_{d'}(x, r)$, $d'(y, x) < r$, thus for any $z \in B_d(y, \frac{r - d'(x,y)}{\sqrt{n}})$, thus $d'(y, z) \leq d(z, y)\sqrt{n} < r - d'(x, y)$, and so we get that $d'(x, z) \leq d'(x, y) + d'(y, z) < d'(x, y) + r - d'(x, y) = r$. It follows that $B_{d'}$ is open under d. We can similarly show that any B_d is open under d'. For any $B_d(x, r)$ and any $y \in B_d(x, r)$, we may take the ball $B_{d'}(y, r - d(x, y))$, then for any $z \in B_{d'}(y, r - d(x, y))$, we have that $d(y, z) \leq d'(y, z) < r - d(x, y)$, so $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$. It follows that the topologies are the same.

20.5

What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology.

Proof:

For any $x \in \mathbb{R}^\omega$, we have that $x \in \overline{\mathbb{R}^\infty}$ iff $\forall\epsilon > 0$, $B(x,\epsilon) \cap \mathbb{R}^\infty \neq \varnothing$. This means that for any $0 < \epsilon < 1$, we have that there is a sequence $y \in \mathbb{R}^\infty$ such that $\sup |x_i - y_i| < \epsilon$, which is to say that each term x_i differs from y_i by at most ϵ. This is the case iff there is some N such that for all $n > N$, $|x_n| < \epsilon$. This happens iff x converges to 0.

21.2

Let X and Y be metric spaces with metric d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. Show that f is an imbedding. It is called an isometric imbedding of X in Y.

Proof:

We want to show that f is injective, continuous, and has continuous inverse. Suppose that $x, y \in X$, then $d_X(x, y) = 0$ iff $d_Y(f(x), f(y)) = 0$, thus $f(x) = f(y)$ iff $x = y$, so f is injective. For any $\epsilon > 0$, then if $y \in B_{d_X}(x,\epsilon)$ we get that $d_Y(f(x), f(y)) = d_X(x, y) < \epsilon$, so f is continuous. f is bijective onto its image, so we can consider $f^{-1} : f(X) \to X$, then for any $\epsilon > 0$, we have that for $y \in B_{d_Y}(x,\epsilon)$ it follows that $d_X(f^{-1}(x), f^{-1}(y)) = d_Y(f(f^{-1}(x)), f(f^{-1}(y))) = d_Y(x, y) < \epsilon$, so f^{-1} is continuous. Therefore f is an isometric imbedding.

Suggested Exercises 11

For $A \subseteq X$, define $\chi_A : A \to \{0, 1\}$ (with the discrete topology) to be 1 on A and 0 otherwise. Show the boundary of A is the set of discontinuities of χ_A.

Proof:

Suppose that $x \in \mathrm{Bd}A$, then for any open set U containing x, we have that $U \cap A \neq \varnothing$ and similarly, $U \cap A^c \neq \varnothing$, thus $\chi_A(U) = \{0, 1\}$, but since $\{0, 1\}$ has the discrete topology, then $\{\chi_A(x)\}$ is open, but no open set containing x maps into it. Therefore χ_A is discontinuous on the boundary of A. If χ_A is discontinuous at x, then for any open set U containing x, we must have that $\chi_A(U)$ does not map inside of $\{\chi_A(x)\}$ and so we must have that $\chi_A(U) = \{0, 1\}$. It follows that there is some $a \in U$ such that $\chi_A(a) = 0$ and so $U \cap A \neq \varnothing$ and also that there is some $b \in U$ such that $\chi_A(b) = 1$ and so $U \cap A^c \neq \varnothing$, thus $x \in \mathrm{Bd}A$. We have then shown that the discontinuities of χ_A is exactly A.

22.1

Show that the quotient topology on $A = \{a, b, c\}$ induced by $p : \mathbb{R} \to A$ given by $p(x) = a$ if $x < 0$, $p(x) = b$ if $x < 0$ and $p(x) = c$ if $x = 0$ is exactly $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

Proof:

We have that UA is open iff $p^{-1}(U)$ is open. We may look at each subset of A. Clearly, $\varnothing$ and A are open. $p^{-1}(a) = (-\infty, 0)$ is open, $p^{-1}(b) = (0, \infty)$ is open, $p^{-1}(c) = \{0\}$ is not open. For two element sets, $p^{-1}(\{a, b\}) = (0, \infty) \cup (-\infty, 0)$ is open, $p^{-1}(\{a, c\}) = [0, \infty)$ is not open, $p^{-1}(\{b, c\}) = (-\infty, 0]$ is not open.

22.2

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f = \mathrm{Id}_Y$, then p is a quotient map.
(b) If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction map is a quotient map.

Proof:

a) For any $y \in Y$, $p(f(y)) = y$, so p is surjective. p is continuous, so $p^{-1}(U)$ is open if U is open. We need only show the converse, that U is open if $p^{-1}(U)$ is open. Since U is open, then $U = \mathrm{Id}_Y(U) = \mathrm{Id}_Y^{-1}(U) = (p \circ f)^{-1}(U)$ is open. Notice that $(p \circ f)^{-1} = f^{-1} \circ p^{-1}$, so $(f^{-1} \circ p^{-1})(U)$ is open, but $f^{-1}(p^{-1}(U))$ is open only if $p^{-1}(U)$ is open since f is continuous, so $p^{-1}(U)$ is open. Therefore, p is a quotient map.

b) Consider the inclusion map $i : A \to X$, then i is continuous and $(r \circ i)(x) = r(i(x)) = r(x) = x$ for all $x \in A$, thus $r \circ i = \mathrm{Id}_A$, so r is a quotient map.

22.3

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points (x, y) for which either either $x \geq 0$ or $y = 0$ or both; let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

Proof:

q is continuous since it is the restriction of a continuous function. For any $x \in \mathbb{R}$, we have that $(x, 0) \in A$ and $q(x, 0) = x$, so q is surjective. We want to show that if U is a set in $\mathbb{R}$ such that $q^{-1}(U)$ is open, then U must be open. Notice that $q^{-1}(U) = (U \times \mathbb{R}) \cap A$. Since $q^{-1}(U)$ is open, then for any $x \in U$, we may find an open set V in A such that $(x, 0) \in V$. Since V is open in A, then $V = V' \cap A$, where V' is open in $\mathbb{R}^2$. It follows that there is some $\epsilon > 0$ such that $(x, 0) \in B((x, 0), \epsilon) \cap A \subseteq V' \cap A \subseteq q^{-1}(U) \subseteq A$. Therefore, $(x - \epsilon, x + \epsilon) \subseteq U$ since $(a, 0) \in q^{-1}(U)$ for all $a \in (x - \epsilon, x + \epsilon)$. Thus, q is a quotient map.

Notice that $q(B((0,1),1) \cap A) = [0,1)$, so q is not an open map. Similarly, for $U = \{(x,y) \big| x \geq \frac{1}{1+y^2}\}$, U is closed, but $q(U) = (0,\infty)$ is open.

22.4

(a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows: $(x,y) \sim (x',y')$ if $x + y^2 = x' + (y')^2$. Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it?
(b) Repeat a for the equivalence relation $(x,y) \sim (x',y')$ if $x^2+y^2 = (x')^2+(y')^2$.

Proof:

a) Consider the function $g(x,y) = x + y^2$, the fibres of g are the equivalence classes of X. Notice that g is continuous since it a polynomial and furthermore it is surjective since $g(x,0) = x$. We also want to show that g is a quotient map. We also have the map $i : \mathbb{R} \to X$ given by $i(x) = (x,0)$ which is the inclusion of $\mathbb{R}$ into X and is hence continuous. Notice, however that $g(i(x)) = g((x,0)) = x$, thus $g \circ i$ is the identity on $\mathbb{R}$ and so g is a quotient map. Consider the following diagram:

$$\begin{tikzcd} X = \mathbb{R}^2 \arrow[d, "q"] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

We have that g is constant on the fibres of q and thus induces a map $f : X^* \to \mathbb{R}$. For any two fibres $\{(x,y) \big| x + y^2 = c\}$ and $\{(x,y) \big| x + y^2 = c'\}$, we have that $f(\{(x,y) \big| x + y^2 = c\}) = g(x,y) = c$ and $f(\{(x,y) \big| x + y^2 = c'\}) = g(x,y) = c'$. It follows that if $f(\{(x,y) \big| x + y^2 = c\}) = f(\{(x,y) \big| x + y^2 = c'\})$, then $c = c'$ and hence the fibres are the same. Thus f is injective. f is also surjective since $f(\{(x,y) \big| x + y^2 = c\}) = c$ for any $c \in \mathbb{R}$. It follows that f is a homeomorphism, thus the quotient space is just $\mathbb{R}$.

b) We proceed in the same manner by noticing that $g(x,y) = x^2 + y^2$ has fibres which are the equivalence classes of X, is continuous (polynomial) and is surjective onto $\mathbb{R}_{\geq 0}$. Again, if we consider the continuous map f from $\mathbb{R}_{\geq 0}$ into X by $f(x) = (\sqrt{x}, 0)$, then we see that $g \circ i = \mathrm{Id}_{\mathbb{R}_{\geq 0}}$ and so g is a quotient map. Again, the induced map is a bijection and so it is a homeomorphism.

Question 5

Let $p : X \to Y$ be an open map. Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.

Proof:

For any open set U in A, U is open in X since A is open, thus $q(U) = p(U)$ is open in Y and it is contained in $p(A)$ and is thus open in $p(A)$. It follows that q is an open map.

Suggested Exercise 13

Show that $\mathbb{R}^{n+1} \setminus \{0\}$ is homeomorphic to $S^n \times \mathbb{R}$.

Proof:

Consider the map $f(x) = (\frac{x}{||x||}, ||x||)$, this map is continuous (norm is continuous and nonzero) and its inverse $f^{-1}(x, r) = rx$ is also continuous. Note that these are inverses since $f(f^{-1}(x, r)) = f(rx) = (\frac{rx}{r||x||}, r||x||) = (x, r)$ since $x \in S^n$ so $||x|| = 1$. And similarly, $f^{-1}(f(x)) = f^{-1}(\frac{x}{||x||}, ||x||) = ||x|| \frac{x}{||x||} = x$. It follows that $\mathbb{R}^{n+1} \setminus \{0\} \cong S^n \times \mathbb{R}^+$. Since $\mathbb{R}^+ \cong \mathbb{R}$ by $x \mapsto \ln(x)$, then by the next exercise, $\mathbb{R}^{n+1} \setminus \{0\} \cong S^n \times \mathbb{R}$.

Suggested Exercise 14

Suppose A, B is homeomorphic to C, D respectively, then show $C \prod D$ is a product of A and B and $C \coprod D$ is a coproduct of A and B.

Proof:

Consider the following diagrams:

$$\begin{tikzcd}
C \arrow[r, shift left, "f^{-1}"] \arrow[dr, "\pi_A \circ f^{-1}"] \arrow[dd, "\pi_C"'] & A \arrow[l, shift left, "f"] \arrow[r, "p"] \arrow[d, "\pi_A"] \arrow[ddl, bend right, "\pi_C \circ f"'] & Z \\
& A \coprod B \arrow[ur, "\exists ! h"] & B \arrow[u, "q"'] \arrow[l, "\pi_B"'] \arrow[d, shift left, "g"] \arrow[dll, bend left, "\pi_D \circ g"] \\
C \coprod D \arrow[ur, "\exists ! m"] \arrow[uurr, bend left, "h \circ m"] & & D \arrow[u, shift left, "g^{-1}"] \arrow[ul, "\pi_B \circ g^{-1}"'] \arrow[ll, "\pi_D"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, shift left, "f^{-1}"] \arrow[dr, "f \circ \pi_A"] & A \arrow[l, shift left, "f"] & Z \arrow[l, "p"'] \arrow[d, "q"] \arrow[dl, "\exists ! h"'] \arrow[ddll, bend right, "m \circ h"'] \\
& A \prod B \arrow[u, "\pi_A"] \arrow[r, "\pi_B"] \arrow[dl, "\exists ! m"] \arrow[dr, "g \circ \pi_B"] & B \arrow[d, shift left, "g"] \\
C \prod D \arrow[uu, "\pi_C"] \arrow[uur, bend left, "f^{-1} \circ \pi_C"] \arrow[urr, bend right, "g^{-1} \circ \pi_D"'] \arrow[rr, "\pi_D"] & & D \arrow[u, shift left, "g^{-1}"]
\end{tikzcd}$$

QED.

Suggested Exercise 15

Suppose $q_1 : A \to B$, $q_2 : C \to D$ are quotient maps, and A is homeomorphic to C by a map that preserves fibres of the quotient maps. Then B is homeomorphic to D.

Proof:

Consider the diagram:

$$\begin{tikzcd}
A \arrow[r, "q_1"'] \arrow[d, shift right, "h"'] \arrow[dr, "q_2 \circ h"] & B \arrow[d, shift left, "m"] \\
C \arrow[r, "q_2"'] \arrow[u, shift right, "h^{-1}"'] & D \arrow[u, shift left, "n"]
\end{tikzcd}$$

Notice that h is a homeomorphism and hence a quotient map, thus $q_2 \circ h$ is a quotient map. q_1 is also a quotient map. For any fibre $q_1^{-1}(x)$, we have that $h(q_1^{-1}(x)) = q_2^{-1}(y)$ since h preserves the fibres of the quotient maps, thus for any $z \in q_1^{-1}(x)$, we have that $q_2(h(z)) \in q_2(q_2^{-1}(y)) = \{y\}$, so $q_2 \circ h$ is constant on the fibres of q_1. For any fibre $h^{-1}(q_2^{-1}(x))$, we have that $h^{-1}(q_2^{-1}(x)) = q_1^{-1}(y)$ and so for any $z \in q_1^{-1}(y)$, we have that $q_1(z) = y$, so q_1 is constant on the fibres of $q_2 \circ h$. Since these are quotient maps which are constant on each others fibres, then we get continuous maps n and m which make the diagram commute. We want to show that these maps are inverses of each other. For any $x \in B$, we have that $x = q_1(y)$, thus

$$n(m(x)) = n(m(q_1(y))) = n(q_2(h(y)) = q_1(y) = x$$

And for any $x \in D$, we have that $x = q_2(h(y))$, thus

$$m(n(x)) = m(n(q_2(h(y)))) = m(q_1(y)) = q_2(h(y)) = x$$

Exercises 23.1

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supseteq \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

Proof:

Since $\mathcal{T}' \supseteq \mathcal{T}$, the identity function is continuous, thus $\mathcal{T}$ is connected if $\mathcal{T}'$ is connected. We cannot say anything about the connectedness of $\mathcal{T}$ if $\mathcal{T}'$ is disconnected. For instance, take the discrete topology on X and the indiscrete topology on X.

Exercises 23.2

Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Proof:

Let $X_n = \bigcup_{i=1}^n A_i$, then we have that X_1 is connected. Since $X_{n+1} = X_n \cup A_{n+1}$ and $X_n \cap A_{n+1} \neq \varnothing$, then since X_n and A_{n+1} are connected, then X_{n+1} is connected. Since $A_1 \subseteq X_n$ for all n and $\bigcup A_n = \bigcup X_n$, then $\bigcup X_n$ is connected.

Exercises 23.3

Let$\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

Proof:

Let $Y = A \cup (\bigcup A_\alpha)$. Suppose that $Y = U \cup V$ is a separation. WLOG, let $U \cap A \neq \varnothing$, then $A \subseteq U$. Since $V \neq \varnothing$ and $V \cap A \subseteq V \cap U = \varnothing$, then $V \cap A = \varnothing$. It follows that some $U_\alpha \cap V \neq \varnothing$ and thus $U_\alpha \subseteq V$, but $U_\alpha \cap U \neq \varnothing$, thus $V \cap A \neq \varnothing$ which is a contradiction. It follows that no such separation exists.

Exercises 23.4

Show that if X is an infinite set, it is connected in the finite complement topology.

Proof:

Suppose that $X = U \cup V$ is a separation by open sets, then $U^c = V$ and $V^c = U$, so U and V are closed and hence finite (or X, but since one is non-empty, that is impossible). It follows that U and V are finite, so X is finite.

Exercises 23.5

A space is **totally disconnected** if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Proof:

If $A \subseteq X$, then if A has more than one element, then for any $x \in A$, $\{x\}, A \setminus \{x\}$ are open and a separation of A. It follows that the only connected subspaces are one-point sets (note that one-point sets are trivially connected since they cannot be written as the union of two disjoint non-empty sets). The converse is not true, as $\mathbb{Q}$ is totally disconnected under the standard topology, however one-point sets are not open. If $A \subseteq \mathbb{Q}$ contains p, q distinct, then there is an irrational number $r \in \mathbb{R}$ s.t. $p < r < q$, then $A \cap \{x \big| x < r\}$ and $A \cap \{x \big| x > r\}$ is a separation of A.

Exercises 23.6

Let $A \subseteq X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects $\mathrm{Bd}A$.

Proof:

$(\overline{A} \cap C) \cup (\overline{X \setminus A} \cap C) = C$ and are each closed, so they must intersect non-trivially, thus $\overline{A} \cap \overline{X \setminus A} \cap C \neq \varnothing$, thus $\mathrm{Bd}A \cap C \neq \varnothing$.

Exercises 23.9

Let A be a proper subset of X and let B be a proper subset of Y. if X and Y are connected, show that (X $\times Y) \setminus A \times B)$ is connected.

Proof:

Choose $x \notin A$ and $y \notin B$, then let $Z = (X \times Y) \setminus (A \times B)$. We have that $x \times Y, X \times y \subseteq Z$. Let $W = x \times Y \cup X \times y$, then for any point $(a, b) \in Z$, we have that either $a \times Y$ or $X \times b$ is contained in Z and they intersect W, thus $Z = \bigcup_{(a,b) \in Z} (a, b)$ is connected.

Exercises 23.10

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of connected spaces; let X be the product space $X = \prod_{\alpha \in J} X_\alpha$. Let $\mathbf{a} = (a_\alpha)$ be a point in X.
(a) Given any finite subset K of J, let X_K denote the subspace of X consisting of all points $\mathbf{x} = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show that X_K is connected.
(b) Show that the union Y of the spaces X_K is connected.
(c) Show that X equals the closure of Y; conclude that X is connected.

Proof:

(a): X_K homeomorphic to $\prod_{\alpha \in K} X_\alpha$ which is connected since it is a finite product of connected spaces.

(b): Each X_K contains $\mathbf{a}$, so their union is connected.

(c): For any point $(x_\alpha)_{\alpha \in J}$ and any open set U containing (x_α), $U = \prod_{i=1}^n U_{\alpha_i} \times \prod_{\alpha \in J \setminus \{\alpha_1, \cdots, \alpha_n\}} X_\alpha$, then $\cap X_{\{}\alpha_1, \cdots, \alpha_n\}$ is non-empty, so (x_α) is in the closure of Y. Since the closure of a connected set is connected, then X is connected.

Exercises 23.11

Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Proof:

Suppose we have a separation $X = U \cup V$, then for any $x \in U$, $f^{-1}(\{f(x)\})$ is connected and thus entirely contained in U, so U is saturated, similarly V is saturated, thus $p(U)$ and $p(V)$ are open, then since p is surjective, $Y = p(U) \cup p(V)$ is a separation. This is a contradiction, thus X is connected.

Exercises 24.1

(a): Show that no two of the spaces $(0, 1), (0, 1]$, and $[0, 1]$ are homeomorphic.

(b): Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

(c): Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Proof:

(a): $(0, 1)$ has 0 cut points, $(0, 1]$ has 1 cut point, and $[0, 1]$ has 2 cut points.

(b): Consider the spaces $(0, 1)$ and $[0, 1]$. $(0, 1)$ embeds into $[0, 1]$ by inclusion and $[0, 1]$ embeds into $(0, 1)$ by $x \mapsto \frac{x+1}{3t}$.

(c): Since $\mathbb{R}^n$ has no cut points for $n > 1$, whereas $\mathbb{R}$ has uncountably many cut points, then they are not homeomorphic.

Exercises 24.2

Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.

Proof:

Let $g(x) = f(x) - f(-x)$, then pick some $x \in S$, either $g(x) = 0$, in which case we have $f(x) = f(-x)$. If $g(x) \neq 0$, then $g(x) = -g(x)$, so by connectedness of S^1, there is some $c \in S^1$ such that $g(c) = 0$, then $f(c) = f(-c)$.

Exercises 24.3

Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a fixed point of f. What happens if $X = [0, 1)$ or $(0, 1)$?

Proof:

Consider the function $g(x) = x - f(x)$, then $g(0) = -f(0) \leq 0$ and $g(1) = 1 - f(1) \geq 0$. By IVT, there is some $c \in [0, 1]$ such that $g(c) = 0 = c - f(c)$, so $c = f(c)$. This is not true if $X = [0, 1]$ or $X = (0, 1)$, since $f(x) = \frac{1+x}{2}$ has no fixed point in either.

Exercises 24.9

Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus A$ is path connected.

Proof:

Consider all lines that pass through x, and all lines that pass through y. There are uncountably many lines through each of x and y. For any line through x, there is a unique line passing through y to which it is perpendicular. These two lines must intersect. The path from x to the intersection to y gives a set of disjoint paths and there are uncountably many, thus not all of them can contain a point from A, so one of them must be a path in $\mathbb{R}^2 \setminus A$.

Exercises 24.10

Show that if U is an open connected subspace of $\mathbb{R}^2$, then U is path connected.

Proof:

If U is empty, then U is trivially path connected. If U is not empty, pick $x \in U$ and consider the set $A = \{y \in U \big| \exists \gamma \text{ a path from } x \text{ to } y\}$. We want to show that $A = U$. For any $y \in A$, since U is open, then there is a ball $B(y, \epsilon) \subseteq U$. Since x is path connected to y and y is path connected to any point in $B(y, \epsilon)$, then there is a path from x to any point in $B(y, \epsilon)$, it follows that $B(y, \epsilon) \subseteq A$, so A is open. For any $y \in \overline{A}$, there is a ball $B(y, \epsilon) \subseteq U$, then there is some $z \in A \cap B(y, \epsilon)$, it follows that there is the straight line path from z to y and a path from x to z, thus there is a path from x to z. It follows that $A = \overline{A}$, so A is closed. Since A is a non-empty clopen subset of a connected space, then $A = U$, so U is path connected.

Exercises 24.11

If A is a connected subspace of X, does it follow that Int$A$ and BdA are connected? Does the converse hold?

Proof:

$\overline{B}((0,0),1) \cup \overline{B}((2,0),1)$ is connected, but its interior is $B((0,0),1) \cup B((2,0),1)$ is not connected. $[0,1] \times \mathbb{R}$ has boundary $\{0,1\} \times \mathbb{R}$ which is not connected.

$\mathbb{Q}$ has empty interior which is connected and Bd$\mathbb{Q} = \mathbb{R}$ is connected, but $\mathbb{Q}$ is not connected.

Suggested Exercise 19

Suppose $A_1, \cdots, A_n$ are closed subspaces of X such that $\bigcup A_i = X$. Suppose $f_i : Y_i \to A_i$ are homeomorphisms. Define $g_i : Y_i \to X$ to be the composition of f_i with the inclusion. Show that $\coprod Y_i / \sim$ is homeomorphic to X where $a \sim b$ if $(\coprod g_i)(a) = (\coprod g_i)(b)$.

Proof:

We know that $X \cong \coprod A_i/\sim$ where $x \sim y$ if $x = y$ in X is homeomorphic to X. Since Y_i is homeomorphic to A_i, then the coproducts are homeomorphic. We want to show that homeomorphisms from the coproducts respect the quotients of the coproduct. The homeomorphism from $\coprod A_i \to \coprod Y_i$ is given by $h(y^{(i)}) = [f_i(y^{(i)})]$ for any $y^{(i)} \in Y_i$. We want to show that the maps $\coprod Y_i/\sim \to X$ given by $[y] \xrightarrow{p} g_i(y_i)$ for any $y_i \in [y]$ and $q : \coprod A_i/\sim \to X$ by $[a] \xrightarrow{q} a_i$ where $a_i \in [y]$ have bijective fibres under h. For any $x \in X$, we have that $p^{-1}(\{x\}) = [y]$ where $y_i \in y$ has $y_i = g_i^{-1}(x) = f_i^{-1}(x)$. Similarly, $q^{-1}(x) = [a]$ where $a_i \in [a]$ has $a_i = x$. We then have that h yields a bijection between the fibres (they are single elements, so we need only check that $h([y]) = [a]$. Since $h([y]) = \{f_i(y_i)|y_i \in [y]\} = \{f_i(\iota_i^{-1} \circ f_i^{-1}(x))\} = \{x^{(i)}\} = [a_i] = [a]$, then the fibres are preserved, so the quotients are homeomorphic, thus $X \cong \coprod Y_i/\sim$.

Exercise 26.1

(a) Let $\mathcal{T}, \mathcal{T}'$ be two topologies on the set X; Suppose that $\mathcal{T}' \subseteq \mathcal{T}$. What does the compactness of X under one of these topologies imply about compactness under the other?
(b): Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T}$ and $\mathcal{T}'$ are equal of they are not comparable.

Proof:

(a): If X is compact under $\mathcal{T}$, then for any open cover $\mathcal{A}$ in $\mathcal{T}'$, it is also an open cover in $\mathcal{T}$ and thus admits a finite subcover, thus X is compact under $\mathcal{T}'$. If X is compact under $\mathcal{T}'$, then X need not be compact under $\mathcal{T}$, for example, $\mathbb{R}$ is compact under the indiscrete topology, but not under the discrete topology, since $\{\{x\}\big|x \in \mathbb{R}\}$ is an open cover with no finite subcover.

(b): If $\mathcal{T}$ and $\mathcal{T}'$ both yield compact Hausdorff topologies, then if they are comparable, wlog let $\mathcal{T} \subseteq \mathcal{T}'$, then the identity map $Id : X \to X$ is a continuous bijection from X under $\mathcal{T}'$ to X under $\mathcal{T}$, so it is a continuous bijection from a compact to Hausdorff space, and thus is a homeomorphism, so $\mathcal{T} = \mathcal{T}'$. It follows that either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

Exercise 26.2

(a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.
(b): If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} \setminus A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

Proof:

(a): Let $\mathcal{A}$ be an open cover of $X \subseteq \mathbb{R}$ under the finite complement topology, then pick a non-empty open set $U \in \mathcal{A}$. It follows that $U^c = \{x_1, \cdots, x_n\}$. Let $U_1, U_2, \cdots, U_n \in \mathcal{A}$ such that $x_i \in A_i$, then $\{U, U_1, \cdots, U_n\}$ is a finite subcover of $\mathcal{A}$.

(b): $[0,1]$ is not compact under this topology. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ and let $U_i = (\mathbb{R} \setminus X) \cup \{1, \frac{1}{2}, \cdots, \frac{1}{i}\}$, then $\{U_i\}$ forms an open cover of $[0,1]$, however for any finite subcover, there is a largest i such that $\frac{1}{i}$ is contained in the finite subcover, and thus it cannot be all of $[0,1]$.

Exercise 26.3

Show that a finite union of compacts subspaces of X is compact.

Proof:

Let $A_1, \cdots, A_n \subseteq X$ be subspaces, then let $\mathcal{A}$ be an open cover of $A = A_1 \cup \cdots \cup A_n$. It follows that $\mathcal{A}$ is an open cover of each A_i, so it admits a finite subcover $\mathcal{A}_i \subseteq \mathcal{A}$ which covers A_i. Since unions of finite sets are finite, then $\mathcal{A}' = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ is a finite subcover of A.

Exercise 26.4

Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed and bounded subspace is compact.

Proof:

Since metric spaces are Hausdorff, then any compact subspace is closed. Consider the open cover $\{B(0,n) \big| n \in \mathbb{R}\}$ of our space X, then if Y is a compact subspace of X, then it is contained in a finite number of these balls, and hence contained in a largest such ball, thus it is bounded.

In $\mathbb{R}^\infty$, the set $\{e_1, e_2, \cdots\}$ is closed and bounded, but not compact. For proof refer to 257 A3/A2.

Exercise 26.5

Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B respectively.

Proof:

Fix some $x \in A$, then for any $y \in B$, since X is Hausdorff, there are open sets U_y, V_y that separate x and y. Since $\{V_y | y \in B\}$ is an open cover of B, then it admits a finite subcover, $\{V_1, V_2, \cdots, V_n\}$ and a corresponding set $\{U_1, \cdots, U_n\}$. It follows that $U = U_1 \cap \cdots \cap U_n$ is disjoint from each $V_1, \cdots, V_n$ and hence disjoint from their union $V = V_1 \cup \cdots \cup V_n$ which covers B. We have established that we may separate a point $x \in A$ from B by open sets. For each $x \in A$, let U_x, V_x separate x and B, then since $\{U_x \big| x \in A\}$ forms a cover of A, it admits a finite subcover $U_1, U_2, \cdots, U_n$ and a corresponding set $V_1, \cdots, V_n$. Let $V = V_1 \cap \cdots \cap V_n$, then V is disjoint from each U_i and thus disjoint from their union $U = U_1 \cup \cdots \cup U_n$. Since $B \subseteq V_i$ for each i, then $B \subseteq V$ and $A \subseteq U$ since U is the union of a cover of A and V and U are disjoint as desired.

Exercise 26.6

Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map.

Proof:

Let C be a closed set in X, then since it is a closed subset of a compact space, it is compact and thus $f(C)$ is compact. Since $f(C)$ is a compact subset of a Hausdorff space, then it is closed, thus f is a closed map.

Exercise 26.7

Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Proof:

For any closed set C in $X \times Y$, for any $x \notin \pi_1(C)$, we have that $x \times Y \in C^c$ which is open, thus by the tube lemma, there is an open set $x \in W \subset X$ such that $W \times Y \in C^c$, thus $x \in W \subseteq \pi_1(C)^c$, so $\pi_1(C)^c$ is open and thus $\pi_1(C)$ is closed, hence π_1 is a closed map.

Exercise 26.8

Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the graph of f, $G_f = \{(x, f(x)) \big| x \in X\}$ is closed in $X \times Y$.

Proof:

$\Rightarrow$ If f is continuous, then for any point $(x, y) \notin G_f$, we have that $y \neq f(x)$, so we may separate them by open sets U, V with $y \in U, f(x) \in V$. Since $f(x)$ is continuous, then the preimage of V is open, then consider $f^{-1}(V) \times U$. For any $(x, y) \in f^{-1}(V) \times U$, we have that $f(x) \in U$ and $y \in V$, which are disjoint, thus $(x, y) \neq (x, f(x))$, so $f^{-1}(V) \times U \in G_f^c$. It follows that G_f^c is open and hence G_f is closed.

$\Leftarrow$ If G_f is closed, then if V is an open neighborhood of $f(x_0)$, the intersection of G_f and $X \times (Y \setminus V)$ is closed, let C denote this intersection. Since Y is compact, then the projection onto X is closed. The projection onto x is the set of all x such that $f(x) \in Y \setminus V$. It follows that the complement of this set is open, thus $\{x \in X \big| f(x) \in V\} = f^{-1}(V)$ is open. Therefore, f is continuous.

Exercise 26.9

Let A, B be subspaces of X, Y respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y respectively such that $A \times B \subseteq U \times V \subseteq N$.

Proof:

Let $\{B_x \times B_y\}$ be a covering of $A \times B$ by basis elements contained in N, then there is a finite subcover $\{B_{x_i} \times B_{y_i}\}_{i=1}^n$. We construct U and V as follows. To make U, we consider each $x \in A$, then let $U_x = \bigcap_{x \in B_{x_i}} B_{x_i}$, i.e. the intersection of all basis elements containing x. These are all open since we intersect only finitely many open sets. Let $U = \bigcup_{x \in A} U_x$. We construct V analogously. Clearly, $A \times B \subseteq U \times V$ and for any $(x, y) \in U \times V$ we have that there is some $(x', y') \in A \times B$ such that $x \in \bigcap_{x' \in B_{x_i}} B_{x_i}$ and $y \in \bigcap_{y' \in B_{y_i}} B_{y_i}$. Since $(x', y') \in A \times B$, then there is some i s.t. $(x, y) \in B_{x_i} \times B_{y_i}$, thus $(x, y) \in B_{x_i} \times B_{y_i} \in N$, so $U \times V \in N$.

Exercise 26.11

Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$ (such a map is called a **perfect map**). Show that if Y is compact, then X is compact.

Proof:

If U is an open set containing $p^{-1}(\{y\})$, then there is an open neighborhood W of y such that $p^{-1}(W)$ is contained in U. This is because U^c is closed and thus $f(U^c)$ is closed and $y \in f(U^c)^c$. We then have that $f^{-1}(f(U^c)^c)$ is open in X and for any $x \in f^{-1}(f(U^c)^c)$, we have that $f(x) \in f(U^c)^c$, thus $f(x) \notin f(U^c)$, thus $x \notin U^c$, so $x \in U$. It follows that $f^{-1}(f(U^c)^c)$ is the open set we were looking for. Let $\mathcal{A}$ be an open cover of X, then for each $y \in Y$, we have that $p^{-1}(\{y\})$ is covered by a non-empty finite subcover $\mathcal{A}_y \subseteq \mathcal{A}$ (non-empty because p is surjective). We then let $U_y = \bigcup \mathcal{A}_y$. It follows that $\{U_y\}_{y \in Y}$ is an open cover of X. Since U_y contains $p^{-1}(\{y\})$, then there is a $W_y \subseteq Y$ open such that $y \in W_y$ and $p^{-1}(W_y) \subseteq U_y$. The set $\{W_y\}_{y \in Y}$ is an open cover of Y and by compactness, there is a finite subcover $W_1, \cdots, W_n$. Let $U_1, \cdots, U_n$ be the corresponding U's, then we have that $p^{-1}(W_1 \cup \cdots \cup W_n) = p^{-1}(Y) = X$, but also $p^{-1}(W_1 \cup \cdots \cup W_n) \subseteq \bigcup_{i=1}^n U_i$, thus $U_1, \cdots, U_n$ covers X. Let $\mathcal{A}_1, \cdots, \mathcal{A}_n$ correspond to $U_1, \cdots, U_n$, then $\mathcal{A}' = \bigcup_{i=1}^n \mathcal{A}_i$ is a finite subcover of $\mathcal{A}$.

Exercise 27.2

Let X be a metric space with metric d; let $A \subseteq X$ be nonempty.
(a) Show that $d(x, A) = 0$ iff $x \in \overline{A}$.
(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.
(c) Define the ϵ-neighborhood of A in X to be the set $U(A, \epsilon) = \{x \big| d(x, A) < \epsilon\}$. Show that $U(A, \epsilon)$ is the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.
(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.
(e) Show the result (d) need not hold if A is closed but not compact.

Proof:

(a): If $d(x, A) = 0$, then $\inf_{y \in A} d(x, y) = 0$, thus for any $\epsilon > 0$, there is some

$y \in A$ such that $y \in B(x,\epsilon) \cap A$. For any open set U containing x, we have that $x \in B(x,\epsilon) \subseteq U$, thus $U \cap A \neq \varnothing$, so $x \in \overline{A}$. If $x \in \overline{A}$, then for any ball $B(x,\epsilon)$, $B(x,\epsilon) \cap A \neq \varnothing$, thus $\inf_{y\in A} d(x,y) < \epsilon$ for all $\epsilon > 0$, so $d(x,A) = 0$.

(b): Since $d_x(y) = d(x,y)$ is continuous, then restricted to A, we have that $d_x : A \to \mathbb{R}_{\geq 0}$ is continuous and thus attains its minimum value at some point $a \in A$, then $d(x,A) = \inf_{y\in A} d(x,y) = d(x,a)$.

(c): For any $x \in U(A,\epsilon)$, we have that $d(x,A) < \epsilon$, thus there is a point $y \in A$ such that $d(x,y) < \epsilon$, so $x \in B(y,\epsilon)$. It follows that $U(A,\epsilon)$ is a subset of the given union. For any $x \in B(a,\epsilon)$, we have that $d(x,a) < \epsilon$, thus $d(x,A) < \epsilon$, so $x \in U(A,\epsilon)$. Therefore, the union of balls and the ϵ-neighborhood are equal.

(d): Consider the function $d(x,U^c) : A \to \mathbb{R}_{\geq 0}$. Since A is compact and this function is continuous, then it attains a minimum, thus there is some $a \in A$ such that $d(a,U^c) = \int_{y\in A} d(y,U^c)$. If this minimum were 0, then we would have that $d(a,U^c) = 0$ and thus $a \in \overline{U^c} = U^c$, but $a \in A \subseteq U$, so $a \notin U^c$. It follows that the minimum is greater than 0, then $U(A, d(a,U^c))$ is such an ϵ-neighborhood.

(e): Consider the set $A = \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$ and the open set $U = \{(x,y) \in \mathbb{R}^2 \big| x < \frac{1}{1+y^2}$, then $A \subseteq U$, but for any $\epsilon > 0$, $(\frac{\epsilon}{2}, \sqrt{\frac{2}{\epsilon} - 1}) \in U(A,\epsilon)$, but for this point, we have that $x \geq \frac{1}{1+y^2}$, thus it is not in U. It follows that no ϵ-neighborhood of A is contained in U.

Exercise 27.4

Show that a connected metric space having more than one point is uncountable.

Proof:

Let (X,d) be a connected metric space. Suppose that X were countable, then let $X = \{x_1, x_2, \cdots\}$. Since $X \times X$ is countable, then it cannot surject onto $\mathbb{R}$, thus there is some element $r \in \mathbb{R}$ such that $d^{-1}([0,r))$ and $d^{-1}((r,\infty))$ are non-empty and $d^{-1}(\{r\}) = \varnothing$. Since X is connected, then $X \times X$ is connected, however $[0,r)$ and (r,∞) are open and d is continuous, so $d^{-1}([0,r))$ and $d^{-1}((r,\infty))$ are open. Furthermore, $d^{-1}([0,r)) \cap d^{-1}((r,\infty)) = \varnothing$ and since $d^{-1}(\{r\})$ is empty, then $X \times X = d^{-1}([0,r)) \cup d^{-1}((r,\infty))$ is a separation, which is a contradiction. It follows that X must be uncountable.

Exercise 27.6

Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third", $(\frac{1}{3}, \frac{2}{3})$. In general, define A_n by the equation $A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$. The intersection $C = \bigcap_{n=1}^{\infty} A_n$ is called the Cantor set.

(a) Show that C is totally disconnected.

(b) Show that C is compact.

(c) Show that each set A_n is the union of finitely many disjoint closed intervals

of length $\frac{1}{3^n}$; and show that the end points of these intervals lie in C.
(d) Show that C has no isolated points.
(e) Conclude that C is uncountable.

Proof:

(a): Suppose that $A \subseteq C$ is connected and contains two points a, b, then it contains the interval $[a, b]$. In any closed interval of length $\frac{1}{3^n}$, it must contain a point $\frac{3+6k}{2 \cdot 3^n} \in (\frac{1+3k}{3^n}, \frac{2+3k}{3^n})$. There is some n such that $\frac{1}{3^n} < b - a$, thus $[a, b]$ contains such a point, and thus $[a, b] \not\subseteq C$. It follows that A has at most 1 point, thus C is totally disconnected.

(b): Since A_0 is closed and at each step, we remove a closed set from A_{n-1} to get A_n, then each A_n is closed. Since C is the intersection of closed sets, then C is closed. Since C is a subset of $[0, 1]$, then it is bounded. It follows that C is a closed and bounded subset of $\mathbb{R}$ and is thus compact.

(c): We want to show that the endpoints of A_{n-1} are endpoints of A_n and that each A_n is the union of disjoint closed intervals of length $\frac{1}{3^n}$ whose endpoints are of the form $\frac{k}{3^n}$ where $k \in \mathbb{Z}_{\geq 0}$. Clearly A_0 satisfies these conditions (note that A_{-1} doesn't exist). Since $A_n = \bigcup_{i=1}^{m} [\frac{k_i}{3^n}, \frac{k_i+1}{3^n}]$, then when we remove $(\frac{3k_i+1}{3^{n+1}}, \frac{3k_i+2}{3^n})$, we get two closed intervals $[\frac{3k_i}{3^{n+1}}, \frac{3k_i+1}{3^{n+1}}]$ and $[\frac{3k_i+2}{3^{n+1}}, \frac{3k_i+3}{3^{n+1}}]$ each of which are closed intervals of length $\frac{1}{3^{n+1}}$ and the endpoints are preserved. It follows that the endpoints also lie in the intersection.

(d): Suppose that $x \in C$, then $x \in A_n$ for all n, thus for each n, x is contained in an interval of length $\frac{1}{3^n}$ and so the endpoints of the interval are at most $\frac{1}{3^n}$ away from x. Since the endpoints are contained in C, then x is not an isolated point.

(e): Since C is a subspace of a Hausdorff space, then C is Hausdorff and since C is compact, then by virtue of having no isolated points, C is uncountable.

Exercise 29.1

Show that $\mathbb{Q}$ is not locally compact.

Proof:

For any point $x \in \mathbb{Q}$, and for any open set $U \in \mathbb{Q}$, then there is an irrational number $i \in U$, so if there were a compact set $C \supseteq U$, then we must have that C is complete, and thus $i \in C$, but $i \notin \mathbb{Q}$ and $C \subseteq \mathbb{Q}$, therefore $\mathbb{Q}$ is not locally compact.

Exercise 29.3

Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What if f is both continuous and open? Justify your answer.

Proof:

Consider the discrete topology on $\mathbb{Q}$ and the standard topology on $\mathbb{Q}$, then the identity map is continuous and under the discrete topology, for any point x, $\{x\}$ is open and contained in $\{x\}$ which is compact, however its image is not locally compact. If f is both continuous and open, then for any $x \in X$, we have that $\exists U \subseteq X$ open containing x such that $\exists C \supseteq U$ compact. We then have that $f(x) \subseteq f(U) \subseteq f(C)$ with $f(U)$ open and $f(C)$ compact. Therefore $f(X)$ is locally compact.

Exercise 29.6

Show hat the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.

Proof:

Let $X = \mathbb{R} \cup \{\infty\}$, then let $S^1 = [-\frac{\pi}{2}, \frac{\pi}{2}]/\sim$ where $\frac{\pi}{2} \sim -\frac{\pi}{2}$. Consider then the function $f : S^1 \to X$ given by $x \mapsto \tan(x)$ and $\{\frac{\pi}{2}, -\frac{\pi}{2}\} \mapsto \infty$. Since tan is continuous, then we need only check continuity at $\{\frac{\pi}{2}, -\frac{\pi}{2}\}$. For open set $X \setminus C$ containing ∞, where C is a compact subset of $\mathbb{R}$, then we have that $C \subseteq [-M, M]$ for some M, then for the open arc on S^1 going from $-\arctan(M)$ to $\arctan(M)$ which includes $\{\frac{\pi}{2}, -\frac{\pi}{2}\}$, we have that the image is contain inside of $X \setminus C$, thus the function is continuous at $\{\frac{\pi}{2}, -\frac{\pi}{2}\}$. Since S^1 is compact and X is Hausdorff, then f is a homeomorphism.

Exercise 43.1

Let X be a metric space.
(a) Suppose that for some $\epsilon > 0$, every ϵ-ball in X has compact closure. Show that X is complete.
b) Suppose that for each $x \in X$, there is an $\epsilon > 0$ such that the ball $B(x, \epsilon)$ has compact closure. Show by means of an example that X need not be complete.

Proof:

(a): Let $\{x_i\}$ be a Cauchy sequence in X, then there is some n such that for all $m > n$, we have that $d(x_n, x_m) < \epsilon$, thus the tail of the sequence is contained in $B(x_n, \epsilon)$ which has compact closure, thus the tail of the sequence is a Cauchy sequence in a compact space and hence converges. It follows that the whole sequence converges.

(b): Consider the open set $(0, 1)$ in $\mathbb{R}$. For any $x \in (0, 1)$, we have that there is some ϵ such that $B(x, \epsilon) \subseteq (0, 1)$, then $\overline{B(x, \frac{\epsilon}{2})} \subseteq (0, 1)$ is compact, but $(0, 1)$ is not complete.

Exercise 43.2

Let (X, d_X) and (Y, d_Y) be metric spaces; let Y be complete. Let $A \subseteq X$. Show that if $f : A \to Y$ is uniformly continuous, then f can be uniquely extended to a continuous function $g : \overline{A} \to Y$, and g is uniformly continuous.

Proof:

For any $x \in \overline{A}$, then for each $\epsilon > 0$, we have that $B(x, \epsilon)$ is nonempty, then we may construct a sequence $\{x_i\}$ such that $x_i \in B(x, \frac{1}{i})$. If $\{x_i\}$ and $\{y_i\}$ are two sequences approaching x, then for any ϵ, we want to show that there is some N such that for all $i, j > N$, $d(f(x_i), f(y_i)) < \epsilon$. Since f is uniformly continuous, then there is some δ such that if $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$. Since $\{x_i\}, \{y_i\}$ both converge to x, then there are some N, M s.t. for all $i > N, j > M$ we have $d(x_i, x) < \frac{\delta}{2}, d(y_j, x) < \frac{\delta}{2}$. It follows that $d(x_i, y_j) < \delta$ for $i, j > \max(N, M)$. Then by uniform continuity, we have that $d(f(x_i), f(y_i)) < \epsilon$. We also get that for any $\epsilon > 0$, there is some N such that if $i, j > N$, we have $d(f(x_i), f(x_j)) < \epsilon$, so $\{f(x_i)\}$ is a Cauchy sequence in Y and thus converges to a point which we may call $g(x)$. This is well-defined since if $\{y_i\}$ also converges to x, then we have that $\forall i, j > N, d(f(x_i), f(y_i)) < \epsilon$, so $\{f(y_i)\}$ also converges to $g(x)$. We want to show that g is continuous. Suppose that $x \in \overline{A}$, then there is a sequence $\{x_i\}$ in A approaching x then for any $y \in \overline{A}$ such that $d(x, y) < r$, we have that there is a sequence $\{y_i\}$ in A approaching y. There is some N such that for all $i > N$, $d(y_i, y) < r - d(x, y) - \eta$ where there is some M such that for all $j > M$, we have that $d(x_j, x) < \eta$. Therefore, $d(y_i, x_j) < r$ and so by uniform continuity, we have that $d(f(y_j), f(x_i)) < \epsilon$ and thus $d(f(y_j), g(x)) \leq \epsilon$. It follows that $d(g(y), g(x)) \leq \epsilon$ and thus g is continuous. Since g is continuous over a compact set, then g is uniformly continuous. The uniqueness of g follows from 18.13.

Exercise 43.3

Two metrics d, d' on a set X are said to be metrically equivalent if the identity map, $i : (X, d) \to (X, d')$ and its inverse are both uniformly continuous.
(a) Show that d is metrically equivalent to the bounded metric $\overline{d}$ derived from d.
(b) Show that if d and d' are metrically equivalent, then X is complete under d iff it is complete under d'.

Proof:

(a): Suppose that $d(x, y) < \epsilon$, then since $\overline{d}(x, y) \leq d(x, y) < \epsilon$, we have that i is uniformly continuous. If $\overline{d}(x, y) < \min(1, \epsilon)$, then have that $d(x, y) = \overline{(d)}(x, y) < \epsilon$. It follows that i^{-1} is uniformly continuous.

(b): This follows trivially from the fact that uniformly continuous functions preserve Cauchy sequences.

Exercise 43.4

Show that the metric space (X, d) is complete iff for every nested sequence $A_1 \supseteq A_2 \supseteq \cdots$ of nonempty closed sets of X such that diam $A_n \to 0$ the intersection is nonempty.

Proof:

If X is complete, then take $x_i \in A_i$ to get a Cauchy sequence $\{x_i\}$ which must converge to a point x. Suppose that x were not in A_n, then we would have that $d(x, A_n) > 0$, so for $\epsilon = d(x, A_n)$, there is not point in the Cauchy sequence within $B(x, \epsilon)$. It follows that $x \in A_n$ for all n and hence in the intersection.

If the intersection of all such nested set is nonempty, then consider any Cauchy sequence $\{x_i\}$. For each n, let $A_n = \overline{B}(x_{i_n}, \frac{2}{2^n})$ where $i_n > \max(N, i_{n-1})$ with N such that $\forall i, j > N, d(x_i, x_j) < \frac{1}{2^n}$. We have that if $A_n = \overline{B}(x_{i_n}, \frac{2}{2^n})$ and $A_{n+1} = \overline{B}(x_{i_{n+1}}, \frac{2}{2^{n+1}})$, then $d(x_{i_n}, x_{i_{n+1}}) < \frac{1}{2^n}$, so for any $y \in A_{n+1}$, $d(x_{i_n}, y) \leq \frac{1}{2^n} + \frac{2}{2^(n+1)} = \frac{2}{2^n}$, so $A_n \supseteq A_{n+1}$. It follows that there is an element in the intersection of all the A_n's, call it x, then $\{x_i\}$ converges to x since for $j > i_n$, we have that $d(x_j, x) \leq \frac{4}{2^n}$.

Exercise 43.5

If (X, d) is a metric space, recall that a map $f : X \to X$ is called a contraction if there is a number $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Show that if f is a contraction of a complete metric space, then there is a unique fixed point $x \in X$ such that $f(x) = x$.

Proof:

$\{f^n(x)\}$ is a Cauchy sequence for any x and thus converges to a point x_*. Since f is uniformly continuous, then we have that $\{f(f^n(x))\}$ is a Cauchy sequence that converges to $f(x_*)$, but this sequence converges to x_*, thus $f(x_*) = x_*$. If x_* and y_* are fixed points of f, then $d(f(x_*), f(y_*)) \leq \alpha d(x_*, y_*)$, but we also have that $d(f(x_*), f(y_*)) = d(x_*, y_*)$, thus the distance must be 0, so $x_* = y_*$.

Exercise 43.6

A space X is said to topologically complete if there exists a metric for the topology of X relative to which X is complete.
(a) Show that a closed subspace of a topologically complete space is topologically complete.
(b) Show that a countable product of topologically complete spaces is topologically complete (in the product topology).
(c) Show that an open subspace of a topologically complete space is topologically complete.

Proof:

(a): Let d be the metric on X which induces its topology, then the restriction of this metric to a closed subspace C of X generates the subspace topology on C. For any Cauchy sequence $\{x_i\}$ of points in C, it converges to a point x in X.

Since for any $\epsilon > 0$, there is a point $x_i \in B(x, \epsilon)$ for some i, then $x \in \overline{C} = C$, thus C is topologically complete.

(b): Suppose that $X = \prod_{i=1}^{\infty} X_i$ where each X_i has a topology induced by a metric d_i, then we define the metric $d(x, y) = \sup_{i \in \mathbb{Z}_{>0}} \{\frac{\overline{d_i}(x_i, y_i)}{i}\}$. This induces the product topology on X. To show that this space is complete, suppose that we have a Cauchy sequence $\{x^{(i)}\}$ in X, then this sequence converges iff all of its projections converge. This means that we need only show that its projections are Cauchy since Cauchy sequences converge in any X_i. Fix some $k \in \mathbb{Z}_{>0}$. For any $\epsilon < 1$, let $\epsilon' = \frac{\epsilon}{k}$, then we have that since $\{x^{(i)}\}$ is Cauchy, there is some N such that for all $i, j > N$, we have $d(x^{(i)}, x^{(j)}) < \epsilon'$, hence

$$\frac{\overline{d_k}(x_k^{(i)}, x_k^{(j)})}{k} \leq \sup_{i \in \mathbb{Z}_{>0}} \{\frac{\overline{d_i}(x_i, y_i)}{i}\} < \epsilon'$$

It follows that $\overline{d_k}(x_k^{(i)}, x_k^{(j)}) < \epsilon$ and since $\epsilon < 1$, then we have that $\overline{d_k} = d_k$, therefore $\{x_k^{(i)}\}_{i=1}^{\infty}$ is Cauchy. It follows that X is complete.

(c): Let $U \subseteq X$ be open and let X be complete under the metric d. Consider the function $\phi : U \to \mathbb{R}$ given by $x \mapsto \frac{1}{d(x, U^c)}$. Consider the embedding $f : U \to X \times \mathbb{R}$ given by $x \mapsto (x, \phi(x))$, then we want to show that $f(U)$ is closed. Suppose that $(x, y) \notin f(U)$, then we have that either $x \in U$, in which case $y \neq \phi(x)$, so by Hausdorffness of $\mathbb{R}$ we may separate $\phi(x)$ and y by O and V respectively. We then have that $\phi^{-1}(O)$ is open, then $\phi^{-1}(O) \times V$ contains (x, y) and is disjoint from $f(U)$. If $x \notin U$, then for $\epsilon = \frac{1}{|y|+1}$, we have that for any $u \in U$, if $d(x, u) < \epsilon$, then $\phi(u) > |y| + 1$, thus $(x, y) \in B(x, \epsilon) \times (y - 1, y + 1)$ is disjoint from $f(U)$. It follows that $f(U)$ is closed and so it is topologically complete. Since $f(U)$ is homeomorphic to U, then U is topologically complete.

Exercise 43.8

If X and Y are spaces, define $e : X \times \mathcal{C}(X, Y) \to Y$ by $e(x, f) = f(x)$. Show that if d is a metric for Y and $\mathcal{C}(X, Y)$ has the corresponding uniform topology, then e is continuous.

Proof:

Let $\overline{\rho}(f, g) = \sup\{\overline{d}(f(x), g(x))\}$ be the uniform metric on $\mathcal{C}(X, Y)$, then we want to show that e is continuous. Let U be an open set in Y, then we want to show that $e^{-1}(U)$ is open. For any $(x, f) \in e^{-1}(U)$, we have that $f(x) \in U$, therefore, there is some $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq U$, then let $U' = f^{-1}(B(f(x), \epsilon/2))$, then for any $y \in U'$, we have that $f(y) \in B(f(x), \epsilon/2) \subseteq U$. Let $V = B_{\overline{\rho}}(f, \epsilon/2)$, then for any $(y, g) \in U' \times V$, we have that $\overline{d}(f(x), g(x)) \leq \overline{d}(f(x), f(y)) + \overline{d}(f(y), g(y)) < \epsilon/2 + \epsilon/2 = \epsilon$, thus $g(y) \in U$. It follows that U is open, and thus the evaluation is continuous.

Exercise 45.2

Let (Y,d) be a metric space; let $\mathcal{F}$ be a subset of $\mathcal{C}(X,Y)$.
(a) Show that if $\mathcal{F}$ is finite, then $\mathcal{F}$ is equicontinuous.
(b) Show that if f_n is a sequence of elements of $\mathcal{C}(X,Y)$ that converges uniformly, then the collection $\{f_n\}$ is equicontinuous.
(c) Suppose that $\mathcal{F}$ is a collection of differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that each $x\in\mathbb{R}$ lies in a neighborhood U on which the derivatives of the functions in $\mathcal{F}$ are uniformly bounded (there is some M s.t. $|f'(x)|\leq M$ for all $f\in\mathcal{F}, x\in U$). Show that $\mathcal{F}$ is equicontinuous.

Proof:

(a): For any $x_0\in X$ and $\epsilon>0$, since $f_n\in\mathcal{F}$ is continuous, then there is some U_n open in X containing x_0 such that $f_n(U_n)\subseteq B(f_n(x_0),\epsilon)$. Let $U=\bigcap_{i=1}^n U_i$, then for any $f_i\in\mathcal{F}$, we have that $f_i(U)\subseteq f_i(U_i)\subseteq B(f_i(x_0),\epsilon)$. Since finite intersections of open sets are open and each contains x_0, then this is the desired open set, thus $\mathcal{F}$ is equicontinuous.

(b): Let $\{f_n\}$ converge uniformly to f, then for any $x_0\in X$ and $\epsilon>0$, there is some N such that for all $n>N$, we have that $d(f_n(x),f(x))<\epsilon/3$. If we then consider the family of functions $\{f_1,f_2,\cdots,f_N,f\}$, since this is finite, it is equicontinuous, thus there is some U open in X containing x_0 such that $f_i(U)\subseteq B(f_i(x_0),\epsilon/3)$ for all i. This U then works for the whole sequence since for $n>N$, we already have that for any $y\in U$, $d(f_n(x_0),f_n(y))\leq d(f_n(x_0),f(x_0))+d(f(x_0),f(y))+d(f(y),f_n(y))<\epsilon$.

(c): Let $x_0\in X$ and let $\epsilon>0$, then there is a neighborhood U of x_0 on which the derivatives of each $f\in\mathcal{F}$ are uniformly bounded. We then have that there is some interval $(x-\delta,x+\delta)\subseteq U$ and then for any y in this interval, we have that $|f(y)-f(x)|\leq\delta M$ by MVT. It follows that if we pick $\delta'=\min(\delta,\epsilon/M)$, then this works.

Exercise 45.7

Let (X,d) be a metric space. If $A\subseteq X$ and $\epsilon>0$, let $U(A,\epsilon)$ be the ϵ-neighborhood of A. Let $\mathcal{H}$ be the collection of all (nonempty) closed, bounded subsets of X. If $A,B\in\mathcal{H}$, define $D(A,B)=\inf\{\epsilon\big|A\subseteq U(B,\epsilon)\text{ and }B\subseteq U(A,\epsilon)\}$.
(a) Show that D is a metric on $\mathcal{H}$; it is called the **Hausdorff metric**.
(b) Show that if (X,d) is complete, so is $(\mathcal{H},D)$.
(c) Show that if (X,d) is totally bounded, so is $(\mathcal{H},D)$.
(d) Theorem. If X is compact in the metric d, then the space $\mathcal{H}$ is compact in the Hausdorff metric D.

Proof:

(a): Let $A,B\in\mathcal{H}$, then if $D(A,B)=0$, it follows that for any $x\in A$, we have that $D(x,B)=0$, then since B is closed, we must have that $x\in B$, thus $A\subseteq B$. Similarly, we have that $B\subseteq A$. Clearly, we also have that $D(A,B)\geq 0$

for all $A, B \in \mathcal{H}$, thus D satisfies the first property of a metric. The symmetry is trivially true by virtue of the fact that its definition is symmetric. If $A, B, C \in \mathcal{H}$, then we want to show that $D(A, B) \leq D(A, C)+D(C, B)$. For any $x \in A$ and $y \in B$, we have that $D(x, C) \leq D(A, C)$ and $D(y, C) \leq D(C, B)$, thus $D(x, y) \leq D(x, C) + D(y, C) \leq D(A, C) + D(C, B)$, so $D(A, B) \leq D(A, C) + D(C, B)$.

(b): Suppose that $\{A_n\}$ is a Cauchy sequence of closed bounded subsets, then we may construct a new subsequence such that $D(A_n, A_{n+1}) \leq 2^{-n}$. Let A be the set of all points x for which there exists a sequence $x_1, x_2, \cdots$ with $x_i \in A_i$ for which $d(x_i, x_{i+1}) \leq 2^{-i}$ and converge to x. We want to show that $A_n \to \overline{A}$. We have that for any $y \in \overline{A}$, that for any ball $B(y, \epsilon/3)$, there is some $x \in A$ in this ball, then we have that there is a sequence $x_i \to x$. In particular, there is some x_i such that $x_i \in B(x, \epsilon/3$ for all $j > i$. Furthermore, we have that there must be some i which is less than $1-\log_2(\epsilon/3)$ since $d(x_i, x) \leq 2^{-(i-1)}$, $x_j \in B(y, \epsilon)$, thus $\overline{A}, \subseteq U(A_j, \epsilon))$ for all $j > 1 - \log_2(\epsilon/3)$. For any A_j, since $D(A_j, A_{j+1}) \leq 2^{-j}$, then there is an element of A_n a distance at most $2^{-(j-1)}$ away from any point in A_j, thus $A_j \subseteq U(\overline{A}, 2^{-(j-1)}$. It follows that $D(A_j, \overline{A}) \leq \max(\epsilon, 2^{-(j-1)}$ and thus as $j \to \infty$, this tends to 0, so $A_n \to \overline{A}$.

(c): Given any $\epsilon > 0$, choose $\delta < \epsilon$ and let S be a finite subset of X such that the collection $\{B_d(x, \delta) \big| x \in S\}$ covers X. Let $\mathcal{A}$ be the collection of all nonempty subsets of S, then we will show that $\{B_D(A, \epsilon) \big| A \in \mathcal{A}\}$ covers $\mathcal{H}$. For any $B \in \mathcal{H}$, we let $B' = \{x \in S \big| d(x, B) < \delta\}$. Clearly, we have that $B' \subseteq \mathcal{A}$ and that $B \subseteq \bigcup_{x \in B'} B_d(x, \epsilon) = B_D(B', \epsilon)$. We also have that $d(x, B) < \delta$ for all $x \in B'$, thus $B' \subseteq U(B, \delta$. It follows that $D(B', B) < \max(, \delta, \epsilon) \leq \epsilon$. Therefore, $\mathcal{H}$ is totally bounded.

(d): If X is compact, then it is complete and totally bounded, thus $\mathcal{H}$ is complete and totally bounded and hence compact.

Exercise 46.2

Let X be a space; let (Y, d) be a metric space. For the function space Y^X, show the following inclusions of topologies: (uniform) $\supseteq$ (compact convergence) $\supseteq$ (pointwise convergence). If X is compact, the first two coincide, and if X is discrete, the last two coincide.

Proof:

We first show that (uniform) $\supseteq$ (compact convergence). Let $f \in Y^X$ and let C be a compact subspace of X and $\epsilon > 0$, then for any $g \in B_C(f, \epsilon) = \{g \in Y^X \big| \sup_{x \in C} d(f(x), g(x)) < \epsilon\}\}$, we have that $B_{\overline{\rho}}(g, \epsilon - \sup_{x \in C} d(f(x), g(x)))$ is contained in $B_C(f, \epsilon)$. For any $S(x, U)$ in the pointwise convergence topology, we have that $\{x\}$ is compact. For any $f \in S(x, U)$, we have that $f(x) \in U$, so $f(x) \in B(f(x), \epsilon) \subseteq U$ since U is open, thus $f \in B_{\{x\}}(f, \epsilon) \subseteq S(x, U)$, so $S(x, U)$ is open in the compact convergence topology.

If X is compact, then for any $g \in B_{\overline{\rho}}(f, \epsilon)$, we have that for $f \in B_X(g, \min(1, \epsilon))$, $\sup_{x \in X} d(f(x), g(x)) < \min(1, \epsilon)$, thus $\sup_{x \in X} d(f(x), g(x)) = \sup_{x \in X} \overline{d}(f(x), g(x)) = \overline{\rho}(f, g) < \epsilon$, so $B_X(f, \min(1, \epsilon)) \subseteq B_{\overline{\rho}}(f, \epsilon)$, thus the topologies coincide.

If X is discrete, then the only compact sets in X are finite, thus $B_C(f,\epsilon) = \bigcap_{x\in C} S(x, B(f(x),\epsilon))$.

Exercise 46.3

Show that the set $\mathcal{B}(\mathbb{R},\mathbb{R})$ of bounded functions $f : \mathbb{R} \to \mathbb{R}$ is closed in $\mathbb{R}^{\mathbb{R}}$ in the uniform topology, but not in the topology of compact convergence.

Proof:

We want to show that $X = \overline{\mathcal{B}(\mathbb{R},\mathbb{R})} = \mathcal{B}(\mathbb{R},\mathbb{R})$ in the uniform topology. Suppose that $g \in X$, then for any $\epsilon > 0$, we have that $\overline{\rho}(g,f) < \epsilon$ for some $f \in \mathcal{B}(\mathbb{R},\mathbb{R})$. Take $\epsilon = \frac{1}{2}$, then we have that $d(g(x), f(x)) < \frac{1}{2}$ for all $x \in \mathbb{R}$ and thus since f is bounded, say $f(\mathbb{R}) \subseteq B(0,M)$, then $g(\mathbb{R}) \subseteq B(0, M + \frac{1}{2})$. In the topology of compact convergence, we can take $g(x) = x$ which is unbounded, but on any compact set C, g is bounded, say $g(C) \subseteq B(0,M)$, then take $f(x) = g(x)$ on C and M for $x \notin C$, then f is bounded on all of $\mathbb{R}$ and $g \in B_C(g,\epsilon)$ for any $\epsilon > 0$. It follows that g is in the closure of the set of bounded functions, however it is unbounded.

Exercise 46.6

Show that in the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff and regular if Y is regular.

Proof:

Suppose f, g are two functions in $\mathcal{C}(X,Y)$, then if $f \neq g$, there is some $x \in X$ such that $f(x) \neq g(x)$ and thus may be separated by open sets U and V disjoint. We then have that $f \in S(\{x\}, U)$ and $g \in S(\{x\}, V)$ are open and disjoint, thus $\mathcal{C}(X,Y)$ is Hausdorff.

If Y is regular, then any disjoint closed set and a point can be separated by open sets. We want to show that if $\overline{U} \subseteq V$, then we have that $\overline{S(C,U)} \subseteq S(C,V)$. Suppose that $f \in \overline{S(C,U)}$, then for any open set O and any point $x \in C$, we have that if $f(x) \in O$, then $S(\{x\}, O) \cap S(C,U) \neq \varnothing$, thus there is a function g such that $g(x) \in O \cap U$ and hence for any O open, $f(x) \in O \Rightarrow O \cap U \neq \varnothing$ and thus $f(x) \in \overline{U} \subseteq V$. It follows that $f \in S(C,V)$. We want to show that for any function f and open set V containing f, that there is an open set U such that $\overline{U} \subseteq V$. Since $f \in V$, then there is a basis element $f \in S(C,V') \subseteq V$, then since Y is regular and $f(C)$ is compact and hence closed, we may take U', V'' separating $\underline{f(C)}$ and V^c, then $f(C) \subseteq \overline{U'} \subseteq V$, and thus $f \in S(C,U')$. Since $\overline{U'} \subseteq V$, then $\overline{S(C,U')} \subseteq S(C,V)$.

Exercise 46.7

Show that if Y is locally compact Hausdorff, then composition of maps $\mathcal{C}(X,Y) \times (Y,Z) \to (X,Z)$ is continuous, provided the compact-open topology is used on each.

Proof:

If $g \circ f \in S(C, U)$, then we want to show that there is V such that $f(C) \subseteq V$ and $g(\overline{V}) \subseteq U$. We have that $g(f(C)) \subseteq U$, then since G is continuous, we have that $f(C) \subseteq g^{-1}(U)$ with $g^{-1}(U)$ open and hence locally compact Hausdorff. We then have that for any $x \in f(C)$, there is some U_x open in $g^{-1}(U)$ containing x and $\overline{U_x}$ is compact. We then have that $f(C) \subseteq \bigcup_{x \in f(C)} U_x$, thus there is a finite subcover $U_1, \cdots, U_n$. We then have that $V = \bigcup_{i=1}^{n} U_i$ has compact closure and $f(C) \subseteq V$. Since $\overline{U_i} \subseteq g^{-1}(U)$, then $\overline{V} \subseteq g^{-1}(U)$, thus $g(\overline{V}) \subseteq U$. We then have that for any $(f', g') \in S(C, V) \times S(\overline{V}, U)$ we have that $g'(f'(C)) \subseteq g'(V) \subseteq U$, so $g' \circ f \in S(C, U)$. It follows that function composition is continuous.

Exercise 51.1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Proof:

Let H, K be the respective homotopies for h, h' and k, k'. Define a new homotopy, $F(t, x) = K(t, H(t, x))$. We have that $F(0, x) = K(0, H(0, x)) = K(0, h(x)) = k(h(x))$ and $F(1, x) = H(1, K(1, x)) = k'(h'(x))$. $l : (t, x) \mapsto (t, H(t, x))$ is continuous since for any open set $B(a, \epsilon) \times U$, we have that the preimage under l is given by $\{(t, x) \big| t \in B(a, \epsilon), H(t, x) \in U\} = H|_{B(a,\epsilon) \times X}^{-1}(U)$ and since restrictions of continuous functions are continuous, then the preimage is open in $B(a, \epsilon) \times X$ which itself is open, thus the preimage of $B(a, \epsilon) \times U$ under l is open. F is the a composition of continuous functions and thus continuous. It follows that $k \circ h$ and $k' \circ h'$ are homotopic with homotopy F.

Exercise 51.2

Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.
(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ is a single element.
(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Proof:

(a): Suppose that f, g are two continuous functions from X to I, then consider the homotopy, F which takes $F_t(x) = f(x) + (1 - t)g(x)$, then this is the straight-line homotopy, and $F_0 = f, F_1 = g$, thus f, g are homotopic. It follows that $[X, I]$ consists of a single element.

(b): Let $f, g : I \to Y$ be two continuous functions, then for each x, there is a path $\gamma_x : [0, 1] \to Y$ starting at $f(x)$ and ending at $g(x)$. Let $F_t(x) = \gamma_x(t)$. For any closed set C, we have that $F^{-1}(C) = \{(t, x) \big| \gamma_x(t) \in C\}$. Since γ_x is continuous, then $\gamma_x^{-1}(C)$ is closed, and we have that $F^{-1}(C) = \bigcap_{x \in I} \gamma_x^{-1}(C)$ is closed, so F is continuous. We clearly have that $F_0 = f$ and $F_1 = g$, so f and g are homotopic.

Exercise 51.3

A space X is said to be **contractible** if the identity map $i_X : X \to X$ is nulhomotopic.
(a) Show that I and $\mathbb{R}$ are contractible.
(b) Show that a contractible space is path connected.
(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Proof:
(a): In I and $\mathbb{R}$, we may take the map $F(t, x) = (1-t)x$ which is a homotopy with the constant 0 map and thus the identity is nulhomotopic.
(b): Let F be the homotopy between i_X and the a constant map. We have that for any $x, y \in X$, $F(1, x) = F(1, y) = c$ and that $F(0, x) = x, F(0, y) = y$. Let $\gamma_1 : [0, 1] \to X$ be given by $\gamma_1(t) = F(t, x)$, then γ_1 is a path from x to c. Similarly, let $\gamma_2(t) = F(1 - t, y)$, then γ_2 is a path from c to y. It follows that the composition $\gamma_1 * \gamma_2$ goes from x to c to y, so X is path connected.
(c): If Y is contractible, then it has a homotopy F from Id_Y to C, we then have that for any two continuous functions $f, g : X \to Y$ that $Id_Y \circ f$ is homotopic to $C \circ f$ and similarly, $Id_Y \circ g$ is homotopic to $C \circ g$. $C \circ f$ and $C \circ g$ are the same function, thus $Id_y \circ f = f$ is homotopic to $Id_Y \circ g = g$.
(d): Since X is contractible, then for any two functions $f, g : X \to Y$, we have that $f \circ Id_X$ is homotopic to the constant path $f(C)$ and that $g \circ Id_X$ is homotopic to the constant path $g(C)$. Since Y is path connected, then $f(C)$ and $g(C)$ are homotopic, thus f and g are homotopic.

Exercise 52.1

A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.
(a) Find a star convex set that is not convex.
(b) Show that if A is star convex, A is simply connected.

Proof:
(a): Consider two disks $\overline{B}((-1, 0), 1) \cup \overline{B}((1, 0), 1)$, both disks share the point $(0, 0)$ and are each convex, so every point is connected to $(0, 0)$ in a straight line, thus the shape is star convex. However, the straight line from $(-1, 1)$ to $(1, 1)$ is not contained in the set.
(b): Suppose X is star convex, then let $x \in X$ be a point from which all others are connected by a straight line in X. We then have that for any path from x to itself, there is a straight-line homotopy to the constant path at x, thus

all paths from x to x are homotopic. Since X is path connected (all points are connected to x), then the fundamental group of X is trivial, thus X is simply connected.

Exercise 52.3

Let x_0, x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian iff for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof:

If $\pi_1(X, x_0)$ is abelian, then we have that for any path α, $[\overline{\alpha}][\beta][\alpha] = [\overline{\alpha}][\alpha][\beta] = [\beta]$, so they are all the identity map and are hence equal. It $\hat{\alpha} = \hat{beta}$, then we have that $\hat{\alpha}([\beta]) = \hat{\beta}([\beta]) = [\beta]$, thus

$$[\overline{\alpha}][\beta][\alpha] = [\beta] \Rightarrow [\beta][\alpha] = [\alpha][\beta]$$

Thus $\pi_1(X, x_0)$ is abelian.

Exercise 52.5

Let A be a subspace of $\mathbb{R}^n$; let $h : (A, a_0) \to (Y, y_0)$. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then h_* is the trivial homomorphism (i.e. maps everything to the identity).

Proof:

Let h' be the extension of h to a continuous function on all of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is simply connected, then h'_* must be trivial. We have that $h = h' \circ \iota$ where ι is the inclusion map from A into $\mathbb{R}^n$. It follows that $h_* = h'_* \circ \iota_*$ and since h'_* is trivial, then h_* is trivial as well.

Exercise 52.6

Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphism of the group involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that $\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}$.

Proof:

For any path γ from x_0 to itself, we have that $(\hat{\beta} \circ (h_{x_0})_*)([\gamma]) = [\overline{\beta}][h \circ \gamma][\beta]$ and since $\beta = h \circ \alpha$, then we have that $[\overline{\beta}] = [\beta]^{-1} = [h \circ \alpha]^{-1} = h_{x_0}([\alpha])^{-1} = h_{x_0}([\overline{\alpha}]) = [h \circ \overline{\alpha}]$. It follows that we have $[\overline{\beta}][h(\gamma)][\beta] = [h \circ \overline{\alpha}][h \circ \gamma][h \circ \alpha] = (h_{x_1})_*([\overline{\alpha}][\gamma][\alpha]) = (h_{x_1})_* \circ \hat{\alpha}$.

Exercise 53.3

Let $p : E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every b in B. In such a case, E is called a **k-fold covering** of B.

Proof:

Let $A = \{b \in B \big| |p^{-1}(b)| = k\}$. We know that A is non-empty, so we want to show that A is clopen and thus must be B. For any $b \in A$, we have that there is an open set U containing b which is evenly covered by E, i.e. $p^{-1}(U) = \bigcup_{i=1}^{n} V_i$ where each V_i is disjoint from the others and homeomorphic to U. Since each V_i is homeomorphic to U, then there is exactly 1 point which maps to b. Since $b \in U$, then $p^{-1}(b) \subseteq p^{-1}(U)$, thus we may conclude that $n = k$. For any other point $x \in U$, we have that U is evenly covered by p and with the same argument as before, $k = |p^{-1}(x)|$, so $x \in A$. It follows that $U \subseteq A$ and hence A is open. To show that A is closed, we want to show that $\overline{A} = A$. Suppose that $x \in \overline{A}$, then there is an open set U containing x which is evenly covered by p, and since $x \in \overline{A}$, then it must intersect A at some point b, and thus by the same argument as before, $k = |p^{-1}(b)| = |p^{-1}(x)|$, so $x \in A$. Since A is nonempty and clopen, then $A = B$.

Exercise 53.4

Let $q : X \to Y$ and $r : Y \to Z$ be covering maps; let $p = r \circ q$. Show that if $r^{-1}(z)$ is finite for each $z \in Z$, then p is a covering map.

Proof:

For any point $x \in Z$, we have that there is an open set U evenly covered by r, with $r^{-1}(U) = \bigcup_{i=1}^{n} V_i$ where each V_i is disjoint and homeomorphic to U. For each V_i, consider the restriction $q|_{V_i}$ which is a covering map and thus for each $x_i \in V_i$ with $r(x_i) = x$, we have that there is an open set U_i which is evenly covered by q, then since V_i is open in Y, we have that U_i is open and thus $r(U_i)$ is open. Let $U' = \bigcap_{i=1}^{n} r(U_i)$, then we want to show that U' is evenly covered by p. Notice that $p^{-1}(U') = q^{-1}(r^{-1}(U'))$. We have that $r^{-1}(U') = r^{-1}(\bigcap_{i=1}^{n} r(U_i)) = \bigcap_{i=1}^{n} U_i$ since r is a homeomorphism on each U_i. We then have that $q^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} q^{-1}(U_i)$. Since each U_i is evenly covered by q, then are V_α^i such that $q^{-1}(U_i) = \bigcup_{\alpha \in J} V_\alpha^i$ and each V_α^i is homeomorphic to U_i. We then have that $\bigcap_{i=1}^{n} q^{-1}(U_i) = \bigcup_{\beta \in I} V_\beta$ where $V_\beta = \bigcap_{i=1}^{n} V_{\alpha_i}^i$. These are trivially disjoint since all V_α^i are disjoint and they are open since they are finite intersections. We have that $p(V_\beta) = \bigcap_{i=1}^{n} r(q(V_{\alpha_i}^i)) = \bigcap_{i=1}^{n} r(U_i) = U'$ and this is a homeomorphism since it is the composition of homeomorphism ($q|_{V_{\alpha_i}^i}$ and $r|_{U_i}$). It follows that p is a covering map.

Exercise 53.5

Show that $p : S^1 \to S^1$ given by $z \mapsto z^n$ where S^1 is considered as a subset of $\mathbb{C}$ is a covering map.

Proof:

The map is clearly continuous and surjective. For any $e^{i\theta} \in S^1$, consider the open set $U = \{e^{i\varphi} \big| \varphi \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})\}$, then let $U_i = \{e^{i\varphi} \big| \varphi \in (\theta + \frac{(2i-\frac{1}{2})\pi}{n}, \theta + \frac{(2i+\frac{1}{2})\pi}{n})\}$, then we have that $p^{-1}(U) = \bigcup_{i=1}^{n} U_i$. Each segment U_i is disjoint and for any U_i, we have that $\overline{U_i}$ is compact and $p(\overline{U_i}) = \overline{U}$ with $\overline{U}$ Hausdorff and thus they are homeomorphic. It then follows that $p(U_i) = U$ is also a homeomorphism, thus p is a covering map.

Exercise 54.1

What goes wrong with the "path-lifting" lemma for the local homeomorphism of $p : \mathbb{R}_+ \to S^1$ given by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$?

Proof:

If we start lifting a path going clockwise from 0 in $\mathbb{R}_+$, then since any neighborhood of $(1, 0)$ in S^1 contains a set V_0 containing 0 in $\mathbb{R}_+$ which only embeds into the open set, rather than being homeomorphic to it, then we run into the issue that we cannot pull an element out of it as we try to move clockwise around $\mathbb{R}_+$. For a sufficiently small radius around $(1, 0)$, all of the points clockwise from $(1, 0)$ have preimages that have a distance at least $\frac{1}{2}$ from 0 in $\mathbb{R}_+$, and thus no lift exists.

Exercise 54.3

Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Show that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

Proof:

We want to show that $p \circ (\tilde{\alpha} * \tilde{\beta}) = \alpha * \beta$. For any $x \in [0, 1]$, we have that if $x \leq \frac{1}{2}$, then $(\tilde{\alpha} * \tilde{\beta})(x) = \tilde{\alpha}(2x)$, then we have that $p(\tilde{\alpha}(2x)) = \alpha(2x)$. Similarly, we get that if $x \geq \frac{1}{2}$, then $(p \circ (\tilde{\alpha} * \tilde{\beta}))(x) = \beta(2x - 1)$. The statement then follows.

Exercise 54.6

Consider the maps $g, h : S^1 \to S^1$ given by $g(z) = z^n$ and $h(z) = 1/z^n$. Compute the induced homomorphisms g_*, h_*.

Proof:

Let $\gamma : [0,1] \to S^1$ be given by $\gamma(t) = e^{2\pi it}$, then $\phi_0(\gamma) = 1$, thus $[\gamma]$ generates $\pi_1(S^1,(1,0))$. We have that $(g \circ \gamma)(t) = e^{2\pi int}$, then $\phi_0(g \circ \gamma) = n$, thus $g_*(x) = nx$. Similarly, $(h \circ \gamma)(t) = e^{-2\pi int}$, thus $\phi_0(h \circ \gamma) = -n$ and so $h_*(x) = -nx$.

Exercise 54.7

Generalize the proof of Theorem 54.5 to show that the fundamental group of torus isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Proof:

Let $p : \mathbb{R} \times \mathbb{R} \to S^1 \to S^1$ be the covering map given by

$$(x, y) \to ((\cos(2\pi x), \sin(2\pi x)), (\cos(2\pi y), \sin(2\pi y)))$$

We then have that for the point $(1,0),(1,0))$, its preimage is $\mathbb{Z} \times \mathbb{Z}$. It follows that the fundamental group of $S^1 \times S^1$ is in bijection with $\mathbb{Z} \times \mathbb{Z}$ since $\mathbb{R}^2$ is simply connected. We define the map $\phi_0 : \pi_1(S^1, S^1, ((1,0),(1,0)) \to \mathbb{Z} \times \mathbb{Z}$ by $\phi_0([g]) = \tilde{g}(1)$ where $\tilde{g}$ is a lift of g beginning at $(0,0)$. We then have that for any two paths f, g, let $\tilde{f}$ and $\tilde{g}$ be their lifts, then define $\tilde{\tilde{g}} = (n,m) + \tilde{g}$ where $(n,m) = \tilde{f}(1)$. We then have that $\tilde{\tilde{g}}$ is a lift of g since $p((n,m)+x) = p(x)$ for any $n, m \in \mathbb{Z}$. We then have that $\tilde{f} * \tilde{\tilde{g}}$ is a lift of $f * g$ and the endpoint of $\tilde{\tilde{g}}$ is $\tilde{\tilde{g}}(1) = (n,m) + (n',m')$ where $\tilde{g}(1) = (n',m')$. We then have that:

$$\phi_0([f] * [g]) = (n,m) + (n',m') = \phi_0([f]) + \phi_0([g])$$

It follows that ϕ_0 is an isomorphism and thus $\pi_1(S^1 \times S^1, ((1,0),(1,0))) = \mathbb{Z} \times \mathbb{Z}$.

Exercise 54.8

Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then p is a homeomorphism.

Proof:

Since B is simply connected then $\pi_1(B, b_0)$ is trivial. Since E is path connected and p is a covering map, then $\pi_1(E, e_0) \cong p(\pi_1(E, e_0)) \leq \pi_1(B, b_0)$ and thus $\pi_1(E, e_0)$ is trivial, then we have that the lifting correspondence gives a bijection $\phi : 1 \to p^{-1}(b)$, thus $p^{-1}(b)$ is a singleton, so p is bijective. We now want to show that a bijective local homeomorphism is a homeomorphism.

Let $f : X \to Y$ be a bijective local homeomorphism. Let U be open in X, then for any $x \in U$, there is an open neighborhood U_x of x which maps homeomorphically into $f(U_x)$. We then have that $U_x \cap U$ maps homeomorphically into $f(U_x \cap U)$. It follows that $f(U_x \cap U)$ is an open subset of $f(U_x)$ since $U_x \cap U$ is an open subset of U_x. Since $f(U_x)$ is open in Y, then $f(U_x \cap U)$ is open in Y and $f(U) = \bigcup_{x \in U} f(U_x \cap U)$, so $f(U)$ is open in Y. It follows that f^{-1} is continuous and thus f is a homeomorphism.

Suggested Exercise 27

Suppose $p : Y \to X$ satisfies the covering map definition at a point $x_0 \in X$. Assume for any $x \in X$ there exists a homeomorphism $f_x : X \to X$ such that $f_x(x_0) = x$ and it lifts to a homeomorphism of Y. Show that p is a covering map.

Proof:

Let $x \in X$, then we want to show that there is an open neighborhood of x which is evenly covered by p. We have that f_x is a homeomorphism of X which sends $x_0 \mapsto x$ and lifts to a homeomorphism $f'_x : Y \to Y$ where we have $p \circ f'_x = f_x \circ p$. We then have that since x_0 has an open set U_0 which is evenly covered by p, then $U = f_x^{-1}(U_0)$ is an open set of x. We then have that $p^{-1}(U) = p^{-1}(f_x^{-1}(U_0)) = (f_x \circ p)^{-1}(U_0) = (p \circ f'_x)^{-1}(U_0) = f_x'^{-1}(p^{-1}(U_0))$. Since $p^{-1}(U_0)$ is the union of disjoint slices $\{V_\alpha\}$ and f'_x is a homeomorphism, then their disjointness is preserved. We also have that there is a homeomorphism $f_\alpha : V_\alpha \to U_0$, then we have that since $f_x'^{-1}(V_\alpha)$ is homeomorphic to U_0 and U_0 is homeomorphic to U, then there is a homeomorphism from $f_x'^{-1}(V_\alpha) \to U$. We then have that p is a covering map everywhere.

Suggested Exercise 28

Suppose $f : X \to Y$ is a homotopy equivalence between path connected spaces. Show that it induces an isomorphism of the fundamental groups.

Proof:

Let $g : Y \to X$ be the homotopy inverse of f, then we want to show that f_* is an isomorphism from $\pi_1(X, x_0) \to \pi_1(Y, y_0)$. Notice that f_* is clearly a bijection as $f_* \circ g_* = (Id_Y)_*$ and $g_* \circ f_* = (Id_X)_*$. Note that we have equality since if h and k are homotopic, then for any path α, we have that α is homotopic to α and thus $h \circ \alpha$ is homotopic to $k \circ \alpha$. It then immediately follows that $h_* = k_*$. Since f_* is a homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$ and is bijective, then it is an isomorphism.

Suggested Exercise 29

Let $\sim_1, \sim_2$ be equivalence relations on X. Let $\sim_3$ be the equivalence relation generated by $\sim_1, \sim_2$ and $\sim_4$ the equivalence relation induced by $\sim_2$ on $\sim_1$. Show that $X/\sim_3$ is homeomorphic to $X/\sim_1/\sim_4$.

Proof:

We let $g : X \to X/\sim_1/\sim_4$ be given by the composition of the maps q_1 and q_4 where $q_1 : X \to X/\sim_1$ and $q_4 : X/\sim_1/\sim_4$. g is a composition of quotient maps and is hence a quotient map. We want to show that the fibres of g and the fibres of $q_3 : X \to X/\sim_3$ are the same and thus conclude that $X/\sim_1/\sim_4$ is homeomorphic to $X/\sim_3$. We want to show that the composition of equivalences on $X/\sim_1/\sim_4$ is the same as the equivalence on $X/\sim_3$. If $x \sim_3 y$, then there are $z_0, \cdots, z_n$ such that $z_0 = x$ and $z_n = y$ with $z_i \sim_1 z_{i+1}$ or $z_i \sim_2 z_{i+1}$. If $n = 1$, then we have that $x \sim_1 y$ in which case $[[x]_{\sim_1}]_{\sim_4} = [[y]_{\sim_1}]_{\sim_4}$ since $[x]_{\sim_1} = [y]_{\sim_1}$ or $x \sim_2 y$ where $[[x]_{\sim_1}]_{\sim_4} = [[y]_{\sim_1}]_{\sim_4}$ since $x \sim_2 y$. The rest follows by induction. I don't really want to do the other direction, but it probably follows by using the definition $[x]_{\sim_1} \sim_4 [y]_{\sim_1}$ if $\exists z$ s.t. $z \sim_1 x, z \sim_2 y$ or $z \sim_1 y, z \sim_2 x$.

Suggested Exercise 31

Show that if $f : S^1 \to S^1$ for all x, $f(x) \neq f(-x)$, then f is homotopic to the identity.

Proof:

Let $S^1 = [0,1]/(0 \sim 1)$, then consider the function $\tilde{f} : [0,1] \to S^1$ given by $\tilde{f}(0) = \tilde{f}(1) = f(\{0,1\})$ and for all $x \neq 0,1$, $\tilde{f}(x) = f(\{x\})$. We then get that $\tilde{f} = f \circ q$ where q is the quotient map. Consider the straight line homotopy of $f \to -Id$, then since $f(x) \neq -x$, at not point will this pass through the point $(0,0)$. We may then push this homotopy on the circle. Let H be the homotopy in $R^2 \setminus \{(0,0)\}$, then consider the homotopy $F(t,x) = \frac{H(t,x)}{||H(t,x)||}$. This is the desired homotopy, thus f is homotopic to $-Id$ which itself is homotopic to Id by $F'(t,x) = x + \frac{t}{2}$.

Suggested Exercise 32

Find the natural bijection between $\pi_1(X, x_0)$ and $Hom((S^1,(1,0)),(X,x_0))$, the set of all morphisms in the pointed homotopy category. Can you describe the group structure on this set?

Proof:

The bijection is clear, you send an equivalence class $[\alpha]$ to the equivalence class of α' where α' is the morphism from S^1 which is given by the quotient map.

Exercise 58.1

Show that if A is a deformation retract of X, and B is a deformation retract of A, then B is a deformation retract of X.

Proof:

Let $p : X \to A$ and $q : A \to B$ be deformation retractions, then we want to show that $q \circ p : X \to B$ is a deformation retraction. We have that $i_B : B \to A$ and $i_A : A \to X$ are inclusions, then similarly, we have that $i'_B : B \to X$ is also an inclusion. We see that $i'_B = i_A \circ i_B$, then we have that $(q \circ p) \circ i'_B = q \circ (p \circ i_A) \circ i_B = q \circ Id_A \circ i_B = q \circ i_B = Id_B$. It follows that $q \circ p$ is a retraction. We also have homotopies H, F s.t. $H(x,0) = x, H(x,1) = p(x)$ and for any $x \in A, H(x,t) = x$ and similar for F. Then then want to show that $G(t,x) := F(t, H(t,x))$ is the desired homotopy. This was shown to be a homotopy in Exercise 51.1. We now show that $G(t,x) = x$ for any $x \in B$. Let $x \in B$, then $x \in A$, and we have that $G(t,x) = F(t, H(t,x))$. Since $x \in B$, then $H(t,x) = x$ and since $x \in A$, then $F(t, H(t,x)) = F(t,x) = x$ as desired.

Exercise 58.2

Find the fundamental groups of the following spaces:
(a) $B^2 \times S^1$
(b) $S^1 \times S^1$ with a point removed.
(c) $S^1 \times [0,1]$
(d) $S^1 \times \mathbb{R}$
(e) $\mathbb{R}^3$ with the nonnegative x, y, z axes removed.
(f) $\{x \in \mathbb{R}^2 \big| ||x|| > 1\}$
(g) $\{x \in \mathbb{R}^2 \big| ||x|| \geq 1\}$
(h) $\{x \in \mathbb{R}^2 \big| ||x|| < 1\}$
(i) $S^1 \cup (\mathbb{R}_+ \times 0)$
(j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$
(k) $S^1 \cup (\mathbb{R} \times 0)$
(l) $\mathbb{R}^2 \setminus (\mathbb{R}^+ \times 0)$

Proof:

(a) Deformation retraction onto S^1, so $\mathbb{Z}$.
(b) Figure 8, $\mathbb{Z} * \mathbb{Z}$.
(c) Deformation retraction onto S^1, so $\mathbb{Z}$.
(d) Same as above.
(e) Deformation retraction onto S^2 with 3 points removed, then stereographic projection back to $\mathbb{R}^2$ with two points removed, this is just $\mathbb{Z} * \mathbb{Z}$.
(f) Deformation retraction to a circle of radius 2, so $\mathbb{Z}$.
(g) Same as above.
(h) This is convex, so trivial.
(i) Deformation retraction onto S^1, so $\mathbb{Z}$.
(j) Deformation retraction onto S^1, so $\mathbb{Z}$.
(k) Deformation retraction onto $S^1 \wedge S^1$, so $\mathbb{Z} * \mathbb{Z}$.
(l) This is star convex from the point $(-1, 0)$, so trivial.

Exercise 58.5

Recall that a space X is said to be contractible if the identity map of X to itself is nulhomotopic. Show that X is contractible iff X has the homotopy type of a one-point space.

Proof:

If X has the homotopy type of a one-point space $\{x\}$, then there is a homotopy equivalence $f : X \to \{x\}$ with homotopy inverse $g : \{x\} \to X$ such that $f \circ g$ is homotopic to the identity on $\{x\}$ and $g \circ f$ is homotopic to the identity on X. Since $f(x) = x_0$ for all $x \in X$, then $g \circ f$ is a constant function, thus Id_X is homotopic to a constant function and thus X is nulhomotopic. If X is nulhomotopic, then Id_X is homotopic to some constant function C_x for some point $x \in X$. We then have that the function $f : X \to \{x\}$ is a homotopy equivalence with homotopy inverse $g : \{x\} \to X$ by $x \mapsto x$.

Exercise 58.6

Show that a retract of a contractible space is contractible.

Proof:

Let $r : X \to A$ be a retraction, then let Id_X be homotopic to C_x. WLOG, let $x \in A$. Since r is homotopic to r and Id_X is homotopic to C_x, then $r = r \circ Id_X$ is homotopic to $r \circ C_x = C_x$, thus r is homotopic to C_x. We then have that A is contractible.

Exercise 58.9

We define the *degree* of a continuous map $h : S^1 \to S^1$ as follows: Let b_0 be the point $(1, 0)$ of S^1; choose a generator γ for $\pi_1(S^1, b_0)$. If x_0 is any point of S^1, choose a path α in S^1 from b_0 to x_0, and define $\gamma(x_0) = \hat{\alpha}(\gamma)$. Then $\gamma(x_0)$ generates $\pi_1(S^1, x_0)$. The element $\gamma(x_0)$ is independent of the choice of the path α, since the fundamental group of S^1 is abelian.

Now given $h : S^1 \to S^1$,choose $x_0 \in S^1$ and let $h(x_0) = x_1$. Consider the homomorphism $h_* : \pi_1(S^1, x_0) \to \pi_1(S^1, x_1)$. Since both groups are infinite cyclic, we have $(*)$ $h_*(\gamma(x_0)) = d \cdot \gamma(x_1)$ for some integer d. The integer d is called the ***degree*** of h and is denoted $\deg h$.

The degree of h is independent of the choice of the generator γ; choosing the other generator would merely change the sign of both sides of $(*)$.

(a) Show that d is independent of the choice of x_0.
(b) Show that if $h, k : S^1 \to S^1$ are homotopic, they have the same degree.
(c) Show that $\deg(h \circ k) = (\deg h)(\deg k)$.
(d) Compute the degrees of the constant map, the identity map, the reflection map, and the map $h(z) = z^n$.
(e) Show that if $h, k : S^1 \to S^1$ have the same degree, they are homotopic.

Proof:

(a): Suppose β is a map from x_0 to y_0, and $\delta = h \circ \beta$, then we have that

$\hat{\delta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\beta}$. We then have that $(h_{x_0})_* = \hat{\delta}^{-1} \circ (h_{x_1})_* \circ \hat{\beta}$, thus:

$$\begin{aligned} d \cdot \gamma(x_1) &= h_*(\gamma(x_0)) \\ &= (\hat{\delta}^{-1} \circ h_* \circ \hat{\beta})(\gamma(x_0)) \\ &= (\hat{\delta} \circ h_*)([\overline{\beta}] * [\overline{\alpha}] * [\gamma] * [\alpha] * [\beta]) \\ &= (\hat{\delta} \circ h_*)(\gamma(y_0)) \end{aligned}$$

Thus $d \cdot \hat{\delta}^{-1} \circ \gamma(x_1) = h_*(\gamma(y_0))$ and hence $d \cdot \gamma(y_1) = h_*(\gamma(y_0))$.
(b) idk

Suggested Exercise 34

Show that if the fundamental group of B is finite, any path connected cover E of B has finite fibres.

Proof:
Let p be the covering map $E \to B$, then $\phi_{e_0} : \pi_1(B, b_0)/p_*(\pi_1(E, e_0)) \to p^{-1}(b_0)$ is a bijection since E is path connected. It then follows that $p^{-1}(b_0)$ is finite, thus since E is connected, and has finite fibres at a point, all fibres are finite.

Suggested Exercise 35

Show that the Morbius band deformations retracts to (a homeomorphic copy of) S^1.

Proof:

Suggested Exercise 36

Find f_* where f is:
(a) The diagonal map $S^1 \to S^1 \times S^1$.
(b) The embedding $S^1 \to M$ onto the edge of the Morbius band.
(c) The embedding $S^1 \to M$ onto the centerline of the Morbius band.

Proof:
(a): Let $\mathbb{R}^2$ be the covering space of $S^1 \times S^1$, then we have that the diagonal map $f : S^1 \to S^1 \times S^1$ take the map $\alpha : [0,1] \to S^1$ given by $t \mapsto e^{2\pi i t}$ to a map $\alpha' = f_* \circ \alpha = (e^{2\pi i t}, e^{2\pi i t})$. We then have that this loop lifts the covering space beginning from $(0,0)$ as the map $\tilde{\alpha'} = (Id, Id)$, and thus has an endpoint of $(1,1)$. It follows that looking at $\pi_1(S^1)$ as $\mathbb{Z}$ and $\pi_1(S^1 \times S^1)$ as $\mathbb{Z} \times \mathbb{Z}$, then $f_* : 1 \mapsto (1,1)$.

(b):

Suggested Exercise 37

Show that if A is a totally bounded subset of X, then so is $\overline{A}$.

Proof:

Let $\epsilon > 0$, then we may cover A with balls of size $\epsilon/2$. Call these balls $B(x_1, \epsilon/2), \cdots, B(x_n, \epsilon/2)$, then we have that for any $x \in \overline{A}$, the distance $d(x, A) = 0$, thus there is a point $y \in A$ s.t. $d(x, y) < \epsilon/2$ and hence there is a ball $B(x_i, \epsilon/2)$ s.t. $y \in B(x_i, \epsilon/2)$ and hence $d(x, x_i) \leq d(x, y) + d(y, x_i) < \epsilon$. It follows that $B(x_1, \epsilon), \cdots, B(x_n, \epsilon)$ covers $\overline{A}$, so $\overline{A}$ is totally bounded.

Suggested Exercise 38

Suppose that $X = \bigcup_{i=1}^{\infty} X_i$ where X_i are open and totally bounded, X is a metric space. Let $Y \subseteq X$. Show that $C_{lip}(S^2, Y) = \bigcup_{i=1}^{\infty} K_i$ where K_i are some totally bounded sets w.r.t. $\overline{\rho}$ and C_{lip} means Lipschitz continuous functions.

Proof:

For any X_i, we have that $X_i \cap Y$ is open and totally bounded. For any function $f : S^2 \to Y$, since S^2 is compact and $X_i \cap Y$ is an open cover of Y, then $f(S^2)$ is covered by finitely many $X_i \cap Y$. Suppose that $f(S^2)$ is covered by $X_1 \cap Y, \cdots, X_n \cap Y$, then we have that $f \in C_{lip}(S^2, \bigcup_{i=1}^{n} X_i \cap Y)$. Since each $X_i \cap Y$ is totally bounded, then a finite union of them is totally bounded. Furthermore, $C_{lip}(S^2, \bigcup_{i=1}^{n} X_i \cap Y) \subseteq C_{lip}(S^2, Y)$. Suppose that X is compact and Y is totally bounded, then we want to show that $C_{lip}(X, Y)$ is a countable union of totally bounded sets. Let C_i be the set of all Lipschitz functions from X to Y with Lipschitz constant at most i. We then have that C_i is equicontinuous and from X compact to Y totally bounded. We now show that if $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ with $\mathcal{F}$ equicontinuous and X compact, Y totally bounded, then $\mathcal{F}$ is totally bounded.

Let $\epsilon > 0$, then choose $\delta = \epsilon/3$. We then have that given any $a \in X$, we have some U_a containing a s.t. $d(f(x), f(a)) < \delta$ for all $x \in U_a, f \in \mathcal{F}$ by equicontinuity. These sets cover X and thus by compactness admit a finite subcover. Call this subcover $U_1, \cdots, U_k$ with corresponding points $a_1, \cdots, a_k$. We may then cover Y be sets $V_1, \cdots, V_m$ of diameter less than δ by total boundedness.

Let J be the set of all functions from $\alpha : \{1, \cdots, k\} \to \{1, \cdots, m\}$. Let J' be the set of all α for which there is a function $f_\alpha \in \mathcal{F}$ such that $f(U_i) \subseteq V_{\alpha(i)}$. Let $\{f_\alpha\} \subseteq \mathcal{F}$ denote a set of such functions, one for each $\alpha \in J'$. We then want to show that $\{B(f_\alpha, \epsilon) \big| \alpha \in J'\}$ covers $\mathcal{F}$.

Let $f \in \mathcal{F}$. For each $i = 1, \cdots, k$ there is some j s.t. $f(a_i) \in V_j$ and let $\alpha(i) = j$. This then gives one α in J'. We then want to show that $f \in B(f_\alpha, \epsilon)$. Let $x \in X$, then there is some i s.t. $x \in U_i$. We then have that:

$$d(f(x), f(a_i)) < \delta \tag{1}$$
$$d(f(a_i), f_\alpha(a_i)) < \delta \tag{2}$$
$$d(f_\alpha(a_i), f_\alpha(x)) < \delta \tag{3}$$

Thus we get that $d(f(x), f_\alpha(x)) < 3\delta = \epsilon$, thus $\sup\{\overline{d}(f, f_\alpha)\} < \min(1, \epsilon) < \epsilon$.

It follows that C_i is totally bounded and that $C_{lip}(S^2, \bigcup_{i=1}^n X_i \cap Y) = \bigcup_{i=1}^\infty C_i$ and $C_{lip}(S^2, Y)$ is a union of $C_{lip}(S^2, Y \cap \bigcup K)$ where K is an finite subset of $\{X_1, X_2, \cdots\}$ which is countable. It then follows that $C_{lip}(S^2, Y)$ is a countable union of totally bounded sets.

Suggested Exercise 39

Let $X = \mathbb{R}^2 \setminus \{(0,0),(2,0)\}$ Show that X retracts onto S^1. Explicitly show why your retraction is not homotopic to Id_X via the straight line homotopy. Show that it is impossible for any retraction $X \to S^1$ to be a deformation retraction.

Proof:

X retracts onto S^1 by $x \mapsto \frac{x}{||x||}$. This is not homotopic to the identity by a straight line homotopy since if it were, then we would have that $F(t, x) = tx + (1-t)\frac{x}{||x||}$ and so $F(\frac{1}{2}, (3, 0)) = (3/2, 0) + (1/2, 0) = (2, 0)$ which is not in X. We cannot have any deformation retraction $X \to S^1$ since X deformation retracts to $S^1 \wedge S^1$ and so X has fundamental group $Z * Z$ while S^1 has fundamental group S^1. Since deformation retractions induce isomorphisms of the fundamental group, then this cannot exist.

Suggested Exercise 40

Find π_1 of the one point compactification of $\mathbb{R}^2 \setminus B^2$. What is this space homeomorphic to?

Proof:

The one point compactification of $\mathbb{R}^2 \setminus B^2$ is homeomorphic to $\overline{H}^2 = S^2 \cap \{(x, y, z) \in \mathbb{R}^3 \big| z \geq 0\}$. This homeomorphism is via the stereographic projection. The fundamental group of $\overline{H}^2$ is trivial since we may take the retraction map $r : S^2 \to \overline{H}^2$ given by $(x, y, z) \mapsto (x, y, z)$ if $z \geq 0$ and $(x, y, z) \mapsto (x, y, -z)$ if $z \leq 0$. This is continuous since both maps are continuous and they are equal when $z = 0$. Let $i : \overline{H}^2 \to S^2$ be the inclusion, then we have that $r \circ i = Id_{\overline{H}^2}$ and thus $r_* \circ i_* = Id$. It follows that $i_* : \pi_1(\overline{H}^2) \to \pi_1(S^2)$ is injective, thus $\pi_1(\overline{H}^2)$ has at most 1 element and is thus trivial.