Hartshorne

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II.1.1

Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Proof:

We want to show that the sheaf of locally constant functions satisfies the universal property of the sheafification. The map from the constant presheaf $\mathscr{F}$ to $\mathscr{A}$ is given by mapping $a \in \mathscr{F}(U)$ to the constant function $s(x) = a$ which is a function from $s : U \to A$. This gives a function $\theta : \mathscr{F} \to \mathscr{A}$. We now want to show that for any sheaf $\mathscr{G}$, and any morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism $\psi : \mathscr{A} \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$.

For any open set $U \subseteq X$, then for any element $s \in \mathscr{A}(U)$, we have that $s : U \to A$ is a continuous function. We may cover U by open sets of the form $U_a = s^{-1}(\{a\})$. Suppose U_a is non-empty, then we have that $s|_{U_a} = \theta(U_a)(a)$. It follows that since θ is injective, then $\varphi = \psi \circ \theta$ implies that $\varphi(U_a)(a) = \psi(U_a)(s|_{U_a})$. Since ψ is a sheaf morphism, then it commutes with restriction, thus $\psi(U_a)(s|_{U_a}) = \psi(U)(s)|_{U_a}$. Since U_a cover U, then we know that $\psi(U)(s)$ must be the element of $\mathscr{G}(U)$ restricting to $\psi(U)(s)|_{U_a}$ on U_a, thus ψ is uniquely defined. Furthermore, it is a sheaf morphism since for any $V \subseteq U \subseteq X$ open, we have that

$$\begin{aligned} \psi(V)(s|_V)|_{V_a} &= \psi(U_a \cap V)(s|_{U_a \cap V}) \\ &= \psi(U_a)(s|_{U_a})|_{U_a \cap V} \\ &= \psi(U)(s)|_{V_a} \end{aligned}$$

Since equality holds on each V_a which cover V, then since $\mathscr{G}$ is a sheaf, we have that $\psi(V)(s|_V) = \psi(U)(s)|_V$ as desired.

II.1.2

(a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\mathrm{im}\varphi)_P = \mathrm{im}(\varphi_P)$.

(b) Show that φ is injective (respectively, surjective) iff the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $\cdots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \cdots$ of sheaves and morphisms is exact iff for each $P \in X$, the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof:

(a) For any $[U, s]$ in $(\ker\varphi)_P$ we have that $\varphi_P([U,s]) = [U, \varphi(U)(s)] = [U, 0] = 0$, and for any $[U, s]$ in $\ker(\varphi_P)$, we have that $[U, \varphi(U)(s)] = [U, 0]$, so there is some $W \subseteq U$ s.t. $P \in W$ and $\varphi(U)(s)|_W = 0$, then $\varphi(W)(s|_W) = 0$, therefore $[U, s] = [W, s|_W] \in (\ker\varphi)_P$. We have that $\theta : \text{im}\varphi \to (\text{im}\varphi)^+$ yields isomorphisms on stalks by sending $[U, s]$ to $[U, s']$ where $s'(P) = s_P$. Now, $\theta_P([U, \varphi(U)(s)]) = [U, \theta(U)(\varphi(U)(s))]$ which is in $(\text{im}\varphi)_P$. For any $[U, s'] \in (\text{im}\varphi)_P$, we have that $[U, s'] = \theta_P([W, t])$ with $t \in \text{im}\varphi(W)$, then $[W, t] \in \text{im}(\varphi_P)$.

(b) If φ is injective, then each $\varphi(U)$ is injective, thus $\ker\varphi$ is trivial and hence $\ker\varphi_P = (\ker\varphi)_P$ is trivial. If $\ker\varphi_P$ is trivial, then $(\ker\varphi)_P$ is trivial, thus $\ker\varphi$ is trivial. If φ is surjective, then $\text{im}\varphi(U)$ is $\mathscr{G}(U)$, therefore $\text{im}(\varphi_P) = \mathscr{G}_P$. If φ_P is surjective, then $\text{im}(\varphi_P) = \mathscr{G}_P$ and therefore θ induces an isomorphism between $\mathscr{G}_P$ and $(\text{im}\varphi)_P$ and therefore θ induces an isomorphism from $\mathscr{G} \to \text{im}\varphi$.

(c) Let $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{A}$ be an exact sequence of sheaves, then we have that $\text{im}\varphi = \ker\psi$ and therefore $(\text{im}\varphi)_P = (\ker\psi)_P$ and hence up to θ_P, we have that $\text{im}(\varphi_P) = \ker(\psi_P)$, thus the sequence of stalks is exact. If all of the exact sequences of stalks are exact, then we have that $\theta_P^{-1}(\text{im}(\varphi_P)) = \ker\psi_P$, and therefore θ induces an isomorphism between $\text{im}\varphi$ and $\ker\psi$ and thus the sequence is exact.

II.1.3

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective iff the following holds: for every open $U \subseteq X$, and any $s \in \mathscr{G}(U)$, there is a cover $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(U_i)(t_i) = s|_{U_i}$ for all i.
(b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U); \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Proof:

(a) If φ is surjective, then φ is surjective on stalks, thus for any $U \subseteq X$ and $s \in \mathscr{G}(U)$ and any $P \in U$, there is some $W \subseteq U$ with $P \in W$ and $t^P \in \mathscr{F}(W)$ with $\varphi(W)(t^P) = s|_W$. These W cover U. Conversely, choose any $P \in X$, then for any $[U, s] \in \mathscr{G}$, there is some U_i in the cover of U such that $P \in U_i$ and $\varphi(U_i)(t_i) = s|_{U_i}$, thus $[U, s] = [U_i, s|_{U_i}] = [U_i, \varphi(U_i)(t_i)]$, therefore $\text{im}(\varphi_P) = \mathscr{G}_P$ and hence φ is surjective.

(b) Let $X = \{a, b, c\}$ with $\{b\}, \{c\}$ closed points and generic point a, the consider $\mathscr{F}$ to be the constant sheaf on $\mathbb{Z}$ and let $\mathscr{G}$ be given by $\mathscr{G}(U) = \mathbb{Z}^n$ where n is $|U \cap \{b, c\}|$ with the restrictions collapsing the powers that vanish. We then have that $\mathscr{F}_a = \mathscr{F}_b = \mathscr{F}_c = \mathbb{Z}$ and $\mathscr{G}_a = 0, \mathscr{G}_b = \mathbb{Z}, \mathscr{G}_c = \mathbb{Z}$, then the map given by sending $s \in \mathscr{F}(U)$ to $(s(b), s(c))$ (if both b, c are in U, otherwise restrict to only those that are in U) is clearly surjective on stalks, however on the global section, we have that X is connected since it is the closure of a point, thus $\mathscr{F}(X) = \mathbb{Z}$, but $\mathscr{G}(X) = \mathbb{Z}^2$, thus it is not surjective on global sections.

II.1.4

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

Proof:

(a) We define φ^+ by letting $\varphi^+(U)(s)(P) = \varphi_P(s(P))$. We check that this is a morphism of sheaves. First note that for any $Q \in U$, there is some $V \subseteq U$ containing Q and $t \in \mathscr{F}(V)$ such that $s(P) = t_P$ for all $P \in V$, then $\varphi^+(U)(s)(P) = \varphi_P(t_P) = \varphi(V)(t)_P$, hence $\varphi^+(U)(s) \in \mathscr{G}^+(U)$. If $V \subseteq U$, then $\varphi^+(U)(s)|_V(P) = \varphi_P(s(P)) = \varphi_P(s|_V(P)) = \varphi^+(V)(s|_V)(P)$. To show that this is injective, we note that $\varphi^+(U)(s) = 0$ means that $\varphi_P(s(P)) = 0$ for all P in U and since φ_P is injective, then $s(P) = 0$ for all P in U, so $s = 0$.

(b) Let $\mathscr{A}$ be the presheaf image of φ, then there is an injective morphism of presheaves from $\mathscr{A} \to \mathscr{G}$ by inclusion and therefore, $\mathscr{A}^+$ includes into $\mathscr{G}^+ = \mathscr{G}$ injectively.

II.1.5

Show that a morphism of sheaves is an isomorphism iff it is both injective and surjective.

Proof:

If $\varphi : \mathscr{F} \to \mathscr{G}$ is an isomorphism iff $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for all P iff φ_P is both injective and surjective for all P iff φ is both injective and surjective.

II.1.6

(a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence:

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Proof:

(a) We have that the map from $\mathscr{F}$ to the quotient presheaf $\mathscr{F}/\mathscr{F}'$ is surjective and therefore it is surjective on stalks and thus the induced map into the actual sheaf $\mathscr{F}/\mathscr{F}'$ is surjective. The stalk of the kernel at P is the same as the kernel of the stalk map on the presheaves, which is exactly $\mathscr{F}'_P$ since the stalk of $\mathscr{F}/\mathscr{F}'$ at P is $\mathscr{F}_P/\mathscr{F}'_P$. Therefore, the kernel is $\mathscr{F}'$. Since $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$, then we obtain the described exact sequence.

(b) Let $\varphi(U_i)(s_i) \in \varphi(U_i)(\mathscr{F}'(U_i))$, then if $\varphi(U_i)(s_i)|_{U_i \cap U_j} = \varphi(U_j)(s_j)|_{U_i \cap U_j}$ and thus $\varphi(U_i \cap U_j)(s_i|_{U_i \cap U_j} = \varphi(U_i \cap U_j)(s_j|_{U_i \cap U_j}$. Since $\varphi(U_i \cap U_j)$ is injective, then we have that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and therefore there is a unique $s \in \mathscr{F}(\bigcup U_i)$ such that $s|_{U_i} = s_i$. Then we have that $\varphi(\bigcup U_i)(s)|_{U_i} = \varphi(U_i)(s_i)$ since φ is a morphism of sheaves. It follows that $\mathrm{im}\varphi$ is already a sheaf and φ is both injective and surjects onto $\mathrm{im}\varphi$, thus $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$. Let $\psi : \mathscr{F} \to \mathscr{F}''$, then on stalks, we have that ψ_P is surjective and $(\mathrm{im}\psi)_P = \mathrm{im}\psi_P = \mathscr{F}''_P$ up to isomorphism by sheafification. ψ induces an morphism from $\mathscr{F}/(\ker\psi) \to \mathscr{F}''$ given by the sheafification of the map $\widetilde{\psi}(U)(\overline{s}) = \psi(U)(s)$. On stalks, this map is given by $\widetilde{\psi}_P([U, \overline{s}]) = [U, \psi(U)(s)] = \widetilde{\psi_P}$. Since ψ_P is surjective, then $\widetilde{\psi_P}$ is an isomorphism, and thus $\mathscr{F}/(\ker\psi)$ is isomorphic to $\mathscr{F}''$ by $\widetilde{\psi}$. Note then that $\ker\psi = \mathrm{im}\varphi$ (and $\mathrm{im}\varphi$ is already a sheaf, thus no sheafification is required).

II.1.7

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.
(a) Show that $\mathrm{im}\varphi \cong \mathscr{F}/\ker\varphi$.
(b) Show that $\mathrm{coker}\varphi \cong \mathscr{G}/\mathrm{im}\varphi$.

Proof:

(a) Consider the sequence $0 \to \ker\varphi \to \mathscr{F} \to \mathrm{im}\varphi \to 0$. Taking stalks at P, we get $0 \to \ker\varphi_P \to \mathscr{F}_P \to \mathrm{im}\varphi_P \to 0$ where $\mathrm{im}\varphi_P$ is isomorphic to $(\mathrm{im}\varphi)_P$ by sheafification. This sequence is then trivially exact. By 6b, it follows that $\mathrm{im}\varphi \cong \mathscr{F}/\ker\varphi$.

(b) For any $[U, \overline{s}] \in (\mathrm{coker}\varphi)_P$, we map this to $\overline{[U,s]} \in \mathrm{coker}\varphi_P$. If $[U,\overline{s}] = [V,\overline{t}]$, then $\overline{s}|_W = \overline{t}_W$ hence $s|_W - t|_W \in \mathrm{im}\varphi(W)$, therefore $s|_W - t|_W = \varphi(W)(x)$ for some $x \in \mathscr{F}(W)$. It follows that $\overline{[U,s]} - \overline{[V,t]} = \overline{[W, s|_W - t|_W]} = \overline{[W, \varphi(W)(x)]} = \overline{\varphi_P([W,x])} = 0$, hence the map is well-defined. The inverse is given by $\overline{[U,s]} \mapsto [U,\overline{s}]$ for which well-definedness is proven in essentially the same manner. It follows that $(\mathrm{coker}\varphi)_P = \mathrm{coker}\varphi_P$ and $(\mathscr{G}/\mathrm{im}\varphi)_P = \mathscr{G}_P/\mathrm{im}\varphi_P = \mathrm{coker}\varphi_P$. It follows that $\mathrm{coker}\varphi$ and $\mathscr{G}/\mathrm{im}\varphi$ have the same stalks (up to the sheafification). It follows that they are isomorphic.

II.1.8

For any subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e. if $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups.

Proof:

We already know that $\Gamma(U, \cdot)$ preserves injectivity. Let $x \in \ker\psi(U)$ and $P \in U$, then the sequence of stalks is exact and therefore $x_P \in \ker\psi_P$ means that there is some $s_P \in \mathscr{F}'_P$ such that $\varphi_P(s_P) = x_P$. It follows that there is some $V \subseteq U$ containing P and $s(P) \in \mathscr{F}'(V)$ such that $s_P = [V, s(P)]$ and $\varphi(V)(s(P)) = x|_V$. Now, cover U by V_i, and let $s(P_i) = s_i$, then we have that $\varphi(V_i)(s_i)|_{V_i \cap V_j} = x|_{V_i \cap V_j} = \varphi(V_j)(s_j)|_{V_i \cap V_j}$. Since $\varphi(V_i \cap V_j)$ is injective, then we get that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. It follows that the s_i glue to some $s \in fs'(U)$ and thus $x = \varphi(U)(s)$. Therefore, the sequence is exact.

II.1.9

Direct Sum. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and direct product in the category of sheaves of abelian groups on X.

Proof:

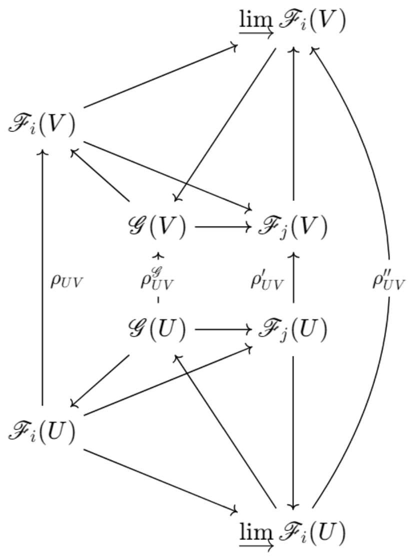
Let $U = \bigcup U_i$ and let $(s_i, t_i) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$, and suppose that $(s_i, t_i)|_{U_i \cap U_j} = (s_j, t_j)|_{U_i \cap U_j}$, i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. It follows that there is a unique $s \in \mathscr{F}(U)$ and a unique $t \in \mathscr{G}(U)$ such that $(s,t)|_{U_i} = (s_i, t_i)$ and therefore the direct sum forms a sheaf. To check that this is indeed the product in the category of sheaves, suppose that we have a morphism $\varphi : \mathscr{A} \to \mathscr{F}$ and $\varphi' : \mathscr{A} \to \mathscr{G}$, then for each $U \subseteq X$, there is a unique $\psi(U) : \mathscr{A}(U) \to \mathscr{F}(U) \oplus \mathscr{G}(U)$. Furthermore we have that ψ commutes with restrictions since $\psi(U)(s)|_W = (\varphi(U)(s), \varphi'(U)(s))|_W = (\varphi(W)(s|_W), \varphi'(W)(s|_W)) = \psi(W)(s|_W)$. It follows that $\mathscr{F} \oplus \mathscr{G}$ satisfies the universal property of the direct sum.

II.1.10

Direct Limit. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$ and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

Proof:

A morphism of sheaves is determined by the morphisms on stalks. Given a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we have that for any $s \in \mathscr{F}(U)$, $\varphi(s)_P = \varphi_P(s_P)$, thus we know $\varphi(s)_P$ for all $P \in U$. It follows that we have $V_i \subseteq U$ and $t_i \in \mathscr{G}(V_i)$ such that $t_i = \varphi(s)|_{V_i}$, then $\varphi(s)$ is obtained by gluing t_i. Therefore, morphisms of sheaves are determined entirely by the morphisms on stalks. The stalks of the direct limit sheaf are given by $\varinjlim_{P \in U} \varinjlim \mathscr{F}_i(U)$. For any group G with maps $\varinjlim \mathscr{F}_i(U) \to G$ which commute with the restriction on the direct limit sheaf, we are equivalently given maps $\varphi_{i,U} : \mathscr{F}_i(U) \to G$ which commute with both restriction and the maps from $\mathscr{F}_i \to \mathscr{F}_j$. Since $\varphi_{i,U}$ commute with restrictions, then we get a unique map $\varphi_{i,P} : \mathscr{F}_{i,P} \to G$ and the maps $\varphi_{i,P}$ commute with the morphisms between $\mathscr{F}_i$ and thus the morphisms between $\mathscr{F}_{i,P}$. It follows that we obtain a unique map from $\varinjlim \mathscr{F}_{i,P}$. Therefore, the stalks of the direct limit sheaf is the direct limit of the stalks. Suppose we have morphisms $\mathscr{F}_i \to \mathscr{G}$ which commute with the maps from $\mathscr{F}_i \to \mathscr{F}_j$, then we obtain maps $\mathscr{F}_{i,P} \to \mathscr{G}_P$ which commute with the maps from $\mathscr{F}_{i,P}$ to $\mathscr{F}_{j,P}$ and therefore obtain a unique map from the stalk of the direct limit sheaf to $\mathscr{G}$. It follows that any morphism of sheaves obtained from $\mathscr{F}_i \to \mathscr{G}$ must be unique. To show that it exists, consider the following diagram. We obtain morphisms from $\varinjlim \mathscr{F}_i(U) \to \mathscr{G}(U)$ and since this diagram commutes for each i (and any $j > i$), then we have that the morphism commutes with restriction.



II.1.11

Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case, show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

Proof:

We first show that noetherian topological spaces are hereditarily compact. Let $\{U_i\}_{i\in I}$ be a cover of X, then $X = \bigcup_{i\in I} U_i$, thus $\varnothing = \bigcap_{i\in I} U_i^c$. If no finite subset of $\{U_i^c\}_{i\in I}$ intersects to $\varnothing$. Pick an arbitrary $i_1 \in I$, then suppose we have chosen $i_1, \cdots, i_n$ such that $\bigcap_{k=1}^n U_{i_k}^c \subsetneq \bigcap_{k=1}^{n-1} U_{i_k}^c$. Since $\bigcap_{i\in I} U_i^c = \varnothing$, then there is some i_{n+1} such that $\bigcap_{k=1}^{n+1} U_{i_k}^c \subsetneq \bigcap_{k=1}^{n} U_{i_k}^c$. This is a descending sequence of closed sets, and therefore it must terminate, however it may only terminate at the empty set, thus there is a finite subset intersecting to $\varnothing$ and thus we have a finite subcover. We now show that subspaces of noetherian spaces are noetherian. Let $Z \subseteq X$ and let $C_1 \supsetneq C_2 \supsetneq \cdots$. Since C_i is closed in Z, $C_i = W_i \cap Z$. Let $Y_k = \bigcap_{i\leqslant k} W_i$, then $Y_i \supsetneq Y_{i+1}$ and thus this sequence terminates and hence the sequence terminates in Z as well. Let $U \subseteq X$ open and let $U = \bigcup_{j\in J} U_j$ with $s_j \in \varinjlim \mathscr{F}_i(U)$. Let $\{U_1, \cdots, U_n\}$ be a finite subcover, and suppose that $s_j \in \mathscr{F}_{i_j}(U)$, then there is some $k > i_j$ for all j since the poset is directed and there are only finitely many i_j. Furthermore, we have that $\rho_{U_j, U_j\cap U_m}(s_j) = \rho_{U_m, U_j\cap U_m}(s_m)$, thus they agree for some $n_{jm} > i_j, i_m$, suppose that $k > n_{jm}$ for all j, m as well. Let $\varphi_{i_j,k}$ be the map from $\mathscr{F}_{i_j} \to \mathscr{F}_k$, then $s_j = \varphi_{i_j,k}(U_j)(s_j)$ in the direct limit and:

$$\begin{aligned}
\rho^{i_j}_{U_j, U_j\cap U_m}(s_j) &= \rho^{k}_{U_j, U_j\cap U_m}(\varphi_{i_j,k}(U_j)(s_j)) \\
&= \varphi_{i_j,k}(U_j \cap U_m)(s_j|_{U_j\cap U_m}) \\
&= \varphi_{i_m,k}(U_j \cap U_m)(s_m|_{U_j\cap U_m}) \\
&= \rho^{k}_{U_m, U_j\cap U_m}(\varphi_{i_m,k}(U_m)(s_m)) \\
&= \rho^{i_m}_{U_m, U_j\cap U_m}(s_m)
\end{aligned}$$

It follows that the s_j glue in $\mathscr{F}_k$ to give some $s \in \mathscr{F}_k(U)$ such that $s|_{U_j} = s_j$. Furthermore, note that for any other $i \in I$, we have that $s|_{U_i}$ is obtained by gluing $s_j|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j} = s_i|_{U_i\cap U_j}$, thus $s|_{U_i}$ and s_i restrict to the same elements on an open cover, thus they are the same. It follows that the direct limit is already a sheaf.

II.1.12

Inverse Limit. Let $\{\mathscr{F}_i\}_{i\in I}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof:

Let $U = \bigcup_{j\in J} U_j$ and let $s_j \in \varprojlim \mathscr{F}_i(U)$ such that $s_j|_{U_j\cap U_m} = s_m|_{U_j\cap U_m}$. For each $i \in I$, we have $\pi_i(U) : \varprojlim \mathscr{F}_i(U) \to \mathscr{F}_i(U)$ which commute with the restriction maps. For each i, we have $\pi_i(s_j)$, which glue to an element of $s_i \in \mathscr{F}_i(U)$. We then define $s \in \varprojlim \mathscr{F}_i(U)$ to have components $\pi_i(s) = s_i$. The morphisms $\varphi_{ij}(U) : \mathscr{F}_i(U) \to \mathscr{F}_j(U)$ must commute with $\pi_{i,U}$, i.e. $\varphi_{ij}\pi_i = \pi_j$. We need to show that for $k \leqslant i$, we have that $s_i = \varphi_{ik}(U)(s_k)$, however this can be checked on each open set, i.e. $s_i|_{U_j} = \varphi_{ik}(U)(s_k)|_{U_j} = \varphi_{ik}(U_j)(s_k|_{U_j})$. Note that $s_i|_{U_j} = \pi_i(s_j)$ and $s_k|_{U_j} = \pi_k(s_j)$. The equality $\pi_i(s_j) = \varphi_{ik}(U_j)(\pi_k(s_j))$ holds since $s_j \in \varprojlim \mathscr{F}_i(U_j)$. Therefore $\varprojlim \mathscr{F}_i$ is a sheaf. To check that it has the universal property of an inverse limit, suppose that we are given morphisms $\psi_i : \mathscr{G} \to \mathscr{F}_i$ which commute with φ_{ij}, then for each U, we obtain a unique morphism $\psi(U) : \mathscr{G}(U) \to \varprojlim \mathscr{F}_i(U)$ such that $\psi_i(U) = \pi_i(U) \circ \psi(U)$. We now need only check that ψ commutes with restriction. Since $\psi(U)$ is determined by $\pi_i(U) \circ \psi(U)$, then we have that $\pi_i(U)(\psi(U)(s)|_V) = \psi_i(U)(s)|_V = \psi_i(V)(s|_V) = \pi_i(V)(\psi(V)(s|_V))$, thus $\psi(U)(s)|_V = \psi(V)(s|_V)$ as desired.

II.1.13

Espace Étalé of a Presheaf. Given a presheaf $\mathscr{F}$ on X, we define a topological space $\text{Spé}(\mathscr{F})$, called the *espace étalé* of $\mathscr{F}$, as follows. As a set, $\text{Spé}(\mathscr{F}) = \bigcup_{P \in X} \mathscr{F}_P$. We define a projection map $\pi : \text{Spé}(\mathscr{F}) \to X$ by sending $s \in \mathscr{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map $\overline{s} : U \to \text{Spé}(\mathscr{F})$ by sending $P \mapsto s_P$. This maps has the property that $\pi \circ \overline{s} = \text{id}_U$, in other words, it is a "section" of π over U. We now make $\text{Spé}(\mathscr{F})$ into a topological space by giving it the finest topology such that all the maps $\overline{s} : U \to \text{Spé}(\mathscr{F})$ for all U, and all $s \in \mathscr{F}(U)$, are continuous. Now show that the sheaf $\mathscr{F}^+$ associated to $\mathscr{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of all *continuous* sections of $\text{Spé}(\mathscr{F})$ over U. In particular, the original presheaf $\mathscr{F}$ was a sheaf iff for each U, $\mathscr{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathscr{F})$ over U.

Proof:

Let $f : U \to \text{Spé}(\mathscr{F})$ be a continuous section, then for any $P \in U$, $f(P) \in \mathscr{F}_P$, thus $f(P) = [V, t]$ for some $t \in \mathscr{F}(V)$ and $V \subseteq U$ containing P. For any $\overline{s} : W \to \text{Spé}(\mathscr{F})$, $Q \in \overline{s}^{-1}(\text{im}\overline{t})$ iff $s_Q = t_Q$, i.e. iff $(s|_{W \cap V} - t|_{W \cap V})_Q = 0$. Note that if $(s|_{W \cap V} - t|_{W \cap V})_Q = 0$, then the difference is 0 on a neighborhood of Q, thus the set is open. It follows that $\text{im}\overline{t}$ is open in $\text{Spé}(\mathscr{F})$, then $f^{-1}(\text{im}\overline{t})$ is open in U and thus $f(Q) = t_Q$ on some neighborhood of P. Conversely, given a section $f : U \to \text{Spé}(\mathscr{F})$ such that for each $P \in U$, there is a neighborhood V of P and $t \in \mathscr{F}(V)$ such that $f|_V = \overline{t}$, then f is continuous by the pasting lemma.

II.1.14

Support. Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\text{Supp}(s)$, is defined to be $\{P \in U | s_P \neq 0\}$. Show that $\text{Supp}(S)$ is a closed subset of U. We define the *support* of $\mathscr{F}$, $\text{Supp}\mathscr{F}$, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

Proof:

To show that $\text{Supp}(s)$ is closed, we will show that its complement is open. Suppose that $s_P = 0$, then there is an open set containing P contained in U such that s is 0 on that set, thus the complement is open and hence $\text{Supp}(s)$ is closed in U. Note that $\text{Supp}(\mathscr{F}) = \bigcup_{s \in \mathscr{F}(U), U \subseteq X} \text{Supp}(s)$.

II.1.15

Sheaf $\mathscr{H}om$. Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\text{hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves as a natural abelian group structure. Show that the presheaf $U \mapsto \text{hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

Proof:

Let $U = \bigcup U_i$ be an open subset of X and suppose that we have morphisms $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. Let $V \subseteq U$ open and $s \in \mathscr{F}(V)$, $s|_{V \cap U_i} \in \mathscr{F}(V \cap U_i)$, then

$\varphi_i(V \cap U_i)(s|_{V \cap U_i}) \in \mathscr{G}(V \cap U_i)$ and furthermore, for any i, j, we have that

$$\begin{aligned}\varphi_i(V \cap U_i)(s|_{V \cap U_i})|_{V \cap U_i \cap U_j} &= \varphi_i(V \cap U_i \cap U_j)(s|_{V \cap U_i \cap U_j}) \\ &= \varphi_j(V \cap U_i \cap U_j)(s|_{V \cap U_i \cap U_j}) \\ &= \varphi_i(V \cap U_j)(s|_{V \cap \cap U_j})|_{V \cap U_i \cap U_j}\end{aligned}$$

Thus these elements glue to some $t \in \mathscr{G}(V)$, we then define $\varphi : \mathscr{F}|_U \to \mathscr{G}|_U$ by $\varphi(V)(s) = t$. For any $W \subseteq V$, we have that

$$\begin{aligned}\varphi(V)(s)|_{W \cap U_i} &= \varphi_i(V \cap U_i)(s|_{V \cap U_i})|_{W \cap U_i} \\ &= \varphi_i(W \cap U_i)(s|_{W \cap U_i})\end{aligned}$$

therefore, $\varphi(V)(s)|_W = \varphi(W)(s|_W)$ by uniqueness of gluing in $\mathscr{G}$. Note that for any morphism $\varphi : \mathscr{F}|_U \to \mathscr{G}|_U$ such that $\varphi|_{U_i} = \varphi_i$, then we have that $\varphi(V)(s)|_{V \cap U_i} = \varphi(V \cap U_i)(s|_{V \cap U_i}) = \varphi_i(V \cap U_i)(s|_{V \cap U_i})$, thus by gluing, this morphism is exactly the φ we constructed. It follows that $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf.

II.1.16

Flasque Sheaves. A sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.
(a) Show that a constant sheaf on an irreducible topological space is flasque.
(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.
(c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.
(d) If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.
(e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of *discontinuous sections* of $\mathscr{F}$ as follows. For each open set $U \subseteq X$, $\mathscr{G}(U)$ is the set of maps $s : U \to \bigcup_{P \in U} \mathscr{F}_P$ such that for each $P \in U, s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Proof:

(a) For $U \subseteq X$ open, U is irreducible, thus $\mathscr{F}(U) = A$ for all $U \subseteq X$ non-empty, hence the restriction maps are trivially surjective.

(b) The section functor is left exact, so we only need to check that it preserves surjectivity. Let $x \in \mathscr{F}''(U)$, then for any $P \in U$, we have some $y \in \mathscr{F}(V)$ such that $\varphi_P(y_P) = x_P$ for some neighborhood V of P. Let

$$S = \{(W, s) \big| V \subseteq W \subseteq U, s \in \mathscr{F}(W), \varphi(W)(s) = x|_W\}$$

Ordered by $(W, s) \leqslant (M, t)$ if $W \subseteq M$ and $t|_W = s$. Suppose that we have a chain $\{(W_i, s_i)\}_{i \in I}$ in S, then $W = \bigcup_{i \in I} W_i$ is open and the s_i glue to some $s \in \mathscr{F}(W)$ since $s_i|_{W_j} = s_j$ for $W_j \subseteq W_i$. Furthermore, $\varphi(W)(s)|_{W_i} = \varphi(W_i)(s_i) = x|_{W_i}$, hence $\varphi(W)(s) = s|_W$. Therefore chains in S have upper bounds and S is non-empty since $(P, y) \in S$, therefore by Zorn's lemma, S contains a maximal element. Let (W, s) be a maximal element. Suppose that $W \neq U$, then choose $P \in U \backslash W$, then there is a set V containing P in U and $y \in \mathscr{F}(V)$ such that $\varphi(V)(y) = x|_V$ as before. Now consider the set $W' = W \cup V$. On $V \cap W$, we have that $\varphi(V \cap W)(y|_{V \cap W} - s|_{V \cap W}) = 0$, then by exactness, there is some $z \in \mathscr{F}'(V \cap W)$ such that $\psi(V \cap W)(z) = y|_{V \cap W} - s|_{V \cap W}$. Since $\mathscr{F}'$ is flasque, then there is some $z' \in \mathscr{F}'(V)$ such that $z'|_{V \cap W} = z$. Now s and $y - \psi(V)(z')$ agree on $V \cap W$, thus they glue to an element $u \in \mathscr{F}(V \cup W)$ and $\varphi(V)(s|_V) = \varphi(V)(y) - \varphi(V)(\psi(V)(z')) = \varphi(V)(y) = x|_V$ and similarly, $\varphi(W)(s|_W) = \varphi(W)(s) = x|_W$, hence $\varphi(V \cup W)(u) = x|_{V \cup W}$. Furthermore, $u|_W = s$, thus contradicting the maximality of (W, s). It follows that $W = U$ and hence $\varphi(U) : \mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective.

(c) Since $\rho)XV = \rho_{UV} \circ \rho_{XU}$, then if ρ_{XU} is surjective for all U, then ρ_{UV} is surjective. Since $\mathscr{F}'$ is flasque, then $\mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective as is $\mathscr{F}(U) \to \mathscr{F}''(U)$. Since $\mathscr{F}$ is flasque, then $\rho^{\mathscr{F}}_{XU}$ is surjective. Now, $\rho^{\mathscr{F}''}_{XU} \circ \varphi(X) = \varphi(U) \circ \rho^{\mathscr{F}}_{XU}$ is surjective, thus $\rho^{\mathscr{F}''}_{XU}$ is surjective and hence $\mathscr{F}''$ is flasque.

(d) Let $U \subseteq Y$, then $\rho^{f_*\mathscr{F}}_{YU} = \rho^{\mathscr{F}}_{X, f^{-1}(U)}$ is surjective, thus $f_*\mathscr{F}$ is flasque.

(e) $\mathscr{G}$ is clearly a sheaf. Let $U \subseteq X$, then for any $s : U \to \bigcup_{P \in U} \mathscr{F}_P$, let $s' \in \mathscr{G}(X)$ be given by $s'(P) = s(P)$ for $P \in U$ and $s'(P) = 0$ for $P \notin U$, then clearly $\rho_{XU}(s') = s$. Let $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ be given by $\varphi(U)(s) = \overline{s}$, where $\overline{s}(P) = s_P \in \mathscr{F}_P$, then for any $V \subseteq U$, we have that $\varphi(U)(s)|_V = \overline{s}|_V = \overline{s|_V} = \varphi(V)(s|_V)$. Let $s \in \mathscr{F}(U)$ and suppose that $\varphi(U)(s) = 0$, then for all $P \in U$, $s_P = 0$ and thus there is a cover of U on which s restricts to 0, thus $s = 0$. It follows that φ is injective.

II.1.17

Skyscraper Sheaves. Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$ and 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \mathrm{cl}\{P\}$, and 0 elsewhere. Hence the name "skyscraper sheaf". Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\mathrm{cl}\{P\}$, and $i : \mathrm{cl}\{P\} \to X$ is the inclusion.

Proof:

If $Q \notin \mathrm{cl}\{P\}$, then there exists V containing Q open s.t. $P \notin V$, then $i_P(A)_Q = \varinjlim_{P \in U} i_P(A)(U) = \varinjlim_{P \in U \subseteq V} i_P(A)(U) = 0$, and if $Q \in \mathrm{cl}\{P\}$, then $i_P(A)_Q = \varinjlim_{P \in U} i_P(A)(U) = A$. $i_*(A)(U) = A(U \cap \mathrm{cl}\{P\})$. Suppose $U \cap \mathrm{cl}\{P\} \neq \varnothing$ and $P \notin U$, then $\mathrm{cl}\{P\} \cap U^c$ is a closed subset of X containing P, thus it contains $\mathrm{cl}\{P\}$, so U^c contains $\mathrm{cl}\{P\}$, a contradiction. It follows that $U \cap \mathrm{cl}\{P\} \neq \varnothing \iff P \in U$, thus $i_*(A)(U) = i_P(U)$.

II.1.18

Adjoint Property of f^{-1}. Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y, there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$\hom_X(f^{-1}\mathscr{G}, \mathscr{F}) = \hom_Y(\mathscr{G}, f_*\mathscr{F})$$

Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

Proof:

$f^{-1}f_*\mathscr{F}V$ is the sheaf associate to $V \mapsto \varinjlim_{U\supseteq f(V)} \mathscr{F}(f^{-1}(U))$. For any $U \supseteq f(V)$, we have that $f^{-1}(U)$ contains V, thus from each $\mathscr{F}(f^{-1}(U))$, there is a restriction map $\rho^{\mathscr{F}}_{f^{-1}(U),V}$. For any $f(V) \subseteq W \subseteq U$, we have restriction maps $\rho^{\mathscr{F}}_{f^{-1}(U),f^{-1}(W)}$ which commute with $\rho^{\mathscr{F}}_{f^{-1}(U),V}$. It follows that $\rho^{\mathscr{F}}_{f^{-1}(U),V}$ determines a morphism $\varphi(V)$ from $\varinjlim_{U\supseteq f(V)} \mathscr{F}(f^{-1}(U))$ to $\mathscr{F}(V)$. For $W \subseteq V$ and $s \in \mathscr{F}(f^{-1}(U)) \subseteq \varinjlim_{U\supseteq f(W)} \mathscr{F}(f^{-1}(U))$, we have that

$$\begin{aligned}
\varphi(V)(s|_V) &= \rho^{\mathscr{F}}_{f^{-1}(U),V}(\rho^{f^{-1}f_*\mathscr{F}}_{WV}(s)) \\
&= \rho^{\mathscr{F}}_{f^{-1}(U),V}(s) \\
&= \rho^{\mathscr{F}}_{W,V}\rho^{\mathscr{F}}_{f^{-1}(U),W}(s) \\
&= \varphi(W)(s)|_V
\end{aligned}$$

Therefore the sheafification of φ, φ^+ is the desired morphism.

For any $U \subseteq Y$, define $\psi(U) : \mathscr{G}(U) \to f_*f^{-1}\mathscr{G}(U)$ (the presheaf) by sending $s \in \mathscr{G}(U)$ to s in the direct limit. Note that $\mathscr{G}(U)$ is part of the direct limit since $f(f^{-1}(U))$ contains U. Let $V \subseteq U$ and $s \in \mathscr{G}(U)$, then $\psi(U)(s)|_V = \psi(V)(s|_V)$ since the restriction in the direct limit is the restriction of $\mathscr{G}$. We then get a map to the sheafification, $\psi^+ : \mathscr{G} \to f_*f^{-1}\mathscr{G}(U)$ by composition with θ.

For any $\nu : f^{-1}\mathscr{G} \to \mathscr{F}$, we get $f_*\nu : f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$, then $f_*\nu \circ \psi^+ : \mathscr{G} \to f_*\mathscr{F}$. Similarly for any $\mu : \mathscr{G} \to f_*\mathscr{F}$, applying f^{-1}, we get $f^{-1}\mu : f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F}$, then composing with φ, we get $\varphi^+ \circ f^{-1}\mu : f^{-1}\mathscr{G} \to \mathscr{F}$. We first check that $\varphi \circ f^{-1}f_*\nu \circ f^{-1}\psi = \nu$. Let $[s] \in (f^{-1}\mathscr{G})(U)$, then $s \in \mathscr{G}(W)$ for some $W \supseteq f(U)$. Now we have:

$$\begin{aligned}
(\varphi(U) \circ f^{-1}f_*\nu(U) \circ f^{-1}\psi(U))([s]) &= (\varphi(U) \circ f^{-1}f_*\nu(U))([\psi(W)(s)]) \\
&= \varphi(U)([(f_*\nu)(W)(\psi(W)(s)]) \\
&= \varphi(U)([\nu(f^{-1}(W))(\psi(W)(s))] \\
&= \nu(f^{-1}(W))(s)|_U \\
&= \nu(U)([s]|_U) \\
&= \nu(U)([s])
\end{aligned}$$

Similarly, we check that $f_*\varphi \circ f_*f^{-1}\mu \circ \psi = \mu$. Let $s \in \mathscr{G}(U)$, then:

$$\begin{aligned}
(f_*\varphi(U) \circ f_*f^{-1}\mu(U) \circ \psi(U))(s) &= (f_*\varphi(U) \circ f_*f^{-1}\mu(U))([s]) \\
&= f_*\varphi(U)((f^{-1}\mu)(f^{-1}(U))(s)) \\
&= f_*\varphi(U)([\mu(U)(s)]) \\
&= \varphi(f^{-1}(U))([\mu(U)(s)]) \\
&= \rho_{f^{-1}(U),f^{-1}(U)}(\mu(U)(s)) \\
&= \mu(U)(s)
\end{aligned}$$

Note that I only checked this on presheaves, however, this is fine, since the morphisms from a presheaf to a sheaf are in bijection with the sheafification to a sheaf.

II.1.19

Extending a Sheaf by Zero. Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complement and let $j : U \to X$ be the inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."
(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$ and $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$ and 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!$ the sheaf obtained by *extending* $\mathscr{F}$ *by zero* outside U.
(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} i_*(\mathscr{F}|_Z) \to 0$$

Proof:

(a) If $P \notin Z$, then there is an open set containing V not meeting Z, therefore $(i_*\mathscr{F})_P = \varinjlim_{P \in U \subseteq V} (i_*\mathscr{F})(U) = 0$. If $P \in Z$, then for any $U \subseteq X$ with $P \in U$, we have that $(i_*\mathscr{F})(U) = \mathscr{F}(U \cap Z)$. Note that for any $U, V \subseteq X$ with $U \cap Z = V \cap Z$, then $\mathscr{F}(V \cap Z)$ and $\mathscr{F}(U \cap Z)$ are equivalent in the direct limit, therefore $(i_*\mathscr{F})_P = \varinjlim_{P \in U} \mathscr{F}(U \cap Z)$ can be taken over all distinct $U \cap Z$ which is exactly the open subset of Z containing P, thus $(i_*\mathscr{F})_P = \mathscr{F}_P$.

(b) The stalk of a sheaf is the same as the stalk of the sheafification (up to the sheafification isomorphism). For any $P \in U$, we have some neighborhood V of P contained in U, thus $(j_!\mathscr{F})_P = \varinjlim_{P \in U \subseteq V} j_!\mathscr{F}(U) = \mathscr{F}_P$ and if $P \notin U$, then $(j_!\mathscr{F})_P = \varinjlim_{P \in V} j_!\mathscr{F}(V) = \varinjlim 0 = 0$. Let $\mathscr{G}$ be an other sheaf satisfying $\mathscr{G}_P = \mathscr{F}_P$ for $P \in U$ and 0 for $P \notin U$ and $\mathscr{G}|_U = \mathscr{F}$. We have a morphism $j_!\mathscr{F}$ to $\mathscr{G}$ which is obtained by the sheafification of the map $\varphi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ by inclusion if $V \subseteq U$ and the 0 map if $V \nsubseteq U$. This map then gives isomorphisms on stalks, and thus it is an isomorphism.

(c) We first compute the stalks of $i_*i^{-1}\mathscr{F}$ and show that the map given by sending $a \in \mathscr{F}(U)$ to $[a] \in i_*i^{-1}\mathscr{F}(U)$ induces isomorphisms of stalks for $P \in Z$. $(i_*i^{-1}\mathscr{F})_P = \varinjlim_{P \in W} \varinjlim_{V \supseteq i(W)} \mathscr{F}(V)$. Consider the following diagram,

$$\begin{tikzcd}
X_{ik} \arrow[rr, "\psi_{ikl}"] \arrow[dr] & & X_{il} \arrow[rr, "\xi_{ijlk'}"] \arrow[dl] & & X_{jk'} \arrow[rr, "\psi_{jk'l'}"] \arrow[dr] & & X_{jl'} \arrow[dl] \\
& X_i \arrow[rrrr, "\phi_{ij}"] \arrow[drr] \arrow[ddrr] & & & & X_j \arrow[dll] \arrow[ddll] & \\
& & & X \arrow[d, "\exists!"] & & & \\
& & & Y & & &
\end{tikzcd}$$

X_i are direct limits of $\{X_{ik}, \psi_{ikl}\}$. Suppose furthermore, that for any i and k and $j \geqslant i$, that there is some l and a morphism $\xi_{ijkl} : X_{ik} \to X_{jl}$ which composes properly with the ψ_{ijk}, then X is the direct limit of the directed system $\{X_{ik}, \psi_{ikl}, \xi_{ijkl}\}$. Suppose we are given morphisms $f_{ik} : X_{ik} \to Y$ which commutes with all ψ_{ijk} and ξ_{ijkl}, then f yields unique morphisms $f_i : X_i \to Y$. Furthermore, since the above diagram commutes, we know that φ_{ij} are the morphisms making the maps from X_{il}

to X_j through $\xi_{ijlk'}$ and $X_{jk'} \to X_j$ commute and thus $f_i = f_j \circ \varphi_{ij}$. Therefore we obtain a unique map from $X \to Y$, hence X is the direct limit of $\{X_{ik}, \psi_{ikl}, \xi_{ijkl}\}$.

We now see that in your case, for any $K \subseteq W$ and $V \supseteq i(W)$, then $V \supseteq i(K)$, thus there is a map from $\mathscr{F}(V)$ in the direct limit for $i^{-1}\mathscr{F}(W)$ to $\mathscr{F}(V)$ in the direct limit for $i^{-1}\mathscr{F}(K)$, namely the identity. This clearly commutes with the restrictions. It follows that $(i_* i^{-1}\mathscr{F})_P = \varinjlim_{P \in W, i(W) \subseteq V} \mathscr{F}(V)$. We now see that this is just the stalk of $\mathscr{F}$ at P since for any open V containing P, we may simply take $W = Z \cap V$, then we obtain $\mathscr{F}(V)$ in the limit and conversely any V in the limit contains P. Additionally, the map ψ_P maps an element s_P to s_P and is thus an isomorphism.

We now check that the sequence is exact. Let $P \in Z$, then the sequence becomes $0 \to 0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0$ with the only nontrivial map being the identity. If $P \in U$, then the sequence becomes $0 \to \mathscr{F}_P \to \mathscr{F}_P \to 0 \to 0$. The map φ is the sheafification of the inclusion and thus the stalk map is the identity and is hence an isomorphism as required.

II.1.20

Subsheaf with Supports. Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support is contained in Z.
(a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}_Z^0(\mathscr{F})$.
(b) Let $U = X - Z$, and et $j : U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Proof:

(a) Let $U = \bigcup U_i$ and $s_i \in \Gamma_{Z \cap U_i}(V, \mathscr{F}|_V)$, with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. $s\_i \in \mathscr{F}(V)$ and $\mathrm{Supp}(s_i) \subseteq U_i \cap Z$. Since $\mathscr{F}$ is a sheaf, then the s_i glue to some $s \in \mathscr{F}(U)$ and $\mathrm{Supp}(s) = \{P \in U \big| s_P \neq 0\}$. For any $P \notin Z \cap U$, let $P \in U_i$, then since $\mathrm{Supp}(s_i) \subseteq Z \cap U_i$ and $(s_i)_P = s_P$, then $P \notin \mathrm{Supp}(s)$. Therefore $\mathscr{H}_Z^0(\mathscr{F})$ is a sheaf.

(b) The map from $\mathscr{H}_Z^0(\mathscr{F})(V) \to \mathscr{F}(V)$ is the inclusion map and $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is given by mapping $s \in \mathscr{F}(V)$ to $[s] \in j_*(j^{-1}\mathscr{F})(V)$ and $[s] = s|_{V \cap U}$ since U is open. The inclusion map is trivially injective. If $s \in \mathscr{F}(V)$ maps to 0 under the restriction to $V \cap U$, then $\mathrm{Supp} s \subseteq Z \cap V$ in V, hence $s \in \mathscr{H}_Z^0(\mathscr{F})$. If $\mathscr{F}$ is flasque, then since the map from $\mathscr{F}(V) \to j_*(\mathscr{F}|_U)(V) = \mathscr{F}(V \cap U)$ is just restriction, then it is trivially surjective.

II.1.21

Some Examples of Sheaves on Varieties. Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.
(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \to \mathscr{I}_Y(U)$ is a sheaf. It is called the sheaf of ideals $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.
(b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.
(c) Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact

sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathscr{I}_Y \to \mathcal{O} \to \mathscr{F} \to 0$$

Show however, that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F})$ is not surjective. This shows that the global section functor $\Gamma(X, \cdot)$ is not exact.
(d) Again, let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injective $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\bigoplus_{P\in X} i_P(I_P)$ where I_P is the group K/O_P, and $i_P(I_P)$ denotes the skyscraper sheaf given by I_P at the point P.
(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}(\to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0$$

is exact.

Proof:

(a) Suppose $U = \bigcup U_i$ and let $s_i \in \mathscr{I}_Y(U_i)$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then there is some $s \in \mathcal{O}_X(U)$ such that $s|_{U_i} = s_i$. We then have that for any $P \in Y \cap U$, let $P \in U_i$, then $s(P) = s_i(P) = 0$, thus s vanishes on Y and hence $s \in \mathscr{I}_Y(U)$. Therefore $\mathscr{I}_U$ is a sheaf.

(b) Let X be a quasi-projective variety, and let f be a regular function on $U \subseteq X$ open, then for any $P \in U$, there is some $V \subseteq U$ containing P and homogeneous polynomials $f, g \in k[x_0, \cdots, x_n]$ such that $f = g/h$ on U. It follows that $f|_{U\cap Y}$ is regular on $U \cap Y$ in Y, thus we get an element in $i_*\mathcal{O}_Y(U)$. Furthermore, $s \in \mathcal{O}_X(U)$ is 0 in $i_*|O_Y(U)$ iff $s|_{U\cap Y}$ is 0, i.e. iff $s \in \mathscr{I}_Y(U)$, thus we obtain an isomorphism between $\mathcal{O}_X(U)/\mathscr{I}_Y(U)$ and $i_*\mathcal{O}_Y(U)$ for all U and thus $\mathcal{O}_X/\mathscr{I}_Y \cong i_*\mathcal{O}_Y$.

(c) We have that $\mathcal{O}_X/\mathscr{I}_Y \cong i_*\mathcal{O}_Y$ and $Y = \{P, Q\}$, thus $\mathcal{O}_Y = k$, the constant sheaf on Y, and hence for inclusions $j_P : P \to Y$ and $j_Q : Q \to Y$, we have that $O\_Y = (j_P)_*\mathcal{O}_P \oplus (j_Q)_*\mathcal{O}_Q$ and hence $i_*\mathcal{O}_Y = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$. The map on global sections is not surjective, since $\Gamma(X, \mathcal{O}_X) = k$ and maps into $\Gamma(X, \mathscr{F}) = k \oplus k$ by sending $x \mapsto (x, x)$ which is not surjective.

(d) For any $f \in \mathcal{O}(U)$, f is regular on U and therefore it is locally a quotient of homogeneous polynomials, thus it is an element of $\mathscr{K}(U)$. From the quotient presheaf, we obtain a map into $\bigoplus_{P\in X} i_P(I_P)$ given by mapping an element $f = g/h$ in a neighborhood of P to $\overline{g/h}$ in $\mathcal{O}_P$, then note that if $\overline{g/h} = 0$, then the kernel is exactly the locally rational functions on U such that $f_P \in \mathcal{O}_P$ for all $P \in U$, i.e. where $f = g/h$ locally with $(g/h)_P \in \mathcal{O}_P = k[x, y]_{(\mathfrak{m}_P)}$, hence $h(P) \neq 0$ on a neighborhood of P and thus f is regular and hence $f \in \mathcal{O}(U)$. We then obtain a quotient map $\mathscr{K}/\mathcal{O}_Y \to \bigoplus_{P\in X} i_P(I_P)$. To show that it is surjective, looking at the stalk at P, we see that $\mathscr{K}_P = K$ and $(\bigoplus_{P\in X} i_P(I_P))_P = K/O_P$, thus the map is clearly surjective.

(e) We have that $\mathcal{O}(X) = k$ and $\mathscr{K}/\mathcal{O}(X) = \bigoplus_{P\in X} K/\mathcal{O}_P$ and $\mathscr{K}(X) = K$. We need to show that $K \to \bigoplus_{P\in K} K/\mathcal{O}_P$ is surjective. For any $s_1, \cdots, s_n$ with $s_i \in K/\mathcal{O}_{P_i}$, we have that $s_i = f_i/g_i + O_{P_i}$. Let $h_i \in K$ be 0 at P_i and non-zero at all other P_j, then let $h_i = \prod_{j\neq i} 1/h_j$, then we have $\sum_{i=1}^n f_i h_i/g_i$ which maps to s_i at each P.

II.1.22

Glueing Sheaves. Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ such that $phi_{ii} = \mathrm{id}$, and for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by glueing the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Proof:

Let $\mathscr{F}(U) \subseteq \prod_{i\in I} \mathscr{F}_i(U_i \cap U)$ with $(s_i)_{i\in I} \in \mathscr{F}(U)$ if each i, j $\varphi_{ij}(U \cap U_i \cap U_j)(s_i|_{U\cap U_i\cap U_j}) = s_j|_{U\cap U_i\cap U_j}$. Let $V = \bigcup V_j$, then given $(s_{ji})_{i\in I} \in \mathscr{F}(V_i)$ with $(s_{ji})|_{V_k\cap V_j} = (s_{ki})|_{V_k\cap V_j}$, then for any i, we have that $s_{ji}|_{V_k\cap V_j\cap U_i} = s_{ki}|_{V_k\cap V_j\cap U_i}$, hence they glue to some s_i which restricts properly. We check that $(s_i)_{i\in I} \in \mathscr{F}(U)$. For any i, j, k, we have

$$\begin{aligned}
\varphi_{ij}(U \cap U_i \cap U_j)(s_i|_{U\cap U_i\cap U_j})|_{U_i\cap U_j\cap V_k} &= \varphi_{ij}(V_k \cap U_i \cap U_j)(s_i|_{V_k\cap U_i\cap U_j}) \\
&= \varphi_{ij}(V_k \cap U_i \cap U_j)(s_{ik}) \\
&= s_{jk}|_{V_k\cap U_i\cap U_j} \\
&= s_j|_{V_k\cap U_i\cap U_j}
\end{aligned}$$

therefore $(s_i)_{i\in I} \in \mathscr{F}(s)$ and hence $\mathscr{F}$ is a sheaf. The projection map $\pi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$ is the desired isomorphism and $\varphi_{ij}(U \cap U_i \cap U_j)(s_i|_{U\cap U_i\cap U_j}) = s_j|_{U\cap U_i\cap U_j}$ guarantees that $\pi_j = \varphi_{ij} \circ \pi_i$. If we have another such sheaf $\mathscr{G}$, then we obtain a map from $\mathscr{G} \to \iota_*\mathscr{F} \to \iota_*\mathscr{F}_i$ for each i where ι is the inclusion of U_i into X. We therefore get a map into the product, $\chi : \mathscr{G} \to \prod \mathscr{F}_i$ and since $\psi_j = \varphi_{ij} \circ \psi_i$, then the image must satisfy the same requirements as those given for $\mathscr{F}(U)$ and since $\mathscr{G}$ is a sheaf, it will be surjective onto the image, and thus $\mathscr{G} \cong \mathscr{F}$.

II.2.1

Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

Proof:

Let $\varphi : D(f) \to \operatorname{Spec} A_f$ be given by sending $\mathfrak{p}$ to the localization $\mathfrak{p}A_f$. This is a bijection and furthermore, for any $\mathfrak{a} \subseteq A$, we have that $\mathfrak{p} \supseteq \mathfrak{a}$ iff $\mathfrak{p}A_f \supseteq \mathfrak{a}A_f$ for $\mathfrak{p}$ not containing f. It follows that φ yields a homeomorphism of topological spaces. For any $\mathfrak{p} \in D(f)$, we have that $(\mathcal{O}_X|_{D(f)})_\mathfrak{p} = A_\mathfrak{p}$ and $\mathcal{O}_{\operatorname{Spec} A,\varphi(\mathfrak{p})} = (A_f)_{\mathfrak{p}A_f}$. There is an isomorphism $\varphi^\#_P$ from $A_\mathfrak{p} \to (A_f)_{\mathfrak{p}A_f}$ which takes $\frac{g}{q}$ and maps it to $\frac{\left(\frac{g}{1}\right)}{q}$. It is clearly well-defined and surjective. To show that it is injective, suppose that $\frac{\left(\frac{g}{1}\right)}{q} = 0$, then $x\frac{g}{1} = 0$ for some $x \notin \mathfrak{p}A_f$. Let $x = \frac{a}{f^n}$ with $a \notin \mathfrak{p}$, then $\frac{ax}{f^n} = 0$ in A_f and therefore $f^m ag = 0$ for some m. Note that $f \notin \mathfrak{p}$ and $a \notin \mathfrak{p}$, thus $f^m a \notin \mathfrak{p}$ and hence $g = 0$ in $A_\mathfrak{p}$. It follows that the map is injective and thus an isomorphism. Given $U \subseteq D(f)$ and $s \in \mathcal{O}_X(U)$, then for any $P \in U$, there is a neighborhood V of P on which $s = \frac{g}{h}$ with $g, h \in A$ and $V \subseteq D(h)$. We may then map this through $\varphi^\#_P$ to obtain the map $\varphi^\#(U)$. $\varphi^\#$ is an isomorphism on stalks, and thus an isomorphism.

II.2.2

Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

Proof:

Let $P \in U$, then there is an affine neighborhood V with $\varphi : V \xrightarrow{\sim} \operatorname{Spec} A$ of P in X, and $U \cap V$ is open neighborhood of P in U. Since $U \cap V$ is open in V, then $\varphi(U \cap V)$ is open in $\operatorname{Spec} A$, thus there is some $D(f) \subseteq \operatorname{Spec} A$ containing $f(P)$ contained in $\varphi(U \cap V)$, then $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong \operatorname{Spec} A_f$.

Therefore $\varphi^{-1}(D(f))$ is an open neighborhood with $(\varphi^{-1}(D(f)), \mathcal{O}_X|_{\varphi^{-1}(D(f))}) = (\varphi^{-1}(D(f)), (\mathcal{O}_X|_V)|_{\varphi^{-1}(D(f))})$ which under the isomorphism φ corresponds exactly to $(D(f), \mathcal{O}_{\text{Spec } A}|_{D(f)})$ which is affine.

II.2.3

Reduced Schemes. A scheme $(X, \mathcal{O}_X)$ is reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.
(a) Show that $(X, \mathcal{O}_X)$ is reduced iff for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.
(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \to \mathcal{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its nilradical. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. We call it the *reduced scheme* associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.
(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof:

(a) Suppose that $(X, \mathcal{O}_X)$ is reduced. Let $[U, s] \in \mathcal{O}_{X,P}$, and suppose that $[U, s^n] = 0$, then there is an open set V such that $s^n|_V = 0$, thus $s|_V^n = 0$ in $\mathcal{O}_X(V)$, thus $s|_V = 0$ and hence $[U, s] = 0$. Suppose that all stalks are reduced, and let $s \in \mathcal{O}_X(U)$ be nilpotent, then $s^n = 0$ and for all $P \in U$, $s_P^n = 0$, so $s_P = 0$, hence $s = 0$.

(b) Let X be covered by $U_i \cong \text{Spec } A_i$, then for each $\text{Spec } A_i$, the reduced scheme is $\text{Spec } A_{\text{red}}$. The homeomorphism from $\text{Spec } A_{\text{red}} \to \text{Spec } A$ is given by the quotient map $q^\# : A \to A/\mathfrak{N}(A)$. This gives a homeomorphism since the quotient map is exactly the inclusion preserving correspondence of prime ideals containing the nilradical and prime ideals in the quotient, however all primes contain the nilradical.We have that $(q_* \mathcal{O}_{\text{Spec } A_{\text{red}}})_{q(\mathfrak{p})} = (A_{\text{red}})_{q^\#(\mathfrak{p})}$. Furthermore, note that $(A_{\text{red}})_{q^\#(\mathfrak{p})} = A_\mathfrak{p}/(\mathfrak{N}(A)A_\mathfrak{p})$. Note that for $x\frac{a}{b} \in \mathfrak{N}(A)A_\mathfrak{p}$, we have that there is some n such that $x^n = 0$ and thus $x^n \frac{a^n}{b^n} = 0$, hence $\mathfrak{N}(A)A_\mathfrak{p} \subseteq \mathfrak{N}(A_\mathfrak{p})$. Furthermore, for any $\frac{a}{b} \in \mathfrak{N}(A_\mathfrak{p})$, we have some n such that $\frac{a^n}{b^n} = 0$ and hence $sa^n = 0$ for some $s \in A - \mathfrak{p}$. It follows that sa is nilpotent in A, and thus $sa\frac{1}{sb} = \frac{a}{b} \in \mathfrak{N}(A)A_\mathfrak{p}$ and hence $\mathfrak{N}(A)A_\mathfrak{p} = \mathfrak{N}(A_\mathfrak{p})$. We then have that $(A_{\text{red}})_{q^\#(\mathfrak{p})} = (A_\mathfrak{p})_{\text{red}}$.

We now check that $\text{Spec } A_{\text{red}}$ is the reduced scheme of $\text{Spec } A$. Let $X = (\text{Spec } A)_{\text{red}}$, then define a map φ from $\text{Spec } A_{\text{red}}$ to X which on the level of points is the map q^{-1} and $\varphi^\# : (\mathcal{O}_{\text{Spec } A})_{\text{red}} \to q_*^{-1}\mathcal{O}_{\text{Spec } A_{\text{red}}}$ given by the sheafification of the quotient of the following map: for $s \in \mathcal{O}_{\text{Spec } A}(U)$ and any $P \in U$, we have some $V \subseteq U$ containing P and elements $f, g \in A$ with $V \subseteq D(g)$ and $s = \frac{f}{g} \in \mathcal{O}_{\text{Spec } A, \mathfrak{q}} = A_\mathfrak{q}$ for all $\mathfrak{q} \in V$, then map s to the section which is given locally by $\overline{\frac{f}{g}} \in (A_\mathfrak{q})_{\text{red}} = \mathcal{O}_{\text{Spec } A_{\text{red}}, q(\mathfrak{q})}$. Note that if $s \in \mathfrak{N}(\mathcal{O}_{\text{Spec } A}(U))$, then $\frac{f^n}{g^n} = 0$ in $A_\mathfrak{q}$ and thus $\overline{\frac{f}{g}} = 0 \in (A_\mathfrak{q})_{\text{red}}$. It follows that this map descends to a map on $(\mathcal{O}_{\text{Spec } A}(U))_{\text{red}}$. Given $W \subseteq U$, the restriction $s|_W$ does not change the local definition of the map and thus $\varphi^\#$ is a morphism of presheaves. $\varphi^\#$ then sheafifies to give a morphism of sheaves. Given a direct system (A_i, φ_{ij}), with direct limit A and maps $\iota_i' : A_i \to A$ along with maps $\psi_i : (A_i)_{\text{red}} \to R$ commuting with $\overline{\varphi}_{ij} : (A_i)_{\text{red}} \to (A_j)_{\text{red}}$, then we obtain maps $\psi_i' : A_i \to R$ obtained by composing with the quotient. For $i \leqslant j$,

$$\psi_j' \circ \varphi_{ij} \circ q_i = \psi_j \circ q_j \circ \varphi_{ij} \circ q_i = \psi_j \circ \overline{\varphi}_{ij} = \psi_i = \psi_i' \circ q_i$$

Since q_i is surjective, we have that $\psi_j' \circ \varphi_{ij} = \psi_i'$. For any i, let $\iota_i : (A_i)_{\text{red}} \to A_{\text{red}}$ be the quotient of ι, i.e. $q \circ \iota_i' = \iota_i \circ q_i$, then for $i \leqslant j$,

$$\iota_j \circ \overline{\varphi}_{ij} \circ q_i = \iota_j \circ q_j \circ \varphi_{ij} = q \circ \iota_j' \circ \varphi_{ij} = q \circ \iota_i' = \iota_i \circ q_i$$

Since q_i is surjective, $\iota_j \circ \overline{\varphi}_{ij} = \iota_i$. We obtain a unique $\psi' : A \to R$. If $\iota_i(s) \in \mathfrak{N}(A)$, then $\iota_i(s^n) = 0 \in A$ and thus there is some $j \geqslant i$ where $\varphi_{ij}(s)^n = 0$ in A_j, hence $\varphi_{ij}(s) \in \mathfrak{N}(A_j)$. For any $s \in \mathfrak{N}(A)$ with $s = \iota_j(t)$, we have that $\psi'(s) = \psi'(\iota_j(t)) = \psi'_j(t) = 0$ since $t \in \mathfrak{N}(A_j)$. $\psi' : A \to R$ descends to a map $\psi : A_{\text{red}} \to R$ with $\psi' = \psi \circ q$. For and i, we have that $\psi \circ \iota_i \circ q_i = \psi \circ q \circ \iota'_i = \psi' \circ \iota'_i = \psi'_i = \psi_i \circ q_i$, thus $\psi \circ \iota_i = \psi_i$ and hence A_{red} is the direct limit of the reduced directed system. It follows that the stalks of the reduced sheave are the reduction of the stalks of the original sheaf. On stalks, $\varphi_P^{\#}$ induces the map $[U, \overline{s}] \mapsto [V, \overline{\frac{f}{g}}]$ where s is locally $\frac{f}{g}$ around P and thus induces the identity morphism.

Let $\mathscr{F}$ be a sheaf on X and let $U \subseteq X$, then we show that $(\mathscr{F}|_U)_{\text{red}} = \mathscr{F}_{\text{red}}|_U$. Let θ be the sheafification map for $\mathscr{F}_{\text{red}}|_U$. Let $V \subseteq U$ and $s \in \mathscr{F}(V)$, then we map $s \mapsto \theta(V)(s)$. The sheafification of this map then defines a morphism from $(\mathscr{F}|_U)_{\text{red}} \to \mathscr{F}_{\text{red}}|_U$. Note that the stalks of both sheaves are $(\mathscr{F}_P)_{\text{red}}$ and that θ induces the identity map, and thus this is an isomorphism. Now suppose we have a scheme $(X, \mathcal{O}_X)$, with $X = \bigcup U_i$ and $U_i \cong \text{Spec } A_i$, then $((\mathcal{O}_X)_{\text{red}})|_{U_i} = (\mathcal{O}_X|_{U_i})_{\text{red}}$ and as we have seen $(U_i, (\mathcal{O}_X|_{U_i})_{\text{red}}) \cong \text{Spec } (A_i)_{\text{red}}$. From each U_i, we obtain a map $\varphi_i : (U_i)_{\text{red}} \to (\text{Spec } A_i)_{\text{red}} \to \text{Spec } (A_i)_{\text{red}} \to \text{Spec } A_i \to U_i$. We want to show that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. First note that if you have a morphism $\psi : X \to Y$, then you obtain a morphism ψred : $X_{\text{red}} \to Y_{\text{red}}$ given by the sheafification of $\psi_{\text{red}}(U) : (\mathcal{O}_X(U))_{\text{red}} \to ((\varphi_* \mathcal{O}_Y)(U))_{\text{red}}$, the reduction of the map $\psi(U)$. Let $\psi_i : U_i \to \text{Spec } A_i$ be an isomorphism. Let $P \subseteq U_i \cap U_j$, then we want to check that $\varphi_{i,P} = \varphi_{j,P}$. We have that $\varphi_{i,P}^{\#} : \mathcal{O}_{X,P} \to (A_i)_{\mathfrak{p}} \to ((A_i)_{\mathfrak{p}})_{\text{red}} \to ((A_i)_{\mathfrak{p}})_{\text{red}} \to (\mathcal{O}_{X,P})_{\text{red}}$. This is then a composition of $\psi_{i,P}^{\#} \circ q \circ (\psi_{i,P}^{\#})_{\text{red}}^{-1}$ which is just the reduction map on $\mathcal{O}_{X,P}$. It follows that all of the maps of stalks are the same, thus they glue to a morphism from $X_{\text{red}} \to X$.

(c) $X = X_{\text{red}}$ since X is reduced. As noted above, we obtain a map $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ by the sheafification of the reduced maps. Composing f_{red} with the map $\psi : Y_{\text{red}} \to Y$, we get morphisms on stalks given by $(f_{\text{red}}^{\#} \circ \psi^{\#})_P = f_{\text{red},P}^{\#} \circ \psi_{f(P)}^{\#} : \mathcal{O}_{Y,f(P)} \to f_* \mathcal{O}_{X,P}$, which is just $f_P^{\#}$. The uniqueness of f_{red} follows from the uniqueness of the induced quotient map.

II.2.4

Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \text{Spec } A$, we have an associated map on sheaves $f^{\#} : \mathcal{O}_{\text{Spec } A} \to f_* \mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \text{Hom}_{\mathfrak{Sch}}(X, \text{Spec } A) \to \text{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

Show that α is bijective.

Proof:

We want to construct an inverse to α. Let $X = \bigcup U_i$ with U_i open affines. A homomorphism $f : A \to \mathcal{O}_X(X)$ induces homomorphisms $\rho_i \circ f^{\#} : A \to \mathcal{O}_X(U_i)$. These induce unique morphisms f_i : Spec $\mathcal{O}_X(U_i) \to \text{Spec } A$. Since U_i are open affine, there are isomorphisms $\varphi_i : U_i \to \text{Spec } \mathcal{O}_X(U_i)$, then we obtain morphisms $h_i = f_i \circ \varphi_i : U_i \to \text{Spec } A$. Given U, V affine open subsets of X, we have the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{O}_X(V) \arrow[rr, "-_{\varphi_V(P)}"'] \arrow[dr, "-_P"] & & \mathcal{O}_X(V)_{\varphi_V(P)} \arrow[dl, "\varphi_{V,P}"'] \\
\mathcal{O}_X(X) \arrow[u, "\rho_{XV}"] \arrow[r, "-_P"] \arrow[d, "\rho_{XU}"'] & \mathcal{O}_{X,P} & \\
\mathcal{O}_X(U) \arrow[rr, "\varphi_{U,P}"'] \arrow[ur, "-_P"'] & & \mathcal{O}_X(U)_{\varphi_U(P)} \arrow[ul, "\varphi_{U,P}"']
\end{tikzcd}
$$

Where $\varphi_V : V \to \text{Spec } \mathcal{O}_X(V)$ and $\varphi_U : U \to \text{Spec } \mathcal{O}_X(U)$ are isomorphisms. $\varphi_{V,P} : \mathcal{O}_X(V)_{\varphi_V(P)} \to \mathcal{O}_{X,P}$ is a morphism of local rings, thus $\varphi_V(P) = \varphi_{V,P}^{-1}(\mathfrak{m}_P) \cap \mathcal{O}_X(V) = (\varphi_{V,P} \circ -_{\varphi_V(P)})^{-1}(\mathfrak{m}_P)$ where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{X,P}$. It follows that $\rho_{XV}^{-1}(\varphi_V(P)) = (\varphi_{V,P} \circ -_{\varphi_V(P)} \circ \rho_{XV})^{-1}(\mathfrak{m}_P)$. From the above diagram, this is the same as $(\varphi_{U,P} \circ -_{\varphi_U(P)} \circ \rho_{XU})^{-1}(\mathfrak{m}_P) = \rho_{XU}^{-1}(\varphi_U(P))$. Given $P \in U_i \cap U_j$, $h_i(P) = f_i(\varphi_i(P)) = (f^\#)^{-1}(\rho_i^{-1}(\varphi_i(P))) = (f^\#)^{-1}(\rho_j^{-1}(\varphi_j(P))) = h_j(P)$ and hence the h_i's glue as morphisms of topological spaces. Furthermore, $(h_i)_P^\# = (\varphi_i)_P \circ (f_i^\#)_{\varphi_i(P)} \circ (-_{h(P)})$, then notice that this is just $-_P \circ f$, thus the morphisms on stalks are the same.

Let X be a scheme and $X = \bigcup U_i$ and let $\varphi_i : U_i \to Y$ be morphisms of schemes. Let $\rho : \mathcal{O}_X \to i_* \mathcal{O}_X|_{U_i}$ with $i : U \to X$ the inclusion be given by $\rho(V) = \rho_{V,V \cap U}$. For any morphism $\varphi : X \to Y$, we then get a morphism $\varphi|_{U_i}$ given by $(\varphi|_{U_i})^\# = \varphi_* \rho \circ \varphi^\#$. Suppose that for any $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there exists a morphism $\varphi : X \to Y$ such that $\varphi|_{U_i} : U_i \to Y$ is exactly φ_i. Furthermore, $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ holds exactly when $(\varphi_i)_P^\# = (\varphi_j)_P^\#$ for all $P \in U_i \cap U_j$. The second fact follows from the adjunction in II.1.19 and noting the fact that $\varinjlim_{P \in V} \varinjlim_{\varphi(V) \subseteq U} \mathcal{O}_Y(U)$ is just $\mathcal{O}_{Y,P}$ (for any open set U containing $\varphi(P)$, $\varphi^{-1}(U)$ contains P, thus the double direct limit becomes a direct limit). $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ tells us that the topological maps glue to a map $\varphi : X \to Y$. For any $s \in \mathcal{O}_Y(U)$, we get elements $t_i = \varphi_i^\#(U)(s) \in \mathcal{O}_X(\varphi^{-1}(U) \cap U_i)$ noting that $\varphi_i^{-1}(U) = \varphi^{-1}(U) \cap U_i$. The fact that $t_i|_{\varphi^{-1}(U) \cap U_i \cap U_j} = t_j|_{\varphi^{-1}(U) \cap U_i \cap U_j}$ follows immediately from the definition of restriction and that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. It follows that the t_i glue to give an element $t \in \mathcal{O}_X(\varphi^{-1}(U)) = (\varphi_* \mathcal{O}_X)(U)$. Finally,

$$
\begin{aligned}
(\varphi|_{U_i})(V) &= \rho(\varphi^{-1}(V)) \circ \varphi^\#(V) \\
&= \varphi^\#(V)|_{\varphi^{-1}(V) \cap U_i} \\
&= \varphi_i(V)
\end{aligned}
$$

It follows form the what we have established, that the morphisms h_i glue to a morphism $h : X \to \text{Spec } A$. To check that this is the inverse of α, let $f : A \to \mathcal{O}_X(X)$ induce the morphism $h : X \to \text{Spec } A$, with $h|_{U_i} = h_i$, then let $Y = \text{Spec } A$, and thus $h^\#(Y)|_{U_i} = \rho(h^{-1}(Y)) \circ h^\#(Y) = (h|_{U_i})^\#(Y) = (h_i)^\#(Y) = \rho_i \circ f$ and hence by the uniqueness of glueing, $h^\#(Y) = f$. Conversely, given $h : X \to \text{Spec } A$, then we get $f = h(Y) : A \to \mathcal{O}_X(X)$. We then have that $h|_{U_i} : U_i \to \text{Spec } A$ is a morphism of affine schemes and $\rho_i \circ f : A \to \mathcal{O}_X(U_i)$. Taking global sections, we get that $(h|_{U_i})^\#(Y) = h^\#(Y)|_{U_i} = \rho_i \circ f$. Therefore, $h|_{U_i} = h_i$ and thus h exactly what we obtain through the above construction.

II.2.5

Describe Spec $\mathbb{Z}$, and show that it is a final object for the category of schemes, i.e. each scheme X admits a unique morphism to Spec $\mathbb{Z}$.

Proof:

Spec $\mathbb{Z}$ consists of one closed point for each prime number p and a generic point (0). The topology is the cofinite topology. Furthermore, any open set is of the form $D(n)$ for some $n \in \mathbb{Z}$ with $D(n)$ being the complement of the prime divisors of n. The functions on $D(n)$ are all numbers with powers of n in the denominator. Spec $\mathbb{Z}$ is a final object since $\mathbb{Z}$ is initial in the category of unital rings.

II.2.6

Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes.

Proof:

The spectrum of the zero ring is empty since the zero ring has no prime ideals. It is initial in the category of schemes since the only morphism out of it is the trivial map topologically and the zero map on sheaves.

II.2.7

Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof:

Let $f : \text{Spec } K \to X$ be a morphism, then Spec K has a single point (0) which maps to some point $x \in X$, then we obtain a map $f^{\#}_{(0)} : \mathcal{O}_x \to \mathcal{O}_{\text{Spec } K,(0)}$, however $\mathcal{O}_{\text{Spec } K,(0)} = K_{(0)} = K$. Furthermore, we have that $\ker(f^{\#}_{(0)}) = (f^{\#}_{(0)})^{-1}(0) = \mathfrak{m}_x$ and hence we obtain an injection $\overline{f^{\#}_{(0)}} : k(x) \to K$. Conversely, given a point x and an inclusion $i : k(x) \to K$, we obtain a morphism Spec $K \to X$ by sending φ mapping (0) to x and the sheaf morphism is given by the map corresponding to $f : \varphi^{-1}\mathcal{O}_X \to \text{Spec } K$ which has only two morphisms $f(\varnothing) = 0$ and $f(\text{Spec } K) : \mathcal{O}_x \to K$ with $f(\text{Spec } K) = i \circ q$ where q is the quotient map $q : \mathcal{O}_x \to k(x)$. We then get a morphism of sheaves $\widetilde{f} : \mathcal{O}_X \to \varphi_* \text{Spec } K$ with $\widetilde{f}_{(0)} = f_x = i \circ q$ which then descends to i under the quotient map.

II.2.8

Let X be a scheme. For any point $x \in K$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)-$vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of Spec $k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e. such that $k(x) = k$), and an element of T_x.

Proof:

Let f : Spec $k[\epsilon]/\epsilon^2 \to X$ be a k-morphism, then Spec $k[\epsilon]/\epsilon^2$ contains only a single point (ϵ) which must map to a point $f((\epsilon)) = x \in X$. We have morphisms φ : Spec $k[\epsilon]/\epsilon^2 \to$ Spec k and

$\psi : X \to \text{Spec } k$ such that $\psi \circ f = \varphi$, then taking the stalk at $y = (\epsilon)$, we get $\varphi_y^\# = f_y^\# \circ \psi_x^\#$. Let $q : \mathcal{O}_{X,x} \to k(x)$ be the quotient map, then since $(f_y^\#)^{-1}(y) = \mathfrak{m}_x$, then we obtain a morphism from the quotient, $\overline{f_y^\#} : k(x) \to (k[\epsilon]/\epsilon^2)_{(\epsilon)} = k[\epsilon]/\epsilon^2$ and furthermore, $\text{im}(\overline{f_y^\#})$ contains k since $f_y^\# \circ \psi_x^\# = \varphi_y$ which is just the inclusion from k into $k[\epsilon]/\epsilon^2$. Since $\overline{f_y^\#}$ is injective ($k(x)$ is a field) and contains k, then it must be exactly k, thus $k(x) = k$. Additionally, we obtain an element of T_x by noting that $f_y^\#$ induces a morphism $\mathfrak{m}_y \to k[\epsilon]/\epsilon^2$ and since $f_y^\#(\mathfrak{m}_x) \subseteq (\epsilon)$, then $f^\#(\mathfrak{m}_x^2) = 0$, thus we get a morphism $\overline{f_y^\#} : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ (this is actually to $(\epsilon) \cong k$ as k-modules). It follows that $\overline{f_y^\#}$ is the dual of the element we want in T_x.

Conversely, given a point $x \in X$ rational over k, and an element $z \in T_x$, then we construct the desired morphism $\varphi : \text{Spec } k[\epsilon]/\epsilon^2 \to X$ by sending $(\epsilon) = y$ to x. We then define $\varphi^\#$ by the adjunction of the morphism $\varphi^{-1}\mathcal{O}_X \to \mathcal{O}_{\text{Spec } k[\epsilon]/\epsilon^2}$ given by the global section $s : \mathcal{O}_{X,y} \to k[\epsilon]/\epsilon^2$ by mapping $f \in \mathcal{O}_{X,y}$ to $f(0) + z^*(f - f(0))\epsilon$ where $f(0)$ is the inclusion of the element $\overline{f} \in \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$ into $\mathcal{O}_{X,x}$ from the Spec k structure on X. For any $f, g \in \mathcal{O}_{X,x}$, we have

$$\begin{aligned}
z^*(fg - f(0)g(0)) &= z^*(fg - fg(0) + fg(0) - f(0)g(0)) \\
&= z^*(f(g - g(0))) + z^*(f - f(0))g(0) \\
&= z^*((f - f + f(0))(g - g(0))) + z^*(f - f(0))g(0) \\
&= z^*(g - g(0))f(0) + z^*(f - f(0))g(0)
\end{aligned}$$

Therefore z^* is a derivation as expected. Now given $f, g \in \mathcal{O}_{X,x}$, we have that

$$\begin{aligned}
s(fg) &= f(0)g(0) + (z^*(g - g(0))f(0) + z^*(f - f(0))g(0))\epsilon \\
&= f(0)g(0) + (z^*(g - g(0))f(0) + z^*(f - f(0))g(0))\epsilon + z^*(f - f(0))z^*(g - g(0))\epsilon^2 \\
&= (f(0) + z^*(f - f(0))\epsilon) \cdot (g(0) + z^*(g - g(0))) \\
&= s(f)s(g)
\end{aligned}$$

Therefore s defines the global section and the desired scheme morphism.

II.2.9

If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \text{cl}\{\zeta\}$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof:

Let Z be a nonempty irreducible closed subset of X and let $P \in Z$, then there is an affine open neighborhood U of P, and $U \cap Z$ is a closed subset of U. Let $U \cong \text{Spec } R$, then $U \cap Z$ corresponds under this isomorphism to a closed irreducible subset of Spec R and therefore corresponds to $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ in R. Since $\text{cl}\{\mathfrak{p}\} = Z \cap U$ in Spec R, then let $\mathfrak{p}$ correspond to the point $\zeta \in X$, then $\text{cl}\zeta \subseteq Z$ and furthermore, it contains a non-empty open subset of Z which is irreducible and therefore $\text{cl}\{\zeta\} = Z$. Suppose that there were two generic points of Z, ζ and ζ', then let U be an affine open subset of X containing ζ. If $\zeta' \notin U$, then U^c is a closed subset containing ζ' and not ζ, but $\text{cl}\{\zeta\} = Z$, so $\zeta \in Z$ and hence $\zeta \in \text{cl}\{\zeta'\}$, therefore U^c cannot contain ζ' or else, $\zeta \notin \text{cl}\{\zeta'\}$. It follows that ζ and ζ' are both in U, then they correspond to prime ideals $\mathfrak{p}, \mathfrak{q}$ in U such that $V(\mathfrak{p}) = V(\mathfrak{q})$ and thus $\sqrt{\mathfrak{p}} = \sqrt{\mathfrak{q}}$ and hence $\mathfrak{p} = \mathfrak{q}$, thus $\zeta = \zeta'$.

II.2.10

Describe Spec $\mathbb{R}[x]$. How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?

Proof:

The topological space will consist of all linear polynomials and irreducible quadratics which correspond to conjugate pairs. This means that the topological space of Spec $\mathbb{R}[x]$ is a quotient of the complex plane identifying z and $\overline{z}$. The topology is cofinite and thus open sets are all of the form $D(f)$ for some polynomial f and hence the sheaf is simple.

II.2.11

Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe Spec $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof:

Spec $k[x]$ is all irreducible polynomials over $\mathbb{F}_p$. The topology is cofinite, and thus the sheaf is simple like $\mathbb{Z}$ and $\mathbb{R}[x]$. The residue field at a polynomial of degree n will be $\mathbb{F}_{p^n}$. The number of such points is a classic result and is given by $\frac{1}{n}(\mu * p^x)(n)$

II.2.12

Glueing Lemma. Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, let it have the induced scheme structure (II.2.2). Suppose also given for each $i \neq j$, an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each i, j $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ij} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto the open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by *glueing* the schemes X_i along the isomorphisms U_{ij}.

Proof:

Let $X = \coprod X_i / \sim$ where $x^{(i)} \sim y^{(j)}$ if $x^{(i)} \in U_{ij}$ and $\varphi_{ij}(x^{(i)}) = y^{(j)}$. We then have maps $\psi_i : X_i \to X$ given by mapping into the coproduct and then through the quotient map. Let $V \subseteq X$ be open, then define:

$$\mathcal{O}_X(V) := \left\{ (s_i) \in \prod \mathcal{O}_{X_i}(\psi_i^{-1}(V)) \middle| \forall i, j, \varphi_{ij}(s_i|_{\psi_i^{-1}(V) \cap U_{ij}}) = s_j|_{\psi_j^{-1}(V) \cap U_{ji}} \right\}$$

To check that this is a sheaf, let $V = \bigcup_{l \in I} V_l$, then let $(s_{il}) \in \mathcal{O}_X(V_l)$ s.t. $(s_{il})|_{V_l \cap V_m} = (s_{il}|_{\psi_i^{-1}(V_l \cap V_m)}) = (s_{im}|_{\psi_i^{-1}(V_l \cap V_m)}) = (s_{im})|_{V_l \cap V_m}$, then we may glue $(s_{il})_{l \in I}$ to give some $s_i \in \mathcal{O}_{X_i}(\psi_i^{-1}(V))$, then combining these we get the desired element $s \in \mathcal{O}_X(V)$. Furthermore, the condition on s to be in $\mathcal{O}_X(V)$ holds since it holds for the s_i. We now show that ψ_i is an isomorphism. Let $W_i = \psi_i(X_i)$, then we have that $\mathcal{O}_X|_{W_i}$ maps to $\mathcal{O}_{X_i}$ by sending $(s_j) \to s_i$ and the inverse map is sending $s \mapsto (s_j)$ where $s_j = \varphi_{ij}(s|_{\psi^{-1}(V) \cap U_{ij}})$. Therefore the $X_i \cong (W_i, \mathcal{O}_X|_{W_i})$, thus $(X, \mathcal{O}_X)$ is a scheme since W_i has a cover by affines. Properties 3 and 4 are topological and trivial.

II.2.13

A topological space is *quasi-compact* if every open cover has a finite subcover.
(a) Show that a topological is noetherian iff ever open subset is quasi-compact.
(b) If X is an affine scheme, show that $\text{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is *quasi-compact* if $\text{sp}(X)$ is.
(c) If A is a noetherian ring, show that $\text{sp}(\text{Spec } A)$ is a noetherian topological space.
(d) Give an example to show that $\text{sp}(\text{Spec } A)$ can be noetherian even when A is not.

Proof:

(a) We show in II.1.11 that noetherian implies hereditarily compact. Suppose every open subset of X is compact, then let $Z_1 \supseteq Z_2 \supseteq \cdots$ be a descending chain of closed sets, then $Z_1^c \subseteq Z_2^c \subseteq \cdots$ is an ascending chain of open sets, whose union is open and compact and thus we need only finitely many of these sets, and hence some maximal one, thus the chain terminates.

(b) Suppose $\text{Spec } A = \bigcup U_i$, then let $U_i = \bigcup D(f_{ij})$, therefore we may assume WLOG, that $\text{Spec } A = \bigcup D(f_i)$. We have that $\bigcup D(f_i) = \bigcup V((f_i))^c = (\bigcap V((f_i)))^c = V(\sum (f_i))^c$. We then have that $V(\sum(f_i)) = \varnothing = V(1)$ thus $\sqrt{\sum(f_i)} = A$ and hence $\sum(f_i) = A$, thus there is a finite sum $\sum_{\text{fin}} a_i f_i = 1$ and hence $\text{Spec } A = \bigcup_{\text{fin}} D(f_i)$. Therefore $\text{Spec } A$ is quasi-compact. Let $A = \mathbb{Q}[x_1, x_2, \cdots]$, then consider the descending chain of closed subsets $V(x_1) \supseteq V(x_1, x_2) \supseteq V(x_1, x_2, x_3) \cdots$. This chain does not terminate, thus $\text{Spec } \mathbb{Q}[x_1, x_2, \cdots]$ is not noetherian.

(c) Let $Z_1 \supseteq Z_2 \supseteq \cdots$ be a descending chain of closed subsets, then $Z_i = V(I_i)$ for some ideal I_i, thus we obtain an ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ which must terminate since A is noetherian, thus $\text{Spec } A$ is noetherian.

(d) Let $A = \mathbb{Q}[x_1, x_2, \cdots]/I$ where $I = (x_1, x_2, \cdots)^2$, then A is a 1-point space with prime ideal I, however, we have $(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq \cdots$ as an infinite ascending chain of ideals of A.

II.2.14

(a) Let S be a graded ring. Show that $\text{Proj } S \neq \varnothing$ iff every element of S_+ is nilpotent.
(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \text{Proj } T | \mathfrak{p} \nsupseteq \varphi(S_+)\}$. Show that U is an open subset of $\text{Proj } T$, and show that φ determines a natural morphism $f : U \to \text{Proj } S$.
(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \text{Proj } T$ and the morphism $f : \text{Proj } T \to \text{Proj } S$ is an isomorphism.
(d) Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong \text{Proj } S$

Proof:

(a) If $\text{Proj } S = \varnothing$, then for any $f \in S_+$, $D_+(f) \cong \text{Spec } S_{(f)}$ is empty, and thus $\text{Spec } S_{(f)}$ contains no points, hence $S_{(f)}$ contains no prime ideals which means $S_{(f)} = 0$, however, $1 \in S_0$ and hence $1/1 \in S_{(f)}$ and is equal to 0, hence $f^n = 0$ for some n, and thus f is nilpotent. If $S_+ \subseteq \sqrt{0}$, then all primes contain $\sqrt{0}$, thus $\text{Proj } S$ is empty.

(b) $U = \{\mathfrak{p} \in \text{Proj } T | \mathfrak{p} \supseteq \varphi(S_+)\}^C$, then note that $\mathfrak{p} \supseteq \varphi(S_+)$ iff $\mathfrak{p} \supseteq (\varphi(S_+))$, thus $U^C = V((\varphi(S_+)))$ is closed. Consider $f : U \to \text{Proj } S$ given by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. This is a well-defined map since $\varphi^{-1}(\mathfrak{p})$ is prime and if $f = \sum f_i$ in S and $f \in \varphi^{-1}(\mathfrak{p})$, then $\varphi(f) = \sum \varphi(f_i)$ and since $\mathfrak{p}$ is homogeneous, then $\varphi(f_i) \in \mathfrak{p}$ for all i, thus $f_i \in \varphi^{-1}(\mathfrak{p})$. We also have that $f^{-1}(V(I)) = V(\varphi(I))$

since

$$\begin{aligned} f^{-1}(V(I)) &= \{\mathfrak{p} \in U \big| f(\mathfrak{p}) \supseteq I\} \\ &= \{\mathfrak{p} \in U \big| \varphi^{-1}(\mathfrak{p}) \supseteq I\} \\ &= \{\mathfrak{p} \in U \big| \mathfrak{p} \supseteq \varphi(I)\} \\ &= V(\varphi(I)) \end{aligned}$$

thus f is continuous. Let $V \subseteq \text{Proj } S$, then we define $f^\# : \mathcal{O}_{\text{Proj } S}(V) \to (f_* \mathcal{O}_{\text{Proj } T}|_U)(V)$ by mapping $s \in \mathcal{O}_{\text{Proj } S}(V)$ which locally looks like g/h around P, and map it to $\varphi(g)/\varphi(h)$ which works since φ preserves degree, thus the map is well-defined. Note furthermore that if $P \subseteq W$, then $W \subseteq D(h)$ and $f^{-1}(W) \subseteq f^{-1}(D(h)) = D(\varphi(h))$, thus $1/\varphi(h)$ is well defined around $\varphi(P)$. We also check that this is a morphism of sheaves. Let $W \subseteq V$, then let $s \in \mathcal{O}_{\text{Proj } S}(V)$, then for any $P \in W$, $s = g/h$ locally, and $s|_W = g/h$ as well, hence $s|_W$ maps to the restriction of $f^\#(V)(s)$ as desired.

(c) We first check that $U = \text{Proj } T$. Suppose $\mathfrak{p} \in \text{Proj } T$ but $\mathfrak{p} \notin U$, then $\varphi(S_+) \subseteq \mathfrak{p}$. Let t be any homogeneous element in T_+, then there is an element $s \in S$ such that $\varphi(s) = t^{d_0}$, furthermore, since φ is degree preserving, we may take s to be homogeneous. This is because if $s = \sum_{i=1}^n s_i$ with $s_i \in S_i$, then $\varphi(s) = \sum_{i=1}^n \varphi(s_i)$, so all of these elements must be 0 except for the one with the same degree as t^{d_0}. Since $\varphi(S_+) \subseteq \mathfrak{p}$, then $\varphi(s) = t^{d_0} \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime and hence radical, $t \in \mathfrak{p}$. Therefore $T_+ \subseteq \mathfrak{p}$ and hence $\mathfrak{p} \notin \text{Proj } T$. It follows that $U = \text{Proj } T$. For any prime in Proj S, there is some $s \in S_+$ not contained in $\mathfrak{p}$ and thus there is some homogeneous component s' of s not contained in $\mathfrak{p}$. Therefore $\mathfrak{p} \not\supseteq (s')$ and hence $\mathfrak{p} \in D_+(s')$. It follows that $D_+(s)$ ranging over all homogeneous s of degree $\geqslant 1$ covers Proj S. Furthermore, note that $s \in \mathfrak{p}$ iff $s^n \in \mathfrak{p}$ for any $n > 0$ and prime $\mathfrak{p}$, thus $D_+(s^n) = D_+(s)$, hence $D_+(s)$ may range over all homogeneous elements s of degree $\geqslant d_0$. Furthermore, for any homogeneous element $t \in T_+$ of degree $\geqslant d_0$, there is some homogeneous $s \in S_+$ of degree $\geqslant d_0$ such that $\varphi(s) = t$, hence $f^{-1}(D_+(s)) = D_+(t)$. Since the $D_+(t)$ for all homogeneous elements of degree $\geqslant d_0$ cover Proj T, then the preimages of all $D_+(s)$ covers Proj T. Let $s \in S_+$ be homogeneous of degree $\geqslant d_0$, then $f^{-1}(D_+(s)) = D_+(\varphi(s))$, thus f restricts to a morphism from $D_+(\varphi(s))$ to $D_+(s)$. We know that $D_+(\varphi(s)) \cong \text{Spec } T_{(\varphi(s))}$ and that $D_+(s) \cong \text{Spec } S_{(s)}$. The morphism induced by $f|_{D_+(\varphi(s))} :\cong \text{Spec } T_{(\varphi(s))} \to \text{Spec } S_{(s)}$ is given by a ring map $\varphi' : S_{(s)} \to T_{(\varphi(s))}$ which takes an element a/s^n with a and $\deg(a) - \deg(s^n) = 0$, and mapping it to $\varphi(a)/\varphi(s)^n$. Suppose that $\varphi(a)/\varphi(s)^n = 0$, then $\varphi(as)/\varphi(s)^{n+1} = 0$, thus $\varphi(s)^k \varphi(as) = 0$ for some $k > 0$, hence $\varphi(as^{k+1}) = 0$. Since φ is an isomorphism on elements of degree $\geqslant d_0$ and $\deg(as^{k+1}) \geqslant \deg(s) \geqslant d_0$, then $as^{k+1} = 0$, thus $a/s^n = 0$. Given any element $t/\varphi(s)^n = t\varphi(s)/\varphi(s)^{n+1}$, the degree of $t\varphi(s)$ is $\geqslant d_0$, thus there is some $a \in S$ mapping to $t\varphi(s)$, and hence $\varphi(a/s^{n+1}) = t/\varphi(s)^n$, hence phi' is an isomorphism. Since f restricts to isomorphisms on open covers, then f is an isomorphism.

(d) Let V be a projective variety over an algebraically closed field k with homogeneous coordinate ring S, then we show that $t(V) \cong \text{Proj } S$. We first define a map $\beta : V \to \text{Proj } S$ by mapping a point $P \in V$ to the maximal ideal of functions vanishing at P. Note that I.3.4 tells us that this is has a contraction which is an ideal of $S(V)$ given by the homogeneous $f \in S(V)$ with $f(P) = 0$. The ideal is maximal and trivially homogeneous. Given two distinct points in V, there is a homogeneous f vanishing at one and not the other, thus this gives an injective map of V into Proj S. To show that this is continuous, let $D_+(h) \subseteq \text{Proj } S$, then $\beta^{-1}(D_+(h))$ is all points P in V such that $\mp_P$ does not h, i.e. all points for which h does not vanish, thus it is $V(h)$ complement and hence open. Furthermore, given any homogeneous maximal ideal $\mathfrak{m}$, $V(\mathfrak{m})$ in V is a quotient of the vanishing of a proper ideal in $k[x_0, \cdots, x_n]$ and thus cannot be empty, so we have a homeomorphism between V and the maximal ideals of Proj S. We now need to define a homomorphism $\mathcal{O}_{\text{Proj } S}(U) \to (\beta_* \mathcal{O}_V)(U) = \mathcal{O}_V(\beta^{-1}U)$. Given a section $s \in \mathcal{O}_{\text{Proj } S}(U)$ and a point $P \in \mathcal{O}_V(\beta^{-1}U)$, we define $s(P) = \overline{s} \in \mathcal{O}_{\text{Proj } T, \beta(P)}/\mathfrak{m}_{\beta(P)} \cong k$. To see that this is a regular function, notice that $s = \frac{g}{h}$ with g, h homogeneous in S, around $\beta(P)$, and then $s(P) = \overline{\frac{g_P}{h_P}} \in k$. Note that we may pick some x_i such that $P \in D_+(x_i) \subseteq V$, then $D_+(x_i)$ is affine with coordinate ring A, so we have that $\mathcal{O}_{\text{Proj }T, \beta(P)}/\mathfrak{m}_{\beta(P)} = A_{\mathfrak{m}}/\mathfrak{m}$ where $\mathfrak{m}$ is a point in affine space and thus actual evaluation, hence maximal ideals in the projective coordinate ring correspond to actual points and the map corresponds to actual evaluation of the homogeneous polynomials. It follows that we map s to a regular function. Furthermore, for any regular function, we can simply take the local ratios of polynomials to get a section back, thus $\mathcal{O}_{\text{Proj }S}(U) \cong \mathcal{O}_V(\beta^{-1}V)$. Finally, we get that the homogeneous prime ideals of S are in bijection with the irreducible closed subsets of V and the homogeneous elements of S vanishing on an irreducible closed subset vanish at all points contained therein. Therefore, the morphism β extends to all of $t(V)$ and then the isomorphism on the sheaves still holds, thus $t(V) \cong \text{Proj } S$.

II.2.15

(a) Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point iff its residue field is k.
(b) If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.
(c) Now show that if V, W are any two varieties over k, then the natural map

$$\text{Hom}_{\mathfrak{Var}}(V, W) \to \text{Hom}_{\mathfrak{Sch}/k}(t(V), t(W))$$

Proof:

(a) Let $P \in t(V)$, then let $P \subseteq U$ with U affine open in $t(V)$, then $P \sim \mathfrak{p}$ for some prime ideal of A with $U \cong \text{Spec } A$. The residue field of $\mathfrak{p}$ in Spec A is isomorphic to the residue field in $t(V)$. By the Nullstellensatz, we have that if $\mathfrak{p}$ is maximal, then $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$ is a finite extension of k and thus k since k is algebraically closed. We now show that if the transcendence degree of $\kappa(\mathfrak{p})$ over k is 0, then $\mathfrak{p}$ is maximal. Note that $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p} = (A/\mathfrak{p})_{(0)}$ and since V is a variety, then A is an integral domain which is a finitely generated k-algebra as is $A/\mathfrak{p}$ and the dimension of $A/\mathfrak{p}$ is $\dim A - \text{ht } \mathfrak{p}$, thus the transcendence degree of $\kappa(\mathfrak{p})$ which is the transcendence degree of $\text{Frac}(A/\mathfrak{p})$ is the dimension of $A/\mathfrak{p}$ which is $\dim A - \text{ht } \mathfrak{p}$, and thus it is 0 iff ht $\mathfrak{p} = 0$, i.e. when $\mathfrak{p}$ is maximal.

(b) $f_P^{\#} : \mathcal{O}_{Y, f(P)} \to \mathcal{O}_{X,P}$ is a morphism of k-algebras (since these are morphisms over k) as well as a local homomorphism, thus it induces a morphism $\overline{f_P^{\#}} : \kappa_Y(f(P)) \to \kappa_X(P)$ which is a morphism of k-algebras and since $\kappa_X(P) = k$, then $\kappa_Y(f(P)) = k$ as well.

(c) Suppose we have a morphism of varieties (locally ringed spaces), $f : (V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$, then we obtain a morphism $t(f) : t(V) \to t(W)$ as follows. For any irreducible subset $Y \subseteq V$, we map it to the closure of its image $\text{cl}(f(Y)) \subseteq W$. To show that $t(f)$ is continuous, let $Z \subseteq t(W)$ be closed, then Z is a finite union of irreducible subsets and the irreducible sets above them. WLOG, let $Z \subseteq t(W)$ be the irreducible closed subsets above a closed subset $Z' \subseteq W$, then $t(f)^{-1}(Z)$ is all of the irreducible closed subsets above $f^{-1}(Z')$, thus $t(f)$ is continuous. Since α_* induces a bijection on open subsets of V and $t(V)$, then we obtain a morphism $t(f)^{\#} = \alpha_* f^{\#}$ which gives morphisms of k-algebras, and thus a morphism of schemes over k. If we have $t(f) = t(g)$, then the sheaf maps of f and g are the same, and furthermore, for any $x \in V$, $t(f)(x) = \text{cl}\{f(x)\} = \text{cl}\{g(x)\} = t(g)(x)$. Since $f(x)$ an $g(x)$ are closed in W, then this give $f(x) = g(x)$. Now suppose we have a morphism of k-schemes, $\varphi : t(V) \to t(W)$, then φ induces a map on the closed points of $t(V)$ and $t(W)$ by (a) and (b). The closed points of $t(V)$ and $t(W)$ are exactly V and W, thus φ induces a map of topological spaces $\varphi' : V \to W$. For any open set $U \subseteq t(V)$, $\alpha^{-1}(U)$ is an open subset of V, and any U open in V is some preimage. We then define $(\varphi')^{\#}(\alpha^{-1}(U)) = \varphi^{\#}(U) : (\alpha_*\mathcal{O}_V)(U) \to (\alpha_*\mathcal{O}_W)(U)$ as a morphism of k-algebras. It follows that φ' is a morphisms of varieties as it is a morphism of locally ringed spaces from the structure of the scheme morphism.

II.2.16

Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_{X,x}$.
(a) If $U = \text{Spec } B$ is an open *affine* subscheme of X, and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.
(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.
(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact (This hypothesis is satisfied, for example, if $\text{sp}(X)$ is noetherian). Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.
(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Proof:

(a) $U \cap X_f$ is all $P \in U$ such that $f_P \notin \mathfrak{m}_P$, however, $U \cong \text{Spec } B$, thus $f_P = (f|_U)_P$ and $\mathfrak{m}_P$ is a prime ideal of B, thus $U \cap X_f = \{\mathfrak{p} \subseteq B \big| (f|_U)_\mathfrak{p} \notin \mathfrak{p}B_\mathfrak{p}\}$. We have that $D(f|_U) = \{\mathfrak{p} \subseteq B \big| \mathfrak{p} \not\ni f|_U\}$. Note that if $f \in \mathfrak{p}$, then $f \in \mathfrak{p}B_\mathfrak{p}$ and if $f \in B_\mathfrak{p}$, then $f = \frac{a}{s}$ with $s \notin \mathfrak{p}$ and $a \in \mathfrak{p}$, thus $fs \in \mathfrak{p}$ and hence $f \in \mathfrak{p}$. Therefore $f \in \mathfrak{p}$ iff $f_\mathfrak{p} \in \mathfrak{p}B_\mathfrak{p}$, hence $U \cap X_f = D(f|U)$.

(b) Let $X = \bigcup_{i=1}^k U_i$, then $a|_{U_i \cap X_f} = 0$, and hence $a|_{D_i(\overline{f})} = 0$, hence $\overline{a} \in \Gamma(D_i(f), \mathcal{O}_X|_{U_i})$ is 0 and hence $\overline{a}$ is 0 in $(B_i)_{f|_{U_i}}$, hence $f|_{U_i}^{n_i} a|_{U_i} = 0$ for some n_i. Let n be the max of all n_i, then $(f^n a)|_{U_i} = f|_{U_i}^{n} a|_{U_i} = 0$, so $f^n a = 0$.

(c) We have that $b \in \mathcal{O}_X(X_f)$, then for any i, $b|_{U_i \cap X_f} \in \mathcal{O}_X(X_f \cap U_i)$. Let $B_i = \mathcal{O}_X(U_i)$, then $\mathcal{O}_X(X_f \cap U_i) = \mathcal{O}_{\text{Spec } B_i}(D(f|_{U_i})) = (B_i)_{f|_{U_i}}$. It follows that $b|_{U_i \cap X_f} = \frac{a_i}{f|_{U_i}^{n_i}}$ for some $a_i \in B_i$, thus $(f|_{X_f}^{n_i} b)|_{U_i \cap X_f} = a_i$ in $(B_i)_{f|_{U_i}}$, hence there is some k_i such that $f|_{U_i \cap X_f}^{k_i}((f|_{X_f}^{n_i} b)|_{U_i \cap X_f} - a_i|_{U_i \cap X_f}) = 0$. Let $n_i = n_i + k_i$, then we have that $f|_{U_i \cap X_f}^{n_i} b|_{U_i \cap X_f} = f|_{U_i \cap X_f}^{k_i} a_i|_{U_i \cap X_f}$. Let n be the max of all such n_i and rename a_i to be $f|_{U_i}^{k_i+n-n_i} a_i$, then we have that $f|_{U_i \cap X_f}^{n} b|_{U_i \cap X_f} = a_i|_{U_i \cap X_f}$. It follows that $a_i|_{U_i \cap U_j \cap X_f} = a_j|_{U_i \cap U_j \cap X_f}$, then $a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j} = 0$ on X_f, thus there is some n_{ij} such that $f^{n_{ij}}|_{U_i \cap U_j} a_i|_{U_i \cap U_j} = f^{n_{ij}}|_{U_i \cap U_j} a_j|_{U_i \cap U_j}$. Let n' be the maximum of all n_{ij}, then the elements $f^{n'}|_{U_i} a_i$ glue to some element $s \in \mathcal{O}_X(X)$. Furthermore, $s|_{U_i \cap X_f} = f^{n'}|_{U_i \cap X_f} a_i|_{U_i \cap X_f} = f^{n'+n}|_{U_i \cap X_f} b|_{U_i \cap X_f}$, hence $s|_{X_f} = f^{n'+n}|_{X_f} b$ as desired.

(d) For any element $b \in \mathcal{O}_{X_f}(X_f)$, there is some n such that there is an $a \in \mathcal{O}_X(X)$ with $f|_{X_f}^n b = a|_{X_f}$, we then map b to $a/f^n \in A_f$. If $a/f^n = 0$, then there is some k such that $f^k a = 0$ in $\mathcal{O}_X(X)$, thus on each U_i, we have that $f^{n+k} a = f^k b = 0$ in $U_i \cap X_f$, thus $b|_{U_i \cap X_f} = 0$ in $\mathcal{O}_X(U_i \cap X_f)$, hence $b = 0$ in $\mathcal{O}_X(X_f)$. For any element $a/f^n \in A_f$, we have that $a|_{U_i \cap X_f}/f|_{U_i \cap X_f}^n \in (B_i)_{f|_{U_i \cap X_f}}$ and thus glues to an element in $\mathcal{O}_X(X_f)$.

II.2.17

A Criterion for Affineness
(a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i,

such that for each i, then induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.
(b) A scheme X is affine iff there is a finite set of elements $f_1, \cdots f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that the open subsets X_{f_i} are affine, and $f_1, \cdots, f_r$ generate the unit ideal in A.

Proof:

(a) Let U be any open subset of X, then $U = \bigcup U \cap f^{-1}(U_i)$ and $f(U) = \bigcup f(U \cap f^{-1}(U_i))$. Since $f : f^{-1}(U_i) \to U_i$ is a homeomorphism, then each $f(U \cap f^{-1}(U_i))$ is open and thus f is a homeomorphism from X to Y. We now check that the morphisms on stalks are isomorphisms. The morphism $f|_{f^{-1}(U_i)} = i^{-1}(f_*\rho \circ f^\#)$ where $i : U_i \to Y$ is the inclusion and $\rho : \mathcal{O}_X \to j_*\mathcal{O}_X|_{f^{-1}(U_i)}$ is the restriction morphism. We have that $f_*\rho : f_*\mathcal{O}_X \to f_*j_*\mathcal{O}_X|_{f^{-1}(U_i)} = (f \circ j)_*\mathcal{O}_X|_{f^{-1}(U_i)}$. Then $f_*\rho \circ f^\# : \mathcal{O}_Y \to (f \circ j)_*\mathcal{O}_X|_{f^{-1}(U_i)}$ and finally, $i^{-1}(f_*\rho \circ f^\#) : i^{-1}\mathcal{O}_Y \to i^{-1}(f \circ j)_*\mathcal{O}_X|_{f^{-1}(U_i)}$. Clearly $i^{-1}\mathcal{O}_Y = \mathcal{O}_Y|_{U_i}$. For $V \subseteq U$, we have $i^{-1}(f \circ j)_*\mathcal{O}_X|_{f^{-1}(U_i)}(V) = \mathcal{O}_X|_{f^{-1}(U_i)}(j^{-1}(f^{-1}(V))) = \mathcal{O}_X|_{f^{-1}(U_i)}(f^{-1}(V)) = f_*\mathcal{O}_X|_{f^{-1}(U_i)}$ as expected. We now have that for any $P \in U_i$, $(i^{-1}(f_*\rho \circ f^\#))_P$ is an isomorphism, thus $(f_*\rho \circ f^\#)_{i(P)} = (f_*\rho \circ f^\#)_P$ is an isomorphism and this is just $(f_*\rho)_P \circ f^\#_P$. Since $(f_*\rho)_P$ is an isomorphism, then so if $f^\#_P$, thus f is an isomorphism at every stalk and hence an isomorphism.

(b) Suppose such $f_1, \cdots, f_r$ exist and generate the unit ideal and X_{f_i} are all affine, then we have the identity map from $\mathcal{O}_X(X) \to \mathcal{O}_X(X)$ which induces a map φ from X to Spec $\mathcal{O}_X(X)$ by II.2.4. Furthermore, we have that on an affine open cover, the morphism $\varphi|_{X_{f_i}}$ is the morphism induced by the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X_{f_i}) = \mathcal{O}_X(X)_{f_i}$. The image of this map is then all the primes in $\mathcal{O}_X(X)$ which do not contain f_i, i.e. $D(f_i) \subseteq$ Spec $\mathcal{O}_X(X)$. Furthermore, we have that $\varphi|_{X_{f_i}}$ is injective, thus φ is injective. We now need only check that $\varphi|_{X_{f_i}} : X_{f_i} \to D(f_i)$ are isomorphisms. This is clear. Suppose that X is affine, then $D(f)$ forms a basis of the topology on X, and since X is quasi-compact we may take finitely many which cover it, then these f generate the unit ideal (II.2.13b) and $D(f)$ are all affine.

II.2.18

In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.
(a) Let A be a ring, $X =$ Spec A, and $f \in A$. Show that f is nilpotent iff $D(f)$is empty.
(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y \to$ Spec $B \to X =$ Spec A be the induced morphism of affine schemes. Show that φ is injective iff the map of sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e. $f(Y)$ is dense in X.
(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.
(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective.

Proof:

(a) $D(f) = X - V(f)$, thus $D(f) = \varnothing$ iff $V(f) = V(0) = X$, i.e. iff $\sqrt{(f)} = \sqrt{(0)}$. Note that if $\sqrt{(f)} = \sqrt{(0)}$, then $f \in \sqrt{(0)}$, thus f is nilpotent. If f is nilpotent, then $0 \in (f)$ and $\sqrt{(f)} \subseteq \sqrt{(0)}$, thus $V(f) = X$.

(b) If φ is injective, then for any prime ideal $\mathfrak{p} \in A$, we obtain a morphism $(f^\#)_\mathfrak{p} : \mathcal{O}_{X,\mathfrak{p}} \to (f_*\mathcal{O}_Y)_\mathfrak{p}$

II.3.1

Show that a morphism $f : X \to Y$ is locally of finite type iff for *every* open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.

Proof:

We use the affine communication lemma. We have the property P that for any open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by affine opens which are finitely generated B-algebras. If f is locally of finite type, then we have a cover of Y by open affines with this property. Suppose $V = \text{Spec } B$ is an open affine in Y, and has P, then let s be an element of B, then we have $f^{-1}(V)$ is covered by $\text{Spec } A_i = U_i$ with each A_i a finitely generated B algebra. We then have that $f^{-1}(D(s))$ is covered by $\text{Spec } (A_i)_{\varphi_i(s)}$ where $\varphi_i(s)$ is the image of s in A_i. To show this equality of the preimage, we note that φ_i is the map induced by f as a morphism of affine schemes. We now have that $A_i = B[x_1, \cdots, x_n]$ since it is a finitely generated B algebra, then $(A_i)_{\varphi(s)} = B_s[x_1, \cdots, x_n]$, thus $(A_i)_{\varphi(s)}$ is a finitely generated B_s-algebras. Suppose conversely that B_s has finitely-generated preimages for some $(s_1, \cdots, s_n)$ generating the unit ideal in B. Let $f^{-1}(B)$ be covered by $\text{Spec } A_i$. Furthermore, let $f^{-1}(D(s_i))$ be covered by $\text{Spec } A_{ij}$ with A_{ij} a finitely generated B_{s_i}-algebra. It follows that each $\text{Spec } A_k$ has a cover by $\text{Spec } A_k \cap \text{Spec } A_{ij}$ with $\text{Spec } A_{ij}$ being finitely-generated $B_{\varphi(s_i)}$-algebras. We then cover the intersection $U_{ijk} = \text{Spec } A_k \cap \text{Spec } A_{ij}$ by $D(f_{ijk})$ with $(A_{ij})_{\overline{f_{ijk}}} \cong (A_i)_{f_{ijk}}$ which is a finitely generated $B_{s_i}-$algebra.

It follows that we need only prove the following fact: Let A be a B-algebra and $(s_1, \cdots s_n)$ generate the unit ideal of B such that $A_{\varphi(s_i)}$ are finitely generated B_{s_i}-algebras, then A is finitely-generated. Since $(s_1, \cdots, s_n) = B$, then there is some $\sum b_i s_i = 1$. Since $A_{\varphi(s)}$ is finitely generated, let $x_{ij}/\varphi(s)^{n_{ij}}$ generate $A_{\varphi_i(s)}$, then we will show that A is generated by all of the x_{ij} as a B-algebra. Let $y \in A$ be an arbitrary element, then $y = \frac{1}{\varphi(s_i)^{n_i}} p_i(x_{i1}, \cdots, x_{ij})$ where p_i is a polynomial with coefficients in B and n_i is the power of $\varphi(s_i)$ factored out from the polynomial. Then $\varphi(s_i)^{k_i+n_i} y = \varphi(s_i)^{k_i) p_i(x_{i1}, \cdots, x_[ij}})$ for some k_i, let p_i be the RHS and $n_i = n_i + k_i$, then we have that $\varphi(s_i)^{n_i} y = p_i(x_{i1}, \cdots, x_{ij})$. Note that since $(s_1, \cdots, s_n) \in \sqrt{(s_1^{n_1}, \cdots, s_n^{n_n})}$ and the prior is B, then so is the latter, and thus a B-linear combination of the $s_i^{n_i}$ give 1 and thus y is a B-linear combination of $p_i(x_{i1}, \cdots, x_{ij})$. Therefore the x_{ij} do indeed generate A as a B-algebra.

II.3.2

A morphism $f : X \to Y$ of schemes is quasi-compact if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact iff for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof:

We use the affine communication lemma. Suppose that $V \subseteq Y$ is an open affine and $f^{-1}(V)$ is quasi-compact, then $f^{-1}(V) = \bigcup \text{Spec } A_i$ up to isomorphisms in the specs. As we saw in the previous exercise, we then have that for $D(s) \subseteq V$, we have that $f^{-1}(D(s)) = \bigcup \text{Spec } (A_i)_{\varphi_i(s_i)}$ and is thus quasi-compact since it is a finite union of quasi-compact. Conversely, suppose that $\text{Spec } A \subseteq Y$ up to iso, and that $(s_1, \cdots, s_n) = A$ with $\text{Spec } A_{s_i} \subseteq Y$ having quasi-compact preimages. Since the preimage of $\text{Spec } A$ is covered by the primes of $\text{Spec } A_{s_i}$ and there are only finitely many s_i, then the preimage of $\text{Spec } A_i$ is quasi-compact.

II.3.3

(a) Show that a morphism $f : X \to Y$ is of finite type iff it is of locally finite type and quasi-compact.
(b) Conclude from this that f is of finite type iff for *every* open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.
(c) Show also if f is of finite type, then for *every* open affine subset $V = \text{Spec } B \subseteq Y$, and for *every* open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Proof:

(a) If f is of finite type, then it is clearly of locally finite type and furthermore since the preimages have finite covers by quasi-compact, then they themselves are quasi-compact. Conversely, if f is quasi-compact and of locally finite type, then the preimage of an open affine is covered by affines which are f.g. and by quasi-compactness, the preimage is quasi-compact, thus we may choose finitely many.

(b) f is of finite type iff it is of locally finite type and quasi-compact iff the preimage of every open affine is covered by specs of f.g. algebras and is quasi-compact i.e. iff the preimage of every open affine is covered by finitely many f.g. algebras.

(c) We may cover $U = \text{Spec } A \subseteq f^{-1}(V) = \text{Spec } A'$ by finitely many open sets of the form $D(s_i)$ which are distinguished in both Spec A and Spec A', then $(A')_{s_i}$ are f.g. B-algebras since A' is a f.g. B-algebra by virtue of f being finite type and hence covering Spec A' by specs of f.g. B-algebras, thus $A_{\varphi(s_i)}$ are f.g. B-algebras, and $(\varphi(s_1), \cdots, \varphi(s_n)) = A$, thus A is a f.g. B-algebra.

II.3.4

Show that a morphism $f : X \to Y$ is finite iff for *every* open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ is affine, equal to Spec A, where A is a finite B-module.

Proof:

We use the affine communication lemma on the property that $f^{-1}(\text{Spec } B) = \text{Spec } A$ and A is a f.g. B-module. Suppose Spec $B \subseteq Y$ has this property, and let $s \in B$, then we have that $f^{-1}(\text{Spec } B_s) = \text{Spec } A_{\varphi(s)}$ where φ is the morphism on rings induced by f. The generators of A as a B-module clearly generate A as a B_s-module. Suppose that we have $(s_1, \cdots, s_n)$ which generate B, and $f^{-1}(\text{Spec } B_{s_i})$ are all affine, then let $X = f^{-1}(\text{Spec } B)$. We have a map $X \to \text{Spec } B$, thus there is a ring map φ from B to $\mathcal{O}_X(X)$ and the $X_{\varphi(s_i)}$ cover X and generate the whole ring $\mathcal{O}_X(X)$, thus by II.2.18, X is affine, isomorphic to Spec $\mathcal{O}_X(X)$. Let $\mathcal{O}_X(X) = A$, then $X_{\varphi(s_i)} = \text{Spec } A_{\varphi(s_i)}$. Furthermore, we have that $X_{\varphi(s_i)} = f^{-1}(B_{s_i}) = \text{Spec } A_{\varphi(s_i)}$ are all finite B_{s_i}-modules.

We have reduced this problem to the purely algebraic fact as follows: If A is a B-module with structure morphism φ and $(s_1, \cdots, s_n)$ generate B such that $A_{\varphi(s_i)}$ are all finitely generated B_{s_i}-modules, then A is a finitely generated B-module. Since $A_{\varphi(s)}$ is finitely generated, let $x_{ij}/\varphi(s)^{n_{ij}}$ generate $A_{\varphi_i(s)}$, then we will show that A is generated by all of the x_{ij} as a B-module. Let $y \in A$ be an arbitrary element, then $y = \sum_j a_{ij} x_{ij}/\varphi(s)^{n_{ij}}$ for each i, then by multiplying by $n_i = max_j\{n_{ij}\}$, we have that $y = \frac{1}{\varphi(s)^{n_i}} \sum_j b_{ij} x_{ij}$, therefore there is some k_i such that $\varphi(s)^{n_i+k_i} y = \varphi(s)^{k_i} \sum_j b_{ij} x_{ij}$, then letting n_i be $n_i + k_i$, we get that $\varphi(s^{n_i})y = \sum c_{ij} x_{ij}$. Note that since $(s_1, \cdots, s_n) \in \sqrt{(s_1^{n_1}, \cdots, s_n^{n_n})}$ and the prior is B, then so is the latter and hence there is a combination $\sum a_i s_i^{n_i} = 1$, then $y = \varphi(1)y = \sum \varphi(a_i)\varphi(s^{n_i})y = \sum \varphi(a_i) \sum_j c_{ij} x_{ij}$. Therefore A is a finite B-algebra.

II.3.5

A morphism $f : X \to Y$ is *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.
(a) Show that a finite morphism is quasi-finite.
(b) Show that a finite morphism is *closed*, i.e., the image of any closed subset is closed.
(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof:

(a) Let $y \in Y$, then there is an affine open Spec $B = V \subseteq Y$ containing y, and $f^{-1}(y) \subseteq f^{-1}(V)$. Furthermore, we have that $f^{-1}(V) =$ Spec A with A a finitely generated B-module with the structure map from $\varphi : B \to A$ induced by the morphism f. Additionally, we know that f acts by taking preimages along φ, i.e. $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ where $pf \subseteq A$ is a prime ideal. It follows that $f^{-1}(\mathfrak{p})$ with $\mathfrak{p}$ in B prime are all primes $\mathfrak{q} \in$ Spec A with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$, i.e. all primes lying over $\mathfrak{p}$.

We now prove a general fact: Let A be an integral extension of B with structure map $\varphi : B \to A$ and let $\mathfrak{p}$ in B be prime, then there are only finitely many primes lying over $\mathfrak{p}$. In this context, $\mathfrak{q}$ lying over $\mathfrak{p}$ means $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. We have that the primes lying over $\mathfrak{p}$ in B are in bijection with the primes lying over $\mathfrak{p}B_\mathfrak{p}$ in $A_{\mathfrak{p}^e}$ where $\mathfrak{p}^e$ denotes the extension of the ideal $\mathfrak{p}$ into A, i.e. $\mathfrak{p}^e = \varphi(\mathfrak{p})A$. Furthermore, any prime in $A_{\mathfrak{p}^e}$ lying over $\mathfrak{p}B_\mathfrak{p}$ contains $\mathfrak{p}^e A_{\mathfrak{p}^e}$. It follows that the primes lying over $\mathfrak{p}$ are in bijection with the primes lying over $(0) \subseteq B_\mathfrak{p}/(\mathfrak{p}B_\mathfrak{p})$ in $A' = A_{\mathfrak{p}^e}/(\mathfrak{p}^e A_{\mathfrak{p}^e})$. Let $k = B_\mathfrak{p}/(\mathfrak{p}B_\mathfrak{p})$, then A' is a finite k-algebra. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ be ideals in A' corresponding to those lying above $\mathfrak{p}$, then we have a map $p : A' \to \prod A'/\mathfrak{q}_i$ given by the quotient in each component. Since $A'/\mathfrak{q}_i$ are fields and finite k-algebras, then they are finite extensions of k. Furthermore, by no nesting, we may pick $a_{ij} \in \mathfrak{q}_i$ and not in $\mathfrak{q}_j$, then let $a_j = \prod_{i \neq j} a_{ij}$, then a_j is in $\mathfrak{q}_i$ for each $i \neq j$ and not in $\mathfrak{q}_j$ since it is prime. it follows that a_j maps to an element of the form $(0, 0, \cdots, 0, \neq 0, 0, \cdots, 0)$ in the product. Therefore, the dimension of $p(A')$ as a subspace of the k-vector space $\prod A'/\mathfrak{q}_i$ is at least n. This then yields an ascending chain of primes ideals of length at least n by taking preimages of successive products. Therefore we need only show that finite algebras over a field have finitely many maximal ideals. This is because such rings have the no nesting property for all their prime ideals and hence they are all maximal, and ascending chains of ideals must terminate since they are finite, thus such algebras are Artinian.

(b) Let X be a topological space covered by $\{U_i\}$, and let $C \subseteq X$ such that $C \cap U_i$ is closed in U_i for all i. We then have that $U_i \backslash C \cap U_i = (X \backslash C) \cap U_i$ is open and thus $X \backslash C$ is a union of open sets and thus open, hence C is closed. Let Y be covered by open affines $U_i =$ Spec A_i, then we want to show that given any closed subset C of X, the image of C is closed in each A_i. Furthermore, note that $f(C) \cap U_i = f(C \cap f^{-1}(U_i))$. We therefore have reduced the problem to showing that if $f :$ Spec $A \to$ Spec B is finite, then it is closed. Let $V(I) \subseteq$ Spec A, then we want to show that $f(V(I))$ is closed. f corresponds to a map $\varphi : B \to A$ and A is a finite B-module under this structure map. What we then want to show is that $f(V(I)) = \{\varphi^{-1}(\mathfrak{p}) \big| \mathfrak{p} \supseteq I\}$ is closed. If this is closed, then it will be equal to $V(J)$ where $J = \bigcap \varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\bigcap \mathfrak{p}) = \varphi^{-1}(\sqrt{I}) = \sqrt{\varphi^{-1}(I)}$. It follows that we need to show that $f(V(I)) = V(\varphi^{-1}(I))$. Let $\mathfrak{q}$ be any prime ideal containing $\varphi^{-1}(I)$ in B, then A/I is integral over B, thus there exists a prime ideal $\overline{\mathfrak{p}}$ in A/I containing I lying over $\mathfrak{q}$, i.e. $(q \circ \varphi)^{-1}(\overline{\mathfrak{p}}) = \varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$ with $\mathfrak{p}$ containing I.

(c) Consider the map from $k[x]$ to $k[x, x^{-1}]$ by inclusion, then this is a finite type morphism quasi-finite morphism, but not finite. Unfortunately it is not surjective, however we may amend this by taking Spec $k \cup$ Spec $k[x, x^{-1}] \to$ Spec $k[x]$ and further mapping Spec k to the ideal (x).

II.3.6

Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the function field of X, and is denoted by $K(X)$. Show also that if $U = \text{Spec } A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof:

Let $\xi \in U = \text{Spec } A$ for some open affine subset U, then ξ in Spec A is the generic point (0), and furthermore, since the any open subset of X contains the generic point, then the collection of open subsets taken in the direct limit for $\mathcal{O}_\xi$ may be restricted to only those in U, and thus $\mathcal{O}_\xi \cong A_{(0)} = \text{Frac}(A)$ is a field.

II.3.7

A morphism $f : X \to Y$, with Y irreducible, is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is *dominant* if $f(X)$ is dense in Y. Now let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.

Proof:

We may assume that X and Y are affine since if $f : X \to Y$ is dominant means that it is dominant on affines and affine open subsets of integral are integral and generically finite. If we can show that there is a dense subset of each affine of Y such that the induced morphism is finite, then we may cover Y by affines and then the finiteness property holds since it is a local condition. Let $X = \text{Spec } B$ and $Y = \text{Spec } A$, $f : X \to Y$, let ξ be the generic point of X, then $f(\text{cl}\{\xi\}) \subseteq \text{cl}\{f(\xi)\}$. Since f is dominant, then $f(\text{cl}\{\xi\})$ is dense in Y, thus taking closure, we have that $Y \subseteq \text{cl}\{f(\xi)\}$, hence $f(\xi)$ is the generic point of Y. It follows that the preimage of the generic point is nonempty and it is finite by generic finiteness. We have $\varphi : A \to B$ induced by f is of finite type, i.e. B is a finitely generated A algebra. Furthermore, B has only finitely many prime ideals lying over (0) including (0). Furthermore, since A is integral, and $\varphi^{-1}(0) \subseteq \mathfrak{N}(A) = 0$, then φ is injective. Let $S = A - (0)$, then $S^{-1}B$ is a f.g. algebra over $\text{Frac}(A)$. Furthermore, we have localized at $\varphi(A)\backslash\{0\}$. Therefore, prime ideals of $S^{-1}B$ are in correspondence with those of B which do not intersect $\varphi(A)\backslash\{0\}$, i.e. those lying over only (0). Since there are finitely many prime ideals of B lying over (0), then $S^{-1}B$ contains only finitely many prime ideals.

Let S be a finite type k-algebra, then $\dim(S) = 0$ iff S has finitely many prime ideals. S is noetherian, thus $(0) = \mathfrak{N}(S)$ has a primary decomposition into prime ideals, which are all prime ideals of S since S has dimension 0. Since S is a finite type k-algebra, then it is a Jacobson ring (any radical ideal is the intersection of all maximal ideals above it) by the nullstellensatz. Therefore, $\mathfrak{p} \in \text{Spec } S$ is a finite intersection of maximal ideals, then $\mathfrak{p}$ contains one of them (primes containing finite intersections, contain one of the ideals), hence $\mathfrak{p}$ is maximal. Therefore $\dim(S) = 0$.

From the above, we see that $S^{-1}B$ has dimension 0, however it is also an integral domain, thus $S^{-1}B$ is a field. Since $S^{-1}B$ is a finitely generated $\text{Frac}(A)$-algebra which is a field, then by the nullstellensatz, it is a finite extension, thus $\text{Frac}(B) = \text{Frac}(S^{-1}B)$ is a finite extension of $\text{Frac}(A)$. Let $x_1, \cdots, x_n$ generate B as an A-algebra, then since $\text{Frac}(B)$ is algebraic over $\text{Frac}(A)$, then there are monic polynomials p_i in $\text{Frac}(A)[x]$ such that $p_i(x_i) = 0$, then these monic polynomials are of the form $p_i = \frac{1}{a_i}\overline{p_i}$ where $\overline{p_i}$ is a polynomial in $A[x]$ and $a \in A$. It follows that for $a = \prod a_i$, p_i are polynomials in $A_a[x]$. It follows that once we localize B at $\varphi(a)$ and A at a, $B_{\varphi(a)}$ is finite over A_a, thus $f : \text{Spec } B_{\varphi(a)} \to \text{Spec } A_a$ is finite. Furthermore, $\text{Spec } A_a = D(a) \subseteq \text{Spec } A$ is an open dense subset.

II.3.8

Normalization. A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \text{Spec } A$of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \text{Spec } \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the *normalization* of X. Show also that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Proof:

We first show that if $f \in A$, then $\widetilde{A_f} = (\widetilde{A})_f$. Let $\frac{x}{yf^k} \in (\widetilde{A})_f$, then $\frac{x}{y}$ satisfies a monic polynomial in A:

$$\left(\frac{x}{y}\right)^n + a_{n-1}\left(\frac{x}{y}\right)^{n-1} + \cdots + a_0 = 0$$

Then dividing by f^{kn}, we see that $\frac{x}{yf^k}$ satisfies a monic polynomial in A_f, thus $\widetilde{A_f} \subseteq (\widetilde{A})_f$. Conversely, let $\frac{x}{y} \in \widetilde{A_f}$, then $\frac{x}{y}$ satisfies a monic polynomial in A_f,

$$\left(\frac{x}{y}\right)^n + \frac{a_{n-1}}{f^{k_{n-1}}}\left(\frac{x}{y}\right)^{n-1} + \cdots + \frac{a_0}{f^{k_0}} = 0$$

Then let $k = \max\{k_0, \cdots, k_{n-1}\}$ and multiply by f^{kn} to get that $\frac{xf^k}{y} \in \widetilde{A}$, thus $\frac{x}{y} \in (\widetilde{A})_f$. Therefore $\widetilde{A_f} = (\widetilde{A})_f$.

Let X be covered by $U_i \cong \text{Spec } A_i$ with overlaps $U_{ij} = U_i \cap U_j$ and isomorphisms $\varphi_i : U_i \to \text{Spec } A_i$. Let $V_i = \varphi_i(U_i)$ and likewise $V_{ij} = \varphi_i(U_{ij})$, then we have isomorphism $V_{ij} \to V_{ji}$ given by composition of φ_i with φ_j^{-1}. Suppose we have Spec A and Spec B with $U \subseteq \text{Spec } A$ isomorphic to $V \subseteq \text{Spec } B$ by some $\psi : U \to V$. Let $i : A \to \widetilde{A}$ and $j : B \to \widetilde{B}$ be inclusion, then we wish to show that $(i')^{-1}(U)$ is isomorphic to $(j')^{-1}(V)$ where i', j' are the induced maps Spec $\widetilde{A} \to$ Spec A and Spec $\widetilde{B} \to$ Spec B respectively. We may cover U and V by subsets $D_A(f_i)$ and $D_B(g_i)$ with $D_A(f_i) \cong D_B(g_i)$ by ψ. It follows that $A_{f_i} \cong B_{g_i}$ and thus $(\widetilde{A})_{f_i} \cong (\widetilde{B})_{g_i}$. Note furthermore, that for distinguished open subsets $D_A(f) \subseteq \text{Spec } A$, the preimage $(i')^{-1}(D_A(f))$ is just $D_{\widetilde{A}}(f)$ since any prime $\mathfrak{p} \subseteq \widetilde{A}$ containing f we have that $i'(\mathfrak{p}) = \mathfrak{p} \cap A$ contains f thus $D_{\widetilde{A}}(f) \subseteq (i')^{-1}(D_A(f))$ and for any prime $\mathfrak{p}$ such that $i'(\mathfrak{p}) = \mathfrak{p} \cap A$ contains f, then $\mathfrak{p}$ contains f. It follows that $(i')^{-1}(U)$ is a union of $D_{\widetilde{A}}(f_i)$ with the same f_i. For each i we have an isomorphism $\psi_i' : (\widetilde{A})_{f_i} \to (\widetilde{B})_{g_i}$, thus we need only check that these glue to an isomorphism $(i')^{-1}(U)$ to $(j')^{-1}(V)$.

To check that these isomorphisms glue, we have to check that they agree as topological maps. Let $\mathfrak{p} \in (i')^{-1}(V)$, and consider i, j, then we wish to show that $\psi_i'^{-1}(\mathfrak{p}\widetilde{B}_{g_i}) \cap \widetilde{A} = \psi_j^{-1}(\mathfrak{p}\widetilde{B}_{g_j}) \cap \widetilde{A}$. We first have to figure out what the map $\psi_i' : (\widetilde{A})_{f_i} \to (\widetilde{B})_{g_i}$ actually is. We know that $\psi : V \to U$ is an isomorphism, and that we further obtain a restricted isomorphism $D_B(g_i) \to D_A(f_i)$ which correspond to the map given by $\psi_i = \psi^{\#}(D_A(f_i))$ an isomorphism from $A_{f_i} \to B_{g_i}$. We then get the map $\overline{\psi_i}$: Frac$(A) \to$ Frac(B) sending a/b to $\psi_i(a)/\psi_i(b)$, then this induces a morphism $\psi_i' : \widetilde{A_{f_i}} \to \widetilde{B_{g_i}}$ by sending a/b to $\psi_i(a)/\psi_i(b)$. To show that ψ_i' and ψ_j' agree, we need to show that they induce the same map on $D_A(f_i) \cap D_A(f_j) = D_A(f_i f_j)$, i.e. the same map $(\widetilde{A})_{f_i f_j} \to (\widetilde{B})_{g_i g_j}$, however this is obvious since they induce the same map on $A_{f_i f_j} \to B_{g_i g_j}$. It follows that the ψ_i glue to a map from $(i')^{-1}(U)$ to $(i')^{-1}(V)$.

Let X have a cover by $U_i \cong \text{Spec } A_i$ by φ_i and $U_{ij} = U_i \cap U_j$, then let $V_{ij} = \varphi_i^{-1}(U_{ij})$, then we have isomorphisms from $V_{ij} \to V_{ji}$ by composing φ_i and φ_j^{-1}. It follows that these extend to isomorphisms $\psi'_{ij} : \widetilde{V_{ij}} \to \widetilde{V_{ji}}$ where $\widetilde{U}$ denotes the preimage of U in Spec $\widetilde{A}$ one may then check that these compose correctly in order to allow for the gluing of the schemes, however I don't want to do that. $\widetilde{X}$ is clearly normal since $\widetilde{A_i}$ are all normal and thus their local rings are normal and the local rings of $\widetilde{X}$ will be the same as those in the specs.

To get the morphism from $\widetilde{X} \to X$, we note that for each affine open Spec $A \subseteq X$, there is a natural morphism from Spec $\widetilde{A} \to$ Spec A induced by the inclusion from $A \to \widetilde{A}$. On our open cover Spec $A_i \subseteq X$ we get an open cover Spec $\widetilde{A_i} \subseteq \widetilde{X}$ and have morphisms $\varphi_i :$ Spec $\widetilde{A_i} \to$ Spec A_i. For any i, j, we have $\psi'_{ij} : \widetilde{V_{ij}} \to \widetilde{V_{ji}}$, we want to check that $\psi_{ji} \circ \varphi_i|_{\widetilde{V_{ij}}} = \varphi_j \circ \psi'_{ij}$. To do so, we want to check this on each $D_{\widetilde{A}}(f_i) \subseteq \widetilde{V_{ij}}$, i.e. we want to check that the maps $\psi_{ji} \circ \varphi_i|_{D_{\widetilde{A}}(f_i)} = \varphi_j \circ \psi'_{ij}|_{D_{\widetilde{A}}(f_i)}$. To check this, we check that they correspond to the same maps from $(A_j)_{g_i} \to (\widetilde{A})_{f_i}$. The left hand side corresponds to first mapping $x \in (A_j)_{g_i}$ to $\psi_{ji}(x) \in A_{f_i}$, then including it into $(\widetilde{A})_{f_i}$ and the right hand side corresponds to mapping $x \in (A_j)_{g_i}$ into $(\widetilde{A_j})_{g_i}$, then applying the map ψ'_{ij} which just maps it into $\widetilde{A_i}_{f_i}$. Therefore the maps glue to a map from $\widetilde{X} \to X$.

For any normal integral scheme Z, and any dominant morphism $f : Z \to X$, we may extend f to $\widetilde{f} : Z \to \widetilde{X}$ by first covering X with $U_i \cong$ Spec A_i, then $f : Z \to X$ determines $f_i : f^{-1}(U_i) \to$ Spec A_i corresponding to maps $f_i^\# : A_i \to \mathcal{O}_Z(f^{-1}(U_i))$ which are injective since f is dominant.

We now show the purely algebraic fact that if $\varphi : A \to B$ is injective with B integrally closed and A an integral domain, then this uniquely determines a map $\widetilde{\varphi} : \widetilde{A} \to B$ such that $\widetilde{\varphi} \circ i = \varphi$ where $i : A \to \widetilde{A}$ is the inclusion map. Let $a/b \in \widetilde{A}$, then we must have that $\widetilde{\varphi}(a/b)\widetilde{\varphi}(a)\widetilde{\varphi(b)}^{-1} = \varphi(a)/\varphi(b)$ which uniquely determines $\widetilde{\varphi}$. Since φ is injective, then we will never divide by 0. We now need only check that $\varphi(a)/\varphi(b) \in B$. Clearly $\varphi(a) \in B$, thus it suffices to check that $\varphi(b)^{-1} \in B$. Since $1/b$ is in $\widetilde{A}$, then it satisfies a monic polynomial in A, $b^{-n} + \sum a_i b^{-i} = 0$, applying $\widetilde{\varphi}$ to this equation, we get that $\varphi(b)^{-1}$ satisfies a monic polynomial in B, and since B is integrally closed, then $\varphi(b)^{-1} \in B$.

It follows that we get unique $\widetilde{f_i^\#} : \widetilde{A_i} \to \mathcal{O}_Z(f^{-1}(U_i))$ extending $f_i^\#$. The fact that we may glue these again follows from composing with ψ_{ij} and I really don't care to do it. **ACTUALLY CHECK THESE GLUE AT SOME POINT.**

If furthermore X is of finite type over a field k, then the morphism $\widetilde{X} \to X$, then using theorem I.3.9A, we have that the preimages of the $U_i =$ Spec A_i are Spec $\widetilde{A_i}$ and $\widetilde{A_i}$ are finite over A_i.

II.3.9

The Topological Space of a Product. Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology. Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.
(a) Let k be a field, and let $\mathbb{A}^1_k =$ Spec $k[x]$ be the affine line over k. Show that $\mathbb{A}^1_k \times_{\text{Spec } k} A^1_k \cong A^2_k$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).
(b) Let k be a field, let s and t be indeterminates over k. Then Spec $k(s)$, Spec $k(t)$, and Spec k are all one-point spaces. Describe the product scheme Spec $k(s) \times_{\text{Spec } k}$ Spec $k(t)$.

Proof:

(a) This is a product of affine schemes, so it will be the spectrum of the tensor product $k[x] \otimes_k k[y] = k[x, y]$, thus it is $\mathbb{A}^2_k$. The product of the point sets of the factors is not the same as the point set of $\mathbb{A}^2_k$ since if we map pairs of points $(f(x)), (g(y))$ to $(f(x), g(y))$, then we can never get $(x - y)$.

(b) This is again a tensor product $k(s) \otimes_k k(t) = S^{-1}k[s] \otimes_k T^{-1}k[t]$ where S and T are all nonzero polynomials in s and t respectively. It follows that this is just $(ST)^{-1}k[s, t]$ where ST

consists of all products of nonzero polynomials purely in s and purely in t, i.e. elements of the form $f(s)g(t)$. Therefore the points in Spec $k(s) \times_{\text{Spec } k}$ Spec $k(t)$ correspond to irreducible polynomials in $k[s,t]$ which are not pure products of polynomials in s and t.

II.3.10

Fibres of a Morphism.
(a) If $f : X \to Y$ is a morphism, and let $y \in Y$ a point, show that $\text{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.
(b) Let $X = \text{Spec } k[s,t]/(s-t^2)$, let $Y = \text{Spec } k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fiber X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme whose residue field is an extension of degree two of the residue field of η. (Assume k is algebraically closed).

Proof:

The induced topology on $f^{-1}(y)$ is the subspace topology from X. $X_y = X \times_Y \text{Spec } k(y)$ with morphisms $f : X \to Y$ and $i : k(y) \to Y$ sending $(0) \to y$ and inducing the identity map on residue fields. Let $X = \bigcup U_i$ with $U_i \cong \text{Spec } A_i$ by isomorphisms φ_i. We may then cover the fibre product $X \times_Y \text{Spec } k(y)$ by $U_i \times_Y \text{Spec } k(y)$. Suppose now that $f|_{U_i} : U_i \to Y$ has $\text{sp}((U_i)_y)$ homeomorphic to $f|_{U_i}^{-1}(y) = U_i \cap f^{-1}(y)$, then we need only check that the homeomorphisms glue to a homeomorphism between both spaces. Do so, we need to know the homeomorphism in the affine case. In fact, let $V = \text{Spec } B$ be an affine open subset of Y containing y, then $X \times_V \text{Spec } k(y)$ satisfies the universal property of $X \times_Y \text{Spec } k(y)$ since the image of anything mapping through Spec $k(y)$ into Y must have image $y \in Y$. It follows that we may in fact assume $X = f^{-1}(V)$.

Consider the case of $f : \text{Spec } A \to \text{Spec } B$, then $\text{Spec } A \times_{\text{Spec } B} \text{Spec } k(y) = \text{Spec } A \otimes_B k(y)$. Let $\varphi : B \to A$ be the morphism induced by f. Let $y = \mathfrak{q} \in \text{Spec } B$, then $A \otimes_B k(y) = A \otimes_B B_\mathfrak{q}/(\mathfrak{q}B_\mathfrak{q}) = A \otimes_B B_\mathfrak{q} \otimes_B B/\mathfrak{q} = A_\mathfrak{q} \otimes_B B/\mathfrak{q} = A_\mathfrak{q}/(\mathfrak{q}A_\mathfrak{q})$. The prime ideals of $A_\mathfrak{q}$ correspond to primes in A not meeting $\varphi(B \backslash \mathfrak{q})$, i.e. the primes whose preimages under φ are contained in $\mathfrak{q}$. The primes ideals in $A_\mathfrak{q}/(\mathfrak{q}A_\mathfrak{q})$ are the primes of $A_\mathfrak{q}$ containing $\mathfrak{q}A_\mathfrak{q} = \varphi(\mathfrak{q})A_\mathfrak{q}$, thus such primes are exactly those containing $\varphi(\mathfrak{q})$. It follows that the prime ideals of $A_\mathfrak{q}/(\mathfrak{q}A_\mathfrak{q})$ are in correspondence with the prime ideals of A above $\mathfrak{q}$. Furthermore, this correspondence is inclusion preserving, thus we obtain a homeomorphism between the fibre of f over y and $f^{-1}(y)$. **FIGURE OUT HOW TO GLUE**

(b) The fibre is given by $k[s,t]/(s-t^2)$ localized at $k[s] \backslash (s-a) = S$, then modded out by the ideal generated by $(s-a)$ in $S^{-1}k[s,t]/(s-t^2)$,

$$\frac{S^{-1}k[s,t]/(s-t^2)}{(s-a)S^{-1}k[s,t]/(s-t^2)} \cong k[s,t]/(s-a, s-t^2) \cong k[s,t]/(s-a, a-t^2) \cong k[t]/(a-t^2)$$

Since k is algebraically closed, we can factor $(a-t^2) = (\sqrt{a}-t)(\sqrt{a}+t)$, thus $k[t]/(a-t^2) \cong k^2$ which clearly has two points with residue field k. When $a = 0$, we get the ring $k[t]/(t^2)$ which is nonreduced and has a single point (t). When $y = \eta$, we have that the fibre is $k[s,t]/(s-t^2) \otimes_{k[s]} k(s) = k(s)[t]/(s-t^2)$. $s - t^2$ is an irreducible polynomial over $k(s)$, thus we get an extension of $k(s)$ of degree 2.

II.3.11

Closed Subschemes
(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if

$X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.
(b) If Y is a closed subscheme of an affine scheme $X = \text{Spec } A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\text{Spec } A/\mathfrak{a} \to \text{Spec } A$.
(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.
(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the *scheme-theoretic image* of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Proof:

(a) We first show that being a closed immersion if affine local on the target. That is to say, for $f : X \to Y$ a closed immersion, then for any affine open $V \subseteq Y$, $f : f^{-1}(V) \to V$ is a closed immersion and if Y has a cover V_i such that each $f : f^{-1}(V_i) \to V_i$ is a closed immersion, then f is a closed immersion. To prove the prior, we note that if $f : X \to Y$ is a homeomorphism onto its image and the image is closed, then $f(X) \cap V = f(f^{-1}(V))$ is closed in V and the restriction of a homeomorphism is still a homeomorphism. Furthermore, $f^{\#} : \mathcal{O}_Y|_V \to f_*\mathcal{O}_X|_{f^{-1}(V)}$ has the same stalks for $P \in V$, thus it is still surjective. Now conversely, suppose that we have a cover of Y by V_i open affine such that $f : f^{-1}(V_i) \to V_i$ are all closed immersions, then f is a closed immersion. Since the image of f is locally closed, then it is closed as seen in a previous exercise. Furthermore, since f is locally a homeomorphism onto its image, then it is a homeomorphism onto its image. To show this, suppose $f : X \to Y$ and $Y = \bigcup V_i$ with $f : f^{-1}(V_i) \to V_i$ homeomorphisms onto their images, then f is clearly bijective and continuous and for any U open in X, $f(U)$ is covered by $f(U) \cap V_i$ which are each open in $f(X) \cap V_i$, and since this is an open cover of $f(X)$, then $f(U)$ is open in $f(X)$. We now have to show that f is surjective. We have that $f|_{V_i}^{\#} : \mathcal{O}_Y|_{V_i} \to f|_{V_i,*}\mathcal{O}_X|_{f^{-1}(V_i)}$ is surjective. Letting $i : V_i \to Y$, we have that $f|_{V_i}^{\#} = i^{-1}(f_*\rho \circ f^{\#})$, then for $P \in V_i$ in the image of f, we have that $(f|_{V_i}^{\#})_P = (f_*\rho)_P \circ (f^{\#})_P = \rho_{f^{-1}(P)} \circ (f^{\#})_P = (f^{\#})_P$. Therefore each $(f^{\#})_P$ is surjective.

Let $g : X' \to X$. We now show that assuming part b, the closed immersions are preserved under fibre products. We wish to show that $Y \times_X X' \to X'$ is a closed embedding. Since closed embedding are affine local on the target as shown above, then we may take an affine open subset $\text{Spec } A$ of X'. Let $p_2 : Y \times_X X' \to X'$, then we have that $p_2^{-1}(\text{Spec } A)$ is the fibre product $Y \times_X \text{Spec } A$. Let $V = \text{Spec } C \subseteq X$, then by part b, we have that closed immersions are affine morphisms, thus $f^{-1}(V) = \text{Spec } B \subseteq Y$, then we need only show that p_2 is a closed immersion to all $g^{-1}(V)$, however, we have that $p_2^{-1}(g^{-1}(V)) = p_1^{-1}(f^{-1}(V)) = \text{Spec } B \times_X \text{Spec } A$, however now both $f(\text{Spec } B)$ and $g(\text{Spec } A)$ are contained in $V = \text{Spec } C$, thus we have an affine fibre product $\text{Spec } B \times_{\text{Spec } C}$ $\times \text{Spec } A = \text{Spec } B \otimes_C A$. It follows that we need only show that $\text{Spec } B \otimes_C A \to \text{Spec } A$ is a closed embedding. By part b, we have that $B = C/I$, thus we get $\text{Spec } C/I \otimes_C A \to \text{Spec } A$ and $C/I \otimes_C A = A/I^e$ which is a closed embedding. We now have to actually prove part b.

(b) Let $f : Y \to \text{Spec } A$ be a closed embedding, then Y is homeomorphic to a subspace of a quasi-compact space and is therefore quasi-compact. Let $Y = \bigcup U_i$ with U_i affine open, then each $f(U_i)$ are covered by open sets of the form $D(f_{ij}) \cap Y$, therefore we may choose finitely many of them. Additionally, the preimage of a distinguished set under a morphism between affines is distinguished, thus Y is covered by finitely many U_i which are covered by finitely many distinguished open sets.

It follows that Y is covered by finitely many affine open subsets of the form $f^{-1}(D(f_i) \cap Y)$. Furthermore, the complement of $f(Y)$ is open in Spec A, so we may cover it with elements of the form $D(f_j)$, then the collections $\{D(f_i)\}$ together with $\{D(f_j)\}$ cover all of Spec A which is quasi-compact, thus we may pick a finite subcover which contains all of the elements of the form $\{D(f_i)\}$. Therefore we have finitely many elements $f_1, \cdots, f_r$ such that $f^{-1}(D(f_i))$ are an open affine cover of Y and $D(f_i)$ are an open cover of Spec A and hence $f_1, \cdots, f_r$ generate A. $f^{\#} : \mathcal{O}_{\text{Spec } A} \to f_* \mathcal{O}_Y$, then taking global sections, we have that $f^{\#}(X) : A \to \mathcal{O}_Y(Y)$. Let $g_i = f^{\#}(X)(f_i)$, then we have that $Y_{g_i} = D(f_i) \cap Y$ are affine and generate $\mathcal{O}_Y(Y)$ (since some combination of the f_i gives 1), thus Y is affine. Since Y is affine and $f : Y \to X$ is a homeomorphism onto a closed subset and surjective, then $f^{\#}(X)$ is surjective, so $\mathcal{O}_Y(Y) = A/I$ for some ideal I, hence $Y = \text{Spec } A/I$.

(c) Let $Y \subseteq X$ be a closed subset with the reduced induced subscheme structure and let Y' be another closed subscheme of X with the same topological space as Y, then on any open affine subset, we have that $Y' \cap \text{Spec } A = V(I)$, however, $Y \cap \text{Spec } A = V(J) = V(I)$ with J radical thus $I \subseteq J$ and hence there is a map from $A/I \to A/J$ which induces a map $Y \cap \text{Spec } A \to Y' \cap \text{Spec } A$. As Spec A varies over X, it covers Y, thus the map is entirely defined. We want to check that it is also well-defined. Suppose that Spec $A_f \subseteq$ Spec A in X, then $Y \cap \text{Spec } A = V(J)$, hence $Y \cap \text{Spec } A_f = V(I_f)$, thus we obtain a map $A_f/I_f \to A_f/J_f$ which is exactly the localization of the map $A/I \to A/J$, thus it is well-defined.

(d) Let Spec $A \subseteq X$, then we get a morphism $f : f^{-1}(\text{Spec } A) \to \text{Spec } A$ which comes from a map $f^{\#} : A \to \mathcal{O}_Z(f^{-1}(\text{Spec } A))$, then we define Y to be the Spec $A/\ker(f^{\#})$ in Spec A. We now wish to check that this is well-defined. Suppose that Spec $A_a \subseteq$ Spec $A \subseteq X$, then $f|_{\text{Spec } A_a} : f^{-1}(\text{Spec } A_a) \to \text{Spec } A_a$, thus $f|^{\#}_{\text{Spec } A_a} : A_a \to \mathcal{O}_Z(f^{-1}(\text{Spec } A_a)) = \mathcal{O}_Z(f^{-1}(\text{Spec } A))_{f^{\#}(a)}$. We then have that $0 \to \ker(f^{\#}) \to A \to B$ is exact and localizing at a gives $0 \to \ker(f^{\#})A_a \to A_a \to B_{f^{\#}(a)}$ is exact, thus the kernel is the localization, so we do obtain a scheme. Suppose f factors through Y' a closed subscheme of X, then on any open affine Spec $A \subseteq X$, we have that $Y' \cap \text{Spec } A = \text{Spec } A/I$ and furthermore, we have a map $f|_{\text{Spec } A} : f^{-1}(\text{Spec } A) \to Y' \cap \text{Spec } A =$ Spec A/I which induces a map $\widetilde{f}|^{\#}_{\text{Spec } A} : A/I \to \mathcal{O}_Z(f^{-1}(\text{Spec } A))$ such that $f^{\#}|_{\text{Spec } A} = \widetilde{f}^{\#}|_{\text{Spec } A} \circ q$ where q is the quotient map. It follows that $\ker(f^{\#}) \subseteq I$, thus Spec $A/I \to$ Spec A factors through Spec $A/\ker(f|^{\#}_{\text{Spec } A})$.

II.3.12

Closed Subschemes of Proj S
(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex 2.14) is equal to Proj T, and the morphism $f : \text{Proj } T \to \text{Proj } S$ is a closed immersion.
(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \text{Proj } T$ defined as the image of the closed immersion Proj $S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \oplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

Proof:

(a) We first show that $U = \text{Proj } T$. Let $\mathfrak{p} \in \text{Proj } T$, and suppose that $\mathfrak{p} \notin U$, that is to say that $\mathfrak{p} \supseteq \varphi(S_+)$. Let $t \in T_+$ be a homogeneous element, then there is an element $s \in S_+$ of the same degree such that $\varphi(s) = t$, hence $\mathfrak{p}$ contains T_+, so $\mathfrak{p} \notin \text{Proj } T$. This is a contradiction, therefore $U = \text{Proj } T$. We now show that $f : \text{Proj } T \to \text{Proj } S$ is a closed immersion by showing that it is locally a closed immersion. Let $D_+(s)$ cover Proj S with s homogeneous elements of degree

$\geqslant 1$. We then have that $f^{-1}(D_+(s)) = D_+(\varphi(s))$ as was shown in II.2.14. Furthermore, we have that $f|_{D_+(\varphi(s))}$: Spec $T_{(\varphi(s))} \to$ Spec $S_{(s)}$ corresponds to the map $\varphi' : S_{(s)} \to T_{(\varphi(s))}$ by sending $a/s^n \mapsto \varphi(a)/\varphi(s)^n$ which is in $T_{(\varphi(s))}$ since φ preserves degrees (Note that this is all shown very explicitly in II.2.14). Furthermore, for any $t/\varphi(s)^n \in T_{(\varphi(s))}$, we have some $a \in S$ of the same degree as t such that $\varphi(a) = t$, thus φ' is surjective. It follows that im$\varphi' \cong S_{(s)}/\ker\varphi'$ and since φ' is surjective, we have that im$\varphi' = T_{(\varphi(s))}$. Since isomorphisms are closed immersions, and the quotient map induces a closed immersion, then $f|_{D_+(\varphi(s))}$: Spec $T_{(\varphi(s))} \to$ Spec $S_{(s)}$ is a closed immersion. Therefore the whole map is a closed immersion.

(b) If we let $I' = \oplus_{d \geqslant d_0} I_d$, then S/I and S/I' are isomorphic in degrees higher than d_0, thus Proj S/I' and Proj S/I are isomorphic. Furthermore, we have that the map $S \to S/I'$ followed by $S/I' \to S/I$ is the same as the map from $S \to S/I$, thus the morphisms from S are the same.

II.3.13

Properties of Morphisms of Finite Type (a) A closed immersion is a morphism of finite type.
(b) A quasi-compact open immersion is of finite type.
(c) A composition of two morphisms of finite type is of finite type.
(d) Morphisms of finite type are stable under base extension.
(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.
(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.
(g) If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

Proof:

(a) We need only show that it is finite type locally. Let $f : Y \to X$ be a closed immersion, then for any Spec $A \subseteq X$, we have that $f|_{\text{Spec } A} : f^{-1}(\text{Spec } A) \to \text{Spec } A$ is a closed immersion, therefore $f^{-1}(\text{Spec } A) \cong \text{Spec } A/I$ for some ideal I, therefore not only are closed immersion of finite type, in fact they are finite.

(b) If $f : X \to Y$ is a quasi-compact open immersion, then f induces an isomorphism with an open subscheme of $U \subseteq Y$. Since f is quasi-compact, it suffices to show that f is locally of finite type. Isomorphisms are of locally finite type, thus it suffices to show that the inclusion $U \subseteq Y$ is locally of finite type. Let Spec $A \subseteq Y$, then cover Spec $A \cap U = \bigcup D(a_i)$ with $a_i \in A$, then $D(a_i) \to$ Spec A induces the morphism $A \to A_{a_i}$ which is localization and thus of finite type.

(c) Let $f : X \to Y$ and $g : Y \to Z$ be of finite type, then let Spec $A \subseteq Z$, thus $g^{-1}(\text{Spec } A) = \bigcup_{i=1}^n \text{Spec } B_i$ with B_i of finite type over A, then $f^{-1}(\text{Spec } B_i) = \bigcup_{j=1}^{n_i} \text{Spec } C_j$ with C_j of finite type over B_i and thus of finite type over A. Since there are only finitely many C_i, then $(g \circ f)^{-1}(\text{Spec } A) = \bigcup C_i$ with C_i of finite type over A.

(d) Let $f : X \to Y$ be of finite type and let $g : Y' \to Y$, then we obtain a morphism $\widetilde{f}$: $X \times_Y Y' \to Y'$. Finite type is an affine local property, thus let Spec $A \subseteq Y'$, then we have the $\widetilde{f}|_{\text{Spec } A} : X \times_Y \text{Spec } A \to \text{Spec } A$. Cover Y by open affine subset V_i, then $g^{-1}(V_i)$ cover Spec A and furthermore, $g^{-1}(V_i)$ is open in Spec A and therefore has a cover by open affines of the form Spec A_{a_j}. It follows that we need only show that $\widetilde{f}|_{\text{Spec } A_a} : X \times_Y \text{Spec } A_a \to \text{Spec } A_a$ is of finite type. Since $g \circ \widetilde{f}(X \times_Y \text{Spec } A_a) \subseteq V$ is open affine, then we really need only show that $\widetilde{f} : f^{-1}(V) \times_V \text{Spec } A_a \to \text{Spec } A_a$ is of finite type. Simplifying notation, we need to show that if $f : X \to \text{Spec } B$ is of finite type and $g : \text{Spec } A \to \text{Spec } B$, then $\widetilde{f} : X \times_{\text{Spec } B} \text{Spec } A \to \text{Spec } A$ is of finite type. Now $\widetilde{f}^{-1}(\text{Spec } A) = X \times_{\text{Spec } B} \text{Spec } A$ and X has a finite cover $X = \bigcup \text{Spec } C_i$ where C_i is a finitely generated B-algebra. Therefore, we need only show that Spec $C_i \times_{\text{Spec } B}$ Spec A

are of finite type over Spec A which corresponds to showing that $C \otimes_B A$ is a finitely generated A-algebra.

Let C be of finite type over B and let A be a B-algebra, then we wish to show that $C \otimes_B A$ is of finite type over A. Let C be generated by $x_1, \cdots, x_n$ over B and let $x \in C \otimes_B A$, i.e. $x = \sum c_i \otimes a_i = \sum p_i(x_1, \cdots, x_n) \otimes a_i = \sum p_i(x_1 \otimes 1, \cdots, x_n \otimes 1) \cdot 1 \otimes a_i$, thus $C \otimes_B A$ is generated by $x_i \otimes 1$.

(e) Let $f : X \to S$ and $g : Y \to S$ be of finite type, then we have $h = f \times g : X \times_S Y \to S$. Let Spec $A \subseteq S$, then $h^{-1}(\text{Spec } A) = f^{-1}(\text{Spec } A) \times_{\text{Spec } A} g^{-1}(\text{Spec } A)$. Since f and g are of finite type, then we may cover $f^{-1}(\text{Spec } A)$ by Spec B_i with B_i of finite type over A and similarly cover $g^{-1}(\text{Spec } A)$ by Spec C_i and thus we may cover $f^{-1}(\text{Spec } A) \times_{\text{Spec } A} g^{-1}(\text{Spec } A)$ by Spec $B_i \times_{\text{Spec } A}$ Spec C_j. It follows that we need only show that if B and C are of finite type over A, then so is $B \otimes_A C$. Let B be generated by x_i and C by y_j, then clearly $B \otimes_A C$ is generated by $\{x_i \otimes y_j\}$.

(f) Since f is quasi-compact, we need only show that it is of locally finite type. Let Spec $A \subseteq Z$, then $(g \circ f)^{-1}(\text{Spec } A)$ has a cover by Spec C_i with C_i of finite type over A. Let $g^{-1}(\text{Spec } A)$ be covered by Spec B_j, then $f^{-1}(\text{Spec } B_j)$ is also covered by Spec $C_i \cap$ Spec B_j which themselves are covered by Spec $(C_i)_{c_{ijk}}$. Therefore we need to show that Spec C_c is of finite type over Spec B given that Spec C is of finite type over A.

We want to show the following algebraic fact: Suppose $f : A \to C$ and $g : A \to B$ are of finite type and $h : B \to C$ is such that $h \circ g = f$, then h is of finite type. Let C be generated by $x_1, \cdots, x_n$ over A, then for any $x \in C$, we have that $x = p(x_1, \cdots, x_n)$ where p is a polynomial in A with n indeterminates. Since $h \circ g = f$, then $g(p)(x_1, \cdots, x_n) = p(x_1, \cdots, x_n) = x$ and $g(p)$ is obtained by applying g to each coefficient of p and thus C is finitely generated over B.

(g) Let $Y = \bigcup \text{Spec } A_i$ be a finite cover of Y, then A_i are all noetherian, and $f^{-1}(\text{Spec } A_i) = \bigcup \text{Spec } B_{ij}$ a finite union. Since B_{ij} are finitely generated algebras over a noetherian ring A_i, then by Hilbert's basis theorem, all B_{ij} are noetherian. Since there are only finitely many B_{ij} all noetherian which cover X, then X is noetherian.

II.3.14

If X is a scheme of finite type over a field, show that the closed points of X are dense. Given an example to show that this is not true for arbitrary schemes.

Proof:

Since X is of finite type over a field, then we may cover X by finitely many Spec A_i with A_i all finitely generated k-algebras. A point $x \in X$ is closed iff its residue field is a finite extension of k. To show this, we note that if x is closed in X, then $\text{cl}_U\{x\} = \text{cl}\{x\} \cap U$, thus x is closed in each affine open containing it. In the affine case, we saw in II.2.15 that closed is equivalent to the residue field being a finite extension. It follows that the residue field of x is a finite extension of k. Now suppose that $k(x)$ is a finite extension of k, then x is a closed point in each affine open, hence $\text{cl}_{U_i}\{x\} = \{x\}$, then since $\text{cl}_{U_i}\{x\} = \text{cl}\{x\} \cap U_i$ and U_i cover x, then x is closed. Therefore, a point is closed in X iff it is closed in any Spec A_i. It follows that if the closed points in Spec A_i are dense, then they are dense in X as well. To show that the closed points are dense in Spec A with A a finitely generated k-algebra, we need to show that for any radical ideal $I \subseteq A$, there is some maximal ideal $\mathfrak{m} \in \text{Spec } A \backslash V(I)$. $\mathfrak{m} \in \text{Spec } A \backslash V(I)$ is equivalent to saying that $\mathfrak{m}$ does not contain I. Since A is a finitely generated k-algebra, then A is Jacobson, thus I is the intersection of all maximal ideals containing I. If all maximal ideals contained I, then I would be the Jacobson radical. Since the nilradical is the intersection of all maximal ideals containing it and all maximal ideals contain the nilradical, then the nilradical is the Jacobson radical. Therefore, I is the nilradical, and thus Spec $A \backslash V(I) = \varnothing$. It follows that closed points are dense in Spec A. **SHOW THAT THIS IS**

NOT TRUE FOR ARBITRARY SCHEMES

II.3.15

Let X be a scheme of finite type over a field k (not necessarily algebraically closed).
(a) Show that the following three conditions are equivalent (in which case we say that X is *geometrically irreducible*).

1. $X \times_k \overline{k}$ is irreducible, where ovk denotes the algebraic closure of k.
2. $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.
3. $X \times_k K$ is irreducible for every extension field K of k.

(b) Show that the following three conditions are equivalent (in which case we say X is *geometrically reduced*).

1. $X \times_k \overline{k}$ is reduced.
2. $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.
3. $X \times_k K$ is reduced for all extension fields K of k.

(c) We say that X is geometrically integral if $X \times_k \overline{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

Proof:

DO THIS QUESTION

II.3.16

noetherian Induction. Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

Proof:

Let S be the set of all closed subsets of X for which $\mathscr{P}$ does not hold. We show that X noetherian implies that any set of closed subsets contains a minimal element. Let T be a nonempty collection of closed subsets of X and suppose that T contains no minimal elements, then pick $C_1 \in T$ and choose $C_{i+1} \in T$ such that $C_{i+1} \subsetneq C_i$ which can be done since C_i cannot be minimal. It follows that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ is an infinite descending chain of closed subsets of X which is a contradiction since X is noetherian. It follows that our set S contains a minimal element $C \in S$, then for all closed subsets $Y \subseteq C$, $\mathscr{P}$ holds for Y, thus $\mathscr{P}$ holds for C, so $C \notin S$ which is a contradiction. It follows that $\mathscr{P}$ holds for all closed subsets of X including X.

II.3.17

Zariski Spaces. A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point.
For example, let R be a DVR, and let $T = \text{sp}(\text{Spec } R)$. Then T consists of two points t_0 =the

maximal ideal, t_1 =the zero ideal. The open subsets are $\varnothing, \{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.
(a) Show that if X is a noetherian scheme, then $\mathrm{sp}(X)$ is a Zariski space.
(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points*.
(c) Show that a Zariski space X satisfies the axiom T_0: given any two distinct points of X, there is an open set containing one but not the other.
(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.
(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \mathrm{cl}\{x_1\}$, then we say that x_1 *specializes* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization* of x_1, or that x_1 is a *generization* of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are *stable under specialization*). Similarly, open subsets are *stable under generization*.
(f) Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space iff the map $\alpha : X \to t(X)$ is a homeomorphism.

Proof:

(a) If X is noetherian, then $\mathrm{sp}(X)$ is noetherian and by II.2.9, every irreducible closed subset of X has a unique generic point.

(b) Suppose $C \subseteq X$ is a minimal nonempty closed subset of X with X a Zariski space. C is irreducible since it is minimal, thus there is a unique $x \in C$ such that $\mathrm{cl}\{x\} = C$. Suppose that $y \in C$, then $\mathrm{cl}\{y\} = C$ and hence by uniqueness, $x = y$, thus $C = \{x\}$.

(c) Let X be a Zariski space and let $x, y \in X$, then either $y \notin \mathrm{cl}\{x\}$ or $x \notin \mathrm{cl}\{y\}$ for otherwise, we have that $\mathrm{cl}\{x\} = \mathrm{cl}\{y\}$ and hence $x = y$ by uniqueness of the generic points. It follows that $\mathrm{cl}\{x\}^C$ or $\mathrm{cl}\{y\}^C$ works.

(d)Let X be an irreducible Zariski space, and let x be its generic point. Let $U \subseteq X$ be any open subset of X, then U is noetherian since X is and for any irreducible closed subset $C \subseteq U$, $\mathrm{cl}_X(C) \cap U = \mathrm{cl}_U(C)$, thus $\mathrm{cl}_X(C)$ is irreducible and hence has a generic point $y \in \mathrm{cl}_X(C)$. If $y \notin C$, then $y \in \mathrm{cl}_X(C) \cap (X \backslash U)$ which is a closed subset containing y, thus $\mathrm{cl}_X\{y\} \subseteq \mathrm{cl}_X(C) \cap (X \backslash U)$ which does not contain C, a contradication. It follows that $y \in C$. In particular, for $C = X \cap U$, we get that the generic point of X is contained in U. Furthermore, we also get that U is a Zariski space.

(e) Let X be a Zariski space, and let x be minimal w.r.t. specialization ($x > y$ iff $x \rightsquigarrow y$). Let $y \in \mathrm{cl}\{x\}$, then $x \rightsquigarrow y$, thus $y < x$, so x was not minimal. Conversely, let x be maximal w.r.t. specialization. Let $\mathrm{cl}\{x\} \subseteq C$ with C irreducible, then $C = \mathrm{cl}\{y\}$ for some y, thus $y \rightsquigarrow x$, so $x < y$ which is a contradiction. Therefore $\mathrm{cl}\{x\}$ is a maximal irreducible subset of X, i.e. an irreducible component of X. If $C \subseteq X$ is closed, then for any $y \in C$ and $x \in \mathrm{cl}\{y\}$, we have that $x \in \mathrm{cl}\{y\} \subseteq \mathrm{cl}(C) = C$, so $x \in C$. Conversely, if $U \subseteq X$ is open and $y \in U$ such that $y \in \mathrm{cl}\{x\}$ for some x. Suppose that $x \notin U$, then $x \in U^C$ and y specializes x, so $y \in U^C$ which is a contradication. Therefore U is closed under generization.

(f) Let $t(C) \subseteq t(X)$ be an irreducible closed subset of $t(X)$, then C must be irreducible since if $C = Z \cup Z'$, then $t(C) = t(Z) \cup t(Z')$. For any irreducible closed subset $Y \subseteq C$, we and any closed subset $t(Z)$ containing $t(C)$, we have that Z contains C and thus contains Y, so $Y \subseteq \mathrm{cl}_{t(X)}\{C\}$. If C is not the unique generic point of $t(C)$, then suppose there is some Z closed irreducible in X such that $\mathrm{cl}_{t(X)}\{Z\} = t(C)$, then $\mathrm{cl}_{t(X)}\{Z\} = t(Z) = t(C)$, thus $C \subseteq Z$ and $Z \subseteq C$, so $Z = C$. For any descending chain of closed subsets $t(C_1) \supseteq t(C_2) \supseteq \cdots$, we have that C_i, we get that $C_1 \supseteq C_2 \supseteq \cdots$ terminates since X is noetherian, thus the prior terminates as well. Therefore $t(X)$ is a Zariski space. If $\alpha : X \to t(X)$ is a homeomorphism, then since $t(X)$ is a Zariski space, then so is X. If X is a Zariski space, then for any element $Z \in t(X)$, Z is an irreducible closed subset of X and thus there is some point $x \in X$ such that $\mathrm{cl}_X\{x\} = Z$ and hence $\alpha(x) = Z$, so α is surjective. By uniqueness, α is injective, and since t is order preserving, then α is continuous and closed.

II.3.18

Constructible Sets. Let X be a Zariski topological space. A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.
(a) A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible iff it can be written as a finite disjoint union of locally closed subsets.
(b) Show that a constructible subset of an irreducible Zariski space X is dense iff it contains the generic point. Furthermore, in that case it contains a nonempty open subset.
(c) A subset S of X is closed iff it is constructible and stable under specialization. Similarly, a subset T of X is open iff it is constructible and stable under generization.
(d) If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

Proof:

(a) Locally closed subsets of X are subsets of the form $\bigcup_{i=1}^n \mathrm{cl}_U\{x_i\}$. Consider now finite unions of locally closed subsets, i.e. sets of the form $\bigcup_{i=1}^n \mathrm{cl}_{U_i}\{x_i\}$, these are closed under finite unions. Furthermore, since a finite union of locally closed subset is of the form $\bigcup_{i=1}^n U \cap C$, then their complement is $\bigcap_{i=1}^n U^C \cap C^C$ which is just $(\bigcap_{i=1}^n U^C) \cap (\bigcap_{i=1}^n C^C)$ which is locally closed. Thus finite unions of locally closed subsets are exactly the constructible subsets. Additionally, if we have $U \cap C$ and $V \cap D$ not disjoint, then let L be their intersection, then we may write their union as $(U \cap C)\backslash L \cup (V \cap D)\backslash L \cup L$ which is a finite union of disjoint locally closed subsets. It follows that we may take the locally closed subsets to be disjoint.

(b) Suppose that $Z = \bigcup_{i=1}^n (U_i \cap C_i)$ is dense in X. Let U be a minimal element in the set of all intersections of U_i's, then $Z \cap U$ is closed in U and is dense, thus $Z \cap U = U$ for if $Z \neq U$, then Z^C is nonempty and open and does not intersect Z. It follows that Z contains U and thus contains the generic point.

(c) One direction of these is trivial since open and closed subsets are constructible and have the desired properties. If S is constructible and stable under specialization, let $S = \bigcup_{i=1}^n U_i \cap C_i$, then there are points $x_i \in U_i$ with $C_i = \mathrm{cl}_{U_i}\{x_i\}$. Since S is stable under specialization, then $\mathrm{cl}_X\{x_i\} \subseteq S$, thus $\overline{S} = \bigcup_{i=1}^n \overline{U_i \cap C_i} = \bigcup_{i=1}^n \mathrm{cl}_X\{x_i\} \subseteq S$, so $S = \overline{S}$ is thus closed. If $S \subseteq X$ is stable under generization, then we want to show that S^C is stable under specialization. Let $x \in S^C$ and suppose that $y \in \mathrm{cl}\{X\}$ but $y \notin S^C$, then $y \in S$ and $x \notin S$, but x generizes y which is a contradiction. It follows that S^C is constructible and closed under specialization and thus closed, so S is open.

(d) Let $Z = \bigcup_{i=1}^n U_i \cap C_i$ be constructible, then $f^{-1}(Z) = \bigcup_{i=1}^n f^{-1}(U_i) \cap f^{-1}(C_i)$. Since f is continuous, $f^{-1}(Z)$ is constructible.

II.3.19

Let $f : X \to Y$ be a morphism of finite type of noetherian schemes.
(a) Show that $f(X)$ is constructible by reducing to the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.
(b) In the above case, show that $f(X)$ contains a nonempty open subset of Y.
(c) Now use noetherian induction on Y to complete the proof.
(d) Given some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

Proof:

(a) Cover Y by open affine subschemes $U_i = \text{Spec } A_i$, then showing that $f(X)$ is constructible in Y is equivalent to showing that $f(X)$ is constructible in each U_i, for if $f(X) \cap U_i = \bigcup_{j=1}^{n_i} V_{ij} \cap C_{ij}$, then $f(X) = \bigcup_{i,j} V_{ij} \cap C_{ij}$ is constructible. Therefore, it suffices to show that if $f : X \to \text{Spec } A$ is a morphism of finite type, with $X, \text{Spec } A$ noetherian, then $f(X)$ is constructible. Let $X = \bigcup V_i$ where $V_i = \text{Spec } B_i$, then if $f(B_i)$ are all constructible in $\text{Spec } A$, then $f(X)$ is constructible, thus it suffices to show that $f : \text{Spec } B \to \text{Spec } A$ with A, B noetherian and f of finite type, then $f(\text{Spec } B)$ is constructible. We may furthermore restrict only to the irreducible components of $\text{Spec } B$ and $\text{Spec } A$ and additionally since $f(X)$ depends only on the topological map, we may assume that A and B are reduced. It follows that we may take $\text{Spec } A$ and $\text{Spec } B$ to be noetherian, irreducible, and reduced, i.e. noetherian integral schemes. Furthermore, if $f(\text{Spec } B)$ is constructible in $ovf(\text{Spec } B)$, then it is constructible in $\text{Spec } A$, thus we may factor this map through the image subscheme and therefore assume that f is dominant.

(b) Since $f : \text{Spec } B \to \text{Spec } A$ is dominant, then the map on global sections $\varphi : A \to B$ is injective. We have an element $a \in A$ such that for any $\psi : A \to k$ with k algebraically closed, we have that if $\psi(a) \neq 0$, then ψ extends to $\widetilde{\psi} : B \to k$ with $\widetilde{\psi}(1) \neq 0$ (Atiyah-Macdonald 5.23). Let $\mathfrak{p}A_a \in \text{Spec } A_a$, then we obtain a map $\psi : A \to \overline{\text{Frac}(A/\mathfrak{p})}$ and $\ker \psi = \mathfrak{p}$, then we get $\widetilde{\psi}$ with $\widetilde{\psi} \circ \varphi = \psi$, hence $\psi^{-1}(0) = \mathfrak{p} = \varphi^{-1}(\widetilde{\psi}^{-1}(0))$ and since $\widetilde{\psi}(1) \neq 0$, then $\widetilde{\psi}^{-1}(0)$ is a prime ideal of B and thus $f(\text{Spec } B)$ contains $D(a) \subseteq \text{Spec } A$.

(c) Let $\mathscr{P}$ be that $f(X) \cap C$ is constructible for closed $C \subseteq Y$. This trivially holds for the empty set. Suppose that for all $D \subsetneq C$ closed, $f(X) \cap D$ is constructible. If C is not irreducible, then it is a finite union of proper closed subsets and thus the induction holds trivially. If $f(X) \cap C$ is not dense in C, then there is a nonempty open set $U \subseteq C$ such that $f(X) \cap C \cap U = \varnothing$, thus $f(X) \cap C = f(X) \cap (C \backslash U)$ is constructible since $C \backslash U$ is a proper closed subset of C.

If $f(X) \cap C$ is dense in C, then $f : f^{-1}(C) \to C$ is dominant. Furthermore, $f^{-1}(C) \cong \text{Spec } B \times_{\text{Spec } A} \text{Spec } A/I \cong \text{Spec } B/I^e$. Let $f^{-1}(C) = \bigcup_{i=1}^n Z_i$ where Z_i are its irreducible components, then $f(f^{-1}(C)) = \bigcup_{i=1}^n f(Z_i)$ and taking closures, we get that $C = \bigcup_{i=1}^n \text{cl}(f(Z_i))$ and since C is irreducible, then we have that $\text{cl}(f(Z_i)) = C$ for some Z_i since C is irreducible. Z_i is integral by factoring through the reduced induced subscheme structure thus we may apply part (b) and get that $f(Z_i) \subseteq f(f^{-1}(C))$ contains an open subset of C.

Therefore $f(X) \cap C$ contains some $U \subseteq C$ open and hence $f(X) \cap C = f(x) \cap (U \cup C \backslash U) = f(X) \cap U \cup f(x) \cap (C \backslash U)$ and $f(X) \cap U = U$ is open and $C \backslash U$ is a proper closed subset and thus $f(X) \cap (C \backslash U)$ is constructible.

(d) Consider the map $\text{Spec } A[x]_{x(x-1)} \to \text{Spec } A[x]_x \to \text{Spec } A[x, y]$ obtained by inclusion as the first map and the second map being $x \mapsto (x, 1/x)$.

II.4.1

Show that a finite morphism is proper.

Proof:

Let $f : X \to Y$ be a finite morphism, then since being proper is affine local on the target, we need only show that for any Spec $A \subseteq Y$, $f : f^{-1}(\text{Spec } A) \to \text{Spec } A$ is proper. Since f is finite, then $f^{-1} = \text{Spec } B$. Since f is a morphism of affine schemes, then it is separated and since it is finite, then it is of finite type and closed (by II.3.5). It remains to show that f is universally closed, which may be done by showing that finite morphisms are stable under base change.

Let $f : X \to Y$ be finite and let $g : Y' \to Y$ be arbitrary. Let $\widetilde{f} : X \times_Y Y' \to Y'$. By II.3.4, it suffices to show that for an affine cover of Y', $\widetilde{f}$ is finite. To get such an affine cover, let $Y = \bigcup \text{Spec } A_i$, then $g^{-1}(\text{Spec } A_i)$ is covered by Spec B_{ij} which is our affine cover of Y'. It follows that it suffices to show that $\widetilde{f} : g^{-1}(\text{Spec } B_{ij}) \to Y'$ are all finite. To do so, note that $g^{-1}(\text{Spec } B_{ij}) = X \times_Y \text{Spec } B_{ij}$. Furthermore, any map into $X \times_Y \text{Spec } B_{ij}$ factors through $f^{-1}(\text{Spec } A_i) = \text{Spec } C_i$ in the X projection, thus $X \times_Y \text{Spec } B_{ij} = \text{Spec } C_i \times_{\text{Spec } A_i} \times B_{ij}$. It follows that we need only show that $\text{Spec } C_i \otimes_{A_i} B_{ij} \to \text{Spec } B_{ij}$ is finite.

Let C be finite over A and B be any A-algebra, then we want to show that $C \otimes_A B$ is a finite B-module. Let C be generated by $x_1, \cdots, x_n$ as an A-module, then for any $\sum c_i \otimes b_i$, we may expand as follows: $\sum c_i \otimes b_i = \sum(\sum a_{ij}x_j) \otimes b_i = \sum_j x_j \otimes (\sum_i a_{ij}b_i)$. Therefore $C \otimes_A B$ is generated by $x_j \otimes 1$.

II.4.2

Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two $S - morphisms$ of X to Y which agree on an open dense subset of X. Show that $f = h$. Give examples to show that this fails if either (a) X is nonreduced, or (b) Y is nonseparated.

Proof:

Since f and g are both S-morphisms, then the composition of either f or g with the structure morphism of Y gives the structure morphism of X. It follows that we obtain a morphism $h : X \to Y \times_S Y$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$. Now let $Z = X \times_{Y \times_S Y} Y$ with the maps h and Δ. Since Δ is a closed immersion then the map $Z \to X$ is a base change of a closed immersion and hence a closed immersion. Let $U \subseteq X$ be the dense open set on which f and g agree, then we have a map $U \xrightarrow{f} Y$ and $U \subseteq X$, thus after composition with Δ and h respectively, we get the same map, thus we induce a morphism $U \to Z$, hence the image of Z in X contains U. Since U is dense, then $Z \to X$ is surjective. We then have a surjective closed embedding $Z_{\text{red}} \to X$ of reduced schemes. We show any such morphism is an isomorphism. Locally, it is of the form $\text{Spec } A/I \to specA$. The algebraic fact we need to prove reduces to showing that if I is a radical ideal in a reduced ring A and $q : A \to A/I$ has for all primes $\mathfrak{p}$, there is some $\mathfrak{q}$ with $\mathfrak{p} = q^{-1}(\mathfrak{q})$, then we want to show that $I = 0$. Notice that $I = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \bigcap_{\mathfrak{q} \subseteq A/I} f^{-1}(\mathfrak{q}) = \bigcap_{\mathfrak{p} \subseteq A} \mathfrak{p} = \sqrt{0} = 0$. Therefore $Z \to X$ is an isomorphism and thus f and g agree on all of X.

If Y is the line with two origins, then we have two embeddings of $\mathbb{A}^1$ which agree everywhere but at the origin. If $X = \text{Spec } k[x,y]/(xy, x^2)$, then consider maps from X to Spec $k[x,y]$ given by $k[x,y] \mapsto k[x,y]/(xy, x^2)$ with $x \mapsto \overline{x}, y \mapsto 0$ and $x \mapsto ovx, y \mapsto \overline{y}$. Localizing these maps at y, they are the same, and hence they agree on $X \backslash (x)$.

II.4.3

Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Proof:

We first show that $U \cap V$ is isomorphic to $U \otimes_X V$. Let $i_U : U \cap V \to U$ be inclusion and define i_V likewise, then let $j_U : U \to X$ be inclusion and define j_V likewise. Suppose $f : Z \to U$ and $g : Z \to V$ agree up to j_U and j_V, then $\mathrm{im}(f) \subseteq U \cap V$ and $\mathrm{im}(g) \subseteq U \cap V$, thus we get $f' : Z \to U \cap V$ and $g' : Z \to U \cap V$ and $f' = g'$, and this is the unique map into $U \cap V$. Therefore $U \cap V \cong U \times_X V$. We now show that $U \times_X V \cong \Delta^{-1}(U \times_S V)$ where $U \times_S V \to X \times_S X$ is an open immersion and thus may be identified with an open subscheme of $X \times_S X$ which we then pull back through Δ. We show that the latter has the same universal property. Let $f : Z \to U$ and $g : Z \to V$ agree up to j_U and j_V. Let $f' = j_U \circ f$ and $g' = j_V \circ g$, then $f' = g'$ and thus we may consider $Delta \circ f' : Z \to X \times_S X$. Then through the projections p_1, p_2, we get that $p_1 \circ \Delta f' = f'$ and $p_2 \circ \Delta f' = g'$, thus the image of $\Delta \circ f'$ is contained in $U \times_S X \cap X \times_S V$ which is exactly the image of the open immersion $U \times_S V \to X \times_S X$. It follows that $f' : Z \to \Delta^{-1}(U \times_S V)$. Furthermore, $\Delta^{-1}(U \times_s V)$ has maps $i_U \circ \Delta$ and $i_V \circ \Delta$ into U and V. Furthermore, f' is the only map such that $i_U \circ \Delta \circ f' = f$ and $i_V \circ \Delta \circ f' = g$. It follows that $U \cap V \cong U \times_X V \cong \Delta^{-1}(U \times_S V)$ and since Δ is a closed immersion, it is affine, and thus $U \cap V$ is affine.

Let X be the affine plane with two origins, then let U and V be the two immersion of the affine plane, then their intersection is the affine plane remove the origin which is not affine.

II.4.4

Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S. We refer to this result by saying that "the image of a proper scheme is proper".

Proof:

We obtain the graph morphism as the product of the morphisms $\mathrm{id} : X \to X$ and $f : X \to Y$ to get $\Gamma_f : X \to X \times_S Y$. Consider the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\
X \times_S Y \arrow[r, "(f \circ p_1, p_2)"'] & Y \times_S Y
\end{tikzcd}$$

Notice that giving a morphism $g : W \to X \times_S Y$ and $h : W \to Y$ which compose correctly with Δ and $(f \circ p_1, p_2)$ yields a morphism $p_1 \circ g : W \to X$ such that $f \circ p_1 \circ g = q_1 \circ (f \circ p_1, p_2) \circ g$ where $q_1 : Y \times_S Y \to Y$. Then $q_1 \circ (f \circ p_1, p_2) \circ g = q_1 \circ \Delta \circ h = h$. Furthermore, $\Gamma_f \circ p_1 \circ g = (p_1 \circ g, f \circ p_1 \circ g)$, thus we wish to show tat $f \circ p_1 \circ g = p_2 \circ g$. Notice that $(f \circ p_1, p_2) \circ g = (f \circ p_1 \circ g, p_2 \circ g) = \Delta \circ h = \Delta \circ f \circ p_1 \circ g$, hence applied q_2 yields $p_2 \circ g = f \circ p_1 \circ g$ as desired. It follows that the morphism

$p_1 \circ g$ makes the diagram commute. Furthermore, if we have a morphism $l : Z \to X$ making the diagram commute, then we immediately have that $p_1 \circ \Gamma_f \circ l = l$ and also composes to $p_1 \circ g$, thus $l = p_1 \circ g$. It follows that $X \cong Y \times_{Y \times_S Y} (X \times_S Y)$. Therefore, Γ_f is the base change of Δ and since Y is separated, then Δ and hence Γ_f are closed immersions. We may replace X with Z, then $p_2 : Z \times_S Y \to Y$ is a base change of the proper morphism $Z \to S$ and hence is proper and thus closed. $f(Z) = (p_2 \circ \Gamma_f)(Z)$. Since Γ_f is a closed immersion and p_2 is closed, then $f(Z)$ is closed in Y.

$f(Z) \to S$ is of finite type and separated since it is the composition of the morphism from $f(Z)$ to Y is a closed immersion and hence finite type and separated, and $Y \to S$ is finite type and separated. $f : Z \to f(Z)$ is surjective and continuous and $Z \to S$ is closed since it is proper. For any $C \subseteq f(Z)$ closed, let $s_{f(Z)}$ denote the structure map of $f(Z)$ and s_Z denote that of Z, then $s_{f(Z)}(C) = s_{f(Z)}(f(f^{-1}(C))) = s_Z(f^{-1}(C))$ is closed. For any morphism $g : X \to S$, we obtain a base change $\widetilde{s_{f(Z)}} f(Z) \times_S X \to X$ and we additionally get a base change of f and s_Z, $\widetilde{f} : Z \times_S X \to f(Z) \times_S X$ and $\widetilde{s_Z} : Z \times_s X \to X$ which is still closed since $Z \to S$ is universally closed. It remains to be shown that $\widetilde{f}$ remains surjective.

Let L/K and E/K be extensions of some field K, then let $R = E \otimes_K L$. $R \neq 0$ since it is a tensor product of nonzero vector spaces. Let $\mathfrak{m} \subseteq R$ be a maximal ideal, then $E \to R/\mathfrak{m}$ and $L \to R/\mathfrak{m}$ are inclusions, thus for any two field extensions of K, there is a common extension of both.

Let $f : X \to Y$ be a surjective morphism, then for any field K and morphism $\mathrm{Spec}\, K \to Y$ with image $y \in Y$, the morphism factors through $\mathrm{Spec}\, K \to \mathrm{Spec}\, \kappa(y)$ by inclusion of fields. Since f is surjective, there is some point $x \in X$ with $f(x) = y$, then $f_x^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a local homomorphism and thus induces an inclusion $\kappa(y) \to \kappa(x)$. Let L be a common extension of both K and $\kappa(x)$ over $\kappa(y)$, then we have the following commutative diagram,

$$\begin{tikzcd}
\mathrm{Spec}\, L \arrow[r] \arrow[d] & \mathrm{Spec}\, \kappa(x) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}\, \mathrm{K} \arrow[r] & \mathrm{Spec}\, \kappa(y) \arrow[r] & Y
\end{tikzcd}$$

Conversely, if for any $\mathrm{Spec}\, K \to Y$ we have such a morphism $\mathrm{Spec}\, L \to X$ making the above diagram commute, then f is clearly surjective. In fact, it suffices for such L to exist for $K = \kappa(y)$ for all points $y \in Y$.

Now let $f : X \to Y$ be a surjective morphism of S-schemes and let $g : Z \to S$ be any morphism. We want to show that $\widetilde{f} : X \times_S Z \to Y \times_S Z$ is surjective. Let $z \in Y \times_S Z$, then we obtain morphisms $\mathrm{Spec}\, \kappa(z) \to Y$ and $\mathrm{Spec}\, \kappa(z) \to Z$ which agree up to the structure morphisms of Y and Z. Since we have $\mathrm{Spec}\, \kappa(z) \to Y$ and f is surjective, then there is some $\mathrm{Spec}\, L \to X$ such that the following commutes:

$$\begin{tikzcd}
\text{Spec } L \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\text{Spec } \kappa(z) \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & S
\end{tikzcd}$$

And therefore we obtain a morphism Spec $L \to X \times_S Z$ such that the map to Spec $L \to$ Spec $\kappa(z)$ and the morphism Spec $\kappa(z) \to Y \times_S Z$ and $\widetilde{f} : X \times_S Z \to Y \times_S Z$ commute. Therefore $\widetilde{f}$ is surjective.

II.4.5

Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k has *center* x on X if its valuation ring dominates the local ring $\mathcal{O}_{X,x}$.
(a) If X is separated over K, then the center of any valuation K/k on X (if it exists) is unique.
(b) If X is proper over k, then every valuation of K/k has a unique center on X.
(c) Prove the converses of (a) and (b).
(d) If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$.

Proof:

(a) X is of finite type over a field and thus noetherian. Suppose that R is valuation ring centered on x and y and let $\xi \in X$ be the generic point. x and y are specializations of ξ, then $\overline{\xi} = X$ and R dominates $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$, thus we have two morphisms Spec $R \to X$. Furthermore, since R is valuation ring of K/k, then we obtain a morphism from inclusion Spec $k \to$ Spec R. By the valuative criterion, both morphisms Spec $R \to X$ must be the same, hence $x = y$.

(b) By the valuative criterion, there exists a map Spec $R \to X$, thus R dominates some $\mathcal{O}_{X,x}$ and hence has center x for some $x \in X$.

(c) Suppose that for any integral closed subscheme $Z \subseteq X$, Z has property (a), then let Spec $K \to X$ and f, g : Spec $R \to X$, then f, g factor through Z since their images are contained in closure of the image of Spec K which we will say is Z. We then have that f, g correspond to the ring R dominating local rings $\mathcal{O}_{Z,x}$ and $\mathcal{O}_{Z,y}$ and hence having centers on x and y. R is a valuation ring of K/k, however since Spec $K \mapsto \xi$, the generic point of Z, then $K(Z) \subseteq K$ and $\mathcal{O}_{Z,x}$ and $\mathcal{O}_{Z,y}$ are subrings of $K(Z)$ and thus $R \cap K(Z)$ is a valuation ring for $K(Z)/k$ (for any $x \in K(Z)$, either x or x^{-1} is in R). Therefore $x = y$ and thus $f = g$ since R dominates the same local rings and the morphisms from U are the same. It remains to be proven that all integral closed subschemes have property (a).

(d) Let $a \in \Gamma(X, \mathcal{O}_X)$ and suppose that $a \notin k$, then $\mathcal{O}_X(X) \subseteq K$ since if for some $s \in \mathcal{O}_X(X)$, $[s] = 0$, then there is an affine open subset Spec $A \subseteq X$ such that $[s|_{\text{Spec } A}] = 0$, but $s \in A$ and

$K = \mathrm{Frac}(A)$, so $s = 0$ on Spec A and hence $s = 0$ since Spec A is an open dense subset of X.

Since k is algebraically closed, then k is integrally closed in K, and since $a \notin k$, then there is some valuation ring R of K/k with $a \notin R$ and thus $a^{-1} \in R$ and $a \notin R$ thus a^{-1} is not a unit in R and hence $a^{-1} \in \mathfrak{m}_R$. By (b), there is a point $x \in X$ such that R dominates $\mathcal{O}_{X,x}$. $a \in \mathcal{O}_X(X)$, hence the germ of a at x is in $\mathcal{O}_{X,x}$. However, $\mathcal{O}_{X,x} \subseteq R$ and $a \notin R$ which is a contradiction.

II.4.6

Let $f : X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism.

Proof:

Let f : Spec $B \to$ Spec A be a proper morphism of affine varieties over k. We then have that f : Spec $B \to f(\text{Spec } B)$ is a proper morphism of affine varieties. Showing that f is finite is then equivalent to showing that f : Spec $B \to f(\text{Spec } B)$ is finite. It follows that we may assume that f is dominant. Let f induce $\varphi : A \to B$ which is injective. Then $A \subseteq \mathrm{Frac}(B)$ and $\overline{A}$ is the intersection of all valuation rings of $\mathrm{Frac}(B)$ containing A. Let R be a valuation ring in $\mathrm{Frac}(B)$ containing A, then since f is proper, we get a morphism Spec $R \to$ Spec B and hence a morphism $B \to R$. Furthermore, since it must commute with $B \subseteq \mathrm{Frac}(B)$ and $R \subseteq \mathrm{Frac}(B)$, then it is an actual inclusion $B \subseteq R$ and hence $B \subseteq \overline{A}$. Since φ is of finite type and B is integral over A, then it follows that φ is finite and thus f is a finite morphism.

II.4.7

Schemes Over $\mathbb{R}$. For any scheme X_o over $\mathbb{R}$, let $X = X_0 \times_{\mathbb{R}} \mathbb{C}$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to $\mathbb{C}$. Then X is a scheme over $\mathbb{C}$, and σ is a *semi-linear* automorphism, in the sense that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ \mathbb{C} & \xrightarrow[\alpha]{} & \mathrm{Spec}\ \mathbb{C} \end{array}$$

Since $\sigma^2 = \mathrm{id}$, we call σ an *involution.*

(a) Now let X be a separated scheme of finite type over $\mathbb{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. Show that there is a unique separated scheme X_0 of finite type over $\mathbb{R}$ such that $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbb{R}} \mathbb{C}$ described above.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbb{R}$, and X, σ will denote the corresponding scheme with involution over $\mathbb{C}$.

(b) Show that X_0 is affine iff X is.

(c) If X_0, Y_0 are two such schemes over $\mathbb{R}$, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involution, i.e. $f \circ \sigma_X = \sigma_Y \circ f$.

(d) If $X \cong \mathbb{A}^1_{\mathbb{C}}$, then $X_0 \cong \mathbb{A}^1_{\mathbb{R}}$.

(e) If $X \cong \mathbb{P}^1_{\mathbb{C}}$, then either $X_0 \cong \mathbb{P}^1_{\mathbb{R}}$ or X_0 is isomorphic to the conic in $\mathbb{P}^2_{\mathbb{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

Proof:

II.4.8

Let $\mathscr{P}$ be a property of morphisms of schemes such that:
(a) a closed immersion has $\mathscr{P}$.
(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$.
(c) $\mathscr{P}$ is stable under base extension.
Then show that:
(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$.
(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$.
(f) If $f : X \to Y$ has $\mathscr{P}$, then $f_{red} : X_{red} \to Y_{red}$ has $\mathscr{P}$.

Proof:
(d) Consider the following diagram:

$$\begin{tikzcd}
& X \times_S Y \arrow[r] \arrow[dl] & X' \times_S Y \arrow[r] \arrow[dl] \arrow[dr] & X' \times_S Y' \arrow[dr] & \\
X \arrow[rr, "\mathscr{P}"] & & X' \quad Y \arrow[rr, "\mathscr{P}"] & & Y'
\end{tikzcd}$$

Note that $X \times_S Y \cong (X \times_{X'} X') \times_S Y \cong X \times_{X'} (X' \times_S Y)$ is a fibre product and similarly, $X' \times_S Y \cong Y \times_S X' \cong (Y \times_{Y'} Y') \times_S X' \cong Y \times_{Y'} (Y' \times_S X)$. Therefore the morphisms $X \times_S Y \to X' \times_S Y$ and $X' \times_S Y \to X' \times_S Y'$ have $\mathscr{P}$ and thus their composition which is the desired product has $\mathscr{P}$.

(e) Since g is separated, then $Y \to Y \times_Z Y$ is a closed immersion and therefore has $\mathscr{P}$. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Gamma_f"'] & Y \arrow[d, "\Delta"] \\
X \times_Z Y \arrow[r, "(f \circ p_1, p_2)"'] & Y \times_Z Y
\end{tikzcd}$$

As shown in II.4.4, the above is a Cartesian square, i.e. X is the fibre product $Y \times_{Y \times_Z Y} (X \times_Z Y)$. It follows that Γ_f has $\mathscr{P}$, then $f = p_2 \circ \Gamma_f$ by definition of the fibre product $X \times_Z Y$, thus it remains to show that $p_2 : X \times_Z Y \to Y$ has $\mathscr{P}$. This follows from the following diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d, "p_2"'] & X \arrow[d, "g \circ f"] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Therefore $f = p_2 \circ \Gamma_f$ has $\mathscr{P}$.

(f) We have that $X_{red} \to Y$ is given by composing $X_{\mathrm{red}} \to X$ which has $\mathscr{P}$ since it is a closed immersion and then $X \to Y$ which has $\mathscr{P}$ by assumption. We then get $X_{\mathrm{red}} \to Y_{\mathrm{red}} \to Y$ with $Y_{\mathrm{red}} \to Y$ a closed immersion and hence separated and the composition has $\mathscr{P}$, thus $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$.

II.4.9

Show that a composition of projective morphisms is projective. Conclude that projective morphisms have properties (a)-(f) of II.4.8 above.

We first construct the Segre embedding from $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$ by mapping $U_i \times U_j \cong \text{Spec } \mathbb{Z}[x_0/x_i, \cdots x_r/x_i] \otimes_{\mathbb{Z}} \mathbb{Z}[x_0/x_j, \cdots, x_s/x_j] \to \text{Spec } \mathbb{Z}[x_{00}/x_{ij}, \cdots x_{rs}/x_{ij}] \cong U_{ij}$ by the morphism $\mathbb{Z}[x_{00}/x_{ij}, \cdots x_{rs}/x_{ij}] \to \mathbb{Z}[x_0/x_i, \cdots x_r/x_i, x_0/x_j, \cdots, x_s/x_j]$ by $x_{kl}/x_{ij} \mapsto \frac{x_k}{x_i}\frac{x_l}{x_j}$. These then glue to give a morphism $\sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{rs+r+s}$. To check that it is a closed immersion, we note first that the maps on affine a surjective and hence on the morphism is a closed immersion, assuming that $\sigma^{-1}(U_{ij})$ is just $U_i \times U_j$. Let $p \in U_k \times U_l$ map into U_{ij} and suppose that $p \notin U_i \times U_j$, then $p \in V(\frac{x_i}{x_k}) \times V(\frac{x_j}{x_l}) = V(\frac{x_i x_j}{x_k x_l})$. We have that $U_{kl} \backslash U_{ij} = V(x_{ij}/x_{kl})$ and $\sigma^{-1}(V(x_{ij}/x_{kl}) = V(\sigma(\frac{x_{ij}}{x_{kl}}) = V(\frac{x_i x_j}{x_k x_l})$ in $U_k \times U_l$, but notice that this means that $p \notin U_{ij}$.

Let $f : X \to Y$ and $g : Y \to Z$ be projective morphisms. Then consider the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \mathbb{P}^r_Y \arrow[r] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s_Z \arrow[r] \arrow[d] & \mathbb{P}^{rs+r+s}_Z \arrow[ddl] \\
 & Y \arrow[r] \arrow[dr] & \mathbb{P}^s_Z \arrow[d] & \\
 & & Z &
\end{tikzcd}$$

The morphism $\mathbb{P}^r_Y \to \mathbb{P}^r \times \mathbb{P}^s_Z$ is the product of the identity and the closed immersion $Y \to \mathbb{P}^s_Z$ and is thus a closed immersion. Similarly, we then get $\mathbb{P}^r \times P^s \times Z \to \mathbb{P}^{rs+r+s}_Z$ by the Segre embedding and the identity, hence another closed immersion. Thus we obtain a closed immersion $X \to \mathbb{P}^{rs+r+s}_Z$ which $X \to Z$ factors through as desired.

Notice for any closed immersion $Z \to X$, it trivially factors through $Z \to \mathbb{P}^0_X = X$. For any $X \to Y$ factoring through a closed immersion $X \to \mathbb{P}^n_Y$, then we get that $X \times_Y Y' \to Y'$ factors through $X \times_Y Y' \to \mathbb{P}^n_{Y'} = P^n_Y \times_Y Y'$ is thus a product of closed immersions which is a closed immersion as desired.

II.4.10

Chow's Lemma. This results says that proper morphisms are fairly closed to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism $g^{-1}(U) \to U$. Prove this result in the following steps.
(a) Reduce to the case X irreducible.
(b) Show that X can be covered by a finite number of open subsets $U_i, i = 1, \cdots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.
(c) Let $U = \bigcap U_i$, and consider the map

$$f : U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure $\text{cl}(f(U))$. Let $f : X' \to X$ be the projection onto the first factor, and let $h : X' \to P = P_1 \times_s \cdots \times_S P_n$

be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.
(d) Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

ACTUALLY DO THIS QUESTION, IT IS INTERESTING. The first part is done by cutting out the intersections of the irreducible components and the second is done by mapping an affine covering (quotient of f.g. polynomials) into its projective space in 1 more variable. For c, you show that for $f : X \to Y$ and $U \subseteq X$ open such that $f : U \to Y$ is an open immersion, then if f is separated, then $f : \overline{U} \to Y$ is closed and if f is proper, then it is a closed immersion (use the valuative criterion to show stable under specialization).

II.5.1

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathscr{E}$, denoted $\check{\mathscr{E}}$, to be the sheaf $\mathscr{H}om(\mathscr{E}, \mathcal{O}_X)$.

((a) Show that $(\check{\mathscr{E}})\check{} = \mathscr{E}$

((b) For any $\mathcal{O}_X$-module $\mathscr{F}$, $\mathscr{H}om(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.

((c) For any $\mathcal{O}_X$-module $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om(\mathscr{E}, \mathscr{G}))$

((d) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.

Proof:

(a) We define a morphism from $(\check{\mathscr{E}})\check{} \to \mathscr{E}$ and then show that it is locally an isomorphism and hence an isomorphism. Let $U \subseteq X$ be open, then let $\varphi \in (\check{\mathscr{E}})\check{}(U)$, i.e. $\varphi : \mathscr{H}om(\mathscr{E}, \mathcal{O}_X)|_U \to \mathcal{O}_X|_U$, then $\varphi(U) : \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathscr{E}|_U, \mathcal{O}_X|_U) \to \mathcal{O}_X(U)$. We then cover U by affine open subsets U_i, and since $\mathscr{E}$ is locally free of finite rank, then we will have that $\mathscr{E}|_{U_i} \cong \mathcal{O}_X^{n_i}|_{U_i}$ by some isomorphism $\psi_i : \mathcal{O}_X^{n_i}|_{U_i} \to \mathscr{E}|_{U_i}$. Restricting $\varphi(U)$ to U_i, we get $\varphi(U)|_{U_i} : \mathrm{Hom}_{\mathcal{O}_X|_{U_i}}(\mathcal{O}_X^{n_i}|_{U_i}, \mathcal{O}_X|_{U_i}) \to \mathcal{O}_X|_{U_i}$. Let π_j be the projection from $\mathcal{O}_X^{n_i}|_{U_i}$ to the j^{th} component, this is clearly a morphism of $\mathcal{O}_X|_{U_i}$-modules. Let $\varphi(U)|_{U_i}$ map π_j to s_{ij}. Let $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ in $\mathcal{O}_X^{n_i}|_{U_i}$, then we map φ to $\sum_j s_{ij}\psi_i(U_i)(e_j) = \psi_i(U_i)(\sum_j s_{ij}e_j)$. We now check that these elements in $\mathscr{E}(U_i)$ glue to an element in $\mathscr{E}(U)$.

Checking that they glue amounts to checking the following: Let $\psi_1, \psi_2 : \mathcal{O}_X^n \to \mathscr{F}$ be isomorphisms of $\mathcal{O}_X$-modules and let $\varphi_1, \varphi_2 : \mathrm{Hom}(\mathcal{O}_X^n, \mathcal{O}_X) \to \mathcal{O}_X(X)$ such that $\varphi_2(f) = \varphi_1(f \circ \psi_2^{-1} \circ \psi_1)$, then $\sum_i \varphi_1(\pi_i)\psi_1(X)(e_i) = \sum_i \varphi_2(\pi_i)\psi_2(X)(e_i)$. We may apply $\psi_1^{-1}(X)$ to both sides and expand φ_2 to get the following on the RHS, $\sum_i \varphi_1(\pi_i \circ \psi_2^{-1} \circ \psi_1)\psi_1(X)^{-1}\psi_2(X)(e_i)$. For any morphism $g : \mathcal{O}_X^n \to \mathcal{O}_X$, we have that $g(X) = \sum s_i\pi_i(X)$ for some s_i, then for any $U \subseteq X$ and $x \in \mathcal{O}_X^n(U)$, we have that $g(U)(x) = g(U)(\sum x_ie_i) = \sum x_ig(U)(e_i) = \sum x_ig(X)(e_i)|_U = \sum x_i(s_i|_U) = \sum s_i|_U\pi_i(U)(x)$, hence $g(U) = \sum_i s_i|_U\pi_i(U)$. It follows that any morphism $\mathcal{O}_X^n \to \mathcal{O}_X$ may be obtained by taking dot products with the restrictions of some element $s \in \mathcal{O}_X(X)^n$. Furthermore, any morphism $\mathcal{O}_X^n \to \mathcal{O}_X^n$ is given by some matrix $M \in \mathcal{O}_X(X)^{n \times n}$. It follows that we may expand $\pi_i \circ \psi_2^{-1} \circ \psi_1$

as $\sum_j (\pi_i \circ \psi_2^{-1}\psi_1)(X)(e_j)\pi_j$. It follows that the RHS may be expanded as:

$$\sum_j \varphi_1(\pi_j)\left[\sum_i (P_i M^{-1} e_j) M e_i\right]$$

Here P_i is the $1 \times n$ matrix for the projections and M is the $n \times n$ matrix for the $\psi_1^{-1} \circ \psi_2$. We may evaluate the inside sum by expanding all of the terms and we end up with e_j, thus the RHS is $\sum_j \varphi(\pi_j) e_j$ as desired.

It is now clear that this induces an isomorphism, since it induces an isomorphism on open subsets on which $\mathscr{E}$ is free (it is the classical isomorphism of the double dual of a module).

(b) In the case of modules, we have that for any M, N being R-modules, then for any $x \in \operatorname{Hom}(M, R) \otimes_R N$, it has the form $x = \sum \varphi_i \otimes n_i$ and we map it to $\varphi : M \to N$ given by $\varphi(s) = \sum \varphi_i(s) n_i$. Analogously, for any x in the sections over U of the presheaf associated $\check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$, we have that $x \in \check{\mathscr{E}}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$. $\check{\mathscr{E}}(U) = \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{E}|_U, \mathcal{O}_X|_U)$, so $x = \sum \varphi_i \otimes f_i$ where $\varphi_i : \mathscr{E}|_U \to \mathcal{O}_X|_U$. Similar to what was done for modules, we now define $\varphi : \mathscr{E}|_U \to \mathscr{F}|_U$. For any $V \subseteq U$ and $s \in \mathscr{E}(V)$, let $\varphi(U)(s) = \sum \varphi_i(V)(s) f_i|_V \in \mathscr{F}(V)$. We now show that this map is an isomorphism. We need only show that the map is an isomorphism locally and thus we may assume that $\mathscr{E} = \mathcal{O}_X^n$, then may described the map from $\mathscr{H}om(\mathcal{O}_X^n, \mathscr{F}) \to \widetilde{\mathcal{O}_X^n} \times_{\mathcal{O}_X} \mathscr{F}$ by taking an open set $U \subseteq X$ and $\varphi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$, then we map it to an element of $\operatorname{Hom}(\mathcal{O}_X^n|_U, \mathcal{O}_X|_U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$. We map φ to $\sum \pi_i|_U \otimes \varphi(U)(e_i)$. Showing that this is indeed an inverse to the previous map amounts to showing that morphisms from $\mathcal{O}_X^n \to \mathscr{F}$ are entirely determined by their global section and this is almost identical to what was shown in a. It follows that these sheaves are isomorphic.

(c) For R-modules M, N, T, we have a map $\operatorname{Hom}(M \otimes_R N, T) \to \operatorname{Hom}(N, \operatorname{Hom}(M, T))$ given taking a morphism $\varphi : M \otimes_R N \to T$ and mapping it to ψ where ψ such that $\psi(s) : M \to T$ is given by taking an element $x \in M$ and $\psi(s)(x) = \varphi(s \otimes x)$. This is then an isomorphism if M is free and of finite rank. In the case of $\mathcal{O}_X$-modules, we take a morphism $\varphi : \mathscr{E} \otimes \mathscr{F} \to \mathscr{G}$, then φ yields a morphism from the presheaf of $\mathscr{E} \otimes \mathscr{F}$ to $\mathscr{G}$. For any $U \subseteq X$, we have $\varphi(U) : \mathscr{E}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U) \to \mathscr{G}(U)$. We then define $\psi : \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{G})$, let $U \subseteq X$, then for any $x \in \mathscr{F}(U)$, we obtain a morphism $\psi_x : \mathscr{E}|_U \to \mathscr{G}|_U$ given by $\psi(U)(x)(V)(s) = \varphi(V)(s \otimes x)$. We check that ψ is in fact a morphism of $\mathcal{O}_X$-modules. Each $\psi(U)$ is a morphism of $\mathcal{O}_X(U)-$modules by the properties of tensor product. For any $W \subseteq U$, we have that $\psi(U)(x)|_W$ is obtained by just restricting $\varphi(U)(x)$ to $\mathscr{E}|_W \to \mathscr{G}|_W$, and thus it is clear that $\psi(U)(x)|_W(V)(s) = \psi(U)(x)(V)(s)$ when $V \subseteq W$, and $\psi(U)(x)(V)(s) = \psi(W)(x)(V)(s)$, thus it indeed commutes with restriction. We now show that we can go back from $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om(\mathscr{E}, \mathscr{G}))$. Given $\psi : \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{G})$, we want to define $\varphi : \mathscr{E} \otimes \mathscr{F} \to \mathscr{G}$ by $\varphi(U)(e \otimes x) = \psi(U)(x)(U)(e)$. Showing that these are inverse to each other amounts again to showing that morphisms form locally free modules are determined by their global sections which is true.

(d) For any $\mathcal{O}_Y$-module $\mathscr{G}$, we have a morphism $\mathscr{G} \to f_* f^* \mathscr{G}$. For any $U \subseteq Y$, $f^{-1}\mathscr{G}(f^{-1}(U)) = \varinjlim_{V \supseteq f(f^{-1}(U))} \mathscr{G}(V)$ and notice that $U \supseteq f(f^{-1}(U))$, thus we may map $x \in \mathscr{G}(U)$ to $[x] \in f^{-1}\mathscr{G}(f^{-1}(U))$ and then to $[x] \otimes 1 \in f^{-1}\mathscr{G}(f^{-1}(U)) \otimes_{f^{-1}\mathcal{O}_Y(f^{-1}(U))} \mathcal{O}_X(f^{-1}(U))$ thus obtaining a morphism $\mathscr{G} \to f_* f^* \mathscr{G}$. We now give a morphism $f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E} \to f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^* \mathscr{E})$. For any $U \subseteq Y$, consider the map $\mathscr{F}(f^{-1}(U)) \times \mathscr{E}(U) \to \mathscr{F}(f^{-1}(U)) \otimes_{\mathcal{O}_X(f^{-1}(U))} f_* f^* \mathscr{E}(U)$ given by the product of the identity and the map given before. This is bilinear and thus defines a morphism $\mathscr{F}(f^{-1}(U)) \otimes_{\mathcal{O}_Y(U)} \mathscr{E}(U) \to \mathscr{F}(f^{-1}(U)) \otimes_{\mathcal{O}_X(f^{-1}(U))} f_* f^* \mathscr{E}(U)$. This then defines a morphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^* \mathscr{E}) \to f_* \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{E}$. One then has to check that when $\mathscr{E} = \mathcal{O}_Y^n$, this is an isomorphism. Notice that the RHS is $f_* \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^n \cong (f_* \mathscr{F})^n \cong f_*(\mathscr{F}^n)$. The LHS is $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^* \mathcal{O}_Y^n)$ and $f^* \mathcal{O}_Y^n = f^{-1} \mathcal{O}_Y^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong (f^{-1}\mathcal{O}_Y)^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X^n$, thus we get $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \cong f_*(\mathscr{F}^n)$ as desired.

II.5.2

Let R be a discrete valuation ring with quotient field K, and let $X = \text{Spec } R$.

((a) To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

((b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Proof:

A DVR has two points, a maximal ideal generated by some element t of value 1 and the zero ideal. Since $D(t) = \text{Spec } R \backslash \{\mathfrak{m}\}$, then $\mathcal{O}_X(D(t)) = \mathcal{O}_X(\{0\}) = R_t = \text{Frac}(R)$ and $\mathcal{O}_X(X) = R$. Note that the open sets in spec of a DVR are $\varnothing$, $\{0\}$ and X. Given an $\mathcal{O}_X$-module $\mathscr{F}$, we get an R-module $\mathscr{F}(X) = M$ and a $\text{Frac}(R) = K$ vector space $\mathscr{F}(\{0\}) = L$ and a restriction map $\rho : \mathscr{F}(X) \to \mathscr{F}(\{0\})$, i.e. $\rho : M \to L$ which is a morphism of R-modules such that for any $x \in R$ and $m \in M$, $x\rho(m) = x|_{\{0\}}\rho(m)$, thus we have $\rho : M \times K \to L$ given by $(m, x) \mapsto x\rho(m)$. Since this is bilinear as a map of R-modules (note how R acts on K), then we obtain a morphism $\rho : M \otimes_R K \to L$.

Conversely, given M, L, and ρ, we may define an $\mathcal{O}_X$-module $\mathscr{F}$ with $\mathscr{F}(X) = M, \mathscr{F}(\{0\}) = L$ and restriction $M \to L$ given by $m|_{\{0\}} = \rho(m \otimes 1)$ we need to check that given an element $m \in M$ and $x \in R$ that $(xm)|_{\{0\}} = x|_{\{0\}}m|_{\{0\}}$. $\rho((xm) \otimes 1) = x\rho(m \otimes 1) = xm|_{\{0\}}$ and R acts on L by $xm|_{\{0\}} = x|_{\{0\}}m|_{\{0\}}$.

$\mathscr{F}$, an $\mathcal{O}_X$-module, is quasi-coherent iff $\mathscr{F} \cong \widetilde{M}$ iff $L \cong M_t = M \otimes_R R_t = M \otimes_R K$. We need only show the morphism $M \otimes_R K \to L$ is actually ρ. It is obtained through the definition of quasi-coherence. If $\mathscr{F}$ is quasi-coherent, then $\mathscr{F}(\{0\})$ is all elements m/x where $m \in M, x \in R_t = K$, i.e. all elements xm. The restriction in this case from $M \to L$ is $m \mapsto 1m$ which is injective and thus ρ is injective since K is flat over R by virtue of being a localization. By definition of L in the quasi-coherent case, ρ is surjective and thus an isomorphism. If ρ is an isomorphism, then every $L \cong M_t$ and we are done.

II.5.3

Let $X = \text{Spec } A$ be an affine scheme. Show that the functors $\tilde{}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$, there is a natural isomorphism

$$\text{Hom}_A(M, \Gamma(X, \mathscr{F})) \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$$

Proof:

Given any $\varphi : M \to \mathscr{F}(X)$, we get a morphism of $\mathcal{O}_X$-modules $\widetilde{\varphi} : \widetilde{M} \to \mathscr{F}$ such that for any $U = D(f) \subseteq X$, $\widetilde{\varphi}(U) : M_f \to \mathscr{F}(D(f))$ is given by $\widetilde{\varphi}(U)(\frac{x}{f^n}) = \frac{1}{f^n}\varphi(x)|_U$. The inverse is given by taking $\widetilde{\varphi} : \widetilde{M} \to \mathscr{F}$ and taking the global section. These are inverses to each other since $\widetilde{\psi}$ is a morphism of $\mathcal{O}_X$-modules, thus the map must be as defined on $D(f)$.

II.5.4

Show that a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

Proof:

If $\mathscr{F}$ is quasi-coherent and $x \in X$, then there is an affine open neighborhood U of x and $\mathscr{F}|_U \cong \widetilde{M}$. Let $\{a_i\}_{i \in I}$ be a generating set of M, then let N be the kernel of the map from $\mathcal{O}_X(U)^I \to M$, then we get the following exact sequence,

$$0 \to N \to \mathcal{O}_X(U)^n \to M \to 0$$

and localizing at $f \in \mathcal{O}_X(U)$, we get,

$$0 \to N_f \to \mathcal{O}_X(D(f))^I \to M_f \to 0$$

It follows that $\widetilde{M}(D(f)) \cong (\mathcal{O}_X^I/\widetilde{N})(D(f))$, hence

$$0 \to \widetilde{N} \to \mathcal{O}_X^I \to \widetilde{M} \to 0$$

We may now do the same for $\widetilde{N}$ to get an exact sequence,

$$\mathcal{O}_X^J \to \mathcal{O}_X^I \to \widetilde{M} \to 0$$

Thus $\widetilde{M}$ is the cokernel of a morphism of free $\mathcal{O}_X$-modules.

Conversely, suppose that $\mathscr{F}$, an $\mathcal{O}_X$-module with X affine, is the cokernel of a morphism of free $\mathcal{O}_X$-modules, i.e. we have an exact sequence,

$$\mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathscr{F} \to 0$$

Then taking the sections over $D(f)$, we get the exact sequence

$$\mathcal{O}_X(D(f))^J \to \mathcal{O}_X(D(f))^I \to \mathscr{F}(D(f)) \to 0$$

Let $X = \text{Spec } A$, then we have,

$$A_f^J \to A_f^I \to \mathscr{F}(D(f)) \to 0$$

Which one then recognizes as the localization of

$$A^J \to A^I \to \mathscr{F}(X) \to 0$$

and since localization commutes with quotient, it follows that $\mathscr{F}(D(f)) = \mathscr{F}(X)_f$, thus $\mathscr{F} \cong \widetilde{\mathscr{F}(X)}$

The statement for coherent sheaves is analogous. The only reason we require that X be noetherian is to use 5.4 to get that for any open affine, you have such an equivalence.

II.5.5

Let $f : X \to Y$ be a morphism of schemes.

(a) Show by example that if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism.

(c) If f is a finite morphism of noetherian schemes, and if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ is coherent on Y.

Proof:

(a) Let f : Spec $k[x,y]/(xy-1) = X \to$ Spec $k[x] = Y$, then $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module, however $f_*\mathcal{O}_X(Y) = \mathcal{O}_X(X) = k[x,y]/(xy-1)$ which is not a finitely generated $k[x]$-module .

(b) Closed immersions are locally quotients and thus trivially finite.

(c) Finite morphisms are affine morphisms. Let Spec $A = U \subseteq Y$ be affine, then Spec $B = f^{-1}(U)$ is affine in X and furthermore B is finite over A and thus any module which is finite over B is finite over A. Since $\mathscr{F}(f^{-1}(U))$ is a finite B-module, then $f_*\mathscr{F}(U)$ is a finite A-module and thus $f_*\mathscr{F}$ is quasi-coherent.

II.5.6

Support. Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).

(a) Let A be a ring, let M be an A-module, let $X = \text{Spec } A$, and let $\mathscr{F} = \widetilde{M}$. For any $m \in M = \Gamma(X, \mathscr{F})$, show that $\text{Supp}(m) = V(\text{Ann}(m))$, where $\text{Ann}(m)$ is the *annihilator* of $m = \{a \in A | am = 0\}$.

(b) Now suppose that A is noetherian, and M finitely generated. Show that $\text{Supp}(\mathscr{F}) = V(\text{Ann}(M))$.

(c) The support of a coherent sheaf on a noetherian scheme is closed.

(d) For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_\mathfrak{a}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\widetilde{\Gamma}_\mathfrak{a} \cong \mathscr{H}^0_Z(\mathscr{F})$, where $Z = V(\mathfrak{a})$ and $\mathscr{F} = \widetilde{M}$.
[*Hint:* Use (Ex 1.20) and (5.8) to show a priori that $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. Then show that $\Gamma_\mathfrak{a}(M) \cong \Gamma_Z(\mathscr{F})$.]

(e) Let X be a noetherian scheme, and let Z be a closed subset. If $\mathscr{F}$ is a quasi-coherent (respectively, coherent) ($\mathcal{O}_X$-module, then $\mathscr{H}^0_Z(\mathscr{F})$ also quasi-coherent (respectively, coherent).

Proof:

(a) $\text{Supp}(m)$ is all $\mathfrak{p} \in X$ such that $m_\mathfrak{p} \neq 0$, which is equivalent to saying that $\forall a \notin \mathfrak{p}, am \neq 0$ iff $\text{Ann}(m) \cap (A \backslash \mathfrak{p}) = \varnothing$ iff $\text{Ann}(m) \subseteq \mathfrak{p}$ iff $\mathfrak{p} \in V(\text{Ann}(m))$. Therefore $\text{Supp}(m) = V(\text{Ann}(m))$.

(b) Let M be generated by $x_1, \cdots, x_n$ over A. For $\mathfrak{p} \in \text{Spec } A$, $\mathscr{F}_\mathfrak{p} = 0$ iff $M_\mathfrak{p} = 0$ iff for each i there exists some $s_i \notin \mathfrak{p}$ such that $s_i x_i = 0$ iff there is some $s \notin \mathfrak{p}$ such that $sx_i = 0$ for all i (just take the product of the s_i's since there are only finitely many) iff there is $s \notin \mathfrak{p}$ such that $s \in \text{Ann}(M)$. It follows that $\mathfrak{p} \in \text{Supp}(\mathscr{F})$ iff for all $s \notin \mathfrak{p}, s \notin \text{Ann}(M)$, i.e. $\text{Ann}(M) \cap (A \backslash \mathfrak{p}) = \varnothing$ i.e. $\text{Ann}(M) \subseteq \mathfrak{p}$ i.e. $\mathfrak{p} \in V(\text{Ann}(M))$.

(c) To show that a set is closed, we need only show that it is closed in an open cover. Cover X by open affine subsets U_i and let $\mathscr{F}$ be a coherent sheaf on X. It follows that $\mathscr{F}|_{U_i} \cong \widetilde{M}_i$ where M_i is a finitely generated module. Furthermore, stalks are local, thus $\mathscr{F}_x = (\mathscr{F}|_{U_i})_x$ for $x \in U_i$ and hence $\mathrm{Supp}(\mathscr{F}) \cap U_i = \mathrm{Supp}(\mathscr{F}|_{U_i})$. Since each $\mathrm{Supp}(\mathscr{F}|_{U_i}) = V(\mathrm{Ann}(M_i))$ is closed in U_i by (b), it follows that all of $\mathrm{Supp}(\mathscr{F})$ is closed.

(d) We have an exact sequence,

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U) \to 0$$

where $j : U \to X$ is inclusion and $U = X - Z$. For any point $x \in U$, there is an affine open neighborhood V of x in X with $V \subseteq U$, then $(\mathscr{F}|_U)|_V = \mathscr{F}|_V$ is isomorphic to some $\widetilde{M}$, thus $\mathscr{F}|_U$ is quasi-coherent, and hence $f_*(\mathscr{F}|_U)$ is quasi-coherent, since $f_*(\widetilde{N}) = ({}_AN)\tilde{}$. It follows that $\mathscr{H}_Z^0(\mathscr{F})$ is the kernel of a morphism of quasi-coherent sheaves and thus quasi-coherent.

$\mathscr{H}_Z^0(\mathscr{F})(X) = \Gamma_Z(\mathscr{F})$, the submodule of $\mathscr{F}(X) = M$ consisting of all elements whose supports are contained in Z. For any $m \in M$, $\mathrm{Supp}(m) \subseteq Z$ iff for all $\mathfrak{p} \notin Z$, $m_\mathfrak{p} = 0$, iff $\forall \mathfrak{p} \notin V(\mathfrak{a}), m_\mathfrak{p} = 0$ iff $\forall \mathfrak{p} \not\supseteq \mathfrak{a}, \exists s \notin \mathfrak{p}, sm = 0$, i.e. if for all $\mathfrak{p} \not\supseteq \mathfrak{a}$, $\mathrm{Ann}(m) \cap (A\backslash\mathfrak{p}) \neq \varnothing$, iff $\forall \mathfrak{p} \not\supseteq \mathfrak{a}, \mathrm{Ann}(m) \subseteq \mathfrak{p}$ iff $U \subseteq V(\mathrm{Ann}(m))$ iff $V(\mathrm{Ann}(m)) \subseteq V(\mathfrak{a})$ iff $\mathfrak{a}^n \subseteq \mathrm{Ann}(m)$ for some n iff $m \in \Gamma_\mathfrak{a}(M)$. Therefore $\mathscr{H}_Z^0(\mathscr{F}) = \widetilde{\Gamma}_\mathfrak{a}$.

(e) Let U_i be an affine open cover of X, then $Z \cap U_i$ are closed in X, thus we need only show that $\mathscr{H}_Z^0(\mathscr{F})|_{U_i}$ are quasi-coherent (resp. coherent). We first show that $\mathscr{H}_Z^0(\mathscr{F})|_{U_i} \cong \mathscr{H}_{Z\cap U_i}^0(\mathscr{F}|_{U_i})$. We have the exact sequence,

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U) \to 0$$

Then restricting this to U_i, we obtain the exact sequence,

$$0 \to \mathscr{H}_Z^0(\mathscr{F})|_{U_i} \to \mathscr{F}|_{U_i} \to j_*(\mathscr{F}_U)|_{U_i} \to 0$$

Clearly, $j_*(\mathscr{F}|_U)|_{U_i} = j'_*(\mathscr{F}_{U\cap U_i})$ where $j' : U \cap U_i \to U_i$, then we have the following exact sequence,

$$0 \to \mathscr{H}_{Z\cap U_i}^0(\mathscr{F}|_{U_i}) \to \mathscr{F}|_{U_i} \to j'_*(\mathscr{F}|_{U\cap U_i})$$

Furthermore,$U_i\backslash(Z \cap U_i) = (U_i\backslash Z) \cap U_i = U \cap U_i$, thus we obtain an isomorphism $\mathscr{H}_{Z\cap U_i}^0(\mathscr{F}|_{U_i}) \cong \mathscr{H}_Z^0(\mathscr{F})$ as desired. It follows immediately that $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent. To show that it is coherent if $\mathscr{F}$ is coherent, we need to show that in the affine case, if M is finitely generated, then so is $\Gamma_\mathfrak{a}(M)$, however this is trivially true since $\Gamma_a f(M)$ is a submodule of a finitely generated module over a noetherian ring (hence a noetherian module).

II.5.7

Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.

(a) If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathscr{F}|_U$ is free.

(b) $\mathscr{F}$ is locally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_{X,x}$-modules for all $x \in X$.

(c) $\mathscr{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} = \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$).

Proof:

(a) Let $\mathscr{F}|_U = \widetilde{M}$ where U is an irreducible open affine neighborhood of x. We may then assume that $X = \mathrm{Spec}\ A$ with $\mathfrak{N}(A)$ prime and $\mathscr{F} = \widetilde{M}$ with M a finitely generated A-module. Let $x = \mathfrak{p}$, then $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. Let $x_1, \cdots, x_n$ generate M as an A-module, then $x_1/1, \cdots, x_n/1$ generate $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module, therefore we may take a linearly independent spanning subset. Assume WLOG that $x_1/1, \cdots, x_r/1$ are a basis of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. We then have that x_i for $i > r$ may be expressed as $x_i = \sum_{j \leqslant r} \frac{a_{ij}x_j}{s}$ with some $s \notin \mathfrak{p}$ working for all i, j. We now show that $\widetilde{M}$ is free on $D(s)$. Clearly, we may still express x_i with $i > r$ in terms of the x_j in M_s as an A_s-module. Suppose there is a relation among x_j with $j \leqslant r$, then we have $\sum \frac{a_j x_j}{s^n} = 0$ and hence $s^k(\sum_j a_j x_j) = 0$ in M. However, since $s^k \notin \mathfrak{p}$, then this relation still holds in $M_\mathfrak{p}$ which contradicts the fact that $x_1, \cdots, x_r$ are a basis of $M_\mathfrak{p}$. It follows that $x_1, \cdots, x_r$ are a basis of M_s and thus $\widetilde{M}$ is free on $D(s)$.

(b) If $\mathscr{F}$ is free, then for any $x \in X$, there is an affine open neighborhood U of x on which $\mathscr{F}|_U \cong \mathcal{O}_X|_U^n$ and thus $\mathscr{F}_x \cong \mathcal{O}_{X,x}^n$ is free. Conversely, if all stalks are free, then by (a), around every point, there is a neighborhood on which $\mathscr{F}$ is free and thus $\mathscr{F}$ is locally free.

(c) If $\mathscr{F}$ is invertible, we will have that $\mathscr{G} = \check{\mathscr{F}}$, then by II.5.1a, $\mathscr{F} \otimes_{\mathcal{O}_X} \check{\mathscr{F}} \cong \mathscr{H}om(\mathscr{F}, \mathscr{F})$. Given any $U \subseteq X$ open, and $x \in \mathcal{O}_X(U)$, consider the map $\varphi_x : \mathscr{F}|_U \to \mathscr{F}|_U$ given by $\varphi_x(V)(s) = x|_V \cdot s$. To show that this is an isomorphism, it suffices to show it locally, thus we may assume that $\mathscr{F} \cong \mathcal{O}_X$, in which case we have that $\mathcal{O}_X \mapsto \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X)$ has inverse given by taking $\varphi : \mathcal{O}_X|_U \to \mathcal{O}_X|_U$ and mapping it to $\varphi(U)(1) \in \mathcal{O}_X(U)$. Note that morphisms between free sheaves are determined by their global sections which shows that these are inverses to each other. Conversely, suppose that there is some quasi-coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} \cong \mathcal{O}_X$. Taking stalks at $x \in X$, we get that $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x \cong \mathcal{O}_{X,x}$. If this implies that $\mathscr{F}_x$ is free, then we are done by part b.

Therefore we have reduced the problem to showing that if M, N are A-modules with N finitely generated and $M \otimes_A N \cong A$, then $M \cong A$. Localizing at $\mathfrak{p} \in \operatorname{Spec} A$, we get that $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \cong A_\mathfrak{p}$ implies that $M_\mathfrak{p} \cong A_\mathfrak{p}$ for all $\mathfrak{p}$ and hence $M \cong A$. Therefore we need only show this holds when A is a local ring. Let K be the residue field of A, then $(M \otimes_A N) \otimes_A K \cong K$, thus $M_K \otimes_K N_K \cong K$, therefore M_K and N_K are both 1-dimensional vectors spaces. Furthermore, $M_K = M \otimes_A K = M \otimes_A A/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A, thus $M_K \cong M/\mathfrak{m}M$ as A-modules and M_K is generated by some x as a K-vector space. Note however, that for any $\bar{a} \in K$, $\bar{a}x = ax$ for $x \in M/\mathfrak{m}M$, thus $M/\mathfrak{m}M$ is generated by x as an A-module. Since $\mathfrak{m}$ is the Jacobson radical of A, then by Nakayama's lemma, M is generated by x and is therefore free.

II.5.8

Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function $\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$, where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is *upper semi-continuous*, i.e., for any $n \in \mathbb{Z}$, the set $\{x \in X | \varphi(x) \geqslant n\}$ is closed.

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

Proof:

(a) We want to show that $U = \{x \in X | \varphi(x) < n\})$ is open. Suppose that $x \in U$, then $\varphi(x) < n$. We want to show that there is an open neighborhood of x on which $\varphi < n$. Since this question is local, we may assume that $X = \operatorname{Spec} A$ with A noetherian, $x = \mathfrak{p} \in \operatorname{Spec} A$, and that $\mathscr{F} = \widetilde{M}$ for some finitely generated A-module M. $\mathscr{F}_x \otimes_{\mathcal{O}_x} k(x) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A/\mathfrak{p} \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong (M/\mathfrak{p}M)_\mathfrak{p}$. Let M

be generated by $x_1, \cdots, x_m$, then $(M/\mathfrak{p}M)_\mathfrak{p}$ is generated by $\overline{x}_1, \cdots, \overline{x}_m$ and assume that the first $r < n$ are linearly independent and hence a basis of the $k(x)$-vector space $(M/\mathfrak{p}M)_\mathfrak{p}$. It follows that for $i > r$, $\overline{x}_i = \sum_{j \leqslant r} \frac{a_{ij}\overline{x}_j}{s}$ for some $s \notin \mathfrak{p}$. It follows that $\overline{x}_1, \cdots, \overline{x}_r$ generate $(M/\mathfrak{p}M)_s = M_s/\mathfrak{p}M_s$ as a A_s-module. By Nakayama's lemma, there is some $f \in A_s$ such that $f = 1 \pmod{\mathfrak{p}}$ and M_{sf} is generated by $x_1, \cdots, x_r$ as an A_{sf}-module. It follows that on $D(s)$, $\varphi(x) \leqslant r < n$.

(b) Choose $x \in X$ and let $n = \varphi(x)$, then let $U = \varphi^{-1}(\{n\})$. We want to show that U is open and closed. Let $x \in U$, then there is an open affine neighborhood V of x such that $\mathscr{F}|_V \cong \mathcal{O}_X^n$, since which we trivially have that on V, $\varphi = n$. Therefore U is open. Let $x \in \mathrm{cl}(U)$, then there is an open neighborhood V of x such that $\mathscr{F}|_V \cong \mathcal{O}_X^m$. Since $x \in \mathrm{cl}(U)$, then $V \cap U \neq \varnothing$, thus there is some point $y \in V \cap U$ on which $\varphi(y) = n$ and $\varphi(y) = m$, thus $n = m$, so $\varphi(x) = n$. It follows that U is closed and open and since X is connected, then $U = X$ and hence $\varphi = n$ everywhere.

(c) We need only show that at every point $x \in X$, $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module. Since this is a local condition, we may assume that $X = \mathrm{Spec}\ A$ with A reduced. Furthermore, we may assume that Spec A is irreducible, since irreducible components are open, and hence Spec A is integral so A is an integral domain. Let $\mathscr{F} = \widetilde{M}$, and let $\mathfrak{p} \in \mathrm{Spec}\ A$. Now let $x_1, \cdots, x_n \in M$ whose images generate $(M/\mathfrak{p}M)_\mathfrak{p}$ as a $\kappa(\mathfrak{p})$-vector space. Then by a, we have some $s \notin \mathfrak{p}$ such that $x_1, \cdots, x_n$ generate M_s. If M_s is not free, then there is some relation $\sum_i a_i x_i = 0$. Let f be the product of the nonzero a_i, then let $\mathfrak{q}$ be a maximal ideal of A_s containing f, then $(M_s)_\mathfrak{q} = M_\mathfrak{q}$ is generated by $n-1$ elements, thus $\varphi(\mathfrak{q}) \leqslant n-1$ which is a contradiction.

II.5.9

Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \mathrm{Proj}\ S$.

(a) Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$.

(b) Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbb{Z}$ such that for all $d \geqslant d_0, \alpha_d : M_d \to \Gamma(X, \widetilde{M}(d))$ is an isomorphism. [Hint: Use the methods of the proof of (5.19).]

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geqslant d} \cong M'_{\geqslant d}$ Here $M_{\geqslant d} = \bigoplus_{n \geqslant d} M_n$. We will say that a graded S-module M is quasi-finitely generated if it is equivalent to a finitely generated module. Now show that the functors $\tilde{}$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

Proof:

(a) Let $M = \bigoplus_{d \geqslant 0} M_d$ and $\Gamma_*(\widetilde{M}) = \bigoplus_{d \geqslant 0}(\widetilde{M}(d))(X) = \bigoplus_{d \geqslant 0}(M(d))\tilde{}(X)$. Let $x \in M$ be homogeneous of degree of d, then x has degree 0 in $M(d)$, thus we may map it to the function $\overline{x}$ which takes in $\mathfrak{p}$ and gives $x_\mathfrak{p}$.

(b) We want to show that $M \cong \Gamma_*(\widetilde{M})$ in sufficiently large degrees. Let $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$ be a filtration of M with $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(n_i)$. We then have the following exact sequences,

$$0 \to M^{i-1} \to M^i \to M^i/M^{i-1} \to 0$$

Which then give rise to short exact sequences,

$$0 \to \widetilde{M}^{i-1} \to \widetilde{M}^i \to \widetilde{M}^i/\widetilde{M}^{i-1} \to 0$$

Since the twisting functor is exact, we obtain an exact sequence of graded modules,

$$0 \to \Gamma_*(\widetilde{M}^{i-1}) \to \Gamma_*(\widetilde{M}^i) \to \Gamma_*(\widetilde{M}^i/\widetilde{M}^{i-1})$$

And note that $\widetilde{M}^i/\widetilde{M}^{i-1} \cong \widetilde{(S/\mathfrak{p}_i)}(n_i)$. Now suppose that in sufficiently high degrees, $\widetilde{M}^{i-1}(d)(X) \cong M_d^{i-1}$ and $\widetilde{(S/\mathfrak{p}_i)}(n_i)(d)(X) \cong (S/\mathfrak{p}_i)(n_i)_d = (S/\mathfrak{p}_i)_{n_i+d}$. We then get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \widetilde{M}^{i-1}(d)) \arrow[r] & \Gamma(X, \widetilde{M}^i(d)) \arrow[r] & \Gamma(X, \widetilde{S/\mathfrak{p}_i}(n_i+d)) & \\
0 \arrow[r] & M_d^{i-1} \arrow[r] \arrow[u, "\alpha_d^{i-1}"] & M_d^i \arrow[r] \arrow[u, "\alpha_d^i"] & (S/\mathfrak{p}_i)_{n_i+d} \arrow[r] \arrow[u, "\alpha_{n_i+d}^{i,i-1}"] & 0
\end{tikzcd}$$

And the morphisms α_d^{i-1} and $\alpha_{n_i+d}^{i,i-1}$ are isomorphisms. Then by diagram chasing, α_d^i is an isomorphism and thus $M_d^i \cong \Gamma(X, \widetilde{M}^i(d))$. It follows that we need only show the statement for $M = S/\mathfrak{p}$. We need to check that $\Gamma(\text{Proj } S, \widetilde{S/\mathfrak{p}}(n)) = \Gamma(\text{Proj } S/\mathfrak{p}, \widetilde{S/\mathfrak{p}}(n))$. To do so, we first check that $\widetilde{S/\mathfrak{p}} \cong i_*\mathcal{O}_Y$ where $i : Y \to X$, with $Y = \text{Proj } S/\mathfrak{p}$. This follows from the first isomorphism theorem applied to $\mathcal{O}_X \to i_*\mathcal{O}_Y$ and noting that $\mathfrak{p}$ is the ideal sheaf of $\mathcal{O}_Y$. The fact about their twists being the same follows from the fact that

$$\begin{aligned}
\widetilde{S/\mathfrak{p}}(n) = \widetilde{S/\mathfrak{p}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) &\cong i_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \\
&\cong i_*(\mathcal{O}_Y \otimes_{O_Y} i^*\mathcal{O}_X(n)) \\
&\cong i_*(i^*\mathcal{O}_X(n)) \\
&\cong i_*(\mathcal{O}_Y(n))
\end{aligned}$$

The equality of the first and second lines is from the projection formula and the third and fourth follows from the fact that $i^*(\mathcal{O}_X(n)) \cong (S(n) \otimes_S S/\mathfrak{p})\tilde{} \cong (S/\mathfrak{p})\tilde{}(n) \cong \mathcal{O}_Y(n)$. Thus we may assume that S is integral and we need only show that $\Gamma(X, \mathcal{O}_X(d)) \cong S_d$ for sufficiently large d.

Let $S' = \Gamma_*(\mathcal{O}_X)$, then we need to show that $S' = S$ in sufficiently high degree. To do so, we first note that the map in (a) is injective in this case for if an element $x \in S$ of degree d, then for any prime ideal $\mathfrak{p}$, if $x = 0$ in $S(d)_{(\mathfrak{p})}$, then there is some $s \notin \mathfrak{p}$ such that $sx = 0$ in $S(d)$ iff $sx = 0$ in S. Since S is an integral domain, this means that either $x = 0$ or $s = 0$ and $x \neq 0$, so $s = 0$ and thus $s \in \mathfrak{p}$ which is a contradiction. It follows that α is injective. Let $f_1, \cdots, f_n$ generate S_1 as an A-module, then we show that $S' \subseteq S_{f_i}$ for each i. This can be done exactly as in 5.13 and noting that none of the f_i are zero-divisors since S is an integral domain.

By 5.19, S' is finitely generated, let $s_1, \cdots, s_k$ generate S' as S-module with each s_j homogeneous. Then since $s_j \in S_{f_i}$, there is some l_{ij} such that $f_i^{l_{ij}} s_j \in S$. Since $f_1, \cdots, f_n$ generate S as an S_0-algebra, then let $l = \max_{i,j} l_{ij}$, then for any element $y \in S_{\geqslant l}$, we have that $ys_j \in S$. Let l' be the maximum degree of any s_j, then any element in $S'_{\geqslant l+l'}$ must be a linear combination of s_j's and element in S of degree at least l, thus $S'_{\geqslant l+l'} \subseteq S$. Therefore in degrees larger than $l + l'$, S' and S are the same.

(c) We first show that if M is quasi-finitely generated, then $\widetilde{M}$ is coherent. Let $f \in S_1$, then we need only show that $\widetilde{M}|_{D_+(f)}$ is finitely generated. Let M be equivalent to some finitely generated

M'. Note that $\widetilde{M}|_{D_+(f)} \cong (M_{(f)})\tilde{}$, then we have that for any $\frac{x}{f^n} \in M_{(f)}$, let x be homogeneous of degree n and suppose that $M_{\geqslant d} \cong M'_{\geqslant d}$, then $\frac{x}{f^n} = \frac{xf^{d-n}}{f^{d+n}} \in (M')_{(f)}$, thus $M'_{(f)} \cong M_{(f)}$ and hence $\widetilde{M}$ is coherent. Furthermore, we have that $\Gamma_*(\widetilde{M})\tilde{} \cong \widetilde{M}$ by 5.15, thus one of the compositions is actually the identity. Conversely, if $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, then by 5.19, $\Gamma_*(\mathscr{F})$ is a finitely-generated A-module.

II.5.10

Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \text{Proj } S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

(a) For any homogeneous ideal $I \subseteq S$, we define the saturation $\bar{I}$ of S to be $\{s \in S|$ for each $i = 0, \ldots, r$ there is an n such that $x_i^n s \in I\}$. We say that I is saturated if $I = \bar{I}$. Show that $\overline{I}$ is a homogeneous ideal of S.

(b) Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

(c) If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathscr{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.

(d) There is a 1-1 correspondence between saturated ideals of S and closed schemes of X.

Proof:

(a) Suppose that $f = \sum f_i$ is in $\overline{I}$, then we have that $x_i^n f \in I$ for all i (note that we may choose some n that works for all of them). Therefore $x_i^n f_j \in I$ since I is homogeneous, thus $f_j \in \overline{I}$ for all j, so $\overline{I}$ is homogeneous.

(b) Let I_1, I_2 be two ideals of S defining the same closed subscheme of X, then we have the following commutative diagram,

$$\begin{tikzcd}
\text{Proj } S/I_1 \arrow[d, "f"] \arrow[dr, "i"] & \\
\text{Proj } S/I_2 \arrow[r, "j"] & \text{Proj } S
\end{tikzcd}$$

where f is an isomorphism and i, j are closed immersions. The morphisms, as given in 3.12 are given, on points, by taking the preimage of primes under the quotient. Furthermore, we have that $i^{-1}(D_+(x_i)) = D_+(\overline{x}_i)$. We obtain morphisms $i|_{D_+(\overline{x}_i)} : \text{Spec } (S/I_1)_{(\overline{x}_i)} \to \text{Spec } S_{(x_i)}$ corresponding to morphisms sending $s/x_i^n \mapsto \overline{s}/\overline{x}_i^n$. The kernel of this morphism is all s/x_i^n where $\overline{x}_i^k \overline{s} = 0$ for some k, i.e. all s such that $x_i^k s \in I$ for some s. More succinctly, the kernel is $(I_1)_{x_i} \cap S_{(x_i)}$. Since I_1, I_2 define the same closed subscheme of Proj S, we have that $(I_1)_{x_i} \cap S_{(x_i)} = (I_2)_{x_i} \cap S_{(x_i)}$ for all i. Let $f \in \overline{I_1}$ be a homogeneous element, then $x_i^n f \in I_1$. Let $d = \deg(f)$, then f/x_i^d has degree 0 and thus is in $S_{(x_i)}$ and since $x_i^{n+d} f \in I_1$, then $f/x_i^d \in (I_1)_{x_i} \cap S_{(x_i)} = (I_2)_{x_i} \cap S_{(x_i)}$, thus there is some k such that $x_i^{k-d} f \in I_2$. We may take the max k ranging over all i to see that $f \in \overline{I_2}$. It follows that $\overline{I}_1 \subseteq \overline{I}_2$ and we may repeat the argument to show that $\overline{I_2} \subseteq \overline{I_1}$, thus the saturations are the same.

If the saturations are the same, we want to show that they define the same closed subscheme. It suffices to show this on open affines. This comes down to showing that if $\overline{I_1} = \overline{I_2}$, then $(I_1)_{x_i} \cap S_{(x_i)} = (I_2)_{x_i} \cap S_{(x_i)}$. Let $f/x_i^k \in (I_1)_{x_i}$, then $x_i^n f \in I_1$ for some n, then let $f' = x_i^n f$, such that $f'/x_i^{k+n} \in S_{(x_i)}$ and $f' \in I_1$ and hence $f' \in \overline{I_1} = \overline{I_2}$, so there is some m such that $x_i^m f' \in I_2$, i.e. $x_i^{m+n} f \in I_2$, thus $f/x_i^k \in (I_2)_{x_i}$ as desired.

(c) Let $I = \Gamma_*(\mathscr{I}_Y)$. Let $f \in \overline{I}$ be homogeneous of degree d, then $x_j^n f \in I$ for some k and all j. We have $i : Y \to X$, and a morphism $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$. We then get $i_d^\# : \mathcal{O}_X(d) \to i_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ and $i_*\mathcal{O}_Y = \widetilde{S/I}$, thus $i_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) \cong (S/I \otimes_S S(d))\tilde{} \cong i_*(\mathcal{O}_Y(d))$. Furthermore, the kernel of $i_d^\#$ is $I(d)$. Let $\mathcal{O}_{X,*} = \bigoplus_{d \geqslant 0} \mathcal{O}_X(d)$ and similarly, let $\mathcal{O}_{Y,*} = \bigoplus_{d \geqslant 0} \mathcal{O}_Y(d)$, then the $i_d^\#$ induce a morphism $i_*^\# : \mathcal{O}_{X,*} \to i_*\mathcal{O}_{Y,*}$. $\mathcal{O}_{X,*}$ is a sheaf of rings since for any U, and $x \in \mathcal{O}_X(n), y \in \mathcal{O}_X(m)$, locally we have that $x = \frac{f}{s}$ and $y = \frac{g}{t}$ where $\deg(f) - \deg(s) = n$ and $\deg(g) - \det(t) = m$, then we define xy to be $\frac{fg}{st}$ locally. It follows that $i_*^\#$ is a morphisms of sheaves of rings, and thus $i_*^\#(X)(x_j^k f) = i_*^\#(X)(x_j)^k \cdot i_*^\#(X)(f)$. Since $x_j^k f \in I$, then the left hand side is 0. We then have that $i_*^\#(D_+(x_j))(x_j)^k \cdot i_*^\#(D_+(x_j))(f) = 0$ for all j and $i_*^\#(D_+(x_j)) : \bigoplus_{d \geqslant 0}(S_{x_j})_d \to \bigoplus_{d \geqslant 0}((S/I)_{\overline{x}_j})_d$, hence $i_*^\#(D_+(x_j)) : S_{x_j} \to (S/I)_{\overline{x}_j}$ is just quotienting. It follows that each $i_*^\#(D_+(x_j))(x_j)$ is invertible and thus $i_*^\#(D_+(x_j))(f) = 0$ for each j, and hence $i_*^\#(X)(f) = 0$, so $f \in I(d) \subseteq I$.

(d) There is a $1-1$ correspondence between closed subschemes of X and ideal sheaves. We now know that each ideal sheaf gives a saturated ideal. For any saturated ideal, we get a closed subscheme and the ideal sheaf has $\Gamma_*(\mathscr{I}_Y)$ equal to that saturated ideal by (b).

II.5.11

Let S and Y be two graded rings with $S_0 = T_0 = A$. We define the *Cartesian product* $S \times_A T$ to be the graded ring $\bigoplus_{d \geqslant 0} S_d \otimes_A T_d$. If $X = \text{Proj } (S)$ and $Y = \text{Proj } T$, show that $\text{Proj } (S \times_A T) \cong X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\text{Proj } (S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

The Cartesian product of rings is related to the *Segre* embedding of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1 over A corresponding to a projective embedding $X \hookrightarrow \mathbb{P}_A^r$ and if $y_0, \ldots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \hookrightarrow \mathbb{P}_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and hence defines a projective embedding $\text{Proj } (S \times_A T) \hookrightarrow \mathbb{P}_A^N$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbb{P}^r \times \mathbb{P}^s$ in its Segre embedding.

Proof:

Proj $(S \times_A T)$ is covered by open affine sets of the form $D_+(f \otimes g)$ where f, g are homogeneous elements of the same degree in S and T respectively. We then have that $D_+(f \otimes g) \cong \text{Spec } (S \times_A T)_{(f \otimes g)}$. Furthermore, we have a map $S_{(f)} \times T_{(g)} \to (S \times_A T)_{(f \otimes g)}$ given by $(x/f^n, y/g^k) \mapsto (f^{\max\{n,k\}-n} x \otimes g^{\max\{n,k\}-k} y)/(f \otimes g)^{\max\{n,k\}}$. This is bilinear, and thus we obtain a morphism from $S_{(f)} \otimes_A T_{(g)} \to (S \times_A T)_{(f \otimes g)}$ and hence from each of $S_{(f)}$ and $T_{(g)}$. Checking that this glues amounts to checking that

$$\begin{array}{ccc} S_{(f)} & \longrightarrow & (S \times_A T)_{(f\otimes g)} \\ \downarrow & & \downarrow \\ S_{(ff')} & \longrightarrow & (S \times_A T)_{(ff'\otimes gg')} \end{array} \qquad \begin{array}{ccc} x/f^n & \longrightarrow & (x, g^n)/(f\otimes g)^n \\ \downarrow & & \downarrow \\ (f')^n x/(ff')^n & \longrightarrow & ((f')^n x, (gg')^n)/(ff'\otimes gg')^n \end{array}$$

commutes. The diagram on the right is the check. Since this commutes, then we obtain morphisms Proj $(S \times_A T) \to$ Proj S and Proj $(S\times_A T) \to$ Proj T over A. It follows that we obtain a morphism from Proj $(S \times_A T) \to X \times_A Y$. We now check that it is an isomorphism locally. Let $D_+(f) \times_A D_+(g) \subseteq X \times_A Y$. The preimage in Proj $(S \times_A T)$ is $D_+(f \otimes g)$. We then recover the morphism Spec $(S \times_A T)_{(f\otimes g)} \to$ Spec $S_{(f)} \otimes_A$ Spec $T_{(f)}$ = Spec $(S_{(f)} \otimes_A T_{(g)})$. It follows that we need only check that $S_{(f)} \otimes_A T_{(g)} \to (S \times_A T)_{(f\otimes g)}$ is an isomorphism. This amounts to finding an inverse to thee morphism given above. Let $(x \otimes y)/(f \otimes g)^n \in (S \times_A T)_{(f\otimes g)}$ and map this to the element $x/f^n \otimes y/g^n \in S_{(f)} \otimes_A T_{(g)}$. We now check that these are inverses. Let $m = \max\{n, k\}$,

$$x/f^n \otimes y/g^k \mapsto (f^{m-n}x \otimes g^{m-k}y)/(f \otimes g)^m \mapsto x/f^n \otimes y/g^k$$
$$(x \otimes y)/(f \otimes g)^n \mapsto x/f^n \otimes y/g^n \mapsto (x \otimes y)/(f \otimes g)^n$$

Therefore we have an isomorphism. We want to check that $D_+(f) \times_A D_+(g)$ actually cover $X \times_A Y$. Let $(x, y, s, \mathfrak{p})$ be a point in $X \times_A Y$ where $x \in X, y \in Y, s \in$ Spec A and $\mathfrak{p}$ a prime ideal of $\kappa(x) \times_{\kappa(s)} \kappa(y)$. We want to show that we can pick f, g of the same degree such that the points x and y are contained in $D_+(f)$ and $D_+(g)$ respectively. Since x and y do not contain S_+ and T_+ respectively, then we may find a homogeneous element $f \in S_+$ and $g \in T_+$ of degrees n and m such that $f \notin x$ and $g \notin y$. Since x and y are prime, then $f^m \notin x$ and $g^n \notin y$ and $\deg(f^m) = m\deg(f) = mn = n\deg(g) = \deg(g^n)$, thus $x \in D_+(f^m)$ and $y \in D_+(g^n)$ as desired.

We now show that $\mathcal{O}(1)$ on Proj $(S\times_A T)$ is isomorphic to $p_1^*(\mathcal{O}_X(1))\otimes_{\mathcal{O}} p_2^*(\mathcal{O}_Y(1))$. We define the map globally first and then check that it is locally an isomorphism. Let $m \in \mathcal{O}(1)(U)$, then $m = \frac{x\otimes y}{f\otimes g}$ locally and we map this to $\frac{x}{f} \otimes \frac{y}{g}$ in $p_1^*(\mathcal{O}_X(1)) \otimes_{\mathcal{O}} p_2^*(\mathcal{O}_Y(1))$. $p_1^*(\mathcal{O}_X(1))(U) = (p_1^{-1}\mathcal{O}_X(1) \otimes_{p_1^{-1}\mathcal{O}_X} \mathcal{O})(U)$, then $p_1^{-1}\mathcal{O}_X(1)(U) = \varinjlim_{V\supseteq p_1(U)} \mathcal{O}_X(1)(V)$. It follows that the above map makes sense since there will always be some V containing any point in the projection of U. The given morphism maps into the presheaf of the RHS, so we fix this by sheafifying. We now have to check that it is locally an isomorphism. Since $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are quasi-coherent, then so are $p_1^*(\mathcal{O}_X(1))$ and $p_2^*(\mathcal{O}_Y(1))$ and ultimately $p_1^*(\mathcal{O}_X(1))\otimes_{\mathcal{O}} p_2^*(\mathcal{O}_Y(1))$. Let $f\otimes g$ be any homogeneous element of degree > 0 in $S\times_A T$, then we have that $\mathcal{O}(1)(D_+(f\otimes g)) = ((S\times_A T)(1))_{(f\otimes g)}$ and $(p_1^*(\mathcal{O}_X(1))\otimes_{\mathcal{O}} p_2^*(\mathcal{O}_Y(1)))(D_+(f\otimes g))$ is given by a tensor product. $p_1^*(\mathcal{O}_X(1))(D_+(f \otimes g)) = (\mathcal{O}_X(1)(D_+(f)) \otimes_{S_{(f)}} S_{(f)} \times_A T_{(g)})$

II.5.12

(a) Let X be a scheme over a scheme Y and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L} \otimes \mathscr{M}$ is also very ample. [Hint: Use a Segre embedding.]

(b) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L} \otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

Proof:

II.5.13

Let S be a graded ring, generated by S_1, as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geqslant 0} S_n^{(d)}$, where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(l)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.

This construction is related to the d-uple *embedding* (I, Ex. 2.12) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \hookrightarrow \mathbb{P}_A^r$, then the set of monomials of degree d in the x_1 is a set of generators for $S_1^{(d)} = S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbb{P}_A^r$.

Proof:

II.5.14

Let A be a ring, and let X be a closed subscheme of $\mathbb{P}_A^r$. We define the homogeneous coordinate ring $S(X)$ of X for the given embedding to be $A[x_0, \ldots, x_r]/I$, where I is the ideal $\Gamma_*(\mathscr{I}_X)$ constructed in the proof of (5.16). (Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)!) Recall that a scheme X is normal if its local rings are integrally closed domains. A closed subscheme $X \subseteq \mathbb{P}_A^r$ is *projectively normal* for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}_k^r$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.

(a) Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n \geqslant 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure. [Hint: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathscr{I} = \oplus_{n \geqslant 0} \mathcal{O}_X(n)$ on X, and show that $\mathscr{I}$ is a sheaf of integrally closed domains.]

(b) Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

(c) Show that $S(d)$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.

(d) As a corollary of (a), show that a closed subscheme $X \subset \mathbb{P}r_A$ projectively normal if and only if it is normal, and for every $n \geqslant 0$ the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

Proof:

II.6.1

Let X be a scheme satisfying (*). Then $X \times \mathbb{P}^n$ also satisfies (*), and $\operatorname{Cl}(X \times \mathbb{P}^n) \cong (\operatorname{Cl} X) \times \mathbb{Z}$.

Proof:

Let $Y = \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ be the hyperplane with $x_0 = 0$, then $\mathbb{P}^n \backslash Y = \mathbb{A}^n$, thus we obtain an exact sequence, $\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X \times \mathbb{A}^n) \to 0$ And since $\mathrm{Cl}(X \times \mathbb{A}^1) = \mathrm{Cl}(X)$, then by induction, $\mathrm{Cl}(X \times \mathbb{A}^n) = \mathrm{Cl}(X)$ and thus our exact sequence is actually,

$$\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^n) \to \mathrm{Cl}(X) \to 0$$

If we can show that $Z \to \mathrm{Cl}(X \times \mathbb{P}^n)$ is injective, i.e. that $k \cdot X \times Y$ is not principal, then we need only construct a splitting of the sequence. This comes from the map back from $\mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^n)$ by sending $Y \mapsto Y \times \mathbb{P}^n$.

Suppose that $k \cdot X \times Y$ is principal, then it is linearly equivalent to some (f) with $f \in K(X \times \mathbb{P}^n)$. Note that $X \times \mathbb{A}^n$ is an open affine subset of $X \times \mathbb{P}^n$, thus $K(X \times \mathbb{P}^n) = K(X \times \mathbb{A}^n) = K(X)(x_1, \cdots, x_n)$. Furthermore, we may identify $K(X \times \mathbb{P}^n)$ with ratios of homogeneous polynomials of equal degree in $x_0, \cdots, x_n$ over $K(X)$. We have that $f = g/h$ where g and h are homogeneous polynomials of the same degree in $K(X)[x_0, \cdots, x_n]$. Upon projection $\pi : X \times \mathbb{P}^n \to X$ to X, $X \times Y$ maps to X and thus the generic point x of $X \times Y$ maps to $\eta \in X$. It follows that $x \in \pi^{-1}(\eta) = \mathrm{Proj}\ K(X)[x_0, \cdots, x_n]$ and x corresponds to the point (x_0). The prime divisors of (g) and (h) in $\mathrm{Proj}\ K(X)[x_0, \cdots, x_n]$ correspond exactly to the irreducible factors of g and h since they are homogeneous and thus the degree of g must be k more than h which is a contradiction.

II.6.2

Varieties in Projective Space. Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one. For any divisor $D = \sum n_i Y_i$ on X, we define the *degree* of D to be $\sum n_i \deg Y_i$ where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself.

(a) Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbb{P}^n$ for which $Y_i \cap U_i \neq \varnothing$, and let $n_i = \nu_{Y_i}(\overline{f}_i)$ where $\overline{f}_i$ is the restriction of f_i to $U_i \cap X$. Then we define the divisor $V \cdot X = \sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of $\mathrm{Div}\, \mathbb{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\, X$.

(b) If D is a principal divisor on $\mathbb{P}^n$, for which $D \cdot X$ is defined as in (a), show that $D \cdot X$ is principal on X. Thus we get a homomorphism $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, X$.

(c) Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y)$ defined in (I,§7). Then use the generalized Bézout theorem to show that for any divisor D on $\mathbb{P}^n$, none of whose components contain X,

$$\deg(D \cdot X) = (\deg D) \cdot (\deg X)$$

(d) If D is a principal divisor on X, show that there is a rational function f on $\mathbb{P}^n$ such that $D = (f) \cdot X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism

$\deg : \mathrm{Cl}\, X \to \mathbb{Z}$. Finally, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Cl}\ \mathbb{P}^n \arrow[r] \arrow[d, "\deg", "\cong"'] & \mathrm{Cl} X \arrow[d, "\deg"] \\
\mathbb{Z} \arrow[r, "\cdot(\deg X)"] & \mathbb{Z}
\end{tikzcd}$$

and in particular, we see that the map $\mathrm{Cl}\,\mathbb{P}^n \to \mathrm{Cl}\, X$ is injective.

Proof:

(a) We need only show that the n_i are independent of the choice of U_i. It suffices to show this for each i separately. Let U, W be two open sets such that $Y_i \cap U$ and $Y_i \cap W$ are nonempty. Furthermore, we have local equations f_U, f_W for V on U and W respectively, i.e. global sections of $\mathcal{O}|_U$ and $\mathcal{O}|_W$ which generate the ideal sheaf $\mathscr{I}_V|_U$ and $\mathscr{I}_V|_W$. We have that $K(U \cap X) = K(W \cap X) = K(X)$ and letting y be the generic point of Y_i, we have that $\mathcal{O}_y = (\mathcal{O}|_U)_y = (\mathcal{O}|_W)_y$. Note that $f_U = f_U|_{U \cap W}$ in $\mathcal{O}_{X,y}$. We then have a surjective map from $\mathcal{O}_y \to \mathcal{O}_{X,y}$. Thus, it suffices to show that $f_U|_{U\cap W}$ and $f_W|_{U \cap W}$ differ by a unit. Since V is a hypersurface, then it corresponds to an ideal (f) where f is some homogeneous element of $k[x_0, \cdots, x_n]$. It follows that the ideal sheaf of V is generated by the global section f. It follows that $f_U \in f|_U \mathcal{O}(U)$, so $f_U = xf|_U$ and similarly, since f_U generates $\mathscr{I}_V|_U$, then $f|_U = x' f_U = xx' f|_U$. Since $\mathcal{O}(U)$ is an integral domain, then it follows that $xx' = 1$, so $f|_U$ and f_U differ by a unit. Therefore, $f|_{U\cap W}$ and $f_U|_{U \cap W}$ differ by a unit. Similarly, $f|_{U \cap W}$ and $f_W|_{U\cap W}$ differ by units. Therefore, $f_U|_{U\cap W}$ and $f_W|_{U \cap W}$ differ by units and thus have the same valuation. It follows that the valuation is independent of choice of U, V and local equation.

(b) Suppose that $D = (f)$ is a principal divisor on $\mathbb{P}^n$ where $f = g/h$ with g, h homogeneous polynomials of the same degree in $S = k[x_0, \cdots, x_n]$. Let X be given by the homogeneous prime ideal $I \subseteq S$, then we obtain an element $\overline{f} \in K(X)$ given by the map $\mathcal{O}_X \to \mathcal{O}$, however we also know explicitly what this is. It is given by $\overline{f} = \overline{g}/\overline{h} \in \mathrm{Frac}(S/I)$. Note that we require that (f) not contain X which means that $h \notin I$ and we are able to define a map from $(S - I)^{-1} S \to \mathrm{Frac}(S/I)$. We now want to show that $(f) \cdot X = (\overline{f})$. It suffices to show that $(g) \cdot X = (\overline{g})$. Note here that g is *not* an element of $K(X)$. We define (g) and $(\overline{g})$ such that $(g/h) = (g) - (h)$ and $(\overline{g}/\overline{h}) = (\overline{g}) - (\overline{h})$. We can do this by taking a factorization $g = p_1 \cdots p_n$ into irreducibles, then let $(g) = \sum \mathrm{cl}\{(p_i)\}$ and $(\overline{g}) = \sum_{\mathfrak{q} \text{ min over } (p_i)+I} \mathrm{cl}\{\mathfrak{q}\}$. We may similarly extend valuations from the degree 0 elements to all of $\mathrm{Frac}(S)$ and $\mathrm{Frac}(S/I)$. We have the following,

$$(g) \cdot X = \sum_{\substack{\mathrm{ht}\ \mathfrak{p}=1 \\ \mathfrak{p} \in \mathrm{Proj}\ S \\ X \not\subseteq \mathrm{cl}\{\mathfrak{p}\}}} \sum_{\substack{\mathfrak{q} \supseteq \mathfrak{p}/I \\ \mathfrak{q} \in \mathrm{Proj}\ S/I \\ \mathfrak{q}\ \text{minimal}}} \nu_{\mathfrak{p}}(g) \nu_{\mathfrak{q}}(\overline{f_{\mathfrak{p}}}) \mathrm{cl}\{\mathfrak{q}\}$$

$$(\overline{g}) = \sum_{\substack{\mathrm{ht}\ \mathfrak{q}=1 \\ \mathfrak{q} \in \mathrm{Proj}\ S/I}} \nu_{\mathfrak{q}}(\overline{g}) \mathrm{cl}\{\mathfrak{q}\}$$

Let $\varphi : S \to S/I$ be the quotient map. Note that we may assume g is irreducible since taking the divisor associated to a product is the sum. It follows that $\nu_{\mathfrak{p}}(g)$ is nonzero iff $\mathfrak{p} = (g)$, therefore,

$$(g) \cdot X = \sum_{\substack{\overline{g} \in \mathfrak{q} \\ \mathfrak{q} \in \mathrm{Proj}\ S/I \\ \mathfrak{q}\ \text{minimal}}} \nu_{\mathfrak{q}}(\overline{g})$$

Now, we need only show that such $\mathfrak{q}$ are exactly the height 1 prime ideals for which $\nu_{\mathfrak{q}}(\overline{g}) \neq 0$. By Krull's principal ideal theorem, since $\mathfrak{q}$ is minimal over a principal ideal and S/I is noetherian, then the height of $\mathfrak{q}$ is at most 1. Furthermore, $\mathfrak{q} \neq 0$ since $g \notin I$, thus $\mathfrak{q}$ has height 1. Conversely, suppose that $\mathfrak{q}$ has height 1 and contains $\overline{g}$. Suppose that $\mathfrak{q}$ is not minimal over $\overline{g}$, then $\mathfrak{q}' \subseteq \mathfrak{q}$ and $\overline{g} \in \mathfrak{q}'$, thus the height of $\mathfrak{q}$ is at least 2 since we have $(0) \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$.

(c) Let X have homogeneous ideal $\mathfrak{p}$ in S and let Y have homogeneous ideal $\mathfrak{q}$ in S. The intersection multiplicity $i(X, V; Y)$ is the length of the $S_{\mathfrak{q}}$-module $(S/(\mathfrak{p} + (f)))_{\mathfrak{q}}$ where $V = (f)$. Let $S' = S/\mathfrak{p}$. Let $n = \nu_{\overline{\mathfrak{q}}}(\overline{f})$, then we have that $\overline{f} \in \mathfrak{q}^l$ for all $l \leqslant n$. The $S_{\mathfrak{q}}$-submodules of the module $S'_{\overline{\mathfrak{q}}}$ are precisely its ideals since $S_{\mathfrak{q}}$ acts by the action of the quotient $S'_{\overline{qf}}$. Since $\overline{\mathfrak{q}}$ is a height 1 prime ideal of S', then $S'_{\overline{\mathfrak{q}}}$ is a one dimensional noetherian local ring, so it is a DVR. Thus $S'_{\overline{\mathfrak{q}}}$ admits a unique maximal filtration,

$$\cdots \subseteq \overline{\mathfrak{q}}^2 S'_{\overline{\mathfrak{q}}} \subseteq \overline{\mathfrak{q}} S'_{\overline{\mathfrak{q}}} \subseteq S'_{\overline{\mathfrak{q}}}$$

The submodules of $S'_{\overline{\mathfrak{q}}}/(\overline{f})S'_{\overline{\mathfrak{q}}}$ are exactly those which contain $(\overline{f})S'_{\overline{\mathfrak{q}}}$, i.e. all $\overline{\mathfrak{q}}^n S'_{\overline{\mathfrak{q}}}$ such that $f \in \overline{\mathfrak{q}}^n$. It follows that we obtain a unique maximal filtration,

$$0 \subseteq \frac{\overline{\mathfrak{q}}^n S'_{\overline{\mathfrak{q}}}}{(\overline{f})S'_{\overline{\mathfrak{q}}}} \subseteq \cdots \subseteq \frac{\overline{\mathfrak{q}} S'_{\overline{\mathfrak{q}}}}{(\overline{f})S'_{\overline{\mathfrak{q}}}} \subseteq q \frac{S'_{\overline{\mathfrak{q}}}}{(\overline{f})S'_{\overline{\mathfrak{q}}}}$$

Therefore the length of $\frac{S'_{\overline{\mathfrak{q}}}}{(\overline{f})S'_{\overline{\mathfrak{q}}}}$ as an $S_{\mathfrak{q}}$ or equivalently, as an $S'_{\overline{\mathfrak{q}}}$ module is exactly $\nu_{\overline{\mathfrak{q}}}(\overline{f})$.

The generalized Bézout theorem tells us that given the irreducible components Y_i of $X \cap V$, we have that,

$$\sum_{Y_i} i(X, V; Y_i) \deg Y_i = (\deg X) \cdot (\deg V)$$

Now trivially, we may change out $i(X, V; Y_i)$ to n_i, and recalling that $\deg(V \cdot X) = \sum n_i \deg Y_i$, we get that $\deg(V \cdot X) = (\deg X) \cdot (\deg V)$. By linearity of deg, we obtain the desired result.

(d) Let D be a principal divisor on X, then $D = (g/h)$ for some $g, h \in S/I$ homogeneous and of the same degree. Let g', h' be representatives of g, h in S, then by part b, $(g'/h') \cdot X = (g/h)$. The degree of (g'/h') is 0, so by part c, the degree of (g/h) is 0. The commutative diagram commutes because of part c and $\operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} X$ is injective because $\operatorname{Cl} \mathbb{P}^n \to \mathbb{Z}$ is injective and $\cdot(\deg X)$ is injective.

II.6.3

Cones. In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex 2.10). So let V be a projective variety in $\mathbb{P}^n$, which is of dimension $\geqslant 1$ and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in $\mathbb{A}^{n+1}$, and let $\overline{X}$ be its projective closure in $\mathbb{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi : \overline{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \operatorname{Cl} V \to \operatorname{Cl}(\overline{X} - P)$ is an isomorphism. Since $\operatorname{Cl} \overline{X} \cong \operatorname{Cl}(\overline{X} - P)$, we have also $\operatorname{Cl} V \cong \operatorname{Cl} \overline{X}$.

(b) We have $V \subseteq \overline{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\operatorname{Cl} \overline{X}$ is equal to π^*(class of $V \cdot H$) where H is any hyperplane not containing V. Thus conclude using (6.5) that there is an exact sequence,

$$0 \to \mathbb{Z} \to \operatorname{Cl} V \to \operatorname{Cl} X \to 0$$

where the first arrow sends $1 \mapsto V \cdot H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in X.

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a UFD iff (1) V is projectively normal, and (2) $\operatorname{Cl} V \cong \mathbb{Z}$ and is generated by the class of $V \cdot H$.

(d) Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\operatorname{Cl} X \to \operatorname{Cl}(\operatorname{Spec} \mathcal{O}_P)$.

Proof:

(a) Let V be given by the homogeneous ideal $\mathfrak{p}$, then X is $V(\mathfrak{p}) \subseteq \mathbb{A}^{n+1}$. The map $\overline{X} - P \to V$ is given by mapping $(x_0 : \cdots : x_{n+1}) \mapsto (x_0 : \cdots : x_n)$. Notice that $\overline{X}$ is given by the homogeneous ideal $\mathfrak{p}[x_{n+1}]$ in $\mathbb{P}^{n+1}$. Now consider the open subset $V \cap D_+(x_i)$, given by the ideal $\mathfrak{p}_i \in \operatorname{Spec} k[x_0/x_i, \cdots, x_n/x_i]$ obtained from $\mathfrak{p}$ in the obvious way. The preimage of $V \cap D_+(x_i)$ is $\overline{X} \cap D_+(x_i)$, given by the ideal $\mathfrak{p}_i[x_{n+1}/x_i]$. Notice then that $\operatorname{Spec} k[x_0/x_i, \cdots, x_{n+1}/x_i]/\mathfrak{p}_i[x_{n+1}/x_i] = \operatorname{Spec} k[x_{n+1}/x_i] \times \operatorname{Spec} k[x_0/x_i, \cdots, x_n/x_i]/\mathfrak{p}_i \cong \mathbb{A}^1 \times (V \cap D(x_i))$. $\pi^* : \operatorname{Cl} V \to \operatorname{Cl}(\overline{X} - P)$ is given by taking a prime divisor Y and mapping it to $\pi^{-1}(Y)$ and extending linearly. We first show that it takes principal divisors to principal divisors. Let $(f) \in \operatorname{Div} V$ be principal. The function field of V is given by $(S/\mathfrak{p})_{((0))}$, i.e. the degree 0 component of the localization, here $S = k[x_0, \cdots, x_n]$. We then get that the function field of $\overline{X} - P$ is the same as the function field of X, i.e., it is $(S[x]/\mathfrak{p})_{((0))}$, then note that f is an element in the function field of X. We can check locally that the divisor $(f) \in \operatorname{Div}(\overline{X} - P)$ is $\pi^*((f))$. This problem therefore reduces to showing that given an element $(f) \in \operatorname{Div} X$, $(f) = \pi^*((f))$ in $\operatorname{Div} \mathbb{A}^1 \times X$. This is exactly what is stated without proof in 6.6. Again, this question is local, and thus we may assume that X is in fact affine. It then reduces to a purely algebraic fact.

Let A be an integral domain, regular in codimension 1, and let $f \in \operatorname{Frac}(A)$. There is a map $\pi : A \to A[x]$ by inclusion. For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A[x]$ of height 1, $\nu_{\mathfrak{p}}(f) = \nu_{\pi^{-1}(\mathfrak{p})}(f)$. By the properties of valuations, we may assume that $f \in A$. Since $A \subseteq A[x]$, then $\pi^{-1}(\mathfrak{p}) = A \cap \mathfrak{p}$. We then have the trivial observation that since $f \in A$, then $f \in \mathfrak{p}^k \iff f \in \mathfrak{p}^k \cap A \iff f \in (\mathfrak{p} \cap A)^k$, thus the valuations are the same.

Suppose that $D \in \operatorname{Div} V$ and $\pi^* D = (f)$ for some $f \in K(\overline{X} - P)$, then on $U_i = V \cap D_+(x_i)$, we get that $\pi^* D \cap \pi^{-1}(U_i) = \pi^*(D \cap U_i) = (f|_{U_i})$ where $D \cap U_i$ denotes intersecting each component of D with U_i and throwing away those that do not meet U_i. Since the prime divisors which do not meet U_i are exactly those whose generic points are not in U_i, then we get the equality with $(f|_{U_i})$. Since $\pi^{-1}(U_i) = \mathbb{A}^1 \times U_i$, then by 6.6, we get that $f \in K(U_i) = K(V)$. It follows that $D = (f)$ in $\operatorname{Cl} V$ and thus π^* is injective. To show that π^* is surjective, it suffices to show that any prime divisor in $\overline{X} - P$ is linearly equivalent to one of the form $\pi^*(D)$. Let $Y \subseteq \overline{X} - P$ be a prime divisor, then $Y = \operatorname{cl}\{y\}$ for some $y \in \overline{X} - P$. It follows that $y \in \mathbb{A}^1 \times U_i$ for some i since these cover $\overline{X} - P$. By 6.6, $\operatorname{cl}\{y\}$ in $\mathbb{A}^1 \times U_i$ is linearly equivalent to $\pi^*(D)$ for some divisor D and this equivalence still holds in $\overline{X} - P$, thus π^* is surjective. It follows that π^* is an isomorphism. Furthermore, since P is contained in a line in $\overline{X}$ which is contained in $\overline{X}$ ($\overline{X}$ has dimension $\geqslant 2$, so it is not a line), thus P has codimension $\geqslant 2$ and thus $\operatorname{Cl} V \cong \operatorname{Cl}(\overline{X} - P) \cong \operatorname{Cl} \overline{X}$.

(b) Let H be a hyperplane in $\mathbb{P}^n$ not containing V, then we wish to show that $V - \pi^*(V \cdot H)$ is principal. Let $H = V(t)$ for some linear homogeneous polynomial t. By a change of coordinates, we may assume that $t = x_0$, thus $H = V(x_0)$ and $V \nsubseteq V(x_0)$. Let V be given by the homogeneous ideal $\mathfrak{p}$ in $S = k[x_0, \cdots, x_n]$. On each $U_i \subseteq \mathbb{P}^n$, H is given by the local equation x_0/x_i. Let $\mathfrak{q}_j$ be the minimal homogeneous primes over $(x_0) + \mathfrak{p}$, then $H \cdot V$ is given by a sum of $\nu_{\overline{\mathfrak{q}_j}}(\overline{x_0}/\overline{x_i})$ times $\operatorname{cl}\{\mathfrak{q}_j\}$. After applying π^*, we get $\nu_{\overline{\mathfrak{q}_j}}(\overline{x_0}/\overline{x_i})$ times $\operatorname{cl}\{\mathfrak{q}_j[x]\}$. We now need to construct an element of $K(\overline{X})$

with a zero along V and poles of the appropriate orders on the prime divisors of $\pi^*(H \cdot V)$. $K(\overline{X})$ is exactly the degree 0 elements in $\mathrm{Frac}((S/\mathfrak{p})[x_{n+1}])$. Consider the element $f = \overline{x_{n+1}}/\overline{x_0} \in K(\overline{X})$. In $\overline{X}$, V is the hyperplane section at infinity, i.e. $V(x_{n+1}) \cap \overline{X}$, thus V is the prime divisor given by the prime ideal $(\overline{x_{n+1}}) \in \overline{X}$. It follows that $\nu_V(f) = 1$. Furthermore, in each $\pi^{-1}(U_i)$, we get that $\nu_{\overline{\mathfrak{q}_j[x_{n+1}]}}(f|_{\pi^{-1}(U_i)}) = -\nu_{\overline{\mathfrak{q}_j}}(\overline{x_0}/x_i)$ as desired. It follows that V is linearly equivalent to $\pi^*(H \cdot V)$. We know that $X = \overline{X} - V$ and that $\mathrm{Cl}\, V = \mathrm{Cl}\, \overline{X}$, thus we obtain an exact sequence,

$$\mathbb{Z} \to \mathrm{Cl}\, V \to \mathrm{Cl}\, X \to 0$$

Where $1 \mapsto V \in \mathrm{Cl}\, \overline{X}$ which is equivalent to $\pi^*(V \cdot H)$ and thus under the isomorphism maps to $V \cdot H$. The map $1 \to V \cdot H$ is injective since it is the map $\mathrm{Cl}\, \mathbb{P}^n \to \mathrm{Cl}\, V$ defined in 6.2. It follows that we may extend the above sequence to a short exact sequence,

$$0 \to \mathbb{Z} \to \mathrm{Cl}\, V \to \mathrm{Cl}\, X \to 0$$

(c) If $S(V)$ is a UFD, then V is projectively normal since UFDs are integrally closed. Showing that $\mathrm{Cl}\, V$ is generated by $V \cdot H$ is equivalent to showing that $\mathrm{Cl}\, X = 0$, however this is true since X is an affine scheme whose coordinate ring is a UFD. Conversely, suppose that V is projectively normal and $\mathrm{Cl}\, V$ is generated by $V \cdot H$, then by the exact sequence above, we have that $\mathrm{Cl}\, X = 0$. By 6.2, it suffices to show that X is normal, i.e. that all of its stalks are normal. Since V is projectively normal, then $X = \mathrm{Spec}\ A$ with A integrally closed. Since localizations of normal domains are normal, then X is normal and thus A is a UFD.

(d) Any divisor in X is obtained from a divisor in $\overline{X}$, which is linearly equivalent to a divisor such that each prime divisor passes through P. Thus, all divisors of X are of the form $\sum n_i \mathrm{cl}\{\mathfrak{p}_i^e\}$. Since $V(\mathfrak{p}_i^e)$ contains P, then localizing at P, we obtain a prime divisor in $\mathrm{Spec}\ \mathcal{O}_P$. Since $K(X) = K(\mathrm{Spec}\ \mathcal{O}_P)$, then we obtain a map from $\mathrm{Cl}\, X \to \mathrm{Cl}(\mathrm{Spec}\ \mathcal{O}_P)$. If a divisor $D \in \mathrm{Div}\, \mathrm{Spec}\ \mathcal{O}_P$ is principal Furthermore, any prime divisor of $\mathrm{Spec}\ \mathcal{O}_P$ corresponds to a prime divisor of X and thus the map is surjective. The important part of this isomorphism comes from the fact that all prime divisors in $\mathrm{Div}\, X$ are linearly equivalent to ones containing P.

II.6.4

Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \cdots, x_n]$ be a *square-free* nonconstant polynomial. Let $A = k[x_1, \cdots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring.

Proof:

z is a unit in $k(x_1, \cdots, x_n)$, so the field of fractions of A is just $k(x_1, \cdots, x_n)[z]/(z^2 - f)$. This is the splitting field of the polynomial $z^2 - f$, thus it is Galois. Since this is a degree 2 extension, then its Galois group is $\mathbb{Z}/2\mathbb{Z}$ with automorphism $z \mapsto -z$. Let $\alpha = g = hz \in K$, with $g, h \in k(x_1, \cdots, x_n)$, then α has minimal polynomial $X^2 - 2gX + (g^2 - h^2 f)$. If g, h are in $k[x_1, \cdots, x_n]$, then using the minimal polynomial, α is integral over $k[x_1, \cdots, x_n]$. Conversely, suppose α is integral over $k[x_1, \cdots, x_n]$, then we want to show that the minimal polynomial has coefficients in $k[x_1, \cdots, x_n]$. This is a result of the following two lemmas:

Let $A \subseteq B$ be domains, and let $\mathfrak{a} \subseteq A$ be an ideal. Let C be the integral closure of A in B. Then, the integral closure of $\mathfrak{a}$ in B is $\sqrt{\mathfrak{a}C}$. One direction is clear: if x is in the integral closure of $\mathfrak{a}$, then $x^n + a_{n-1}x^{n-1} + \cdots a_0 = 0$ with each $a_i \in \mathfrak{a}$ and thus x is integral over A and thus in C and also, $x^n = -(a_{n-1}x^{n-1} + \cdots a_0) \in \mathfrak{a}C$, so $x \in \sqrt{\mathfrak{a}C}$. Conversely, suppose that $x \in \sqrt{\mathfrak{a}C}$, then $x^n \in \mathfrak{a}C$, i.e. $x^n = \sum a_i y_i$ where $a_i \in \mathfrak{a}$ and $y_i \in C$. Let $M = A[y_1, \cdots, y_n]$, then M is finitely generated and integral over A and thus finite over A. Consider $\varphi : M \to M$ by $\varphi(m) = x^n m$, then

$\varphi(M) = x^n M \subseteq \mathfrak{a}M$. By Cayley-Hamilton, we have $\varphi^k + a_{k-1}\varphi^{k-1} + \cdots + a_0\mathrm{id} = 0$, thus applying this to 1, we get that $x^{nk} + a_{k-1}x^{n(k-1)} + \cdots + a_0 = 0$ with each $a_i \in \mathfrak{a}$, thus x is integral over A.

Let $A \subseteq B$ be domains and $\mathfrak{a} \subseteq A$, with A integrally closed (normal) and suppose $x \in B$ is integral over A. Then, x is algebraic over $\mathrm{Frac}(A)$ and its minimal polynomial, $t^n + a_{n-1}t^{n-1} + \cdots a_0$ has $a_i \in \sqrt{\mathfrak{a}}$ for all i. Let K be the splitting field of the minimal polynomial, then K contains all Galois conjugates $x_1, \cdots, x_n$ of x which are each integral over $\mathfrak{a}$ since they satisfy the minimal polynomial of x. Since the a_i are given by symmetric polynomials in the x_i, then they too are integral over A. Note that a priori, the a_i are in $\mathrm{Frac}(A)$, however since they are integral and A is integrally closed, then they are in $\sqrt{\mathfrak{a}}$ by the previous lemma.

We may now apply this to the above problem by noting that $S = k[x_1, \cdots, x_n]$ is a UFD and thus integrally closed, so if α is integral over S, then its minimal polynomial has coefficients in S and thus $g, h \in S$ as desired. It follows that A is the integral closure of S in $k(x_1, \cdots, x_n)[z]/(z^2 - f)$ and hence is integrally closed.

II.6.5

Quadric Hypersurfaces. Let $\operatorname{char} k \neq 2$, and let X be the affine quadric hypersurface Spec $k[x_0, \cdots, x_n]/(x_0^2 + \cdots + x_r^2)$.

(a) Show that X is normal if $r \geqslant 2$.

(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0x_1 = x_2^2 + \cdots + x_r^2$. Now imitate the proof method of (6.5.2) to show that:

(1) If $r = 2$, then $\mathrm{Cl}\, X \cong \mathbb{Z}/2\mathbb{Z}$;

(2) If $r = 3$, then $\mathrm{Cl}\, X \cong \mathbb{Z}$;

(3) If $r \geqslant 4$, then $\mathrm{Cl}\, X = 0$.

(c) Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equation. Show that:

(1) If $r = 2$, $\mathrm{Cl}\, Q \cong \mathbb{Z}$ and the class of a hyperplane section $Q \cdot H$ is twice the generator;

(2) If $r = 3$, $\mathrm{Cl}\, Q \cong \mathbb{Z} \oplus \mathbb{Z}$;

(3) If $r \geqslant 4$, $\mathrm{Cl}\, Q \cong \mathbb{Z}$, generated by $Q \cdot H$.

(d) Prove Klein's theorem, which says that if $r \geqslant 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbb{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection.

Proof:

(a) To show that X is normal, it is necessary and sufficient to just show that $k[x_0, \cdots, x_n]/(x_0^2 + \cdots + x_r^2)$ is normal. We will use the previous exercise by letting $z = x_0$ and $f = -(x_1^2 + \cdots + x_r^2)$, then f is squarefree since it is irreducible if $r \geqslant 3$ and if $r = 2$, then f factors as a difference of squares which has no repeated roots since $\operatorname{char} k \neq 2$.

(b) Consider the change of coordinates $x_0 \mapsto -x_0 - ix_1$ and $x_1 \mapsto x_0 - ix_1$, then $-x_0x_1 \mapsto x_0^2 + x_1^2$ and thus we have found the desired linear change of coordinates. If $r = 2$, then we get $x_0x_1 = x_2^2$ which we see is the surface in example 6.5.2 and thus $\mathrm{Cl}\, X \cong \mathbb{Z}/2\mathbb{Z}$. When $r = 3$, by a suitable linear change of coordinates, we get $x_0x_1 = x_2x_3$ which we see is the affine variety given by the cone of example 6.6.1. By exercise 6.3 and the intersection with a hyperplane is of type $(1,1)$, we get that $\operatorname{Cl} X \cong \mathbb{Z}$. To see that it is of type $(1,1)$, we first note that the intersection with a hyperplane is given by the homogeneous prime ideal $(x_0x_1 - x_2x_3, x_3) = (x_0x_1, x_3)$ which one recognizes as the union of two $\mathbb{P}^1$'s. These $\mathbb{P}^1$'s are exactly the two $\mathbb{P}^1$'s that make up $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. It follows that we need only show that the multiplicity of each $\mathbb{P}^1$ is 1. We know that both multiplicities will be positive and furthermore, the total degree of the divisor will be $\deg(H \cdot V) = \deg(H) \cdot \deg(V) = 1 \cdot 2 = 2$, so both must have multiplicity 1, thus it is of type $(1,1)$. If $r \geqslant 4$, then consider the ideal $(\overline{x}_0)$. To see that this ideal is prime, the preimage of the ideal in $k[x_0, \cdots, x_r]$ is $(x_0, x_2^2 + \cdots + x_r^2)$ which is prime for $r \geqslant 2$. If we remove the closed subset $Y = V(\overline{x}_0) \subseteq X$, then we get $\operatorname{Spec} k[x_0, x_0^{-1}, x_1, \cdots, x_r]/(-x_0x_1 + x_2^2 + \cdots + x_r^2)$. If we consider the map from $(k[x_0, x_0^{-1}, x_2, \cdots, x_r])[x_1] \to k[x_0, x_0^{-1}, x_2, \cdots, x_r]$ by sending $x_1 \mapsto x_0(x_2^2 + \cdots + x_r^2)$, then by the first isomorphism theorem, we obtain an isomorphism,

$$k[x_0, x_0^{-1}, x_1, \cdots, x_r]/(-x_0x_1 + x_2^2 + \cdots + x_r^2) \to k[x_0, x_0^{-1}, x_2 \cdots, x_r]$$

it follows that once we remove Y, we obtain the spectrum of a UFD and thus $\operatorname{Cl}(X - Y) = 0$. It therefore suffices to show that Y is a principal divisor in X, which involves showing that $\nu_Y(\overline{x}_0) = 1$ at which point we conclude that $\operatorname{Cl} X = 0$. To see that the value of $\overline{x}_0$ is 1, we see that it is obviously at least 1 since $\overline{x}_0$ generates the maximal ideal of the local ring. If $\overline{x}_0 \in (\overline{x}_0)^2$, then we have that $\overline{x}_0 = f/g\overline{x}_0^2$ where f, g are in the quotient ring and $\overline{x}_0 \nmid g$. However, since the quotient ring is an integral domain, then one immediately notices that this implies that $g = \overline{x}_0 f$ and thus $\overline{x}_0 \mid g$ which is a contradiction. It follows that $\operatorname{Cl} X = 0$ as desired.

(c) If $r = 2$, then by 6.3 and in light of part b(1), we get an exact sequence,

$$0 \to \mathbb{Z} \to \operatorname{Cl} Q \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Now suppose that there is a divisor $x \in \operatorname{Cl} Q$ such that $H \cdot Q = 2x$, then we obtain the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "1 \mapsto H\cdot Q"'] & \operatorname{Cl} Q \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"'] \arrow[u, "\sim"'] & \mathbb{Z} \arrow[r] \arrow[u, "1 \mapsto x"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[u, "\sim"'] & 0
\end{tikzcd}$$

Which, by the five lemma, would induce an isomorphism $\operatorname{Cl} Q \cong \mathbb{Z}$. It follows that we need only find this divisor x. I claim it is the prime divisor associated to the prime ideal $(\overline{x}_0)$. The proof is as in 6.5.2. If $r = 3$, then we have the case dealt with in 6.6.1. If $r \geqslant 4$, then in light of part b(3) and exercise 6.3, we have an exact sequence,

$$0 \to \mathbb{Z} \to \operatorname{Cl} Q \to 0 \to 0$$

And thus $\mathbb{Z} \to \operatorname{Cl} Q$ is an isomorphism.

(d) Clearly $S(Q)$ is a UFD since by part a, it is normal and by part b, it has divisor class group 0. If Y is an irreducible subvariety of codimension 1 on Q, then Y is the vanishing of some homogeneous prime ideal $\mathfrak{p}$ of height 1 in the homogeneous coordinate ring $S(Q)$. Since $S(Q)$ is a UFD, it follows that $\mathfrak{p} = (\overline{f})$ for some irreducible homogeneous element $\overline{f} \in S(Q)$. It is then immediately clear that Y is the intersection of the vanishing of f in $\mathbb{P}^n$ and Q. It remains to be shown that f itself is an irreducible elements in $k[x_0, \cdots, x_n]$. Since $k[x_0, \cdots, x_n]$ is a UFD, let $f = g_1 \cdots g_n$ where each

g_i is irreducible. Since $(x_0^2 + \cdots + x_r^2)$ is a homogeneous ideal, then we may assume that f is a homogeneous element in $k[x_0, \cdots, x_n]$ of the same degree. Since $\overline{f}$ is irreducible, it follows that all but 1 of the $\overline{g}_i$ are units in $S(Q)$. Since f is homogeneous, then each g_i is homogeneous. WLOG, let $\overline{g}_1 = f$, then by degree considerations, we see that all $g_i, i > 1$ must have degree 1 and therefore are units. It follows that we may take f to be irreducible and thus Y is the intersection of an irreducible hypersurface and Q.

II.6.8

(a) Let $f : X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \to f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphisms of (6.16).

(c) If X is a locally factorial integral closed subscheme of $\mathbb{P}_k^n$, and if $f : X \to \mathbb{P}_k^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in Ex 6.2 via the isomorphisms of (6.16).

Proof:

(a) To show that f^* induces an isomorphism, we need only check that it respects isomorphisms. This can be done by showing that f^* is a functor. Suppose that $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules, then we obtain a morphism $f^*\varphi : f^*\mathscr{F} \to f^*\mathscr{G}$ by taking the sheafification of the morphism $f^{-1}\varphi \otimes \mathrm{id}$. It is clear that if $\varphi = \mathrm{id}$, then $f^{-1}\mathrm{id} \otimes \mathrm{id}$ is the identity morphism on the presheaf of $f^*\mathscr{F}$ and thus its sheafification is the identity morphism on $f^*\mathscr{F}$. The functoriality of f^* then follows from the functoriality of f^{-1}, namely, if we have $\varphi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{G} \to \mathscr{H}$, then $f^{-1}(\psi \circ \varphi) \otimes \mathrm{id} = (f^{-1}\psi \circ f^{-1}\varphi) \otimes \mathrm{id}$ and one then checks on the presheaves that this is the composition of $f^{-1}\varphi \otimes \mathrm{id}$ and $f^{-1}\psi \otimes \mathrm{id}$. It follows that f^* is a functor, hence preserves isomorphisms, and thus induces a morphism $\operatorname{Pic} Y \to \operatorname{Pic} X$.

(b) The morphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ takes any point $Q \in Y$ and considers a local parameter $t \in \mathcal{O}_Q$ such that $\nu_Q(t) = 1$, then $f^*Q = \sum_{f(P)=Q} \nu_P(t)P$. Note that since $f(P) = Q$ and f induces a morphism $f_P^\# : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$, so $\nu_P(t)$ makes sense, the t is really $f_P^\#(t)$. Note that since nonsingular curves are locally factorial since at their generic point, the local ring is a field and any other point will have codimension 1 and thus the local ring with be a DVR and hence a UFD. We first convert f^* on Cl to CaCl. Given any Cartier divisor D on Y, it may be locally given by $\{(U_i, f_i)\}$, then for any prime divisor Z of Y, we choose some i such that $U_i \cap Z \neq \varnothing$, then the coefficient on Z of the Weil divisor associated to D is $\nu_Z(f_i)$. It follows that f^*D is the Cartier divisor associated to $\sum_Z \sum_{f(P)=Z} \nu_Z(f_i)\nu_P(t_Z) \cdot P$. Since f is finite, then the image of f cannot be a closed point, for $f : X \to \operatorname{Spec} k$ cannot be finite, thus f is dominant and we obtain a morphism of function fields, $\widetilde{f} : K(Y) \to K(X)$. In $K(Y)$, $f_i = ut_Z^{\nu_Z(f_i)}$ for some unit u in $\mathcal{O}_Z$, thus,

$$\nu_P(\widetilde{f}(f_i)) = \nu_P(\widetilde{f}(t_Z^{\nu_Z(f_i)})) = \nu_P(f_P^\#(t_Z)^{\nu_Z(f_i)}) = \nu_Z(f_i)\nu_P(t_Z)$$

Thus f^*D is the Cartier divisor associated to $\sum_Z \sum_{f(P)=Z} \nu_P(\widetilde{f}(f_i)) \cdot P = \sum_P \nu_P(\widetilde{f}(f_i)) \cdot P$. It follows that f^*D is the Cartier divisor $\{(f^{-1}(U_i), \widetilde{f}(f_i))\}$. To show that this yields the same morphism on Pic, we need only check that if $\mathscr{L}$ is an invertible subsheaf of $\mathscr{K}_Y$ generated by g^{-1} on U, then $f^*\mathscr{L}$ is given by generated by $(\widetilde{f}(g))^{-1}$ on $f^{-1}(U)$ as a subsheaf of $\mathscr{K}_X$. Since this is a local question, we may assume that $Y = U$ and U is affine and hence $f^{-1}(U) = X$.

If we have $f : X \to Y = \text{Spec } A$ and $\mathscr{L}$ is an invertible sheaf on Y given by multiplying $\mathcal{O}_Y$ by some $g \in K(Y)$, then $f^*\mathscr{L} \cong \widetilde{f}(g)\mathcal{O}_X$. We have,

$$f^*_{\text{pre}}(g\mathcal{O}_Y)(V) = \varinjlim_{U \supseteq f(V)} (g\mathcal{O}_Y)(U) \otimes_{\varinjlim_{W \supseteq f(V)} \mathcal{O}_Y(W)} \mathcal{O}_X(V) = (\widetilde{f}(g)\mathcal{O}_X)(V)$$

By mapping $[g|_U s] \otimes t$ to $\widetilde{f}(g)|_V f^{-1} f^{\#}(V)([s])t$. To check that these morphism glue, it suffices to check for localizations of A. Suppose we have $y \in A$ and $\mathscr{L} = g'\mathcal{O}_Y$ such that $g'/g \in \mathcal{O}_Y(Y)^*$, then we have that $\widetilde{f}(g') = \widetilde{f}(g'/g)\widetilde{f}(g)$ and since $g'/g \in \mathcal{O}_Y(Y)^*$, then $\widetilde{f}(g'/g) \in \mathcal{O}_X(X)^*$ and thus the isomorphisms glue.

(c) We need only check what happens to $D = V(x_0)$. Let Y_i be the irreducible components of $V(x_0) \cap X$, then the divisor f^*D is given by $\sum \nu_{Y_i}(\overline{x}_0)Y_i$ where $\overline{x}_0 \in \text{Frac}(S/\mathfrak{p})$ and $X = V(\mathfrak{p})$. The Cartier divisor of D is given x_0/x_i on $D_+(x_i)$ and we know how the f^* for Pic acts on Cartier divisors, it is given by $\sum_P \nu_P(\widetilde{f}(f_i)) \cdot P$. There is a slight difference, that being that in the case of finite morphisms of nonsingular curves, $\widetilde{f}$ is a morphism of the function fields. In this case, however, we only get a morphism from a subring of the function field of the function field of $\mathbb{P}^n$, the subring whose denominators are not contained in $\mathfrak{p}$. It follows that $\widetilde{f}$ is then the restriction of the morphism $S_{\mathfrak{p}} \to \text{Frac}(S/\mathfrak{p})$ to the degree 0 elements. Now, for any prime divisor P on X such that $P \cap D_+(x_i) \neq \varnothing$, then $\nu_P(\overline{g}) = \nu_P(\overline{g}/\overline{x}_i^{\deg(g)})$, thus $f^*D = \sum_P \nu_P(\overline{x}_0) \cdot P$. If $\nu_P(\overline{x}_0) > 0$, then $\overline{x}_0/\overline{x}_i \in \mathfrak{m}_{X,P}$ and thus $x_0/x_i \in \mathfrak{m}_{\mathbb{P}^n,P}$, so $P \subseteq V(x_0) \cap X$ is an irreducible component. The converse also holds, thus f^*D is the same regardless of the definition used.

II.6.9

Singular Curves. Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\widetilde{X}$ be its normalization, and let $\pi : \widetilde{X} \to X$ be the projection map. For each point $P \in X$, let $\mathcal{O}_P$ be the local ring, and let $\widetilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$. We use an * to denote the group of units in a ring.

(a) Show there is an exact sequence,

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \to \text{Pic}\, X \xrightarrow{\pi^*} \text{Pic}\, \widetilde{X} \to 0$$

(b) Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbf{G}_a \to \text{Pic}\, X \to \mathbb{Z} \to 0$$

and if X is a plane nodal cubic curve, there is an exact sequence,

$$0 \to \mathbf{G}_m \to \text{Pic}\, X \to \mathbb{Z} \to 0$$

Proof:

(a) Let $X \cap D_+(x_i) \neq \varnothing$, then $X \cap D_+(x_i) = \text{Spec } S$ for some one-dimensional k-algebra S. By Noether normalization, we have that S is finite over $k[x]$ for some $x \in S$. Let S be generated by $x_1, \cdots, x_n$, and suppose that S is not free over $k[x]$, then there is a relation $\sum a_i x_i = 0$. WLOG, let $a_1 \neq 0$, then we may eliminate the generator x_1 from S_{a_1} over $k[x]_{a_1}$. By induction, we get some $f \in k[x]$ such that S_f is free over $k[x]_f$. Since S_f is free and an integral domain, then $S_f = k[x]_f$.

It follows that $X \cap D_+(x_i)$ has an open subset isomorphic to an open subset of affine space. X is nonsingular on this open subset since its local rings are all DVRs. Therefore, there are only finitely many P such that $\widetilde{\mathcal{O}}_P \neq \mathcal{O}_P$.

We want to show that given any elements in $\widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^*$ for P, we may find a global section of $\mathscr{K}^*/\mathcal{O}^*$ with the desired germs at each point. It suffices to show that $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is flabby. Note, we define $\widetilde{\mathcal{O}}(U) = \widetilde{\mathcal{O}(U)}$. One fairly easily checks that $\widetilde{\mathcal{O}} = \pi_* \mathcal{O}_{\widetilde{X}}$ by looking on affines. We first show that $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is a direct sum of skyscraper sheaves and then since a direct sum of flabby sheaves is flabby, then we are done. Let $\mathscr{F}^P$ denote the skyscraper sheaf given by $\widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^*$ at P.

Since X is a curve, and the nonsingular points are finite, then we may cover X by open sets U_i such that each U_i contains the i^{th} singular point and U_0 is the complement of the set of singular points. It follows that the intersection of any two U_i, U_j contains no singular points and thus $(\widetilde{\mathcal{O}}^*/\mathcal{O}^*)|_{U_i \cap U_j}$ is identically the 0 sheaf. It follows that $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is isomorphic to the gluing or arbitrary isomorphic copies of $(\widetilde{\mathcal{O}}^*/\mathcal{O}^*)|_{U_i}$ since all isomorphism automatically glue. We now show that $(\widetilde{\mathcal{O}}^*/\mathcal{O}^*)|_{U_i}$ is isomorphic to $\mathscr{F}^P$ where P is the singular point in U_i. To do so, it suffices to show that any sheaf whose support is a single point is a skyscraper sheaf. Let $\mathscr{F}$ have support $\{P\}$ with $\{P\}$ closed, then we want to show that $\mathscr{F} \cong i_P(\mathscr{F}_P)$. Let $U \subseteq X$, then take $[W, s] \in \mathscr{F}_P = i_P(\mathscr{F}_P)(U)$. Since P is a closed point, then $U \cap W - \{P\}$ is open, and s and 0 agree on $U \cap W - \{P\}$ since $\mathscr{F}$ is 0 away from P. It follows that we may extend s to U by gluing 0 on $U - \{P\}$ and s on W. This then induces an isomorphism on stalks and thus $\mathscr{F} \cong i_P(\mathscr{F}_P)$. Since skyscraper sheaves are flabby, then $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is flabby.

Since $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is a subsheaf of $\mathscr{K}^*/\mathcal{O}^*$ whose global sections are $\operatorname{CaCl}(X)$ and $\operatorname{CaCl}(X) \cong \operatorname{Pic}(X)$, then since $\widetilde{\mathcal{O}}^*/\mathcal{O}^*$ is flabby, we get,

$$\bigoplus_{P \in X} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* = \bigoplus_{P \text{ singular}} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* = (\widetilde{\mathcal{O}}^*/\mathcal{O}^*)(X) \subseteq (\mathscr{K}^*/\mathcal{O}^*)(X) = \operatorname{CaCl}(X) \cong \operatorname{Pic}(X)$$

We now show that we have an exact sequence,

$$0 \to \pi_* \mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\mathcal{O}_X^* \to \mathscr{K}^*/\pi_* \mathcal{O}_{\widetilde{X}}^* \to 0$$

then since $\pi_* \mathcal{O}_{\widetilde{X}}^*/\mathcal{O}_X^* = \widetilde{\mathcal{O}}_X^*/\mathcal{O}_X^*$ is flabby, then by 1.16, the sequence remains exact after taking global sections. At which point, we need only show that the term on the right of the sequence is $\operatorname{Pic} \overline{X}$. Looking at the presheaves of each of these, and any open $U \subseteq X$, we see that,

$$0 \to \widetilde{\mathcal{O}}_X^*(U)/\mathcal{O}_X^*(U) \to K^*/\mathcal{O}_X^*(U) \to K^*/\widetilde{\mathcal{O}}_X^*(U) \to 0$$

is exact, thus the sequence is exact at each stalk and thus exact. Taking global sections, and noting that $\widetilde{\mathcal{O}}_X^*/\mathcal{O}_X^*$ is flabby, we have that the following is exact,

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \to (\mathscr{K}^*/\mathcal{O}_X^*)(X) \to (\mathscr{K}^*/\widetilde{\mathcal{O}}_X^*)(X) \to 0$$

Any element of $(\mathscr{K}^*/\widetilde{\mathcal{O}}_X^*)(X)$ is given by $\{(U_i, f_i)\}$ where $f_i \in K^*$, then we obtain a Cartier divisor $\{(\pi^{-1}(U_i), f_i)\}$ on $\widetilde{X}$. This map is injective for if this Cartier divisor is 0, then $f_i \in \mathcal{O}_{\widetilde{X}}^*(\pi^{-1}(U_i)) = \pi_* \mathcal{O}_{\widetilde{X}}^*(U_i)$ for each i and since $\pi_* \mathcal{O}_{\widetilde{X}}^* = \widetilde{\mathcal{O}}_X^*$, then $f_i \in \widetilde{\mathcal{O}}_X^*(U_i)$ for each i and thus the Cartier divisor was 0 on X.

We now need to show that given any Cartier divisor on $\widetilde{X}$, $\{(U_i, f_i)\}$, it arises from the map described above. Since $\widetilde{X}$ is one dimensional noetherian and normal, then its local rings are regular.

We then obtain points on $\widetilde{X}$ where $\nu_P(f_i) \neq 0$. It suffices to be able to obtain the Cartier divisors associated to single points in $\widetilde{X}$. Let $P \in \widetilde{X}$ be any closed point, then if $\pi(P)$ is not a singular point, we have that there is some $f \in K^*$ such that $\nu_P(f) = 1$ and $\nu_Q(f) \neq 0$ for only finitely many other $Q \in \widetilde{X}$. As in 6.11, for each Q, we get an f_Q such that f/f_Q is invertible on open neighborhoods U_Q of each Q. Since P is not a singular point, then it corresponds to the Cartier divisor $\{(U_i, f_i)\}$ where f_i are either f if $i = 0$ or f_Q for the corresponding Q.

If $\pi(P)$ is a singular point of X, then we wish to show that we can construct a Cartier divisor on X which distinguishes the preimages of $\pi(P)$. We wish to find a single element of K^* which has the correct valuations at each $Q \in \pi^{-1}(\pi(P))$, i.e. 0 on all $Q \neq P$ and 1 on P. We therefore have the following algebraic problem: Let A be a k-algebra, and let $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{q}_1, \cdots, \mathfrak{q}_n \in \operatorname{Spec} \overline{A}$ lie over $\mathfrak{p}$, then there is some $s \in \operatorname{Frac}(A)$ such that $\nu_{\mathfrak{q}_1}(s) = 1$ and $\nu_{\mathfrak{q}_i}(s) = 0$ for all $i > 1$. This follows from the prime avoidance lemma on $\mathfrak{q}_1^2, \mathfrak{q}_2, \cdots, \mathfrak{q}_n$. Since $\mathfrak{q}_1$ and $\mathfrak{q}_i$ lie over $\mathfrak{p}$, then $\mathfrak{q}_1 \nsubseteq \mathfrak{q}_i$ and since $\dim_k \mathfrak{q}_1/\mathfrak{q}_1^2 \geqslant \dim(A) = 1$, then $\mathfrak{q}_1 \nsubseteq \mathfrak{q}_1^2$. It follows that there is some $x \in \mathfrak{q}_1$ not contained in $\mathfrak{q}_1^2, \mathfrak{q}_2, \cdots, \mathfrak{q}_n$. Since $\nu_{\mathfrak{q}_i}(x) > 0$ iff $x \in \mathfrak{q}_i$ (since $\mathfrak{q}_i$ is prime), then $\nu_{\mathfrak{q}_i}(x) = 0$ for all $i > 1$ and $\nu_{\mathfrak{q}_1}(x) = 1$.

From the above, we may take such an $s \in K^*$, then there are only finitely many points in X, other than $\pi(P)$ on on which s has non-zero valuation. Removing those points, we obtain an open subset U of X containing $\pi(P)$. For any open affine V contained in $U - \pi(P)$, we have that s has valuation 0 everywhere on $\pi^{-1}(V)$. Since $\pi^{-1}(V)$ is integrally closed, then $\nu_P(s) = 0$ for all $P \in \pi^{-1}(V)$ means that $s \in \mathcal{O}_{\widetilde{X}}(\pi^{-1}(V))$. Furthermore, if s is not invertible, then there is some maximal ideal $\mathfrak{m} \in \mathcal{O}_{\widetilde{X}}(\pi^{-1}(V))$ containing s in which case, $\nu_{\mathfrak{m}}(s) \neq 0$ which is a contradiction. It follows that $s \in \mathcal{O}^*_{\widetilde{X}}(V)$. We may therefore consider the element $\{(X - \pi(P), 1), (U, s)\} \in (\mathscr{K}^*/\widetilde{\mathcal{O}}^*_X)(X)$ which maps to the Cartier divisor associated to P as desired.

To obtain the Cartier divisor class groups, we quotient the middle and last terms by the image of K^*, i.e. the principal Cartier divisors.

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{P \in X} \widetilde{\mathcal{O}}^*_P/\mathcal{O}^*_P \arrow[r] & (\mathscr{K}^*/\mathcal{O}^*_X)(X) \arrow[r] \arrow[d] & (\mathscr{K}^*/\widetilde{\mathcal{O}}^*_X)(X) \arrow[r] \arrow[d] & 0 \\
 & & \frac{(\mathscr{K}^*/\mathcal{O}^*_X)(X)}{\text{im } K^*} \arrow[r] & \frac{(\mathscr{K}^*/\widetilde{\mathcal{O}}^*_X)(X)}{\text{im } K^*} \arrow[r] & 0
\end{tikzcd}$$

We need to show that the morphism $\bigoplus_{P \in X} \widetilde{\mathcal{O}}^*_P/\mathcal{O}^*_P \to \frac{(\mathscr{K}^*/\mathcal{O}^*_X)(X)}{\text{im} K^*}$ is still injective. This amounts to showing that its image contains no principal divisors. From our exact sequence, we know that the image of $\bigoplus_{P \in X} \widetilde{\mathcal{O}}^*_P/\mathcal{O}^*_P$ is exactly the kernel the morphism from $(\mathscr{K}^*/\mathcal{O}^*_X)(X) \to (\mathscr{K}^*/\widetilde{\mathcal{O}}^*_X)(X)$. It follows that we need only show that no principal Cartier divisor maps to 0 under this quotient. If $f \in K^*$ yields a principal Cartier divisor on X, then the associated Cartier divisor on $\widetilde{X}$ is also given by f. Suppose that this Cartier divisor is 0 on $\widetilde{X}$, then f is invertible everywhere, and thus as we saw above, for any affine open $U \subseteq \widetilde{X}$, $f \in \mathcal{O}^*_{\widetilde{X}}(U)$ and thus $f \in \mathcal{O}^*_{\widetilde{X}}(\widetilde{X})$. By exercise II.4.5d, we have that $\mathcal{O}^*_{\widetilde{X}}(\widetilde{X}) = k^*$, thus f is a nonzero element in the base field. Therefore, $f \in \mathcal{O}^*_X(X) = k^*$, so f defines the 0 Cartier divisor. It follows that $\bigoplus_{P \in X} \widetilde{\mathcal{O}}^*_P/\mathcal{O}^*_P \to \frac{(\mathscr{K}^*/\mathcal{O}^*_X)(X)}{\text{im} K^*}$ is still injective. The rest of the exactness remains, and thus, we obtain the exact sequence,

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}}^*_P/\mathcal{O}^*_P \to \operatorname{CaCl}(X) \to \operatorname{CaCl}(\widetilde{X}) \to 0$$

Using the equivalence of the Cartier divisor class group and the Picard group for integral schemes, we obtain the desired exact sequence,

$$0 \to \bigoplus_{P \in X} \tilde{\mathcal{O}}_P^* / \mathcal{O}_P^* \to \operatorname{Pic}(X) \to \operatorname{Pic}(\tilde{X}) \to 0$$

(b) If X is a plane cuspidal cubic, $X = \operatorname{Proj} k[x,y,z]/(zy^2 - x^3)$, then there is a single singular point, at the ideal (x,y). The normalization is a nonsingular, projective curve, so it will be isomorphic to the abstract nonsingular curve over its function field. Thus, we need only compute the function field. We may therefore compute the function field of any open affine subset, namely Spec $k[x,y]/(y^2 - x^3)$. Let $K = \operatorname{Frac}(k[x,y]/(y^2 - x^3))$. Let $t \in K$ be given by y/x, then $t^2 = y^2/x^2 = x^3/x^2 = x$, thus $x = t^2, y = t^3$, so $K = k(t)$ and thus $\tilde{X} \cong \mathbb{P}^1$. It follows that we need only compute the

II.7.1

Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.

Proof:

We need only show that f is an isomorphism on stalks. We have $f_P : \mathscr{L}_P \to \mathscr{M}_P$ and since $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves, then there are isomorphisms $\varphi : \mathcal{O}_P \to \mathscr{L}_P$ and $\psi : \mathscr{M}_P \to \mathcal{O}_P$, thus we obtain a sequence,

$$\mathcal{O}_P \to \mathscr{L}_P \to \mathscr{M}_P \to \mathcal{O}_P$$

Since all maps are surjective, then the composition is a surjective endomorphism of finitely generated modules and thus an isomorphism. It follows that the composition is injective and since the first map, $\mathcal{O}_P \to \mathscr{L}_P$ is surjective, then the middle morphism must be injective.

II.7.2

Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \cdots, s_n\}$ and $\{t_0, \cdots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leqslant m$. Show that the corresponding morphisms $\varphi : X \to \mathbb{P}_k^n$ and $\psi : X \to \mathbb{P}_k^m$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m - n - 1$.

Proof:

We that $\mathscr{L} = \psi^* \mathcal{O}_{\mathbb{P}^m}(1)$ and t_i are the pullbacks of the global sections x_i on $\mathbb{P}^m$. Here, by pullback, we mean to consider $\psi^* \mathcal{O}_{\mathbb{P}^m}(1)$ as the tensor product $\psi^{-1}\mathcal{O}_{\mathbb{P}^m}(1) \otimes_{\psi^{-1}\mathcal{O}_{\mathbb{P}^m}} \mathcal{O}_X$ whose global sections contain the elements $[x_i] \otimes 1$ which are the pullbacks of x_i. Since the t_i and s_j generate the same subspace, then let $\vec{s}$ be a vector of the s_j and similarly $\vec{t}$ be a vector of the t_i and let M be some matrix such that $M\vec{t} = \vec{s}$. We now need to find global sections v_l, where $l = 0, \cdots, n$ of $\mathcal{O}_{\mathbb{P}^m}(1)$ such that the image of X is contained in the open subset of $\mathbb{P}^m$ where the v_l do not all vanish. Suppose we have the v_l, then we get a morphism $\theta : U \to \mathbb{P}^n$ where U is the open subscheme of $\mathbb{P}^m$ on which $\mathcal{O}_{\mathbb{P}^m}(1)|_U$ is generated by the global section v_l. Since the image of X under ψ is contained in U, then we may compose it with θ to obtain a morphism $\theta \circ \psi : X \to \mathbb{P}^n$ and $(\theta \circ \psi)^*(x_i) = \psi^* \theta^*(x_i) = \psi^*(v_i)$. We now want to compose this with automorphism α of $\mathbb{P}^n$ such that $\psi^* \theta^* \alpha^*(x_i) = s_i$. We can represent α^* by a representative matrix A of an element in $\operatorname{PGL}(n,k)$. It follows that we obtain

$\psi^*\theta^*\alpha^*(x_i) = \psi^*\theta^*(Ax_i) = \psi^*\theta^*(\sum_{j=1}^n a_{ji}x_j) = \psi^*(\sum_{j=1}^n a_{ji}v_j) = \sum_{j=1}^n a_{ji}\psi^*(v_j)$. We need this to equal $s_i = \sum_{j=1}^m m_{ij}t_j$. We can write $v_i = \sum_{j=1}^m b_{ij}x_j$, then let B be the $n \times m$ matrix given by these b_{ij}. If one expands $\psi^*(v_j)$, then you find that we require $A^TB = M$. Equivalently, we may find B by $B = (A^T)^{-1}M$, or just $B = AM$ for some invertible $n \times n$ matrix A.

It follows that we need only find some A such that the image of X in $\mathbb{P}^m$ does not intersect the vanishing of v's corresponding to AM. The places where the v_l all vanish are the points $\mathfrak{p}$ in $\mathbb{P}^n$ such that $(v_l)_\mathfrak{p} \in \mathfrak{m}_\mathfrak{p}\mathcal{O}_{\mathbb{P}^m}(1)_\mathfrak{p}$ for all $\mathfrak{p}$. Since these are the degree 1 elements in $S_\mathfrak{p}$ where $S = k[x_0, \cdots, x_m]$, then these are simply the points $\mathfrak{p}$ where $(v_l)_\mathfrak{p} \in S_\mathfrak{p}$. Suppose that some $P \in X$ maps under ψ to a point Q at which all v_l vanish. We then have $\psi_P : \mathcal{O}_{\mathbb{P}^m,Q} \to \mathcal{O}_{X,P}$. Since $\mathcal{O}_{\mathbb{P}^m}(1)$ is invertible, then it is isomorphic to $\mathcal{O}_{\mathbb{P}^m}$ around Q. This isomorphism is achieved by dividing by some nonzero x_i. It follows that $(v_l/x_i)_Q \in \mathfrak{m}_{\mathbb{P}^m,Q}$ for all l. The map ψ_P maps $(x_j/x_i)_Q \to (t_j/t_i)_P$, thus $(v_l/x_i)_Q \mapsto (s_l/t_i)_P$. Since ψ_P is a local homomorphism, then $(v_l/x_i)_Q \in \mathcal{O}_{\mathbb{P}^m,Q}$ means that they map into $\mathcal{O}_{X,P}$, but since s_l generate $\mathscr{L}$ at every point, then we must have that the image of the $(v_l/x_i)_Q$ generate, which is a contradiction. It follows that we may simply take $A = \mathrm{id}$. The fact that L has dimension $m-n-1$ follows from the fact that we have $n+1$ linear polynomials and $\mathbb{P}^m$ has dimension m.

II.7.3

Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$ be a morphism. Then:

(a) either $\varphi(\mathbb{P}^n_k) = pt$ or $m \geqslant n$ and $\dim \varphi(\mathbb{P}^n) = n$.

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a uniquely determined $d \geqslant 1$, (2) as a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m$. Also, φ has finite fibres.

Proof:

(a) We have that $\mathbb{P}^n_k \to \mathbb{P}^m_k \to \operatorname{Spec} k$ is proper and the latter morphism is separated, so φ is proper. Suppose that $m < n$, then we have the invertible sheaf $\varphi^*\mathcal{O}_{\mathbb{P}^m}$ on $\mathbb{P}^n$ and $m+1$ global sections which generate $\varphi^*\mathcal{O}_{\mathbb{P}^m}$. Since $\varphi^*\mathcal{O}_{\mathbb{P}^m}$ is an invertible sheaf, then it is isomorphic to some $\mathcal{O}_{\mathbb{P}^n}(d)$ for some d. The global sections of $\varphi^*\mathcal{O}_{\mathbb{P}^m}$ are not 0. We want to show that $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \geqslant 1$ requires at least $n+1$ generators. Given any element $s \in \mathcal{O}_{\mathbb{P}^n}(d)(X)$, then s does not generate the local rings at $V(s)$, which is a hypersurface. It follows by the dimension theorem for intersections of hypersurfaces in $\mathbb{P}^n$ that we require at least $n+1$ global sections, for if we have only n, then the dimension of the intersection of all $V(s)$ will be 0 and thus nonempty. It follows that $\varphi^*\mathcal{O}_{\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^n}$. We then notice that if $\varphi^*\mathcal{O}_{\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^n}$, then $\varphi^*(x_i)$ are all constant and thus φ is constant.

If $m \geqslant n$, then we again get that $\varphi^*\mathcal{O}_{\mathbb{P}^m} = \mathcal{O}_{\mathbb{P}^n}(d)$ for some d, however since $m \geqslant n$, this needn't be $\mathcal{O}_{\mathbb{P}^n}$. If $d = 0$, then we just have a point again. If $d \neq 0$, then we need only show that $\dim \varphi(\mathbb{P}^n) = n$. We first show that if φ is finite and closed, then this holds. Second, we show that φ is finite. Note that φ is closed since it is proper.

Suppose that $\varphi : X \to Y$ is finite and closed and X is integral, then we wish to show that $\dim \varphi(X) = \dim X$. We may immediately replace Y by the image subscheme of X, onto which φ surjects since the image is closed. Furthermore, the image of X is integral since X is integral. It follows that we may assume that $\varphi : X \to Y$ is surjective and Y is also integral. Now we need only show that a finite surjective morphism preserves dimension. Since φ is finite, then in particular, it is affine. Let $U \subseteq Y$ be an affine open subset, then $\dim U = \dim Y$ and $f^{-1}(U) = \operatorname{Spec} B$ is affine,

$U = \text{Spec } A$ and B over A is finite. Since finite morphisms are integral, then B is integral over A and thus they have the same dimension. It follows that $\dim X = \dim Y$.

To see that f is finite, it suffices to show that it is affine, since, by Ex 4.6, a proper morphism of affine varieties over a field is finite. $\mathbb{P}^m$ is covered by $D_+(x_i)$, then $f^{-1}(D_+(x_i)) = D_+(s_i)$, i.e. all points where s_i does not vanish in $\mathbb{P}^n$. Since $\varphi^*\mathcal{O}_{\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^n}(d)$ for some $d \geqslant 1$ (assume this is not the case where the image is a point), then s_i are homogeneous polynomials of degree d, and thus $D_+(s_i) = \text{Spec } k[x_0, \cdots, x_n]_{(s_i)}$, which are affine as desired.

(b) In light of the previous question, we need only show that the d-uple embedding and $\varphi^*(x_i)$ generate the same subspace of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. The d-uple embedding generates all of the global sections. To . φ has finite fibres since it is a finite morphism.

II.7.4

(a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

(b) Let X be the affine line over field k with the origin doubled. Calculate $\text{Pic } X$, determine which invertible sheaves are generated by global sections, and then show directly that there is no ample invertible sheaf on X.

Proof:

(a) If X admits an ample invertible sheaf, then by 7.6, some power of that ample sheaf is very ample and thus X has an immersion into $\mathbb{P}^n_A$ and since closed immersions and open immersions are both separated and $\mathbb{P}^n_A$ is proper over A, then X is separated over A.

(b) X is an integral scheme, thus $\text{Pic } X \cong \text{CaCl } X$. Since $U = X - \{0_1, 0_2\}$ is an affine open subset, then the function field of X is the same as the function field of U. Since $U \cong \text{Spec } k[x]_x$, then $K(X) = K(U) = k(x)$. Any Cartier divisor is given by $f_1, f_2 \in k(x)$, i.e. the divisor $\{(\mathbb{A}^1_1, f_1), (\mathbb{A}^1_2, f_2)\}$. This is then well-defined up to multiplication by a scalar in each component since $\mathcal{O}_X(\mathbb{A}^1_i) = k[x]$. Quotienting the Cartier divisors by the principal divisors is then equivalent to making one of the $f_i = 1$, say $f_2 = 1$. Then since f_1 and f_2 must agree on U and $\mathcal{O}_X(U) = k[x]_x$, whose units are constant multiples of powers of x, then it follows that f_1 my differ from f_2 by any constant times a power of x. Furthermore, f_1 is only defined up to a unit in $k[x]$, so the Cartier divisor class group is determined solely by the power of x. Now, for any nonzero power of x, the corresponding sheaf is not ample since for no nonzero power of x is any invertible sheaf generated by global sections and thus $\mathscr{L} \otimes \mathscr{L}^n$ is not generated by global sections. If $\mathscr{L} = \mathcal{O}_X$, then simply take $\mathscr{F} = \mathscr{L}$ for some nonzero power of x, then $\mathscr{F} \otimes \mathcal{O}_X^n = \mathscr{F}$ is not generated by global sections. It follows tha there is no ample invertible sheaf on X.

II.7.5

Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.

(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is ample.

(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.

(c) If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.

(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.

(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geqslant n_0$.

Proof:

(a) Let $\mathscr{F}$ be a coherent sheaf on X, then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for all n greater than some n_0. If tensor products of sheaves generated by global sections are generated by global sections, then we are done since this implies that $\mathscr{M}^n$ is generated by global sections and thus $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n = (\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^n$ is generated by global sections. Suppose that $\mathscr{F}, \mathscr{G}$ are two coherent sheaves on X which are generated by global sections $\{s_i\}_{i \in I}$ and $\{t_j\}_{j \in J}$, then $(\mathscr{F} \otimes \mathscr{G})_P \cong \mathscr{F}_P \otimes \mathscr{G}_P$ and thus the global sections $\{s_i \otimes t_j\}_{i \in I, j \in J}$ generate the sheaf at each stalk since the tensor product of two modules is generated by the tensor products of the generators.

(b) Since $\mathscr{M}$ is invertible, and hence coherent, then $\mathscr{M} \otimes \mathscr{L}^n$ is generated by global sections for sufficiently large n. It follows that $\mathscr{L}^{n+1} \otimes \mathscr{M}$ is ample by (a).

(c) Since $\mathscr{F}$ is ample, then $\mathscr{L}^n \otimes \mathscr{M}$ is ample for sufficiently large n. Similarly, $\mathscr{L} \otimes \mathscr{M}^m$ is ample for sufficiently large m since $\mathscr{M}$ is ample. Choose $N \geqslant n, m$, then $\mathscr{L}^N \otimes \mathscr{M}$ and $\mathscr{L} \otimes \mathscr{M}^N$ are both ample. It follows that there is some k such that $\mathscr{L}^{Nk} \otimes \mathscr{M}^k$ is generated by global sections. Furthermore, $\mathscr{L}^k \otimes \mathscr{M}^{Nk}$ is ample, so by (a), $(\mathscr{L}^{Nk} \otimes \mathscr{M}^k) \otimes (\mathscr{L}^k \otimes \mathscr{M}^{Nk})$ is ample, but this is just $\mathscr{L}^{(N+1)k} \otimes \mathscr{M}^{(N+1)k} = (\mathscr{L} \otimes \mathscr{M})^{(N+1)k}$, thus $\mathscr{L} \otimes \mathscr{M}$ is ample.

(d) Let $\mathscr{M}$ be generated by global sections $\{s_i\}_{i=1}^n$, note that we may take this collection to be finite since X is noetherian, thus X_{s_i} is an open cover of X and hence we may take a finite subcover. Since $\mathscr{L}$ is very ample, then

(e) Since $\mathscr{L}$ is ample, then $\mathscr{L}^n$ is generated by global sections for all $n \geqslant n_0$, furthermore, by 7.6, $\mathscr{L}^m$ is very ample for some m and hence by (d), $\mathscr{L}^k$ is very ample for all $k \geqslant m + n_0$.

II.7.6

The Riemann-Roch Problem. Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behaviour for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathscr{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \hookrightarrow \mathbb{P}^n_k$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of X. Thus in this case $\dim |nD|$ is a polynomial function of n, for n large.

(b) If D corresponds to a torsion element of $\operatorname{Pic} X$, of order r, then $\dim |nD| = 0$ if $r | n$, and -1 otherwise. In this case the function is periodic of period r.

Proof:

(a) Let $\mathscr{L} \cong \mathscr{L}(D)$. Since $\mathscr{L}$ is very ample, then we have an immersion $i : X \to \mathbb{P}^n_k$ such that $\mathscr{L} \cong i^*\mathcal{O}(1)$, hence $\mathscr{L}^n \cong i^*\mathcal{O}(1)^n = i^*\mathcal{O}(n)$. Since X is a projective variety, then its image is closed and thus i is a closed immersion. If follows that we obtain a homogeneous coordinate ring $S(X)$ under this closed immersion. We need only show that $\dim \Gamma(X, \mathscr{L}^n) = P_X(n)$ for sufficiently large n. Let $M = S(X)$ be the homogeneous coordinate ring of X, then we have that for sufficiently large

n, $P_X(n) = \dim_k S(X)_n$. Furthermore, by 5.12(c), $i^*(\mathcal{O}_{\mathbb{P}^n_k}(d)) = \mathcal{O}_X(d)$ whose global sections are $S(X)_d$. Furthermore, $\mathscr{L}^n \cong i^*\mathcal{O}(d) \cong \mathcal{O}_X(d)$ and thus $\dim_k \Gamma(X, \mathscr{L}^n) = \dim_k S(X)_n$ as desired,

(b) If $r|n$, then $r = nk$ and thus for $\mathscr{L} \cong \mathscr{L}(D)$, we have that $\mathscr{L}^r \cong \mathcal{O}_X$ and hence $\mathscr{L}^{rk} \cong \mathcal{O}_X^k \cong \mathcal{O}_X$. Since X is a projective variety, then the global sections of $\mathcal{O}_X$ is just k, and thus $\dim |nD| = 1 - 1 = 0$. If r does not divide n, then suppose we have an effective divisor $E \sim nD$, then $rE \sim n(rD) \sim 0$. It follows that rE is principal and thus has degree 0, but it is also effective, so $rE = 0$ and hence $E = 0$. It follows that $E \sim nD \sim 0$, but n is not divisible by the order of D, thus nD is not principal which is a contradiction. It follows that there are no effective divisors equivalent to nD, so $\dim |nD| = -1$.

II.7.7

Some Rational Surfaces. Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface.

(b) Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbb{P}^4$. The image is called the *Veronese surface* in $\mathbb{P}^4$.

(c) Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\widetilde{X}$, then this map extends to give a closed immersion of $\widetilde{X}$ in $\mathbb{P}^4$. Show that $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\widetilde{X}$ which do not meet. $\widetilde{X}$ is the union of all these lines, so we say $\widetilde{X}$ is a *ruled surface*.

Proof:

II.7.8

Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between *sections* of π (i.e. morphisms $\sigma : X \to \mathbb{P}(\mathscr{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

Proof:

II.7.9

Let X be a regular noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf of rank $\geqslant 2$ on X.

(a) Show that $\operatorname{Pic} \mathbb{P}(\mathscr{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

(b) If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$ as X-schemes iff there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.

Proof:

II.8.1

Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the lest also.

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring iff $\Omega_{B/k}$ is free of rank $= \dim B + \text{tr.d.}\, k(B)/k$.

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring iff the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X \big| \mathcal{O}_x \text{ is a regular local ring}\}$ is an open dense subset of X.

Proof:

(a) We wish to use 8.4A to obtain the right exact sequence. In that case, we have $C = k(B) = B/\mathfrak{m}$, thus $I = \mathfrak{m}$ and thus we obtain,

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B k(B) \to \Omega_{k(B)/k} \to 0$$

We wish to show that δ is injective. If we replace B with $B/\mathfrak{m}^2$, then since $d(\mathfrak{m}^2) = 0$ for any derivation d, and $k(B/\mathfrak{m}^2) = k(B)$, then we may assume that $B = B/\mathfrak{m}^2$ which is a complete local ring. As in 8.7, we consider the dual map of $\delta, \delta' : \text{Hom}_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \to \text{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$. Note that δ is a $k(B)$-linear transformation and that $\mathfrak{m}/\mathfrak{m}^2$ is a $B/\mathfrak{m} = k(B)$-vector space.

Given any K-linear transformation $T : V \to W$ if $T' : W^* \to V^*$ is surjective, then for any $x \in V$, we may extend $\{x\}$ to a basis of V and thereby obtain a morphism $f : V \to K$ sending x to $1 \in K$. Since T' is surjective, there is some linear map $g : W \to K$ such that $T'(g) = g \circ T = f$. Then, $g(T(x)) = f(x) \neq 0$, thus $T(x) \neq 0$ and hence T is injective.

It follows that we need only show that δ' is surjective. By 8.2A, letting $B = B$, $A = k$, $A' = k(B)$, $B' = B \otimes_k k(B)$, then we have that $\Omega_{B'/k(B)} \cong \Omega_{B/k} \otimes_B k(B)$. It follows that, $\text{Hom}_{k(B)}(\Omega_{B/k} \otimes_B k(B), k(B)) = \text{Hom}_{k(B)}(\Omega_{B'/k(B)}, k(B))$. Furthermore, for any $k(B)$-derivation $d' : B' \to k(B)$, there is some $f \in \text{Hom}_{k(B)}(\Omega_{B'/k(B)}, k(B))$ such that $d' = f \circ d$ where d is the map from $B' \to \Omega_{B'/k(B)}$. Conversely, any map $f \in \text{Hom}_{k(B)}(\Omega_{B'/k(B)}, k(B))$ yields a $k(B)$-derivation of B' by taking $f \circ d$. It follows that $\text{Hom}_{k(B)}(\Omega_{B'/k(B)}, k(B))$ is exactly $\text{Der}_{k(B)}(B', k(B))$.

Let $d : B' \to k(B)$ be any $k(B)$-derivation. $(\delta' d)(x) = d(\delta(x))$ where $\delta(x) = x \otimes 1 \in B \otimes_B k(B)$. Note that we must have that $d(\mathfrak{m}^2) = 0$, thus We now wish to show that δ' is surjective. Let $h \in \text{Hom}_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$. Since B is a complete local ring containing k and $k(B)$ is a separably generated extension of k, then we may choose a field of representatives $K \subseteq B$ such that $K \cong k(B)$. Let $b \in B$, then since $K \cong k(B)$ and $\bar{b} \in k(B)$, then we get a unique element $\lambda \in K$ such that $b = \lambda + (b - \lambda)$, and since $\overline{b - \lambda} = 0$, then $b - \lambda \in \mathfrak{m}$, thus elements of b may be uniquely written as $b = \lambda + x, \lambda \in K, x \in \mathfrak{m}$. Now define a $k(B)$-derivation on B by $d(b) = h(\overline{x})$ where $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$. Now define a k-bilinear map $f : B \times k(B) \to k(B)$ by $(b, c) \mapsto cd(b)$. This then yields a morphism $\widetilde{d} : B' \to k(B)$ and clearly for any $x \in \mathfrak{m}$ and $b = x + \lambda$ for any $\lambda \in K$, $(\delta'\widetilde{d})(b) = \widetilde{d}(b \otimes 1) = 1 \cdot d(x) = d(b) = h(\overline{x})$ as desired.

(b) Let $R = k[x_1, \cdots, x_n]/I$ be an algebra of finite type over k and $B = S^{-1}R$ for some multiplicatively closed $S \subseteq R$ such that B is a local ring. Furthermore, we may suppose that S contains all units of R since that does not change B. Let $\mathfrak{m} \subseteq B$ be its maximal ideal, then $\mathfrak{m} \cap S = \varnothing$ and thus $\mathfrak{m}$ corresponds to an ideal $\mathfrak{p} \in \operatorname{Spec} R$ (via $S^{-1}\mathfrak{p} = \mathfrak{m}$). We now show that $S = R - \mathfrak{p}$. Let $x \in S$, then $x/1$ is invertible with inverse $1/x \in B$. If $x \in \mathfrak{p}$, then since $S^{-1}\mathfrak{p} = \mathfrak{m}$, then we would have that $x/1 \in \mathfrak{m}$, and would therefore be non-invertible which is a contradiction. Similarly, suppose that $x \notin S$, then $x/1$ is invertible iff x is a unit in R, but S was assumed to contain all units, and thus $x/1$ is not invertible and hence $x/1 \in \mathfrak{m}$. It follows that $x/1 \in S^{-1}\mathfrak{p}$, i.e. $x/1 = a/s$ for some $a \in \mathfrak{p}$ and $s \in S$. It follows that $w(sx - a) = 0$ for some $w \in S$ and thus $(sw)x = aw \in \mathfrak{p}$. Since $sw \in S$ and hence $sw \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, then $x \in \mathfrak{p}$. It follows that $S = R - \mathfrak{p}$.

It follows from the above that we may take $B = R_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ in R. We now wish to show that $k(B)$ is a finitely generated extension of k. $k(B) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = \operatorname{Frac}(R/\mathfrak{p})$. Since $R/\mathfrak{p}$ is generated by the images of the x_i, then $\operatorname{Frac}(R/\mathfrak{p})$ is generated by them as a field and thus $\operatorname{Frac}(R/\mathfrak{p})$ is a finitely generated field extension of k. Since it is also separably generated over k, then by 8.6A, we have that $\dim_{k(B)} \Omega_{k(B)/k} = \text{tr.d.}\, k(B)/k = \text{tr.d.}\operatorname{Frac}(R/\mathfrak{p}) = \dim R/\mathfrak{p}$.

Suppose that $\Omega_{k(B)/k}$ is a free B-module of rank $\dim B$, then $\Omega_{B/k} \otimes_B k(B)$ is a free $k(B)$-vector space of dimension $\dim B + \text{tr.d.}\, k(B)$. Then using the exact sequence from part a, we get that,

$$\dim B + \text{tr.d.}\, k(B) = \dim \Omega_{B/k} \otimes_B k(B) = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \Omega_{k(B)/k} = \dim \mathfrak{m}/\mathfrak{m}^2 + \text{tr.d.}\, k(B)$$

Note that all dimensions are over $k(B)$. It follows that $\dim B = \dim \mathfrak{m}/\mathfrak{m}^2$ and thus B is a regular local ring.

Conversely, suppose that B is a regular local ring of dimension r. We have that $B = R_\mathfrak{p}$ and since B is regular local ring, then in particular it is integral. We wish to show that we may take R to be a integral finitely generated k-algebra. B is the local ring at a point in $\operatorname{Spec} R$ and thus we may take an irreducible component $R' \subseteq R$ such that $\mathfrak{p} \in \operatorname{Spec} R'$. It follows that we may suppose $\operatorname{Spec} R$ is irreducible.

Since the irreducible components of $\operatorname{Spec} R$ are in bijection which minimal primes of R, then R has only one minimal prime and since $\mathfrak{N}(R)$ is the intersection of all primes in R, then $\mathfrak{N}(R)$ is the minimal prime of R. It follows that $\mathfrak{N}(R)$ is prime and thus $R/\mathfrak{N}(R)$ is reduced and hence integral. Now let $\mathfrak{p}' = \mathfrak{p}/\mathfrak{N}(R)$, then $(R/\mathfrak{N}(R))_{\mathfrak{p}'} = R_\mathfrak{p}/(\mathfrak{N}(R)R_\mathfrak{p}) = B/(\mathfrak{N}(R)B)$. Since B is integral, then it has no nonzero nilpotent elements and since every element of $\mathfrak{N}(R)B$ is nilpotent, then $\mathfrak{N}(R)B = 0$, thus $(R/\mathfrak{N}(R))_{\mathfrak{p}'} \cong B$ and $R/\mathfrak{N}(R)$ is an integral f.g. k-algebra.

Since B is a regular local ring, then $\dim \mathfrak{m}/\mathfrak{m}^2 = r = \dim B$. Then using the exact sequence from part a, we get that $\dim \Omega_{B/k} \otimes_B k(B) = r + \dim \Omega_{k(B)/k} = r + \text{tr.d.}_k\, k(B)$ since $k(B)$ is finitely generated as proven above. Furthermore, $k(B) = \operatorname{Frac}(R/\mathfrak{p})$. Now, since R is an integral k-algebra, and $B = R_\mathfrak{p}$, then $\operatorname{ht} \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$, but $\operatorname{ht} \mathfrak{p} = \dim R_\mathfrak{p} = \dim B$. We have that $\text{tr.d.}\operatorname{Frac}(R/\mathfrak{p}) = \dim R/\mathfrak{p} = \dim R - \dim B = \text{tr.d.}\operatorname{Frac}(R) - \dim B = \text{tr.d.}\operatorname{Frac}(B) - \dim B$. Let $K = \operatorname{Frac}(B)$, then it follows that $\text{tr.d.}\, K = \dim B + \text{tr.d.}\, k(B)$. By 8.2A with $S = B - \{0\}$, then we get that $\Omega_{K/k} \cong S^{-1}\Omega_{B/k} = \Omega_{B/k} \otimes_B K$. Furthermore, $\dim \Omega_{K/k} = \text{tr.d.}\, K = \dim B + \text{tr.d.}\, k(B)$.

Since $\dim \Omega_{B/k} \otimes_B k(B) = \dim B + \text{tr.d.}\, k(B)$ and $\dim \Omega_{B/k} \otimes_B K = \dim B + \text{tr.d.}\, k(B)$ and $\Omega_{B/k}$ is finitely generated by 8.5, then by 8.9, $\Omega_{B/k}$ is free of rank $\dim B + \text{tr.d.}\, k(B)$.

(c) Let $U = \operatorname{Spec} R$ be an open affine neighborhood of x, then since X is of finite type over k, R is a finitely generated k-algebra. Furthermore, we have that $\Omega_{X/k}|_U = (\Omega_{R/k})\widetilde{\ }$. Let $x = \mathfrak{p} \in \operatorname{Spec} R$, then $(\Omega_{X/k})_x = (\Omega_{R/k})_\mathfrak{p} = \Omega_{R_\mathfrak{p}/k}$ by 8.2A. It follows that $(\Omega_{X/k})_x$ is exactly the module of relative differentials of $\mathcal{O}_{x,X} = R_\mathfrak{p}$ over k. Since k is perfect and $R_\mathfrak{p}$ is a localization of a R which is a finitely generated k-algebra, then $\mathcal{O}_{x,X}$ is a regular local ring iff $(\Omega_{X/k})_x$ is free of rank $\dim R_\mathfrak{p} + \text{tr.d.}\, k(R_\mathfrak{p})$.

If $\Omega_{x,X}$ is a regular local ring, then it is integral and thus $\dim R_\mathfrak{p} + \text{tr.d.}\, k(R_\mathfrak{p}) = \dim R = \dim X = n$ as in part b and thus $(\Omega_{X/k})_x$ is free of rank n. Conversely, if $(\Omega_{X/k})_x$ is free of rank n, then we wish to show that $\dim R_\mathfrak{p} + \text{tr.d.}\, k(R_\mathfrak{p}) = \dim R = n$. This is true by noting the fact that we may take R such that Spec R is irreducible, i.e. $\mathfrak{N}(R)$ is prime and then we may apply 1.8A by modding out by the nilradical.

(d) For any point $x \in U$, $(\Omega_{X/k})_x$ is free of rank n and thus $\Omega_{X/k}$ is free of rank n on a neighborhood of x by Ex 5.7, thus U is open. It is nonempty for the same reason as in 8.16, that is, the condition holds at the generic point of X. Since X is irreducible and U is a nonempty open set, then U is dense.

II.8.2

Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence,

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is also locally free.

Proof:

Consider $B \subseteq X \times V$ given by $B = \{(x, s) \in X \times V \big| s_x \in \mathfrak{m}_x \mathscr{E}_x\}$. These are the bad points and sections. We want to show that B is closed and has sufficiently small dimension such that upon projecting it onto V, it does not fill up the whole space and thus we may take a section $s \in V$ which does not vanish anywhere.

We can assume that V is finite dimensional. By induction on the dimension of X, suppose that if $\dim X = n-1$ and V is a vector space of global sections generating a locally free sheaf $\mathscr{E}$ everywhere, then V has a finite dimensional subspace which also generates $\mathscr{E}$ everywhere. Now for $\dim X = n$, at the generic point, $\mathscr{E}$ is generated by the rank of $\mathscr{E}$ elements which is finite and thus $\mathscr{E}$ is generated by finitely many elements of V on an open subset of X. The complement of this open subset is then a union of finitely many closed irreducible subsets of X (since X is noetherian). It follows that taking the pullback of $\mathscr{E}$ under the inclusions of those irreducible subsets, $i^*\mathscr{E}$ is generated by a finite subspace of i^*V and thus taking all of the generators, we get that $\mathscr{E}$ is generated by a finite dimensional subspace of V on X.

To show that B is closed, we want to show that it is closed in every open affine of $X \times_k V$. In fact, we may take open affines on which $\mathscr{E}$ is free. Let U be an open affine in X such that $\mathscr{E}|_U$ is free, then $U \times_k V$ is an open affine in $X \times_k V$. Since $\mathscr{E}|_U$ is free, then if we restrict all sections in V to U, denoted by $V|_U$, it still generates $\mathscr{E}|_U$ and through the isomorphism expressing the freeness of $\mathscr{E}|_U$, we get $W \subseteq \mathscr{E}(U)$ which generate $(\mathcal{O}_X|_U)^r$ at every point where r is the rank of $\mathscr{E}$. It follows that we have reduced the problem to showing that if X is affine and V is a subspace of $\mathcal{O}_X(X)^n$ which generates $\mathcal{O}_X^n$ everywhere, then $B = \{(x, s) \big| s_x \in \mathfrak{m}_x \mathcal{O}_{x,X}^n\}$ is closed.

Let $X = \text{Spec } A$, then since X is a variety, we need only consider closed points of X, i.e. maximal ideal of A. We may give V the structure of an $\mathbb{A}_k^l$ for some l. Then we need only find some ideal of Spec $A \otimes_k k[x_1, \cdots, x_l] = \text{Spec } A[x_1, \cdots, x_l] = \mathbb{A}_A^l$ for which B is the corresponding vanishing of. We need to show that the functions which vanish on B form an ideal, this will be the largest ideal vanishing on B. The maximal ideals of $A[x_1, \cdots, x_l]$ are given by pairs of maximal ideals of A

and $k[x_1, \cdots, x_l]$, i.e. a maximal ideal $\mathfrak{m} \in \text{Spec } A$ and any $s \in \mathbb{A}^l_k$ such that at the corresponding maximal ideal in $A[x_1, \cdots, x_l]$ is $\mathfrak{m} + (x_1 - s_1, \cdots, x_l - s_l)$. Suppose that for

Now give B the reduced induced structure. We wish to show that each irreducible component of B has dimension less than $\dim V$. To prove this, note that B maps dominantly onto $\pi_1(B)$ where $\pi_1 : X \times_k V \to X$. We now want to compute the dimension of the fibres. Let $x \in \pi_1(B)$, then the fibre over x is a closed subset of V. In particular, it is the kernel of the map $\varphi_x : V \to \mathscr{E}_x/(\mathfrak{m}_x\mathscr{E}_x)$. This is a surjective morphism of vector spaces since V generates $\mathscr{E}$ everywhere. The dimension of $\mathscr{E}_x/(\mathfrak{m}_x\mathscr{E}_x) = \mathcal{O}^r_{x,X}/(\mathfrak{m}_x\mathcal{O}^r_{x,X}) = k(x)^r$ is r where r is the rank of $\mathscr{E}$. Thus the morphism has fibers of dimension $\dim V - \text{rank } \mathscr{E}$ by rank-nullity. Since $\dim \pi_1(B) \leqslant \dim X = n$, then by Ex. 3.22, we have that $\dim B - \dim X \leqslant \dim V - \text{rank } \mathscr{E}$, so $\dim B \leqslant \dim V + (\dim X - \text{rank } \mathscr{E})$. Since $\dim X$ is strictly less than the rank of $\mathscr{E}$, then $\dim B < \dim V$ and thus $\pi_2(B) \neq V$, so there is some $s \in V$ such that $\pi_2^{-1}(s)$ is empty, i.e. $s_x \notin \mathfrak{m}_x\mathscr{E}_x$ for all x.

Now consider the morphism $\varphi : \mathcal{O}_X \to \mathscr{E}$ given by $\varphi(U) : \mathcal{O}_X(U) \to \mathscr{E}(U)$ mapping $x \mapsto x \cdot s|_U$. The morphism on the level of stalks is given by multiplication by s_x which is nonzero since $s_x \notin \mathfrak{m}_x\mathscr{E}_x$ and thus the multiplication is injective. It follows that we obtain an exact sequence,

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}' = \mathscr{E}/\mathcal{O}_X$. The stalk of $\mathscr{E}'$ at x is given by $\mathscr{E}_x/(s_x\mathcal{O}_{x,X})$. This is isomorphic to $\mathcal{O}^n_{x,X}/(s_x\mathcal{O}_{x,X})$ where s_x is some unit not contained in $\mathfrak{m}_x\mathcal{O}^n_{x,X}$. It follows that we need only show the following algebraic statement: Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $s \in R^n$ not in $\mathfrak{m} \cdot R^n$, then $R^n/(sR) \cong R^{n-1}$. Since $s \notin \mathfrak{m} \cdot R^n$, then some $s_i \notin \mathfrak{m}$ and is thus invertible. It follows that for any element $(x_1, \cdots, x_n) \in R^n$, there is a unique $a \in R$ such that $(as)_1 = x_1$, then we map $(x_1, \cdots, x_n)$ to $(x_1, \cdots, x_n) - (as)_1$ and remove the first element since it is 0. This is clearly a surjective map onto R^{n-1} and its kernel is exactly the multiples of s, so $R^n/(sR) \cong R^{n-1}$ is free.

II.8.3

Product Schemes

(a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X \times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$

(b) If X and Y are nonsingular varieties over a field k, show that $\omega_{X \times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$. This shows that arithmetic genus and geometric genus of a nonsingular projective variety may be different.

Proof:

(a) Let $Z = S$, $Y = X$, $X = X \times_S Y$, $f : X \times_S Y \to X$ be projection, and $g : X \to S$ be the structure map in 8.11, then we obtain an exact sequence,

$$p_1^*\Omega_{X/S} \to \Omega_{X \times_S Y/S} \to \Omega_{X \times_S Y/X} \to 0$$

Since the morphism $X \times_S Y \to X$ is obtained by base extension, of $Y \to S$, then we have by 8.10 that $\Omega_{X \times_S Y/Y} \cong p_2^*\Omega_{Y/S}$. Thus our exact sequence is the following:

$$p_1^*\Omega_{X/S} \to \Omega_{X \times_S Y/S} \to p_2^*\Omega_{Y/S} \to 0$$

We now need to show that the left morphism is injective and the sequence splits. We may do both of these at the same time by showing the sequence splits on the left. We wish to obtain a morphism $\Omega_{X\times_S Y/S} \to p_1^*\Omega_{X/S}$ and then show locally that it is is the inverse of the morphism given by 8.11. To obtain this morphism, we may again use 8.11 identically to before, except we let $Y = Y$ instead of X this time to get,

$$p_2^*\Omega_{Y/S} \to \Omega_{X\times_S Y/S} \to p_1^*\Omega_{X/S} \to 0$$

Therefore we have a morphism $\Omega_{X\times_S Y/S} \to p_1^*\Omega_{X/S}$ which we need to show is the left inverse of the morphism in our first exact sequence.

Since we have our morphisms, showing that we have a left inverse may be done locally. Thus we may restrict to the case of modules. We have the following:

$$\begin{tikzcd}
\Omega_{A/R} \otimes_A (A \otimes_R B) \arrow[r, "a"] \arrow[d, "\sim"'] & \Omega_{A\otimes_R B/R} \arrow[r] \arrow[dl, "b"] & \Omega_{A\otimes_R B/A} \arrow[r] & 0 \\
\Omega_{A\otimes_R B/B} & & &
\end{tikzcd}$$

We obtain the morphisms a and b from 8.3A and the isomorphism from 8.2A. We need only show that the isomorphism is the same as the composition $b\circ a$. The map a is defined by $a(d(x)\otimes(y\otimes z)) = (y \otimes z)d(x \otimes 1)$ and the map b is given by $b(d(x \otimes y)) = d(x \otimes y)$ and thus,

$$b(a(d(x) \otimes (y \otimes z))) = (y \otimes z)d(x \otimes 1)$$

and this is exactly the isomorphism from 8.2A. It follows that the exact sequence splits and hence $\Omega_{X\times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b) Since X and Y are nonsingular, then $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free sheaves. By part (a), we have an exact sequence,

$$0 \to p_1^*\Omega_{X/k} \to \Omega_{X\times_k Y/k} \to p_2^*\Omega_{Y/k} \to 0$$

and thus by 5.16d, we have that the highest exterior power of $\Omega_{X\times_k Y/k}$ is just the tensor products of the other two, i.e. $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$. Note that p_1^* and p_2^* commute with the exterior powers since they are locally tensor products.

(c) We have that $\omega_X \cong p_1^*\omega_Y \otimes p_2^*\omega_Y$. We know that $\omega_Y \cong \mathcal{O}_Y$. Therefore, $\omega_X \cong p_1^*\mathcal{O}_Y \otimes p_2^*\mathcal{O}_Y$. We then have that $p_1^*\mathcal{O}_Y \cong p_1^{-1}\mathcal{O}_Y \otimes_{p_1^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$. Thus $\omega_X \cong \mathcal{O}_X$. Since X is a projective variety, then $\mathcal{O}_X(X) = k$ is 1 dimensional, thus $p_g(X) = 1$. To compute the arithmetic genus of X, we note that $p_a(Y) = \frac{1}{2}(3-1)(3-2) = 1$ by I.7.2b, then using part e, we get that $p_a(Y \times Y) = 1 \cdot 1 - 1 - 1 = -1$. Note that the embedding into $\mathbb{P}^8$ vs $Y \times Y$ which lives in $\mathbb{P}^4$ does not matter by III.5.3 (which I will hopefully prove in a few weeks). pf: trust me bro.

II.8.6

Infinitesimal Lifting Property. The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that Spec A is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence where B' is a k-algebra, I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism

$f : A \to B$, then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram,

$$\begin{tikzcd}
& 0 \arrow[d] \\
& I \arrow[d] \\
& B' \arrow[d] \\
A \arrow[r, "f"] \arrow[ur, "g"] & B \arrow[d] \\
& 0
\end{tikzcd}$$

We call this the *infinitesimal lifting property* for A. We prove this result in several steps.

(a) First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\mathrm{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of a B-module and hence also of an A-module. Conversely, for any $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about Spec A being nonsingular).

(b) Now let $P = k[x_1, \cdots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exists a homomorphism $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h induces an A-linear map $\overline{h} : J/J^2 \to I$.

(c) Now use the hypothesis Spec A is nonsingular and (8.17) to obtain an exact sequence,

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

Show furthermore that applying the functor $\mathrm{Hom}_A(\cdot, I)$ gives an exact sequence,

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0$$

Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\overline{h} \in \mathrm{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B' such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g : A \to B'$.

Proof:

(a) Clearly θ is A-linear, thus we need only show that it is a k-derivation. Let $c \in k$, then since g, g' are $k-$algebra homomorphisms, then we get that $g(c) = g'(c) = c$ and thus $\theta(c) = 0$. If $a, b \in A$, then,

$$\theta(ab) = g(ab) - g'(ab) = g(a)g(b) - g'(a)g'(b)$$

We may rewrite this as the following,

$$\begin{aligned}
\theta(ab) &= g(a)g(b) - g(a)g'(b) + g(a)g'(b) - g'(a)g'(b) \\
&= g(a)(g(b) - g'(b)) + g'(b)(g(a) - g'(a)) \\
&= g(a)\theta(b) + g'(b)\theta(a)
\end{aligned}$$

Since $I^2 = 0$, then $xy = 0$. Note that A acts on I by $ax = f(a)x$ and B acts on I by $\overline{b}x = bx$ where $b \in B'$ such that $\overline{b} = B$ and since $I^2 = 0$, then this is independent of the choice of representative of $\overline{b}$. Since $g(a)$ is a representative of $f(a)$ in B' and similarly for $g'(b)$ and $f(b)$, then,

$$\theta(ab) = a\theta(b) + b\theta(a)$$

Thus θ is a k-derivation of A into I.

Conversely, if θ is a k-derivation of A into I, then clearly $g' = g + \theta$ is k-linear and since θ maps into I, then it lifts f, thus we need only show that it is a ring homomorphism. Let $a, b \in A$, then,

$$\begin{aligned}
g'(ab) &= g(ab) + \theta(ab) \\
&= g(a)g(b) + a\theta(b) + b\theta(a) \\
&= g(a)g(b) + g(a)\theta(b) + g(b)\theta(a) \text{ by def. of the action of a on I} \\
&= g(a)g(b) + g(a)\theta(b) + g(b)\theta(a) + \theta(b)\theta(a) \text{ since } I^2 = 0 \\
&= (g(a) + \theta(a))(g(b) + \theta(b)) \\
&= g'(a)g'(b)
\end{aligned}$$

(b) To give a morphism from $P \to B'$ we need only say where the x_i go, so let the x_i map to arbitrary preimages of $f(\overline{x}_i)$. Now we have that $\overline{h(x_i)} = f(\overline{x}_i)$. It follows that $\overline{h(x)} = f(\overline{x})$ for all $x \in P$. Since for any $x \in J$, $\overline{h(x)} = f(\overline{x}) = f(0) = 0$, then $h(J) \subseteq I$, thus h induces a morphism from J to I and for any $x, y \in J$, $h(xy) = h(x)h(y) \in I^2$, so $h(J^2) = 0$, thus h induces a morphism from $J/J^2 \to I$ as desired.

(c) Since Spec A is affine, then 8.17 may be restated purely in terms of modules, namely that,

$$0 \to J/J^2 \to \Omega_{P/k} \otimes_k A \to \Omega_{A/k} \to 0$$

is exact. Now, $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, I)$, by the tensor-hom adjunction is naturally isomorphic to $\mathrm{Hom}(\Omega_{P/k}, \mathrm{Hom}_A(A, I)) = \mathrm{Hom}_P(\Omega_{P/k}, I)$. $\mathrm{Hom}(\cdot, I)$ is a contravariant left exact functor, thus we need only show that $\mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I)$ is surjective. The map $J/J^2 \to P$ is given by $\delta\overline{x} = dx \otimes 1$, and $\mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I)$ is given by taking $d : P \to I$ and Let $h \in \mathrm{Hom}_A(J/J^2, I)$, then **FINISH THIS**.

III.2.1

(a) Let $X = \mathbb{A}^1_k$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.

(b) More generally, let $Y \subseteq X = \mathbb{A}^n_k$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible result.

Proof:

(a) Let $Y = \{P, Q\}$, then we have a short exact sequence,

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$$

This yields a long exact sequence, the part which we care about being, $H^0(X, \mathbb{Z}) \to H^0(X, \mathbb{Z}_Y)\, to H^1(X, \mathbb{Z}_U) \to H^1(X, \mathbb{Z})$. Since $\mathbb{Z}$ is a constant sheaf on an irreducible space, it is flasque and thus $H^1(X, \mathbb{Z}) = 0$. Furthermore, $\mathbb{Z}_Y$ is the direct sum of two skyscraper sheaves for $\mathbb{Z}$ at P and Q, thus $H^0(X, \mathbb{Z}_Y) = \mathbb{Z} \oplus \mathbb{Z}$ and the map from $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is $x \mapsto (x, x)$. In general, for a closed subset $Y \subseteq X$ included by $i : Y \to X$, we have that $\mathscr{F} \to i_*(\mathscr{F}|_Y) = i_*(i^{-1}\mathscr{F})$ is given by sending $a \in \mathscr{F}(U)$ to $[a] \in i_* i^{-1}\mathscr{F}(U)$. In this case, $i^{-1}\mathscr{F}$ is the sheafification of $U \mapsto \varinjlim_{V \supseteq i(U)} \mathscr{F}(V)$. Now any global section of this sheaf is thus determined by its restrictions to P and Q which are arbitrary since $X - P$ and $X - Q$ are open sets in X around Q and P respectively. It follows that $i^{-1}\mathscr{F}$ is the constant sheaf $\mathbb{Z}$.

We now have that $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is injective, and thus we see that $H^1(X, \mathbb{Z}_U) = (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} = \mathbb{Z} \neq 0$.

(b) This is hard.

III.2.2

Let $X = \mathbb{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathscr{K} \to \mathscr{K}/\mathcal{O} \to 0$ is a flasque resolution of $\mathcal{O}$. Conclude that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Proof:

Since $\mathscr{K}/\mathcal{O}$ is a direct sum of skyscraper sheaves, then it is flasque and $\mathscr{K}$ is a constant sheaf, thus it is flasque. Since we have the above exact sequence (from II.1.21), we see that this is indeed a flasque resolution of $\mathcal{O}$, thus $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

III.2.3

Cohomology with Supports. Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y.

(a) Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H^i_Y(X, \cdot)$. They are the *cohomology groups* of X with supports in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that,

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'') \to 0$$

is exact.

(c) Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X, \mathscr{F}) = 0$ for all $i > 0$.

(d) If $\mathscr{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathscr{F}$, there is a long exact sequence of cohomology groups,

$$\begin{aligned} 0 \to H^0_Y(X, \mathscr{F}) &\to H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) \to \\ = H^1_Y(X, \mathscr{F}) &\to H^1(X, \mathscr{F}) \to H^1(U, \mathscr{F}|_U) \to \\ = H^2_Y(X, \mathscr{F}) &\to \cdots \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there is are natural functorial isomorphisms, for all i and $\mathscr{F}$,

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V)$$

Proof:

We first wish to show that $\Gamma_Y(X, \cdot)$ is a functor at all. The only non-obvious check is that given $\varphi : \mathscr{F} \to \mathscr{G}$ a morphism of sheaves on X, then does $\varphi(X)$ map sections with support in Y to sections with support in Y. Suppose that $s \in \mathscr{F}(X)$ is a global section and $\mathrm{Supp}(s) \subseteq Y$. φ induces morphisms on stalks. For any $x \in X$, we obtain a morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ given by $\varphi_x([U, s]) = [U, \varphi(U)(s)]$. It follows that $\varphi(s)_x = \varphi_x(s_x)$. If $s_x = 0$, then $\varphi_x(s_x) = \varphi_x(0) = 0$, thus $\mathrm{Supp}(\varphi(X)(s)) \subseteq \mathrm{Supp}(s) \subseteq Y$. It follows that $\Gamma_Y(X, \cdot)$ is indeed a functor.

Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of sheaves (of abelian groups) on X. We know that $\Gamma(X, \cdot)$ is left exact, thus we obtain an exact sequence,

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$$

Since $\Gamma_Y(X, \mathscr{F}') \subseteq \Gamma(X, \mathscr{F}')$, and $\Gamma(X, fs') \to \Gamma(X, \mathscr{F})$ is injective,then $\Gamma_Y(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ is injective and furthermore, by the above, it image is contained in $\Gamma_Y(X, \mathscr{F})$. Furthermore, for any $s \in \Gamma_Y(X, \mathscr{F})$ which is mapped to 0 in $\Gamma(X, \mathscr{F}'')$, we have some $t \in \Gamma(X, \mathscr{F}')$ mapping to s. The support of t must be exactly the support of s since the map $\mathscr{F}'_x \to \mathscr{F}_x$ is injective for all x. It follows that $t \in \Gamma_Y(X, \mathscr{F}')$, thus we obtain an exact sequence,

$$0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$$

(b) If $\mathscr{F}'$ is also flasque, then the sequence,

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

is exact by II.1.16b. It follows that for any $s \in \Gamma_Y(X, \mathscr{F}'')$, there is some $t \in \Gamma(X, \mathscr{F})$ mapping to s. It remains to be shown that t may be taken to have support in Y. Let $U = X - Y$. If we restrict t to U, then we have that the image of $t|_U$ has support in $U \cap Y = \varnothing$, thus the image of $t|_U$ is 0. Since the sequence,

$$0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$$

is exact, then there is some $g' \in \Gamma(U, \mathscr{F}')$ mapping to $t|_U$. Since $\mathscr{F}'$ is flasque, then there is some $g \in \Gamma(X, \mathscr{F}')$ mapping to g'. If we then consider $t - g \in \Gamma(X, \mathscr{F})$, its image in $\Gamma(X, \mathscr{F}'')$ is still s

since g maps to 0, however on $U = X - Y$, $(t-g)|_U = 0$, thus the support of $t-g$ is contained in Y. It follows that $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is surjective, thus we obtain the exact sequence,

$$0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'') \to 0$$

(c) Let $\mathscr{F}$ be flasque and embed $\mathscr{F}$ into an injective sheaf $\mathscr{I}$, then $\mathscr{I}$ is flasque by 2.4, and we have,

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$$

is exact where $\mathscr{G}$ is the quotient sheaf, $\mathscr{I}/\mathscr{F}$. Furthermore, $\mathscr{G}$ is flasque since it is a quotient of flasque sheaves. It follows form the previous part that we get an exact sequence,

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{I}) \to \Gamma_Y(X, \mathscr{G}) \to 0$$

Then from the long exact sequence and noting that $\mathscr{I}$ is injective and thus $H^i_Y(X, \mathscr{I}) = 0$ for all $i > 0$, we get,

$$\Gamma_Y(X, \mathscr{I}) \to \Gamma_Y(X, \mathscr{G}) \to H^1_Y(X, \mathscr{F}) \to 0 \to \cdots$$

Which says that $H^1_Y(X, \mathscr{F})$ is the quotient of $\Gamma_Y(X, \mathscr{G})$ by the image of $\Gamma_Y(X, \mathscr{F})$ which is just 0. Thus $H^1_Y(X, \mathscr{F}) = 0$. Then we again get $0 \to H^{i-1}_Y(X, \mathscr{G}) \to H^i_Y(X, \mathscr{F}) \to 0$ in the exact sequence for each $i \geqslant 2$. From this we see that $H^{i-1}_Y(X, \mathscr{G}) \cong H^i_Y(X, \mathscr{F})$ and since $\mathscr{G}$ is also flasque, then by induction $H^{i-1}_Y(X, \mathscr{G}) = 0$ and thus $H^i_Y(X, \mathscr{F}) = 0$. It follows that flasque sheaves are acyclic under Γ_Y.

(d) $\Gamma_Y(X, \mathscr{F})$ injects into $\Gamma(X, \mathscr{F})$ by definition and $\Gamma(X, \mathscr{F})$ surjects onto $\Gamma(X - Y, \mathscr{F})$ by definition of flasque. It follows that we need only show exactness in the middle. Let $s \in \Gamma(X, \mathscr{F})$ which maps to 0 in $\Gamma(X - Y, \mathscr{F})$. It follows that $s|_{X-Y} = 0$ and thus the support of s is contained in Y, so $s \in \Gamma_Y(X, \mathscr{F})$ as desired.

(e) Let $\mathscr{F}$ be a sheaf and pick an injective resolution $0 \to \mathscr{F} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \cdots$. Since injective sheaves are flasque, then by the previous problem we get a diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_Y(X, \mathscr{I}_0) \arrow[r] \arrow[d] & \Gamma_Y(X, \mathscr{I}_1) \arrow[r] \arrow[d] & \Gamma_Y(X, \mathscr{I}_2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma(X, \mathscr{I}_0) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{I}_1) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{I}_2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma(X - Y, \mathscr{I}_0) \arrow[r] \arrow[d] & \Gamma(X - Y, \mathscr{I}_1) \arrow[r] \arrow[d] & \Gamma(X - Y, \mathscr{I}_2) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We need to check that this diagram commutes. To do so, we can consider a simpler case. Let $\mathscr{F}, \mathscr{G}$ be flasque sheaves and $\varphi : \mathscr{F} \to \mathscr{G}$, then we obtain a diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_Y(X, \mathscr{F}) \arrow[r] \arrow[d, "\varphi(X)"'] & \Gamma(X, \mathscr{F}) \arrow[r] \arrow[d, "\varphi(X)"'] & \Gamma(X - Y, \mathscr{F}) \arrow[r] \arrow[d, "\varphi(X-Y)"'] & 0 \\
0 \arrow[r] & \Gamma_Y(X, \mathscr{G}) \arrow[r] & \Gamma(X, \mathscr{G}) \arrow[r] & \Gamma(X - Y, \mathscr{G}) \arrow[r] & 0
\end{tikzcd}$$

Since the maps from $\Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$ and $\Gamma_Y(X, \mathscr{G})$ to $\Gamma(X, \mathscr{G})$ are just inclusion, then the first square clearly commutes. The horizontal morphisms of the second square are just restrictions, so the commutativity of the second square is exactly the requirement for φ to be a morphism of sheaves. It follows that the diagram commutes and thus the previous diagram is a short exact sequence of cochain complexes. It follows that it induces the desired long exact sequence of cohomology.

(f) Let $\mathscr{F}$ be a sheaf on X. For any $s \in \Gamma_Y(V, \mathscr{F}|_V)$, we may extend it by 0 on $X - Y$ to obtain a global section, thus $\Gamma_Y(X, \mathscr{F}) = \Gamma_Y(V, \mathscr{F}|_V)$. It follows that the functors $\Gamma_Y(X, \cdot)$ and $\Gamma_Y(V, \cdot|_V)$ are the same. Now consider an injective resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{I}_0 \to \mathscr{I}_1 \to \cdots$, then $\mathscr{I}_i$ is injective and hence flasque, thus $\mathscr{I}_i|_V$ is also flasque. Since $H^i_Y(V, \mathscr{F}) = 0$ for $i > 0$ for any flasque sheaf, then we have that $H^i_Y(V, \cdot|_V) = R^i\Gamma_Y(V, \cdot|_V)$. It follows that $H^i_Y(V, \cdot|_V)$ is a universal δ-functor and since $H^i_Y(X, \mathscr{F})$ is as well and they are the same for $i = 0$, then for any i we have,

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V)$$

III.2.5

Let X be a Zariski space. Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \mathrm{cl}\{P\}$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j : X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all i, $\mathscr{F}$, we have,

$$H^i_P(X, \mathscr{F}) = H_P(X_P, \mathscr{F}_P)$$

Proof:

For $i = 0$, we have that $H^0_P(X_P, \mathscr{F}_P) = \Gamma_P(X_P, \mathscr{F}_P)$. Any global section may be given by elements $([s_i], U_i)$ such that U_i cover X_P. Since s has support in $\{P\}$, then for any U_i not containing P, we have that $[s_i] = 0$. Since X is noetherian, then X_P is also noetherian, thus we may assume that there are finitely many U_i. Then let U be the intersection of all U_i containing P and let $[\tilde{s}]$ be the restriction of the $[s_i]$ to U. It follows that the element $s \in \Gamma_P(X_P, \mathscr{F}_P)$ is given by 0 on $X_P - \{P\}$ and on U containing P it is given by $[\tilde{s}]$. Furthermore, since the support of s is $\{P\}$, then for any $x \neq P$, we have that $[\tilde{s}]_x = 0$. It follows that

III.3.1

Let X be a noetherian scheme. Show that X is affine iff X_{red} is affine.

Proof:

If X is affine for instance, $X = \text{Spec } R$, then $X_{\text{red}} = \text{Spec } R_{\text{red}}$ is affine. Conversely, let $\mathscr{F}$ be a coherent sheaf on X and let $\mathscr{N}$ be the sheaf of nilradicals on X. We obtain a filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots$. First we show that this filtration must end at some point by showing that the sheaf of nilradicals is itself nilpotent, i.e. $\mathscr{N}^n = 0$ for some n. To do so, let $\text{Spec } A \subseteq X$, then $\mathscr{N}(\text{Spec } A) = \mathfrak{N}(A)$ which is an ideal of A, a noetherian ring and thus is generated by finitely many elements $x_1, \cdots, x_k$, each of which are nilpotent. Let m be the maximum index of each of these elements, i.e. $x_i^m = 0$ for all i. It follows that $\mathfrak{N}(A)^m = 0$. Now since X may be covered by finitely many affine noetherian opens, then picking the maximum of all m's yields an $m > 0$ such that $\mathscr{N}^m = 0$ as a presheaf, and thus to obtain the sheaf $\mathscr{N}^m$ which is the sheafification of $U \mapsto \mathscr{N}(U)^m$ is also 0. Therefore the filtration terminates.

We now show that $\mathscr{G}/\mathscr{N}\cdot\mathscr{G}$ is acyclic for any coherent $\mathscr{G}$. We have a closed immersion $i : X_{\text{red}} \to X$. Furthermore, $\mathscr{G}/\mathscr{N}\cdot\mathscr{G}$, an $\mathcal{O}_X$-module, is also an $\mathcal{O}_{X_{\text{red}}}$-module, since there is a natural action of $\mathcal{O}_X/\mathscr{N} = \mathcal{O}_{X_{\text{red}}}$ on $\mathscr{G}/\mathscr{N}\cdot\mathscr{G}$ (it is locally the sheafification of $M/\mathfrak{N}(A)M = A_{\text{red}} \otimes_A M$). Furthermore, locally $i^*\mathscr{G} = A_{\text{red}} \otimes M = M/\mathfrak{N}(A)M = \mathscr{G}/\mathscr{N}\cdot\mathscr{G}$. It follows that $\mathscr{G}/\mathscr{N}\cdot\mathscr{G}$ is a coherent sheaf on X_{red} since X is noetherian. Since X_{red} is affine, then we have that $H^i(X_{\text{red}}, \mathscr{G}/\mathscr{N}\cdot\mathscr{G}) = 0$ for all $i > 0$. Since $i_*(\mathscr{G}/\mathscr{N}\cdot\mathscr{G}) = \mathscr{G}/\mathscr{N}\cdot\mathscr{G}$, then $H^i(X, \mathscr{G}/\mathscr{N}\cdot\mathscr{G}) = 0$ for all $i > 0$ and therefore is acyclic.

We now climb the filtration to show that $\mathscr{F}$ is acyclic. For each $j > 0$, we have an exact sequence, $0 \to \mathscr{N}^j \cdot \mathscr{F} \to \mathscr{N}^{j-1} \cdot \mathscr{F} \to \mathscr{N}^{j-1} \cdot \mathscr{F}/\mathscr{N}^j \cdot \mathscr{F} \to 0$. This yields a long exact sequence and since the last term is acyclic, we get that $H^i(X, \mathscr{N}^j \cdot \mathscr{F}) = H^i(X, \mathscr{N}^{j-1} \cdot \mathscr{F})$ for all $j > 0$ and thus $H^i(X, \mathscr{F}) = H^i(X, \mathscr{N}^m \cdot \mathscr{F})$ and since $\mathscr{N}^m = 0$, we have that $H^i(X, \mathscr{F}) = 0$ for all $i > 0$. Therefore X is affine.

III.3.2

Let X be a reduced noetherian scheme. Show that X is affine iff each irreducible component is affine.

Proof:

If X is affine, then the irreducible components are obtained by modding by the minimal primes and are therefore affine. Conversely, let $X = \bigcup_{i=1}^n X_i$ with each X_i affine. Note that there are only finitely many since X is noetherian. Let $\mathscr{F}$ be a coherent sheaf on X and we want to show that $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ using Mayer-Vietoris. If $\dim X = 0$, then X is just a finite disjoint union of points and thus if $X_i = \text{Spec } R_i$, then $X = \text{Spec } R_1 \times \cdots \times R_n$. Suppose by induction that if $\dim X < n$ the statement holds. Let $Y_1 = X_1$ and $Y_2 = \bigcup_{i=2}^n X_i$. Then by Mayer-Vietoris, we have a long exact sequence,

$$\cdots \to H^i_{Y_1 \cap Y_2}(X, \mathscr{F}) \to H^i_{Y_1}(X, \mathscr{F}) \oplus H^i_{Y_2}(X, \mathscr{F}) \to H^i_{Y_1 \cup Y_2}(X, \mathscr{F}) \to \cdots$$

Now for any noetherian ring R and nonzero ideal $I \subseteq R$, we have that $\dim R/I + \text{ht } I \leqslant \dim R$ and thus $\dim R/I \leqslant \dim R - \text{ht } I < \dim R$. It follows that the irreducible components of $Y_1 \cap Y_2$ are affine and $Y_1 \cap Y_2$ has dimension strictly less than X. We now want to use induction to show that they are 0 and thus we need to relate $H^i_Y(X, \mathscr{F})$ to $H^i(Y, \mathscr{F}|_Y)$. In particular, we will obtain an injection from the prior into the latter and hence it will be 0.

III.4.1

Let $f : X \to Y$ be an affine morphism of noetherian separated schemes. Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

Proof:

An affine morphism is locally a morphism of affine schemes and thus is locally separated and hence separated. Furthermore, since X and Y are quasi-compact, then f is quasi-compact. It follows that $f_*\mathscr{F}$ is quasi-coherent. It follows that we have isomorphisms between $H^i(Y, f_*\mathscr{F}) = \check{H}^i(\mathfrak{U}, f_*\mathscr{F})$

for some open affine cover $\mathfrak{U}$. Now $f^{-1}(\mathfrak{U}) = \{f^{-1}(U) | U \in \mathfrak{U}\}$ is an affine open cover of X since f is an affine morphism. It follows that we need only show an isomorphism of Čech cohomology.

It is now clear that the Čech cohomology groups are the same since we have that $C^i(f^{-1}(\mathfrak{U}), \mathscr{F}) = C^i(\mathfrak{U}, f_*\mathscr{F})$ and the maps are the same.

III.4.2

Prove Chevalley's theorem let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.

(c) Now prove Chevalley's theorem. First use Ex 3.1 and Ex 3.2 to reduce to the case X and Y are integral. Then use 3.7, Ex 4.1, consider $\ker \beta$ and cokerβ, and use noetherian induction on Y.

Proof:

(a) We first solve this in the case that X and Y are affine, so $f : \text{Spec } B \to \text{Spec } A$ is finite and surjective, therefore $f^\# : A \to B$ is finite and injective. We then have that $f^\#_{(0)} : \text{Frac}(A) \to B_{(0)}$ is finite and injective. It follows that we may pick generating elements $x_1, \cdots, x_r \in B_{(0)}$ and then consider the morphism $\alpha : A^r \to B$ given by $(a_1, \cdots a_r) \mapsto \sum_i a_i x_i$. This is then an isomorphism at the generic point because of the morphism $\text{Frac}(A) \to B_{(0)}$.

When X and Y are not affine, pick an affine open Spec $A \subseteq Y$ with preimage Spec $B \subseteq X$. Now as described above, we have a coherent sheaf $\mathscr{M}$ on Spec B which has the desired properties. Let $i : \text{Spec } B \to X$ be the inclusion, then we have the sheaf $i_*\mathscr{M}$ on $\mathcal{O}_X$. Since our spaces are noetherian, then $i_*\mathscr{M}$ is quasi-coherent on X sine Spec B is noetherian. We now have a morphism $\alpha : \mathcal{O}_Y^r \to f_*(i_*\mathscr{M})$ given on an open U by $\mathcal{O}_Y^r(U) \to \mathscr{M}(\text{Spec } B \cap f^{-1}(U))$ which first restricts $\mathcal{O}_Y^r(U) \to \mathcal{O}_Y^r(U \cap \text{Spec } A)$ and then uses the morphism we have as affine schemes. It follows that this morphism is the same on the generic point, since the generic point is contained in Spec A. However, $i_*\mathscr{M}$ needn't be coherent. To make $i_*\mathscr{M}$ coherent, we wish to take a subsheaf whose stalk at η_X is the same and such that we still have a morphism. We have that α is given by choosing r global sections of $f_*(i_*\mathscr{M})$ which is equivalent to choosing r global sections of $\mathscr{M}$. Now consider the subsheaf generated by these r sections over X. It will now be coherent and α at η_Y will still be an isomorphism as desired.

(b) $\mathscr{H}om(\cdot, \mathscr{F})$ is a contravariant functor, so applying it to α, we obtain $\beta : \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{H}om(\mathcal{O}_Y^r, \mathscr{F})$. Now $\text{Hom}_{\mathcal{O}_Y|_U}(\mathcal{O}_Y|_U^r, \mathscr{F}|_U) = \text{Hom}(\mathcal{O}_Y|_U, \mathscr{F}|_U)^r = \mathscr{F}(U)^r$. It follows that we have a morphism $\beta : \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{F}^r$. We need to show that β is an isomorphism at the generic point. At the generic point η, we have that $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})_\eta = \text{Hom}_{K(Y)}((f_*\mathscr{M})_\eta, \mathscr{F}_\eta)$. Now since β is obtained by α, then since $(f_*\mathscr{M})_\eta \cong \mathcal{O}_{Y,\eta}^r$ by α_η, then β induces an isomorphism at the generic point. Now $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) = f_*\mathscr{H}om(\mathscr{M}, f^*\mathscr{F}))$ and therefore we may take $\mathscr{G} = \mathscr{H}om(\mathscr{M}, f^*\mathscr{F})$. Now on affine opens U, we have that $\mathscr{H}om(\mathscr{M}, f^*\mathscr{F})|_U = \widetilde{\left(\text{Hom}_{\mathcal{O}_X|_U}(\mathscr{M}|_U, (f^*\mathscr{F})|_U)\right)}$ and is therefore coherent.

(c) By 3.1, 3.2, we have that Y is affine iff every irreducible component is affine, therefore we may assume that Y is integral and X becomes the preimage of any given irreducible component. Then by finiteness and surjectivity, there is a point in X mapping to the generic point of Y and this will be the generic point of some irreducible component of X, so we may assume that X is irreducible. Since $X_{\text{red}} \to X$ is finite and surjective, then we may assume X is integral. Since X is integral, its image subscheme structure will be integral (for any $\text{Spec } A \subseteq Y$ and $\text{Spec } B \subseteq f^{-1}(\text{Spec } A)$, we have that the image in $\text{Spec } A$ will be gluings of $\text{Spec } A/\ker(f^{\#})$ and $A/\ker(f^{\#}) \cong \text{im} f \subseteq B$ is integral). It follows that the image of f is the reduction of Y and therefore if it is affine, then Y is affine. It follows that we may assume both X and Y are integral. We now want to show that for any coherent $\mathscr{F}$ on Y, we have $H^i(Y, \mathscr{F}) = 0$ for all $i > 0$ which implies Y is affine by 3.7.

Since $f_*\mathscr{G} \to \mathscr{F}$ is an isomorphism generically, then there is an open set $U \subseteq Y$ such that $(f_*\mathscr{G})|_U \cong \mathscr{F}|_U$. Let $C = Y \backslash U$ which is closed in Y and of strictly smaller dimension than Y since Y is integral. We have an exact sequence,

$$0 \to \ker\beta \to f_*\mathscr{G} \xrightarrow{\beta} \mathscr{F}^r \to \text{coker}\beta \to 0$$

Now if $H^i(Y, \ker\beta) = H^i(Y, \text{coker}\beta) = 0$ for all $i > 0$, then splitting this exact sequence into two short exact sequences,

$$0 \to \ker\beta \to f_*\mathscr{G} \to \text{im}\beta \to 0$$
$$0 \to \text{im}\beta \to \mathscr{F}^r \to \text{coker}\beta \to 0$$

We further obtain two long exact sequences. Since $H^i(Y, f_*\mathscr{G}) = H^i(X, \mathscr{G}) = 0$ then the first long exact sequence yields $H^i(Y, \text{im}\beta) = H^i(Y, \ker\beta) = 0$. Since $H^i(Y, \text{im}\beta) = 0$, then the second long exact sequence yields $H^i(Y, \mathscr{F}^r) = H^i(Y, \text{coker}\beta) = 0$ and since $H^i(Y, \mathscr{F}^r) = H^i(Y, \mathscr{F})^r$, then this implies that $H^i(Y, \mathscr{F}) = 0$ as desired.

We have now reduced the problem to showing that $\ker\beta$ and $\text{coker}\beta$ have 0 cohomology. Notice that the support of $\ker\beta$ is contained in $\text{Supp}(f_*\mathscr{G}) \cap C$ and the support of $\text{coker}\beta$ is contained in $\text{Supp}(\mathscr{F}) \cap C$.

III.4.3

Let $X = \mathbb{A}_k^2 = \text{Spec } k[x, y]$, and let $U = X \backslash \{(0, 0)\}$. Using a suitable cover of U by affine open subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j | i, j < 0\}$. In particular, it is infinite-dimensional.

Proof:

Note that U is a separated noetherian scheme, so we can just compute the Čech cohomology. Let $U_1 = D(x)$ and $U_2 = D(y)$, then $U = U_1 \cup U_2$. Furthermore, we have that $\mathcal{O}_U(U_1) = k[x, y]_x, \mathcal{O}_U(U_2) = k[x, y]_y, \mathcal{O}_U(U_{1,2}) = k[x, y]_{xy}$. It follows that we have a Čech complex,

$$C^0 = k[x, y]_x \times k[x, y]_y, \qquad C^1 = k[x, y]_{xy}$$

We then have that $H^1(U, \mathcal{O}_U) = k[x, y]_{xy}/\text{im}(C^0)$. The image of C^0 is all differences of the form $x^{-i} f(x, y) - y^{-j} g(x, y)$. The collection of all such things is a subspace $k[x, y, x^{-1}, y^{-1}]$. What remains after the quotient is all elements whose terms are products $x^{-i} y^{-j}, i, j > 0$. If U were affine, we would have that $H^1(U, \mathcal{O}_U) = 0$ since $\mathcal{O}_U$ is quasi-coherent, thus U is not affine.

III.4.4

On an arbitrary topological space X with an arbitrary abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j\in J}$, together with a map $\lambda : J \to I$ of the index sets, such that for each $j \in J, V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F})$$

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

Proof:

(a) Give I any partial order and give J a partial order such that λ respects the partial order. We wish to show that λ induces a morphism of cochain complexes, i.e. that $d\lambda = \lambda d$ where given an element of $(s_{i_0 i_1 \cdots i_n})_{i_0<\cdots<i_n} \in C^n(\mathfrak{U}, \mathscr{F})$, we obtain a new element,

$$\lambda((s_{i_0 i_1 \cdots i_n})_{i_0<\cdots<i_n}) = (s_{\lambda(j_0)\cdots\lambda(j_n)})_{j_0<\cdots<j_n} \in C^n(\mathfrak{V}, \mathscr{F})$$

. Let s be our element, then,

$$d(\lambda s)_{j_0\cdots j_{n+1}} = \sum_{i=0}^{n+1} (-1)^i (\lambda s)_{j_0\cdots \hat{j}_i \cdots j_{n+1}} = \sum_{i=0}^{n+1} (-1)^i s_{\lambda(j_0)\cdots\widehat{\lambda(j_i)}\cdots\lambda(j_{n+1})}$$

And in the other direction, we get,

$$(\lambda(ds))_{j_0\cdots j_{n+1}} = (ds)_{\lambda(j_0)\cdots\lambda(j_{n+1})} = \sum_{i=0}^{n+1} (-1)^i s_{\lambda(j_0)\cdots\widehat{\lambda(j_i)}\cdots\lambda(j_{n+1})}$$

It follows that d and λ commute and therefore λ induces a morphism of cohomology groups,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$$

(b) I have no idea what the """natural maps""" are because I haven't read enough homological algebra. **DO THIS QUESTION**. The important part about this result is that it means we may define a morphism out of the direct limit.

(c) As noted above, we do indeed obtain a morphism out of the direct limit. Note that refinement does not yield a poset, it yields only a directed pre-order, but this is enough for a direct limit to exist. Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$ and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$. We now have an exact sequence,

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$$

This does not yield an exact sequence of Čech complexes, however we can still obtain the following exact sequence,

$$0 \to C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{U}, \mathscr{G}) \to D^\bullet(\mathfrak{U}) \to 0$$

where $D^\bullet(\mathfrak{U})$ is the quotient of these complexes. We note that the elements in the complex $D^\bullet(\mathfrak{U})$ are the global sections of presheaf quotient $\mathscr{C}^i(\mathfrak{U}, \mathscr{R})_{\text{pre}} = \mathscr{C}^i(\mathfrak{U}, \mathscr{F})/\mathscr{C}^i(\mathfrak{U}, \mathscr{G})$. It follows that we have natural sheafification maps from $D^\bullet(\mathfrak{U}) \to C^\bullet(\mathfrak{U}, \mathscr{R})$. We have that $\check{H}^0(\mathfrak{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$ and that $\check{H}^i(\mathfrak{U}, \mathscr{G}) = H^i(X, \mathscr{G}) = 0$ for $i > 0$. Taking long exact sequences of cohomology, we have the following two sequences:

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to h^0(D^\bullet(\mathfrak{U})) \to \check{H}^1(\mathfrak{U}, \mathscr{F}) \to 0$$
$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \Gamma(X, \mathscr{R}) \to H^1(X, \mathscr{F}) \to 0$$

We have maps from $\check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ and from $h^0(D^\bullet(\mathfrak{U})) \to \Gamma(X, \mathscr{R})$. I like to believe that these commute if we add them to the above diagram, but I don't know what the first one is, so idk. Call the map $h^0(D^\bullet(\mathfrak{U})) \to \Gamma(X, \mathscr{R})$ $s_\mathfrak{U}$, then clearly $s_\mathfrak{U}$ commutes with refining $\mathfrak{U}$ since we do not change the global section. Furthermore, for any global section $x \in \Gamma(X, \mathscr{R})$, it can be described by elements of the presheaf on an open cover. It follows that if we take $\mathfrak{U}$ to be that open cover, then x is in the image of $s_\mathfrak{U}$. Therefore $s_\mathfrak{U}$ is surjective in the limit. Furthermore, if we have a presheaf section that maps to 0 when taking the corresponding global section of $\mathscr{R}$, then on some cover it is 0, so the presheaf section is 0 up to refining covers. It follows that $s_\mathfrak{U}$ are injective in the limit. Therefore $\varinjlim_\mathfrak{U} h^0(D^\bullet(\mathfrak{U})) \cong \Gamma(X, \mathscr{R})$.

We now have an exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] \arrow[d, "\sim"] & \Gamma(X,\mathscr{G}) \arrow[r] \arrow[d, "\sim"] & \varinjlim_\mathfrak{U} h^0(D^\bullet(\mathfrak{U})) \arrow[r] \arrow[d, "\sim"] & \varinjlim_\mathfrak{U} \check{H}^1(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] & \Gamma(X,\mathscr{R}) \arrow[r] & H^1(X,\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

Since the first three vertical morphisms are isomorphisms, then the fourth must also be an isomorphism, so

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \cong H^1(X, \mathscr{F})$$

III.4.5

For any ringed space $(X, \mathcal{O}_X)$, let $\operatorname{Pic} X$ be the group of isomorphism classes of invertible sheaves. Show that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$ where $\mathcal{O}_X^*$ denotes the sheaf of units of $\mathcal{O}_X$.

Proof:

Let $\mathscr{L} \in \operatorname{Pic} X$ be an invertible sheaf. Let U_i be an open cover of X with $\psi_i : \mathcal{O}_X|_{U_i} \to \mathscr{L}|_{U_i}$ be isomorphisms expressing the freeness of $\mathscr{L}$. We then have isomorphisms $\psi_i^{-1}\psi_j : \mathcal{O}_X|_{U_i \cap U_j} \to \mathcal{O}_X|_{U_i \cap U_j}$. Such an isomorphism is $\mathcal{O}_X|_{U_i \cap U_j}$-linear and thus corresponds to taking a single invertible global section, i.e. an element $s_{ij} \in \mathcal{O}_X(U_i \cap U_j)$. Let $\mathfrak{U}$ be our open cover. To these isomorphisms, we may associate the element $(s_{ij})_{i<j} \in \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Note that on any triple intersection U_{ijk} we want to show that $s_{ij}s_{jk} = s_{ik}$ (i.e. the element gets mapped to 0 by the coboundary map). This is just a restatement of the fact that $\psi_i^{-1}\psi_j\psi_j^{-1}\psi_k = \psi_i^{-1}\psi_k$. On the other hand, this is exactly the cocycle condition required to guarantee the gluing of a sheaf. It follows that any element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$ defines an invertible sheaf in $\operatorname{Pic} X$. We need to make sure that this definition is well-defined, i.e. that if the element is 0 in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$, then we should just get $\mathcal{O}_X$. If this element is 0, then that means there are $s_i \in \mathcal{O}_X^*(U_i)$ such that $s_{ij} = s_i^{-1}s_j$. The morphism ψ_i on U_i is given by a choice of an element $x_i \in \mathscr{L}(U_i)$. These morphisms will not glue, however we may change them such that they are still isomorphisms, but do glue. Consider instead the isomorphisms $s_i^{-1}\psi_i : \mathcal{O}_X|_{U_i} \to \mathscr{L}|_{U_i}$ which maps $1 \mapsto s_i^{-1}x_i$. We now just need to check that $(s_i^{-1}\psi_i)^{-1}(s_j^{-1}\psi_j) : \mathcal{O}_X|_{U_{ij}} \to \mathcal{O}_X|_{U_{ij}}$ is given by mapping 1 to some element. $s_j^{-1}\psi_j$ maps 1 to $s_j^{-1}x_j$, then there is some k such that $x_j = kx_i$, so ψ_i maps k to x_j. It follows that ψ_i^{-1} maps $s_j^{-1}x_j$ to ks_j^{-1}. Therefore the whole map sends 1 to $ks_j^{-1}s_i$. We need to check that this is 1 so the morphisms on the intersection are equal and thus glue. Notice that $\psi_i^{-1}\psi_j$ maps 1 to x_j and then to k, so $s_{ij} = k$, but $s_{ij} = s_i^{-1}s_j$. It follows that $ks_j^{-1}s_i = 1$ and thus glue. Therefore we have that $\operatorname{Pic} X \cong \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$.

By exercise III.4.4 (the previous exercise), we get an isomorphism,

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*)$$

However, from what we just proved above, the LHS is a direct limit of things all isomorphic to $\operatorname{Pic} X$, and upon noting that the construction of the invertible sheaf described above commutes with refinement, we get that the LHS is exactly $\operatorname{Pic} X$. It follows that $H^1(X, \mathcal{O}_X^*) \cong \operatorname{Pic} X$.

III.4.8

Cohomological Dimension. Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\operatorname{cd}(X)$, to be the least integer n such that $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and all $i > n$. Thus for example, Serre's theorem says that $\operatorname{cd}(X) = 0$ iff X is affine. Grothendieck's theorem says that $\operatorname{cd}(X) \leqslant \dim X$.

(a) In the definition of $\operatorname{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X.

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.

(c) Suppose that X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\operatorname{cd}(X) \leqslant r$.

(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude that $\operatorname{cd}(X) \leqslant \dim(X)$.

(e) Let Y be a set-theoretic complete intersection of codimension r in $X = \mathbb{P}_k^n$. Show that $\operatorname{cd}(X - Y) \leqslant r - 1$.

Proof:

(a) By exercise II.5.15, we have that every quasi-coherent sheaf is a direct limit of its coherent subsheaves under inclusion. By 2.9, we have that,

$$\varinjlim H^i(X, \mathscr{F}_\alpha) \cong H^i(X, \varinjlim \mathscr{F}_\alpha)$$

Therefore if the cohomology of all coherent sheaves vanish, then so does the cohomology of quasi-coherent sheaves. Similar if the cohomology of quasi-coherent sheaves vanish, then coherent sheaves are quasi-coherent, so they vanish as well. It follows that we need only consider the cohomology of coherent sheaves.

(b) Let $\mathscr{F}$ be any coherent sheaf on X, then $X \hookrightarrow Y$ where Y is a projective scheme of dimension r over k. By exercise 5.15, we have some coherent sheaf $\mathscr{F}'$ on Y such that $\mathscr{F}'|_X = \mathscr{F}$. It follows that $\mathscr{F}'$ is a quotient of $\mathscr{E} = \bigoplus \mathcal{O}_Y(n_i)$, i.e. we have,

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F}' \to 0$$

Now we obtain a long exact sequence, $H^i(Y, \mathscr{E}) \to H^i(Y, \mathscr{F}') \to H^{i+1}(Y, \mathscr{R}) \to H^{i+1}(Y, \mathscr{E})$. Since $\mathscr{E}$ is locally free of finite rank, then it has 0 cohomology by assumption, thus $H^i(Y, \mathscr{F}') \cong H^{i+1}(Y, \mathscr{R})$ for $i > n$ where n is such that the cohomology of locally free sheaves vanish. Now $\mathscr{R}$ is a coherent sheaf. Then by induction, $H^i(Y, \mathscr{F}') \cong H^k(Y, \mathscr{G})$ for some coherent sheaf $\mathscr{G}$ and $k > i$. Since Y is finite dimensional, then eventually cohomology vanishes, so picking k sufficiently large, $H^i(Y, \mathscr{F}') \cong H^k(Y, \mathscr{G}) = 0$. Then $H^i(X, \mathscr{F}) = H^i(X, \mathscr{F}'|_X) = H^i(Y, \mathscr{F}') = 0$. It follows that the cohomology of coherent sheaves vanishes for $i > n$. Conversely since locally free coherent sheaves are coherent, then if the cohomology of coherent sheaves vanishes, then so does theirs. It follows that we need only consider locally free sheaves of finite rank.

(c) Let $\mathscr{F}$ be a quasi-coherent sheaf. We know that $C^{r+1}(\mathfrak{U}, \mathscr{F}) = 0$ since there are no $r+2$-intersections of an open cover of $r+1$ sets. We now have a sequence,

$$C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}) \to \cdots \to C^r(\mathfrak{U}, \mathscr{F}) \to 0$$

It follows that $\check{H}^i(\mathfrak{U}, \mathscr{F}) = 0$ for $i > r$ and since X is noetherian and separated, then $H^i(X, \mathscr{F}) = \check{H}^i(\mathfrak{U}, \mathscr{F})$ and therefore $\mathrm{cd}(X) \leqslant r$.

(d) Let $X \hookrightarrow Y$ where Y is a projective scheme, and let $Y \hookrightarrow \mathbb{P}^n_k$. Let $Z = Y \backslash X$. It follows that Z is a projective scheme since we have an embedding as a closed subscheme of $\mathbb{P}^n_k$. Let $\dim X = \dim Y = r$, then $\dim Z < r$. For any $f \in k[x_0, \cdots, x_n]$ homogeneous, by Bézout, $Y \cap V(f) \neq \varnothing$, so $D_+(f)$ is affine and thus $Y \cap D_+(f)$ is an affine open subset of Y. For X, we need that $X \cap V(f) \neq \varnothing$ to make sure that $X \cap D_+(f)$ is affine. Notice that $X \cap V(f) = \varnothing$ iff $Y \cap V(f) \subseteq Z$. Let Z be the vanishing of a homogeneous prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ in $k[x_0, \cdots, x_n]$, then we may pick f homogeneous such that $f \notin \mathfrak{p}_i$ for each i (prime avoidance), we have that f does not vanish on all of Z and therefore $\dim Z \cap V(f) < r - 1$ and since f does not vanish on Z and thus not on Y, then $\dim Y \cap V(f) = r - 1$, so $\dim X \cap V(f) = r - 1$ as well. Now we repeat with $Y' = V \cap V(f)$ and $X' = X \cap V(f)$. We may repeat this until $\dim X \cap V(f) = 0$ to obtain $f_1, f_2, \cdots, f_{r+1}$ such that $X \cap \bigcap_{i=1}^{r+1} V(f_i) = \varnothing$ and $V(f_i) \cap X \neq \varnothing$ and thus $X \cap D_+(f_i)$ yields an open affine cover of X by $r + 1$ affines.

Y is the a set-theoretic complete intersection of codimension r if it is the intersection of r hypersurfaces. Let $Y = \bigcap_{i=1}^r V(f_i)$, then $D_+(f_i)$ are all affine and $X - Y = \bigcup_{i=1}^r D_+(f_i)$, so there is an open affine cover by r sets, thus $\mathrm{cd}(X - Y) \leqslant r - 1$ by (c).

III.4.11

This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show that one can recover (4.5) as a corollary of this more general result.

Proof:

We need only show that we still obtain an exact sequence of Čech complexes as in 4.5 at which point the rest of the proof will remain the same. To do so we need to show that $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0$ remains exact upon taking sections over any V. This follows immediately upon taking the long exact sequence of cohomology over V, so we have,

$$0 \to \mathscr{F}(V) \to \mathscr{G}(V) \to \mathscr{R}(V) \to H^1(V, \mathscr{F}|_V) = 0$$

and therefore the proof remains intact.

III.5.1

Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by,

$$\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F})$$

If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Proof:

We obtain a long exact sequence of cohomology,

$$0 \to H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to \cdots$$

Each of these are finite dimensional k vectors spaces by 5.2. Now for any i, we have that $\dim_k H^i(X, \mathscr{F}) = \dim \operatorname{Im}(H^i(X, \mathscr{F})) + \dim \ker(H^i(X, \mathscr{F}))$ by the rank-nullity theorem. Since the sequence is exact, then $\ker(H^i(X, \mathscr{F})) = \operatorname{Im}(H^i(X, \mathscr{F}'))$ and also $\operatorname{Im}(H^i(X, \mathscr{F})) = \ker(H^i(X, \mathscr{F}''))$. Again by the rank-nullity theorem, we have that $\dim H^i(X, \mathscr{F}'') = \dim \ker(H^i(X, \mathscr{F}'')) + \operatorname{Im}(H^i(X, \mathscr{F}''))$ and similarly, $\dim H^i(X, \mathscr{F}') = \dim \ker(H^i(X, \mathscr{F}')) + \operatorname{Im}(H^i(X, \mathscr{F}'))$. It follows that we have the formula,

$$\dim_k H^i(X, \mathscr{F}) = \dim_k H^i(X, \mathscr{F}') - \dim_k \ker(H^i(X, \mathscr{F}')) + \dim_k H^i(X, \mathscr{F}'') - \dim_k \operatorname{Im}(H^i(X, \mathscr{F}''))$$

Now since $\operatorname{Im}(H^i(X, \mathscr{F}'')) = \ker(H^{i+1}(X, \mathscr{F}'))$, then in the alternating sum all of these terms will cancel, and we are left with,

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'') - \dim_k \ker(H^0(X, \mathscr{F}')) - \dim_k \operatorname{Im}(H^n(X, \mathscr{F}''))$$

where n is the number at which the cohomology all vanish, in which case the image will also be 0. Similarly, the kernel of the first map is 0 since it is injective, thus $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

III.5.2

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathcal{O}_X(1)$.

(b) Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \cdots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

Proof:

(a) We will do this by induction on the dimension of $\mathrm{Supp}(\mathscr{F})$. If $\mathrm{Supp}(\mathscr{F})$ is 0-dimensional, then it is a finite collection of closed points of X. It follows that $\mathscr{F}$ is a direct sum of skyscraper sheaves corresponding to the stalks at each point of the support. Tensor product distributes over direct sums and for any skyscraper sheaf $i_P(A)$, we have that $i_P(A) \otimes \mathcal{O}_X(n)$ is 0 away from P and thus is also a skyscraper sheaf, $i_P(A) \otimes \mathcal{O}_X(n) = i_P(A \otimes \mathcal{O}_{X,x}(n)) = i_P((A \otimes \mathcal{O}_{X,x})(n)) = i_P(A(n))$. Note that $A \otimes \mathcal{O}_{X,x} = A$ since A is an $\mathcal{O}_{X,x}$-module. Since $\mathscr{F}(n)$ is a direct sum of skyscraper sheaves which are flasque, then $\mathscr{F}(n)$ is flasque and thus $H^0(X, \mathscr{F}(n)) = \Gamma(X, \mathscr{F}(n))$ and $H^i(X, \mathscr{F}(n)) = 0$ for $i > 0$. It follows that $\chi(\mathscr{F}(n)) = \dim_k \bigoplus_i A_i(n) = \dim_k \bigoplus_i A_i$, so $\chi(\mathscr{F}(n)) = \chi(\mathscr{F})$ is constant and hence a numerical polynomial.

Let $j : X \to \mathbb{P}^n_k$ be the embedding yielding $\mathcal{O}_X(1) = j^*\mathcal{O}(1)$, then let $s = j^*x_i$ such that s does not vanish on all of $\mathrm{Supp}(\mathscr{F})$ (note that if every j^*x_i vanishes everywhere on $\mathrm{Supp}(\mathscr{F})$, then $\mathrm{Supp}(\mathscr{F}) = \varnothing$). Now consider the map $\mathscr{F}(-1) = \mathscr{F} \otimes \mathcal{O}_X(-1) \to \mathscr{F}$ given by multiplication by $s \in \mathcal{O}_X(1)(X)$. This yields an exact sequence,

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \xrightarrow{\times s} \mathscr{F} \to \mathscr{Q} \to 0$$

where $\mathscr{Q}$ is the cokernel and $\mathscr{R}$ is the kernel. Since s does not vanish on all of X, then it follows that multiplication by s is injective on $D_+(s) \subseteq X$. In fact, it is an isomorphism because $\frac{1}{s} \in \mathcal{O}_X(-1)(D_+(s))$ exists. It follows that the supports of $\mathscr{R}$ and $\mathscr{Q}$ are contained in $V(s)$ which is of smaller dimension than $\mathrm{Supp}(\mathscr{F})$. Since tensoring with $\mathcal{O}_X(n)$ is exact, then we obtain an exact sequence,

$$0 \to \mathscr{R}(n) \to \mathscr{F}(n-1) \to \mathscr{F}(n) \to \mathscr{Q}(n) \to 0$$

and then applying χ we get that $\chi(\mathscr{R}(n)) - \chi(\mathscr{F}(n-1)) + \chi(\mathscr{F}(n)) - \chi(\mathscr{Q}(n)) = 0$. Rearranging, we get that,

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{Q}(n)) - \chi(\mathscr{R}(n))$$

Since the dimension of the support of the sheaves on the RHS is smaller than that of $\mathscr{F}$, then we know that the terms on the RHS are numerical polynomials, thus $\chi(\mathscr{F}(n))$ is as well by (I ,7.3).

(b) The Hilbert polynomial is the polynomial associated to $\varphi(l) = \dim_k M_l$. We have that $M = \Gamma_*(\mathscr{F})$, so $M_l = \mathscr{F}(l)(X)$. Now since $H^i(X, \mathscr{F}(n)) = 0$ for $i > 0$ for sufficiently large $n > N$, then we have that,

$$\chi(\mathscr{F}(n)) = \sum_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathscr{F}(n)) = \dim_k \Gamma(X, \mathscr{F}(n)) = \dim_k M_n$$

as desired.

III.5.3

Arithmetic Genus. Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X, \mathcal{O}_X) = k$, so that

$$p_a(X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

(b) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational invariant.*

Proof:

(a) By II.4.5 (the problem about centers) since X is integral and projective (hence proper) over k, then $\Gamma(X, \mathcal{O}_X) = k$. Furthermore, since X has dimension r, then $H^i(X, \mathcal{O}_X) = 0$ for $i > r$. It follows that,

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1) = (-1)^r \sum_{i=1}^{r}(-1)^i \dim_k H^i(X, \mathcal{O}_X) = \sum_{i=0}^{r-1}(-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$$

If X is a curve, then $r = 1$, so we are left with $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

(b) In the previous problem, we showed that the Hilbert polynomial of X was in fact given by $\chi(\mathcal{O}_X(n))$ for all $n \in \mathbb{Z}$, so evaluating the Hilbert polynomial at 0 is the same as $\chi(\mathcal{O}_X)$ and therefore the arithmetic genus is the same. Note that if $X = \operatorname{Proj} S$, then $S = \Gamma_*(\mathscr{I})$ where $\mathscr{I}$ is the ideal sheaf of X. Now $\mathscr{I}$ is the push forward of $\mathcal{O}_X$, so the Euler characteristics are the same, thus $\chi(\mathscr{F}(n))$ computes the Hilbert polynomial of S and therefore yields the arithmetic genus.

(c) Any curve birational to a nonsingular projective curve will be an open subset of one. It follows that the structure sheaf will be a restriction and therefore the cohomology will be the same and hence the arithmetic genus will be the same. For degree $d \geqslant 3$, we have that the arithmetic genus of a projective degree d plane curve is $\frac{1}{2}(d-1)(d-2) \neq 0$ and therefore by part (a), we know the arithmetic genus is $\frac{1}{2}(d-1)(d-2)$ and therefore the plane curve is not rational.

III.5.5

Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection. Then:

(a) For all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

(b) Y is connected.

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$.

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

[Hint: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which (5.1)]

Proof:

(a) Suppose inductively that this is true for a complete intersection Y of codimension l, then we wish to show that for $Z = Y \cap V(f)$ again a complete intersection that this is true. To do so, we want to show that $\mathscr{I}_Z = \mathcal{O}_Y(-d)$ where d is the degree of f. Since Y is a complete intersection, then $Y = \text{Proj } S/I$ and $I = (f_1, \cdots, f_l)$ where $f_1, \cdots, f_l, f$ form a regular sequence. Since f is not a zero divisor in S/I, then $S/I \xrightarrow{\times \overline{f}} S/I$ is injective. Furthermore, it surjects onto the ideal $(\overline{f}) \subseteq S/I$. Now since $Z = Y \cap V(f)$, then $Z = \text{Proj } (S/I)/(\overline{f})$ so $\mathscr{I}_Z = \widetilde{(\overline{f})}$. Since $S/I \xrightarrow{\times \overline{f}} (\overline{f})$ is an isomorphism, then letting $d = \deg(f)$, we have that $(S/I)(-d) \xrightarrow{\times \overline{f}} (\overline{f})$ is a degree preserving isomorphism, so we have that, $\mathscr{I}_Z = \mathcal{O}_Y(-d)$. By part c, we now have that $H^1(Y, \mathscr{I}_Z(n)) = H^1(Y, \mathcal{O}_Y(n-d)) = 0$, and we have an exact sequence,

$$0 \to \mathscr{I}_Z(n) \to \mathcal{O}_Y(n) \to i_*\mathcal{O}_Z(n) \to 0$$

Now since $H^1(Y, \mathscr{I}_Z(n)) = 0$, then taking the long exact sequence of cohomology, we get surjectivity of the map $\Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(Z, \mathcal{O}_Z(n))$. Inductively we have that $\Gamma(X, \mathcal{O}_X(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is surjective and therefore $\Gamma(X, \mathcal{O}_X(n)) \to \Gamma(Z, \mathcal{O}_Z(n))$ is surjective as desired.

(b) Taking $n = 0$, we have that $\Gamma(X, \mathcal{O}_X) = k$, and so $\dim_k \Gamma(Y, \mathcal{O}_Y) \leqslant 1$, but if $Y = \bigsqcup Y_i$, where Y_i are the connected component of Y, then $\mathcal{O}_Y = \bigoplus \mathcal{O}_{Y_i}$ and for each Y_i, we get that $\dim_k \Gamma(Y, \mathcal{O}_{Y_i}) \geqslant 1$, so we must have only one connected component.

(c) Let Y, Z be as in part (a) and assume that the statement holds for Y. Now since Z is a closed subscheme, we have an exact sequence,

$$0 \to \mathscr{I}_Z = \mathcal{O}_Y(-d) \to \mathcal{O}_Y \to i_*\mathcal{O}_Z \to 0$$

Note that twisting is exact, so taking the long exact sequence of cohomology of the twist, we get that for $i0 < i < q - 1$, we have

$$\cdots \to H^i(Y, \mathcal{O}_Y(n)) \to H^i(Z, \mathcal{O}_Z(n)) \to H^{i+1}(Y, \mathcal{O}_Y(n-d)) \to H^{i+1}(Y, \mathcal{O}_Y(n)) \to \cdots$$

Now by induction, we have that $H^i(Y, \mathcal{O}_Y(n)) = H^{i+1}(Y, \mathcal{O}_Y(n-d)) = H^{i+1}(Y, \mathcal{O}_Y(n)) = 0$, so $H^i(Z, \mathcal{O}_Z) = 0$ as desired.

(d) This follows immediately from the previous part and the formula obtained in exercise III.5.3.

III.5.6

Curves on a Nonsingular Quadric Surface. Let Q be the nonsingular quadratic surface $xy = zw$ in $X = \mathbb{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y if Q. These correspond to Cartier divisors on Q. On the other hand, we know that $\operatorname{Pic} Q \cong \mathbb{Z} \oplus \mathbb{Z}$, so we can talk about the *type* (a, b) of Y. Let us denote the invertible sheaf $\mathscr{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbb{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.

(a) Use the special cases $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbb{P}^1$ in Q, to show:

  (i) if $|a - b| \leqslant 1$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$

  (ii) if $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$

  (iii) If $a \leqslant -2$, then $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$

(b) Now use these results to show:

  (i) if Y is a locally principal closed subscheme of type (a, b) with $a, b > 0$, then Y is connected.

  (ii) now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b).

  (iii) an irreducible curve Y of type $(a, b), a, b > 0$ on Q is projectively normal iff $|a - b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type $(1, 3)$, which is just the rational quartic curve (I, Ex 3.18).

(c) If Y is a locally principal subscheme of type (a, b) in Q, show that $p_a(Y) = (a - 1)(b - 1)$

Proof:

(a) Let $Y = V(y, w) \subseteq Q$, then $Y = \mathbb{P}^1$. This is because Q is a Segré embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by $([a : b], [c : d]) \mapsto [ac : ad : bc : bd]$. The image of $\mathbb{P}^1 \times \{[1 : 0]\}$ is exactly all points of the form $[x : 0 : z : 0]$, i.e. $V(y, w)$. We now have an exact sequence,

$$0 \to \mathscr{I}_Y \to \mathcal{O}_Q \to i_* \mathcal{O}_Y \to 0$$

where i is the closed immersion which includes Y into Q. Let $S(Q)$ be the homogeneous coordinate ring of Q. Assume for now that $\Gamma_*(\mathcal{O}_Q) = S(Q)$ and that $\Gamma_*(\mathscr{I}_Y) = I$ where $I = (y, w)S(Q)$. Note that since Q is a hypersurface, then $H^1(Q, \mathcal{O}_Q(n)) = 0$ for all n. Taking twists and the long exact sequence of cohomology, we get,

$$0 \to \Gamma(Q, \mathscr{I}_Y(n)) \to \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Q, i_* \mathcal{O}_Y(n)) \to H^1(Q, \mathscr{I}_Y(n)) \to 0$$

We have that $\dim_k \Gamma(Q, \mathcal{O}_Q(n)) = \dim_k S(Q)_n$ and $\dim_k \Gamma(Q, \mathscr{I}_Y(n)) = \dim_k I_n$. Since $Y \cong \mathbb{P}^1$, then $\dim_k \Gamma(Q, i_* \mathcal{O}_Y(n)) = n + 1$. For I_n, we have an exact sequence,

$$0 \to I \to S(Q) \to S(Q)/I \to 0$$

and notice that $S(Q)/I = k[x, y, z, w]/(y, w, xw - yz) = k[x, y, z, w]/(y, w) = k[x, z]$. It follows that $\dim_k I_n = \dim_k S(Q)_n - (n + 1)$. Since $\dim_k$ is an additive function then we have that,

$$\begin{aligned}
\dim_k H^1(Q, \mathscr{I}_Y(n)) &= \dim_k \Gamma(Q, \mathscr{I}_Y(n)) - \dim_k \Gamma(Q, \mathcal{O}_Q(n)) + \dim_k \Gamma(Q, i_* \mathcal{O}_Y(n)) \\
&= \dim_k S(Q)_n - (n + 1) - \dim_k S(Q)_n + (n + 1) \\
&= 0
\end{aligned}$$

It follows that Now we need only show that $\Gamma_*(\mathcal{O}_Q) = S(Q)$ and that $\Gamma_*(\mathscr{I}_Y) = I$.

In fact, we can put off proving the second fact, since degree-wise, $S(Q) \to S(Q)/I$ is a surjection and so $H^1(Q, \mathscr{I}_Y(n)) = 0$. Then note that $\mathscr{I}_Y = \mathcal{O}_Q(-1, 0)$, so $\mathscr{I}_Y(n) = \mathcal{O}_Q(n-1, n)$ therefore proving the case where $|a-b| = 1$. Note that if $a = b$, then $\mathcal{O}_Q(a, a) = \mathcal{O}_Q(a)$ has vanishing first cohomology by the previous question.

When $a, b < 0$, let $a = -q-k$ and $b = -k$ (assuming $a < b$), then letting Y be a disjoint union of q lines, we get an exact sequence,

$$0 \to \mathcal{O}_Q(-q, 0) \to \mathcal{O}_Q \to i_*\mathcal{O}_Y \to 0$$

now twisting by $-k$ and taking the long exact sequence, we get,

$$0 \to \Gamma(Q, \mathcal{O}_Q(a, b)) \to 0 \to 0 \to H^1(Q, \mathcal{O}_Q(a, b)) \to 0$$

so $H^1(Q, \mathcal{O}_Q(a, b)) = 0$. For $\mathcal{O}_Q(a, 0)$ with $a \leqslant -2$, we have that this is the ideal sheaf corresponding to a disjoint union of $-a$ lines, call it Y, so we have a short exact sequence,

$$0 \to \mathcal{O}_Q(a, 0) \to \mathcal{O}_Q \to i_*\mathcal{O}_Y \to 0$$

Now taking the long exact sequence, we get

$$0 \to \Gamma(Q, \mathcal{O}_Q(a, 0)) \to \Gamma(Q, \mathcal{O}_Q) \to \Gamma(Y, \mathcal{O}_Y) \to H^1(Q, \mathcal{O}_Q(a, 0)) \to 0$$

now noticing that $\dim_k \Gamma(Q, \mathcal{O}_Q) = 1$ and $\dim_k \Gamma(Y, \mathcal{O}_Y) = -a$, we get that $\dim_k H^1(Q, \mathcal{O}_Q(a, 0))$ is either $-a$ or $-a-1$ both of which are nonzero since $a \leqslant -2$.

(b) (i) Follows from the fact that $\Gamma(Q, \mathcal{O}_Q) \to \Gamma(Y, \mathcal{O}_Y)$ is surjective, the prior is k so the latter must have dimension 1 and therefore Y is connected. For (ii), we have that $\mathcal{O}_Q(a, b)$ is very ample, so we have an embedding $i : Q \to \mathbb{P}^N_k$ such that $\mathcal{O}_Q(a, b) = i^*\mathcal{O}(1)$. By Bertini, there is a hyperplane H in $\mathbb{P}^N_k$ such that $H \cap Q$ is nonsingular. Since H has degree 1, so the ideal sheaf of $H \cap i(Q)$ is $\mathcal{O}_{i(Q)}(-1)$ and hence the corresponding invertible sheaf is $\mathcal{O}_{i(Q)}(1)$. Now pulling back $H \cap i(Q)$ we get a curve with the corresponding invertible sheaf being $i^*\mathcal{O}(1) = \mathcal{O}_Q(a, b)$. For (iii), since Y is a nonsingular curve, then in particular it is normal, so Y is projectively normal iff $\Gamma(\mathbb{P}^3, \mathcal{O}_{P^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ are all surjective (II Ex 5.14 d). This is the case iff $|a-b| = 0$ since we have the following exact sequences,

$$\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Q, \mathcal{O}_Q(n)) \to 0 \qquad \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)) \to H^1(Q, \mathscr{I}_Y(n)) \to 0$$

So the composition $\Gamma(\mathbb{P}^3,\ \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ will be surjective iff $H^1(Q, \mathscr{I}_Y(n)) = 0$ for all n. Now $\mathscr{I}_Y(n) = \mathcal{O}_Q(-a, -b) \otimes \mathcal{O}_Q(n, n) = \mathcal{O}_Q(n-a, n-b)$. If $|a-b| \leqslant 1$, then $|(n-a)-(n-b)| = |a-b| \leqslant 1$ and therefore it will be 0 for all n and thus Y will be projectively normal. If $|a-b| \geqslant 2$, then assume WLOG that $a > b$, so for $n = b$, we get $\mathcal{O}_Q(b-a, 0)$ which has nonzero global sections since $b-a \leqslant -2$, so the map is not surjective and therefore Y is not projectively normal.

(c) In the case of a disjoint union of q lines, we have,

$$0 \to \mathcal{O}_Q(-q, 0) \to \mathcal{O}_Q \to i_* \bigoplus_{k=1}^{q} \mathcal{O}_{\mathbb{P}^1} \to 0$$

We compute the Hilbert polynomial of $\mathcal{O}_Q(q, 0)$ as a difference,

$$\begin{aligned}
\chi(\mathcal{O}_Q(n-q, n)) &= \chi(\mathcal{O}_Q(n)) - \chi(\bigoplus_{k=1}^{q} \mathcal{O}_{\mathbb{P}^1}(n)) \\
&= \binom{3+n}{3} - \binom{1+n}{3} - q(n+1) \\
&= (n+1)^2 - q(n+1) = (n+1)(n+1-q)
\end{aligned}$$

Now for any a, b, assume $a < b$, then we have an exact sequence,

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q \to i_*\mathcal{O}_Y \to 0$$

then twisting by $n+b$, we get

$$0 \to \mathcal{O}_Q(n+b-a,n) \to \mathcal{O}_Q(n+b) \to i_*\mathcal{O}_Y(n+b) \to 0$$

Now we compute the Hilbert polynomial of $\mathcal{O}_Y$ as follows:

$$\chi(\mathcal{O}_Y(n+b)) = \binom{3+n+b}{3} - \binom{1+n+b}{3} - (n+1)(n+1+(b-a))$$

Now plugging in $n = -b$, we get,

$$\chi(\mathcal{O}_Y(0)) = 1 - 0 - (-b+1)(-a+1) = (a-1)(b-1)+1$$

It follows that

$$p_a(Y) = -(\chi(\mathcal{O}_Y(0)) - 1) = -(a-1)(b-1)$$

THIS IS OFF BY A FACTOR OF -1

III.5.7

Let X (respectively Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

(b) $\mathscr{L}$ is ample on X iff $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X iff $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f : X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y iff $f^*\mathscr{L}$ is ample on X.

Proof:

(a) By 5.3, $i^*\mathscr{L}$ is ample iff for any coherent $\mathscr{F}$, the cohomology of $H^i(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) = 0$ for $n \gg 0$. We have that $(i^*\mathscr{L})^n = i^*\mathscr{L}^n$ and $H^i(Y, \mathscr{F} \otimes i^*\mathscr{L}^n) = H^i(X, i_*(\mathscr{F} \otimes i^*\mathscr{L}^n))$ then by the projection formula, we get that this is just $H^i(X, i^*\mathscr{F} \otimes \mathscr{L}^n) = 0$ for $n \gg 0$ since $\mathscr{L}$ is ample on X and $i^*\mathscr{F}$ is coherent on X.

(b) If $\mathscr{L}$ is ample on X, then X_{red} is a closed subscheme of $X$and therefore $i^*\mathscr{L} = i^{-1}\mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}} = \mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}} = \mathscr{L}_{\text{red}}$ is ample. Conversely, for any coherent sheaf $\mathscr{F}$ on X, we have a filtration by the powers of the sheaf of nilpotents $\mathscr{N}$ giving,

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \cdots \mathscr{N}^n \cdot \mathscr{F} = 0$$

Note that since X is noetherian, then the sheaf of nilpotents is itself nilpotent. Now each quotient is of the form $\mathscr{G}/(\mathscr{N}\mathscr{G}) = i^*\mathscr{G}$ for some coherent sheaf $\mathscr{G}$ on X. It follows that $\mathscr{G}/(\mathscr{N}\mathscr{G})$ has a natural $\mathcal{O}_{X_{\mathrm{red}}}$-structure. Now we have that

$$\begin{aligned} H^i(X_{\mathrm{red}}, \mathscr{L}_{\mathrm{red}}^n \otimes_{\mathcal{O}_{X_{\mathrm{red}}}} \mathscr{G}/(\mathscr{N}\mathscr{G})) &= H^i(X, (\mathscr{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\mathrm{red}}}) \otimes_{\mathcal{O}_{X_{\mathrm{red}}}} \mathscr{G}/(\mathscr{N}\mathscr{G})) \\ &= H^i(X, \mathscr{L}^n \otimes_{\mathcal{O}_X} (\mathcal{O}_{X_{\mathrm{red}}} \otimes_{\mathcal{O}_{X_{\mathrm{red}}}} \mathscr{G}/(\mathscr{N}\mathscr{G}))) \\ &= H^i(X, \mathscr{L}^n \otimes_{\mathcal{O}_X} \mathscr{G}/(\mathscr{N}\mathscr{G})) \\ &= 0 \end{aligned}$$

For n sufficiently large. Since $\mathscr{L}^n$ is invertible, then $\cdot \otimes_{\mathcal{O}_X} \mathscr{L}^n$ is exact, so from the quotient, we have an exact sequence of $\mathcal{O}_X$-modules given by

$$0 \to \mathscr{N}^{k+1}\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n \to \mathscr{N}^k\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n \to (\mathscr{N}^k\mathscr{F})/(\mathscr{N}^{k+1}\mathscr{F}) \otimes_{\mathcal{O}_X} \mathscr{L}^n \to 0$$

For n sufficiently large the higher cohomology of the last term vanishes. Now assume inductively that $H^i(X, \mathscr{N}^{k+1}\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$. We know this is true when for large enough k since $\mathscr{N}^k$ is eventually 0. From the long exact sequence of cohomology, we then get that

$$H^i(X, \mathscr{N}^k\mathscr{F} \otimes \mathscr{L}^n) \cong H^i(X, (\mathscr{N}^k\mathscr{F})/(\mathscr{N}^{k+1}\mathscr{F}) \otimes_{\mathcal{O}_X} \mathscr{L}^n) = 0$$

Now by induction on k, we get that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$ when n is sufficiently large. Therefore $\mathscr{L}$ is ample.

(c) If $\mathscr{L}$ is ample, then $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is the pullback to X_i under the inclusion and since X_i is an irreducible component and hence a closed subscheme, then it is ample by part a. Conversely, let $\mathscr{F}$ be a coherent sheaf on X. We have a map $\mathcal{O}_X \to \bigoplus_j \mathcal{O}_{X_j}$ and locally it is injective, since for any Spec A, the irreducible components are Spec $A/\mathfrak{p}_j$ where $\mathfrak{p}_j$ are the minimal primes, then $s \in A$ is 0 iff $\bar{s} \in A_{\mathfrak{p}_j}$ is 0 for each j since in that case $s \in \bigcap_j \mathfrak{p}_j = \mathfrak{N}(A) = 0$, so $s = 0$. It follows that $\mathcal{O}_X \to \bigoplus_j \mathcal{O}_{X_j}$ is injective. Furthermore, the kernel of this map is $\bigcap_j \mathscr{I}_{X_j}$ and thus the intersection of the ideal sheaves, and consequently their product, is 0. We now have a filtration of $\mathscr{F}$,

$$\mathscr{F} \supseteq \mathscr{I}_{X_1}\mathscr{F} \supseteq \mathscr{I}_{X_1}\mathscr{I}_{X_2}\mathscr{F} \supseteq \cdots \supseteq \left(\prod_{j=1}^{n-1} \mathscr{I}_{X_j}\right)\mathscr{F} \supseteq 0$$

For any j, the quotient is a sheaf of the form $\mathscr{G}/(\mathscr{I}_{X_j}\mathscr{G})$ and we want to show that this has vanishing cohomology. We first want to show that $\mathscr{G}/(\mathscr{I}_{X_j}\mathscr{G}) = i_*i^*\mathscr{G}$ where i is the closed immersion including X_j into X. We have a natural map $\mathscr{G} \to i_*i^*\mathscr{G}$ given by the sheafification of the map $s \mapsto [s] \otimes 1$, thus we need only show that $\mathscr{I}_{X_j}\mathscr{G}$ is the kernel of this map. We first check that $\mathscr{I}_{X_j}\mathscr{G}$ is contained in the kernel and then on stalks check that this inclusion is an isomorphism. This is equivalent to checking that the sequence $\mathscr{I}_{X_j} \otimes \mathscr{G} \to \mathscr{G} \to i_*i^*\mathscr{G}$ is exact. We can do this on stalks. For any $x \in X$ if $x \notin X_j$ this is trivial so let $x \in X_j$. Given $a \otimes b \in (\mathscr{I}_{X_j} \otimes \mathscr{G})_x$ a simple tensor, it maps to $ab \in \mathscr{G}_x$. Now $(i_*i^*\mathscr{G})_x = (i^*\mathscr{G})_x = \mathscr{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X_j,x}$ and $\mathcal{O}_{X_j,x} = \mathcal{O}_{X,x}/\mathscr{I}_{X_j,x}$. It follows that $(i_*i^*\mathscr{G})_x = \mathscr{G}_x/(\mathscr{I}_{X_j,x}\mathscr{G}_x)$ and therefore $ab \mapsto 0$ and the kernel is exactly $\mathscr{I}_{X,x}\mathscr{G}_x$, the image of $\mathscr{I}_{X_j,x} \otimes \mathscr{G}_x$. It follows that $i_*i^*\mathscr{G} = \mathscr{G}/(\mathscr{I}_{X_j}\mathscr{G})$.

Since $\mathscr{L} \otimes \mathcal{O}_{X_j}$ is ample, then we have that $H^i(X_j, i^*\mathscr{G} \otimes i^*\mathscr{L}^n) = 0$ for all $i > 0$. Applying i_*, we get that $H^i(X, \frac{\mathscr{G}}{\mathscr{I}_{X_j}\mathscr{G}} \otimes \mathscr{L}^n) = 0$. Now by an analogous argument to part (b), we get that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $i > 0$ for some n sufficiently large and therefore $\mathscr{L}$ is ample.

(d) Let $\mathscr{L}$ be ample, then we may repeat the same proof as in part (a), but note that $H^i(Y, f_*\cdot) = H^i(X, \cdot)$ by Ex. 4.1 since f is finite and hence affine and X, Y are proper, so f is proper by II Ex

4.8e. Now suppose that $f^*\mathscr{L}$ is ample on X. We may slightly modify f since $\mathscr{L}$ is ample on Y iff it is ample on Y_{red} so we may compose with the map from $Y \to Y_{\text{red}}$ which is finite and surjective. Furthermore since $\mathscr{L}$ is ample iff it is ample on each component, then we may assume that Y is irreducible and hence integral since it is reduced. Since f is surjective, it must map some point of X to the generic point of Y and since f is finite, it preserves codimension, so this must be the generic point of an irreducible component of X. Since $f^*\mathscr{L}$ is still ample on any irreducible component of X, we may assume that X is integral.

We have now reduced the problem to showing that for X, Y integral and $f : X \to Y$ a proper finite surjective morphism, then if $\mathscr{L}$ is an invertible sheaf on Y with $f^*\mathscr{L}$ ample on X, then $\mathscr{L}$ is ample on Y. Given any coherent sheaf $\mathscr{F}$ on Y

III.5.10

Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \cdots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \cdots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact.

Proof:

Let α_i be the morphism from $\mathscr{F}_i$ to $\mathscr{F}_{i+1}$. We then have short exact sequences,

$$0 \to \ker(\alpha_i) \to \mathscr{F}_i \to \operatorname{im}(\alpha_i) \to 0$$

Now since our original sequence is exact, then $\operatorname{im}(\alpha_i) = \ker(\alpha_{i+1})$. Since twisting is exact, then $\ker(\alpha_i(n)) = \ker(\alpha_i)(n)$. Taking a long exact sequence of the twists, we get,

$$0 \to \Gamma(X, \ker(\alpha_i(n))) \to \Gamma(X, \mathscr{F}_i(n)) \to \Gamma(X, \ker(\alpha_{i+1}(n))) \to H^1(X, \ker(\alpha_i(n))) \to 0$$

Since X is noetherian and $\mathscr{F}_i(n)$ is coherent, then $\ker(\alpha_i(n))$ is quasi-coherent and thus coherent. Since X is projective, then for n sufficiently large, we have that $H^1(X, \ker(\alpha_i(n))) = 0$. Since ker as a presheaf is already a sheaf, then for n sufficiently large we have,

$$0 \to \ker(\alpha_i(n)(X)) \to \Gamma(X, \mathscr{F}_i(n)) \to \ker(\alpha_{i+1}(n)(X)) \to 0$$

is exact. In particular, we see that $\Gamma(X, \mathscr{F}_i(n)) \to \ker(\alpha_{i+1}(n)(X))$ is surjective, so the image of $\Gamma(X, \mathscr{F}_i(n))$ is the kernel of $\alpha_{i+1}(n)(X)$ and therefore the sequence,

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \cdots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact for n so large that the first cohomology of all kernels vanish.

III.6.1

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}', \mathscr{F}'' \in \mathfrak{Mod}(X)$. An extension of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}'$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\operatorname{Hom}(\mathscr{F}'', \cdot)$, in particular the map

$$\delta : \operatorname{Hom}(\mathscr{F}'', \mathscr{F}'') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$$

and let $t \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ be $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$ and elements of the group $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

Proof:

We obtain a long exact sequence by applying $\operatorname{Hom}(\mathscr{F}'', \cdot)$,

$$0 \to \operatorname{Hom}(\mathscr{F}'', \mathscr{F}') \to \operatorname{Hom}(\mathscr{F}'', \mathscr{F}) \to \operatorname{Hom}(\mathscr{F}'', \mathscr{F}'') \xrightarrow{\delta} \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}) \to \cdots$$

Let $t \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ be the image of the identity map in $\operatorname{Hom}(\mathscr{F}'', \mathscr{F}'')$. We now need to show two things: First, given two isomorphic extensions, the image of the identity map is the same and second, every element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is the image of the identity map under some δ.

Suppose we have an isomorphism of extensions,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, "="] & \mathscr{F} \arrow[r] \arrow[d, "\varphi"] & \mathscr{F}'' \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{F}'' \arrow[r] & 0
\end{tikzcd}$$

with φ an isomorphism. Now if we have an isomorphism of short exact sequences, $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$, then given any morphism from the first into a short exact sequence of injective objects, we may extend it to the isomorphic exact sequence,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[d] & A \arrow[d] \arrow[r] \arrow[dl, dashed, "\alpha"'] & I_A \arrow[d] \\
A' \arrow[d] \arrow[urr] & B \arrow[d] \arrow[r] \arrow[dl, dashed, "\beta"'] & I_B \arrow[d] \\
B' \arrow[d] \arrow[urr] & C \arrow[d] \arrow[r] \arrow[dl, dashed, "\gamma"'] & I_C \arrow[d] \\
C' \arrow[d] \arrow[urr] & 0 & 0 \\
0
\end{tikzcd}$$

We get morphisms into the injective objects since α, β, γ are isomorphisms and hence injective. Furthermore since they are surjective, then the commutativity of the original sequence and the injective sequence yields commutativity for the second sequence and the injective sequence. It follows that an injective resolution for $0 \to A \to B \to C \to 0$ also yields an injective resolution for isomorphic short exact sequences. Since the injective resolutions are the same, then the coboundary maps will be the same, so the image of $1_{\mathscr{F}''}$ will be the same in $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

We now need to show that any element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ arises as the image of the identity under the coboundary map for some extension. To do so, look at the proof in Weibel and take the dual when you need a projective object.

III.6.2

Let $X = \mathbb{P}^1_k$, with k an infinite field.

(a) Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$. [Hint: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j : V \to X$ is the inclusion.]

(b) Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$. [Hint: Consider surjections of the form $\mathscr{L} \to \mathscr{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathscr{L}$ is an invertible sheaf on X.]

Proof:

(a) We first show give criterion for when $j_!^{\text{pre}}\mathscr{F}$ is already a sheaf where $j : U \to X$ is an open inclusion. Let X be a topological space, and $U \subseteq X$ open with $j : U \to X$ the inclusion map, then $j_!^{\text{pre}}\mathscr{F}$ is a separated presheaf. Clearly, $j_!^{\text{pre}}\mathscr{F}(\varnothing) = 0$. Let $W \subseteq X$ open and cover W by $W = \bigcup_i V_i$ and let $s \in j_!^{\text{pre}}\mathscr{F}(W)$ such that $s|_{V_i} = 0$ for all i. If $W \cap U^c \neq \varnothing$, then $j_!^{\text{pre}}\mathscr{F}(W) = 0$ and thus $s = 0$. If $W \cap U^c = \varnothing$, then $W \subseteq U$ and therefore $V_i \subseteq U$ for all i. It follows that $s|_{V_i} = 0$ means $s = 0$ since we obtain uniqueness of $\mathscr{F}$.

We now need only understand when we have gluing. If X is not connected, then there is no hope for general $\mathscr{F}$. If X is connected, then if the topology is sufficiently fine there is no hope. If X is irreducible then there is hope. Now using the same setup as before, let $s_i \in j_!^{\text{pre}}\mathscr{F}(V_i)$ agree on intersections. If $W \cap U^c = \varnothing$, then we can glue as on $\mathscr{F}$. We therefore need to show that if $W \cap U^c \neq \varnothing$, then $s_i = 0$ for all i. Let $V_k \cap U^c \neq \varnothing$, then $s_i|_{V_i \cap V_k} = 0$ for all i since $s_k \in j_!^{\text{pre}}\mathscr{F}(V_k)$ is 0. To then conclude that s_i is 0 it suffices to assume that restriction to a nonempty set is injective in $\mathscr{F}$ (note that $V_i \cap V_k$ is not empty since X is irreducible. Therefore $j_!^{\text{pre}}\mathscr{F}$ is a sheaf if X is irreducible and restriction to nonempty sets is injective.

For any open subset $V \subseteq \mathbb{P}^1_k$, we have that $\mathcal{O}_X|_V$ has the property that restriction to a nonempty set is injective since it is just a further localization of integral domains. It follows that $\mathcal{O}_V$ is actually just $\mathcal{O}_V(W) = \mathcal{O}_X(W)$ for $W \subseteq V$ and 0 otherwise. It follows that we have a surjective morphism from $\mathcal{O}_X \to \mathcal{O}_V$ given by mapping by the identity or by 0. Let $\varphi : \mathscr{P} \to \mathcal{O}_X$ be a surjective morphism with φ surjective, then let $U \subseteq X$ be an open set. We want to show that $\varphi(U) = 0$. Let V, W be open sets covering X, then we obtain a surjective morphism $\mathcal{O}_V \oplus \mathcal{O}_W \to \mathcal{O}_X$ given by addition and this yields a morphism $\psi : \mathscr{P} \to \mathcal{O}_V \oplus \mathcal{O}_W$ since $\mathscr{P}$ is projective. Now for any open set U such that $V, W \nsubseteq U$ we have that $\mathcal{O}_V(U) = \mathcal{O}_W(U) = 0$ since these are the same as the presheaves. It follows that the map $\psi(U)$ is 0, but $\varphi(U)$ is a composition of $\psi(U)$ with the map $\mathcal{O}_V \oplus \mathcal{O}_W \to \mathcal{O}_X$ and therefore $\varphi(U) = 0$. Since we may always find such V, W for any U, then it follows that $\mathscr{P}$ cannot have a surjective morphism onto $\mathcal{O}_X$.

(b) We will first prove the coherent case and then the quasi-coherent case will follow with a slight modification. Let $\varphi : \mathscr{P} \to \mathcal{O}_X$ be our surjective map. Now pick a point $x \in X$, we must have that φ_x is surjective. We want to show that in fact $\varphi_x = 0$. To do so, we have the following commutative diagram for any invertible sheaf $\mathscr{L}$,

$$\begin{tikzcd}
\mathscr{P} \arrow[r, "\varphi"] \arrow[d, dashed, "\psi"'] \arrow[dr, dashed] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
\mathscr{L} \arrow[r] & i_*\mathcal{O}_{X,x} \arrow[r] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

Note that $\mathcal{O}_X \to i_*\mathcal{O}_{X,x}$ is surjective since $\{x\}$ is a closed subscheme of X. Furthermore tensoring this map with $\mathscr{L}$ gives $\mathscr{L} \to \mathscr{L} \otimes i_*\mathcal{O}_{X,x}$ and since $\mathscr{L}_x = \mathcal{O}_{X,x}$, then $\mathscr{L} \otimes i_*\mathcal{O}_{X,x} \cong i_*\mathcal{O}_{X,x}$ (the support is just a single point, so this is just a module). Since φ is surjective and $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_{X,x}$ is surjective, then so is the composition and therefore the surjective map $\mathscr{L} \to i_*\mathcal{O}_{X,x}$ yields a map from $\mathscr{P} \to \mathscr{L}$ denoted ψ. We know that $\psi \in \mathrm{Hom}(\mathscr{P}, \mathscr{L}) = \mathrm{Hom}(\mathscr{P} \otimes \mathscr{L}^\vee, \mathcal{O}_X)$. Letting $\mathscr{L} = \mathcal{O}(d)$, we have that $\psi \in \mathrm{Hom}(\mathscr{P}(-d), \mathcal{O}_X) = \mathrm{Hom}(\mathscr{P}(-d+2), \mathcal{O}_X(-2)) = \mathrm{Hom}(\mathscr{P}(-d+2), \omega_X)$. By Serre duality, we have that $\mathrm{Hom}(\mathscr{P}(-d+2), \omega_X) = H^1(X, \mathscr{P}(-d+2))^\vee$. For d sufficiently large since $\mathscr{P}$ is coherent, then $H^1(X, \mathscr{P}(-d+2))$ vanishes. It follows that for d large, the only such map is 0. Now the composition ψ with $\mathscr{L} \to i_*\mathcal{O}_{X,x}$ must be the same as $i^\# \circ \varphi$, so it must be surjective, but since $\psi = 0$, then it cannot be surjective. It follows that $\mathscr{P}$ does not exist.

For the quasi-coherent case, we first prove a cute little lemma. On a finite dimensional noetherian scheme X, for any quasi-coherent sheaf $\mathscr{G}$ and a surjective morphism $\varphi : \mathscr{G} \to \mathscr{F}$ with $\mathscr{F}$ coherent, there is a coherent subsheaf $\mathscr{G}' \subseteq \mathscr{G}$ such that $\varphi|_{\mathscr{G}'} : \mathscr{G}' \to \mathscr{F}$ is surjective. Choose $\eta \in X$ of codimension 0, then $\mathscr{F}_\eta$ is finitely generated and so we may pick elements $x_1, \cdots, x_n \in \mathscr{G}_\eta$ whose images generate $\mathscr{F}_\eta$. Let $x_1, \cdots, x_n$ be sections over an open set $U \subseteq X$. Note that we may take them all in one open set since there are only finitely many. Now let $\mathscr{A}$ be the coherent sheaf on U generated by all x_i. Since $\mathscr{A} \subseteq \mathscr{G}|_U$, then by II Ex 5.15, we have a coherent sheaf $\mathscr{A}' \subseteq \mathscr{G}$ whose restriction to U is $\mathscr{A}$. It follows that $\varphi : \mathscr{A}' \to \mathscr{F}$ is surjective on U. Now we may take such a point η for each irreducible component X_i of X to get $\mathscr{A}_i'$ and let $\mathscr{A}'$ be the subsheaf of $\mathscr{G}$ generated by all $\mathscr{A}_i'$. The component of the union of the U_i has dimension strictly smaller than X and now by taking each irreducible component of $X \backslash \bigcup_i U_i$ and taking corresponding points we get more η and U and $\mathscr{A}$ and then taking the subsheaf of $\mathscr{G}$ generated by those, we can reduce the dimension again. Proceeding in this way, we eventually reduce the dimension to 0, so $\mathscr{A} \subseteq \mathscr{G}$ is coherent and $\varphi : \mathscr{A} \to \mathscr{F}$ is surjective.

Now from our projective $\mathcal{O}_X$ module, our surjective map is obtained from a coherent subsheaf which has only the 0 map ψ into $\mathcal{O}_X(d)$ and so the morphism we get from $\mathscr{P} \to \mathscr{L}$ composed with the inclusion into $\mathscr{P}$ of our coherent sheaf is 0 and therefore the map is not surjective.

III.6.3

Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.

(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}\mathrm{xt}^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geqslant 0$.

(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}\mathrm{xt}^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

Proof:

We have that $\mathscr{E}\mathrm{xt}^i(\mathscr{F}, \mathscr{G})|_U = \mathscr{E}\mathrm{xt}^i(\mathscr{F}|_U, \mathscr{G}|_U)$. Letting U be an open affine of X, we reduce the problem to the case where X is affine. We may now assume that $\mathscr{F} = \widetilde{M}, \mathscr{G} = \widetilde{N}$ with M, N finitely generated, we want to show that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is $\mathrm{Ext}^i(M,N)\tilde{}$. We may take an injective resolution $0 \to N \to I^\bullet$, then we obtain an injective resolution by quasi-coherent sheaves, $0 \to \widetilde{N} \to \widetilde{I}^\bullet$. Since $\mathscr{H}om(\widetilde{M},\widetilde{N}) = \mathrm{Hom}(M,N)\tilde{}$ for M finitely-generated **PROVE THIS** (17.22.4), then since $\widetilde{I}$ are all quasi-coherent, we get that $\mathscr{E}xt^i(\widetilde{M},\widetilde{N}) = \mathrm{Ext}^i(M,N)\tilde{}$ and therefore quasi-coherent. We have now reduced the problem to showing that if M, N are finitely generated R-modules, then $\mathrm{Ext}^i(M,N)$ are all finitely-generated, however in the module case, this becomes easy since we may compute it with a projective resolution and in fact a free resolution. Since R is noetherian, then submodules of finitely generated modules are finitely generated, so we may take a resolution by finitely generated free modules.

III.6.4

Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees*. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $\mathscr{E}xt^i(\cdot,\mathscr{G})$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is coeffaceable (§1) for $i > 0$.]

Proof:

To show that $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is universal, we need only show that it is coeffaceable in each degree $i > 0$, i.e. it sends some epimorphism to 0. Therefore we need to show that for any $\mathscr{F} \in \mathfrak{Coh}(X)$, there is some object $\mathscr{P}$ and a epimorphism $\varphi : \mathscr{P} \to \mathscr{F}$ such that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \xrightarrow{\mathscr{E}xt^i(\varphi,\mathscr{G})} \mathscr{E}xt^i(\mathscr{P},\mathscr{G})$ is 0. Since $\mathfrak{Coh}(X)$ has enough locally frees, then $\mathscr{F}$ is a quotient of a locally free module $\mathscr{P}$ of finite rank. By 6.7 we have that,

$$\mathscr{E}xt^i(\mathscr{P},\mathscr{G}) = \mathscr{E}xt^i(\mathcal{O}_X, \breve{\mathscr{P}} \otimes \mathscr{G}) = 0$$

Therefore the epimorphism is sent to 0. It follows that $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is coeffaceable and hence universal.

III.6.5

Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\mathrm{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:

(a) $\mathscr{F}$ is locally free $\iff$ $\mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(b) $\mathrm{hd}(\mathscr{F}) \leqslant n \iff \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;

(c) $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_{X,x}} \mathscr{F}_x$.

Proof:

(a) Since $\mathscr{F}$ is locally free and coherent (i.e. of finite rank), then by 6.7, we see that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = \mathscr{E}xt^i(\mathcal{O}_X, \breve{\mathscr{F}} \otimes \mathscr{G}) = 0$ for $i > 0$. Conversely, let $0 \to \mathscr{R} \to \mathscr{P} \xrightarrow{\psi} \mathscr{F} \to 0$ be an exact sequence with $\mathscr{P}$ locally free of finite rank, then from the long exact sequence of cohomology we obtain,

$$0 \to \mathscr{H}om(\mathscr{F},\mathscr{R}) \to \mathscr{H}om(\mathscr{F},\mathscr{P}) \to \mathscr{H}om(\mathscr{F},\mathscr{F}) \to 0$$

Now since $\mathscr{F}$ is coherent, we have that, $\mathscr{H}om(\mathscr{F},\mathscr{P})_x = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{P}_x)$, so taking stalks at x, we get an exact sequence and therefore there is some $\varphi \in \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{P}_x)$ mapping to the identity

$\mathrm{id} : \mathscr{F}_x \to \mathscr{F}_x$ under the map which composes with ψ_x, i.e. $\psi_x \circ \varphi = \mathrm{id}$. We get the following (not a priori commutative) diagram,

$$\begin{tikzcd}
\mathscr{P}_x \arrow[r, shift left, "\psi_x"] \arrow[drr, dashed, "\gamma"'] & \mathscr{F}_x \arrow[l, shift left, "\varphi"] \arrow[r, "\alpha"] & A \\
& & B \arrow[u, twoheadrightarrow, "\beta"']
\end{tikzcd}$$

Since $\mathscr{P}_x$ is free, then it is projective, so given a surjective morphism from B to A and a morphism $\alpha : \mathscr{F}_x \to A$, then we obtain a morphism $\mathscr{P}_x$ to A and hence an extension from $\mathscr{P}_x$ to B. By definition of projective, we have that $\beta \circ \gamma = \alpha \circ \psi_x$, then composing both sides with φ on the right and recalling that $\psi_x \circ \varphi = \mathrm{id}$, we get that $\beta \circ \gamma \circ \varphi = \alpha$ and therefore we obtain the morphism $\gamma \circ \varphi : \mathscr{F}_x \to B$. It follows that $\mathscr{F}_x$ is projective and therefore free since projective modules over a local ring are free (10.85).

(b) If the homological dimension of $\mathscr{F}$ is at most n, then by 6.5, we can compute $\mathscr{E}\mathrm{xt}^i(\mathscr{F}, \mathscr{G})$ using a locally free resolution and therefore $\mathscr{E}\mathrm{xt}^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$. Conversely, given an exact sequence,

$$0 \to \mathscr{M} \to \mathscr{L}_n \to \mathscr{L}_{n-1} \to \cdots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

with each $\mathscr{L}_i$ locally free and $\mathscr{M}$ the kernel of the morphism $\mathscr{L}_n \to \mathscr{L}_{n-1}$, then we wish to show that $\mathscr{M}$ is locally free. To do so, we need only show that $\mathscr{E}\mathrm{xt}^1(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{n+1}(\mathscr{F}, \mathscr{G}) = 0$ at which point $\mathscr{M}$ is locally free by (a) and therefore $\mathrm{hd}(\mathscr{F}) \leqslant n$.

To prove that $\mathscr{E}\mathrm{xt}^1(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{n+1}(\mathscr{F}, \mathscr{G})$, we first show that for a sequence $0 \to \mathscr{M} \to \mathscr{L} \to \mathscr{F} \to 0$ we have that $\mathscr{E}\mathrm{xt}^i(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{i+1}(\mathscr{F}, \mathscr{G})$ at which point the rest will follow from splitting into short exact sequences. In the case $0 \to \mathscr{M} \to \mathscr{L} \to \mathscr{F} \to 0$ we obtain a long exact sequence of cohomology (however the order is reversed since $\mathscr{E}\mathrm{xt}^*(\cdot, \mathscr{G})$ is contravariant,

$$\cdots \to \mathscr{E}\mathrm{xt}^i(\mathscr{L}, \mathscr{G}) \to \mathscr{E}\mathrm{xt}^i(\mathscr{M}, \mathscr{G}) \to \mathscr{E}\mathrm{xt}^{i+1}(\mathscr{F}, \mathscr{G}) \to \mathscr{E}\mathrm{xt}^{i+1}(\mathscr{L}, \mathscr{G}) \to \cdots$$

Since $\mathscr{L}$ is locally free, then it is $\mathscr{H}om(\cdot, \mathscr{G})$-acyclic as seen previously, therefore we have that $\mathscr{E}\mathrm{xt}^i(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{i+1}(\mathscr{F}, \mathscr{G})$. Now given the longer exact sequence, we split it into short exact sequences,

$$0 \to \mathscr{M}_i \to \mathscr{L}_i \to \mathrm{im}\mathscr{L}_i \to 0$$

Where $\mathscr{M}_i$ is the kernel of the morphism $\mathscr{L}_i \to \mathscr{L}i - 1$. Since $\mathscr{L}_0 \to \mathscr{F}$ is surjective, then we get $0 \to \mathscr{M}_0 \to \mathscr{L}_0 \to \mathscr{F} \to 0$ and therefore $\mathscr{E}\mathrm{xt}^i(\mathscr{M}_0, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{i+1}(\mathscr{F}, \mathscr{G})$. Since we have a longer exact sequence, then we have that $\mathrm{im}\mathscr{L}_i = \mathscr{M}_{i-1}$, so we get that $\mathscr{E}\mathrm{xt}^i(\mathscr{M}_k, \mathscr{G}) = \mathscr{E}\mathrm{xt}\, i + 1(\mathscr{M}_{k-1}, \mathscr{G})$. Now by induction we have that $\mathscr{E}\mathrm{xt}^i(\mathscr{M}_k, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{i+k+1}(\mathscr{F}, \mathscr{G})$. Since $\mathscr{M} = \mathscr{M}_n$, then $\mathscr{E}\mathrm{xt}^i(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}\, i + n + 1(\mathscr{F}, \mathscr{G})$. In particular, for $i = 1$, we get that $\mathscr{E}\mathrm{xt}^1(\mathscr{M}, \mathscr{G}) = \mathscr{E}\mathrm{xt}^{n+2}(\mathscr{F}, \mathscr{G}) = 0$ so $\mathscr{M}$ is locally free and therefore we have the desired locally free resolution. (Note that n should actually be $n - 1$ so the resolution has length n and not $n + 1$ but I am too lazy to change it now).

(c) We need to show that $\mathrm{hd}(\mathscr{F}) \geqslant \sup_x \mathrm{pd}_{\mathcal{O}_{X,x}} \mathscr{F}_x$ and vice versa. To do so, we need to show that given a locally free resolution of $\mathscr{F}$, we can construct a projective resolution of any $\mathscr{F}_x$ of at most the same length, then we will have $\mathrm{hd}(\mathscr{F}) \geqslant \sup_x \mathrm{pd}_{\mathcal{O}_{X,x}} \mathscr{F}_x$ (this direction is obvious). Conversely, we want to show that for any x, given a projective resolution of $\mathscr{F}_x$, we can make a locally free resolution of $\mathrm{hd}(\mathscr{F})$ of at most the same length.

We deal with the first inequality because it is easy. Given any locally free resolution $\mathscr{L}_\bullet \to \mathscr{F} \to 0$, for any x, since taking stalks is exact we obtain a free resolution $(\mathscr{L}_\bullet)_x \to \mathscr{F}_x \to 0$ of at most the same length (some of the stalks may be 0). Since free modules are projective, then $\mathrm{pd}(\mathscr{F}_x) \leqslant \mathrm{hd}(\mathscr{F})$ for all x and therefore we obtain the first inequality.

For the other direction, we have that $\operatorname{hd}(\mathscr{F})$ is the largest n such that there is some $\mathscr{G} \in \mathfrak{Mod}(X)$ such that $\mathscr{E}xt^n(\mathscr{F}, \mathscr{G}) \neq 0$ by part (b). It follows that there is some x such that $\mathscr{E}xt^n(\mathscr{F}, \mathscr{G})_x = \operatorname{Ext}^n(\mathscr{F}_x, \mathscr{G}_x) \neq 0$ and therefore $\operatorname{pd}_{\mathcal{O}_{X,x}} \mathscr{F}_x \geqslant n$, so $\operatorname{hd}(\mathscr{F}) \leqslant \operatorname{pd}(\mathscr{F}_x)$ and therefore equality holds.

III.6.6

Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

(a) M is projective if and only if $\operatorname{Ext}^i(M, A) = 0$ for all $i > 0$. [Hint: Use (6.11 A) and descending induction on i to show that $\operatorname{Ext}^i(M, N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p. 129]).]

(b) Use (a) to show that for any $n, \operatorname{pd} M \leqslant n$ if and only if $\operatorname{Ext}^i(M, A) = 0$ for all $i > n$.

Proof:

III.6.7

Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$

and

$$\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)\widetilde{\ }$$

Proof:

Refer to III.6.3 where I prove this.

III.6.8

Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X. [Hint: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathscr{L}, s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}x, f \notin \mathcal{O}\zeta$. Let $D = (f)_\infty$, and let $s \in \Gamma(X, \mathscr{L}(D))$ correspond to D (II, §6).]

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ various invertible sheaves $\mathscr{L}_i$, and various integers n_i.

Proof:

III.6.9

Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is regular if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\epsilon : K_1(X) \to K(X)$. Show that ϵ is an isomorphism (Borel and Serre [1, §4]) as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$. Then use (6.11 A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum (-1)^i \gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

Proof:

III.6.10

Duality for a Finite Flat Morphism.

(a) Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{F})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{F}$ (II, Ex. 5.17e).

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*, \mathscr{F}, \mathscr{G})$$

(c) For each $i \geqslant 0$, there is a natural map

$$\varphi_i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

[*Hint:* First construct a map

$$\operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{F}).$$

Then compose with a suitable map from $f_*f^!\mathscr{G}$ to $\mathscr{G}$.]

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat - see §9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y. [*Hints*: First do $i = 0$. Then do $\mathscr{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathscr{F}$ locally free. Do the general case by induction on i, writing $\mathscr{F}$ as a quotient of a locally free sheaf]

Proof:

III.7.1

Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathscr{L}$ be an ample sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$

Proof:

Since $\mathscr{L}$ is ample, then $\mathscr{L}^n$ is very ample, so it is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^N_k$ by a morphism $i : X \to \mathbb{P}^N_k$. Any such morphism requires at least $\dim X + 1$ global sections of $\mathscr{L}^n$ to define and therefore $\dim_k \mathscr{L}^n(X) \geqslant \dim X + 1$. If $\mathscr{L}^{-1}(X) \neq 0$, then we can take some global section $s \in \mathscr{L}^{-1}(X)$ and consider the map $\mathscr{L}^n \to \mathcal{O}_X$ given by taking an element $x \in \mathscr{L}^n(U)$ and then mapping to $x \otimes s^{\otimes n} \in \mathcal{O}_X$. We then get that $\dim_k \mathcal{O}_X(X) \geqslant \dim X + 1 \geqslant 2$ which is a contradiction. It follows that no such s exists.

III.7.2

Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f^! \omega_Y^\circ$ is a dualizing sheaf for X, where $f^!$ is defined as in III Ex 6.10.

(b) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_* \omega_X \to \omega_Y$.

Proof:

III.8.1

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_* \mathscr{F} = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_* \mathscr{F})$$

Proof:

Consider an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$, then we obtain a sequence $0 \to f_* \mathscr{F} \to f_* \mathscr{I}^\bullet$. Since $\mathscr{I}^i$ are injective, then they are flasque. Now if $\mathscr{G}$ is flasque, then for any $U \subseteq V$, we have that $f^{-1}(U) \subseteq f^{-1}(V)$ and therefore $\mathscr{G}(f^{-1}(V))$ surjects onto $\mathscr{G}(f^{-1}(U))$ and hence $f_* \mathscr{G}$ is flasque. It follows that $f_* \mathscr{I}^\bullet$ is flasque. To show that we still have a resolution, we need to show that f_* preserves the exactness of the resolution. We have that a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact iff $0 \to \ker \beta \to B \to \text{coker}\alpha \to 0$ is exact. Since f_* is left exact, it will preserve exactness at $\mathscr{F}$, so we need only preserve exactness at each $\mathscr{I}^i$. Let $\alpha^i : \mathscr{I}^i \to \mathscr{I}^{i+1}$ and $\varepsilon : \mathscr{F} \to \mathscr{I}^0$. We now have short exact sequences,

$$0 \to \ker \alpha^0 \to \mathscr{I}^0 \to \text{coker}\varepsilon \to 0 \qquad 0 \to \ker \alpha^{i+1} \to \mathscr{I}^i \to \text{coker}\alpha^i \to 0$$

Let $\mathscr{I}_i^\bullet$ be the terms after the i^{th} term in $\mathscr{I}^\bullet$. We then obtain injective resolutions, $0 \to \ker \alpha^i \to \mathscr{I}_i^\bullet$ where $\ker \alpha^i \to \mathscr{I}^i$ is inclusion. It follows that $R^j f_* \ker \alpha^i = R^{i+j} f_* \mathscr{F}$ for $j > 0$ and therefore $R^1 f_* \ker \alpha^i = 0$ for all i. It follows that

$$0 \to f_* \ker \alpha^0 \to f_* \mathscr{I}^0 \to f_* \text{coker}\varepsilon \to 0 \qquad 0 \to f_* \ker \alpha^{i+1} \to f_* \mathscr{I}^i \to f_* \text{coker}\alpha^i \to 0$$

is exact and therefore $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ is a flasque resolution, so it computes cohomology on Y. Now the result follows immediately since applying the global sections functor to $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ with $\Gamma(X, \cdot)$ or $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ with $\Gamma(Y, \cdot)$ yields the same sequence.

III.8.2

Let $f : X \to Y$ be an affine morphism of schemes with X noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that Ex 8.1 is satisfied and therefore $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geqslant 0$.

Proof:

Since f is affine, then f_* is exact, so $R^i f_* = 0$ for $i > 0$ and hence $R^i f_*\mathscr{F} = 0$ for $i > 0$ so $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for $i > 0$.

III.8.3

Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula,

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$$

Proof:

Take an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$, then since $f^*\mathscr{E}$ is locally free, we have that tensoring preserves exactness, so $0 \to \mathscr{F} \otimes f^*\mathscr{E} \to \mathscr{I}^\bullet \otimes f^*\mathscr{E}$ is exact. We now want to show that this is an acyclic resolution of $\mathscr{F} \otimes f^*\mathscr{E}$. Therefore we need to show that for $\mathscr{I}$ injective, $R^i f_*(\mathscr{I} \otimes f^*\mathscr{E}) = 0$, however it suffices to show this locally, so we may assume that $\mathscr{E} = \mathcal{O}_Y^n$. Then,

$$\begin{aligned} R^i f_*(\mathscr{I} \otimes f^*\mathcal{O}_Y^n) &= R^i f_*(\mathscr{I} \otimes \mathcal{O}_X^n) \\ &= R^i f_*(\mathscr{I}^n) \\ &= (R^i f_*(\mathscr{I}))^n \\ &= 0 \end{aligned}$$

It follows that $0 \to \mathscr{F} \otimes f^*\mathscr{E} \to \mathscr{I}^\bullet \otimes f^*\mathscr{E}$ is an acyclic resolution and therefore computes cohomology, so we get a cochain complex,

$$0 \to f_*(\mathscr{I}^0 \otimes f^*\mathscr{E}) \to f_*(\mathscr{I}^1 \otimes f^*\mathscr{E}) \to \cdots$$

now we use the projection formula to get,

$$0 \to f_*\mathscr{I}^0 \otimes \mathscr{E} \to f_*\mathscr{I}^1 \otimes \mathscr{E} \to \cdots$$

Then since $\cdot \otimes \mathscr{E}$ is exact, it distributes over quotients, so $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) = R^i f_*(\mathscr{F}) \otimes \mathscr{E}$.

III.9.1

A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open, i.e. for every open subset $U \subseteq X$, $f(U)$ is open in Y

Proof:

Any open set U is trivially constructible, and f is a morphism of finite type of noetherian schemes, so by II Ex 3.19, we have that $f(U)$ is constructible in Y. To show that $f(U)$ is open, we need only show that it is stable under generization. Let $y \in Y$ be a generization of a point $y' \in f(U)$, i.e. $y' \in \mathrm{cl}\,\{y\}$ then we want to show that $y \in U$. Let Spec $A = V \subseteq Y$ be an open affine neighborhood of y', then since open sets are closed under generization, we have that $y \in U$.

We now have that the preimage of V under f is covered by finitely many $U_i =$ Spec B_i such that each B_i is a finitely generated A-algebra. Since $y' \in f(U)$, then there is some $x \in U$ mapping to y', then $x \in U_i$ for some i. We have a map $f : U_i \to V$ of affine schemes. We want to show that can now go down. Let $U_i =$ Spec $B_i =$ Spec B, then f corresponds to a finite type flat morphism $\varphi : A \to B$. Let $y = \mathfrak{p}$ and $y' = \mathfrak{q}$ be the corresponding prime ideals in A, then we have $\mathfrak{q} \subseteq \mathfrak{p}$ since $\mathfrak{q}$ generizes $\mathfrak{p}$. We have some $\mathfrak{p}'$ lying over $\mathfrak{p}$ since $f(\mathfrak{p}') = \varphi^{-1}(\mathfrak{p}') = \mathfrak{p}$ so now we want to go down to show that there is some $\mathfrak{q}'$ lying over $\mathfrak{q}$ contained in $\mathfrak{p}$. Thus we want to show that flat morphisms have going down.

We show that for $f : A \to B$ flat and $\mathfrak{q} \subseteq B, \mathfrak{p} \subseteq A$ with $\mathfrak{q}^c = \mathfrak{p}$, then $f^{\#}$ is surjective from Spec $A_{\mathfrak{p}} \to$ Spec $B_{\mathfrak{q}}$. To do so, we show that $f : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is faithfully flat. Once we have this, then since taking preimages under localization is bijective on prime ideals and preserves containment, then we have going-down and hence openness. Note that $B_{\mathfrak{q}}$ is flat over $B_{\mathfrak{p}}$ since it is a further localization and $B_{\mathfrak{p}} = B \otimes A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ since it is a tensor product of flat rings. To show faithful flatness, we need to show that for any maximal ideal $\mathfrak{m}$ of $A_{\mathfrak{p}}$, $\mathfrak{m}^e \neq B_{\mathfrak{q}}$. $A_{\mathfrak{p}}$ contains a unique maximal ideals, namely $\mathfrak{p}A_{\mathfrak{p}}$, so $f(\mathfrak{p}A_{\mathfrak{p}}) \subseteq \mathfrak{q}B_{\mathfrak{p}}$ which generates $\mathfrak{q}B_{\mathfrak{q}}$, so $f(\mathfrak{p}A_{\mathfrak{p}})$ is contained in the maximal ideal of $B_{\mathfrak{q}}$. It follows that $B_{\mathfrak{q}}$ is faithfully flat over $A_{\mathfrak{p}}$ as desired.

III.9.2

Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

Proof:

III.9.3

Some examples of flatness and nonflatness

(a) If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

(b) Let X be the union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y =$ Spec $k[x, y]$ and $X =$ Spec $k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.

(c) Again let $Y =$ Spec $k[x, y]$, but take $X =$ Spec $k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{\mathrm{red}} \cong Y, X$ has no embedded points, but that f is not flat.

Proof:

III.9.4

Open nature of flatness. Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X | f \text{ is flat at } x\}$ is an open subset of X.

Proof:

I have to learn more about flatness.

III.9.10

A scheme X_0 over a field k is *rigid* if it has no infinitesimal deformations.

(a) Show that $\mathbb{P}^1_k$ is rigid, using (9.13.2).

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f : X \to \mathbb{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbb{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbb{A}^2$ for which $f^{-1}(U) \cong U \times \mathbb{P}^1_k$.

(c) Show however, that one can trivialize a global deformation of $\mathbb{P}^1_k$ after a flat base extension, in the following sense: let $f : X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t = \mathbb{P}^1_k$. Then there exists a nonsingular curve T', and a flat morphism $g : T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f' : X' \to T'$ is isomorphic to $\mathbb{P}^1_{T'} \to T'$.

Proof:

(a) We know that the infinitesimal deformations of a scheme X are in bijection with $H^1(X, \mathscr{T}_X)$. Now for $X = \mathbb{P}^1_k$, we know that $\mathscr{T}_X$ is the dual of $\Omega_{X/k} = \omega_X$ since X is one dimensional, so $\mathscr{T}_X$ is the dual of $\mathcal{O}_X(-2)$ which is $\mathcal{O}_X(2)$. Now by Serre duality, we have that $H^1(X, \mathcal{O}_X(2)) \cong H^0(X, \mathcal{O}_X(-2) \otimes \omega_X)^{\vee} = H^0(X, \mathcal{O}_X(-4))^{\vee} = 0$.

III.9.11

Let Y be a nonsingular curve of degree d in $\mathbb{P}^n_k$ over an algebraically closed field k. Show that,

$$0 \leqslant p_a(Y) \leqslant \frac{1}{2}(d-1)(d-2)$$

Proof:

The fact that $0 \leqslant p_a(Y)$ is clear since $p_a(Y) = -(\chi(\mathcal{O}_Y) - 1) = \dim_k H^1(Y, \mathcal{O}_Y) \geqslant 0$. Now let $\varphi_a : \mathbb{P}^n_k \to \mathbb{P}^n_k$ be given by $\varphi_a(x_0 : \cdots x_n) = (x_0 : x_1 : x_2 : ax_3 : \cdots : ax_n)$. Then let $Y_a = \varphi_a(Y)$, so the family Y_a over all $a \in \mathbb{A}^1 \setminus \{0\}$ is isomorphic to $Y \times (\mathbb{A}^1 \setminus \{0\})$ and therefore flat. Now we have a unique flat limit of this family at 0, call it Y_0 such that Y_a is a flat family over all of $\mathbb{A}^1$. Furthermore, $Y_0 = Y_{(0)}$ set-theoretically, where $Y_{(0)}$ is the reduced induced structure on the fiber over 0. We the have that $Y_{(0)}$ is contained in $\mathbb{P}^2 \subseteq \mathbb{P}^n$ and is therefore a plane curve. By flatness we have that the Hilbert polynomial $P_{Y_0} = P_Y$, so letting $\widetilde{d} = \deg Y_{(0)}$, we want to show that,

$$0 \leqslant p_a(Y) \leqslant p_a(Y_{(0)}) = \frac{1}{2}(\widetilde{d}-1)(\widetilde{d}-2) \leqslant \frac{1}{2}(d-1)(d-2)$$

the middle equality follows from the fact that $Y_{(0)}$ is a plane curve. It follows that we need only show two things: First, show that $\widetilde{d} \leqslant d$ and then that $p_a(Y) \leqslant p_a(Y_{(0)})$. The first is not too hard.

Since $Y_0 = Y_{(0)}$ set-theoretically, then $\mathcal{O}_{Y_{(0)}}$ and $\mathcal{O}_{Y_0}$ are sheaves on the same space. In fact, we have that $(\mathcal{O}_{Y_0})_{\text{red}} = \mathcal{O}_{Y_{(0)}}$ since $Y_{(0)}$ has the reduced structure, so we have a short exact sequence,

$$0 \to \mathscr{N} \to \mathcal{O}_{Y_0} \to \mathcal{O}_{Y_{(0)}} \to 0$$

where $\mathscr{N}$ is the sheaf of nilpotents on $\mathcal{O}_{Y_0}$. Since the Hilbert polynomial is additive, then $P_{Y_0} = P_{Y_{(0)}} + P_{\mathscr{N}}$ and $P_{\mathscr{N}}$ has a leading coefficient which is $\geqslant 0$, so $d = \widetilde{d} + c$ where c is some nonnegative number, therefore $d \geqslant \widetilde{d}$ as desired. For the arithmetic genus, we know that $p_a(Y) = (-1)^r(P_Y(0)-1)$ in our case this is just $1 - P_Y(0)$ since Y is one dimensional. Showing that $p_a(Y) \leqslant p_a(Y_{(0)})$ is equivalent to showing that $1 - P_{Y_0}(0) \leqslant 1 - P_{Y_{(0)}}(0)$ or equivalently, showing that $P_{Y_0}(0) \geqslant P_{Y_{(0)}}(0)$. Since $P_{Y_0}(0) = P_{Y_{(0)}}(0) + P_{\mathscr{N}}(0)$, then we need to show that $P_{\mathscr{N}}(0) \geqslant 0$. To do so, we have that $P_{\mathscr{N}}(0) = \dim H^0(Y, \mathscr{N}) - \dim H^1(Y, \mathscr{N})$ since Y is one dimensional, so all higher cohomology vanish. So it suffices o show that $\dim_k H^1(Y, \mathscr{N}) = 0$ which is true if $\mathscr{N}$ is supported on a finite set (since then it is a direct sum of skyscraper, i.e. flasque sheaves). To do so we need to choose a suitable projection. This will be done by computing the dimension of the secant variety of Y in order to obtain a birational map onto a curve in $\mathbb{P}^2$. **DO THIS**.

IV.1.1

Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere except at P.

Proof:

Since k is algebraically closed, then X has infinitely many closed points, so we may pick another closed point P', so the divisor $\mathcal{O}_X(2P - P')$ has degree 1, thus $\mathcal{O}_X(2nP - nP')$ has degree n. For $n > 2g - 2$, we know that $h^0(X, \mathcal{O}_X(2nP - nP') > 1$, and therefore we can pick an effective divisor $D \in |2nP - nP'|$. We now have $D - 2nP + nP' = \operatorname{div}(f)$ for some $f \in K(X)$ since D is linearly equivalent to $2nP - nP'$. D is effective and $\deg D = n$, so D has a zero of at most order n at P, thus f must have a pole of order at least n at P. Since D and nP' are effective, then f is regular everywhere else.

IV.1.2

Again let X be a curve, and let $P_1, \cdots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Proof:

We again pick another point $P' \in X$ for which we want to put a zero of large order. We want to essentially do the same thing as in question 1, but we need to make sure the poles at the P_i do not cancel with any zeros. To do so, in question 1, f had a zero of at most order n, and we managed a pole of order $2n$ at P, so in total the function had a pole of order at least n at P. We want to do something similar, but we will want f_i to have at most n zeros, but a pole of order $(r+1)n$ at P_i.

Consider the divisors $Q_i = (r+1)P_i - rP'$. The degree of Q_i is 1, so nQ_i has degree n. We may choose n sufficiently large such that $l(nQ_i) > 1$ for all i and thus there are effective divisors D_i of degree n such that D_i is linearly equivalent to nQ_i. It follows that $D_i - nQ_i = \operatorname{div}(f_i)$, then f_i has a pole of order at least rn at P_i and at most n zeros elsewhere. It follows now that $\prod_{i=1}^r f_i$ is the desired rational function. Note that this rational function has a zero of large order at P'.

IV.1.3

Let X be an integral, separated, regular, one-dimensional scheme of finite type over k which is not proper over k. Then X is affine.

Proof:

By prop I.6.7, X is a subvariety of the abstract nonsingular curve of $K(X)$. Now let f be a rational function which is regular on X and and has poles on the complement of X in $\overline{X} = C_{K(X)}$. Since X is not proper, then the complement has at least 1 point, so f is nonconstant. f induces a finite map $\overline{X} \to \mathbb{P}^1$ and since on the complement of X, f has poles, then it maps to ∞ in $\mathbb{P}^1$, thus $X = f^{-1}(\mathbb{A}^1)$. Since $f : \overline{X} \to \mathbb{P}^1$ is finite, then in particular, it is affine, thus $X = f^{-1}(\mathbb{A}^1)$ is affine.

IV.1.4

Show that a separated, one-dimension scheme of finite type over k, none of whose irreducible components is proper over k is affine.

Proof:

By III.3.1 and 3.2, we know that a scheme is affine iff each irreducible component of its reduction is affine. Thus we may assume that X is integral. The only thing stopping X from being affine is now that it may not be regular, so consider the normalization map $\pi : \widetilde{X} \to X$. Now $\widetilde{X}$ is normal and thus non-singular, so it is a curve and π is finite. It follows from question 3 that $\widetilde{X}$ is affine, thus by Chevalley's theorem (III.4.2), we have that X is affine as desired.

IV.1.5

For an effective divisor D on a curve X of genus g, show that $\dim |D| \leqslant \deg D$. Furthermore, equality holds iff $D = 0$ or $g = 0$.

Proof:

By definition, $\dim |D| = \dim H^0(X, \mathcal{O}_X(D)) - 1$, then by Riemann-Roch,

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Since $\dim H^1(X, \mathcal{O}_X(D)) \geqslant 0$, then moving it to the other side and subtracting one gives exactly the desired inequality. If $D = 0$, then equality holds since both $\dim |D|$ and $\deg D$ are 0. Similarly, if $g = 0$, then we have $\dim |D| = \deg D + \dim H^1(X, \mathcal{O}_X(D))$. By Serre duality, $\dim H^1(X, \mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(K-D))$. The degree of $K-D$ is < 0 since D is effective and $\deg K = -2$. Since $K-D$ is not effective, then $\dim H^0(X, \mathcal{O}_X(K-D)) = 0$, otherwise $K-D$ would be linearly equivalent to an effective divisor and so would have degree at least 0. It follows that $\dim |D| = \deg D$.

Now suppose that equality holds, let $D = \sum_i x_i \neq 0$, then the dimension of h^0 must drop by at least 1 for each point we remove but by the previous part we know it does drop. The degree also drops by 1, so $h^0(\mathcal{O}(x_1)) = 2$ and therefore there is another point $p \neq x_1$ such that $p \sim x_1$ and thus X is rational so X has genus 0.

IV.1.6

Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leqslant g+1$

Proof:

Pick a point $P \in X$ and consider the divisor $D = (g+1)P$, then $\deg D = g+1$ and so by the Riemann-Roch inequality, we have that $h^0(X, \mathcal{O}_X(D)) \geqslant g+1+1-g = 2$, thus there is an effective divisor $Q \sim D$ with $Q \neq D$, then $Q - D = \mathrm{div}(f)$ and thus f is nonconstant since $Q \neq D$ so it has poles and zeros of order at most $g+1$ but at least 1. It follows that f defines a degree $g+1$ map from X into $\mathbb{P}^1$.

IV.1.7

A curve X is called *hyperelliptic* if $g \geqslant 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1 without base points. Use (II.7.8.1) to conclude that X is hyperelliptic.

(b) Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geqslant 2$.

Proof:

a) Since X has genus 2, then $\deg K = 2g - 2 = 2$. Similarly, we have that,

$$l(K) - l(0) = \deg K + 1 - 2$$

Therefore $l(0) = 1$, so $l(K) = 2$, therefore $|K|$ has dimension $l(K) - 1 = 1$. To show that $|K|$ is basepoint free, we want to show that $\dim |K - P| = 0$ for any point P. If $\dim |K - P| = 0$, then there is a unique effective divisor $D \sim K + P$ and since $\deg D = 1$, then D is just some point Q. If $\dim |K - P| \neq 0$, then $\dim |K - P| = 1$, and thus D is not unique. It follows that we need only show that there cannot be two points Q, Q' such that $K \sim P + Q \sim P + Q'$. If there were, then we would have $Q - Q' \sim 0$ and therefore X would be rational which is a contradiction. It follows that there is a morphism $X \to \mathbb{P}^1$ defined by $|K|$. The degree of this morphism is exactly $\deg K = 2$.

b) Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbb{P}^3$, then we know that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ by the Segre embedding. We have projections p_1, p_2 from Q to $\mathbb{P}^1$ in either coordinate. Now let C be a curve of type $(g+1, 2)$, then we want to show that $p_2 : C \to \mathbb{P}^1$ is a map of degree 2.

IV.1.8

p_a *of a Singular curve.* Let X be an integral projective scheme of dimension 1 over k, and let $\widetilde{X}$ be its normalization. Then there is an exact sequence of sheaves on X,

$$0 \to \mathcal{O}_X \to f_* \mathcal{O}_{\widetilde{X}} \to \sum_{P \in X} \widetilde{\mathcal{O}}_P / \mathcal{O}_P \to 0$$

where $\widetilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = l(\widetilde{\mathcal{O}}_P / \mathcal{O}_P)$.

(a) Show that $p_a(X) = p_a(\widetilde{X}) + \sum_{P \in X} \delta_P$.

(b) If $p_a(X) = 0$, show that X is already nonsingular and in fat isomorphic to $\mathbb{P}^1$.

(c) If P is a node or an ordinary cusp, show that $\delta_P = 1$.

Proof:

a) Applying the Euler characteristic to the exact sequence, we get:

$$\chi(f_*\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_X) + \sum_{P\in X} \chi(\widetilde{\mathcal{O}}_P/O_P)$$

Now X and $\widetilde{X}$ are separated Noetherian schemes, and f is finite so in particular it is affine. Therefore by Ex. III.4.1 $\chi(f_*\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_{\widetilde{X}})$. Now taking 1 minus both sides of our equation we get,

$$p_a(\mathcal{O}_{\widetilde{X}}) = p_a(\mathcal{O}_X) - \sum_{P\in X} \chi(\widetilde{\mathcal{O}}_P/\mathcal{O}_P)$$

Since $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ are all supported at points, then $\chi(\widetilde{\mathcal{O}}_P/\mathcal{O}_P) = \dim_k \widetilde{\mathcal{O}}_P/\mathcal{O}_P$. FINISH THIS

IV.2.1

Use (2.5.3) to show that $\mathbb{P}^n$ is simply connected.

Proof:

Let $f : X \to \mathbb{P}^n$ be an étale covering of $\mathbb{P}^n$. We may assume that X is connected, since otherwise we may consider each connected component (étale is a local property). Taking any hyperplane $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$, we get $f^{-1}\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is étale and therefore $f^{-1}\mathbb{P}^{n-1} = \mathbb{P}^{n-1}$. Note that $f^{-1}\mathbb{P}^{n-1}$ is connected for n at least 2.

IV.2.2

Classification of Curves of Genus 2. Fix an algebraically closed field k of characteristic $\neq 2$.

(a) If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to an automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to an automorphism.

(b) Conversely, given six distinct elements $\alpha_1, \cdots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x-\alpha_1)\cdots(x-\alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbb{P}^1$ and nowhere else.

(c) Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbb{P}^1$, then there exists a unique $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\varphi(P_1) = 0, \varphi(P_2) = 1, \varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbb{P}^1$ and then normalize by sending the first three to $0, 1, \infty$, respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$ where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k \neq 0, 1$.

(d) Let Σ_6 be the symmetric group on 6 elements. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of $k \neq 0, 1$ as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element $\sigma \in \Sigma_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

(e) Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct $\beta_1, \beta_2, \beta_3$ of $k \neq 0, 1$ modulo the action of Σ_6 described in (d). We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}^3_k$ modulo a finite group.

Proof:

a) The morphism is clearly finite since it is a morphism of curves and $f^*_{|K|}\mathcal{O}(1) = \mathcal{O}(K)$ has degree $2g - 2 = 2$ and is therefore a morphism of degree 2. Furthermore $K(X)/K(\mathbb{P}^1)$ is a Galois extension since it is degree 2 over a field with characteristic $\neq 2$. In particular, f is separable. We know that $g(X) = 2, g(\mathbb{P}^1) = 0$, so by Hurwitz, we have that the degree of the ramification divisor is exactly $2(2) - 2 - 2(2(0) - 2) = 6$. To see that the ramification index is 2 at each point, we note that $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L}$ for some line bundle $\mathscr{L}$ and furthermore $X = \text{Spec }_{\mathbb{P}^1}(\mathcal{O} \oplus \mathscr{L})$ where the algebra structure is given by a map $\mathscr{L}^2 \to \mathcal{O}$, i.e. multiplication by some polynomial. Therefore we have a map of $\mathcal{O}_X$ algebras, $\text{Sym}\,\mathscr{L} \to \mathcal{O} \oplus \mathscr{L}$ which induces a closed immersion $X \to \text{Spec }_{\mathbb{P}^1} \text{Sym}\,\mathscr{L}$. We may choose an open subset $U = D_+(t) \subseteq \mathbb{P}^1$ such that $\mathscr{L}|_U \cong \mathcal{O}_U$, then $X|_U$ is cut out by some polynomial in $\mathbb{A}^1$ over U. In particular, we have a polynomial $g \in \Gamma(\mathcal{O}_{\mathbb{P}^1}, \mathscr{L}^{-2})$ which gives the map $\mathscr{L}^2 \to \mathcal{O}_{\mathbb{P}^1}$. It follows that $\mathcal{O} \oplus \mathscr{L} = \mathcal{O}[x]/(x^2 - g)$ and therefore $X|_U$ is a closed subscheme of $\mathbb{A}^1_U$ cut out by $x^2 - g$. It follows that at any point of ramification $p \in \mathbb{P}^1$, we have that X is of the form $\mathcal{O}_{\mathbb{P}^1,p}[x]/(x^2 - g)$. Since g is separable and irreducible, it follows that each point of ramification has index 2.

b) K is a Galois extension of degree 2 by Ex. II.6.4. It follows that X is hyperelliptic. Since f is a degree 2 map out of a hyperelliptic which is unique up to a automorphism of $\mathbb{P}^1$. We know then that $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L}$ for some line bundle $\mathscr{L}$ on $\mathbb{P}^1$. If we show that f is ramified over the six points α_i with index of ramification 2 at each, then by Riemann-Hurwitz, we will know that $g(X) = 2$. Letting $R = k[x, y, z]/(z^2y^4 - (x - \alpha_1 y) \cdots (x - \alpha_6 y))$, then we know that R is integrally closed and on the open subset of Proj R where $y = 1$, we have an affine scheme $R' = \text{Spec } k[x, z]/(z^2 - (x - \alpha_1) \cdots (x - \alpha_6))$ whose function field is exactly K. Then to see that it is nonsingular, we recall that an integrally closed one dimensional Noetherian local ring is regular and R' is already integrally closed, so all its local rings are too. Checking that it is integrally closed when $z = 1$ and $x = 1$ is also easy (just check for non-vanishing derivatives). It follows that Proj R is a complete curve and the map $k[x, y] \to R$ gives the map $f : \text{Proj } R \to \text{Proj } k[x, y] = \mathbb{P}^1$. K is separable over $k(x)$. Since f is a morphism of degree 2, then the ramification at any point will be of index at most 2. Since $\text{char}\, k \neq 2$, it follows that all ramification is tame. The ramification occurs where the two preimages of a point in $\mathbb{P}^1$ are the same in X. This occurs exactly at the roots of $(x - \alpha_1) \cdots (x - \alpha_6)$, i.e. at the points $\alpha_1, \cdots, \alpha_6$ in $\mathbb{P}^1$. At these points, the pullback of their divisors has degree 2, so the ramification is of index 2.

c) The existence and uniqueness of this map is an elementary exercise in analytic geometry. These are just the fractional linear transformations. We can define them pointwise since k is algebraically closed.

d) We can check that the action defined is in fact a group action by checking that composing elements in Σ_6 commutes with the action. This is a computation that I do not want to do.

e) Given any 3 distinct points $\beta_1, \beta_2, \beta_3$, by part (b) there is a curve of genus 2 which is ramified at $0, 1, \infty, \beta_1, \beta_2, \beta_3$. Permuting these elements does not change the curve and applying an automorphism to $\mathbb{P}^1$ changes only the choice of 2-1 map to $\mathbb{P}^1$. Given any curve of genus 2, we get 6 points up to an automorphism of $\mathbb{P}^1$ and ordering, so we have a unique triple of points under the equivalence relation. To see that this is a bijection, we saw in part a) that our curve is of the form $\text{Spec }_{\mathbb{P}^1}\mathcal{O}[x]/(x^2 - g)$ and we know that g is a separable irreducible polynomial of degree 6 as desired.

IV.2.3

Plane Curves. Let X be a curve of degree d in $\mathbb{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P. Considering $T_P(X)$ as a point in the dual projective plane $(\mathbb{P}^2)^*$, the map $P \mapsto T_P(X)$ gives a morphism of X to its dual curve X^* in $(\mathbb{P}^2)^*$. Note that even though X is nonsingular, X^* in general will have singularities. We assume $\operatorname{char} k = 0$ below.

(a) Fix a line $L \subseteq \mathbb{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$ for each point $P \in X$. Show that φ is ramified at P iff either (1) $P \in L$, or (2) P is an *inflection point* of X, which means that the intersection multiplicity of $T_P(X)$ with X at P is $\geqslant 3$. Conclude that X has only finitely many inflection points.

(b) A line of $\mathbb{P}^2$ is a *multiple tangent* of X if it is tangent to X at more than one point. It is a *bitangent* if it is tangent at exactly two points. If L is a multiple tangent of X, tangent to X at the points $P_1, \cdots, P_r$ and if none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an *ordinary r-fold point*, which means a point of multiplicity r with distinct tangent directions. Conclude that X has only finitely many multiple tangents.

(c) Let $O \in \mathbb{P}^2$ be a point which is not on X, nor any inflectional or multiple tangent of X. Let L be a line not containing O. Let $\psi : X \to L$ be the morphism defined by projection from O. Show that ψ is ramified at a point $P \in X$ iff the line OP is a tangent to X at P, and in that case the ramification index is 2. Use Hurwitz's theorem and (I, Ex 7.2) to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence the degree of the dual curve (sometimes called the *class* of X) is $d(d-1)$.

(d) Show that for all but a finite number of points of X, a point O of X lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O.

(e) Show that the degree of the morphism φ of (a) is $d(d-1)$. Conclude that if $d \geqslant 2$, then X has $3d(d-2)$ inflection points, properly counted. (If $T_P(X)$ has intersection multiplicity r with X at P, then P should be counted $r-2$ times as an inflection point. If $r = 3$ we call it an *ordinary inflection point*). Show that an ordinary inflection point of X corresponds to an ordinary cusp of the dual curve X^*.

(f) Now let X be a plane curve of degree $\geqslant 2$, and assume that the dual curve X^* has only nodes and ordinary cusps as singularities (which should be true for sufficiently general X). Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents. [Hint: Show that X is the normalization of X^*. Then calculate $p_a(X^*)$ in two ways: once as a plane curve of degree $d(d-1)$, and once using Ex 1.8]

(g) For example, a plane cubic has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one.

(h) A plane quartic curve has exactly 28 bitangents.

Proof:

We first want to produce the map from X to $L \cong \mathbb{P}^1$. By way of an automorphism of $\mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$ we may assume that L is just $V(y)$. On the affine open $D_+(z) = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ where $z = 1$, we choose a point $p = (p_1, p_2) \in X \cap D_+(z)$. $X \cap D_+(z)$ is an irreducible closed subset, then it is $V(I)$ for some height 1 prime ideal I. Since $k[x, y]$ is a UFD, then $I = (f)$ for some irreducible element $f \in k[x, y]$. The tangent line at the point p is given by some $T_P(X) =$

Spec $k[x,y]/(\ell)$ where ℓ is a linear polynomial. ℓ must be 0 at p, so we need that $\ell \in \mathfrak{m}_p$, the maximal ideal of $\mathcal{O}_{X,p}$. It follows that we may write $\ell = a(x-p_1) + b(y-p_2)$. Since we require that $T_P(X)$ be tangent at p, then we must have $\ell \in \mathfrak{m}_p^2$. The only linear combination of constant terms contained in $\mathfrak{m}_p^2$ is the linearization of f at p, i.e. $\frac{\partial f}{\partial x}(p)\cdot(x-p_1) + \frac{\partial f}{\partial y}(p)\cdot(y-p_2)$. It follows that $\ell = \frac{\partial f}{\partial x}(p)\cdot(x-p_1) + \frac{\partial f}{\partial y}(p)\cdot(y-p_2)$. We want to find where this line intersects L. This will be a point (x,y) such that $y=0$ and $\ell(x,y)=0$, i.e. $(x,0)$ s.t. $\frac{\partial f}{\partial x}(p)(x-p_1) = p_2 \cdot \frac{\partial f}{\partial y}(p)$. Therefore the intersection point is at $p_1 + \frac{f_y(p)}{f_x(p)}p_2 \in L$. Since $L = \mathbb{P}^1$, then this is really the point $[f_x(p)p_1 + f_y(p)p_2 : f_x(p)]$. This is given by a map $\mathcal{O}_X^2 \to \omega_X(1)$ on $D_+(z)$ by taking a basis e_1, e_2 of $\mathcal{O}_X^2$ and mapping by $e_1 \mapsto xdx + ydy, e_2 \mapsto dx$. Now one constructs the map of sheaves on the other affine opens and they glue to give a map $\mathcal{O}_X^2 \to \omega_X(1)$ which is surjective and thus we obtain a map $X \to \mathbb{P}^1$ which maps a point $P \in X$ to $T_P(X) \cap L$.

We now want to show that φ is ramified at exactly $X \cap L$ and the inflection points of X. To do this, since ramification is a local property, then we may assume that $X = \text{Spec } k[x,y]/(f)$ and $L = \text{Spec } k[t]$ embedded into Spec $k[x,y]$ by $x \mapsto t, y \mapsto 0$. The map φ, given a point $\mathfrak{p} = (x-a, y-b)$, sends $(a,b) \mapsto a + \frac{f_y}{f_x}(p)b = a'$ and thus sends $\mathfrak{p}$ to $\mathfrak{q} = (t-a')$ in $k[t]$. Since $t-a'$ is a uniformizer in $k[t]_{\mathfrak{q}}$, then φ is ramified at $\mathfrak{p}$ iff the image of $t-a'$ is in $\mathfrak{m}_{\mathfrak{p}}^2$. The image of $t-a'$ under the map on coordinate rings induced by φ is $A = x + \frac{f_y}{f_x}y - a'$. We now write

$$\frac{f_y}{f_x} = \frac{f_y}{f_x}(p) + \left(\frac{f_y}{f_x}\right)_x (p)(x-a) + \left(\frac{f_y}{f_x}\right)_y (p)(y-b) + O(\mathfrak{m}_{\mathfrak{p}}^2)$$

Therefore,

$$A = (x-a) + \frac{f_y}{f_x}(p)(y-b) - \left(\frac{f_y}{f_x}\right)_x (p)y(x-a) - \left(\frac{f_y}{f_x}\right)_y (p)y(y-b) \pmod{\mathfrak{m}_{\mathfrak{p}}^2}$$

Notice that $(x-a)+\frac{f_y}{f_x}(p)(y-b)$ is already in $\mathfrak{m}_{\mathfrak{p}}^2$ since it is $\frac{1}{f_x(p)}$ times ℓ. We can write $y = (y-b)+b$, therefore,

$$A = b\left[\left(\frac{f_y}{f_x}\right)_x (p)(x-a) - \left(\frac{f_y}{f_x}\right)_y (p)(y-b)\right] \pmod{\mathfrak{m}_{\mathfrak{p}}^2}$$

If $b=0$, then $A = 0 \pmod{\mathfrak{m}_{\mathfrak{p}}^2}$, so φ is ramified at $X \cap L$. If $b \neq 0$, then we need that the remaining terms are 0 mod $\mathfrak{m}_{\mathfrak{p}}^2$. This means that it must be a multiple of the relation $f_x(p)(x-a)+f_y(p)(y-b)$. Therefore there exists some c such that,

$$\left(\frac{f_y}{f_x}\right)_x (p) = \frac{f_{xy}f_x - f_yf_{xx}}{f_x^2}(p) = cf_x(p)$$
$$\left(\frac{f_y}{f_x}\right)_y (p) = \frac{f_{yy}f_x - f_yf_{xy}}{f_x^2}(p) = cf_y(p)$$

We now multiply the first equation by $f_x^2(p)f_y(p)$ and the second by $f_x(p)^3$. This yields,

$$f_{xy}(p)f_x(p)f_y(p) - f_y(p)^2f_{xx}(p) = cf_x(p)^3f_y(p)$$
$$f_{yy}(p)f_x(p)^2 - f_x(p)f_y(p)f_{xy}(p) = cf_x(p)^3f_y(p)$$

Now taking the difference we see that $2f_x(p)f_y(p)f_{xy}(p) = f_x(p)^2 f_{yy}(p) + f_y(p)^2 f_{xx}(p)$. We will show that this is equivalent to saying that X has an inflection point at p.

An inflection point of x is a point where $T_pX \cap X$ has multiplicity $\geqslant 3$ at p. Letting $\ell = f_x(x-a) + f_y(y-b)$, this is the same as saying that $\ell \in \mathfrak{m}_\mathfrak{p}^3$. We know that $\ell \in \mathfrak{m}_\mathfrak{p}^2$ and furthermore we know that:

$$0 = f(p) + \ell + f_{xx}(p)(x-a)^2 + 2f_{xy}(p)(x-a)(y-b) + f_{yy}(p)(y-b)^2 \pmod{\mathfrak{m}_\mathfrak{p}^3}$$

We know that $f(p) = 0$, so $-\ell = f_{xx}(p)(x-a)^2 + f_{xy}(p)(x-a)(y-b) + f_{yy}(p)(y-b)^2$ modulo $\mathfrak{m}_\mathfrak{p}^3$. This expresses ℓ in terms of the generating set for $\mathfrak{m}_\mathfrak{p}^2$. There are two linearly independent relations in $\mathfrak{m}_\mathfrak{p}^3$ given by $\ell \cdot (x-a)$ and $\ell \cdot (y-b)$. Therefore $\ell \in \mathfrak{m}_p^3$ iff,

$$f_{xx}(p)(x-a)^2 + 2f_{xy}(p)(x-a)(y-b) + f_{yy}(p)(y-b)^2 = c_1\ell(x-a) + c_2\ell(y-b)$$

Expanding ℓ we see that this yields three requirements,

$$\begin{aligned} f_{xx}(p) &= c_1 f_x(p) \\ f_{yy}(p) &= c_2 f_y(p) \\ 2f_{xy}(p) &= c_1 f_y(p) + c_2 f_x(p) \end{aligned}$$

Now multiplying the third equality by $f_x(p)f_y(p)$ we see that this is equivalent to requiring that $2f_xf_yf_{xy} = f_{xx}f_y^2 + f_{yy}f_x^2$ at p. This is exactly what we had before, so the points of ramification are exactly $X \cap L$ and the inflection points of X. Since the points of ramification are finite since φ is separable, then there are only finitely many inflection points.

b) Let $S = k[x,y,z]/(f)$ and $X = \text{Proj } S$, then we want to compute the tangent line at $p \in X$. As before, we expand f around p,

$$f = f(p) + f_x(p)(x-p_1) + f_y(p)(y-p_2) + f_z(p)(z-p_3) + O((x-p_1, y-p_2, z-p_3)^2)$$

Since $p \in X$, then $f(p) = 0$. Since f is homogeneous, then $f(\lambda x) = \lambda^d f(x)$, then taking the derivative with respect to λ, we get that $\sum_i \frac{\partial f}{\partial x_i}(x)x_i = d\lambda^{d-1}f(x)$. It follows that for $x \in X$, we have that $\sum_i \frac{\partial f}{\partial x_i}(x)x_i = 0$. In particular, $f_x(p)p_1 + f_y(p)p_2 + f_z(p)p_3 = 0$. It follows that our expansion becomes,

$$f = f_x(p)x + f_y(p)y + f_z(p)z + O((x-p_1, y-p_2, z-p_3)^2)$$

Let $\ell = f_x(p)x + f_y(p)y + f_z(p)z$ and suppose wlog that $p_1 \neq 0$. Let $p = \mathfrak{p} \in \text{Proj } S$, then x is a unit in $S_\mathfrak{p}$, so ℓ/x is a degree 0 element in $S_\mathfrak{p}$ and therefore is an element of $\mathcal{O}_{X,p} = S_{(\mathfrak{p})} = (S_\mathfrak{p})_0$. Furthermore, we have that $f = \ell + O(\mathfrak{m}_p^2)$. Since $f = 0$ in $\mathcal{O}_{X,p}$, then ℓ/x intersects X with multiplicity $\geqslant 2$. Thus $V(\ell)$ is the tangent line at X.

It follows that the image of the map $p \mapsto [f_x(p) : f_y(p) : f_z(p)]$ from $X \to \mathbb{P}^2$ is exactly the dual curve of X.

There are only finitely many inflection points in X, then any multitangent containing an inflection point is the tangent line at that inflection point, so there are only finitely many multitangents containing an inflection point. We now want to show that the r-multitangents of X are exactly in correspondence with points $\xi \in X^*$ with r distinct tangent directions through ξ. If we can prove this, then X^* is a curve over k and $\operatorname{char} k = 0$, so X^* is generically smooth. Since a multitangent requires distinct tangent directions, then ξ must be a singular point of X^*. Since X^* is generically smooth, it is smooth on a non-empty open subset and therefore regular on that subset. It follows that singularities can appear only on a proper closed subset of X^* which is finite. Therefore there are only finitely many multitangents.

We have now simplified the problem to showing that an r-multitangent of X intersecting no inflection points has image in X^* which is a singular point.

IV.2.5

Automorphisms of a Curve of Genus $\geqslant 2$. Prove the theorem of Hurwitz that a curve of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$ automorphisms. We will see later that the group $G = \operatorname{Aut}(X)$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves of degree n.

(a) If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}(f(P))$ consists of exactly n/r points, each having ramification index r. Let $P_1, \cdots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that

$$(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i)$$

(b) Since $g \geqslant 2$, the left hand side of the equation is > 0. Show that if $g(Y) \geqslant 0$, $s \geqslant 0, r_i \geqslant 2$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i) > 0$$

then the minimum value of this expression is $1/42$. Conclude that $n \leqslant 84(g-1)$.

Proof:

a) Note that since L is the fixed field of $G = \operatorname{Aut}_k(K(X))$, then $K(C)^G = L$ and $K(C)/L$ is Galois with $\operatorname{Gal}(K(C)/L) = \operatorname{Aut}_k(K(X)) = \operatorname{Aut}_k(X)$. Since $f : X \to Y$ is degree n and $f(P) = Q$ is degree 1 in Y, then $f^*Q = \sum_{f(P_i)=Q} \nu_{P_i}(t)P_i$ is degree n, i.e. $\sum_{P_i} \nu_{P_i}(t) = n$ where t is a uniformizer at Q. We now need only show that $\nu_{P_i}(t) = \nu_{P_j}(t)$ for all i, j since we know that $\nu_P(t) = r$ by definition of ramification.

Let $U = \operatorname{Spec} A$ be an open affine in Y containing Q, then since f is finite, then affine and integral, so $f^{-1}(U) = \operatorname{Spec} B \subseteq X$ with $P_i \in \operatorname{Spec} B$ for all i. It follows that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is integral, i.e. $f^\# : A \to B$ is integral. Since $f^\#$ is integral, then B is contained in the integral closure of A in $\operatorname{Frac}(B)$. Since X and Y are curves, then they are locally normal, thus A and B are integrally closed, so in fact B is exactly the integral closure of A in $\operatorname{Frac}(B)$. Since P_i all lie over Q, it follows from [MA 9.3] that there is an automorphism σ of $\operatorname{Frac}(B) = K(X)$ over $\operatorname{Frac}(A) = L$ sending P_i to P_j for any i, j. Since this automorphism fixes A, then we have that $\nu_{P_i}(t) = \nu_P(t)$ for all i. Therefore $n = r\left|f^{-1}(Q)\right|$, so there are exactly n/r points in the preimage of Q all with ramification index r. It follows that letting $e_{P_i} = r_i$, then since we are over characteristic 0 then all ramification is tame, so,

$$\deg R = \sum_i \frac{n}{r_i}(r_i - 1) = n\sum_i (1 - 1/r_i)$$

Since we are in characteristic 0, then $K(X)/L$ is automatically separable and therefore Hurwitz's formula gives that,

$$2g - 2 = n \cdot (2g(Y) - 2) + n\sum_i (1 - 1/r_i)$$

Dividing by n gives the desired result.

b) If $g(Y) = 1$, then the RHS becomes $\sum_{i=1}^{s}(1 - 1/r_i)$ which is $> 1/2$ since the LHS is > 0 so we must have $s > 0$. It follows that the minimum occurs when $g(Y) = 0$, so we want to minimize $\sum_{i=1}^{s}(1 - 1/r_i)$. We must have $s \leqslant 4$ since for $s > 4$, this sum is greater than $2 + \frac{1}{42}$. We require that the minimum be $\geqslant 2$ so that $-2+\sum_{i=1}^{s}(1-1/r_i)>0$. It follows that we require $s\geqslant 3$, so either $s=3$ or $s=4$. If $s=4$, then the smallest positive value is $-2+\frac{3}{2}+\frac{1}{3}=\frac{1}{6}>\frac{1}{42}$. It follows that we must have $s=3$. We want to minimize $1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}$ with $a,b,c\geqslant 2$ such that the sum remains positive. It follows that we cannot have more than one of them be 2, so assume that $a\geqslant 2$, $b,c\geqslant 3$. We can get $\frac{1}{42}$ with $a=2,b=3,c=7$. Now we want to show that $42>\frac{42}{a}+\frac{42}{b}+\frac{42}{c}>41$ is impossible. If $a\geqslant 3$, then if $b=c=3$ we would have 42 so one of a,b,c must be larger than 3, but we have that $\frac{42}{3}+\frac{42}{3}+\frac{42}{4}<41$, so we must have that $a=2$. Now we want $21>\frac{42}{b}+\frac{42}{c}>20$ with $b,c\geqslant 3$. If $b,c\geqslant 4$, then $\frac{42}{4}+\frac{42}{4}=21$, so we must have that one of them is at least 5, say $c\geqslant 5$, then we have that $\frac{42}{4}+\frac{42}{5}<20$. It follows that one of b,c must be 3, so wlog let $b=3$, then we need $7>\frac{42}{c}>6$, so $6<c<7$ which is impossible. Therefore $\frac{1}{42}$ is the minimum.

It follows that $\frac{2g-2}{n}\geqslant\frac{1}{42}$, so $n\geqslant 84(g-1)$ as desired.

IV.3.1

If X is a curve of genus 2, show that a divisor D is very ample iff $\deg D\geqslant 5$.

Proof:

Suppose that D is very ample, then D is effective. If $\deg D\leqslant 2$ we have that $h^0(\mathcal{O}(D))\leqslant \deg(D)/2+1\leqslant 2$ by Clifford's theorem. It follows that $\mathcal{O}(D)$ embeds X into $\mathbb{P}^1$ or $\mathbb{P}^0$. The latter is clearly impossible and X cannot be a closed subscheme of $\mathbb{P}^1$ since then it would be isomorphic to $\mathbb{P}^1$ and hence genus 0. If $\deg D\geqslant 3$, then we have that $h^1(\mathcal{O}(D))=0$, so $h^0(\mathcal{O}(D))=\deg D-1$. For the same reason we therefore cannot have $\deg D=3$. If $\deg D=4$, then $\mathcal{O}(D)$ embeds X as a plane curve. Since D is degree 4, then it must embed as a plane quartic. We know that plane quartics have genus $\frac{1}{2}(4-1)(4-2)=3$, but X has genus 2, thus $\deg D\geqslant 5$ as desired.

IV.3.5

Let X be a curve in $\mathbb{P}^3$, which is not contained in any plane.

(a) If $O\notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbb{P}^2$, show that $\varphi(X)$ *must* be singular. [Hint: Calculate $\dim H^0(X,\mathcal{O}_X(1))$ in two different ways.]

(b) If X has degree d and genus g, conclude that $g<\frac{1}{2}(d-1)(d-2)$. [Use Ex 1.8]

(c) Now let $\{X_t\}$ be the flat family of curve induced by the projection whose fibre over $t=1$ is X, and whose fibre X_0 over $t=0$ is a scheme with support $\varphi(X)$. Show that X_0 always has nilpotent elements. Thus the example (III, 9.8.4) is typical.

Proof: