Cartier Equality

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1 Introduction

We want to show that for k a perfect field, K an extension of k and L a finitely generated extension of K, then

$$\dim_L \Omega_{L/K} = \operatorname{tr.d.}_K L + \dim_L \Gamma_{L/K/k}$$

In particular, if L/K is a separable extension, then we have that $\dim_L \Omega_{L/K} = \operatorname{tr.d.}_K L$. We first need to understand derivations.

2 Derivations

Def: Let A be a ring and M and A-module. A map $D : A \to M$ is called a *derivation* if $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$. For a subring $k \to A$ we say that D is a *k-derivation* if $D|_k = 0$.

For any ring A and A-module M the collection of all derivations $D : A \to M$ forms an A-module called $\operatorname{Der}(A, M)$ where $(D + D')(a) = D(a) + D'(a)$ and $(aD)(b) = aD(b)$. For any $k \to A$, the k-derivations are denoted $\operatorname{Der}_k(A, M)$. For fixed A, the functor $M \mapsto \operatorname{Der}_k(A, M)$ is representable, i.e. there is a module $\Omega_{A/k}$ such that the functor $M \mapsto \operatorname{Der}_k(A, M)$ is naturally isomorphic to $\operatorname{Hom}_A(\Omega_{A/k}, M)$. In particular, we have a derivation $d_{A/k} : A \to \Omega_{A/k}$ such that for any derivation $D : A \to M$, there is a unique A-linear map $f : \Omega_{A/k} \to M$ such that $D = f \circ d_{A/k}$. We construct this module $\Omega_{A/k}$ called the module of differentials now:

Module of Differentials

Let A be a ring and $k \to A$ a ring homomorphism. Then there exists such a module $\Omega_{A/k}$ and derivation $d_{A/k} : A \to \Omega_{A/k}$.

Proof. Let $\mu : A \otimes_k A \to A$ be given by extending $\mu(x \otimes y) = xy$ linearly to all of A. Let $I = \ker \mu$. Let $\Omega_{A/k} = I/I^2$ and let $d_{A/k}(x) = x \otimes 1 - 1 \otimes x \pmod{I^2}$. Note that $\mu(d_{A/k}(x)) = x - x = 0$ so $d_{A/k}$ does indeed map into I/I^2. $d_{A/k}$ is obviously linear. To see that $d_{A/k}$ is a k-derivation let $x, y \in A$ notice that $(x \otimes 1 - 1 \otimes x)(y \otimes 1 - 1 \otimes y) = 0 \bmod I^2$, i.e. $x \otimes y + y \otimes x = xy \otimes 1 + 1 \otimes xy$, then

$$
\begin{aligned}
d_{A/k}(xy) &= xy \otimes 1 - 1 \otimes xy \\
&= -2 \otimes xy + (xy \otimes 1 + 1 \otimes xy) \\
&= -2 \otimes xy + x \otimes y + y \otimes x \\
&= -2 \otimes xy + (1 \otimes y)(x \otimes 1 - 1 \otimes x) + (1 \otimes x)(y \otimes 1 - 1 \otimes y) + 2 \otimes xy \\
&= y d_{A/k}(x) + x d_{A/k}(y)
\end{aligned}
$$

Furthermore, for any $x \in k$ we have that $d_{A/k}(x) = x \otimes 1 - 1 \otimes x = (x - x)(1 \otimes 1) = 0$. It follows that $d_{A/k}$ is a k-derivation. We now need to show that $\Omega_{A/k}$ has the stated universal property. Let $D : A \to M$ be a derivation.

Let $D \in \mathrm{Der}_k(A, M)$. Let $A * M$ be the k-algebra given set theoretically by $A \times M$ with multiplication $(a, x)(b, y) = (ab, ay + bx)$. We then have that $f : A \otimes_k A \to A * M$ given by extending $f(x \otimes y) = (xy, xDy)$ linearly. Notice by definition of I we have for any $s \in I$, $f(s) = (\mu(s), \cdots) = (0, \cdots)$ so $f : I \to M$. Furthermore f is a k-algebra homomorphism. To check this we need only check on products of simple tensors since it then extends by linearity:

$$f((x \otimes y)(z \otimes w)) = (xyzw, xyzDw + xzwDy) = (xy, xDy)(zw, zDw)$$

It follows that $f(I^2) \subseteq M^2$ but notice that $M^2 = 0$ in $A * M$. Therefore $f : I/I^2 \to M$ and $I/I^2 = \Omega_{A/k}$. Now $-f(d_{A/k}(x)) = -f(x \otimes 1 - 1 \otimes x) = -xD(1) + 1D(x) = D(x)$. If f, f' both satisfy $D = f \circ d_{A/k} = f' \circ d_{A/k}$ then $0 = (f - f') \circ d_{A/k}$. For any x, y we have that $x \otimes y = (1 \otimes y)(x \otimes 1 - 1 \otimes x) + 1 \otimes xy = y d_{A/k}(x_i) + 1 \otimes xy$, then for any $\sum x_i \otimes y_i \in I$ we have:

$$\sum x_i \otimes y_i = \sum y_i d_{A/k}(x_i) + 1 \otimes \sum x_i y_i = \sum y_i d_{A/k}(x_i) + \mu(\sum x_i \otimes y_i) = \sum y_i d_{A/k}(x_i)$$

It follows that $\Omega_{A/k}$ is generated by elements of the form dx for $x \in A$. It follows that $0 = (f - f') \circ d_{A/k}$ implies that $f - f' = 0$ and thus f is unique. Therefore $\Omega_{A/k}$ has the desired universal property. $\square$

Def: Let $A \to B$ be a ring homomorphism. B is said to be 0-smooth over A if the following property holds: For any A-algebra C and any ideal $I \subseteq C$ with $I^2 = 0$ and any A-algebra homomorphism $u : B \to C/N$ we have a lifting $v : B \to C$ such that the following diagram commutes:

$$
\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, "\exists v"] & C/N \\
A \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}
$$

First Fundamental Exact Sequence

Given ring homomorphisms $k \xrightarrow{f} A \xrightarrow{g} B$ we have an exact sequence:

$$\Omega_{A/k} \otimes B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \to 0$$

where $\alpha(d_{A/k}(a) \otimes b) = b d_{B/k}(g(a))$ and $\beta(d_{B/k}(b)) = d_{B/A}(b)$.
Furthermore, if B is 0-smooth over A, then the sequence is split exact.

Proof. To show that the sequence is exact, it suffices to show that it is exact after applying $\mathrm{Hom}(-, T)$ for all B-modules T. Therefore we have to show that:

$$0 \to \mathrm{Hom}_B(\Omega_{B/A}, T) \xrightarrow{\beta^*} \mathrm{Hom}_B(\Omega_{B/k}, T) \xrightarrow{\alpha^*} \mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, T)$$

By tensor-hom we have that $\mathrm{Hom}_B(N \otimes_A B, T) = \mathrm{Hom}_A(N, \mathrm{Hom}_B(B, T)) = \mathrm{Hom}_A(N, T)$ where T is now considered as an A-module through g. For any $f : \Omega_{B/k} \to T$ corresponding to a derivation $D = f \circ d_{B/k}$ we have it is mapped to $\alpha^* f = f \circ \alpha : \Omega_{A/k} \otimes_A B \to T$. Under the isomorphism $\mathrm{Hom}_B(\Omega_{A/k} \otimes_A B, T) = \mathrm{Hom}_A(\Omega_{A/k}, T)$ this corresponds to the map $f \circ \alpha \circ u : \Omega_{A/k} \to T$ where $u(x) = x \otimes 1$. This corresponds to the derivation $D' = f \circ \alpha \circ u \circ d_{A/k}$. Computing this we get:

$$\begin{aligned} D'(a) &= f(\alpha(u(d_{A/k}(a)))) \\ &= f(\alpha(d_{A/k}(a) \otimes 1)) \\ &= f(d_{B/k}(g(a))) \\ &= (D \circ g)(a) \end{aligned}$$

Therefore α^* corresponds to g^* on derivations. Similarly for any $f : \Omega_{B/A} \to T$ corresponding to a derivation $D = f \circ d_{B/A}$ we get $\beta^* f = f \circ \beta$ giving the derivation $D' = f \circ \beta \circ d_{B/k}$. We compute:

$$\begin{aligned} D'(a) &= f(\beta(d_{B/k}(a))) \\ &= f(d_{B/A}(a)) \\ &= D(a) \end{aligned}$$

Therefore β^* corresponds to the inclusion $\mathrm{Der}_A(B, T) \to \mathrm{Der}_k(B, T)$. It follows that upon rewriting our sequence in terms of derivations, we have:

$$0 \to \mathrm{Der}_A(B, T) \to \mathrm{Der}_k(B, T) \xrightarrow{g^*} \mathrm{Der}_k(A, T)$$

As noted, $\mathrm{Der}_A(B, T)$ is a submodule of $\mathrm{Der}_k(B, T)$ and the first map is just the inclusion map so it is injective. Now for $D : B \to T$ a k-derivation if $g^* D = 0$, then $D(g(x)) = 0$ for all $x \in A$ but this is exactly the statement that D is an A-derivation and thus the sequence is exact. It follows that the original sequence is exact.

Now if B is 0-smooth over A, then let $T = \Omega_{A/k} \otimes_A B$ and let $D \in \mathrm{Der}_k(A, T)$. Consider again $A * T$, so T is an ideal of $B * T$ with square 0. We have a map $\varphi : A \to B * T$ given by $\varphi(a) = (g(a), D(a))$ making $B * T$ into an A algebra. Then we have a commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[dr, dashed, "\exists v"] & B \\ A \arrow[u, "g"] \arrow[r, "\varphi"'] & B * T \arrow[u] \end{tikzcd}$$

From 0-smoothness of B over A we get $v : B \to B * T$. Let $D' : B \to T$ be given by $\mathrm{pr}_2 \circ v : B \to T$ where $\mathrm{pr}_2 : B * T \to T$. Since $\mathrm{pr}_1 \circ v = \mathrm{id}$, then $v(x) = (x, D'(x))$. We have that D' is a derivation since for $x, y \in B$ we have that $v(xy) = v(x)v(y) = (x, D'(x))(y, D'(y)) = (xy, xD'(y) + yD'(x))$. Therefore $D'(xy) = xD'(y) + yD'(x)$ as desired. Furthermore, we have that $v \circ g = \varphi$ so $D' \circ g = D$. Now since $T = \Omega_{A/k} \otimes_A B$ we may take $D = d_{A/k} \otimes 1$. Now D' corresponds to a map $f : \Omega_{B/k} \to \Omega_{A/k} \otimes_k T$. Then $D' = f \circ d_{B/k}$. Therefore $f \circ d_{B/k} \circ g = d_{A/k} \otimes 1$. By definition of α we have that $d_{B/k} \circ g = \alpha \circ d_{A/k} \otimes 1$. It follows that $f \circ \alpha \circ d_{A/k} \otimes 1 = d_{A/k} \otimes 1$. Since $\mathrm{Hom}_B(M \otimes_A B, T) = \mathrm{Hom}_A(M, T)$ for any A-module M and B-module T, then $(f \circ \alpha) \circ d_{A/k} = \mathrm{id} \circ d_{A/k}$. By the universal property of the module of differentials, we have that $f \circ \alpha = \mathrm{id}$ as desired. $\square$

Second Fundamental Exact Sequence

Let $k \xrightarrow{f} A \xrightarrow{g} B$ be a sequence of ring homomorphisms with g surjective and $B = A/I$, then there is an exact sequence:

$$I/I^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \to 0$$

where $\alpha(d_{A/k}(a) \otimes b) = bd_{B/k}(a)$ and $\delta(x \pmod{I^2}) = d_{A/k}(a) \otimes 1$.

Proof. Since g is surjective, then $\Omega_{B/A} = 0$ since any A-derivation $B \to M$ is 0 on $g(A) = B$. It follows from the first fundamental exact sequence that we have $\Omega_{A/k} \otimes_A B \to\xrightarrow{\alpha} \Omega_{B/k} \to 0$. We now need only show exactness at $\Omega_{A/k} \otimes_A B$. To do so, we again take $\mathrm{Hom}_B(-, T)$ for arbitrary B-modules T. This gives:

$$\mathrm{Der}_k(B, T) \xrightarrow{\alpha^*} \mathrm{Der}_k(A, T) \xrightarrow{\delta^*} \mathrm{Hom}_B(I/I^2, T)$$

Recall from the first fundamental exact sequence that α^* is just the pullback g^*. To understand δ^*, let D be a k-derivation corresponding to the map $f : \Omega_{A/k} \to T$, i.e. $D = f \circ d_{A/k}$, then f corresponds to $f = \tilde{f} \circ u$ where $u : \Omega_{A/k} \to \Omega_{A/k} \otimes_A B$ and $\tilde{f} : \Omega_{A/k} \otimes_A B \to T$ given by $\tilde{f}(x \otimes b) = bf(x)$. Therefore D is mapped to $\tilde{f} \circ \delta$. Now suppose that $\tilde{f} \circ \delta = 0$. Then $\tilde{f}(x) = 0$ for any $x \in I$. We want to find $D' : B \to T$ a k-derivation such that $D = D' \circ g$. Since g is surjective, let $D'(g(a)) = D(a)$. If $g(a) = g(b)$, then $a - b \in I$ and therefore $D(a - b) = 0$ so D' is well-defined. Therefore we have exactness. $\square$

Remark

For $A = K[X_1, \cdots, X_n]$ we have that $\Omega_{A/K} = A^n$ spanned by $dX_1, \cdots, dX_n$. To see this, we first show it for one variable, i.e. $\Omega_{K[X]/K} = K[X]dX$. Let $d : K[X] \to K[X]dX$ be given by $d(f(X)) = f'(X)dX$. Let $D \in \mathrm{Der}_K(K[X], T)$ be any K-derivation, then we have $D(X^s) = sX^{s-1}D(X)$. We want to find $g : K[X]dX \to T$, $K[X]$-linear such that $D = g \circ d$. We must have that $g(d(X)) = g(dX) = D(X)$, therefore we get that $g(f(X)dX) = f(X)D(X)$. g must be of this form so it is unique and for such g we do indeed have g. Notice that we do not need K to be a field for this to be true. Now $K[X]$ is 0-smooth over K since for any C a K-algebra and $I \subseteq C$ with $I^2 = 0$ and $u : K[X] \to C/I$ is given by $X \mapsto \bar{a}$, then we extend this to $u : K[X] \to C$ by $X \mapsto a$. Therefore by induction letting $k = K, A = K[X_1, \cdots, X_{n-1}], B = A[X_n]$ assume that $\Omega_{A/K} = AdX_1 \oplus \cdots \oplus AdX_{n-1}$. By the one variable case we have $\Omega_{B/A} = BdX_n$, then since B is 0-smooth over A the first fundamental exact sequence gives:

$$0 \to \Omega_{A/K} \otimes_A B \to \Omega_{B/K} \to \Omega_{B/A} \to 0$$

is split exact and the LHS is $BdX_1 \oplus \cdots \oplus BdX_{n-1}$ and the RHS is BdX_n so $\Omega_{B/K} = \bigoplus_{i=1}^n BdX_i$.

Remark

For $B = K[X_1, \cdots, X_n]/(f_1, \cdots, f_r)$ we have a surjection $A = K[X_1, \cdots, X_n]$ to B with kernel $I = (f_1, \cdots, f_r)$. By the second fundamental exact sequence we have:

$$I/I^2 \to \Omega_{A/K} \otimes_A B \to \Omega_{B/K} \to 0$$

We know that $\Omega_{A/K} = \bigoplus_{i=1}^n AdX_i$. The image of I/I^2 is the B span of df_i, therefore:

$$\Omega_{B/K} = \bigoplus_{i=1}^n BdX_i/(Bdf_1 + Bdf_2 + \cdots + Bdf_n)$$

Remark

Let $A \to B$ be a ring homomorphism and $S \subseteq B$ a multiplicatively closed subset, then we want to show that $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$. Since the map $B \to S^{-1}B$ has image whose $S^{-1}B$-span is all of $S^{-1}B$, then $\Omega_{S^{-1}B/B} = 0$. Furthermore for any B-algebra C given by a map $f : B \to C$ and ideal $I \subseteq C$ with $I^2 = 0$ and map $g : S^{-1}B \to C/I$ satisfying $g(a/1) = \overline{f(a)}$ we want to extend g to a map $S^{-1}B \to C$. To do so, we simply take the map $S^{-1}f : S^{-1}B \to C$ given by $S^{-1}f(a/b) = f(a)/f(b)$. For this to work, we need that $f(b)$ is a unit in C for all $b \in S$. This follows from the fact that b is a unit in $S^{-1}B$ and so $\overline{f(b)} = g(b)$ is a unit in C/I and I is nilpotent so $f(b)$ is a unit in C. Then $S^{-1}f$ extends g as desired. It follows that the first fundamental exact sequence reduces to $0 \to \Omega_{B/A} \otimes_B S^{-1}B \to \Omega_{S^{-1}B/A} \to 0$. This is the desired result.

Def: For ring homomorphisms $k \to A \to B$ we let $\Gamma_{A/B/k}$ denote the kernel of $\alpha : \Omega_{A/k} \otimes B \to \Omega_{B/k}$. $\Gamma_{A/B/k}$ is called the *imperfection module* of the A-algebra B over k.

Lemma

Let $k \to K \to L \to L'$ be field homomorphisms. Then there is an exact sequence

$$0 \to \Gamma_{L/K/k} \otimes_L L' \to \Gamma_{L'/K/k} \to \Gamma_{L'/L/k} \to \Omega_{L/K} \otimes_L L' \to \Omega_{L'/K} \to \Omega_{L'/L} \to 0$$

Proof. Using the first fundamental exact sequence and adding the imperfection module on left we get an exact sequence. We use this for $k \to K \to L$ and $k \to K \to L'$. We can then construct a morphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{L/K/k} \otimes_L L' \arrow[r] \arrow[d] & \Omega_{K/k} \otimes_K L' \arrow[r] \arrow[d, equal] & \Omega_{L/k} \otimes_L L' \arrow[r] \arrow[d] & \Omega_{L/K} \otimes_L L' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_{L'/K/k} \arrow[r] & \Omega_{K/k} \otimes_K L' \arrow[r] & \Omega_{L'/k} \arrow[r] & \Omega_{L'/K} \arrow[r] & 0
\end{tikzcd}$$

Note that $(\Omega_{K/k} \otimes_K L) \otimes_L L' = \Omega_{K/k} \otimes_K L'$. The two vertical maps on the right are given by the α map from the first fundamental exact sequence. The leftmost map is the map induced by the inclusion into $\Omega_{K/k} \otimes_K L'$ and then consequently. The diagram is commutative (although I will not check that). We will abbreviate the names and rewrite the diagram as:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

We can now shorten this diagram to get a morphism of short exact sequences by replacing A by A/X and A/Y respectively. This gives:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/X & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\
0 & \longrightarrow & A/Y & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

Now we use the snake lemma to get:

$$0 \to \ker q \to \ker \alpha_1 \to \ker \alpha_2 \to \text{coker} q \to \cdots$$

Since A/Y is a further quotient of A/X and q comes from the identity on A and is thus the quotient map, then q is surjective so $\text{coker} q = 0$. It follows that we have a short exact sequence:

$$0 \to \ker q \to \ker \alpha_1 \to \ker \alpha_2 \to 0$$

Now $\ker q = Y/X$. We have the inclusion map $\ker \alpha_1 \to C$ and the map $\alpha_2 : C \to E$ has kernel $\ker \alpha_2$ which is the image of $\ker \alpha_1$ under the map $B \to C$ by the SES that we have above. It follows the following sequence is exact:

$$0 \to X \to Y \to \ker f_2 \to C \to E \to \text{coker} \alpha_2 \to 0$$

. From the first fundamental exact sequence we know that $\text{coker} \alpha_2 = \Omega_{L'/L}$ and by definition of the imperfection module we have that $\ker f_2 = \Gamma_{L'/L/k}$. Therefore replacing the above exact sequence by what we know the terms are, we get:

$$0 \to \Gamma_{L/K/k} \otimes_L L' \to \Gamma_{L'/K/k} \to \Gamma_{L'/L/k} \to \Omega_{L/K} \otimes_L L' \to \Omega_{L'/K} \to \Omega_{L'/L} \to 0$$

This is exactly the desired exact sequence. □

The Cartier Equality

Let k be a perfect field. Let K be an extension of k and L a finitely generated extension of K. Then

$$\dim_L \Omega_{L/K} = \text{tr.d.}_K L + \dim_L \Gamma_{L/K/k}$$

Proof. We want to reduce this to the case where L is generated by a single element over K. To do so consider $k \to K \to L \to L'$ with L f.g. over K and L' f.g. over L. Suppose the theorem holds for $k \to L \to L'$ and $k \to K \to L$, then we want to show that the theorem holds for $k \to K \to L'$ as well. This essentially lets us take L' to be L with one more element adjoined. From the lemma we have an exact sequence,

$$0 \to \Gamma_{L/K/k} \otimes_L L' \to \Gamma_{L'/K/k} \to \Gamma_{L'/L/k} \to \Omega_{L/K} \otimes_L L' \to \Omega_{L'/K} \to \Omega_{L'/L} \to 0$$

Since this is exact, then we know that the alternating sum of the dimensions over L' is 0. This gives:

$$\dim_L \Gamma_{L/K/k} - \dim_{L'} \Gamma_{L'/K/k} + \dim_{L'} \Gamma_{L'/L/k} - \dim_L \Omega_{L/K} + \dim_{L'} \Omega_{L'/K} - \dim_{L'} \Omega_{L'/L} = 0$$

rearranging the above we get that:

$$\dim_{L'} \Omega_{L'/K} - \dim_{L'} \Gamma_{L'/K/k} = \left(\dim_{L'} \Omega_{L'/L} - \dim_{L'} \Gamma_{L'/L/k}\right) + \left(\dim_L \Omega_{L/K} - \dim_L \Gamma_{L/K/k}\right)$$

since we have already assumed the theorem is true for $k \to L \to L'$ and $k \to K \to L$, then we know that the RHS is just:

$$\dim_{L'} \Omega_{L'/K} - \dim_{L'} \Gamma_{L'/K/k} = \text{tr.d.}_L\, L' + \text{tr.d.}_K\, L$$

Putting together the transcendence basis of L' over L and L over K we get a transcendence basis of L' over K, therefore we get:

$$\dim_{L'} \Omega_{L'/K} = \text{tr.d.}_K\, L' + \dim_{L'} \Gamma_{L'/K/k}$$

Therefore the theorem holds for $k \to K \to L'$.

Since any finitely generated extension is obtain by a sequence of extensions generated by a single element, then by induction on the number of generators we need only prove the case where L is obtained by adjoining a single element, $L = K(\alpha)$. Any extension can be obtained by performing repeated extensions by one element of one of the following three kinds:

1. $L = K(\alpha)$ where α is transcendental over K.
2. $L = K(\alpha)$ where α is separable algebraic over K.
3. $L = K(\alpha)$ where $\operatorname{char} K = p$ and $\alpha^p = a \in K$ and $\alpha \notin K$.

In the first case we have that $\Omega_{K(\alpha)/K} = S^{-1}\Omega_{K[\alpha]/K} = S^{-1}K[\alpha]d\alpha$ where $S = K[\alpha]\backslash\{0\}$, therefore $\Omega_{L/K} = Ld\alpha$. We also have that $\text{tr.d.}_K\, L = 1$ so we need to show that the imperfection module $\Gamma_{L/K/k}$ is 0. This follows since $K[\alpha]$ is 0-smooth over K.

In the second case we have that L is separable algebraic over K so L is 0-smooth over K so the imperfection module is trivial. I will not prove that L is 0-smooth over K, but the idea is similar to the proof that $K[X]$ is 0-smooth over K, however, when lifting the image of X we use separability to choose an appropriate lifting. We also have that $\text{tr.d.}_K\, L = 0$. It therefore remains to show that $\Omega_{L/K} = 0$. Letting $A = K[X]$, $B = K[X]/(f(X))$ where f is the minimal polynomial of α and $I = f(X)A$, from the second fundamental exact sequence we get:

$$I/I^2 \to \Omega_{A/K} \otimes_A B \to \Omega_{B/K} \to 0$$

We know that $\Omega_{A/K} = AdX$ and the image of I/I^2 is the span of $df(X) = f'(X)$. Therefore $\Omega_{B/K} = (B/f'(X)B)dX$. Since α is separable, then f' is coprime to f so $f'(X)$ is a unit in B thus $B/f'(X)B = 0$ thus $\Omega_{L/K} = 0$ as desired.

Now in the third case, we have that $\text{tr.d.}_K\, L = 0$. Furthermore, $L = K[X]/(X^p - a)$ so as above, by the second fundamental exact sequence we have that $\Omega_{L/K} = LdX$ is one dimensional over L. Therefore we need to show that $\Gamma_{L/K/k}$ is also one dimensional. From the second fundamental exact sequence with $k \to K[X] \to L$ and $\mathfrak{m} = (X^p - a)$ we get:

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{K[X]/k} \otimes_{K[X]} L \to \Omega_{L/k} \to 0$$

The image of the map from $\mathfrak{m}/\mathfrak{m}^2$ is just $Ld(X^p - a)$. Since $d(X^p) = pX^{p-1} = 0$,then this is just Lda. Therefore:

$$\Omega_{L/k} = \Omega_{K[X]/k} \otimes_{K[X]} L/Lda$$

Now consider the maps $k \to K \to K[X]$ then from the first fundamental exact sequence we get:

$$\Omega_{K/k} \otimes_K K[X] \to \Omega_{K[X]/k} \to \Omega_{K[X]/K} \to 0$$

From the second fundamental exact sequence we get that $\Omega_{K[x]/K} = K[X]dX$. Since $K[X]$ is 0-smooth over K then this splits, so we have that $\Omega_{K[X]/k} = \Omega_{K/k}[X] \oplus K[X]dX$. It follows that,

$$\Omega_{L/k} = (\Omega_{K/k}[X] \oplus K[X]dX) \otimes_{K[X]} L/Lda$$

Now notice that $K[X]dX = \Omega_{K[X]/K}$ so $da = 0$ in $\Omega_{K[X]/K}$ thus da has no component in $K[X]dX$ so

$$\Omega_{L/k} = (\Omega_{K/k}[X] \otimes_{K[X]} L)/Lda \oplus K[X]dX \otimes_{K[X]} L$$

Now $\Omega_{K/k}[X] \otimes_{K[X]} L = \Omega_{K/k} \otimes_K L$ and $k[X]dX \otimes_{K[X]} L = Ld\alpha$. It follows that the map $\Omega_{K/k} \otimes_K L \to \Omega_{L/k}$ is the quotient by Lda. Therefore the kernel, i.e. the imperfection module $\Gamma_{L/K/k}$ is exactly Lda. Therefore we need only check that $da \neq 0$ in $\Omega_{K/k} \otimes_K L$, or equivalently check that it is nonzero in $\Omega_{K/k}$. To see this, since k is perfect, then $k = k^p$ and so $K^p = K^p(k)$. Now consider $K^p(a)$, then since $a \notin K^p$, then we have that $1, a, a^2, \cdots, a^{p-1}$ is a K^p-basis of $K^p(a)$. By Zorn's lemma, we may find a collection $\{x_\alpha\}$ with $x_0 = a$ such that the set of products of powers of x_α with exponents less than p forms a basis of K over K^p. Then choose $y \neq 0$ in K and for any finite collection $x_0, x_1, \cdots, x_n$ of the x_α's with $x_0 = a$ we let,

$$D(x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \alpha_0 x^{\alpha_0 - 1} x_1^{\alpha_1} \cdots x_n^{\alpha_n} y$$

Then we extend D K^p-linearly to all of K. Then D is a K^p-derivation on K. Since k is perfect, then $k \subseteq K^p$ so D is a k-derivation of K. Therefore there exists $f : \Omega_{K/k} \to K$ such that $D = f \circ d_{K/k}$ and therefore $y = D(a) = f(da)$ and $y \neq 0$ so $da \neq 0$ as desired.

It follows that the imperfection module $\Gamma_{L/K/k} = Lda$ has dimension 1 and $\text{tr.d.}_K L = 0, \dim_L \Omega_{L/K} = 1$ thus the theorem holds. □

Note that taking k to be the prime subfield Π of L and K we have that L/K is separable iff it is separably generated iff $\Omega_{K/\Pi} \otimes_K L \to \Omega_{L/\Pi}$ is injective iff $\Gamma_{L/K/\Pi} = 0$. Now Π is perfect so the Cartier equality holds giving the statement in Hartshorne:

Hartshorne Thm 8.6A

Let L/K be a finitely generated extension of fields, then $\dim_L \Omega_{L/K} \geqslant \text{tr.d.}_K L$ with equality iff L is separably generated over K.