MAT1191 Presentation Write-up

Ethan Bottomley-Mason

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Def: A k-algebra Q is a quaternion algebra if it has k-basis $1, i, j, k$ with $i^2 = a, j^2 = b, ij = -ji = k$ for $a, b \in k^\times$. We denote this quaternion algebra by $Q = (a, b)$.

Def: A quaternion algebra Q is called split if $Q \cong M_2(k)$.

Def: The conic associated to $Q = (a, b)$ is $C = V(ax^2 + by^2 - z^2) \subseteq \mathbb{P}^2$.

I will now list a series of facts about quaternion algebras:

1. If $Q \cong Q'$, then the associated conics C, C' are isomorphic.
2. Q is split iff the associated conic is iso to $\mathbb{P}^1$.
3. $(a, b) \cong (ac^2, b)$ for any $c \in k^\times$.
4. $(1, a)$ is split.
5. $(a, b) \otimes_k k(\sqrt{a})$ is split.
6. $(a, b) \otimes_k k(C)$ is split.
7. If $(a, b) \otimes_k k(t)$ is split, then (a, b) is split.
8. If $Q \otimes k(\sqrt{a})$ is split, then $Q \cong (a, c)$ for some c with $c = N_{k(\sqrt{a})/k}(f)$ for some $f \in k(\sqrt{a})$.
9. $(a, b) = (a, bc)$ for any $c = N_{k(\sqrt{a})/k}(f)$ with $f \in k(\sqrt{a})$.

We want to classify conics, so we want to show the converse of 1. Isomorphisms of quaternion algebras are simple, they are linear isomorphisms which respect the multiplication rules. Isomorphisms of curves are a priori more complicated.

Theorem (Witt): If $Q_1 = (a_1, b_1), Q_2 = (a_2, b_2)$ with associated conics C_1, C_2, then $k(C_1) \cong k(C_2)$ implies that $Q_1 \cong Q_2$.

Pf: If one of Q_1 or Q_2 is split, this is not very hard to show. Suppose that Q_1, Q_2 are both non-split. Let $C = C_1$ and $L = k(\sqrt{a_1})$, then $Q_1 \otimes_k L$ is split by 5, so the associated conic $C_L = C \times_k L$ is isomorphic to $\mathbb{P}^1_L$ by 2. By 6, $Q_2 \otimes_k k(C_2)$ is split. Since $k(C_2) \cong k(C)$, then letting $L(C) = L \otimes_k k(C) = k(C_L)$, we have that $Q_2 \otimes L(C)$ is split. Since $C_L \cong \mathbb{P}^1_L$, then $L(C) \cong L(t)$, so

$(Q_2 \otimes_k L) \otimes_L L(t)$ split implies that $Q_2 \otimes_k L$ is split by 7. Since $L = k(\sqrt{a_1})$, then by 8, $Q_2 = (a_1, c)$ with $c \in k^\times$ some norm. We want to use information about our curve so notice $Q_2 \otimes_k k(C) = (a_1, c)$ with $c = N_{L(C)/k(C)}(f)$ also by 8 for some $f \in L(C)$.

We want to understand f. Since $C_L \cong \mathbb{P}^1_L$, then we have:

$$0 \to L(C)^\times / L^\times \to \mathrm{Div}(C_L) \to \mathbb{Z} \to 0$$

The group $G = \mathrm{Gal}(L/k) = \{1, \sigma\}$ acts on this SES equivariantly. We have that $\sigma^2 = 1$, so we get:

$$\mathrm{Div}(C_L) = \bigoplus_{P \in C_L} \mathbb{Z}P = \left(\bigoplus_{P = \sigma(P)} \mathbb{Z}P \right) \oplus \left(\bigoplus_{P \neq \sigma(P)} \mathbb{Z}P \right)$$

write $\mathrm{div} f = E_1 + E_2$ according to this direct sum decomposition. Then we have:

$$(1 + \sigma)\mathrm{div} f = \mathrm{div}(f\sigma(f)) = \mathrm{div}(c) = 0$$

Therefore $E_1 + E_2 = -\sigma(E_1) - \sigma(E_2)$. We know that $\sigma(E_1) = E_1$, so by the direct sum decomposition, $E_1 = -E_1$ and $E_2 = -\sigma(E_2)$, therefore $E_1 = 0$ and E_2 is of the form $E_2 = \sum_i P_i - \sigma(P_i)$. Let $D = \sum_i P_i$ so that $\mathrm{div} f = (1 - \sigma)D$. Let $d = \deg D$, then consider the point $P = (1 : 0 : \sqrt{a_1})$ in C_L. It follows that $\deg(D - dP) = 0$, so by the SES, we have that $D - dP = \mathrm{div} g$ for some g. Let $f' = fg^{-1}\sigma(g)$, then

$$N_{L(C)/k(C)}(f') = fg^{-1}\sigma(g)\sigma(f)\sigma(g^{-1})g = f\sigma(f) = c$$

Now we have that:

$$\mathrm{div} f' = \mathrm{div} f - \mathrm{div} g + \sigma \mathrm{div} g = (1 - \sigma)D - D + dP + \sigma D - d\sigma P = d(1 - \sigma)P$$

Now consider $h = \frac{z - \sqrt{a_1}x}{y} \in L(C)^\times$. A fairly easy computation shows that $\mathrm{div} h = (1 - \sigma)P$, therefore $\mathrm{div} f' = d\mathrm{div} h = \mathrm{div} h^d$, so by our SES, $f' = h^d c_0$ for some $c_0 \in L^\times$. It follows that:

$$c = N_{L(C)/k(C)}(f') = N_{L(C)/k(C)}(h)^d N_{L(C)/k(C)}(c_0) = \left(\frac{(z - \sqrt{a_1}x)(z + \sqrt{a_1}x)}{y^2} \right)^d N_{L/k}(c_0)$$

Notice that in $L(C)$ we have $z^2 - a_1x^2 = b_1y^2$, so $N_{L(C)/k(C)}(h) = b_1$, therefore $c = b_1^d N_{L/k}(c_0)$. It follows that $Q_2 = (a_1, b_1^d N_{L/k}(c_0)) = (a_1, b_1^d)$ by 9, and by 3, $(a_1, b_1^d) = (a_1, b_1^{d \pmod 2})$. If $2 \mid d$, then $Q_2 = (a_1, 1)$ is split, which contradicts our assumption, thus $Q_2 = (a_1, b_1) = Q_1$ as desired.

Corollary: There are only two quaternion algebras over $\mathbb{R}$. Since positive numbers are all squares in $\mathbb{R}$, then (a, b) is isomorphic to one of $(1, 1), (1, -1)$, or $(-1, -1)$. We have that $(1, 1) \cong (1, -1) \cong M_2(\mathbb{R})$ and $(-1, -1)$ corresponds to the pointless curve. Therefore the only curves of genus 0 over $\mathbb{R}$ are $\mathbb{P}^1_{\mathbb{R}}$ and the pointless conic $V(x^2 + y^2 + z^2)$.