Étale Cohomology Seminar: Cohomology of Curves and Surfaces

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1 Introduction

We will cover the following material:

1. Poincaré duality for curves.
2. Lefschetz trace formula.
3. Lefschetz pencils.
4. Picard-Lefschetz formula.

2 Poincaré Duality

Poincaré Duality

Let $k = \overline{k}$ and $n, \operatorname{char} k$ coprime. Let X be a smooth projective curve over k. Let $\operatorname{Ext}^r_U(F, F') = \operatorname{Ext}^r_{\mathbf{S}(U_{ét}, \underline{\mathbb{Z}/n\mathbb{Z}})}(F, F')$. Then,

a) For any nonempty open $U \subseteq X$, there is a canonical isomorphism $H^2_c(U, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$.

b) For any constructible sheaf F of $\mathbb{Z}/n\mathbb{Z}$-modules on such a U, the groups $H^r_c(U, F)$ and $\operatorname{Ext}^r_U(F, \mu_n)$ are finite for all r and zero for $r > 2$; the canonical pairings

$$H^r_c(U, F) \times \operatorname{Ext}^{2-r}_U(F, \mu_n) \to H^2_c(U, \mu_n) = \mathbb{Z}/n\mathbb{Z}$$

are nondegenerate.

We recall the definition of cohomology with compact support and the canonical pairing on Ext.

For any separated variety X, suppose we have an embedding $j : X \hookrightarrow \overline{X}$ with $\overline{X}$ a proper variety. Then we define $H^r_c(X, F) = H^r(\overline{X}, j_! F)$. $j_!$ is the extension by 0 functor defined as the left adjoint of j_*. We can equivalently say $j_! F = \ker(j_* F \to i_* i^* j_* F)$ where i is the inclusion of $\overline{X} - X \to \overline{X}$.

We now construct the canonical pairing on Ext groups. Let A be a sheaf of rings on $X_{ét}$. Let F, G, R be sheaves of A-modules. Let $\operatorname{Ext}^r = \operatorname{Ext}^r_{S(X_{ét}, A)}$ be the left derived functors of $\operatorname{Hom}_A(F, -)$. We can describe an element of $\operatorname{Ext}^r(F, G)$ as follows: Let $G \to I^\bullet$ be an injective resolution of G by injective A-modules. Let F denote the complex concentrated in degree 0 with F. Let $I^\bullet[r]$

be the chain complex with I^{r+s} in degree s and differential $d[r]^s = (-1)^r d^{r+s}$. A morphism of chain complexes $f : F \to I^\bullet[r]$ is a morphism $f : F \to I^r$ such that $d[r]^0 f = 0$. It follows that $f \in \ker((d^r)^*)$ where $(d^r)^* : \mathrm{Hom}(F, I^r) \to \mathrm{Hom}(F, I^{r+1})$, therefore f defines an element of $\mathrm{Ext}^r(F,G)$. Similarly any element of $\mathrm{Ext}^r(F,G)$ is a morphism $f : F \to I^r$ such that $d[r]^0 f = 0$ and thus defines a chain morphism. f is chain homotopic to 0 iff $f = d[r]^{-1}s$ for some morphism $s : F \to I^{r-1}$, i.e. iff f is in the image of the map $\mathrm{Hom}(F, I^{r-1}) \to \mathrm{Hom}(F, I^r)$. It follows that Ext is homotopy classes of morphism $f : F \to I^r$ with $d[r]^0 f = 0$.

If instead we have a resolution $F \xrightarrow{\epsilon} F^\bullet$, then a morphism $f^\bullet : F^\bullet \to I^\bullet$ yields a morphism $f = f^0\epsilon$ such that $d[r]^0 f = f^0 d^0 \epsilon = 0$. Therefore we obtain an element of $\mathrm{Ext}^r(F,G)$. For any $f \in \mathrm{Ext}^r(F,G)$, we have that $f : F \to I^r$, since $\epsilon : F \to F^0$ is injective, then by injectivity of I^r, f lifts to $f^0 : F^0 \to I^r$, then since $I^\bullet[r]$ is injective, we obtain a morphism $f^\bullet : F^\bullet \to I^\bullet$ with f^0 as before. If $s : F^\bullet \to I^\bullet[r-1]$ is a chain homotopy, then $f^0 = d[r]^{-1}s^0 + s^1 d^0$, so $f = f^0\epsilon = d[r]^{-1}s^0\epsilon + s^1 d^0 \epsilon = d[r]^{-1}s^0\epsilon$ is in the image of $(d[r]^{-1})^*$ and thus 0. Therefore $\mathrm{Ext}^r(F,G)$ is homotopy classes of chain maps $f^\bullet : F^\bullet \to I^\bullet[r]$.

Now we define the canonical pairing on Ext. We want a pairing $\mathrm{Ext}^r(F,G) \times \mathrm{Ext}^s(G,R) \to \mathrm{Ext}^{r+s}(F,R)$. Let $G \to G^\bullet, R \to R^\bullet$ be injective resolutions, the given $f \in \mathrm{Ext}(F,G), g^\bullet \in \mathrm{Ext}(G,R)$, $f : F \to G^\bullet[r]$, then $g^\bullet : G^\bullet \to R^\bullet[s]$, then $g^\bullet[r] : G^\bullet[r] \to R^\bullet[r+s]$ so we get $g^\bullet[r] \circ f : F \to R^\bullet[r+s]$ which is an element of $\mathrm{Ext}^{r+s}(F,R)$. One checks that if $f \sim f', g \sim g'$, then $gf \sim g'f'$.

If $F = A$, then $\mathrm{Hom}_A(A,F)$ is naturally isomorphic to $\Gamma(X,F)$ since any morphism $A \to F$ is determined by global sections (A is a rank 1 free A-module). Since $\mathrm{Ext}^r(A,-)$ and H^r are both derived functors, it follows that $\mathrm{Ext}^\bullet(A,-)$ is isomorphic to $H^\bullet$ as δ-functors. Therefore we obtain the canonical pairing:

$$H^s(X_{\acute{e}t}, F) \times \mathrm{Ext}^r(F,G) \to H^{r+s}(X_{\acute{e}t}, G)$$

I will now describe the other map in Poincaré duality. We need an isomorphism $\eta : H^2_c(U_{\acute{e}t}, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$. We have a short exact sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$. Taking the LES in cohomology we get:

$$H^1(X_{\acute{e}t}, \mathbb{G}_m) \xrightarrow{n} H^1(X_{\acute{e}t}, \mathbb{G}_m) \to H^2(X_{\acute{e}t}, \mu_n) \to H^2(X_{\acute{e}t}, \mathbb{G}_m)$$

For curves over algebraically closed fields Tsen's theorem implies that $H^2(X_{\acute{e}t}, G_m) = 0$. Furthermore $H^1(X_{\acute{e}t}, \mathbb{G}_m) = H^1(X_{Zar}, \mathbb{G}_m) = \mathrm{Pic}(X)$ since $H^1(X_{Zar}, \mathbb{G}_m)$ are $\mathbb{G}_m$-torsor, e.g. line bundles, up to isomorphism. We thus have the following diagram:

$$\begin{tikzcd}
 & H^1(X_{\acute{e}t}, \mathbb{G}_m) \arrow[r, "n"] \arrow[d, "\sim"'] & H^1(X_{\acute{e}t}, \mathbb{G}_m) \arrow[r] \arrow[d, "\sim"'] & H^2(X_{\acute{e}t}, \mu_n) \arrow[r] \arrow[d, "\eta"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times n"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

And η is given by the quotient map composed with the morphism $H^1 \to \mathbb{Z}$. η is clearly surjective and by the four lemma is an isomorphism.

For $U \subseteq X$ open, let $Z = X - U$ with the reduced induced subscheme structure, so $Z = \coprod_{i=1}^n k$. Let $i : U \to X, j : Z \to X$ be the inclusions, then we have a SES:

$$0 \to i_! i^* \mu_n \to \mu_n \to j_* j^* \mu_n \to 0$$

Taking cohomology we get:

$$H^{r-1}(X_{\acute{e}t}, j_* j^* \mu_n) \to H^r(X_{\acute{e}t}, i_! i^* \mu_n) \to H^r(X_{\acute{e}t}, \mu_n) \to H^r(X_{\acute{e}t}, j_* j^* \mu_n)$$

Since f is affine, then $R^s j_* = 0$ for $s \geqslant 1$, so $H^r(X_{ét}, j_* j^* \mu_n) = H^r(Z_{ét}, \mu_n) = 0$ for $r \geqslant 1$ since Z is a disjoint union of points over a separably closed field (hence the cover $\{x_i\}$ for each point $x_i \in Z$ refines any étale cover). It follows that plugging in $r = 2$ into our exact sequence, $H_c^r(U_{ét}, \mu_n) \cong H^r(X_{ét}, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$.

3 Lefschetz Trace Formula

Let $F = \{F_n\}$ be an l-adic sheaf, then $H^r(X, F \otimes \mathbb{Q}_l) = H^r(X, F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Furthermore we let $H^r(X, \mathbb{Q}_l) = H^r(X, \mathbb{Z}_l \otimes \mathbb{Q}_l)$ where $\mathbb{Z}_l = \left\{ \underline{\mathbb{Z}/l^n\mathbb{Z}} \right\}$ with the natural quotient maps. For any affine morphism $\varphi : X \to Y$, for any sheaf F on Y, we have a map $F \to \varphi_* \varphi^* F$ which yields a morphism $H^r(Y_{ét}, Y) \to H^r(Y_{ét}, \varphi_* \varphi^* F)$. Since φ is affine, then $R^i \varphi_* = 0$ for $i > 0$, therefore $H^r(Y, \varphi_* \varphi^* F) = H^r(X, \varphi^* F)$. Now suppose $\varphi : X \to X$ nonconstant with X a curve, then $\varphi^* \mathbb{Z}_l = \mathbb{Z}_l$, so we get a morphism $\varphi^* : H^r(X, \mathbb{Q}_l) \to H^r(X, \mathbb{Q}_l)$. We let $\mathrm{Tr}_l^r(\varphi)$ be the trace of this morphism of finite dimensional $\mathbb{Q}_l$-vector spaces.

Lefschetz Trace Formula

Let $k = \overline{k}$, X a smooth projective curve over k, $\varphi : X \to X$ nonconstant $\Gamma_\varphi \cdot \Delta$ is the number of fixed points of φ. Then for any $l \neq \operatorname{char} k$, we have,

$$\Gamma_\varphi \cdot \Delta = \sum_{r=0}^{2} (-1)^r \, \mathrm{Tr}_l^r(\varphi)$$

Notice that by definition of φ^*, it is the identity on $H^0(X, \mathbb{Q}_l) = \mathbb{Q}_l$, so $\mathrm{Tr}_l^0(\varphi) = 1$. Since k is algebraically closed, we may fix an isomorphism $\mu_{l^n} \to \mathbb{Z}/l^n\mathbb{Z}$ for each n such that $\mathbb{Z}_l(1) = \varprojlim \mu_{l^n} \cong \mathbb{Z}_l$. It follows that $H^r(X, \mathbb{Q}_l(1)) = H^r(X, \mathbb{Q}_l) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(1) \cong H^r(X, \mathbb{Q}_l)$ and this isomorphism leaves the trace intact. According to Milne, $H^1(X, \mathbb{Q}_l(1)) \cong V_l(J(k)) := \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, J(k)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ where J is the Jacobian of X, e.g. the kernel of $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$. The map φ^* is the map $J(\varphi)$ under this isomorphism. As for $H^2(X, \mathbb{Q}_l(1))$, we have that $H^2(X_{ét}, \mu_{l^n}) \cong \mathrm{Pic}(X)/l^n \mathrm{Pic}(X)$ since we have the exact sequence

$$H^1(X_{ét}, \mathbb{G}_m) \xrightarrow{l^n} H^1(X_{ét}, \mathbb{G}_m) \to H^2(X_{ét}, \mu_n) \to 0$$

Furthermore, according to Mumford, $\mathrm{Pic}_0(X)$ is q-divisible for all q coprime to $\operatorname{char} k$, so in particular, $l^n : \mathrm{Pic}_0(X) \to \mathrm{Pic}_0(X)$ is surjective, therefore $\mathrm{Pic}(X) = \mathrm{Pic}(X) \times \mathbb{Z}$ implies that $\mathrm{Pic}(X)/l^n \mathrm{Pic}(X) \cong \mathbb{Z}/l^n\mathbb{Z}$. Therefore $H^2(X, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l$ and the map φ^* is given by its action on the degree, e.g. φ^* acts by multiplication by $\deg \varphi$ on $H^2(X, \mathbb{Q}_l(1))$. It follows that:

$$\Gamma_\varphi \cdot \Delta = 1 - \mathrm{Tr}(J(\varphi)) + \deg \varphi$$

This is very reminiscent of the case for elliptic curves, where $\Gamma_\varphi \cdot \Delta = \deg(1 - \varphi) = 1 - \mathrm{Tr}(\varphi_l) + \deg \varphi$ where φ_l is the action on φ on the Tate modules of the elliptic curve.

4 Lefschetz Pencils

Let $\check{\mathbb{P}}^m$ be the dual space of hyperplanes in $\mathbb{P}^m$. For any line D in $\check{\mathbb{P}}^m$ the axis of D is the intersection $\bigcap_{H \in D} H$ of all hyperplanes in D. Given a line D in $\check{\mathbb{P}}^m$, the family $\{H \cap X\}_{H \in D}$ is called a Lefschetz pencil on $X \hookrightarrow \mathbb{P}^m$ if

1. The axis of D intersects X transversely
2. There exists a dense open subset U of D such that for all $H \in U$, H intersects X transversely.
3. For all H in $D - U$, $H \cap X$ is smooth except at a single node.

Existence of Lefschetz Pencils

Let X be a smooth projective surface over $k = \overline{k}$. There exists a surface X^* given by a finite sequence of blowups of X and a map $\pi : X^* \to \mathbb{P}^1$ such that π is proper,flat and has a section. The generic fiber of π is a smooth curve and the closed fibers are connected with at most a single node as singularity.

Proof. Embed $X \hookrightarrow \mathbb{P}^m$ and let $V \subseteq X \times \check{\mathbb{P}}^m$ be the subvariety of pairs (x, H) where $x \in X$ and H is a hyperplane of $\mathbb{P}^m$ containing the tangent space $T_x X$. Since X is smooth, then this guarantees that $X \cap H$ is singular at x for all $(x, H) \in V$. Let $\check{X}$ be the image of V under the projection $X \times \check{\mathbb{P}}^m \to \check{\mathbb{P}}^m$. $\check{X}$ is irreducible, proper, and of codimension $\geqslant 1$. The projection $V \to \check{\mathbb{P}}^m$ is unramified at (x, H) iff x is a node on $X \cap H$. We may assume that $V \to \check{\mathbb{P}}^m$ is generically unramified, therefore the set F of points of $\check{X}$ where $H \cap X$ has more than a node is codimension $\geqslant 2$ in $\check{\mathbb{P}}^m$. Choose a hyperplane H_0 such that $H_0 \cap X$ is smooth and irreducible (Bertini). There is an open subset of $\check{\mathbb{P}}^m$ intersecting $H_0 \cap X$ transversely. There is an $m-1$ dimensional collections of lines through H_0 and only an $m-2$ dimensional collection of lines through H_0 which intersect F since F has codimension $\geqslant 2$, therefore we may choose an H_∞ such that the line D through H_0 and H_∞ does not intersect F. The collection $\{H \cap X\}_{H \in D}$ is our Lefschetz pencil. The intersection of any two distinct elements $H, H' \in D$ is the axis of D which is codimension 2, It follows that $X \cap L$ where L is the axis of D is a finite collection of points $Q_1, \cdots, Q_n$. Blowing up at those points we get a surface X^* which extends the rational map $X \to D$ given by taking a point $x \in X - L$ and sending it to the unique hyperplane in D containing x. This map is well-defined away from the axis of D, where all hyperplanes in D meet. It follows that upon blowing up, we obtain a well-defined map $X^* \to D$. This is the desired map.

5 Picard-Lefschetz Formula

Let $\pi : X \to \mathbb{P}^1$ be a Lefschetz pencil, e.g. proper, flat, has a section, the generic fiber is a smooth curve and the closed fibers are connected with at most a single node. Let T be the closed subset of $\mathbb{P}^1$ where the fibers are not smooth.

Let $F = R^1\pi_*\mu_n$. Then $F_{\overline{s}} = \mathrm{Pic}(X_{\overline{s}})[n]$.Let $V_s = \ker(F_s \to \mathrm{Pic}(X'_s))$ where X'_s is the normalization of X_s. V_s are called the vanishing cycles at s. If $s \notin T$, then $F_{\overline{s}} = \mathrm{Pic}(X_s)$ and X_s is already normal, so $V_s = 0$, therefore $V_s \neq 0$ only for $s \in T$. We have the cospecialization map $\varphi_s : F_{\overline{s}} \to F_{\overline{\eta}}$ where η is the generic point of $\mathbb{P}^1$. We may therefore identify V_s and $F_{\overline{s}}$ with their images in $F_{\overline{\eta}}$. In fact, given our embedding $\mathcal{O}^{sh}_{\mathbb{P}^1,s} \to k(\overline{\eta})$ used for the cospecialization map, we get the inertia group $I_s \subseteq \mathrm{Gal}(k(\overline{\eta})/k(\eta))$ and $\varphi_s(F_{\overline{s}}) = F_{\overline{\eta}}^{I_s}$. There is a non-degenerated, skew-symmetric pairing $e_n : F_{\overline{\eta}} \times F_{\overline{\eta}} \to \mu_n(k)$ (I believe this comes from the Weil pairing).

Now suppose that π in fact has irreducible fibers, then for $s \in T$ we get that $V_s = \mu_n(k)$ and so

$$V_s(-1) = \mu_n(k) \otimes_{\mathbb{Z}/n\mathbb{Z}} \mu_n(k)\check{} = \mathbb{Z}/n\mathbb{Z}$$

Therefore there is a canonical element $1 \in V_s(-1)$. The map $\varphi_s : F_{\overline{s}} \to F_{\overline{\eta}}$ yields a map $\varphi(-1)_s : F(-1)_{\overline{s}} \to F(-1)_{\overline{\eta}}$. The image of $1 \in V_s(-1)$ under this map is denoted δ_s and is called the canonical vanishing cycle at s. The pairing e_n yields a pairing $F(-1)_{\overline{\eta}} \times F(-1)_{\overline{\eta}} \to (\mathbb{Z}/n\mathbb{Z})(-1)$. We write $(\gamma, \gamma') \mapsto \gamma \cdot \gamma'$.

Let t be a uniformizer at s, then for any $\sigma \in I_s$, we have that $\sigma(t^{1/n}) = \varepsilon_s(\sigma) t^{1/n}$. This gives a character $\varepsilon_s : I_s \to \mu_n(k)$.

Picard-Lefschetz Formula

There exists a unit $\lambda_X = (\lambda_X(n)) \in \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z}$ such that

$$\sigma(\gamma) = \gamma(y) - \lambda_X(n) \varepsilon_s(\sigma)(\gamma \cdot \delta_s)\delta_s$$

for all $\sigma \in I_s, \gamma \in F(-1)_{\overline{\eta}}$.