MAT417 Suggested Exercises

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1.2.7

Show that

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$

Proof:

We note that $\mu(n)$ is a multiplicative function and thus $|\mu(n)|$ is also a multiplicative function, so,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} &= \prod_p (1 + |\mu(p)|p^{-s} + |\mu(p^2)|p^{-2s} + \cdots) \\
&= \prod_p (1 + p^{-s}) \\
&= \prod_p \frac{1 - p^{-2s}}{1 - p^{-s}} \\
&= \frac{\zeta(s)}{\zeta(2s)}
\end{aligned}$$

1.4.1

Show that the average order of $d(n)$ is $\log(n)$.

Proof:

We wish to show that $A(x) = \sum_{n \leqslant x} d(n)$ satisfies:

$$\lim_{x\to\infty} \frac{A(x) - x\log(x)}{x\log(x)} = 0$$

We may rewrite $\sum_{n\leqslant x} d(n)$ as $\sum_{ab\leqslant x} 1$. We may again rewrite this as $\sum_{a=1}^{x} \left\lfloor \frac{x}{a} \right\rfloor = \sum_{a=1}^{x} \left(\frac{x}{a} - \left\{ \frac{x}{a} \right\} \right) = xH_x + O(x)$. Now $H_x - \log x = O(1)$, so,

$$\lim_{x\to\infty} \frac{A(x) - x\log(x)}{x\log(x)} = \lim_{x\to\infty} \frac{xH_x + O(x) - x\log(x)}{x\log(x)} = \lim_{x\to\infty} \frac{H_x - \log(x)}{\log(x)} = \lim_{x\to\infty} \frac{O(1)}{\log(x)} = 0$$

1.5.1

Prove that

$$\sum_{\substack{n\leqslant x\\(n,k)=1}} \frac{1}{n} \sim \frac{\varphi(k)}{k}\log x$$

as $x \to \infty$

Proof:

Note that $\frac{1}{x+n} - \frac{1}{x} = O(\frac{1}{x^2})$ We have that,

$$\sum_{\substack{n\leqslant x\\(n,k)=1}} \frac{1}{n} = \sum_{\substack{m\leqslant k\\(m,k)=1}} \sum_{n\leqslant\lfloor x/k\rfloor} \frac{1}{kn+m} + O(1) = \frac{1}{k}\sum_{\substack{m\leqslant k\\(m,k)=1}} \sum_{n\leqslant\lfloor x/k\rfloor} \frac{1}{n} + O(1) = \frac{\varphi(k)}{k}\sum_{n\leqslant\lfloor x/k\rfloor}\frac{1}{n} + O(1) \sim \frac{\varphi(k)}{k}\log x$$

1.5.5

Let $d_k(n)$ be the number of ways of writing n as the product of k positive numbers. Show that,

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta^k(s)$$

Proof:

$d_k(n)$ is the number of ways of splitting each prime factor p^r into k parts, i.e. there are $\binom{r+k-1}{k-1}$ ways for each prime factor, and then a product is taken over each of the prime factors. It follows immediately that $d_k(n)$ is multiplicative, but since adding r's in $\binom{r+k-1}{k-1}$ does not multiply, then it is not completely multiplicative. This yields,

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \prod_p (1 + d_k(p)p^{-s} + d_k(p^2)p^{-2s} + \cdots)$$

As was just stated, $d_k(p^r) = \binom{r+k-1}{k-1}$. It follows that,

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \prod_p (1 + \binom{1+k-1}{k-1} p^{-s} + \binom{2+k-1}{k-1} p^{-2s} + \cdots)$$

Now notice that the coefficient on p^{-ls} in $(1 + p^{-s} + p^{-2s} + \cdots)^k$ is also just the number of ways of adding k numbers to l and thus is $\binom{l+k-1}{k-1}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \prod_p (1 + p^{-s} + p^{-2s} + \cdots)^k = \left(\prod_p (1-p^{-s})^{-1}\right)^k = \zeta^k(s)$$

1.5.9

Prove that

$$\sum_{\substack{d|n\\ \nu(d)\leqslant r}} \mu(d) = (-1)^r \binom{\nu(n)-1}{r}$$

where $\nu(n)$ denotes the number of distinct prime factors of n.

Proof:

We will first find the sum when $\nu(d) = r$. For $r = 0$, this the sum is just 1, so let $f(0, n) = 1$. Now for $r + 1$, let $n = p_1^{a_1} \cdots p_k^{a_k}$ where $k = \nu(n)$, then

$$\sum_{\substack{d|n \\ \nu(d)=r+1}} \mu(d) = \frac{1}{r+1} \sum_{i=1}^{k} \sum_{j=1}^{a_i} \sum_{\substack{d|n/p_i^{a_i} \\ \nu(d)=r}} \mu(dp_i^j)$$

Note that we have a $\frac{1}{r+1}$ since there are $r + 1$ ways to obtain a number with $r + 1$ distinct prime divisors from ones with r distinct prime divisors. $\mu(dp_i^j) = 0$ if $j \geqslant 2$ and $-\mu(d)$ if $j = 1$, so we may simplify the sum as follows:

$$\sum_{\substack{d|n \\ \nu(d)=r+1}} \mu(d) = \frac{-1}{r+1} \sum_{i=1}^{k} \sum_{\substack{d|n/p_i^{a_i} \\ \nu(d)=r}} \mu(d) = \frac{-1}{r+1} \sum_{i=1}^{k} f(r, n/p_i^{a_i})$$

Now assume that $f(r, n)$ depends only on $\nu(n)$ and not n itself, then $\nu(n/p_i^{a_i}) = \nu(n) - 1$, so we get that,

$$f(r + 1, n) = \frac{-\nu(n)}{r + 1} f(r, n/p_1^{a_1})$$

Notice that this assumption of dependence only on $\nu(n)$ is fine since in this recurrence, we only have the value of $\nu(n)$, not n itself. Define $g(r, l) = f(r, n_l)$ where n_l is any number with l prime factors, then our recurrence above shows that,

$$g(r + 1, k) = \frac{-k}{r + 1} g(r, k - 1)$$

We know that $g(0, k) = 1 = (-1)^0 \binom{k}{0}$. Assume inductively that $g(r, k) = (-1)^r \binom{k}{r}$, then,

$$g(r + 1, k) = (-1)^r \frac{-k}{r + 1} \binom{k - 1}{r} = (-1)^{r+1} \binom{k}{r + 1}$$

Therefore the induction holds. It follows that,

$$\sum_{\substack{d|n \\ \nu(d)\leqslant r}} \mu(d) = \sum_{i=0}^{r} (-1)^i \binom{k}{i} = (-1)^r \binom{k - 1}{r}$$

This can be shown by a simple induction which I will omit.

1.5.10

Let $\pi(x, z)$ denote the number of $n \leqslant x$ coprime to all primes $p \leqslant z$. Show that

$$\pi(x, z) = x \prod_{p \leqslant z} \left(1 - \frac{1}{p}\right) + O(2^z)$$

Proof:

The number of $n \leqslant x$ divisible by any k is exactly $\lfloor x/k \rfloor$. Let P_z be the product of all primes $\leqslant z$. By the inclusion-exclusion principle, we get,

$$\pi(x,z) = \sum_{k|P_z} \mu(k) \lfloor x/k \rfloor = \sum_{k|P_z} \mu(k) \left(\frac{x}{k} - \left\{ \frac{x}{k} \right\} \right) = \sum_{k|P_z} \mu(k) \frac{x}{k} - \sum_{k|P_z} \mu(k) \left\{ \frac{x}{k} \right\}$$

We can multiply the first sum by $\frac{P_z}{x}\frac{x}{P_z}$ and the second sum is $O(\sum_{k|P_z} |\mu(k)|)$ since $\left| - \left\{ \frac{x}{k} \right\} \right| \leqslant 1$. It follows that,

$$\pi(x,z) = \frac{x}{P_z} \sum_{k|P_z} \mu(k) \frac{P_z}{k} + O(\sum_{k|P_z} |\mu(k)|)$$

Now $\sum_{k|P_z} |\mu(k)| = \sum_{k|P_z} 1 = 2^{\pi(z)}$. Since $\pi(z) \leqslant z$, then this is $O(2^z)$. By the Morbius inversion formula, we know $n = \sum_{d|n} \varphi(d)$, thus, we have that,

$$\sum_{k|P_z} \mu(k) \frac{P_z}{k} = \varphi(P_z)$$

Thus we obtain the desired result,

$$\pi(x,z) = x \frac{\varphi(P_z)}{P_z} + O(2^z) = x \prod_{p \leqslant z} \left(1 - \frac{1}{p} \right) + O(2^z)$$

2.3.2

Show that for $s > 1$,

$$\sum_{\chi \,(\mathrm{mod}\ q)} \log(L(s,\chi)) = \varphi(q) \sum_{n \geqslant 1} \sum_{p^n \equiv 1 \,(\mathrm{mod}\ q)} \frac{1}{np^s}$$

Proof:

We have that $\log(L(s,\chi)) = -\prod_p \log(1 - \chi(p)p^{-s})$. Since $s > 1$, then $p^{-s} < 1$ and hence $|\chi(p)p^{-s}| < 1$, so we have $\log(1 - \chi(p)p^{-s}) = -\sum_{n \geqslant 1} \frac{\chi(p)^n}{np^{ns}} = \sum_{n \geqslant 1} \frac{\chi(p^n)}{np^{ns}}$. Now taking a sum over χ, we get,

$$\sum_{\chi \,(\mathrm{mod}\ q)} \log(L(s,\chi)) = \sum_{n \geqslant 1} \frac{1}{np^{ns}} \sum_{\chi \,(\mathrm{mod}\ q)} \chi(p^n)$$

Since $\sum_{\chi(\mathrm{mod}\ q)} \chi(n)$ is $\varphi(q)$ if $n \equiv 1 (\mathrm{mod}\ q)$ or 0 otherwise, then we get that,

$$\sum_{\chi \,(\mathrm{mod}\ q)} \log(L(s,\chi)) = \varphi(q) \sum_{n \geqslant 1} \sum_{p^n \equiv 1 \,(\mathrm{mod}\ q)} \frac{1}{np^{ns}}$$

2.3.7

If $L(1,\chi) \neq 0$ for every $\chi \neq \chi_0$, deduce that,

$$\lim_{s \to 1^+} (s-1) \prod_{\chi \,(\mathrm{mod}\ q)} L(s,\chi) \neq 0$$

and hence,

$$\sum_{p\equiv 1 \pmod q} \frac{1}{p} = +\infty$$

Proof:

We need only show that $\lim_{s\to 1^+}(s-1)L(s,\chi_0) \neq 0$. We know that $L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-p^{-s})$, then since $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$, and the rest is a finite product, we have the result. From the previous result (2.3.2), it follows that,

$$\lim_{s\to 1^+}\left(\log(s-1) + \sum_{\chi \pmod q} \log(L(s,\chi))\right)$$

is finite and nonzero. Therefore if we remove $\log(s-1)$, then it must be $+\infty$. It follows that,

$$\lim_{s\to 1^+}\sum_{n\geqslant 1}\sum_{p^n\equiv 1 \pmod q} \frac{1}{np^{ns}} = \infty$$

Now for each p, we have that $\sum_{n\geqslant 2}\frac{1}{np^{ns}} \leqslant \sum_{n\geqslant 2} p^{-n} = \frac{p^{-2}}{1-p^{-1}} \leqslant p^{-2}$ and $\sum_p p^{-2}$ converges. It follows that $\lim_{s\to 1^+}\sum_{p\equiv 1 (\mathrm{mod}\ q)}\frac{1}{p} = +\infty$ and therefore,

$$\sum_{p\equiv 1 \pmod q} \frac{1}{p} = \infty$$

2.3.9

Fix $(a,q) = 1$. If $L(1,\chi) \neq 0$, show that

$$\lim_{s\to 1^+}(s-1)\prod_{\chi \pmod q} L(s,\chi)^{\overline{\chi(a)}} \neq 0$$

. Deduce that

$$\sum_{p\equiv a \pmod q} \frac{1}{p} = +\infty$$

Proof:

Since each $L(1,\chi) \neq 0$, then $L(1,\chi)^{\overline{\chi(a)}}$ will have a nonzero product ranging over $\chi \neq \chi_0$. For χ_0, $\chi_0(a) = 1$ since $(a,q) = 1$ and therefore $(s-1)L(s,\chi_0) \to 1$ as $s \to 1^+$, thus the product is nonzero.

We have that

$$\begin{aligned}\sum_{\chi \pmod q} \log(L(s,\chi)^{\overline{\chi(a)}}) &= -\sum_{\chi \pmod q} \overline{\chi(a)}\sum_p \log(1-\chi(p)p^{-s}) \\ &= \sum_{\chi \pmod q}\sum_{n\geqslant 1}\sum_p \frac{\overline{\chi(a)}\chi(p^n)}{np^{ns}} \\ &= \sum_{n\geqslant 1}\sum_p \frac{1}{np^{ns}}\sum_{\chi \pmod q}\chi(p^n a^{-1})\end{aligned}$$

We have that $\chi(p^n a^{-1})$ is $\varphi(q)$ if $p^n \equiv a (\text{mod } q)$ and 0 otherwise. It follows that this sum is simply,

$$\varphi(q) \sum_{n \geqslant 1} \sum_{p^n \equiv a \,(\mathrm{mod}\ q)} \frac{1}{np^{ns}}$$

Now following the same reasoning as the previous question, we get that $\sum_{p \equiv a (\mathrm{mod}\ q)} \frac{1}{p} = \infty$.

2.4.3

Let χ be a real character (mod q). Define,

$$f(n) = \sum_{d|n} \chi(d)$$

Show that $f(1) = 1$ and $f(n) \geqslant 0$. In addition, show that $f(n) \geqslant 1$ whenever n is a perfect square.

Proof:

$\chi(1) = 1$, thus $f(1) = \chi(1) = 1$. Since χ is multiplicative, then so is f, thus we need only check that $f(n) \geqslant 0$ for prime powers. Let $n = p^k$, then $f(n) = \chi(1) + \chi(p) + \cdots + \chi(p^k)$. If $p \nmid q$, then we have three cases, $\chi(p) = 1$, then this is $k+1$. If $\chi(p) = -1$, then this is 0 for k odd and 1 for k even. If $p|q$, then $\chi(p) = 0$, so $f(p^k) = 1$. Now if n is a perfect square, then any prime will have an even power and thus $f(p^{2k}) \geqslant 1$, so $f(n) \geqslant 1$.

2.4.7

Let

$$a_n = \sum_{d|n} \chi(d)$$

where χ is a non principal character (mod q). Show that

$$\sum_{n \leqslant x} a_n = xL(1, \chi) + O(\sqrt{x})$$

Proof:

We will use Dirichlet's hyperbola method, with $g = \chi, h = 1, y = \sqrt{x}$, then $a_n = g * h$ and thus,

$$\sum_{n \leqslant x} a_n = \sum_{d \leqslant \sqrt{x}} g(d) \left\lfloor \frac{x}{d} \right\rfloor + \sum_{d \leqslant \sqrt{x}} G\left(\frac{x}{d}\right) - G(\sqrt{x}) \left\lfloor \sqrt{x} \right\rfloor$$

Since χ is non principal, then $|G(x)| \leqslant q$, so the last two terms are $O(\sqrt{x})$. Then we have,

$$\sum_{d \leqslant \sqrt{x}} g(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d \leqslant \sqrt{x}} \frac{\chi(d)}{d} + \sum_{d \leqslant \sqrt{x}} \chi(d) \left\{ \frac{x}{d} \right\}$$

Since $|\chi(d)| \leqslant 1$ and $\left|\frac{x}{d}\right| \leqslant 1$, then the last term is $O(\sqrt{x})$. The first term on the RHS is now $xL(1, \chi) - x \sum_{d > \sqrt{x}} \frac{\chi(d)}{d}$. To obtain the desired result, we need only bound that second term by $O(\sqrt{x})$. To do so, we will prove that $\sum_{d > x} \frac{\chi(d)}{d} = O(x^{-1})$. We have that $\chi(d)$ repeats every q

numbers and that $\sum_{n=0}^{q-1} \chi(n) = 0$, thus we want to use that to bound this sum. Suppose we are given $a_0, \cdots, a_k$ all with absolute value at most M, then,

$$\left| \sum_{i=0}^{k} \frac{a_i}{n+i} - \frac{\sum_{i=0}^{k} a_i}{n} \right| \leqslant M \sum_{i=0}^{k} \left(\frac{1}{n} - \frac{1}{n+i} \right) \leqslant M \sum_{i=0}^{k} \frac{k}{n^2} = \frac{Mk^2}{n^2}$$

We now use this to bound our sum,

$$\sum_{d>x} \frac{\chi(d)}{d} = \sum_{d=\lceil x \rceil}^{q\lceil \frac{x}{q} \rceil - 1} \frac{\chi(d)}{d} + \sum_{k=\lceil \frac{x}{q} \rceil}^{\infty} \sum_{i=0}^{q-1} \frac{\chi(kq+i)}{kq+i}$$

The first sum has q elements bound above in absolute value by $\frac{1}{x}$, thus is $O(x^{-1})$. To bound the second sum, by what we just proved and noting that $|\chi| \leqslant 1$ and $\sum_{i=0}^{q-1} \chi(kq+i) = 0$, we have that,

$$\left| \sum_{i=0}^{q-1} \frac{\chi(kq+i)}{kq+i} \right| \leqslant \frac{q^2}{k^2 q^2} = \frac{1}{k^2}$$

It follows that the second sum is also bound above by $O(x^{-1})$. Therefore $\sum_{d>x} \frac{\chi(d)}{d} = O(x^{-1})$. We now conclude that

$$x \sum_{d \leqslant \sqrt{x}} \frac{\chi(d)}{d} = xL(1,\chi) - xO(x^{-1/2}) = xL(1,\chi) + O(\sqrt{x})$$

as desired.

2.5.3

Let $A(x) = \sum_{n \leqslant x} a_n$. Show that for x a positive integer,

$$\sum_{n \leqslant x} a_n \log\left(\frac{x}{n}\right) = \int_1^x \frac{A(t)}{t} dt$$

Proof:

We can split the integral to evaluate it as a sum,

$$\int_1^x \frac{A(t)}{t} dt = \sum_{n=1}^{x} A(n-1) \int_{n-1}^{n} \frac{1}{t} dt = \sum_{n=1}^{x} A(n-1) \log\left(\frac{n}{n-1}\right)$$

now we can expand the sum and rewrite it in the desired form,

$$= \sum_{n=1}^{x} \sum_{k=1}^{n-1} a_k \log\left(\frac{n}{n-1}\right) = \sum_{k=1}^{x-1} a_k \sum_{n=k+1}^{x} \log\left(\frac{n}{n-1}\right) = \sum_{k=1}^{x-1} a_k \log\left(\frac{x}{k}\right)$$

2.5.9

Let $d(n)$ be the number of divisors of n. Show that for some constant c,

$$\sum_{n\leqslant x}\frac{d(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+c+O(\frac{1}{\sqrt{x}})$$

for $x\geqslant 1$.

Proof:

We proceed in two steps, first with a use of the hyperbola method, then using Euler-Maclaurin summation. Let $g=h=x^{-1}, y=\sqrt{x}$, then notice that,

$$\frac{d(n)}{n}=\sum_{d|n}\frac{1}{n}=\sum_{d|n}\frac{1}{d}\frac{d}{n}=\sum_{d|n}g(d)h\left(\frac{n}{d}\right)$$

Furthermore, let $f(n)=G(n)=H(n)$ be the harmonic numbers, then by the hyperbola method, we have,

$$\sum_{n\leqslant x}\frac{d(n)}{n}=\sum_{d\leqslant\sqrt{x}}h(d)G\left(\frac{x}{d}\right)+\sum_{d\leqslant\sqrt{x}}g(d)H\left(\frac{x}{d}\right)-G(\sqrt{x})H(\sqrt{x})$$

Now recall that $f(x)=\log(x)+\gamma+\frac{1}{2x}+O(x^{-2})$, thus $f(\sqrt{x})^2=\frac{1}{4}\log^2(x)+\gamma\log(x)+\frac{\log(x)}{2\sqrt{x}}+O(x^{-1/2})$. It follows that we may rewrite our sum as,

$$\sum_{n\leqslant x}\frac{d(n)}{n}=2\sum_{d\leqslant\sqrt{x}}\frac{f\left(\frac{x}{d}\right)}{d}-\frac{1}{4}\log^2(x)-\gamma\log(x)-\frac{\log(x)}{2\sqrt{x}}+O(x^{-1/2})$$

We now need only approximate the inner sum as follows,

$$\begin{aligned}\sum_{d\leqslant\sqrt{x}}\frac{1}{d}f\left(\frac{x}{d}\right)&=\sum_{d\leqslant\sqrt{x}}\frac{1}{d}\left(\log\left(\frac{x}{d}\right)+\gamma+\frac{d}{2x}+O(d^2x^{-2})\right)\\&=\log(x)f(\sqrt{x})-\sum_{d\leqslant\sqrt{x}}\frac{\log d}{d}+\gamma f(\sqrt{x})+\frac{1}{2\sqrt{x}}+O(x^{-1})\\&=\frac{1}{2}\log^2(x)+\gamma\log(x)+\frac{\log(x)}{2\sqrt{x}}+\frac{1}{2}\gamma\log(x)+\gamma^2-\sum_{d\leqslant\sqrt{x}}\frac{\log d}{d}+O(x^{-1/2})\end{aligned}$$

We now have that,

$$\sum_{n\leqslant x}\frac{d(n)}{n}=\frac{3}{4}\log^2(x)+2\gamma\log(x)+\frac{\log(x)}{2\sqrt{x}}+2\gamma^2-2\sum_{d\leqslant\sqrt{x}}\frac{\log d}{d}+O(x^{-1/2})$$

We now need only approximate the final remaining term,

$$\sum_{d\leqslant x}\frac{\log d}{d}$$

We will do this with Euler-Maclaurin summation. Let $f(t) = \frac{\log t}{t}$, then let $a = 1, b = x, k = 1$ and we have,

$$\begin{aligned}
\sum_{d \leqslant x} \frac{\log(d)}{d} &= \sum_{1<d\leqslant x} \frac{\log(d)}{d} \\
&= \int_1^x \frac{\log t}{t} dt - \frac{1}{1}\left(\frac{\log x}{x}\right)\left(\frac{-1}{2}\right) + \frac{1}{2}\left(\frac{1-\log x}{x^2} - 1\right)\left(\frac{-1}{6}\right) - \frac{1}{2}\int_1^x B_2(t)\frac{2\log t - 3}{t^3}dt \\
&= \frac{1}{2}\log^2(x) + \frac{\log x}{2x} + \frac{1}{12} - \frac{1-\log(x)}{12x^2} - \frac{1}{2}\int_1^x B_2(t)\frac{2\log t - 3}{t^3}dt
\end{aligned}$$

Now we have that $\int_x^\infty B_2(t)\frac{2\log t-3}{t^3} = O(x^{-1})$. Furthermore, $\int_1^\infty B_2(t)\frac{2\log t-3}{t^3}dt$ converges to some constant C and thus, $\int_1^x B_2(t)\frac{2\log(t)-3}{t^3}dt = C + O(x^{-1})$. It follows that,

$$\sum_{d\leqslant x} \frac{\log(d)}{d} = \frac{1}{2}\log^2(x) + \frac{\log(x)}{2x} + \frac{1}{12} + C + O(x^{-1})$$

Therefore,

$$\sum_{d\leqslant \sqrt{x}} \frac{\log(d)}{d} = \frac{1}{8}\log^2(x) + \frac{\log(x)}{4\sqrt{x}} + \frac{1}{12} + C + O(x^{-1/2})$$

and hence finally,

$$\sum_{n\leqslant x} \frac{d(n)}{n} = \frac{1}{2}\log^2(x) + 2\gamma\log(x) + 2\gamma^2 - \frac{1}{6} - 2C + O(x^{-1/2})$$

Simplifying the constant, we get,

$$\sum_{n\leqslant x} \frac{d(n)}{n} = \frac{1}{2}\log^2(x) + 2\gamma\log(x) + c + O(x^{-1/2})$$

3.1.8

Prove that

$$\sum_{p\leqslant x} \frac{1}{p} = \log\log x + O(1)$$

Proof:

We have that $\sum_{p\leqslant x}\frac{1}{p} = \sum_{p\leqslant x}\frac{\log p}{p}\frac{1}{\log p}$. Now we use partial summation with $a_n = 0$ if n is not prime and $a_n = \frac{\log n}{n}$ for n prime and $f(x) = \frac{1}{\log(x)}$. We know that $\sum_{p\leqslant x}\frac{\log p}{p} = \log x + O(1)$ by the previous exercise, thus it follows that,

$$\sum_{p\leqslant x}\frac{1}{p} = O(1) + \int_e^x \frac{\log(t) + O(1)}{t\log^2(t)}dt = \log\log x + O(1)$$

3.1.10

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and set

$$S(x) = \sum_{n \leqslant x} a_n$$

If

$$\lim_{x \to \infty} \frac{S(x)}{x} = \alpha$$

Show that

$$\sum_{n \leqslant x} \frac{a_n}{n} = \alpha \log x + o(\log(x))$$

as $x \to \infty$

Proof:

By partial summation we have that,

$$\sum_{n \leqslant x} \frac{a_n}{n} = \frac{S(x)}{x} + \int_1^x \frac{S(t)}{t^2} dt = O(1) + \int_1^x \frac{\alpha t + o(t)}{t^2} dt = \alpha \log x + o(\log x)$$

Note that $\frac{o(t)}{t^2} = o(t^{-1})$. Furthermore, if we have $f = o(g)$, then $\int f = o\left(\int g\right)$ assuming $f, g > 0$ and $\int_1^{\infty} g = \infty$. It follows that $\zeta(1 + it) \neq 0$.

3.2.5

Prove that for $\sigma > 1, t \in \mathbb{R}$,

$$\left|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)\right| \geqslant 1$$

Deduce that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}, t \neq 0$. Deduce in a similar way, by considering

$$\zeta(\sigma)^3 L(\sigma, \chi)^4 L(\sigma, \chi^2)$$

that $L(1, \chi) \neq 0$ for χ not real.

Proof:

Multiply our equation by $|\sigma - 1|^3$, then we get that,

$$|\left((\sigma - 1)\zeta(\sigma)\right)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geqslant |\sigma - 1|^3$$

Now let $\sigma \to 1^+$, then $(\sigma - 1)\zeta(\sigma) \to 1$ as $\sigma \to 1^+$. If $\zeta(\sigma + it) \to 0$, then the LHS is approaching 0 at least as fast as $|\sigma - 1|^4$ but that contradicts the inequality since the other terms are bounded.

To prove the case for $L(\sigma, \chi)$, we need to get a similar inequality that I am too lazy to figure out right now.

3.4.10

Show that

$$\nu(n) = O\left(\frac{\log n}{\log\log n}\right)$$

where $\nu(n)$ denotes the number of distinct prime factors of n.

Proof:

We know that $\nu(n)$ is at most $\pi(k)$ where k is the smallest number such that the primorial of k is greater than n. Furthermore, $\prod_{p\leqslant k} p = e^{\psi(k)} = e^{k+o(k)}$, thus we need to have $n = e^{k(1+o(1))}$ and hence $k = O(\log n)$. Then $\nu(n) \leqslant \pi(k) = \pi(O(\log n)) = O\left(\frac{\log n}{\log\log n}\right)$.

3.4.11

Let $\nu(n)$ be as in the previous exercise. Show that

$$\sum_{n\leqslant x} \nu(n) = x\log\log x + O(x)$$

Proof:

Let g denote the prime indicator function, i.e. $g(n) = 1$ if n is prime and 0 otherwise. We then have that $\nu(n) = g * 1$ where 1 is the constant 1 function. Notice that $G(x) = \sum_{n\leqslant x} g(n) = \pi(x)$. By Dirichlet's hyperbola method, we have that,

$$\sum_{n\leqslant x} \nu(n) = \sum_{n\leqslant\sqrt{x}} g(p)\left\lfloor\frac{x}{n}\right\rfloor + \sum_{n\leqslant\sqrt{x}} \pi(n) - \pi(\sqrt{x})\left\lfloor\sqrt{x}\right\rfloor$$

The first sum is just $x\sum_{p\leqslant\sqrt{x}}\frac{1}{p} + O(\sqrt{x})$. By a previous exercise, this is just $x\log\log\sqrt{x} + O(\sqrt{x})$. Furthermore, $x\log\log\sqrt{x} = x\log(\frac{1}{2}\log x) = x\log\log x - x\log 2 = x\log\log x + O(x)$. It follows that the first term is $x\log\log x + O(x)$. The second term is:

$$\sum_{n\leqslant\sqrt{x}} \pi(n) \leqslant \sum_{n\leqslant\sqrt{x}} n = O(x)$$

Finally, $\pi(\sqrt{x})\left\lfloor\sqrt{x}\right\rfloor \leqslant x$, so we have that $\sum_{n\leqslant x}\nu(n) = x\log\log x + O(x)$.

4.1.9

Let

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

be a Dirichlet series absolutely convergent in $\mathrm{Re}(s) > c - \epsilon$. Show that

$$\frac{1}{x^k}\sum_{n\leqslant x} a_n(x-n)^k = \frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f(s)x^s ds}{s(s+1)\cdots(s+k)}$$

for any $k \geqslant 1$

Proof:

Since $f(s)$ is absolutely convergent, we may interchange the integral and the sum,

$$\int_{c-i\infty}^{c+i\infty} \frac{f(s)x^s ds}{s(s+1)\cdots(s+k)} = \sum_{n=1}^{\infty} a_n \int_{c-i\infty}^{c+i\infty} \frac{n^{-s}x^s ds}{s(s+1)\cdots(s+k)} = \sum_{n=1}^{\infty} a_n \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s ds}{s(s+1)\cdots(s+k)}$$

We know that the integral evaluates to 0 when $x/n < 1$ and to $\frac{2\pi i}{k!}\left(1-\frac{n}{x}\right)^k$ when $x/n \geqslant 1$, i.e. when $x \geqslant n$. It follows that,

$$\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{f(s)x^s ds}{s(s+1)\cdots(s+k)} = x^{-k}\sum_{n\leqslant x} a_n(x-n)^k$$

4.3.2

Suppose that for any $\epsilon \geqslant 0$, we have $a_n = O(n^\epsilon)$. Prove that for any $c > 1$ and x not an integer,

$$\sum_{n\leqslant x} a_n = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{f(s)x^s}{s} ds + O\left(\frac{x^{c+\epsilon}}{R}\right) + O\left(\frac{x^\epsilon \log x}{R}\right)$$

where

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Proof:

Let $\delta(x)$ be 1 if $x > 1$, $\frac{1}{2}$ if $x = 1$ and 0 if $0 < x < 1$, then we have that $\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s} ds = \delta(x)$. We wish to bound, $\int_{c-iR}^{c+iR} \frac{x^s}{s} ds$. Suppose $0 < x < 1$, then let $U > c$, then consider the rectangle K_U with vertices $c \pm iR$ and $U \pm iR$, then since $\frac{x^s}{s}$ has no poles in this region, we have that,

$$0 = \int_{K_U} \frac{x^s}{s} ds \leqslant \int_{c-iR}^{c+iR} + \int_{c+iR}^{U+iR} + \int_{U+iR}^{U-iR} + \int_{U-iR}^{c-iR}$$

It follows that,

$$\int_{c-iR}^{c+iR} = -\int_{c+iR}^{U+iR} - \int_{U+iR}^{U-iR} - \int_{U-iR}^{c-iR}$$

We know that $\delta(x) = 0$, so we want to show that the RHS goes to 0 and bound how fast it does so. We have that $\int_{c+iR}^{U+iR} \frac{x^s}{s} ds \ll \frac{1}{R}\int_c^U x^t dt$ since $|s| > R$ and since $x^s = x^{t+iR}$ and $|x^{iR}| = 1$. It follows that $\int_{c+iR}^{U+iR} \ll \frac{1}{R}\frac{x^U - x^c}{\log x}$. Since $x < 1$, then $\log x < 0$, so we have $\ll \frac{1}{R}\frac{x^c - x^U}{|\log x|}$. Furthermore as $U \to \infty$, this becomes $\ll \frac{x^c}{R|\log x|}$. For the remaining integral, we have that $\int_{U-iR}^{U+iR} \ll \frac{1}{U}\int_{-R}^{R} x^{U+it} dt \ll \frac{2Rx^U}{U}$ which goes to 0 as $U \to \infty$. It follows that for $0 < x < 1$, we have

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{x^s}{s} ds - \delta(x) = O\left(\frac{x^c}{R\,|\log x|}\right)$$

For $x > 1$, simply take K_U to have $U < 0$, then you get a residue of $\delta(x)$ and the inequalities remain essentially the same. Alternatively, we may integrate around a half circle to get $\ll x^c$.

Note that the constant that we get for the integral above is independent of x. Therefore,

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds = \sum_n \frac{a_n}{2\pi i}\int_{c-iR}^{c+iR}\frac{(x/n)^s}{s}ds = \sum_n a_n\delta(x/n) + \sum_n a_n O\left(\frac{(x/n)^c}{R\left|\log(x/n)\right|}\right)$$

We now have our main term and error term. From the above, we see that,

$$\sum_{n\leqslant x} a_n = \frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds + \sum_n a_n O\left(\frac{(x/n)^c}{R\left|\log(x/n)\right|}\right)$$

We bound the error term as follows. There are two cases, either $\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}$ or not. If not, then we have that either $n<\frac{x}{2}$ and therefore $\frac{x}{n}>2$, so $|\log(x/n)|>\log 2$, or $n>\frac{3x}{2}$, so $\frac{n}{x}>\frac{3}{2}$ and therefore $-\log(x/n)>\log(3/2)$. It follows that $|\log(x/n)|>\log(3/2)$. We now have that,

$$\sum_{n<\frac{x}{2}\text{ or }n>\frac{3x}{2}} a_n\frac{(x/n)^c}{R\left|\log(x/n)\right|} \ll \frac{x^c}{R}\sum_n\frac{a_n}{n^c} \ll \frac{x^c}{R}\sum_{n>x}\frac{1}{n^{c-\epsilon}} \ll \frac{x^{c+\epsilon}}{R}$$

Since $c>1$, then let $\epsilon=\frac{c-1}{2}$, then we have that $a_n=O(n^\epsilon)$. Therefore, $\frac{a_n}{n^c}=O(n^{\epsilon-c})$ and $\epsilon-c=\frac{-1-c}{2}<-1$. It follows that the above series converges. Now when $\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}$, we have that for $z=1-\frac{n}{x}$, then since $\frac{1}{2}\leqslant\frac{n}{x}\leqslant\frac{3}{2}$, then $\frac{-1}{2}\leqslant z\leqslant\frac{1}{2}$. In particular, $|z|\leqslant\frac{1}{2}$. It follows that $-\log(1-z)=z+\frac{z^2}{2}+\cdots$. We then have that $\left|\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right|\leqslant|z|\sum_{n=1}^\infty|z|^n/2$. Therefore $|\log(1-z)|=|\log(x/n)|\geqslant\frac{1}{2}|z|=\frac{|x-n|}{2|x|}$. Since $\frac{x}{2}\leqslant n$, then $\frac{x}{n}\leqslant 2$ and hence $\left(\frac{x}{n}\right)^c<2^c$. It follows that for any $\epsilon>0$,

$$\sum_{\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}} a_n\frac{(x/n)^c}{R\left|\log(x/n)\right|} \ll \frac{1}{R}\sum_{\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}} a_n\frac{x}{|x-n|} \ll \frac{x^\epsilon}{R}\sum_{\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}}\frac{x}{|x-n|}$$

Note however that we also bounded our integral by x^c as well as $\frac{x^c}{R|\log(x)|}$. It follows that we may add a $\min\left\{1,\frac{1}{R|\log(x/n)|}\right\}$ to get,

$$\ll\frac{x^\epsilon}{R}\sum_{\frac{x}{2}\leqslant n\leqslant\frac{3x}{2}}\min\left\{1,\frac{x}{|x-n|}\right\}\ll\frac{x^{\epsilon+1}\log x}{R}$$

It follows that,

$$\sum_{n\leqslant x}a_n=\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds+O\left(\frac{x^{c+\epsilon'}}{R}\right)+O\left(\frac{x^{1+\epsilon}\log x}{R}\right)$$

Note that since we may take ϵ arbitrarily small, then we may take $1+\epsilon<c+\epsilon'$, and therefore, we actually just get,

$$\sum_{n\leqslant x}a_n=\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds+O\left(\frac{x^{c+\epsilon}}{R}\right)$$

4.3.3

Assuming the Lindelof hypothesis, prove that for any $\epsilon>0$,

$$\sum_{n\leqslant x}d_k(n)=xP_{k-1}(\log x)+O(x^{1/2+\epsilon})$$

where $d_k(n)$ denotes the number of ways of writing n as a product of k natural numbers.

Proof:

We know that the Dirichlet series of d_k is ζ^k, and furthermore we know that $d_k(n) \leqslant d(n)^k = O(n^{k\epsilon}) = O(n^\epsilon)$. It follows by the previous question that for any $c > 1$, we have

$$\sum_{n \leqslant x} a_n = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{\zeta^k(s)x^s}{s}ds + O\left(\frac{x^{c+\epsilon}}{R}\right)$$

$\zeta^k(s)x^s s^{-1}$ has poles at $s = 1$ and $s = 0$. Consider the contour C which is a rectangle connecting $\frac{1}{2} \pm iR$ and $c \pm iR$, then by the residue theorem, we have that

$$\int_C \frac{\zeta^k(s)x^s}{s}ds = \text{res}_{s=1}\frac{\zeta^k(s)x^s}{s} = \frac{1}{(k-1)!}\lim_{s\to 1}\frac{d^{k-1}}{dz^{k-1}}\left((s-1)^k\zeta^k(s)x^s s^{-1}\right)$$

This will be x times some degree $k-1$ polynomial P_{k-1} in $\log x$. We now want to bound the integral along the contour. From $\frac{1}{2} \pm iR$ to $c \pm iR$, we know that $\zeta^k(s)$ is bounded independent of R for R sufficiently large. It follows that these integrals are $\ll \frac{x^c}{R}$. For the integral from $\frac{1}{2} - iR$ to $\frac{1}{2} + iR$, we can bound $\zeta(\frac{1}{2} + it) = O(1)$, and $x^{1/2} \ll x^c$. It follows that this integral is $\ll x^{1/2}\int_0^R \frac{1}{|\frac{1}{2}+it|}dt$. It follows that,

$$\int_C \frac{\zeta^k(s)x^s}{s}ds = \int_{c-iR}^{c+iR}\frac{\zeta^k(s)x^s}{s}ds + O\left(\frac{x^c}{R}\right) + O\left(x^{1/2}\log R\right)$$

Now take $c = 1 + \epsilon$ and $R = x$, then we have that,

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{\zeta^k(s)x^s}{s}ds = xP_{k-1}(\log x) + O(x^\epsilon) + O(x^{1/2}\log x)$$

It follows that,

$$\sum_{n\leqslant x} a_n = xP_{k-1}(\log x) + O(x^{1/2+\epsilon})$$

4.3.4

Show that

$$M(x) := \sum_{n\leqslant x}\mu(n) = O\left(x\exp\left(-c\,(\log x)^{1/10}\right)\right)$$

for some positive constant c.

Proof:

We have that the Dirichlet series of $\mu(n)$ is just $\frac{1}{\zeta(s)}$, and $\mu(n) = O(1) = O(n^0)$. For any $c > 1$ and R, we have that,

$$\sum_{n\leqslant x}\mu(n) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{x^s}{s\zeta(s)}ds + O\left(\frac{x^{c+\epsilon}}{R}\right)$$

Furthermore, we know that $|\zeta(s)| \gg \frac{1}{\log^7 R}$ for $1 \leqslant \text{Im}(s) \leqslant R$ and $\delta = 1 - \frac{c_2}{\log^9 R} \leqslant \text{Re}(s) \leqslant 2$. It follows that $\frac{1}{|\zeta(s)|} \leqslant \log^7 R$. Now consider a contour C which is a rectangle with corners $\delta \pm iR$ and $c \pm iR$. Along the sections from $\delta \pm iR$ to $c \pm iR$, we have that $\frac{x^s}{s\zeta(s)} \ll \frac{x^c\log^7 R}{R}$ and thus the integral will be $\ll \frac{x^c\log^7 R}{R\log x}$. Along the vertical strip, we will have that $\frac{x^c}{s\zeta(s)} \ll \frac{x^\delta\log^7 R}{s}$ and therefore the integral will be $\ll x^{\delta}\log^8 R$. We want these error terms to be the same in order to minimize the error. Choose $c = 1 + \frac{1}{\log x}$, then we have that,

$$M(x) = O\left(\frac{x^{1+\frac{1}{\log x}+\epsilon}}{R}\right) + O\left(\frac{x^{1+\frac{1}{\log x}}\log^7 R}{R\log x}\right) + O\left(x^{1-\frac{c}{\log^9 R}}\log^8 R\right)$$

Note that $x^{\frac{1}{\log x}} = e$. Apparently just put in $R = \exp(-c_2\log^{1/10}(x))$.

4.4.2

Show that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

converges for every s with $\mathrm{Re}(s) = 1$.

Proof:

We will do a partial summation. We have that,

$$\sum_{n=1}^{N}\frac{\mu(n)}{n^{1+it}} = \frac{1}{N^{1+it}}\sum_{n\leqslant N}\mu(n) + (1+it)\int_1^N M(x)x^{-2-it}dx$$

We know the first term converges by the previous problem since it will be $O(\exp(-c(\log x)^{1/10}))$. We want to show that the second term also converges as $N\to\infty$. Since $\exp(-c(\log x)^{1/10})$ is $O(\log(x)^a)$ for all $a<0$, then in particular, it is $O(\log(x)^{-2})$ and therefore $M(x) = O(x/\log^2(x))$, therefore,

$$\int_1^N M(x)x^{-2-it}dx = O\left(\int_2^N \frac{1}{x\log^2(x)}dx\right)$$

This integral converges as $N\to\infty$ and $\frac{1}{N}\sum_{n\leqslant N}\mu(n)\to 0$ as $N\to\infty$. It follows that $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ converges.

4.4.10

Let $Q(x)$ be the number of square-free numbers less than or equal to x. Show that

$$Q(x) = \frac{x}{\zeta(2)} + O\left(x^{1/2}\exp\left(-c\left(\log x\right)^{1/10}\right)\right)$$

Proof:

Notice that $\sum_{d^2|n}\mu(d)$ is actually a sum over only those d which are squarefree with $d^2|n$. If this set is nonempty, the sum is 0, and if it is empty, only $1^2|n$, so the sum is 1 and n is squarefree. It follows that,

$$Q(x) = \sum_{n\leqslant x}\left(\sum_{d^2|n}\mu(d)\right)$$

Let $g(n)$ be $\mu(m)$ where $m = n^2$ or 0 otherwise. It follows that $\sum_{d^2|n} \mu(d) = \sum_{d|n} g(d) = (g * 1)(n)$. We may now use Dirichlet's hyperbola method,

$$Q(x) = \sum_{n \leqslant y} g(n) \left\lfloor \frac{x}{n} \right\rfloor + \sum_{n \leqslant \frac{x}{y}} G\left(\frac{x}{n}\right) - G(y) \left\lfloor \frac{x}{y} \right\rfloor$$

We know that $G(y) = M(\sqrt{y})$ where $M(x) = \sum_{n \leqslant x} \mu(n)$. Let $N(x) = \exp(-c(\log x)^{1/10})$, then $G(y) = O(\sqrt{y}N(\sqrt{y}))$. Since $N(x)$ is decreasing and $n \leqslant \frac{x}{y}$ means that $\frac{x}{n} \geqslant y$, then $G\left(\frac{x}{n}\right) = O(\sqrt{\frac{x}{n}}N(\frac{x}{n})) = O(\sqrt{\frac{x}{n}}N(\sqrt{y})) = \frac{O(\sqrt{x}N(\sqrt{y}))}{\sqrt{n}}$. It follows that,

$$\sum_{n \leqslant \frac{x}{y}} G\left(\frac{x}{n}\right) \ll \sqrt{x}N(y) \sum_{n \leqslant \frac{x}{y}} \frac{1}{\sqrt{n}} \ll \sqrt{x}N(y)\sqrt{\frac{x}{y}} = xy^{-1/2}N(y^{1/2})$$

We also have that $G(y) \left\lfloor \frac{x}{y} \right\rfloor \ll xy^{-1/2}N(y^{1/2})$. For the first term, we split it into two parts,

$$\sum_{n \leqslant y} g(n) \left\lfloor \frac{x}{n} \right\rfloor \ll x \sum_{n \leqslant y} \frac{g(n)}{n} + \sum_{n \leqslant y} g(n) = x \sum_{n \leqslant y} \frac{g(n)}{n} + O(\sqrt{y}N(\sqrt{y}))$$

To estimate the first term, we use partial summation. Notice that,

$$\sum_{n=1}^{\infty} \frac{g(n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$$

Since $\frac{1}{x}G(x) \to 0$ as $x \to \infty$, then we have that,

$$\sum_{n \leqslant y} \frac{g(n)}{n} = \sum_{n=1}^{\infty} \frac{g(n)}{n} - \sum_{n > y} \frac{g(n)}{n} \ll \frac{1}{\zeta(2)} + \int_y^{\infty} \frac{M(\sqrt{t})}{t^2} dt$$

Since $M(\sqrt{t}) \ll \sqrt{t}N(\sqrt{t})$, and $N(\sqrt{t})$ is decreasing, then

$$\int_y^{\infty} \frac{M(\sqrt{t})}{t^2} dt \ll \int_y^{\infty} \frac{N(\sqrt{t})}{t^{3/2}} dt \ll N(\sqrt{y}) \int_y^{\infty} \frac{1}{t^{3/2}} dt \ll y^{-1/2}N(y^{1/2})$$

It follows that,

$$Q(x) = \frac{x}{\zeta(2)} + O(y^{1/2}N(y^{1/2})) + O(xy^{-1/2}N(y^{1/2}))$$

Now picking $y = \sqrt{x}$ yields our bound since $O(N(x)) = O(N(\sqrt{x}))$.

5.4.3

Prove that

$$\pi^{-(s+1)/2} q^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) n^{-s} = \int_0^{\infty} n e^{-\pi n^2 x/q} x^{\frac{s+1}{2}} \frac{dx}{x}$$

and hence deduce that

$$\pi^{-\left(\frac{s+1}{2}\right)} q^{\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi) = \frac{1}{2} \int_0^{\infty} \theta_1(x, \chi) x^{\frac{s+1}{2}} \frac{dx}{x}$$

where

$$\theta_1(x,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n)e^{-n^2\pi x/q}$$

Proof:

(a) Expand $\Gamma\left(\frac{s+1}{2}\right) = \int_0^\infty t^{\frac{s-1}{2}}e^{-t}dx$ Now let $t = n^2\pi x/q$, so we have

$$\Gamma\left(\frac{s+1}{2}\right) = q^{-1}n^2\pi\left(n^2\pi/q\right)^{\frac{s-1}{2}}\int_0^\infty x^{\frac{s+1}{2}}e^{-\pi n^2 x/q}\frac{dx}{x}$$

Now multiply both sides by $n^{-s}\pi^{-\frac{s+1}{2}}q^{\frac{s+1}{2}}$ to get that,

$$\pi^{-(s+1)/2}q^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)n^{-s} = \int_0^\infty ne^{-\pi n^2 x/q}x^{\frac{s+1}{2}}\frac{dx}{x}$$

(b) Multiplying our equation by $\chi(n)$ and summing over n, we get that,

$$\pi^{-(s+1)/2}q^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = \sum_n n\chi(n)\int_0^\infty e^{-\pi n^2 x/q}x^{(s+1)/2}\frac{dx}{x}$$

We want to show that $\sum_n n\chi(n)e^{-\pi n^2 x/q}x^{(s-1)/2}$ converges uniformly to an integrable function. Trust me, it does, so we may swap the integral and sum. Furthermore notice that the sum is exactly $\frac{1}{2}\theta_1(x,\chi)x^{(s-1)/2}$

5.4.5

Prove that for $\chi(-1) = -1$, if we set,

$$\xi(s,\chi) = \pi^{-s/2}q^{s/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)$$

then $\xi(s,\chi)$ is entire and

$$\xi(s,\chi) = w_\chi\xi(1-s,\overline{\chi})$$

where $w_\chi = \tau(\chi)/iq^{1/2}$

Proof:

We multiply $\theta - 1(x,\chi)$ by $\tau(\overline{\chi})$ to get that,

$$\tau(\overline{\chi})\theta_1(x,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n)\tau(\overline{\chi})e^{-n^2\pi x/q}$$

We then have that $\chi(n)\tau(\overline{\chi}) = \sum_{m=1}^{q}\overline{\chi}(m)e\left(\frac{mn}{q}\right)$, therefore, we may rewrite this sum as,

$$\sum_{n=-\infty}^{\infty}\sum_{m=1}^{q} n\overline{\chi}(m)e^{-n^2\pi x/q+2\pi imn/q}$$

Now we interchange the sums and use the identity from 5.4.4 to get that,

$$\tau(\overline{\chi})\theta_1(x,\chi) = i(q/x)^{3/2}\sum_{m=1}^{q}\overline{\chi}(m)\sum_{n=-\infty}^{\infty}\left(n+\frac{m}{q}\right)e^{-\pi(n+m/q)^2 q/x}$$

Now let $l = nq + m$, so we have that,

$$\tau(\overline{\chi})\theta_1(x,\chi) = i(q/x)^{3/2} \sum_{m=1}^{q} \overline{\chi}(m) \sum_{n=-\infty}^{\infty} \frac{l}{q} e^{-\pi l^2/(xq)}$$

Now $\overline{\chi}(l) = \overline{\chi}(m)$ since $l \equiv m \pmod{q}$. It follows that,

$$\tau(\overline{\chi})\theta_1(x,\chi) = iq^{1/2}x^{-3/2} \sum_{l=-\infty}^{\infty} \overline{\chi}(l) l e^{-\pi l^2/(xq)} = iq^{1/2}x^{-3/2}\theta_1(x^{-1},\overline{\chi})$$

We have that,

$$\pi^{-(s+1)/2} q^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \frac{1}{2}\left(\int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x} + \int_0^1 \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x}\right)$$

In the second integral, let $u = x^{-1}$, so $du = -\frac{1}{x^2}dx$, and therefore $dx = -x^2 du = -u^{-2}du$. It follows that,

$$\int_0^1 \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x} = -\int_\infty^1 \theta_1(u^{-1},\chi) u^{-(s+1)/2} \frac{du}{u}$$

Then noting that $\theta_1(u^{-1},\chi) = \frac{iq^{1/2}u^{3/2}}{\tau(\overline{\chi})}\theta_1(u,\overline{\chi})$, we get,

$$\frac{iq^{1/2}}{\tau(\overline{\chi})} \int_1^\infty \theta_1(u,\overline{\chi}) u^{1-s/2} \frac{du}{u}$$

Therefore we may rewrite our original formula as,

$$\pi^{-(s+1)/2} q^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) = \frac{1}{2}\left(\int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x} + \frac{iq^{1/2}}{\tau(\overline{\chi})} \int_1^\infty \theta_1(x,\overline{\chi}) x^{1-s/2} \frac{dx}{x}\right)$$

Now notice that these integrals both exist for all s, so $\xi(s,\chi)$ is entire. If we replace s with $1-s$ and χ by $\overline{\chi}$ we get,

$$\frac{1}{2}\left(\int_1^\infty \theta_1(x,\overline{\chi}) x^{1-s/2} \frac{dx}{x} + \frac{iq^{1/2}}{\tau(\overline{\chi})} \int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x}\right)$$

Now since $\tau(\overline{\chi}) = \sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{m}{q}\right) = -\sum_{m=1}^{q} \overline{\chi}(-m) e\left(\frac{m}{q}\right)$ since χ is odd, so we get that,

$$\tau(\overline{\chi}) = -\sum_{m=1}^{q} \overline{\chi}(m) e\left(\frac{-m}{q}\right) = -\overline{\tau(\chi)}$$

It follows that $\tau(\overline{\chi})\tau(\chi) = -q$. Let $RHS = \pi^{-1+s/2} q^{1-s/2} \Gamma\left(1-\frac{s}{2}\right) L(1-s,\overline{\chi})$, then we get that,

$$\begin{aligned}
&= \frac{1}{2}\left(\int_1^\infty \theta_1(x,\overline{\chi}) x^{1-s/2} \frac{dx}{x} + \frac{iq^{1/2}}{\tau(\overline{\chi})} \int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x}\right) \\
&= \frac{\tau(\chi)}{2iq^{1/2}}\left(\frac{iq^{1/2}}{\tau(\chi)} \int_1^\infty \theta_1(x,\overline{\chi}) x^{1-s/2} \frac{dx}{x} + \frac{-q}{\tau(\chi)\tau(\overline{\chi})} \int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x}\right) \\
&= \frac{\tau(\chi)}{iq^{1/2}} \frac{1}{2}\left(\int_1^\infty \theta_1(x,\chi) x^{(s+1)/2} \frac{dx}{x} + \frac{iq^{1/2}}{\tau(\chi)} \int_1^\infty \theta_1(x,\overline{\chi}) x^{1-s/2} \frac{dx}{x}\right)
\end{aligned}$$

Then one notices that this last line states that,

$$\pi^{-1+s/2}q^{1-s/2}\Gamma\left(1-\frac{s}{2}\right)L(1-s,\overline{\chi}) = \frac{\tau(\chi)}{iq^{1/2}}\pi^{-(s+1)/2}q^{(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)$$

Now multiplying both sides by $\pi^{1/2}q^{-1/2}$ yields,

$$\xi(s,\chi) = w_\chi\xi(1-s,\overline{\chi})$$

as desired.

5.5.1

Let

$$f(y) = \sum_{n=1}^{\infty} a_n e^{-2\pi ny}$$

converge for $y > 0$. Suppose that for some $w \in \mathbb{Z}$,

$$f(1/y) = (-1)^w y^r f(y)$$

and that $a_n = O(n^c)$ for some constant $c > 0$. Let

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

Show that $(2\pi)^{-s}\Gamma(s)L_f(s)$ extends to an entire function and satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^w(2\pi)^{-(r-s)}\Gamma(r-s)L_f(r-s)$$

Proof:

For any n, we have that,

$$(2\pi)^{-s}\Gamma(s)n^{-s} = (2\pi)^{-s}\int_0^{\infty} n^{-s}x^{s-1}e^{-x}dx$$

Let $x = 2\pi ny$, then we have,

$$(2\pi)^{-s}\Gamma(s)n^{-s} = \int_0^{\infty} y^{s-1}e^{-2\pi ny}dy$$

Now multiplying each by a_n we may sum over all n. This converges absolutely since $a_n = O(n^c)$. We get that,

$$(2\pi)^{-s}\Gamma(s)L_f(s) = \int_0^{\infty} f(y)y^{s-1}dy = \int_1^{\infty} f(y)y^{s-1}dy + \int_0^{1} f(y)y^{s-1}dy$$

Then letting $u = y^{-1}, du = -u^{-2}dy$, we obtain,

$$\int_0^1 f(y)y^{s-1}dy = \int_1^\infty f(1/u)u^{-1-s}du$$

Now we use that $f(1/u) = (-1)^w u^r f(u)$ to get that this is just,

$$(-1)^w \int_1^\infty f(u)u^{r-s-1}du$$

It follows that

$$(2\pi)^{-s}\Gamma(s)L_f(s) = \int_1^\infty f(y)y^{s-1}dy + (-1)^w \int_1^\infty f(y)y^{r-s-1}dy$$

Now replacing s with $r-s$, we see that,

$$\begin{aligned}(2\pi)^{-(r-s)}\Gamma(r-s)L_f(r-s) &= \int_1^\infty f(y)y^{r-s-1}dy + (-1)^w \int_1^\infty f(y)y^{s-1}dy \\ &= (-1)^w \left((-1)^w \int_1^\infty f(y)y^{r-s-1}dy + \int_1^\infty f(y)y^{s-1}dy\right) \\ &= (-1)^w (2\pi)^{-s}\Gamma(s)L_f(s)\end{aligned}$$

thus proving our desired functional equation.

5.5.6

(Pólya-Vinogradov inequality) Let χ be a primitive character mod q. Show that for $q > 1$,

$$\left|\sum_{n\leqslant x} \chi(n)\right| \ll q^{1/2}\log q$$

Proof:

We have that, $\left(\sum_{n\leqslant x}\chi(n)\right)\tau(\overline{\chi}) = \sum_{n\leqslant x}\sum_{m=1}^{q}\overline{\chi}(m)e\left(\frac{mn}{q}\right)$. We then have that,

$$\sum_{n\leqslant x} e\left(\frac{mn}{q}\right) = \frac{e\left(\frac{m\lfloor x\rfloor}{q}\right)-1}{1-e\left(\frac{m}{q}\right)}$$

In absolute value, this is bounded above by $\frac{2}{\left|1-e\left(\frac{m}{q}\right)\right|}$. It follows that,

$$|\tau(\overline{\chi})|\left|\sum_{n\leqslant x}\chi(n)\right| \ll \sum_{m=1}^{q-1}\frac{2}{\left|1-e\left(\frac{m}{q}\right)\right|}$$

Note that we may exclude $m = q$ since $\chi(q) = \chi(0) = 0$. We have that $\left|1-e\left(\frac{m}{q}\right)\right| \geqslant |\sin(\pi m/q)|$. For $\frac{q}{4} \leqslant m \leqslant \frac{3q}{4}$ we have that $\left|1-e\left(\frac{m}{q}\right)\right| \geqslant 1$, thus,

$$\sum_{m=1}^{q-1}\frac{2}{\left|1-e\left(\frac{m}{q}\right)\right|} \leqslant 4\sum_{m=1}^{q/4}\frac{1}{|\sin(\pi m/q)|} + q$$

Now $|\sin(\pi x)| \geqslant x$ for $x \leqslant \frac{1}{2}$, so we have a final bound,

$$\leqslant 4 \sum_{m=1}^{q/4} \frac{q}{m} + q = (4\log(q/4) + 1)q \ll q\log(q)$$

Now we can use this for the original problem, noting that $|\tau(\overline{\chi})| = q^{1/2}$ we have that

$$\left| \sum_{n \leqslant x} \chi(n) \right| \ll q^{1/2} \log(q)$$

5.5.7

Show that if χ is a primitive character (mod q), then

$$L(1,\chi) = \sum_{n \leqslant x} \frac{\chi(n)}{n} + O\left(\frac{q^{1/2}\log q}{x} \right)$$

for any $x \geqslant 1$ and $q > 1$

Proof:

We have that,

$$L(1,\chi) = \sum_{n \leqslant x} \frac{\chi(n)}{n} + \sum_{n > x} \frac{\chi(n)}{n}$$

Now we need to bound the second term. Since $\frac{1}{x}\sum_{n \leqslant x} \chi(n) \to 0$ as $x \to \infty$, then we can use partial summation. Let $S(t) = \sum_{n \leqslant t} \chi(t)$, then

$$\sum_{n > x} \frac{\chi(n)}{n} \ll \left| \int_x^\infty \frac{S(t)}{t^2} dt \right| \ll q^{1/2} \log q \int_x^\infty \frac{1}{t^2} dt = \frac{q^{1/2}\log q}{x}$$

Therefore,

$$L(1,\chi) = \sum_{n \leqslant x} \frac{\chi(n)}{n} + O\left(\frac{q^{1/2}\log q}{x} \right)$$

Question 1

Give an outline of proof of the following statement: The Riemann hypothesis is equivalent to $\psi(x) = \sum_{n \leqslant x} \Lambda(n) = x + O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $\epsilon > 0$

[You may use the following estimate under the Riemann Hypothesis: $\left|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right| \ll (\log|t|)^{2-2\sigma}$ for $\sigma > \frac{1}{2}$]

Proof:

We have an explicit formula,

$$\psi(x) = x - \sum_{\rho, |\mathrm{Im}(\rho)| < R} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2}) + O\left(\frac{x}{R} \left(\frac{(\log R)^2}{\log x} + \log(x)^2 \right) \right)$$

where ρ ranges over the non-trivial zeros of $\zeta(s)$. The last two terms are $O(x^{1/2})$, so they may be ignored. We need only show that the sum is also $O(x^{1/2+\epsilon})$. Assuming the Riemann Hypothesis, we have that $\rho = \frac{1}{2} + it$, so $|x^\rho| = \sqrt{x}$. It follows that the sum is bound above in absolute value,

$$\left| \sum_{\rho, |t|<R} \frac{x^\rho}{\rho} \right| \leqslant x^{1/2} \sum_{\rho, |t|<R} \frac{1}{|\rho|}$$

We now show that this sum is $O(\log(R)^2)$. To do so, recall that $N(R) \sim \frac{R \log R}{2\pi}$, so by partial summation we have that,

$$\sum_{\rho, |t|<R} \frac{1}{|\rho|} \ll \frac{N(R)}{R} + \int_1^R N(t) t^{-2} dt = O(\log R) + \int_1^R O\left(\frac{\log t}{t}\right) dt = O((\log R)^2)$$

Now take $R = \sqrt{x}$, and we get that the sum is $O(\sqrt{x}(\log x)^2)$ and the error term in the explicit formula becomes $O(\sqrt{x}(\log(x))^2)$ as well, therefore $\psi(x) = x + O\left(x(\log(x))^2\right) = O(x^{1/2+\epsilon})$ for any $\epsilon > 0$.

Conversely, we have that $\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\psi(t)}{t^{s+1}} dt$ which converges when $\mathrm{Re}(s) > 1$. We can analytically continue this since,

$$\int_1^\infty \frac{\psi(t)}{t^{s+1}} ds = \int_1^\infty \frac{t}{t^{s+1}} ds + \int_1^\infty \frac{O(t^{1/2+\epsilon})}{t^{s+1}} ds$$

Now the second of the two integrals converges for $\mathrm{Re}(s) > \frac{1}{2} + \epsilon$ and since ϵ is arbitrary, it converges for $\mathrm{Re}(s) > \frac{1}{2}$. The first integral is exactly $\int_1^\infty t^{-s} ds = \frac{t^{1-s}}{1-s}\Big|_1^\infty = \frac{1}{s-1}$ when $\mathrm{Re}(s) > 1$. So we analytically continue $\frac{\zeta'}{\zeta}$ to $\mathrm{Re}(s) > \frac{1}{2}$ with the formula,

$$\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} + s \int_1^\infty \frac{\psi(t) - t}{t^{s+1}} ds$$

Since this yields an analytic continuation of ζ'/ζ to $\mathrm{Re}(s) > \frac{1}{2}$ with a pole only appearing at $s = 1$, then there are no poles in $\frac{1}{2} < \mathrm{Re}(s) < 1$ and therefore $\zeta(s)$ has no zeros in that region, thus the Riemann hypothesis holds.

Question 2

Give an outline of proof of Dirichlet's theorem on arithmetic progression in natural density. Namely,

$$\lim_{x \to \infty} \frac{\pi(x, q, a)}{\pi(x)} = \frac{1}{\varphi(q)}$$

where $\pi(x, q, a) = \sum_{\substack{p \leqslant x \\ p \cong a \pmod q}} 1$ and $(q, a) = 1$ [You may prove its equivalent form: $\lim_{x\to\infty} \frac{\psi(x,q,a)}{x} = \frac{1}{\varphi(q)}$, where $\psi(x, q, a) = \sum_{\substack{n \leqslant x \\ n \cong a \pmod q}} \Lambda(n)$]

Proof:

We have that $\psi(x, q, a) = \frac{1}{\varphi(q)} \sum_\chi \overline{\chi(a)} \psi(x, \chi)$. Now $\psi(x, \chi_0) = \psi(x) + O(...) = x + O(...)$. For $\chi \neq \chi_0$, we use the explicit formula for $\psi(x, \chi)$ to get that it is $O\left(x e^{-c\sqrt{\log x}}\right) = O\left(\frac{x}{(\log x)^2}\right)$ for

$q \leqslant (\log x)^{1-\epsilon}$. We now have that,

$$\psi(x,q,a) = \frac{1}{\varphi(q)}\left(x + O(1) + \sum_{\chi \neq \chi_0} \overline{\chi(a)} O(\frac{x}{(\log x)^2}\right) = \frac{x}{\varphi(q)} + O(1) + O\left(\frac{qx}{\varphi(q)(\log x)^2}\right)$$

Therefore as $x \to \infty$, $\frac{\psi(x,q,a)}{x} \to \frac{1}{\varphi(q)}$ as desired.

Question 3

Find a formula for $\zeta(2k)$ in terms of the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}$$

Proof:

Take the logarithmic derivative of $\sin z = z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 n^2}\right)$, so $\cot(z) = \frac{1}{z} - 2\sum_{n=1}^{\infty} \frac{\frac{z}{\pi^2 n^2}}{1 - \frac{z^2}{\pi^2 n^2}}$, therefore $z\cot(z) = 1 - 2\sum_{n=1}^{\infty} \frac{\frac{z^2}{\pi^2 n^2}}{1 - \frac{z^2}{\pi^2 n^2}} = 1 + 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{z^{2k}}{n^{2k}\pi^{2k}}$. Let $x = 2iz$, then $\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x(e^x+1)}{2(e^x-1)} = \frac{x}{2i}\cot\left(\frac{-ix}{2}\right)$, therefore,

$$\frac{x}{e^x - 1} + \frac{x}{2} = z\cot(z) = i\sum_{k=1}^{\infty} \frac{1}{\pi^{2k}}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right) z^{2k}$$

So equating coefficients we see that $(-1)^k 2^{-2k}\zeta(2k) = \frac{\pi^{2k} B_{2k}}{(2k)!}$, therefore,

$$\zeta(2k) = (-1)^k \frac{(2\pi)^{2k} B_{2k}}{(2k)!}$$

Question 4

Let $Q(x)$ be the number of square free integers $\leqslant x$. Under the Riemann hypothesis, show that

$$Q(x) = \frac{x}{\zeta(2)} + O\left(x^{\frac{2}{5}+\epsilon}\right)$$

[You may use the fact that under the Riemann hypothesis, $\sum_{n \leqslant x} \mu(n) = O\left(x^{\frac{1}{2}+\epsilon}\right)$, and $\left|\zeta\left(\frac{1}{4} + \epsilon + it\right)\right| \ll |t|^{\frac{1}{4}-\epsilon}$ and $\left|\zeta\left(\frac{1}{2} + \epsilon + it\right)\right| \gg |t|^{-\epsilon}$]

Proof:

We will use the hyperbola method. Let $g(n) = \mu(\sqrt{n})$ if n is square and 0 otherwise. We then have that $\sum_{d^2|n} \mu(d) = \sum_{d|n} g(d)$. Furthermore, if $\gamma(n)$ is the squarefree part of n i.e. $\gamma(n) = \prod_{p|n} p$, then $\sum_{d^2|n} \mu(d) = \sum_{d|\sqrt{\frac{n}{\gamma(n)}}} \mu(d)$ which is 0 if n is not squarefree and 1 if n is squarefree. Therefore $\sum_{d|n} g(d) = |\mu(n)|$. It follows that,

$$Q(x) = \sum_{n \leqslant x} |\mu(n)| = \sum_{n \leqslant x}\sum_{d|n} g(d) = \sum_{n \leqslant x} (g * 1)(n)$$

We have that $h = 1, g = g$, so $G(x) = \sum_{d \leqslant x} g(d) = \sum_{d \leqslant \sqrt{x}} \mu(d) = O(x^{1/4+\epsilon})$ by assumption. We also get $H(x) = \lfloor x \rfloor$. By the hyperbola method, we get,

$$Q(x) = \sum_{n \leqslant y} g(n) H\left(\frac{x}{n}\right) + \sum_{n \leqslant \frac{x}{y}} G\left(\frac{x}{n}\right) - G(y) H\left(\frac{x}{y}\right)$$

We approximate the middle and last term. The last term is $O(y^{1/4+\epsilon}) O\left(1 + xy^{-1}\right) = O(y^{1/4+\epsilon}) + O(xy^{-3/4+\epsilon})$. The middle term is,

$$\sum_{n \leqslant \frac{x}{y}} O\left(x^{1/4+\epsilon} n^{-1/4-\epsilon}\right) = x^{1/4+\epsilon} \sum_{n \leqslant \frac{x}{y}} n^{-1/4-\epsilon} \ll x^{1/4+\epsilon} \left(\frac{x}{y}\right)^{3/4-\epsilon} = xy^{-3/4+\epsilon}$$

Now the first term is,

$$\ll \sum_{n \leqslant y} g(n) \frac{x}{n} + \sum_{n \leqslant y} |g(n)| \ll x \sum_{n \leqslant y} \frac{g(n)}{n} + Q(y^{1/2})$$

We know that $\sum_{n=1}^{\infty} \frac{g(n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)}$, so,

$$\sum_{n \leqslant y} \frac{g(n)}{n} = \frac{1}{\zeta(2)} - \sum_{n > y} \frac{g(n)}{n}$$

and the final sum can be bound by an integral,

$$\sum_{n > y} \frac{g(n)}{n} \ll \int_y^{\infty} \frac{G(t)}{t^2} dt = \int_y^{\infty} O(t^{-7/4+\epsilon}) dt = O(y^{-3/4+\epsilon})$$

It follows that,

$$Q(x) = \frac{x}{\zeta(2)} + O(xy^{-3/4+\epsilon}) + O(y^{1/4}) + O(Q(y^{1/2}))$$

Since $Q(x) = O(x)$, then we finally have that,

$$Q(x) = \frac{x}{\zeta(2)} + O(xy^{-3/4+\epsilon}) + O(y^{1/2})$$

Let $y = x^{\alpha}$, then we want $x^{1-3/4\alpha} = x^{1/2\alpha}$, so $1 - 3/4\alpha = 1/2\alpha$ which means that $\alpha = \frac{4}{5}$ whence we obtain,

$$Q(x) = \frac{x}{\zeta(2)} + O(x^{2/5+\epsilon})$$

Question 5

Prove Mertens' formula:

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-c}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

For some constant c. Determine c.

Proof:

Let $v(x)$ denote the LHS, then we have that $-\log(v(x)) = \sum_{p\leqslant x} -\log(1-\frac{1}{p})$. Expanding log as a power series we get that $-\log(v(x)) = \sum_{p\leqslant x}\sum_{n=1}^{\infty}\frac{1}{np^n}$. We now split off the first power and the higher powers,

$$-\log(v(x)) = \sum_{p\leqslant x}\frac{1}{p} + \sum_{n\geqslant 2}\sum_{p\leqslant x}\frac{1}{np^n} \leqslant \sum_{p\leqslant x}\frac{1}{p} + \sum_{p\leqslant x}\frac{1}{p^2}\sum_{n\geqslant 0}p^{-n} = \sum_{p\leqslant x}\frac{1}{p} + \sum_{p\leqslant x}\frac{1}{p^2}\frac{1}{1-\frac{1}{p}}$$

The last sum is $\sum_{p\leqslant x}\frac{1}{p(p-1)} \leqslant \sum_{n=2}^{\infty}\frac{1}{n(n-1)} = 1$. It follows that $-\log(v(x)) = \sum_{p\leqslant x}\frac{1}{p} + O(1)$. We know that $A(t) = \sum_{p\leqslant t}\frac{\log p}{p} = \log t + O(1)$, so by partial summation, we get that,

$$\sum_{p\leqslant x}\frac{1}{p} = \sum_{p\leqslant x}\frac{\log p}{p}\frac{1}{\log p} = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t\log(t)^2}dt = \frac{A(x)}{\log x} + \int_2^x \frac{1}{t\log t}dt + \int_2^x \frac{O(1)}{t(\log t)^2}dt$$

The first integral is $\log\log x$ and the second converges as $x \to \infty$ since the indefinite integral is $\frac{-1}{\log x}$. It follows that $\sum_{p\leqslant x}\frac{1}{p} = \log\log x + O(1) + O(\frac{1}{\log x})$. We then get that $-\log(v(x)) = \log\log x + c + O(\frac{1}{\log x})$ and therefore,

$$v(x) = \frac{e^{-c}}{\log x}e^{-O(\frac{1}{\log x})} = \frac{e^{-c}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

Note that we Taylor expand e^x to get the last equality.

To find c, consider the function $f(\sigma) = \log\zeta(1+\sigma) - \sum_p \frac{1}{p^{1+\sigma}}$.

Question 6

Give an outline of the proof of the Polya-Vinogradov inequality: For a primitive character χ mod q, where $q > 1$,

$$\left|\sum_{n\leqslant x}\chi(n)\right| \ll q^{1/2}\log q$$

Proof:

We have that, $\left(\sum_{n\leqslant x}\chi(n)\right)\tau(\overline{\chi}) = \sum_{n\leqslant x}\sum_{m=1}^{q}\overline{\chi}(m)e\left(\frac{mn}{q}\right)$. We then have that,

$$\sum_{n\leqslant x} e\left(\frac{mn}{q}\right) = \frac{e\left(\frac{m\lfloor x\rfloor}{q}\right) - 1}{1 - e\left(\frac{m}{q}\right)}$$

In absolute value, this is bounded above by $\frac{2}{|1-e(\frac{m}{q})|}$. It follows that,

$$|\tau(\overline{\chi})|\left|\sum_{n\leqslant x}\chi(n)\right| \ll \sum_{m=1}^{q-1}\frac{2}{\left|1 - e\left(\frac{m}{q}\right)\right|}$$

Note that we may exclude $m = q$ since $\chi(q) = \chi(0) = 0$. We have that $\left|1 - e\left(\frac{m}{q}\right)\right| \geqslant |\sin(\pi m/q)|$. For $\frac{q}{4} \leqslant m \leqslant \frac{3q}{4}$ we have that $\left|1 - e\left(\frac{m}{q}\right)\right| \geqslant 1$, thus,

$$\sum_{m=1}^{q-1}\frac{2}{\left|1 - e\left(\frac{m}{q}\right)\right|} \leqslant 4\sum_{m=1}^{q/4}\frac{1}{|\sin(\pi m/q)|} + q$$

Now $|\sin(\pi x)| \geqslant x$ for $x \leqslant \frac{1}{2}$, so we have a final bound,

$$\leqslant 4\sum_{m=1}^{q/4} \frac{q}{m} + q = (4\log(q/4)+1)q \ll q\log(q)$$

Now we can use this for the original problem, noting that $|\tau(\overline{\chi})| = q^{1/2}$ we have that

$$\left|\sum_{n\leqslant x} \chi(n)\right| \ll q^{1/2}\log(q)$$

Question 7

The Brun-Titchmarsh theorem states that for $(a,k)=1$, and $k \leqslant x, \pi(x,k,a) \leqslant \frac{(2+\epsilon)x}{\varphi(k)\log(2x/k)}$ for $x > x_0(\epsilon)$. Use it to prove Titchmarsh divisor theorem, namely,

$$\sum_{p\leqslant x} d(p-1) = O(x)$$

where the sum is over primes and $d(n)$ denotes the divisor function.

Proof:

We have that,

$$d(n) \leqslant 2\sum_{\substack{d|n\\ d\leqslant\sqrt{n}}} 1 \Rightarrow \sum_{p\leqslant x} d(p-1) \leqslant 2\sum_{p\leqslant x}\sum_{\substack{d|p-1\\ d\leqslant\sqrt{p}}} 1 = \sum_{d\leqslant\sqrt{x}}\sum_{\substack{d|p-1\\ p\leqslant x}} 1 = 2\sum_{d\leqslant\sqrt{x}}\pi(x,d,1)$$

We now use the Brun-Titchmarsh theorem to get that $\pi(x,d,1) \ll \frac{x}{\varphi(d)\log(2x/d)}$ and since $d \leqslant \sqrt{x}$, then this is $\pi(x,d,1) \ll \frac{x}{\varphi(d)\log(\sqrt{x})} \ll \frac{x}{\varphi(d)\log x}$. We can now put this into our sum,

$$\sum_{p\leqslant x} d(p-1) \ll 2\sum_{d\leqslant\sqrt{x}} \frac{x}{\varphi(d)\log x} \ll \frac{x}{\log x}\sum_{d\leqslant\sqrt{x}}\frac{1}{\varphi(d)}$$

We wish to show that $\sum_{n\leqslant y}\frac{n}{\varphi(n)} = O(y)$. To do so, write,

$$\sum_{n\leqslant y}\frac{n}{\varphi(n)} = \sum_{n\leqslant y}\prod_{p|n}(1-\frac{1}{p})^{-1} = \sum_{n\leqslant y}\sum_{\sigma(d)|n}\frac{1}{d} \ll \sum_{\sigma(d)\leqslant y}\frac{1}{d}\frac{y}{\sigma(d)} \leqslant y\sum_{d=1}^{\infty}\frac{1}{d\sigma(d)} = y\prod_p\left(1+p^{-2}+p^{-3}+\cdots\right)$$

where $\sigma(d)$ is the squarefree d, i.e. $\sigma(d) = \prod_{p|d} p$. We wish to show that the product converges. To do so, rewrite it as follows,

$$\prod_p\left(1+p^{-2}+p^{-3}+\cdots\right) = \prod_p\left(1+p^{-2}\frac{1}{1-\frac{1}{p}}\right) = \prod_p\left(1+\frac{1}{p(p-1)}\right) \leqslant \prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$$

Now we just need to show that the logarithm of the product converges. Recall that $\log(1+x) \leqslant x$, so,

$$\log\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right) \leqslant \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

It follows that $\sum_{d\leqslant y}\frac{d}{\varphi(d)} = O(x)$.

Now $\sum_{d\leqslant y}\frac{1}{\varphi(d)} = \sum_{d\leqslant y}\frac{d}{\varphi(d)}\frac{1}{d}$, then by partial summation, we have that,

$$\sum_{d\leqslant y}\frac{1}{\varphi(d)} \ll \frac{\sum_{d\leqslant y}\frac{d}{\varphi(d)}}{y} + \int_1^y \frac{\sum_{d\leqslant t}\frac{d}{\varphi(d)}}{t^2}dt \ll O(1) + \int_1^y \frac{O(1)}{t}dt$$

It follows that $\sum_{d\leqslant y}\frac{1}{\varphi(d)} = O(\log y)$ and therefore $\sum_{p\leqslant x} d(p-1) \ll x$ as desired.

Question 8

Show that $\sum_{n\leqslant x}\frac{\Lambda(n)}{n} = \log x + O(1)$. Deduce that $\sum_{p\leqslant x}\frac{\log p}{p} = \log x + O(1)$.

Proof:

Let $T(x) = \sum_{n\leqslant x}\log(n)$, so by partial summation, we have that,

$$T(x) = \log x\lfloor x\rfloor - \int_1^x \frac{\lfloor t\rfloor}{t}dt = x\log x + O(\log x)$$

By the Mobius inversion formula, we have that,

$$T(x) = \sum_{n\leqslant x}\log n = \sum_{n\leqslant x}\sum_{d|n}\Lambda(d) = \sum_{d\leqslant x}\Lambda(d)\left\lfloor\frac{x}{d}\right\rfloor = x\sum_{d\leqslant x}\frac{\Lambda(d)}{d} + O(x)$$

By dividing through by x, we see that,

$$\sum_{d\leqslant x}\frac{\Lambda(d)}{d} = \log x + O(1)$$

We now have that

$$\sum_{p\leqslant x}\frac{\log p}{p} = \sum_{n\leqslant x}\frac{\Lambda(n)}{n} - \sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{\log p}{p^k}$$

And the second sum is less than $\sum_p \log p\sum_{k\geqslant 2}p^{-k} = \sum_p\frac{\log p}{p(p-1)}$ which converges since $\log p < p^{1/2}$ so the sum is at most $\zeta(3/2)$. It follows that $\sum_{p\leqslant x}\frac{\log p}{p} = \log x + O(1)$.

Question 9

Show that the sequence $\{x_n = \sqrt{n}\}$ is uniformly distributed mod 1, namely, for $0\leqslant a < b\leqslant 1$,

$$\lim_{N\to\infty}\frac{\#\left\{n\leqslant N\middle|(x_n)\in[a,b]\right\}}{N} = b - a$$

where $(x_n) = x_n - \lfloor x_n\rfloor$.

Proof:

We will use Weyl's criterion. Let m be a nonzero integer and let $f(x) = e^{2\pi im\sqrt{x}}$, so $f'(x) = e^{2\pi i\sqrt{x}}\cdot\frac{m\pi i}{\sqrt{x}}$. Using Euler-Maclaurin summation with $k=0$, we get that,

$$\sum_{n=1}^N f(x) = \int_1^N f(x)dx + \int_1^N B_1(x)f'(x)dx + O(1)$$

Since $|B_1(x)| \leqslant 1$, then we have,

$$\left|\int_1^N B_1(x)f'(x)dx\right| \leqslant \int_1^N \left|f'(x)\right| dx \ll \int_1^N \frac{m}{\sqrt{x}}dx = O(m\sqrt{N})$$

For the first integral, we notice that $f(x) = \frac{\sqrt{x}}{m\pi i}f(x)$, therefore,

$$\int_1^N f(x)dx = \int_1^N f'(x)\frac{\sqrt{x}}{m\pi i}dx$$

Then we can use integration by parts to get,

$$\int_1^N f(x)dx = f(x)\frac{\sqrt{x}}{m\pi i}|_1^N + \int_1^N f(x)\frac{1}{2\sqrt{x}m\pi i}dx$$

The first term is clearly $O(\sqrt{N}/m)$ and the second term is also $O(\sqrt{N}/m)$, the proof of which is analogous to how we bounded the second integral. It follows that,

$$\frac{1}{N}\sum_{n=1}^N e^{2\pi i\sqrt{n}} = O(m/\sqrt{N})$$

therefore the sum is $o(N)$ and so Weyl's criterion holds and thus $\sqrt{n}$ is uniformly distributed modulo 1.

Question 10

Let χ be a real primitive character mod q. Show that $L(1,\chi) \ll \log q$. If there is no Siegel zero, namely, there is no zero in the interval $\left(1 - \frac{c}{\log q}, 1\right)$, then show that $L(1,\chi) \gg \frac{1}{\log q}$.

Proof:

We have that,

$$L(1,\chi) = \sum_{n=1}^q \frac{\chi(n)}{n} + \sum_{n=q+1}^\infty \frac{\chi(n)}{n}$$

The first sum is $\ll \log q$ and the second sum let $S(t) = \sum_{n\leqslant t}\chi(n)$, then since $|S(t)| < q$, we have that $\frac{1}{x}S(x) \to 0$ as $x \to \infty$, therefore,

$$\sum_{n>q}\frac{\chi(n)}{n} \ll \int_q^\infty \frac{|S(t)|}{t^2}dt \ll 1$$

Since the integral converges independent of q, then we see that $L(1,\chi) \ll \log q$.

For the opposite in inequality, we know,

$$\log L(1,\chi) - L(s_0,\chi) = \int_{s_0}^1 \frac{L'}{L}(s,\chi)ds$$

Using the formula for $\log L(s,\chi)$, we get,

$$\log L(s,\chi) = \sum_{n=1}^\infty \frac{\chi(n)\Lambda(n)}{n^s\log n} \geqslant -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s\log n} = -\log\zeta(s)$$

We know that $-\log\zeta(s) = \log\frac{1}{s-1} + O(1)$, so taking $s_0 = 1 + \frac{c}{\log q}$, we have that,

$$\log L(s_0, \chi) \gg -\log\log q$$

Now suppose that we know L'/L is $O(\log q)$ on the interval $(1, s_0)$, then we would have that,

$$\log L(1, \chi) \gg \log L(s_0, \chi) + \int_{s_0}^{1} O(\log q) ds \gg -\log\log q + O(1)$$

And it follows that $\log L(1, \chi) \gg -\log\log q$ as desired. Therefore we need only show that $\log L(s, \chi) = O(\log q)$ on the interval $(1, s_0)$.

Question 11

Give an outline of a proof of a function equation of $\zeta(s)$. [You may use the functional equation of the θ-function: Let $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}, z \in \mathbb{C}, \mathrm{Im}(z) > 0$. Set $w(y) = \theta(iy)$, then $w\left(\frac{1}{y}\right) = y^{1/2}w(y)$]

Proof:

By definition, $\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{s/2-1}e^{-t}dt$. Let $t = n^2\pi x$, then $\Gamma\left(\frac{s}{2}\right) = \int_0^\infty n^s\pi^{s/2}x^{s/2-1}e^{-n^2\pi x}dx$, so $\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}n^{-s} = \int_0^\infty x^{s/2-1}e^{-n^2\pi x}dx$. It follows that,

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta s = \sum_{n=1}^{\infty}\int_0^\infty x^{s/2-1}e^{-n^2\pi x}dx$$

Notice that if we write $s = \sigma + it$ with $\sigma > 1$, then,

$$\sum_{n=1}^{\infty}\int_0^\infty \left|x^{s/2-1}e^{-n^2\pi x}\right| dx = \sum_{n=1}^{\infty}\int_0^\infty x^{\sigma/2-1}e^{-n^2\pi x}dx = \Gamma\left(\frac{\sigma}{2}\right)\pi^{-\sigma/2}\zeta(\sigma) < \infty$$

Since the double integral of the absolute value converges, then we may apply Fubini's theorem to get that,

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \int_0^\infty x^{s/2-1}\sum_{n=1}^{\infty} e^{-n^2\pi x}dx$$

By definition of $w(x)$, we get $\sum_{n=1}^\infty e^{-n^2\pi x} = \frac{w(x)-1}{2}$ and therefore,

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \int_0^\infty x^{s/2-1}\frac{w(x)-1}{2}dx = \int_1^\infty x^{s/2-1}\frac{w(x)-1}{2}dx + \int_0^1 x^{s/2-1}\frac{w(x)-1}{2}dx$$

In the second integral make a substitution, $y = x^{-1}$, so $dy = -y^2dx$, then the second integral becomes,

$$\int_0^1 x^{s/2-1}\frac{w(x)-1}{2}dx = \int_1^\infty y^{-1-s/2}\frac{w(1/y)-1}{2}dy$$

Using the functional equation of w, we can write, $\frac{w(1/y)-1}{2} = \frac{y^{1/2}w(y)-1}{2} = y^{1/2}\frac{w(y)-1}{2} + \frac{y^{1/2}}{2} - \frac{1}{2}$, therefore our integral is now,

$$\int_0^1 x^{s/2-1}\frac{w(x)-1}{2}dx = \int_1^\infty \frac{1}{2}y^{-1/2-s/2} - \frac{1}{2}y^{-1-s/2} + y^{-1/2-s/2}\frac{w(y)-1}{2}dy$$

Computing these integrals we see that,

$$\int_0^1 x^{s/2-1}\frac{w(x)-1}{2}dx = \frac{-1}{2\,(1/2-s/2)} - \frac{-1}{2\,(s/2)} + \int_1^\infty x^{-1/2-s/2}\frac{w(x)-1}{2}dx$$

Therefore,

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(x^{s/2-1} + x^{-1/2-s/2}\right)\frac{w(x)-1}{2}dx$$

Letting $\xi(s) = s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)$, this formula yields,

$$\xi(1-s) = 1 + (1-s)(1-s-1)\int_1^\infty \left(x^{(1-s)/2-1} + x^{-1/2-(1-s)/2}\right)\frac{w(x)-1}{2}dx$$

After some trivial algebra, we see that this is the same as $\xi(s)$, so $\xi(1-s) = \xi(s)$ is the functional equation for $\zeta(s)$.

Question 12

Let $\pi_2(x)$ be the number of primes $p \leqslant x$ such that $p+2$ is also a prime. Give an outline of the proof of $\pi_2(x) = O\left(\frac{x}{(\log(x))^2}\right)$. [You may use Selberg's sieve: Given a sequence a_n, let $N(d) = \#\left\{n \leqslant x \middle| d \mid a_n\right\} = \frac{x}{f(d)} + R_d$ for a multiplicative function f. Let $P(z) = \prod_{p\leqslant z} p$. Then

$$N(x,z) = \#\left\{n \leqslant x \middle| (a_n, P(z)) = 1\right\} \leqslant \frac{x}{U(z)} + O\left(\sum_{d_1,d_2\leqslant x} \left|R_{[d_1,d_2]}\right|\right)$$

where $U(z) = \sum_{d\leqslant z}\frac{\mu(d)^2}{f_1(d)}$, and $f(n) = \sum_{d\mid n} f_1(d)$]

Deduce that $\sum_p' \frac{1}{p}$ converges, where the sum is over twin primes.

Proof:

Let $a_n = n(n+2)$ and let $\mathscr{P}$ be the collection of all primes, so $P(z) = \prod_{p\leqslant z} p$. We have that $p \mid n(n+2)$ iff $n(n+2) \cong 0 \pmod p$ which has 2 solutions if p is odd and one solution if $p = 2$. It follows that for primes, we have $N(d) = 2\frac{x}{p} + O(1)$ Let $\nu(d)$ denote the number of odd prime factors of d, then by the Chinese remainder theorem, we have $N(d) = 2^{\nu(d)}\frac{x}{d} + R_d$ where $R_d = O(2^{\nu(d)})$. It follows that our $f(d) = \frac{d}{2^{\nu(d)}}$. The corresponding completely multiplicative function is $\widetilde{f}(d) = \frac{d}{2^{\omega(d)}}$ where $\omega(d)$ counts the number of odd prime factors with multiplicity. The Selberg sieve then tells us that,

$$N(x,z) \leqslant \frac{x}{U(z)} + O\left(\sum_{d_1,d_2\leqslant x} O(2^{\nu([d_1,d_2])})\right)$$

We can bound the error term since $\nu([d_1,d_2]) \leqslant \nu(d_1) + \nu(d_2)$, so,

$$\sum_{d_1,d_2\leqslant x} O(2^{\nu([d_1,d_2])}) = O\left(\sum_{d_1,d_2\leqslant x} 2^{\nu(d_1)}2^{\nu(d_2)}\right) = O\left(\left(\sum_{d\leqslant x} 2^{\nu(d)}\right)^2\right)$$

Now $\frac{\zeta(s)^2}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s}$ therefore $\sum_{d \leqslant x} 2^{\nu(d)} \sim \frac{x \log x}{\zeta(2)}$. For $U(z)$, we know that $U(z) \geqslant \sum_{n \leqslant z} \frac{1}{\tilde{f}(z)} = \sum_{n \leqslant z} \frac{2^{\omega(n)}}{n} \geqslant \sum_{n \leqslant z} \frac{2^{\nu(n)}}{n}$. By partial summation, we get,

$$U(z) \geqslant \frac{1}{z} \left(\sum_{n \leqslant z} 2^{\nu(n)} \right) + \int_1^z \left(\sum_{n \leqslant t} 2^{\nu(n)} \right) \frac{1}{t^2} dt$$

The integral is $\sim \frac{1}{2\zeta(2)} (\log(z))^2$, and the other term is $\sim \frac{1}{\zeta(2)} \log z$, so in total,

$$U(z) \gg (\log z)^2$$

Any twin prime will either be at most z in which case it will not be coprime to $P(z)$, or p and $p+2$ will be prime and so $p(p+2)$ will be coprime to $P(z)$. It follows that $\pi_2(x) \leqslant z + N(x,z)$. Therefore,

$$N(x,z) \ll \frac{x}{(\log z)^2} + O(z^2 (\log z)^2) = O\left(\frac{x}{(\log z)^2} + z^2 (\log z)^2 \right)$$

Now choosing $z = x^{1/4}$, we get that,

$$\pi_2(x) \leqslant x^{1/4} + O\left(\frac{x}{(\log x)^2} + x^{1/2} (\log x)^2 \right)$$

Therefore $\pi_2(x) = O\left(\frac{x}{(\log x)^2} \right)$. By partial summation, we have that,

$$\sum_p{}' \frac{1}{p} = \lim_{n \to \infty} \frac{1}{n} \pi_2(n) + \int_2^n \frac{\pi_2(t)}{t^2} dt = 0 + O\left(\int_2^{\infty} \frac{1}{t(\log t)^2} dt \right)$$

The integral is then $\frac{1}{\log t}\Big|_2^{\infty}$ which converges, and so the sum of the reciprocals of twin primes converges.