MAT1191 - Algebraic Geometry Curves and Surfaces

Ethan Bottomley-Mason, Cole Franklin

January-Feburary 2024

Contents

Throughout these notes, [HA] refers to Hartshorne, [MA] refers to Commutative Ring Theory by Matsumura, and [STA] refers to the stacks project.

1 Curves

1.1 Lecture 1 - Line bundles and divisors

Def: An "equation" for a scheme X is a map $X \to \mathbb{P}^n$. In turn these correspond to a choice of $n+1$ global sections which globally generate a line bundle $\mathscr{L}$ on X.

Def: A *nice curve* over k is a smooth, proper, geometrically integral k-scheme of dimension 1.

Def: A *curve* over k is a geometrically integral k-scheme of dimension 1.

(The following definition was assumed but will be included for completeness)

Def: A scheme X of finite type over a field k is *geometrically integral* (resp. *reduced, irreducible*) if $X \times_k \bar{k}$ is integral (resp. reduced, irreducible). See [HA, Ex II.3.15] for some equivalent properties.

Def: The *arithmetic genus* of a nice curve X/k, denoted $p_a(X)$ or $g(X)$, is $1 - \chi(\mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \omega_X)$.

Recall that the geometric genus of a scheme X/k is $\dim_k \Gamma(X, \omega_X)$. In general, the arithmetic genus of X/k is given by $p_a(X) = (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1)$. If X is a proper integral curve over an algebraically closed field, then $\Gamma(X, \mathcal{O}_X) = k$ [HA, Ex II.4.5d]. If X is nice, then it is geometrically integral and thus $\Gamma(X, \mathcal{O}_X) = k$ still holds (Why?). Therefore $\dim_k H^0(X, \mathcal{O}_X) = 1$, so $p_a(X) = \dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$ by Serre duality. So for nice curves, arithmetic genus and geometric genus coincide.

Def: A *Weil divisor* on a scheme X is a $\mathbb{Z}$-linear combination of integral codimension 1 subschemes. The free abelian group of Weil divisors is denoted $\operatorname{Div} X$.

If X is a nice curve over $k = \bar{k}$ then $\operatorname{Div}(X) = \bigoplus_{p \in X^{cl}} \mathbb{Z}p$

Def: Let X be is a nice curve$/k$, $D = \sum_{p \in X^{cl}} n_p p$, we define the degree of D to be

$$\deg D = \sum_{p \in X^{cl}} n_p [k(p) : k]$$

Here, X^{cl} is the set of closed points of X.

It follows easily that deg is a morphism and that it is stable under base change with a careful definition of the base change morphism).

Let Y be a normal integral finite type k-scheme, regular in codimension 1 (e.g. a nice curve).

Def: Let Y be as above. A *Cartier divisor* on X is a global section of $\mathscr{K}_X^*/\mathcal{O}_X^*$.

$\mathscr{K}_X^*$ is the sheaf of invertible elements of the field of fractions of $\mathcal{O}_X$. $\mathscr{K}_X$ can alternatively be defined as $\iota_* \iota^{-1} \mathcal{O}_X$ where $\iota : \text{Spec } K(X) \to X$ is the natural inclusion. Cartier divisors can be defined more generally, if you care you can look in [HA II.6].

Theorem 1.1.1 (Equivalence of Divisors)

If Y as above and furthermore is locally factorial, then

1. $\text{Div}(Y) \to \Gamma(Y, \mathscr{K}_Y^*/\mathcal{O}_Y^*)$ is an isomorphism.

2. $\Gamma(Y, \mathscr{K}_Y^*/\mathcal{O}_Y^*) \to \{(\mathscr{L}, s) | \mathscr{L} \text{ line bundle}, s \in \Gamma(Y, \mathscr{L} \otimes \mathscr{K}_Y) - \{0\}\}/\sim$ is an isomorphism. Here $\sim$ is an isomorphism of the line bundle mapping the distinguished sections to each other (???).

Proof. 1) Note: this proof is not the one given in class, since it seems we assumed Y had a cover by UFDs which is not generally true. To get the map $\text{Div}\, Y \to \Gamma(Y, \mathscr{K}^*/\mathcal{O}_X^*)$, we consider a divisor $D \in \text{Div}\, Y$. Now let $p \in Y$ and consider the inclusion $\iota_p : \text{Spec } \mathcal{O}_{Y,p} \to Y$, then this induces a divisor $D_p = \iota_p^* D$ (recall, the pullback is given by taking preimages). Since $\mathcal{O}_{Y,p}$ is a UFD, then all divisors are principal [HA II.6.2], therefore $D_p = \text{div}(f_p)$ for some $f_p \in \mathcal{O}_{Y,p}$. Now $\mathcal{O}_{Y,p} \subseteq K(Y)$, then for any point z of codimension 1, the multiplicity of D along z and of f_p along z agree as long as z passes through p. Now there are only finitely many z of codimension 1 such that z does not pass through p and $\nu_z(f_p) \neq 0$ or D has nonzero multiplicity along z (finiteness is from Y being Noetherian, although this is not a trivial consequence). Therefore excluding these codimension 1 subschemes, we obtain an open set U_z on which $\text{div}(f_p)|_{U_p} = D|_{U_p}$. Now $p \in U_p$, so $\{U_p\}$ covers Y and therefore we want to show $\{(U_p, f_p)\}$ is a Cartier divisor. To do so, we just want to show that for any p, p', we have that $f_p/f_p' \in \mathcal{O}_Y^*$ locally. To do so we cover $U_p \cap U_{p'}$ by affine opens. Now in any open affine Spec R we have some element $h = f_p/f_p' \in \text{Frac}(R)$ such that $\nu_{\mathfrak{p}}(h) = 0$ for all $\mathfrak{p}$ of height 1. Since R is normal, then $R = \bigcap_{\text{ht }(\mathfrak{p})=1} R_{\mathfrak{p}}$ and therefore $h \in R$. Similarly $h^{-1} \in R$, so $h \in \mathcal{O}_Y^*(\text{Spec } R)$. It follows that $\{(U_p, f_p)\}$ is a Cartier divisor.

The inverse map is much easier. Given any Cartier divisor $\{(U_i, f_i)\}$, we let $D = \sum_z \nu_z(f_i) z$ where $z \in U_i$. Note that if $z \in U_i \cap U_j$, then f_i/f_j is a unit in $\mathcal{O}_X(U_i \cap U_j)$, thus $\nu_z(f_i) = \nu_z(f_j)$, so D is well-defined. One (not me) checks that this is the inverse to the previously described map.

2) We have an exact sequence,

$$0 \to \mathcal{O}_Y^* \to \mathscr{K}_Y^* \to \mathscr{K}_Y^*/\mathcal{O}_Y^* \to 0$$

Now notice that $\mathscr{K}_Y^*$ is a constant sheaf and thus flasque, therefore $H^1(X, \mathscr{K}_Y^*) = 0$. Now we can take a long exact sequence of cohomology and get a surjective map $\delta : \Gamma(Y, \mathscr{K}_Y^*/\mathcal{O}_Y^*) \to H^1(Y, \mathcal{O}_Y^*)$. Now taking an affine open cover $\{U_i\}$ of Y, we can compute $H^1(Y, \mathcal{O}_Y^*)$ using Čech cohomology as we have maps

$$\bigoplus_i \mathcal{O}_Y^*(U_i) \to \bigoplus_{i,j} \mathcal{O}_Y^*(U_i \cap U_j) \to \bigoplus_{i,j,k} \mathcal{O}_Y^*(U_i \cap U_j \cap U_k)$$

we compute $H^1(Y, \mathcal{O}_X^*)$ as the kernel of the second modulo the image of the first. It follows that we have sections $f_{ij} \in \mathcal{O}_Y^*(U_i \cap U_j)$ which thus describe isomorphisms $f_{ij} : \mathcal{O}_Y|_{U_i \cap U_j} \to \mathcal{O}_Y|_{U_i \cap U_j}$. Since the collections of sections is in the kernel of the boundary map, then they satisfy the cocycle condition, so $f_{ij} f_{jk} = f_{ik}$ on $U_i \cap U_j \cap U_k$. It follows that these isomorphisms define a line bundle $\mathscr{L}$ on Y. We now want to exhibit a global section s of $\mathscr{L} \otimes \mathscr{K}$ which determines the embedding into $\mathscr{K}$. To do so, consider the global section given by $1 \otimes f_{ij}$ in $\mathcal{O}_X(U_{ij}) \otimes \mathscr{K}(U_{ij})$ and then taking the image in $(\mathscr{L} \otimes \mathscr{K})(U_{ij})$. We need to check that $1 \otimes f_{ij} = 1 \otimes f_{kl}$ on any quadruple intersection, however it suffices to just show it in the case where $k = j$, since then we can use the same argument twice. So we want to show that applying the transition map to $1 \otimes f_{ij}$ yields $1 \otimes f_{jl}$. The transition map is exactly multiplication by the restriction of f_{il} and thus $f_{il}(1 \otimes f_{ij}) = 1 \otimes (f_{il} f_{ij}) = 1 \otimes f_{jl}$ as desired. $\square$

Remark

If Y is a curve, then $\mathscr{K}^*/\mathcal{O}^* \cong \sum_{p \in Y^{cl}} i_* \mathbb{Z}$. The isomorphism comes from the valuation on the stalks $(\mathscr{K}^*/\mathcal{O}^*)_p$ as a normal curve Y is locally a DVR [HA I.6.2A or MA 11.2].

1.2 Lecture 2 - Maps into $\mathbb{P}^n$, Riemann-Roch, and Serre Duality

Let k be a field and X a nice curve $/k$. Note that X is then locally factorial since X smooth, then X is geometrically regular at any point [STA, 33.12.6] and therefore X is regular. Since X is one-dimensional, then one dimensional regular local rings are UFDs (in fact, they are PIDs). Therefore X is locally factorial.

We have the map of thm 1.1.1 from $\operatorname{Div} X \to \left\{(\mathscr{L}, s) \middle| \mathscr{L} \text{ line bundle}, s \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{K}\right\} / \sim$. This is an isomorphism when X is nice by the above discussion. Two line bundles are isomorphic $\mathscr{L} \cong \mathscr{L}'$ iff they differ by an element of k^* (this is true for line bundles contained in $\mathscr{K}$, I think).

How can we explicitly describe this map? We may define it on the generators of $\operatorname{Div} X$, i.e. all closed points of X, then the rest of the map will be given by taking duals and tensor products. We map $p \mapsto \mathcal{O}_X(p) := \check{\mathscr{I}_p}$ where $\mathscr{I}_p$ is the ideal sheaf of p.

To see that $\mathscr{I}_p$ is a line bundle, we want to show that it is locally free of rank 1. To show that a coherent module is locally free, it suffices to show that it is free on stalks (idk if this is true for quasi-coherent modules). For any point $z \in X$, then $(\mathscr{I}_p)_z \subseteq \mathcal{O}_{X,z}$ which is a DVR, then picking a uniformizing parameter t, we have that $(\mathscr{I}_p)_z = (t^n)$ for some n and therefore $(\mathscr{I}_p)_z = t^n \mathcal{O}_{X,z}$ is free. Alternatively, $\mathcal{O}_{X,z}$ is a DVR and thus a PID, and $(\mathscr{I}_p)_z$ is clearly torsion-free ($\mathcal{O}_{X,z}$ is integral) and therefore by the structure theorem for modules over a PID, $(\mathscr{I}_p)_z$ is free.

We have an inclusion $\iota : \mathscr{I}_p \hookrightarrow \mathcal{O}_X$, which is an element of $\check{\mathscr{I}_p}(X) = \mathscr{H}om(\mathscr{I}_p, \mathcal{O}_X)(X)$. The distinguished section to which we map $p \in X$ is the section $\iota \otimes 1 \in \mathscr{I}_p \otimes_{\mathcal{O}_X} \mathscr{K}$ (do you have to sheafify $\mathscr{I}_p \otimes_{\mathcal{O}_X} \mathscr{K}$ or is the tensor product presheaf already a sheaf?).

Def: Let $\mathcal{O}_X(D) = \bigotimes_{p \in X^{cl}} \mathcal{O}_X(p)^{n_p}$ where $D = \sum n_p p$.

Let X be a nice curve, then the map of thm 1.1.1 is given by $D \mapsto \mathcal{O}_X(D)$. Here $\mathcal{O}_X(p)^n$ is the n-fold tensor product or the n-fold tensor product of the dual. For locally free sheaves, the tensor product and dual commute. Given φ, ψ functionals on M, N, $\varphi \otimes \psi$ is a functional on $M \otimes N$ and is an isomorphism in the case of free modules.

For $\mathcal{O}_X(-p) = \mathscr{I}_p$, the distinguished section is the inverse of the morphism $s \otimes \mathrm{id} : \mathscr{I}_p \otimes \mathscr{K} \to \mathcal{O}_X \otimes \mathscr{K}$. Note that $\mathscr{I}_p \to \mathcal{O}_X$ is in general not an isomorphism, but after tensoring with $\mathscr{K}$ it becomes an isomorphism and thus has an inverse $s^{-1} : \mathcal{O}_X \otimes \mathscr{K} \to \mathscr{I}_p \otimes \mathscr{K}$, then this is given by a global section of $\mathscr{I}_p \otimes \mathscr{K}$.

$\Gamma(U, \mathcal{O}_X(D)) = \left\{ t \in \Gamma(U, \mathscr{K}) \middle| \operatorname{Div} t \geq -D|_U \right\}$ and the distinguished global section is $s = 1$.

Def: $D = \sum_p n_p p \in \operatorname{Div}(X)$ is *effective* if $n_p \geq 0$ for all p.

Remark

D is effective iff the distinguished section $s \in \Gamma(X, \mathcal{O}_X(D) \otimes \mathscr{K})$ is regular, i.e. $s \in \Gamma(X, \mathcal{O}_X(D))$

Def: Two divisors D, D' are *linearly equivalent* if $D - D' = div(s)$ for some $s \in \Gamma(X, \mathscr{K})$.

Claim 1.2.1 (Linear Equivalence is Isomorphism)

Let X be a smooth, integral scheme. Two divisors $D \sim_{lin} D'$ iff $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$.

Proof. First suppose that $f \in \hom(\mathcal{O}_X(D), \mathcal{O}_X(D')) \cong \hom(\mathcal{O}_X, \mathcal{O}_X(D' - D))$ is an isomorphism. Then $1/f(1) \in \Gamma(X, \mathscr{K}^*)$ has divisor $D' - D$ since $f(1)$ generates $\mathcal{O}(D' - D)$.
Now suppose that $D \sim_{lin} D'$ so that $D - D' \sim 0$. It suffices to show that $\mathcal{O}(D' - D) \cong \mathcal{O}_X$. There is some $s \in \Gamma(X, \mathscr{K})$ with $D - D' = \operatorname{div}(s)$. Then $\mathcal{O}(\operatorname{div} s)$ is globally generated by $\frac{1}{s}$, so $1 \mapsto s^{-1}$ is the desired isomorphism. $\square$

Def: Let $Pic(X)$ be the group of line bundles on X up to isomorphism with operation given by tensor product.

Corollary:

There exists a diagonal map making the following diagram commute.

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[r] \arrow[d, "\deg"] & \mathrm{Pic}(X) \arrow[dl, dashed] \\ \mathbb{Z} & \end{tikzcd}$$

We content ourselves with a reference [HA, II.6.10]

Corollary:

The set of effective divisors linearly equivalent to a given divisor D is in bijection with $\mathbb{P}\Gamma(X, \mathcal{O}_X(D))(\bar{k})$ (the $\bar{k}$-valued points of the projective space).

I don't know where this $\bar{k}$-values points comes into this, so I will pretend that $k = \bar{k}$. The divisor of zeros D_0 of $D = \sum_p n_p p$ is $D_0 = \sum_{n_p \geqslant 0} n_p p$. [HA, II.7.7]

Proof. First let $s \in \mathbb{P}\Gamma(X, \mathcal{O}_X(D))$ then s corresponds to an element t of $\Gamma(X, \mathcal{O}_X(D)) \subset \Gamma(X, \mathscr{K})$. If D is locally given as a cartier divisor with $\{(U_i, f_i)\}$ with $f_i \in K^*$. Then $\mathscr{L}(D)$ is locally generated by f_i^{-1} so we get a local isomorphism $\varphi : \mathcal{O}_X(D) \to \mathcal{O}_X$ by multiplying s by f_i (this gives a map by the cocyle condition). Then $D = \mathrm{div}_0 t = D' + \mathrm{div} s$ so $D \geqslant 0$ and $D \sim D'$. It is clear that you get the same divisor if you replace t by another representative λt, so it is well defined.
Now, given $D' \geqslant D$ and $D \sim D'$, we have $D' = D + \mathrm{div} f$ for some $f \in \mathscr{K}$. Thus $f \geqslant -D$ since $D' \geqslant 0$. Thus $f \in \Gamma(X, \mathcal{O}_X(D))$ has divisor of zeros D'. You can check that any two f_1, f_2 you get from the second map are related by $f_1 = \lambda f_2$ and that these two maps are inverses. □

Mappings to $\mathbb{P}V$

Thm 1.2.2 (Maps into Projective Space)

Let V be a vector space over a field k and X a scheme, then

$$\mathrm{Hom}(X, \mathbb{P}V) = \left\{ V \otimes \mathcal{O}_X \xrightarrow{\varphi} \mathscr{L} \middle| \varphi \text{ surjection }, \mathscr{L} \in \mathrm{Pic}(X) \right\} / \sim$$

Note: V is a k-vector space, so V is a module on Spec k and we thus denote by V the pull-back along the structure map $X \to \mathrm{Spec}\ k$

Proof. Pick a basis $e_1, \cdots, e_n$ of V with images $s_1, \cdots, s_n \in \Gamma(X, \mathscr{L})$. Let $D_{s_i} = \left\{ p \in X \middle| (s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p \right\}$. In general, such sets are not open, but since $\mathscr{L}$ is locally free, then for any point $p \in D_{s_i}$, take a nbhd U of p such that $\mathscr{L}|_U \cong \mathcal{O}_X|_U$, then let t be the image of s_i in $\mathcal{O}_X|_U$, then $D_{s_i} \cap U = D(t)$ is open. Since we have a surjection, then $\mathscr{L}$ is globally generated by the s_i which is to say that $X = \bigcup D_{s_i}$. On any D_{s_i}, the morphism $\varphi_i : \mathcal{O}_X|_{D_{s_i}} \to \mathscr{L}|_{D_{s_i}}$ given by $x \mapsto x s_i$ is an isomorphism on stalks and thus expresses the freeness of $\mathscr{L}$ on D_{s_i}. It follows that s_j / s_i makes sense as an element of $\mathcal{O}_X(D_{s_i})$ by this isomorphism (it is just $\varphi^{-1}(s_j)$).

Now consider maps $f_i : D_{s_i} \to \mathbb{A}_k^{n-1}$ given by the ring maps $k[x_1, \cdots, x_n] \mapsto \Gamma(D_{s_i}, \mathcal{O}_X)$ by $x_j \mapsto s_j/s_i$. Note that D_{s_i} is not affine, but these ring maps still define scheme maps [HA, Ex II.2.4]. Now we have an open cover of $\mathbb{P}V$ given by $n+1$ copies of $\mathbb{A}_k^{n-1}$, so to check that the f_i yield a map from X to $\mathbb{P}V$, we need only check that they glue. On $D_{s_i} \cap D_{s_k}$ we have two maps f_i and f_k. The transition map from $\mathbb{A}_k^{n-1} = \text{Spec } k[x_0, \cdots, \hat{x}_i, \cdots, x_n] \to \text{Spec } k[y_0, \cdots, \hat{y}_k, \cdots, y_n]$ is given by thinking of x_j and y_j as z_j/z_i and z_j/z_k respectively. With this, the map is then obvious: $y_j = z_j/z_i \cdot z_i/z_k = x_j/x_k$ and $x_j = z_j/z_i = z_j/z_k \cdot z_k/z_i = y_j/y_i$ defined on the localizations at x_k and y_i respectively. To check gluing, we map $x_j \to s_j/s_i$ in $\mathcal{O}_X(D_{s_i} \cap D_{s_k})$ under f_i and under f_k, we apply the transition map $x_j \to y_j/y_i$ which under f_k maps to $s_j/s_k \cdot s_k/s_i = s_j/s_i$. There is slightly more to check than I let on since s_j/s_k and s_k/s_i are defined by isomorphisms, so a priori we don't know that they multiply like rationals, although that is a simple check using the isomorphisms. Therefore the maps glue and we obtain a well-defined morphism $f : X \to \mathbb{P}V$. $\square$

This shows that more properly, $\mathbb{P}V$ corresponds naturally with 1 dimensional quotients of V with the correspondence

$$p \mapsto \text{coker}(V \to \mathscr{L}_p)$$

This should again be thinking of V as an $\mathcal{O}_X$-module given by a pullback.

Def: A *linear system* $(V, \mathscr{L})$ is a line bundle $\mathscr{L}$ along with a subspace $V \subseteq \Gamma(X, \mathscr{L})$. The *complete linear system* $|\mathscr{L}|$ is $(\Gamma(X, \mathscr{L}), \mathscr{L})$. The dimension of $(V, \mathscr{L})$ is $\dim V - 1$.

Given a linear system $(V, \mathscr{L})$, there exists a natural map

$$V \otimes \mathcal{O}_X \hookrightarrow \Gamma(X, \mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L}$$

the second map being given by evaluation, since $\Gamma(X, \mathscr{L}) \cong \text{Hom}(\mathcal{O}_X, \mathscr{L})$ naturally. Equivalently we get the second map from the adjunction $\pi^*\pi_*\mathscr{L} \to \mathscr{L}$ where $\pi : X \to \text{Spec } k$ is the structure map.

Def: $(V, \mathscr{L})$ is *basepoint free* (bpf) if $V \otimes \mathcal{O}_X \to \mathscr{L}$ is surjective. Equivalently, $\forall p \in X, \exists s \in V$ such that s does not vanish at p (i.e. s_p is not in $\mathfrak{m}_p\mathscr{L}_p$ where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$).

Def: $(V, \mathscr{L})$ is *very ample* if the induced map $X \xrightarrow{(V,\mathscr{L})} \mathbb{P}V$ is an embedding.

Riemann-Roch

How can we get a handle on $\dim |\mathscr{L}| = \dim H^0(X, \mathscr{L}) - 1$. When can we figure out if $\mathscr{L}$ is bpf or very ample?

Theorem 1.2.3 (Riemann-Roch)

Let X be a nice curve and $\mathscr{L}$ a line bundle, then,

$$\chi(\mathscr{L}) = \deg \mathscr{L} + 1 - g$$

Equivalently, for a Weil divisor D, $\chi(\mathcal{O}_X(D)) = \deg D + 1 - g$.

Proof. We first show this is true for $\mathcal{O}_X$, then show that it is true when we tensor with $\mathcal{O}_X(p)$ for a point p at which point the rest of the proof will follow easily.

Step 1: If $\mathscr{L} = \mathcal{O}_X$, then $h^0(X, \mathcal{O}_X) = 1, h^1(X, \mathcal{O}_X) = g$ and $\deg \mathcal{O}_X = 0$, so the claim holds trivially.

Step 2: Suppose the claim holds for $\mathcal{O}_X(D)$, then let p be any point. We want to show that the claim also holds for $\mathcal{O}_X(D+p)$. We see that $\deg(D+p) = \deg D + [k(p) : k]$, so we just need to show that $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D+p)) - [k(p) : k]$. We have a short exact sequence,

$$0 \to \mathscr{I}_p = \mathcal{O}_X(-p) \to \mathcal{O}_X \to \iota_* k(p) \to 0$$

where ι is the inclusion $p \to X$. Now $\mathcal{O}_X(D+p)$ is locally free, so tensoring is exact, thus we obtain,

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+p) \to \iota_* k(p) \to 0$$

Note that $\iota_* k(p) \otimes \mathscr{F} = \iota_* k(p)$ for any sheaf $\mathscr{F}$ which is locally free of rank 1 on a nbhd of p. Now ι is a closed immersion, so it induces an isomorphism on cohomology groups, $H^n(X, \iota_* k(p)) = H^n(p, k(p))$, so $\chi(\iota_* k(p)) = [k(p) : k]$. Note that you can see a closed immersion induces isomorphisms since it is locally a quotient and thus affine, so the pushforward is exact and thus $R^i \iota_* = 0$ for $i > 0$ from which the claim follows. Now the Euler characteristic is additive (by the LES of cohomology), so $\chi(\mathcal{O}_X(D+p)) = \chi(\mathcal{O}_X(D)) + [k(p) : k]$ as desired. Now since any line bundle is of the form $\mathcal{O}_X(D)$ an easy induction shows that Riemann-Roch holds for all $\mathcal{O}_X(D)$ as desired. □

Corollary: (Riemann's inequality)

$h^0(X, \mathscr{L}) \geqslant \deg \mathscr{L} + 1 - g$ i.e. $\dim |\mathscr{L}| \geqslant \deg \mathscr{L} - g$.

Theorem 1.2.4 Serre Duality

Let X/k be a smooth, proper, of pure dimension n, geometrically connected. Let $\omega_X = \bigwedge^{top} \Omega_{X/k}$. Then,

1. There is a canonical isomorphism $tr : H^n(X, \omega_X) \to k$.
2. For $\mathscr{E}$ locally free, the cup product,

$$H^i(X, \mathscr{E}) \times H^{n-i}(X, \check{\mathscr{E}} \otimes \omega_X) \to H^n(X, \mathscr{E} \otimes \check{\mathscr{E}} \otimes \omega_X) \to H^n(X, \omega_X) = k$$

is a perfect pairing (i.e. a bilinear map $\varphi : M \times N \to k$ such that $m \mapsto \varphi(m, \cdot)$ is an isomorphism).

A proof of Serre Duality in the projective case is given in [HA III.7.5]. A general proof can probably be found in EGA.

Corollary:

For $\mathscr{E}$ locally free, $h^i(X,\mathscr{E}) = h^{n-i}(X,\check{\mathscr{E}} \otimes \omega_X)$.

Theorem 1.2.5 (Serre Duality for nice curves)

$$h^1(X,\mathscr{L}) = h^0(X,\omega_X \otimes \check{\mathscr{L}}).$$

If $\mathscr{L} = \mathcal{O}_X(D)$ this becomes

$$h^1(X,\mathcal{O}_X(D)) = h^0(X,\omega_X(-D))$$

Daniel alluded to a nice description of the Serre duality cup product in Čech cohomology. I could not find a reference, but in principle you can just find Čech cocycle representations of each group via the natural isomorphisms between sheaf and Čech cohomology.

1.3 Lecture 3 - Computing the Serre Duality maps and some basic application of R.R.

For the rest of the lecture X will be a nice curve $/k$.

R.R. + Serre Duality

If $\mathscr{L} \in \mathrm{Pic}(X)$,

$$\dim H^0(X,\mathscr{L}) - \dim H^0(X,\omega_X \otimes \check{\mathscr{L}}) = \deg \mathscr{L} + 1 - g$$

if $\mathscr{L} = \mathcal{O}(D)$ we can write the left hand side as $\ell(D) - \ell(K - D)$.

How to Compute Serre Duality for Curves:

Let P be a closed point of X, then we get a SES

$$0 \to \omega_X \to \omega_X(P) \to \omega_X(P)|_P \to 0$$

where $\omega_X(P)|_P = \omega_X(P) \otimes k(P)$ is the fibre at P and we often write $(D) = \otimes\mathcal{O}(\mathcal{D})$ for a divisor D and line bundle . Then taking the LES in cohomology we get a boundary map

$$k \cong H^0(X,\omega_X(P)|_P) \to^d H^1(X,\omega_X)$$

which sends $\frac{dz}{z}$ to 1 where z is a local parameter at P.

Claim: (Daniel said this)

$\frac{dz}{z} = \frac{dt}{t}$ in $\omega_X(P)|_P$ for any local parameters z, t at P.

We then consider how to compute the trace map in Čech Cohomology. As we will see in the next lecture, all nice curves are projective, so by Ex III.4.8(d), we can cover X by to open affines $\mathfrak{U} = \{U_1, U_2\}$. Then the Čech complex corresponding to $\mathfrak{U}$

$$C^\bullet(\mathfrak{U}, \omega_X) : H^0(U_1, \omega_X) \oplus H^0(U_2, \omega_X) \xrightarrow{\delta} H^0(U_1 \cap U_2, \omega_X)$$

has only two nontrivial terms, where $\delta(s,t) = s - t$ restricted to $U_1 \cap U_2$. Then $H^0(X, \omega_X) = \ker \delta$ and $H^1(X, \omega_X) = \operatorname{coker}\delta$. So we can define the trace map from the following diagram

$$\begin{tikzcd}
H^1(X, \omega_X) \arrow[r, dashed] & k & \sum_{P \notin U_1 \cap U_2} \operatorname{res}_P \eta \\
H^0(U_1 \cap U_2, \omega_X) \arrow[u, "\pi"] \arrow[ur] & \eta \arrow[ur, mapsto] &
\end{tikzcd}$$

From this point there are a couple of things left to check, which Daniel left to us as a Fun™ Exercise. (e.g. that $\ker(H^0(U_1 \cap U_2, \omega_X) \to k) \supseteq \ker \pi$).

The following is an aside, the contents of which are mainly relevant if you know more about complex manifold theory than scheme theory. (In which case you likely know more than the one writing this.)

Remark

Let X be a compact Riemann Surface and $\mathscr{E}$ a holomorphic vector bundle on X.
Then there is a very natural complex with which we can compute cohomology, namely the Dolbeault complex $C^\bullet_{dol}(X, \mathscr{E})$:

$$C^\infty(X, \mathscr{E}) \xrightarrow{\bar{\partial}} C^\infty(X, \mathscr{E} \otimes \Omega^{0,1}).$$

If $\mathscr{E} = \omega_X$, then the trace map tr is given by

$$C^\infty(X, \mathscr{E} \otimes \Omega^{0,1}) = C^\infty(\text{2-forms}) \to \mathbb{C}$$
$$\eta \mapsto \int_X \eta$$

We now consider the cup product map. In Čech cohomology the map corresponds to

$$H^0(X, \mathscr{E}) \otimes H^0(U_1 \cap U_2, \check{\mathscr{E}} \otimes \omega_X) \to H^0(U_1 \cap U_2, \omega_X)$$

But there are the natural maps

$$H^0(X, \mathscr{E}) \otimes H^0(U_1 \cap U_2, \check{\mathscr{E}} \otimes \omega_X) \to H^0(U_1 \cap U_2, \mathscr{E} \otimes \check{\mathscr{E}} \otimes \omega_X) \xrightarrow{ev} H^0(U_1 \cap U_2, \omega_X) \xrightarrow{tr} k$$

giving us the cup product upon taking the appropriate quotients.

Corollary

$\deg \omega_X + 1 - g = \dim H^0(X, \omega_X) - \dim H^1(X, \omega_X) = g - 1$, so $\deg \omega_X = 2g - 2$.

Def: If $\omega_X = \mathcal{O}_X(D)$, then we say D is a *canonical divisor* and K_X is the linear equivalence class of D.

Theorem 1.4.1 (Cohomology vanishing by degree)

Let $\mathscr{L}$ be a line bundle on X.

1. If $\deg \mathscr{L} < 0$, then $\dim H^0(X, \mathscr{L}) = 0$.
2. If $\deg \mathscr{L} = 0$, then $\dim H^0(X, \mathscr{L}) \neq \iff \mathscr{L} \cong \mathcal{O}_X$.
3. If $\deg \mathscr{L} > 2g - 2$, then $H^1(X, \mathscr{L}) = 0$.
4. If $\deg \mathscr{L} = 2g - 2$, then $H^1(X, \mathscr{L}) = 0 \iff \mathscr{L} \cong \omega_X$

Proof.

1. If $\mathcal{O}_X(D)$ were an effective divisor linearly equivalent to $\mathscr{L}$, then we would have that $\deg \mathscr{L} = \deg \mathcal{O}_X(D) \geqslant 0$ which is impossible. It follows that there are no effective divisors linearly equivalent to $\mathscr{L}$. Since $H^0(X, \mathscr{L})$ is in bijection with effective divisors linearly equivalent to $\mathscr{L}$ up to multiplication by k^*, then $\dim H^0(X, \mathscr{L}) = 0$.
2. If $\mathscr{L} = \mathcal{O}_X$, then $H^0(X, \mathscr{L}) = 1 \neq 0$. Conversely, if $H^0(X, \mathscr{L}) \neq 0$, then there is an effective divisor D linearly equivalent to $\mathscr{L}$, so $\mathscr{L} \cong \mathcal{O}_X(D)$, but then $\mathcal{O}_X(D)$ is a degree 0 effective divisor. Since effective divisors are non-negative sums of prime divisors, then the only degree 0 effective divisor is $\mathcal{O}_X$. It follows that $\mathscr{L} \cong \mathcal{O}_X(D) \cong \mathcal{O}_X$.
3. This is essentially the same as 1 using Serre duality. Since $\deg \mathscr{L} > 2g-2$, then $\deg(\omega_X \otimes \check{\mathscr{L}}) = 2g - 2 - \deg \mathscr{L} < 0$, thus $\dim H^0(X, \omega_X \otimes \check{\mathscr{L}}) = 0$ by 1. By Serre duality, we then get $\dim H^1(X, \mathscr{L}) = 0$.
4. This again follows from 2 using Serre duality. We get that $\deg \omega_X \otimes \check{\mathscr{L}} = 0$, so $\dim H^0(X, \omega_X \otimes \check{\mathscr{L}}) \neq 0$ iff $\omega_X \otimes \check{\mathscr{L}} = \mathcal{O}_X$, i.e. iff $\mathscr{L} \cong \omega_X$. Now we just use Serre duality once more, so $\dim H^1(X, \mathscr{L}) \neq 0 \iff \mathscr{L} \cong \omega_X$.

$\square$

1.4 Lecture 4 - Bpf and very ample divisors and genus 0 curves

Recall that $\operatorname{Hom}(X, \mathbb{P}V) = \left\{V \otimes_k \mathcal{O}_X \twoheadrightarrow \mathscr{L} \middle| \mathscr{L} \text{ line bundle} \right\}$ where $\mathbb{P}V = \operatorname{Proj} \operatorname{Sym}^* V$. Given $V \otimes \mathcal{O}_X \twoheadrightarrow \mathscr{L}$, we want to get a map $X \to \mathbb{P}V$ given by taking a point $x \in X$ and mapping it to $V \otimes k(x) \to \mathscr{L} \otimes k(x)$ which is the fiber over the point x, the cokernel of which is then a point in $\mathbb{P}V$.

The one dimensional subspaces of V are in correspondence with hyperplanes in $\mathbb{P}V$ given by taking a 1-dim subspace $\ell \subseteq V$ and sending it to $\left\{f : V \to k \middle| f(\ell) = 0\right\}$. The quotient map $V \to V/\ell$ induces a map $\mathbb{P}(V/\ell) \to \mathbb{P}V$ which is the map $V \otimes \mathcal{O}_{\mathbb{P}(V/\ell)} \to (V/\ell) \otimes \mathcal{O}_{\mathbb{P}(V/\ell)} \xrightarrow{ev} \mathcal{O}(1)$.

We can specialize to the case of a complete linear system: $H^0(X,\mathscr{L}) \otimes \mathcal{O}_X \xrightarrow{ev} \mathscr{L}$ given by taking $x \mapsto (s \to s(x))$.

There is a bijection,

$$\left\{\begin{matrix}\text{effective divisors}\\ D,\ \mathscr{L} \cong \mathcal{O}(D)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{hyperplanes of functions } H^0(X,\mathscr{L}) \to k\\ \text{vanishing on } s, \text{ where } \operatorname{div}_0 s = D\end{matrix}\right\}$$

The map is given by sending an effective divisor D to H_D, the hyperplane of functions vanishing on D.

If $s(x) = 0$, then $x \in V(s) = D$, so for any $x \in f^{-1}(H_D)$ we have that $s(x) = 0$, thus $x \in D$, so set theoretically, $D = f^{-1}_{|\mathscr{L}|}(H_D)$.

Theorem 1.4.2 (Bpf/Very Ample Criterion)

Let X be a nice $/k$, where k is algebraically closed. Let $\mathscr{L}$ be a line bundle on X then

1. $|\mathscr{L}|$ is base-point-free (bpf) iff $\forall p \in X(k), h^0(X,\mathscr{L}) = h^0(X,\mathscr{L}(-p)) + 1$.
2. $|\mathscr{L}|$ is very ample iff $\forall p,q \in X(k)$, (not necessarily distinct) $h^0(X,\mathscr{L}) = h^0(X,\mathscr{L}(-p-q)) + 2$.

Remark

For $k \neq \bar{k}$ it is enough to base change to $\bar{k}$ then check the conditions ([Daniel mentioned this very loosely, so this is our interpretation] because $\bar{k}/k$ is a faithfully flat base change so we get an iff for surjective before and after the base change).

1. *Proof.* We want $H^0(X,\mathscr{L}) \otimes \mathcal{O}_X \xrightarrow{ev} \mathscr{L}$ to be surjective iff the stated condition holds. Surjectivity of a map of sheaves is equivalent to surjectivity at all closed points.[1] By Nakayama's Lemma, it is thus equivalent to check surjectivity on fibers at all closed points $p \in X$. So

$$ev \text{ surjective} \iff H^0(X,\mathscr{L}) \twoheadrightarrow^{ev_p} \mathscr{L}|_p, \qquad \forall p \in X$$

iff $H^0(X,\mathscr{L}(-p)) = \ker ev_p \subsetneq H^0(X,\mathscr{L})$. But we have a SES

$$0 \to \mathscr{L}(-p) = \mathscr{I}(p) \to \mathscr{L} \to \mathscr{L} \otimes k(p) \to 0$$

which gives the LES

$$0 \to H^0(X,\mathscr{L}(-p)) \to H^0(X,\mathscr{L}) \xrightarrow{ev_p} \mathscr{L} \otimes k(p) \to \cdots$$

So we have $H^0(X,\mathscr{L}(-p)) \subsetneq H^0(X,\mathscr{L})$, $\iff$ $\mathrm{rk} ev_p = 1$ for all $p \in X(k)$ $\iff$ $h^0(X,\mathscr{L}) = h^0(X,\mathscr{L}(-p)) + 1$. ∎

Remark

The proof also shows that it suffices that $h^1(X,\mathscr{L}(-p)) = 0$ for all closed points p.

[1] Every point is a further localization of a closed point, and localization preserves surjections since it is exact.

2. Invoking II.7.3 would substantially simplify this proof, but we do it in detail because that's what Daniel did. We also only prove that the given condition implies that $\mathscr{L}$ is very ample, the other direction being in Hartshorne, for instance.

Proof. We first show that $\mathscr{L}$ is bpf so that $\mathscr{L}$ defines a map from X to a projective space. Let $p \neq q$ be closed points. The assumption gives us that

$$H^0(X, \mathscr{L}) \twoheadrightarrow (k(p) \oplus k(q)) \otimes \mathscr{L}$$

So in particular

$$H^0(X, \mathscr{L}) \twoheadrightarrow \mathscr{L} \otimes k(p)$$

so $\mathscr{L}$ is bpf. Thus $\mathscr{L}$ induces a map $f : X \to \mathbb{P}H^0(X, \mathscr{L})$. To show that $\mathscr{L}$ is very ample, we must show that f is a closed immersion, for which we need to check that (a) f is a homeomorphism onto its image and (b) $\mathcal{O}_{\mathbb{P}} \twoheadrightarrow f_*\mathcal{O}_X$.

(a) Because X is proper, it is enough to check that f is injective and indeed for that it is enough to check that f is injective on closed points. Therefore, given $p \neq q$, we need to show that $f(p) \neq f(q)$ so by "previous remarks" (I'm not clear on what this was), it is ETS that $t \in H^0(X, \mathscr{L})$ with $t(p) = 0$ and $t(q) \neq 0$. But this is the case since by the assumption

$$H^0(X, \mathscr{L}) \twoheadrightarrow (k(p) \oplus k(q)) \otimes \mathscr{L}$$

(b) Since f has closed image, then $f_*\mathcal{O}_X$ is supported on the image so it is enough to check surjectivity on points of $f(X)$. Since f is homeomorphism, for any closed point p the map on stalks is

$$\mathcal{O}_{\mathbb{P}, f(p)} \to (f_*\mathcal{O}_X)_{f(p)} = \mathcal{O}_{X,p}$$

By 7.4, it is enough to check that $\mathfrak{m}_{f(p)} \twoheadrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2$. [2] $\mathcal{O}_{X,p}$ is a DVR since X is a nice curve so it is enough to check that the map is nonzero, i.e. that there is a $s \in H^0(X, \mathscr{L})$ vanishing at P to order 1.

Unwinding definitions: $\mathscr{L} = f^*\mathcal{O}(1)$ so $H^0(\mathbb{P}, \mathcal{O}(1)) = H^0(X, \mathscr{L})$, which is true because $\mathbb{P} = \operatorname{Proj} \operatorname{Sym}_\bullet H^0(X, \mathscr{L})$ and $\Gamma(\mathbb{P}, \mathcal{O}(n))$ is nice for polynomial rings by II.5.13. We know $h^0(X, \mathscr{L}(-2p)) = 2 + h^0(X, \mathscr{L})$ so $h^0(X, \mathscr{L}(-p)) = 1 + h^0(X, \mathscr{L})$ (each twist changes it by at most one). Thus we have

$$H^0(X, \mathscr{L}(-2p)) \subsetneq H^0(X, \mathscr{L}(-p)) \subsetneq H^0(X, \mathscr{L})$$

which is exactly what we need.

□

Corollary

Let X be a nice curve of genus g, then

1. All line bundles $\mathscr{L}$ with $\deg \mathscr{L} > 2g - 1$.
2. All line bundles $\mathscr{L}$ with degree $\deg \mathscr{L} > 2g$ are very ample.

[2] (a) follows since $k = k(f(p)) \to k(p) = k$ is an isomorphism since k is algebraically closed and (c) is trivial since everything is f.g. $/k$

Proof. First pass to the algebraic closure, then for any point p, we have a LES:

$$0 \to H^0(X, \mathscr{L}(-p)) \to H^0(X, \mathscr{L}) \to k(p) \to H^1(X, \mathscr{L}(-p)) \to \cdots$$

Since $\deg \mathscr{L}(-p) = \deg \mathscr{L} - 1 > 2g - 2$, then $H^1(X, \mathscr{L}(-p)) = 0$ by theorem 1.4.1. It follows that we actually have a short exact sequence,

$$0 \to H^0(X, \mathscr{L}(-p)) \to H^0(X, \mathscr{L}) \to k(p) \to 0$$

Now taking the dimension of this, we get $h^0(X, \mathscr{L}) = h^0(\mathscr{L}(-p)) + 1$. Now by the previous theorem, 1.4.2, we get that $\mathscr{L}$ is bpf.

If we have that $\deg \mathscr{L} > 2g$, then $\deg \mathscr{L}(-p) > 2g-1$, so we can twist down again, so $h^0(\mathscr{L}(-p-q)) + 2 = h^0(\mathscr{L}(-p)) + 1 = h^0(\mathscr{L})$. It follows that $\mathscr{L}$ is very ample. □

Corollary

If $\deg \mathscr{L} > 0$, some power is very ample, i.e. $\mathscr{L}$ is ample.

Curves of genus 0

Let X be a nice curve of genus 0. Examples include $\mathbb{P}^1_k$ or if $k = \mathbb{R}$, then we can consider the projective vanishing of $x^2 + y^2 + z^2$ in $\mathbb{P}^2_{\mathbb{R}}$.

Claim

If $s \in H^0(\mathbb{P}^2, \mathcal{O}(2))$ such that $V(s)$ is smooth, then $V(s)$ is a nice curve of genus 0.

Proof. Consider the exact sequence of the ideal and structure sheaf of $X = V(s)$,

$$0 \to \mathcal{O}(-2) \xrightarrow{\times s} \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_X \to 0$$

We can apply the Euler characteristic to get:

$$\chi(\mathcal{O}_{\mathbb{P}^2}) = \chi(\mathcal{O}(-2)) + \chi(\mathcal{O}_X)$$

Now from the computation of cohomology of projective space [HA III.5.1], we know that

$$h^0(X, \mathcal{O}_{\mathbb{P}^2}) = 1, h^1(\mathcal{O}_{\mathbb{P}^2}) = 0, h^2(\mathcal{O}_{\mathbb{P}^2}) = 0$$

the last computation of $h^2(\mathcal{O}_{\mathbb{P}^2})$ is by Serre duality, $h^2(\mathcal{O}_{\mathbb{P}^2}) = h^0(\omega_{\mathbb{P}^2})$ and $\omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. For $\mathcal{O}(-2)$, we have $h^0(\mathcal{O}(-2)) = 0, h^1(\mathcal{O}(-2)) = 0, h^2(\mathcal{O}(-2)) = h^0(\mathcal{O}(-1)) = 0$ the last is again by Serre duality. It follows that $\chi(\mathcal{O}_X) = 1$, so X has genus 0.

To see that X is connected, we take the long exact sequence of cohomology, then we compute the following, $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$ we get $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = H^0(X, \mathcal{O}_X) = k$, so X is connected. Since X is smooth and connected, then X is integral and therefore X is a nice curve of genus 0. □

Theorem 1.4.3 (Coarse classification of genus 0 curves)

Let X/k be a nice curve.

1. If $X(k) \neq \varnothing$, then $X \cong \mathbb{P}^1$
2. X is isomorphic to a conic in $\mathbb{P}^2$, i.e. there is some $\mathscr{L}$ on X which is very ample of degree 2.

Remark

1. There is a finer classification of isomorphism types of genus 0 curves in terms of the Brauer group.
2. Maps $\mathbb{P}^1 \xrightarrow{|\mathscr{L}|} \mathbb{P}^N$ of high degree are still interesting.

Proof. 1) If $X(k) \neq 0$, then choose a point $P \in X(k)$ and consider $\mathscr{L} = \mathcal{O}(p)$, then,

$$h^0(X, \mathcal{O}(P)) - h^1(X, \mathcal{O}(P)) = 1 + 1 - g = 2$$

Since $\deg \mathcal{O}(P) = 1 > 2g - 2$, then $h^1(X, \mathcal{O}(P)) = 0$, thus $h^0(X, \mathcal{O}(P)) = 2$. Similarly $\mathcal{O}(P)$ is very ample since $1 > 0$, so we get $f : X \to \mathbb{P}H^0(X, \mathcal{O}(P)) = \mathbb{P}^1$ which is a closed immersion. Now X is a one-dimensional so it is defined by some ideal sheaf, then since $\mathbb{P}^1$ is reduced we have $X = \mathbb{P}^1$.

2) Let $\mathscr{L} = \check{\omega}_X$, $\deg \omega_X = 2g - 2 = -2$, so $\deg \mathscr{L} = 2$. Then $h^0(X, \mathscr{L}) - h^1(X, \mathscr{L}) = 2 + 1 = 3$ by Riemann-Roch. Since $\deg \mathscr{L} = 2$, then $h^1(X, \mathscr{L}) = 0$. $\mathscr{L}$ is very ample since $\deg \mathscr{L} = 2 > 0$. It follows that $\mathscr{L}$ defines a closed immersion $f_{|\mathscr{L}|} : X \to \mathbb{P}H^0(X, \mathscr{L}) = \mathbb{P}^2$.

By definition, $\mathbb{P}H^0(X, \mathscr{L}) = \operatorname{Proj} \operatorname{Sym}^* H^0(X, \mathscr{L})$. $f_{|\mathscr{L}|} : X \to \mathbb{P}H^0(X, \mathscr{L})$ gives $\mathscr{L} = f_{|\mathscr{L}|}^* \mathcal{O}(1) = \mathcal{O}_X(1)$. Now let $X = \operatorname{Proj} S \subseteq \mathbb{P}H^0(X, \mathscr{L})$, then $\widetilde{S} = \mathcal{O}_X$ and $\widetilde{S}(d) = \mathcal{O}_X(d) = (\mathcal{O}_X(1))^{\otimes d} = \mathscr{L}^{\otimes d}$. By [HA Ex II.5.9b], we have $S_d = H^0(X, \widetilde{S}(d)) = H^0(X, \mathscr{L}^{\otimes d})$ in sufficiently high degrees. It follows by [HA Ex II.2.14c] that $X = \operatorname{Proj} S = \operatorname{Proj} \bigoplus_n H^0(X, \mathscr{L}^{\otimes d})$. It follows that the map $X \to \mathbb{P}^2$ induces a map of graded algebras

$$\operatorname{Sym}^* H^0(X, \mathscr{L}) \to \bigoplus_n H^0(X, \mathscr{L}^n)$$

The equations cutting out X in $\mathbb{P}^2$ are the kernel of this map. We want to show that the kernel is generated by a single degree 2 element.

Notice that $\operatorname{Sym}^2 H^0(X, \mathscr{L})$ has dimension 6 (if x, y, z is a basis of $H^0(X, \mathscr{L})$, then Sym^2 is generated by $x^2, y^2, z^2, xy, xz, yz$). By Riemann-Roch, $\deg \mathscr{L}^2 = 4$, so $h^1(\mathscr{L}^2) = 0$ and therefore $h^0(\mathscr{L}^2) = 4 + 1 - 0 = 5$. It follows that the map $\operatorname{Sym}^2 H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^2)$ has a nontrivial kernel, so we may pick a degree 2 element g in the kernel.

Since g is in the kernel of the morphism of graded algebras, then g vanishes along X, thus $X \subseteq V(g) \subseteq \mathbb{P}^2$. Suppose that $X \neq V(g)$, then we may find another $h \in \operatorname{Sym}^* H^0(X, \mathscr{L})$ such h is not a multiple of g and that h vanishes along X, then $X \subseteq V(h) \cap V(g)$ which by Bézout's theorem has dimension 0 which is a contradiction. Therefore $X = V(g)$ is a plane conic. $\square$

Remark

We still have many questions to ask about curves, even in genus 0. Here are two:

(i) When is K_X bpf / very ample?

(ii) If K is bpf, then how to describe the map

$$f_{\omega_X} : X \to \mathbb{P}^{g-1} = \mathbb{P}H^0(X, \omega_X)$$

Def: $f_{|\omega_X|}$, the map given by the complete linear system of ω_X is called that *Canonical Map.*

Example

1. If $g(X) = 0$, then $\deg \omega_X = 2g - 2 = -2$ so ω_X is not bpf.
2. If $g(X) = 1, \deg \omega_X = 0$ so $h^0(X, \omega_X) = g = 1$ so $\omega_X = \mathcal{O}_X$. Therefore, the canonical map $X \to \mathbb{P}^0$ is trivial.

1.5 Lecture 5 - Introduction to hyperelliptic curves

Theorem 1.5.1 Curves of genus at least 2

Let C/k be a nice curve of genus $g \geqslant 2$,

1. $|\omega_C|$ is bpf
2. ω_C is very ample or then $C_{\bar{k}} := C \times_k \bar{k}$ is hyperelliptic.

Remark

If $k \neq \bar{k}$, in (b) then we can still a degree 2 map from C to some genus 0 curve. (Although this does not quite prove it yet)

Lemma 1.5.2

Let C/k be a nice curve, $\mathscr{L}$ a line bundle on C of degree $\geqslant 0$, then $h^0(\mathscr{L}) \leqslant \deg \mathscr{L} + 1$ with equality iff $g(C) = 0$.

Proof. If $h^0(\mathscr{L}) = 0$, then we are done. If $h^0(\mathscr{L}) \neq 0$, then $\mathscr{L}$ is linearly equivalent to an effective divisor, so we may assume that $\mathscr{L}$ is effective. We may further assume that $k = \bar{k}$. Let $\mathscr{L} = \mathcal{O}(D)$, D effective, then we have a short exact sequence,

$$0 \to \mathcal{O}_X \xrightarrow{\times s} \mathcal{O}(D) \to \mathcal{O}(D)|_D \to 0$$

Where s is an element of $K(X)$ yielding D as its divisor. Taking the LES of cohomology, we get:

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D)|_D)$$

So $h^0(\mathcal{O}(D)) \leqslant 1 + h^0(\mathcal{O}(D)|_D) = 1 + \deg D$.

If we have equality, then let $D = \sum i = 1^n x_i$ where the x_i may be non-distinct. Removing a point from D drops $h^0(\mathcal{O}(D))$ by at most one, therefore $h^0(\mathcal{O}(x_1)) = h^0(\mathcal{O}(D)) - (\deg D - 1) = 2$. Now $\deg \mathcal{O}(x_1 - p - q) = -1$, so $h^0(\mathcal{O}(x_1 - p - q)) = 0 = h^0(\mathcal{O}(x_1)) - 2$, so $\mathcal{O}(x_1)$ is very ample. It follows that we have a closed immersion $f : C \to \mathbb{P}H^0(\mathcal{O}(x_1)) = \mathbb{P}^1$. Since this is a closed immersion and nonconstant, then $C = \mathbb{P}^1$, so C has genus 0. □

Now we prove the theorem.

1. *Proof.* Wlog $k = \bar{k}$ since we can check dimensions of global sections after base change. By a previous result, it suffices to show that $\forall p \in C(k), h^0(C, \omega_C(-p)) = g - 1$. We have the SES

$$0 \to \omega_C(-p) \to \omega_C \to \omega_C|_p \to 0.$$

It suffices to show that $H^0(\omega_C(-p)) \to H^0(\omega_C)$ is not surjective. Looking at a part of the LES in cohomology,

$$\omega_C|_p \to H^1(\omega_C(-p)) \to H^1(\omega_C) \to 0$$

we see that if the map were surjective, then we would have $h^1(\omega_C(-p)) = h^1(\omega_C) + 1 = 2$. Since $h^1(\omega_C(-p)) = h^0(\mathcal{O}(p)) = 2$ it follows by the above lemma that we must have $g = 0$. This contradicts the assumption that $g \geqslant 2$ so we conclude that ω_C is bpf. □

2. *Proof.* Wlog $k = \bar{k}$ since whether the canonical map (which we now know is well-defined) is a closed immersion can be checked after base change to $\bar{k}$. (Hyperellipticity cannot which is why the statement only involves $C_{\bar{k}}$) Assume that ω_C were not very ample. Then there would be $p, q \in C(k)$ with $h^0(\omega_C(-p - q)) \geqslant g - 1$. By base-point freeness, we must have $h^0(\omega_C(-p - q)) = g - 1$ since $h^0(\omega_C(-p - q)) \leqslant h^0(\omega_C(-p)) = g - 1$. Considering the SES

$$0 \to \omega_C(-p - q) \to \omega_C \to \mathcal{O}_{p+q} \to 0$$

we get the LES (using Serre duality)

$$0 \to H^0(\omega_C(-p - q))) \to H^0(\omega_C) \to \mathcal{O}_{p+q} \to H^0(\mathcal{O}(p + q)) \to H^0(\mathcal{O}) \to 0$$

Since $h^0(\omega_C(-p-q)) = g-1$ and $h^0(\omega_C) = g$, the first map has corank 1. The second then has rank 1, so the third does as well since $h^0(\mathcal{O}_{p+q}) = 2$. The last map has rank 1 by surjectivity. Thus we deduce that $h^0(\mathcal{O}(p + q)) = 2$. If $h^0(\mathcal{O}(p + q - r)) = 2 = \deg(\mathcal{O}(p + q - r) + 1$ then by the above lemma we would have $g = 0$, contrary to assumption. Therefore $|\mathcal{O}(p + q)|$ is bpf dimension 1 and of degree 2, which gives us a degree 2 map $\mathbb{P}^1$. That is, C is hyperelliptic. □

Corollary

Let $k = \bar{k}$, then,

- If $g = 2$, then C is hyperelliptic (this works without $k = \bar{k}$, but we did not prove it).
- If $g = 3$, C s hyperelliptic or plane quartic.

Proof. Suppose that $g = 2$. It is enough to show that ω_C is not very ample, for then C will necessarily be hyperelliptic. Notice that $h^0(\omega_C) = 2$ and therefore we obtain $f_{|\omega_C|} : C \to \mathbb{P}^1$. C cannot be a closed subscheme of $\mathbb{P}^1$ since C has genus 2, therefore $f_{|\omega_C|}$ is not a closed immersion and therefore ω_C is not very ample.

If $g = 3$. Either C is hyperelliptic or $f_{|\omega_C|}$ is a closed immersion with image of degree $\deg \omega_C = 4$, i.e. C is a plane quartic. □

1.6 Lecture 6 - Hyperelliptic curves, Riemann-Hurwitz, and cyclic covers

Question

Does there exist a nice of every genus?
Answer: Yes, we will see how below.

Thm

Suppose C/k is nice and C admits a finite degree 2 morphism to $\mathbb{P}^1_k$ and $\operatorname{char} k \neq 2$. We show that C admits an involution over $\mathbb{P}^1_k$.

Proof. The map $f : C \to \mathbb{P}^1_k$ is finite of degree 2 and therefore induces a map $K(\mathbb{P}^1_k) \hookrightarrow K(C)$ which makes $K(C)$ a degree 2 extension. It follows that the extension is Galois. $K(C)$ therefore admits a non-trivial automorphism $\sigma : K(C) \to K(C)$ of order 2. Let η be the generic point of C, then we have a map $\eta \to C$. The map $\eta \to C$ extends to a nbhd U of C since $K(C)$ is a finitely generated field. Since U is open and C is proper, then by the valuative criterion, we may extend the map from $U \to C$ to a map from $C \to C$. This yields a map $\tilde{\sigma} : C \to C$. Furthermore $\sigma^2 = \mathrm{id}$ since it is the identity at η, then looking at an affine open $\operatorname{Spec} R \subseteq C$ with preimage $\operatorname{Spec} S$, we have that $\operatorname{Frac}(R) = \mathcal{O}_{C,\eta} = K(C)$ and so $\tilde{\sigma}^2 : \operatorname{Spec} S \to \operatorname{Spec} R$ inducing the identity map on the generic point and since S, R are integral, then it induces the identity map on U. Since C is reduced and separated, then two maps that agree on an open set are the same, thus $\sigma^2 = \mathrm{id}$. Furthermore, the map $\tilde{\sigma}$ is a map over $\mathbb{P}^1_k$, i.e. $f\tilde{\sigma} = \tilde{\sigma}f$ where $f : C \to \mathbb{P}^1_k$ is our degree 2 finite morphism. This follows from the fact that σ fixes $K(\mathbb{P}^1_k)$ and the equivalence of complete nonsingular curves and function fields. □

Claim:

There exists a $0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_*\mathcal{O}_C \to \mathscr{L} \to 0$.

Proof. Let $\tau : \pi_*\mathcal{O}_C \to \pi_*\mathcal{O}_C$ be map induced by $\tilde{\sigma}$ on $\pi_*\mathcal{O}_C$. Then $\mathcal{O}_{\mathbb{P}^1} = (\pi_*\mathcal{O}_C)^\tau$ (fixed subsheaf), as this is "obviously" true on fibres. Therefore, the exact sequence splits with the splitting map $f \mapsto \frac{1}{2}(f + \tau f)$ using that $\operatorname{char} k \neq 2$. □

Indeed we then have $\pi_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L}$ where $\mathscr{L} = (\pi_*\mathcal{O}_C)^{\tau=-1}$.

Claim:

$\pi_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L}$ as a $\mu_2 = \{\pm 1\}$-graded sheaf of algebras.

Proof. $\mathscr{L}^2 \subset \mathcal{O}_{\mathbb{P}^1}$ since as noted above $\mathcal{O}_{\mathbb{P}^1} = (\pi_*\mathcal{O}_C)^\tau, \tau$ acts as -1 on $\mathscr{L}$, and for any $s, t \in \mathscr{L}(U)$ over an open $U, \tau(st) = \tau(s)\tau(t) = (-s)(-t) = st$. □

Claim:

Let $\pi : C \to \mathbb{P}^1$ be a hyperelliptic curve. Then $C = \mathbf{Spec}_{\ \mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathscr{L})$, where $\mathcal{O} \oplus \mathscr{L}$ has a product structure induces by a map $\mathscr{L}^2 \to \mathcal{O}$ and $V(\mathscr{L}^2 \to \mathcal{O})$ is reduced.

Proof. Here's a sketch of a trace of the idea of the proof. Working (étale)-locally this reduces to the computation that $k[x,y]/(y^2 - f(x)) \supset k[x]$ is smooth when f is separable. This is also true Zariski-locally for the specific case of $\mathbb{P}^1$. $\square$

Def: Let X, Y be nice curves, and $f : X \to Y$ a non-constant map. f is *cyclic* if $k(X)/k(Y)$ is a cyclic Galois extension.

The same "argument" gets you the following.

Thm

Let $f : X \to Y$ be cyclic. If $(\deg f, \operatorname{char} k) = 1$ then $X \cong \mathbf{Spec}\,_Y\mathscr{A}$, where $\mathscr{A} = \mathcal{O} \oplus \mathscr{L} \oplus \mathscr{L}^2 \oplus \cdots \oplus \mathscr{L}^{\deg f - 1}$ is μ_n-graded with a multiplication induced by $\mathscr{L}^{\deg f} \to \mathcal{O}$. X is smooth if $V(\mathscr{L}^{\deg f} \to \mathcal{O})$ is smooth.

Example

Let $\mathscr{L} = \mathcal{O}(-5)$, then $\mathscr{L}^2 = \mathcal{O}(-10)$, so map $\mathscr{L}^2 = \mathcal{O}(-10) \to \mathcal{O}$ is given by an element of $\Gamma(\mathbb{P}^1, \mathcal{O}(10)) = \operatorname{Sym}^{10}\langle x, y\rangle$. Choose $f = x^{10} + y^{10}$ which has no double roots over a field of characteristic not 2 or 5. Therefore $\mathbf{Spec}\,_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-5))$ is a double cover of $\mathbb{P}^1$ "branched" (we will soon learn what this means) at the tenth roots of unity.

How to write equations for the above curve:

Let $f : X \to Y$ be a cyclic cover. We want to embed X in some natural space, namely $\operatorname{Tot}(\check{\mathscr{L}})$, the total space of the line on Y.
We recall that $\operatorname{Tot}(\check{\mathscr{L}}) = \mathbf{Spec}\,_Y(\operatorname{Sym}^\bullet \mathscr{L})$. Thus we have the following property

$$\operatorname{Hom}_Y(Y, \operatorname{Tot} \mathscr{L}) = \operatorname{Hom}(\mathscr{L}, \mathcal{O}_Y) = \operatorname{Hom}(\mathcal{O}_Y, \check{\mathscr{L}})$$

We get a natural map from $\operatorname{Sym}^\bullet \mathscr{L} = \bigoplus_{n \geqslant 0} \mathscr{L}^n$ to $\bigoplus_{0 \leqslant n < \deg f} \mathscr{L}^n$ induced by $id_{\mathscr{L}}$ and $f : \mathscr{L}^{\deg f} \to \mathcal{O}$. By applying the relative spec functor, this gets us a closed immersion $X \hookrightarrow \operatorname{Tot}(\check{\mathscr{L}})$.

Choosing a trivialization U of $\mathscr{L}$, (i.e. $t \in \Gamma(U, \mathscr{L})$ which vanishes nowhere), we have

$$\operatorname{Tot}(\check{\mathscr{L}|_U}) = U \times \mathbb{A}^1_t$$

Then t^n trivializes $\mathscr{L}^n$ over U so we get $f|_U : \mathscr{L}|_U^{\deg f} = \operatorname{Sym}^n \mathcal{O}_U \cong \mathcal{O}_U \to \mathcal{O}_U$ is represented by some polynomial g. Then $X_U \subset U \times \mathbb{A}^1$ is cut out by $t^{\deg f} - g$ (where g is separable if X is smooth) giving us the normal local representation of hyperelliptic curves.

Riemann-Hurwitz formula

Let $f : X \to Y$ be degree d. Using $g(Y)$, d, and some auxiliary data, can we compute $g(X)$? The answer is yes.

Def: Let X, Y be nice curves, then a morphism is *separable* $f : X \to Y$ if it induces a separable extension of function fields $K(X)$ over $K(Y)$.

Claim 1.6.1: SES of Kähler differentials

Let X, Y be nice curves and $f : X \to Y$ a separable morphism, then we have a SES

$$0 \to f^*\omega_Y \to \omega_X \to \omega_{X/Y} \to 0$$

Proof. By [HA II.8.11], we have an exact sequence,

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

and since X, Y are curves, then Ω and ω are the same since everything is dimension 1. We want to show that this is in fact a short exact sequence, i.e. we need to show that it is exact no the left. We can check that the morphism of locally free sheaves $f^*\omega_Y \to \omega_X$ is injective by checking it generically. This follows in greater generality. If $\mathscr{F}, \mathscr{G} \in \mathfrak{Qco}(X)$ with $\mathscr{F}$ torsion-free, if $f : \mathscr{F} \to \mathscr{G}$ is generically injective, then it is actually injective. Suppose that there is some point x such that $f_x : \mathscr{F}_x \to \mathscr{G}_x$ is not injective, then we have two elements $a, b \in \mathscr{F}_x$ such that $f_x(a) = f_x(b)$, then since $\mathscr{F}$ is quasi-coherent, $\mathscr{F}_\eta$ is a further localization of $\mathscr{F}$ and since $\mathscr{F}$ is torsion-free, then $\mathscr{F}_x \hookrightarrow \mathscr{F}_\eta$ and therefore $f_\eta(a) = f_\eta(b)$, so f is not generically injective. It follows that we need only show $f^*\omega_Y \to \omega_X$. $f^*\omega_Y$ is a line bundle, so it is torsion-free. Taking stalks at the generic point, f is separated and $(\omega_{X/Y})_\eta = \Omega_{K(X)/K(Y)} = \operatorname{tr.d.} K(X)/K(Y) = 0$ by [HA II.8.6A]. It follows that $f^*\omega_Y \to \omega_X$ is generically surjective but they are also fields generically, so it is also injective as desired. □

Theorem 1.6.2 Riemann-Hurwitz

Let X, Y be nice curves over k, let $f : X \to Y$ be a degree d separable morphism, then

$$2g(X) - 2 = d\,(2g(Y) - 2) + \ell(\omega_{X/Y})$$

where $\ell(\omega_{X/Y})$ is the length of $\omega_{X/Y}$ i.e. the length of the longest filtration by $\mathcal{O}_X$-modules with nonzero quotients. Furthermore $\ell(\omega_{X/Y}) = h^0(X, \omega_{X/Y})$.

Proof. We have a SES,

$$0 \to f^*\omega_Y \to \omega_X \to \omega_{X/Y} \to 0$$

then taking the Euler characteristic, we have $\chi(\omega_X) = \chi(\omega_{X/Y}) + \chi(f^*\omega_Y)$. $\omega_{X/Y}$ is supported on a dimension 0 set, so $\chi(\omega_{X/Y}) = h^0(\omega_{X/Y})$. By Riemann-Roch,

$$\chi(f^*\omega_Y) = \deg(f^*\omega_Y) + 1 - g(X) = d(2g(Y) - 2) + 1 - g(X)$$

Putting this together, we get that,

$$g(X) - 1 = d(2g(Y) - 2) + 1 - g(X) + h^0(\omega_{X/Y})$$

Rearranging this yields the desired result. It remains just to show that $h^0(\omega_{X/Y}) = l(\omega_{X/Y})$ which follows from noticing that $\omega_{X/Y}$ is a direct sum of skyscraper sheaves at the points of ramification (i.e. $\operatorname{Supp}\omega_{X/Y}$). On a local ring $\mathcal{O}_{X,p}$, a module over the local ring has length equal to the dimension of as a $k = \mathcal{O}_{X,p}/\mathfrak{m}_p$ vector space (Nakayama's lemma). Then since $\omega_{X/Y}$ is a direct sum of skyscraper sheaves, its h^0 and length agree. □

Thm

$f^*\mathcal{O}(D) = \mathcal{O}(f^{-1}(D))$ so $\deg f^*\mathscr{L} = \deg f \deg \mathscr{L}$

Proof. Hartshorne Ex II.6.8 and Prop 6.9. □

Question

What is $\text{len}_R \omega_{X/Y}$?

Example

Since this is a local question we consider

$$\omega_{\mathbb{C}[x^{1/n}]/\mathbb{C}[x]} = \omega_{\mathbb{C}[x,t]/(x^n - t)/\mathbb{C}[x]} = \mathbb{C}[x,t]dt/nt^{n-1}dt$$

which has length $n-1$, as it is supported at $t = 0$ dimension $n - 1/\mathbb{C}$.

Example

Let $S = \mathbb{F}_p[x], R = S[t]/(t^p - t^{p-1} - f(x))$. Then $\omega_{R/S} = Rdt/t^{p-2}dt$ has length $p-2$.

Remark

WARNING: As written this fact is likely wrong. If you have the correct statement, please let us know. If $k = \bar{k}$ is of characteristic 0, all finite extensions of $k((t))$ are of the form $k((t^{1/n}))$ and if R is a domain and $R/k((t))$ is finite flat and normal, then $\omega_{R/k((t))}$ has length $[R : k((t))] - 1$.

Claim:

Let $f : X \to Y, x \in \text{Supp}\omega_{X/Y}, y = f(x)$. Then

$$\omega_{\hat{\mathcal{O}}_{X,x}/\hat{\mathcal{O}}_{Y,y}} = \omega_{X/Y}|_{\hat{\mathcal{O}}_{X,x}} = (\omega_{X/Y}\hat{)}_x$$

Proof. Formation of the module of differentials commutes with base change and $\hat{\mathcal{O}_{X,x}}$ is flat over $\mathcal{O}_{X,x}$. □

The upshot of this is that then

$$l(\omega_{X/Y}) = \sum_{x \in \text{Supp}\omega_{X/Y}} l(\omega_{X/Y})_x = \sum_{x \in \text{Supp}\omega_{X/Y}} l(\omega_{X/Y})\hat{_x}$$

which in characteristic 0 can also be computed as $\sum_{x \in \text{Supp}\omega_{X/Y}} [\text{Frac}\hat{\mathcal{O}_{X,x}} : \text{Frac}\hat{\mathcal{O}_{Y,y}}] - 1$.

Recall: If $f : X \to Y$ is cyclic of degree d, defined by $f : \mathscr{L}^d \to \mathcal{O}_Y$ in local coordinates:

$$R = k[x,t]/(t^n - f(x))$$

We want to figure out what $\omega_{X/Y}$ looks like. If $(n, \operatorname{char} k) = 1$, then

$$\omega_{X/Y} = Rdt/(nt^{n-1}dt)$$

If we choose a point $z \in X$, i.e. a point $(x, t) \in R$ such that $t^n = f(x)$, then

1. 1) If $f(x) \neq 0$, then $t = f(x)$ is invertible in the local ring, so $\omega_{X,Y}$ is 0 at z.
2. 2) If $f(x) = 0$, then t is nilpotent of degree n and the length over $\mathcal{O}_{X,z}$ of $(\omega_{X/Y})_z$ is $n - 1$.

Corollary

Let $X \to \mathbb{P}^1$ hyperelliptic and $X = \mathbf{Spec}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-d))$, then $g(X) = d - 1$.

Proof. We have $\mathcal{O}(-2d) \to \mathcal{O}$ is given by an element $f \in \Gamma(\mathbb{P}^1, \mathcal{O}(2d))$ so there are $2d$ points in $\omega_{X/Y}$, i.e. the zeros of f and at each point, the curve locally looks like $k[x,t]/(t^2 - f(x))$, so the length is $2 - 1 = 1$. By Riemann-Hurwitz,

$$2g(X) - 2 = 2(0 - 2) + 2d = 2d - 4$$

Therefore $2g(X) = 2d - 2$, so $g(X) = d - 1$ as desired. $\square$

Corollary 1.6.3: $\mathbb{P}^1$ is simply connected

If X is a nice curve over k and $f : X \to \mathbb{P}^1$ is étale then f is an isomorphism.

Proof. By Riemann-Hurwitz, $2g(X) - 2 = -2\deg f + \ell(\omega_{X/Y})$. Since f is étale, then $\omega_{X/Y} = 0$, therefore $\deg(f) = 1 - g(X) \leqslant 1$ but we also have that $1 \leqslant \deg(f)$. Therefore f has degree 1 so f is an isomorphism. $\square$

Corollary

There is a bijective correspondence

$$\{f : X \to Y \text{ cyclic étale of degree } d\} \cong \{\mathscr{L} \text{ line bundle } \mathscr{L}^d \cong \mathcal{O}\}$$

Proof sketch: Let σ be an involution of X/Y of order d (which exists for the same reason as the involution from the start of lecture / there is an equivalence of categories between nice curves and their function fields) and ζ a d-th root of unity. Let $f : X \to Y$ as in the left side. Then $f_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathscr{L} \oplus \cdots \oplus \mathscr{L}^{d-1}$ where $\mathscr{L}$ is ζ^d-eigenspace of σ. This gives us $\mathscr{L}^d \to \mathcal{O}$, which is an isomorphism iff f is étale.
Reversing this process combined with the earlier result from lecture gives us the other map. Since every automorphism of $\mathcal{O}$ is given by multiplication by a scalar, the isomorphism $\mathscr{L}^d \to \mathcal{O}$ is just as unique. Hence it suffices to show that scaling the isomorphism gives an isomorphic $f : X \to Y$. $\square$

For line bundles on $\mathbb{P}^1$ we have the following table due to R.R.,

deg	h^0	h^1
-4	0	3
-3	0	2
-2	0	1
-1	0	0
0	1	0
1	2	0
2	3	0
3	4	0
4	5	0

Thm

$\mathcal{O}_{\mathbb{P}^1}(d)$ is very ample for $d \geqslant 1$

Proof. Let $p, q \in \mathbb{P}^1$, then look at $h^0(\mathcal{O}(d) \otimes \mathcal{O}(-p-q))$ in the table above and observe that it goes down by 2. Thus $\mathcal{O}(d)$ is very ample by thm 1.4.2. $\square$

The map $f_{|\mathcal{O}(d)|} : \mathbb{P}^1 \to \mathbb{P}^d$ is a morphism of degree d, i.e. $f^*_{|\mathcal{O}(d)|}\mathcal{O}(1) = \mathcal{O}(d)$ i.e. $f^{-1}H$ is a degree d divisor for a hyperplane H. We get a map $\operatorname{Sym}^d \Gamma(\mathbb{P}^1, \mathcal{O}(1)) \xrightarrow{\sim} \Gamma(\mathbb{P}^1, \mathcal{O}(d))$, i.e. elements in $\Gamma(\mathbb{P}^1, \mathcal{O}(d))$ are degree d two variables polynomials. We then get $f_{|\mathcal{O}(d)|} : \mathbb{P}^1 \to \mathbb{P}^d$ and $[x : y] \mapsto [x^d : x^{d-1}y : \cdots : xy^{d-1} : y^d]$.

Def: The image of $f_{|\mathcal{O}(d)|}$ is a *Rational Normal Curve* (RNC).

1.7 Lecture 7 - Rational normal curves

We want to find a nice surface in which hyperelliptic curves of genus g live. This will turn out to be a so called "ruled surface".

Thm

Suppose $k = \bar{k}$ and $\operatorname{char} k \neq 2$. Let C/k be hyperelliptic, then $f_{|\omega_C|} : C \to \mathbb{P}^{g-1}$ is $2-1$ onto its image, which is a rational normal curve.

We postpone the proof to first see some corollaries and then build up lemmas to tackle the proof.

Corollary

The $2:1$ map $C \to \mathbb{P}^1$ is canonical

Proof. The "canonical" nature of this is up to automorphism of $\mathbb{P}^1$. This will follow by a commutative diagram we will get in thm 1.7.1. $\square$

Corollary

If C is hyperelliptic, then the hyperelliptic involution is in the center of Aut$C$.

Proof. Since the involution over $\mathbb{P}^1$ is uniquely determined by the hyperelliptic 2:1 map, it is sent to the canonical involution of an automorphic image of C. $\square$

Corollary

If $C_{\bar{k}}$ is hyperelliptic, then C admits a $2:1$ map to a smooth conic.

Proof. The canonical map from $C \to \mathbb{P}_k^{g-1}$ base changes to the canonical map out of $C_{\bar{k}}$ which is a $2:1$ map to a rational normal curve. It follows that the image of the canonical map out of C is a genus 0 curve, i.e. a smooth conic. $\square$

Prop

Given coordinates $x_0, \ldots, x_d$ on $\mathbb{P}^d$. A rational normal curve is cut out by the 2x2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}$$

Proof. Let I be the ideal generated by the 2x2 minors of the above matrix. Let $X = V(I)$ be the curve cut out by I. Then we define $\varphi : S(X) = k[x_0, \ldots, x_d]/I \to k[s,t]_{\times d} = S(\mathrm{im} f_{|\mathcal{O}(d)})$ by sending x_i to $s^{d-i}t^i$. φ is well defined by looking at the generators of I and seeing that our definition respects them. It is clear that φ is a graded map so it suffices to show that φ is an isomorphism.
φ is clearly surjective in degree "1" (really degree d of $k[s,t]$ but the homogeneous coordinate ring is $k[s,t]_{\times d}$) and $k[s,t]_{\times d}$ is generated in degree "1."
Therefore we show injectivity. It suffices to show injectivity in degree n. Since the map is surjective we can just compute dimensions where it suffices to show that

$$\dim S(X)_n \leqslant \dim k[s,t]_{nd} = nd + 1$$

It is easy to see using the relations in I that every monomial of degree n in $(k[x_0, \ldots, x_d]/I)_n$ is equivalent to one of the form $x_0^a x_1^{\epsilon_1} \cdots x_{d-1}^{\epsilon_{d-1}} x_d^b$ where $\epsilon_i \in \{0,1\}$ and at most one is non-zero. There are $nd+1$ of there so we are done (this only shows the upper bound to an equality, but that's all we need).

$\square$

Def: A curve $C \subseteq \mathbb{P}^n$ is called *non-degenerate* if it is not contained in any hyperplane.

Thm

Let $k = \bar{k}$. Any non-degenerate geometrically integral curve in $\mathbb{P}^d$ of degree $\leqslant d$ is a rational normal curve.

Proof. Let $C \subseteq \mathbb{P}^d$ satisfy the hypotheses. Let $\widetilde{C}$ be the normalization of C. Then $\widetilde{C}$ is smooth and hence nice since C is geometrically integral. We have $\widetilde{C} \xrightarrow{\pi} C \to \mathbb{P}^d$, so we get a map $f : \widetilde{C} \to \mathbb{P}^d$ given by a line bundle $\mathscr{L}$ on $\widetilde{C}$ and a surjective map $V \otimes \mathcal{O}_X \to \mathscr{L}$. This map being nondegenerate means that the map on global sections $V \to \Gamma(\widetilde{C}, \mathscr{L})$ is an injection. If it is not an injection, then suppose $x \in V$ gets sent to 0, choosing a basis of V containing x, we get that the image of $\widetilde{C}$ in $\mathbb{P}V$ is contained in the hyperplane cut out by x. It follows that $h^0(\mathscr{L}) = d$, but $\deg \mathscr{L} \leqslant d$ (since π is degree 1). It follows by lemma 1.5.2 that $\widetilde{C}$ has genus 0, so $\widetilde{C} = \mathbb{P}^1$. Therefore $\widetilde{C} \to \mathbb{P}^d$ is a d-uple embedding. C has the same scheme theoretic image as $\widetilde{C}$ and is therefore a rational normal curve. ☐

Projections

Given a point $p \in \mathbb{P}^n$, the lines through p correspond exactly to a copy of $\mathbb{P}^{n-1}$.

$$p \in \mathbb{P}V \leftrightarrow H \subseteq V \text{ hyperplane}$$
$$\text{line through } p \in \mathbb{P}V \leftrightarrow H' \subseteq V \text{codim 2 w}/H' \subseteq H \leftrightarrow \mathbb{P}H$$

Suppose we have a map $f_{(V,\mathscr{L})} : C \to \mathbb{P}V$. Let $W \subseteq V$ be a subspace. Then the inclusion $W \subseteq V$ yields an inclusion $W \otimes \mathcal{O}_X \to V \otimes \mathcal{O}_X$. It follows that maps from $V \otimes \mathcal{O}_X$ to $\mathscr{L}$ yield, by way of composition, maps from $W \otimes \mathcal{O}_X \to \mathscr{L}$. Note however, that this map may no longer be surjective, therefore we obtain only a rational map $C \dashrightarrow \mathbb{P}W$.

Recall that a rational map out of a curve can be upgraded to a map from the whole curve. This corresponds to the fact that we may replace $\mathscr{L}$ with the image of $W \otimes \mathcal{O}_X$ which is a torsion free rank 1 subsheaf of $\mathscr{L}$ which is therefore locally free since C is locally a DVR.

Claim 1.7.2: Maps from $\mathbb{P}^1$ to $\mathbb{P}^n$

Any map $\mathbb{P}^1 \to \mathbb{P}^n$ with non-degenerate image is obtained by projection from a rational normal curve.

Proof. Let $f = f_{V,\mathscr{L}}$, the non-degeneracy means that $V \otimes \mathcal{O}_X \to \mathscr{L}$ yields an injection $V \to \Gamma(\mathbb{P}^1, \mathscr{L})$. $\mathscr{L} = \mathcal{O}(d)$ for some d, so we have a map $\Gamma(P^1, \mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L}$ which kills the elements outside of V. Now $V \otimes \mathcal{O}_X \to \mathscr{L}$ factors through the map from $\Gamma(\mathbb{P}^1, \mathscr{L}) \otimes \mathcal{O}_X$ as a projection. It follows that f is a projection after the map $f_{|\mathscr{L}|}$ is a rational normal curve. ☐

1.8 Lecture 8 - g^r_ds and better bounds on $h^0(\mathscr{L})$

Theorem 1.8.1: Hyperelliptic curve canonical map

Let C/k be hyperelliptic with $g \geqslant 2$, i.e. $\pi : C \to \mathbb{P}^1$ is 2-to-1. Then the canonical map $f_{|\omega_C|} : C \to \mathbb{P}^{g-1}$ is 2:1 onto its image - a RNC - and factors through π.

Proof. Let $p+q$ be the fiber over ∞ of π (possibly with $p=q$). Since C is reduced it suffices to show that show that $f_{|\omega_C|}(p) = f_{|\omega_C|}(q)$ (this is essentially by the Reduced-to-Separated theorem).

1. $\pi = f_{|\mathcal{O}(p+q)|}$: $\dim|\mathcal{O}(p+q)| = 1$ because $g > 0$ so $\dim|\mathcal{O}(p+q)| < \deg \mathcal{O}(p+q) = 2$ and $\dim|\mathcal{O}(p+q)| \geqslant 1$ since $\mathcal{O}$ is BPF. π is given by some line-bundle and hence it must be given by $|\mathcal{O}(p+q)|$.

2. We WTS that $p+q$ imposes 1 condition on ω_C, i.e. $h^0(\omega_C(-p-q)) = h^0(\omega_C) - 1 = g-1$. This means any 1-form vanishing at p also vanishes at q (if $p=q$ then take this to mean that any 1-form vanishing at p vanishes to order 2), or equivalently the natural map

$$H^0(\omega_C(-p-q)) \hookrightarrow H^0(\omega_C(-p))$$

is an quality.

Proof of 2. Start with the SES

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{O}_C(p+q) \arrow[r] & \mathcal{O}_{p+q} \arrow[r] & 0 \end{tikzcd}$$

giving the LES

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}_C) \arrow[r] & H^0(\mathcal{O}_C(p+q)) \arrow[r] & H^0(\mathcal{O}_{p+q}) \arrow[dll] \\ & H^1(\mathcal{O}_C) \arrow[r] & H^1(\mathcal{O}_C(p+q)) \arrow[r] & 0 \end{tikzcd}$$

since $p+q$ is 0-dimensional. $h^0(\mathcal{O}_C) = 1$ and $h^0(\mathcal{O}(p+q)) = 2$ because $f_{|\mathcal{O}(p+q)|}$ maps to $\mathbb{P}^1$. Therefore $\operatorname{rk}(H^0(\mathcal{O}_C(p+q)) \to H^0(\mathcal{O}_{p+q})) = 1$ and since $h^0(\mathcal{O}_{p+q}) = \deg p+q = 2$, so does $\operatorname{rk}(H^0(\mathcal{O}_{p+q}) \to H^1(\mathcal{O}_C)) = 1$ hence $h^1(\mathcal{O}_C(p+q)) = h^1(\mathcal{O}_C) - 1 = g-1$. The result follows by Serre duality. □

3. Let $Z = f_{|\omega_C|}(C)$ be the image of C under the cannonical map. We then get the following diagram by the universal property of the normalization.

$$\begin{tikzcd} C \arrow[r] \arrow[dr, dashed] & Z \arrow[r, hook, "\iota"'] & \mathbb{P}^{g-1} \\ & \widetilde{Z} \arrow[u] \arrow[ur, dashed, "\widetilde{\iota}"] & \end{tikzcd}$$

Then $\deg \widetilde{\iota} \leqslant \frac{\deg f_{|\omega_C|}}{\deg(C \to Z)} \leqslant \frac{2g-2}{2} = g-1$ and $\widetilde{\iota}$ has image Z which is nondegenerate. Therefore by the big theorem from lecture 7, Z is a rational normal curve and so $Z = \widetilde{Z}$ and has degree exactly $g-1$. Therefore $\deg f_{|\omega_C|} : C \to Z$ is 2 and the fibers of $f_{|\omega_C|}$ = fibers of π. (Since they are of the same degree / this means scheme theoretically).

□

Corollary

The $2:1$ map π is determined entirely by $f_{|\omega_C|}$, so it is unique up to an isomorphism of $\mathbb{P}^1$.

Originally our goal was to understand maps from curves to $\mathbb{P}^n$. We now understand $2:1$ maps to $\mathbb{P}^1$ at least in the case where $g(C) \neq 1$ and we understand $1:1$ maps to $\mathbb{P}^1$, i.e. genus 0 curves.

Def: We want to more generally understand degree d maps to $\mathbb{P}^n$, i.e. linear systems of dimension r and degree d which we denote g^r_d's

Example

A hyperelliptic curve is a curve with a g^1_2.
A curve with a g^1_3 is called trigonal. For example take $X = \operatorname{Spec}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d) \oplus \mathcal{O}_{\mathbb{P}^1}(-2d))$ where we put an algebra structure by choosing a map $\mathcal{O}_{\mathbb{P}^1}(-3d) \to \mathcal{O}_{\mathbb{P}^1}$, i.e. a degree $3d$ separable homogeneous polynomial.

Claim:

Any non-hyperelliptic genus 3 curve has a g^1_3.

Proof. Let X be a genus 3 non-hyperelliptic, then the canonical map is an embedding. $f_{|\omega_X|} : X \to \mathbb{P}H^0(X, \omega_X) = \mathbb{P}^2$ embeds X as a plane quartic. We can now project away from a point p in X which comes from a map

$$H^0(X, \omega_X(-p)) \otimes \mathcal{O}_X \to \omega_X(-p)$$

This is surjective since for any point q, $h^0(X, \omega_X(-p-q)) = h^0(X, \omega_X(-p)) - 1$ since ω_X is very ample. Therefore there is a section of $H^0(\omega_X(-p))$ vanishing at q, so it surjects. Note that $\omega_X(-p)$ has degree 3 so this is a g^1_3. $\square$

We will later show the following:

Thm

Any non-hyperelliptic $g = 4$ is trigonal.

Question

Which curves have very ample g^2_d's?

Thm

Let $C \subset \mathbb{P}^2$ and ι be the inclusion. Suppose that C is a very nice cuve of degree d. Then $g(C) = \frac{1}{2}(g-1)(g-2)$.

Proof. Let

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-C) = \mathcal{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

be the natural SES for the structure sheaf of C. We then get that $H^1(\mathcal{O}_C) \to H^2(\mathcal{O}_{\mathbb{P}^2}(-d))$ is an isomorphism from the LES in cohomology since $h^1(\mathbb{P}^2) = h^2(\mathbb{P}^2) = 0$ by the cohomology of projective space. But $h^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = h^0(\omega_{\mathbb{P}^2}(d)) = h^0(\mathcal{O}(d-3))$ by Serre Duality. We then recall that $H^0(\mathcal{O}(d-3))$ are the homogeneous polynomials in degree $d-3$ so $h^1(\mathcal{O}_C) = \binom{d-1}{2}$ by stars and bars. $\square$

Better bounds on h^0

For genus $0, h^0(\mathscr{L}) = 1 + h^1(\mathscr{L}) + \deg \mathscr{L}$ and for $\deg \mathscr{L} \geqslant 0$ $h^1(\mathscr{L}) = 0$.

For genus 1:

deg	h^0	h^1
-3	0	3
-2	0	2
-1	0	1
0	$\begin{cases} 1 & \text{if } \mathcal{O} \\ 0 & \text{else} \end{cases}$	$\begin{cases} 1 & \text{if } \mathcal{O} \\ 0 & \text{else} \end{cases}$
1	1	0
2	2	0
3	3	0

In general we understand $h^0(\mathscr{L})$ perfectly for $\deg \mathscr{L} \geqslant 2g-2$. If $\deg \mathscr{L} \geqslant 2g-2$ and $\mathscr{L} \neq \omega_C$ then $h^0(\mathscr{L}) = \deg \mathscr{L} + 1 - g$ since h^1 vanishes. This is linear of slope 1.
In particular, if $\mathscr{L} = 2g-1$ then $h^0(\mathscr{L}) = g$ always holds. We also know that $h^0(\mathscr{L}) \leqslant \deg \mathscr{L}$ for $\deg \mathscr{L} > 0$ and $g > 0$.

Now suppose that $x_1, x_2, \ldots, x_{2g-1} \in X(k)$ are closed points. Consider the sequence

$$1 = h^0(\mathcal{O}(x_1)) \leqslant h^0(\mathcal{O}(x_1 + x_2)) \leqslant \cdots \leqslant h^0(\mathcal{O}(\sum_i x_i)) = g$$

so there are exactly $g-1$ jumps.

Theorem 1.8.2: Clifford

If $0 \leqslant \deg \mathscr{L} \leqslant 2g-2$ then $h^0(\mathscr{L}) \leqslant \frac{\deg \mathscr{L}}{2} + 1$.

We first prove a lemma

Lemma

If D, D' are disjoint (we need to get rid of the disjoint assumption, this is given next lecture) effective divisors, then

$$h^0(\mathcal{O}(D + D')) + 1 \geqslant h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(D'))$$

Proof. The SES $0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}(D)|_D \to 0$ gives a LES:

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D)|_D) \xrightarrow{\varphi} H^1(\mathcal{O}) \to H^1(\mathcal{O}(D)) \to 0$$

We have another SES $0 \to \mathcal{O}(D') \to \mathcal{O}(D + D') \to \mathcal{O}(D + D')|_D \to 0$ which gives another LES,

$$0 \to H^0(\mathcal{O}(D')) \to H^0(\mathcal{O}(D + D')) \to H^0(\mathcal{O}(D + D')|_D) \xrightarrow{\varphi} H^1(\mathcal{O}(D')) \to H^1(\mathcal{O}(D + D')) \to 0$$

It follows that $h^0(\mathcal{O}(D+D')) \geqslant h^0(\mathcal{O}(D)) + \dim\ker\varphi$. We now have $\dim\ker\varphi = \dim h^0(\mathcal{O}(D)) - 1$ so we win if $\dim\ker\psi \geqslant \dim\ker\varphi$.

$$\begin{tikzcd} H^0(\mathcal{O}(D)|_D) \arrow[r, "\varphi"] \arrow[d, "\cdot s_{D'}"] & H^1(\mathcal{O}) \arrow[d, "\cdot s_{D'}"] \\ H^0(\mathcal{O}(D+D')|_D) \arrow[r, "\psi"] & H^1(\mathcal{O}(D')) \end{tikzcd}$$

Where $s_{D'}$ is a global section of D' whose vanishing is exactly D'. Since $D \cap D' = \varnothing$, then the LHS is an isomorphism so the inequality holds. $\square$

We can now prove Clifford's theorem:

Proof. If $h^1(\mathscr{L}) = 0$, then $h^0(\mathscr{L}) = \deg\mathscr{L} + 1 - g$ by Riemann-Roch and $g - 1 \geqslant \frac{\deg\mathscr{L}}{2}$, so $h^0(\mathscr{L}) \leqslant \frac{\deg\mathscr{L}}{2}$.

If $h^1(\mathscr{L}) = h^0(\omega \otimes \check{\mathscr{L}}) \neq 0$, then by Riemann-Roch $h^0(\mathscr{L}) - h^0(\omega \otimes \check{\mathscr{L}}) = \deg\mathscr{L} + 1 - g$.

Claim: If $\mathscr{L} = \mathcal{O}(D)$,then there exists a divisor $D' \in \left|\omega_X \otimes \check{\mathscr{L}}\right|$ disjoint from D.

Assuming the above we get $\mathcal{O}(D + D') = \omega_X$, so

$$h^0(\omega_X) + 1 = g + 1 \geqslant h^0(\mathscr{L}) + h^0(\omega_X \otimes \check{\mathscr{L}})$$

Adding this to Riemann-Roch we get:

$$2h^0(\mathscr{L}) \leqslant \deg\mathscr{L} + 2 \Rightarrow h^0(\mathscr{L}) \leqslant \frac{\deg\mathscr{L}}{2} + 1$$

Thus Clifford's theorem holds, modulo proving the claim that was assumed true, or removing the disjoint requirement. $\square$

Genus 1 curves:

Let E genus 1, $p \in E^{cl}$. $\deg\mathcal{O}(p) = [k(p) : k] =$?????????????? could be anything if $k \neq \bar{k}$.

> **Def:** Gcd($\{\deg D | D \geqslant 0\}$) is index of an genus 1 curve.

Assume that $E(k) \neq \varnothing$ and the let $p \in E(k)$.$|\mathcal{O}(p)| = 0$ and $\dim|\mathcal{O}(2p)| = 1$ so we get a 2-1 map to $\mathbb{P}^1$ by BPF-ness of $\mathcal{O}(2p)$. This map is not necessarily unique as E is not hyper-elliptic. We will soon show that $|\mathcal{O}(3p)|$ gives an embedding to $\mathbb{P}^2$ as a cubic equation.

1.9 Lecture 9 - Some random stuff

We start by removing the disjointness assumption from the previous lecture.

> **Lemma**
>
> Given two effective divisors D, D' we have that $h^0(\mathcal{O}(D+D')) + 1 \geqslant h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(D'))$.

Proof. We first show that $h^0(\mathcal{O}(D)) \geqslant m+1$ if for any m points, $x_1, \cdots, x_m \in X(\bar{k})$ there exists a section of $\mathcal{O}(D)$ vanishing at all x_i. In particular, it suffices to show that $h^0(\mathcal{O}(D)) = m_{\max} + 1$ where $m_{\max}$ is the largest m for which the property holds.

Let $n = h^0(\mathcal{O}(D)) - 1$. We want to show that for point $x_1, \cdots, x_n \in X(\bar{k})$, there is a nonzero section vanishing at all of them. We have a map

$$H^0(X, \mathcal{O}(D)) \to \oplus_i \mathcal{O}(D)_{x_i}$$

of vector vector spaces given by the evaluation map. Since the dimension of the LHS is $n+1$ and the dimension of the RHS is n, then there is a nonzero kernel, i.e. a nonzero section of $\mathcal{O}(D)$ vanishing at each x_i as desired.

Conversely, choose x_1 such that some section of $\mathcal{O}(D)$ doesn't vanish at x_1. Choose x_i such that some nonzero section of $\mathcal{O}(D - x_1 - \cdots - x_{i-1})$ doesn't vanish at x_i. We can continue choosing such points as long as $h^0(X, \mathcal{O}(D - x_1 - \cdots - x_{i-1})) \geqslant 1$. Since each point drops the dimension by at most 1, then we can do this at least $n+1$ times. We have a SES,

$$0 \to \mathcal{O}(-x_1 - \cdots - x_i) \to \mathcal{O}_X \to \mathcal{O}_{x_1 + \cdots + x_i} \to 0$$

Now tensoring with $\mathcal{O}(D)$ and taking the global sections, we get that $H^0(X, \mathcal{O}(D - x_1 - \cdots - x_i))$ injects into $H^0(X, \mathcal{O}(D))$. Let $s_i \in H^0(X, \mathcal{O}(D - x_1 - \cdots - x_{i-1})$ be a section which does not vanish at x_i, then $s_i \in H^0(X, \mathcal{O}(D))$ vanishes at all $x_1, \cdots, x_{i-1}$ and not at x_i. It follows that $\{s_i\}_{i=1}^{n+1}$ forms a basis of $H^0(X, \mathcal{O}(D))$. Therefore an element $s \in H^0(X, \mathcal{O}(D))$ vanishing at all x_i has coefficients of 0 for each s_i and hence $s = 0$.

We can now prove the lemma: Let $h^0(\mathcal{O}(D)) = m_D + 1, h^0(\mathcal{O}(D')) = m_{D'} + 1$. For any $m_D + m_{D'}$ points, there is a nonzero section x of $\mathcal{O}(D)$ vanishing at the first m_D and $y \in H^0(X, \mathcal{O}(D'))$ vanishing at last $m_{D'}$ and therefore $x \otimes y$ is a nonzero section of $\mathcal{O}(D + D')$ vanishing at all the points. It follows that $h^0(\mathcal{O}(D + D')) \geqslant m_D + m_{D'} + 1$. □

Back to Curves of Genus 1:

There are many 2:1 maps from a genus 1 curve, E, to $\mathbb{P}^1$. For example for any $P \in E, f_{|\mathcal{O}(2P)|}$ is ramified at P and 4 other points. In fact (we don't know enough to prove this yet) those 4 points are exactly the points differing from P by a 2-torsion element of E.

If $\deg \mathscr{L} = 3$ then $|\mathscr{L}|$ represents E as a plane cubic by looking at the table in 1.8.1. Similarly, $\deg \mathscr{L} = 4$ represents E as a quartic in $\mathbb{P}^3$. The image of $f_{|\mathscr{L}|}$ in $\mathbb{P}^3$ has homogeneous coordinate ring given by $\bigoplus_{n \geqslant 0} \Gamma(E, \mathscr{L}^{\otimes n})$. We get a surjective graded map

$$\mathbb{C}[x_0, x_1, x_2, x_3] \to \bigoplus_{n \geqslant 0} \Gamma(E, \mathscr{L}^{\otimes n})$$

by the embedding.
For $n = 1$, both sides have dimension 4, LHS by counting and RHS by R.R.
For $n = 2$, the RHS has dimension 10 by counting. The LHS has dim 8 by R.R. Therefore there are two linearly independent quadrics Q_1, Q_2 vanishing on E. Since $\deg E = 4, E = V(Q_1, Q_2)$ since there is a containment and both have the same degree.

Question

Suppose $Q_1, Q_2 \subset \mathbb{P}^3$ are quadrics which intersect in a nice curve X. What is $g(X)$?

Answer. We have the following exact sequence (which is one example of a Koszul complex)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{(Q_2,-Q_1)} \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \xrightarrow{\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Then by additivity of χ we get

$$\chi(\mathcal{O}_X) = 1 - 2\chi(\mathcal{O}(-2)) + \chi(\mathcal{O}(-4)) = 1 - 0 + \chi(\omega_X)) = 1 - 1 = 0.$$

Therefore $g(X) = 1 - \chi(\mathcal{O}_X) = 1$ so X is genus 1. □

We now sketch the construction of the Koszul complex in greater generality.

Claim

Let X be smooth, and $\mathscr{E}$ s vector bundle on X. Let $s \in \Gamma(X, \mathscr{E})$ be such that $\operatorname{codim} V(s) = \operatorname{rk}\mathscr{E}$. Then we can resolve $\mathcal{O}_X$ as follows:

$$0 \to \bigwedge^{top} \mathscr{E}^\vee \xrightarrow{\cdot s} \cdots \xrightarrow{\cdot s} \bigwedge^{2} \mathscr{E}^\vee \xrightarrow{\cdot s} \mathscr{E}^\vee \xrightarrow{\cdot s} \mathcal{O}_X \to \mathcal{O}_{V(s)} \to 0$$

Sketches of an idea of a proof. Since we defined the maps globally we can show exactness locally. Thus wlog $\mathscr{E} = \mathcal{O}_X^{\oplus d}$, $s = (s_1, \ldots, s_d)$. Consider the chain complex (concentrated in degrees 0 and 1)

$$C_i^\bullet = (\mathcal{O}_X \xrightarrow{\cdot s_i} \mathcal{O}_X)$$

we then get another chain complex by setting

$$D^\bullet = \bigotimes_{i=1}^{d} C_i^\bullet$$

which is then obviously exact in degree > 0 as a tensor product of complexes exact in degree > 0. Also $H^0(D^\bullet) = \mathcal{O}_{V(s)}$.

Claim: $D^\bullet$ is the Koszul complex earlier defined.
When $d = 2$, $H^i(C_1^\bullet \otimes C_2^\bullet) = \underline{Tor}_i(\mathcal{O}_{V(s_1)}, \mathcal{O}_{V(s_2)})$ as we have an isomorphism of total complexes between the two (bounded) double chain complexes

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\cdot s_1} & \mathcal{O}_X \\ {\scriptstyle \cdot s_2}\downarrow & & \downarrow{\scriptstyle \cdot - s_2} \\ \mathcal{O}_X & \xrightarrow[\cdot s_2]{} & \mathcal{O}_X \end{array} \qquad\qquad \begin{array}{c} \mathcal{O}_{V(s_1)} \\ \downarrow \\ \mathcal{O}_{V(s_2)} \end{array}$$

here is where we use the assumptions on the codimension of $V(s)$. This is where the lecture ended.

□

1.10 Lecture 10 - Curves of genus 4 and the Picard scheme/functor

Let $C/\mathbb{C}$ be a non-hyperelliptic curve of genus 4, then we have the canonical map $f_{|\omega_C|} : C \to \mathbb{P}^3$. This induces a map of projective schemes which comes from a graded map of rings:

$$\mathbb{C}[x_0, \cdots, x_3] \to \bigoplus_{n \geqslant 0} H^0(C, \omega_C^{\otimes n})$$

If you care to see why C is Proj of the RHS refer to the proof of part 2 of thm 1.4.3. The degree of ω_C is 6 and so for all its tensor powers will have $H^1 = 0$. It follows by Riemann-Roch, that $h^0(\omega_C^{\otimes n}) = 6n - 3$ for $n \geqslant 2$ and for $n = 1$, it $h^0 = 1$, and for $n = 2$, $h^0 = 4$. The dimension of the degree n component of $\mathbb{C}[x_0, \cdots, x_3]$ is $\binom{n+3}{3}$. It follows that in degree 2, we have 10 dimensions on the LHS and 9 on the RHS so there is a quadric Q vanishing on C. Then since the LHS has dimension 4 in degree 1, this quadric gives 4 dimensions worth of elements in degree 3 which vanish on C. Since in degree 3, the LHS has dimension 20 and the RHS has dimension 15 and we already know 4 dimensions of the kernel, then there is at least one cubic F_3 which vanishes on C and not on Q.

Since F_3 is degree 3 and Q is degree 2, then $F_3 \cap Q$ has degree 6 and contains C which also has degree 6. It follows that $F_3 \cap Q = C$. Therefore we have proven

Prop

A canonically embedded genus 4 curve is the intersection of a unique quadric and a non-unique cubic.

Theorem 1.10.1: Curves have a g_d^3

Let C/k be a nice curve with k an infinite field. Then C has a closed embedding into $\mathbb{P}^3$, i.e. C has a very ample g_d^3.

Proof. Let P be a closed point of C, then $\mathcal{O}(P)$ is a line bundle of degree $[k(P) : k] > 0$ and therefore $\mathcal{O}(P)$ is ample, so $C \hookrightarrow \mathbb{P}^N$ for some N. We now want to repeatedly project down from $\mathbb{P}^N$ until we get to $\mathbb{P}^3$.

Claim: If $N > 3$, there is a point $p \in \mathbb{P}^N(k)$ such that $proj_p : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ which induces a closed immersion of C into $\mathbb{P}^{N-1}$. Note that $proj_p$ fails to be a closed immersion iff p lies on a secant/tangent line to C.

To prove this claim, it suffices to find a point which is on neither the secant variety or the tangent variety to C. To do this, it suffices to show that the secant variety has dimension $\leqslant 3$, as then it is a proper closed subscheme of $\mathbb{P}^N$ so there is a rational point not contained in the secant variety by infiniteness of k. To construct $\mathrm{Sec}(C)$, we first consider $\mathbb{G}r(1, N) = Gr(2, N+1)$, the Grassmannian of all lines in $\mathbb{P}^N$. We have $L \subseteq \mathbb{G}r(1, N) \times_k \mathbb{P}^N$, the universal family of lines in $\mathbb{P}^N$, i.e. over any line $\ell \in \mathbb{G}r(1, N)$, the preimage under the projection map of ℓ in L is the line ℓ in $\mathbb{P}^N$.

To construct L, recall that the Grassmannian $Gr(k, n)$ represents the functor

$$F(X) = \left\{ V \otimes \mathcal{O}_X \twoheadrightarrow \mathscr{E} \middle| \mathscr{E} \text{ locally free of rank } k, \dim_k V = n \right\}$$

We can now consider the identity map $\mathrm{id} : \mathbb{G}r(1, N) \to \mathbb{G}r(1, N)$ which by the above corresponds to some map $\varphi : V \otimes \mathcal{O}_{\mathbb{G}r(1,N)} \to \mathscr{E}$ where $\mathscr{E}$ is some rank 2 locally free sheaf on $\mathbb{G}r(1, N)$ and $\dim_k V = N + 1$. We then get a map $\mathrm{Sym}\,\varphi : \mathrm{Sym}(V \otimes \mathcal{O}) \to \mathrm{Sym}\,\mathscr{E}$ which yields a map of relative proj, $f : \operatorname{Proj}_{\mathbb{G}r(1,N)} \operatorname{Sym}(\mathscr{E}) \to \operatorname{Proj}_{\mathbb{G}r(1,N)} \operatorname{Sym}(V \otimes \mathcal{O})$. Since $V \otimes \mathcal{O}_{\mathbb{G}r(1,N)}$ is just a free sheaf of rank $N+1$, then the RHS is exactly $\mathbb{P}^N_{\mathbb{G}r(1,N)} = \mathbb{P}^N_k \times_k \mathbb{G}r(1,N)$. We will call the LHS L. Since the map of $\mathcal{O}_X$-algebras was surjective, then this map is a closed immersion. We want to show that L is this universal family of lines. For any point $\operatorname{Spec} k(\ell) \to \mathbb{G}r(1,N)$ corresponding to a line ℓ in $\mathbb{P}^N_k$, we have the following commutative diagram,

$$\begin{tikzcd}
 & & \mathbb{P}^N_k \\
\operatorname{Spec} k(\ell) \times_{\mathbb{G}r(1,N)} L \arrow[urr, "g"] \arrow[r] \arrow[d] & L \arrow[r] & \mathbb{P}^N_{\mathbb{G}r(1,N)} \arrow[u] \arrow[d] \\
\operatorname{Spec} k(\ell) \arrow[rr] & & \mathbb{G}r(1,N)
\end{tikzcd}$$

The fiber product on the left is the preimage of ℓ under the projection map $L \to \mathbb{G}r(1,N)$. It follows that we want the map g to have image exactly ℓ. Since L is the proj of a rank 2 line bundle on $\mathbb{G}r(1,N)$, then the fibers of the structure map $L \to \mathbb{G}r(1,N)$ will be $\mathbb{P}^1_k$. The map $\operatorname{Spec} k(\ell) \to \mathbb{G}r(1,N)$ corresponds to a surjective map $\psi : k^{N+1} \to k^2$. Now map $\operatorname{Spec} k(\ell) \to \mathbb{P}^1_k$ to an arbitrary point, then consider $\psi \otimes \mathrm{id} : k^{N+1} \otimes \mathcal{O}_{\mathbb{P}^1_k} \to k^2 \otimes \mathcal{O}_{\mathbb{P}^1_k}$. This is a surjective map $\widetilde{\psi} : \mathcal{O}^{N+1}_{\mathbb{P}^1_k} \to \mathcal{O}^2_{\mathbb{P}^1_k}$. The identity map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ corresponds to the map $\mathcal{O}^2_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}(1)$ the sheafification of the map $k[x_0,x_1]^2 \to k[x_0,x_1]$ by $(x,y) \mapsto x_0x + x_1y$. One checks that at the point $\mathfrak{p} = (x_0 - a, x_1 - b)$ $\overline{x}_0 = a, \overline{x}_1 = b$, so the map becomes $k^2 \to k$ by $(x,y) \mapsto ax + by$ which corresponds exactly to the point $[a : b]$ as one expects. Composing with $\widetilde{\psi}$, we obtain a map $\gamma : \mathcal{O}^{N+1}_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}(1)$ corresponding to a map $\mathbb{P}^1_k \to \mathbb{P}^N_k$. The image of this map is exactly the line corresponding to the element $\ell \in \mathbb{G}r(1,N)$. Since $\operatorname{Spec} k(\ell) \times_{\mathbb{G}r(1,N)} L \cong \mathbb{P}^1_k$, we want to show that g is the same map $g : \mathbb{P}^1_k \to \mathbb{P}^N_k$ as ℓ, up to an automorphism of $\mathbb{P}^1$. We may assume that we embed $k(\ell)$ at the point $1 \in \mathbb{P}^1_k$, then $\gamma : k[x_0,x_1]^{N+1} \to k[x_0,x_1]$ maps the standard basis $(e_0, \cdots, e_n)$ to $e_i \mapsto x_0\psi(e_i)_0 + x_1\psi(e_i)_1$. We now want to pull $\mathcal{O}_{\mathbb{P}^N_k}(1)$ back along g, which we can do in steps and show that the pullbacks of x_i agree with $\gamma(e_i)$. I will leave this as an exercise, since it is kind of convoluted.

Now consider $\pi_{1*}(\pi_2^*\mathcal{O}_C \otimes \mathcal{O}_L)$. The rank of this sheaf at any point $[\ell] \in \mathbb{G}r(1,N($ is the multiplicity of the intersection $\ell \in C$. Now consider $Z \subseteq \mathbb{G}r(1,N)$ to be the closed subset $\{[\ell] | \operatorname{rank}_{[\ell]} \pi_{1*}(\pi_2^*\mathcal{O}_C \otimes \mathcal{O}_L) \geqslant 2\}$ with the reduced induced structure. Note that it is closed by upper semicontinuity of rank. We now define $\operatorname{Sec}(C) = \pi_2(\pi_1^{-1}(Z))$. ☐

Picard scheme/functor

The Picard group of a scheme X/k, denoted $\operatorname{Pic}(X)$, is the collection of all line bundles on X up to isomorphism with the composition being the tensor product.

> **Question**
>
> If X/k is sufficiently nice, then is $\operatorname{Pic}(X)$ the points of a k-scheme?

For references on this, look in FGA explained, chapter 9, or Néron Models, chapter 7.

We want to guess what the functor we should be representing is. A reasonable guess would be the following: Let X/k be a scheme and consider the function $T \mapsto \mathrm{Pic}(X \times_k T)$ from k-schemes to Gp. The problem with this functor is that it is not a Zariski sheaf and therefore cannot be representable.

For example, if $X = k$, then if $\mathscr{L}$ is any element of $\mathrm{Pic}_{k/k}(T) = \mathrm{Pic}(T)$, then we may cover T by open sets $\{U_i\}$ such that $\mathscr{L}|_{U_i} = \mathcal{O}_T|_{U_i}$. If $\mathrm{Pic}_{X/k}$ were now a Zariski sheaf, then the fact that $\mathscr{L}$ is trivial on an open cover means that $\mathscr{L} = 0$ in $\mathrm{Pic}_{k/k}(T)$ and so we would have that $\mathrm{Pic}_{k/k}$ is the 0 functor. Since this is not the case, then $\mathrm{Pic}_{X/k}$ is not a Zariski sheaf and therefore cannot be representable.

Another problem is given as follows: Fix T such that $\mathrm{Pic}(T) \neq 0$, then there is an action of $\mathrm{Pic}(T)$ on $\mathrm{Pic}(X \times_k T)$. For any $\mathscr{L} \in \mathrm{Pic}(T)$ and any $\mathscr{E} \in \mathrm{Pic}(X \times_k T)$, we let $\mathscr{L} \cdot \mathscr{E} = \mathscr{E} \otimes \pi_1^* \mathscr{L}$. This action is locally trivial. In other words, we may take an open cover $\{U_i\}$ of T such that $\mathscr{L}|_{U_i} = \mathcal{O}_T|_{U_i}$, then the action of $\mathrm{Pic}(T)$ on $\mathrm{Pic}(X \times_k T)$ restricts to an action $\mathrm{Pic}(U_i) \curvearrowright \mathrm{Pic}(X \times_k U_i)$. Now suppose that the action commutes with restriction, then the fact that the action of each $\mathrm{Pic}(U_i)$ is trivial for $\mathscr{L}$ means that the action of $\mathscr{L} \in \mathrm{Pic}(T)$ should be trivial, but it is not.

Remark

$\mathrm{Hom}_{\mathbf{Sch}/S}(-, Z)$ is a sheaf in the fppf (faithfully flat finite presentation) site:
Given any flat, surjective, finite presented morphism $U \to X$, we get an equalizer diagram

$$\mathrm{Hom}(X, Z) \to \mathrm{Hom}(U, Z) \rightrightarrows \mathrm{Hom}(U \times_X U, Z)$$

That is to say that given a morphism from $U \to Z$ if both projections agree, then it glues to a morphism on X. In the case of the Zariski site, which is just the Zariski topology, if we have an open cover U_i, then taking $U = \bigsqcup_i U_i$, $U \times_X U = \bigsqcup_{i,j} U_i \cap U_j$ and so this is just the statement that the morphisms from the U_i agree on intersections.

This is called fppf descent.

Def: If X/S is an S-scheme, we let $\mathrm{Pic}_{X/S}$ be the sheafification w.r.t. the fppf site of the naïve Picard presheaf we defined above.

Example

If $X = S = \mathrm{Spec}\ k$, then $\mathrm{Pic}_{X/S}$ is represented by $\mathrm{Spec}\ k$ since it is trivial (since any map of a k-scheme into $\mathrm{Spec}\ k$ is just the structure morphism).

Theorem 1.10.2: Existence of Pic scheme

Let $f : X \to S$ be projective, finitely presented, and flat, with integral geometric fibers, then $\mathrm{Pic}_{X/S}$ is representable.

Remark

If f has a section s, e.g. if $S = \operatorname{Spec} k$ and X has a rational point, then everything is much nicer :). For then we can define

$$\operatorname{Pic}^{rig}_{X/S,s}(T) = \left\{\text{line bundles on } X \times_S T \text{ w/ an isomorphism } s_T^* \mathscr{L} \xrightarrow{\sim} \mathcal{O}_T\right\} / \sim$$

Thm

If $f : X \to S$ has a section s and $\operatorname{Pic}_{X/S}$ is representable, then it also represents $\operatorname{Pic}^{rig}_{X/S,s}$.

Example

Let $X/\mathbb{R} = \{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{R}}$.
Q: What represents $\operatorname{Pic}_{X/\mathbb{R}}$?
Warm up: What is $\operatorname{Pic}_{\mathbb{P}^1/\mathbb{C}}$? A: $\bigsqcup_{k\in\mathbb{Z}} \operatorname{Spec} \mathbb{C}$ (one for each degree).

We must have

$$\operatorname{Pic}_{X/\mathbb{R}} \times \mathbb{C} = \operatorname{Pic}_{X\times\mathbb{C}/\mathbb{C}} = \operatorname{Pic}_{\mathbb{P}^1/\mathbb{C}}$$

so therefore

$$\operatorname{Pic}_{X/\mathbb{R}} = \bigsqcup_{k\in\mathbb{Z}} \operatorname{Spec} \mathbb{R}.$$

However, X has no line bundles of degree 1 so this is weird. [Daniel mentioned something about galois actions here that I did not write in great detail]

Quick and dirty detour into riemann surfaces

Let X be a compact Riemann surface. We recall that in a great level of generality $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ by *Čech* cohomology, the same proof works as in the algebraic case. (This is apparently true in very many different topologies on a scheme).
We recall the exponential exact sequence

$$0 \to \underline{2\pi i\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

which yields the LES in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(\underline{\mathbb{Z}}) \arrow[r] & H^0(\mathcal{O}_X) \arrow[r] & H^0(\mathcal{O}_X^*) \arrow[dll] \\
& H^1(\underline{\mathbb{Z}}) \arrow[r] & H^1(\mathcal{O}_X) \arrow[r] & H^1(\mathcal{O}_X^*) \arrow[dll] \\
& H^2(\underline{\mathbb{Z}}) \arrow[r] & 0 &
\end{tikzcd}$$

Where you can see that $H^i(\mathscr{F}) = 0$ for coherent $\mathscr{F}$ and $i > \dim X$ by Dolbeault cohomology (or for projective manifolds by GAGA). We also recall that $H^i(\underline{\mathbb{Z}}) = H^i(X, \mathbb{Z})$ as singular cohomology agrees with sheaf cohomology of constant sheaves for sufficiently nice (locally contractible) spaces (see Voisin Hodge Theory 4.47 or Bredon Sheaf theory III.1.1). It is also (non-trivially) the case that the genus of X in the algebraic sense is the same an in the topological sense. Therefore the above sequence reduces to

$$\mathbb{Z}^{2g} \xrightarrow{\alpha} \mathbb{C}^g \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the map $\mathrm{Pic}(X) \to \mathbb{Z}$ is actually the degree map we have come to know and love. Thus since $\mathbb{Z}$ is free, $\mathrm{Pic}(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \oplus \mathbb{Z}$ as abelian groups.

$$H^1(X, \mathcal{O}_X) = H^0(X, \omega_X)\check{} \supseteq H_1(X, \mathbb{Z})$$

where we get an the inclusion by mapping a cycle γ to integration over γ. The map is well defined by stokes theorem and injective.

Fact

$\mathbb{Z}^{2g} \subset \mathbb{C}^g$ is honestly a lattice so $H^0(X, \omega_X)\check{}/H_1(X, \mathbb{Z})$ is a complex torus T^g and, furthermore, is isomorphic to $(\mathrm{Pic}^0_{X/\mathbb{C}})^{an}$ (if X is algebraic and we consider the associated compact Riemann surface X^{an} in the above).

Question

$X/\mathbb{C}$ a nice curve. What is $T_{[\mathscr{L}]}\,\mathrm{Pic}_{X/\mathbb{C}}$?

Answer. We know it should be $H^1(\mathcal{O}_X)$ by the analytic description. Now we "prove" this algebraically. We first remark that by Hartshorne Ex II.2.8, $T_{[\mathscr{L}]}\,\mathrm{Pic}_{X/\mathbb{C}}$ are just the maps of $\mathbb{C}[\epsilon]/\epsilon^2$ sending the closed point to $[\mathscr{L}]$.

Alternatively, we get the tangent space at a point $p \in Z(k)$ (which is a map from $\mathrm{Spec}\ k \to Z$) as the fiber over p of the map

$$\mathrm{Hom}(\mathrm{Spec}\ k[\epsilon]/\epsilon^2, Z) \to \mathrm{Hom}(\mathrm{Spec}\ k, Z)$$

which sends $v \in T_pZ$ to p.

Question: Can we understand $\mathrm{Pic}_{X/\mathbb{C}}(\mathbb{C}[\epsilon]/\epsilon^2)$?
Yes,

$$\mathrm{Pic}_{X/\mathbb{C}}(\mathbb{C}[\epsilon]/\epsilon^2) = \mathrm{Pic}(X_{\mathbb{C}[\epsilon]/\epsilon^2}) = H^1(X_{\mathbb{C}[\epsilon]/\epsilon^2}, \mathcal{O}^*_{\mathbb{C}[\epsilon]/\epsilon^2}) = H^1(X, \mathcal{O}^*_{\mathbb{C}[\epsilon]/\epsilon^2})$$

We get a SES (the fake exponential exact sequence)

$$a + b\epsilon \longmapsto a$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}^*_{X_{\mathbb{C}[\epsilon]/\epsilon^2}} \longrightarrow \mathcal{O}^*_X \longrightarrow 0$$

$$f \longmapsto 1 + f\epsilon$$

yielding the following exact sequence

$$0 \to H^1(\mathcal{O}_X) \to \mathrm{Pic}(X_{\mathbb{C}[\epsilon]/\epsilon^2}) \to \mathrm{Pic}(X) \to 0$$

□

1.11 Lecture 11 - Abel-Jacobi map

If X is a nice curve over k, with $X(k) \neq \varnothing$, then $\mathrm{Pic}_{X/k} = \mathbb{Z} \times \mathrm{Pic}^0_{X/k}$ where $\mathrm{Pic}^0_{X/k}$ is the connected component of $\mathcal{O}_X$.

More canonically, there is a short exact sequence,

$$0 \to \mathrm{Pic}^0_{X/k} \to \mathrm{Pic}_{X/k} \xrightarrow{\deg} \mathbb{Z} \to 0$$

Where we associate $\mathbb{Z}$ with

$$\mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} \mathrm{Spec}\ k$$

Whence the above SES is a SES of algebraic groups. Moreover, $\mathrm{Hom}(T, \mathrm{Pic}_{X/k})$ is all rigidified line bundles on $X \times_k T$ up to isomorphism. That is to say there is a section $T \to X \times_k T$ since $X(k) \neq \varnothing$.

Given $\mathscr{L} \in \mathrm{Pic}(X \times_k T)$, we get a map $T \to \mathrm{Pic}_{X/k}$ by $\mathscr{L} \otimes (\pi^* s^* \mathscr{L})^{\vee}$.

Def: $\mathrm{Pic}^d_{X/k} = \deg^{-1}(d)$.

Construction: Let X be a nice curve, then we want to find a map $AJ^1 : X \to \mathrm{Pic}^1_{X/k}$ corresponding set theoretically to the map $p \mapsto \mathcal{O}(p)$. A map $X \to \mathrm{Pic}^1_{X/k}$ is a line bundle on $X \times_k X$ by the functorial definition of Pic. We choose the line bundle $\mathscr{L} = \mathcal{O}_{X \times_k X}(\Delta) = \mathscr{I}_{\Delta}^{\vee}$. For any slice $X \times \{p\}$, we have $\mathscr{L}|_{X \times \{p\}} = \mathscr{I}_p^{\vee} = O(p)$.

Hard exercise: $dAJ^1 \to T_pX \to T_{[\mathcal{O}(p)]} \mathrm{Pic}^1_{X/k}$. We know that the latter is $H^1(X, \mathcal{O}_X)$ which by Serre duality means we have a map $(dAJ^1)^{\vee} : H^0(X, \omega_X) \to (T_pX)^{\vee}$ which maps $\eta \mapsto \eta_p$ where η is a differential, so η_p is a derivation at p, i.e. an element in $(T_pX)^{\vee}$.

Since we know what dAJ^1 is, then we obtain AJ^1 by integrating. Fix $p \in X(\mathbb{C})$, then we define $AJ_p : X \to \mathrm{Pic}^0_{X/k}$ by $AJ^1 \otimes \mathcal{O}(-p)$, then this sends p to $\mathcal{O}_X$.

Claim

$AJ_p(q) = \left[\eta \mapsto \int_p^q \eta\right]$.

Proof. It is well-defined, i.e. it is independent of the curve from p to q. This follows from the fact that in the complex construction of $\mathrm{Pic}_{X/\mathbb{C}}$, we mod out by the image of $H^1(X, \mathbb{Z})$, i.e. integrals along loops, so two curves give the same value. Then $dAJ_p = dAJ^1$ by FTOC. It follows that they are equal since they give the same value at $q = p$. □

AJ^1 is a closed immersion for $g \geqslant 1$ since then ω_X is globally generated. Since the cotangent space, $(T_pX)\check{}$ is just derivations at p, then being globally generated means that $H^0(X, \omega_X) \to (T_pX)\check{}$ is surjective and since this is the dual $(dAJ^1)\check{}$, then dAJ^1 is injective. Therefore AJ^1 separates tangent vectors. To show that it is a closed immersion, it therefore suffices to show that it is injective on points. Suppose that $\mathcal{O}(p) = \mathcal{O}(q)$, then the genus would be 0. Therefore AJ^1 is a closed immersion.

Remark

The map AJ^1 exists even if $X(k) = \varnothing$.

We now want to construct a map AJ^d which should map from the space of degree d effective divisors over X to the degree d component, $\mathrm{Pic}^d_{X/k}$.

From AJ^1, we can take $(AJ^1)^d : X^d \to (\mathrm{Pic}^1_{X/k})^d$, then we have a map $\Sigma : (\mathrm{Pic}^1_{X/k})^d \to \mathrm{Pic}^d_{X/k}$ given by taking the tensor product of the line bundles. The map Σ is commutative, i.e. invariant under the natural S_d action on X^d given by permuting the factors. It follows that if X^d/S_d exists, then the map $X^d \to \mathrm{Pic}^d_{X/k}$ given by composing $\Sigma \circ (AJ^1)^d$ factors through X^d/S_d.

Fact

The quotient $\mathrm{Sym}^d X = X^d/S_d$ exists in $\mathbf{Sch}_k$.

In general, the quotient of a sufficiently nice scheme by a finite group will always exist.

Example

$(\mathrm{Spec}\ R)/G \cong \mathrm{Spec}\ R^G$ where R^G is the ring of invariants under the action of G.
$\mathbb{A}^d/S_d = \mathrm{Spec}\ k[x_1, \cdots, x_d]^{S_d} = \mathrm{Spec}\ k[\sigma_1, \cdots, \sigma_n]$ where $\sigma_1, \cdots, \sigma_d$ are the elementary symmetric polynomials. Therefore $\mathbb{A}^d/S_d$ is also a copy of $\mathbb{A}^d$ but the quotient map $\mathbb{A}^d \to \mathbb{A}^d/S_d$ is $d!$ to 1 and given by interpreting a point x in A^d as the roots of a polynomial and then sending it to the point $(\sigma_1(x), \sigma_2(x), \cdots \sigma_d(x))$.

Def: The map AJ^d is the quotient map induced by $\Sigma \circ (AJ^1)^d$ from $\mathrm{Sym}^d X \to \mathrm{Pic}^d_{X/k}$.

Note that the points of $\mathrm{Sym}^d X$ are collections of d points of X without order, so their sum determines a degree d effective divisor. All degree d effective divisors are sums of d points, so the points of Sym^d are exactly the effective divisors over X.

We now look at the Hilbert scheme. The Hilbert functor over X is given by,

$$\mathrm{Hilb}^d(X)(T) = \left\{ \substack{Z \subseteq X \times T \text{ flat } /T \text{ s.t.} \\ Z_{\bar{t}} \text{ has length } d \text{ for all } \bar{t} \in T(\bar{k})} \right\}$$

Grothendieck

If X is projective, then Hilb^d is representable.

FGA explained

X is a nice curve, then $\mathrm{Hilb}^d(X)$ is smooth projective and isomorphic to $\mathrm{Sym}^d(X)$.

It follows that we can understand AJ^d as a map from $\mathrm{Hilb}^d(X) \to \mathrm{Pic}^d_{X/k}$. To understand what this is given by, we have a universal object $Z_{univ} \subseteq X \times \mathrm{Hilb}^d(X)$ which is the closed subscheme, flat over $\mathrm{Hilb}^d(X)$ corresponding to the identity map from $\mathrm{Hilb}^d(X) \to \mathrm{Hilb}^d(X)$. Then we know that $Z_{\bar{t}}$ has length d for all geometric points $\bar{t} \in \mathrm{Hilb}^d(X)(\bar{k})$. That is to say that $Z_{\bar{t}}$ parameterizes the collections of d geometric points in X.

Prop

$(AJ^d)^{-1}([\mathscr{L}]) = \mathbb{P}H^0(X, \mathscr{L})$.

Proof. Set-theoretically we have that $(AJ^d)^{-1}([\mathscr{L}])$ is the collection of all effective divisors linearly equivalent to $\mathscr{L}$, i.e. $|\mathscr{L}| = \mathbb{P}H^0(X, \mathscr{L})$. □

Prop

$\mathrm{Sym}^d(X)$ is a projective space bundle over $\mathrm{Pic}^d_{X/k}$ for $d > 2g-2$ and $X(k) \neq \varnothing$.

Proof. $h^1(\mathscr{L})$ vanishes in degree $> 2g-2$, so $h^0(\mathscr{L}) = d+1-g$ and therefore the fibers at every point are exactly $\mathbb{P}^{d-g}$. Note that we have not defined a projective space bundle. □

Remark

If $X(k) \neq \varnothing$, then AJ^d is a Zariski-locally trivial $\mathbb{P}^{d-g}$-bundle for $d > 2g-2$.

1.12 Lecture 12 - Stronger Vanishing Results and Zeta Functions

Recall the Abel Jacobi map $AJ_d : \mathrm{Sym}^d C \to \mathrm{Pic}^d_{C/k}$ and that $AJ_d^{-1}([\mathscr{L}]) = \mathbb{P}H^0(C, \mathscr{L})$.

Corollary

For $d > 2g-2, C(k) \neq 0, \mathrm{Sym}^d C$ is a $\mathbb{P}^{d-2}$-bundle over $\mathrm{Pic}^d_{C/k}$. (We might come back to this when we know the proper definition of a projective space bundle).

Thm

Let C/k be a nice curve where $k = \bar{k}$ of genus g.

1. A general line bundle $\mathscr{L}$ (i.e. for all points of a nonempty open set of $\mathrm{Pic}^d_{C/k}$) of degree $d < g$ has $h^0(\mathscr{L}) = 0$.
2. A general line bundle $\mathscr{L}$ of degree $d \leqslant g$ with $h^0(\mathscr{L}) \neq 0$ has $h^0(\mathscr{L}) = 1$.

In fact this theorem implies the case where $k \neq \bar{k}$ by base changing.

Slogan 1

For general $\mathscr{L}$ with $\deg \mathscr{L} = d, h^0(\mathscr{L}) = \max(0, d - g + 1)$.

1. *Proof.* Consider $W_d = \{\mathscr{L} \in \mathrm{Pic}^d_{C/k} | h^0(\mathscr{L}) > 0\}$. W_d is closed by upper-semicontinuity of cohomology. Alternatively W_d can be seen to be the image of $\mathrm{Sym}^d C$ under C as

$$(AJ^d)^{-1}(\mathscr{L}) = \mathbb{P}H^0(C, \mathscr{L}) \neq 0 \iff \mathscr{L} \text{ has a global section}$$

Thus W_d closed and irreducible. Therefore to show that W_d^c is generic, it suffices to show that $W_d \neq \mathrm{Pic}^d_{C/k}$.
We have that $\dim W_d \leqslant \dim \mathrm{Sym}^d C = d$ but $\dim \mathrm{Pic}^d_{C/k} = \dim T_{\mathscr{L}} \mathrm{Pic}^d_{C/k} = h^1(\mathcal{O}_C) = g$ (from lecture 11) as $\mathrm{Pic}^d_{C/k}$ is smooth.
As $\mathrm{Pic}^d_{C/k}$ is an algebraic group this is always true in characteristic 0. If $\operatorname{char} k > 0$ then in fact (due to some deformation theory facts and functor of points nonsense) this is also true since $H^2(C, \mathcal{O}_C) = 0$. Thus by dimension considerations $W_d \subsetneq \mathrm{Pic}^d_{C/k}$ is a proper closed subset and so we win. □

2. *Proof.* It is equivalent to show that $\mathrm{Sym}^d C \to W_d$ is injective on an open subset as $(AJ^d)^{-1}([\mathscr{L}]) = \mathbb{P}H^0(\mathscr{L})$ is finite iff it is a single point iff $h^0(\mathscr{L}) = 1$. But (by Hartshorne Ex II.3.22(e)) dim general fiber $= \dim \mathrm{Sym}^d C - \dim \dim W_d = d - \dim W_d$ it suffices to show that $\dim W_d \geqslant d$ (and therefore exactly d).

   (a) If $d = g$ then R.R. says that $\chi(\mathscr{L}) = 1$ so $h^0(\mathscr{L}) > 0$ for all $\mathscr{L}$. Therefore $W_g = \mathrm{Pic}^g_{C/k}$ so we win.

   (b) It suffices to prove that $\dim W_d > \dim W_{d-1}$ for $1 \leqslant d \leqslant g$, for then $\dim W_d \geqslant d + \dim W_0 = d$. As notes above W_d is irreducible as it is the image of the irreducible spaces $\mathrm{Sym}^d C$ under a continuous map. Now choose $p \in C(k)$. Let $\iota_p : \mathrm{Pic}^{d-1}_{C/k} \to \mathrm{Pic}^d_{C/k}$ be the map induced by twisting with p. Then ι_p is an isomorphism (with inverse given by twisting down by p). In particular ι_p maps W_{d-1} into W_d as effective line bundles remain so after being twisted by a point. Since ι_p is injective, then, $\dim W_d \geqslant \dim W_{d-1}$ with equality iff $W_d = \iota_p(W_{d-1})$ as $\iota_p(W_{d-1})$ is a closed subset of the irreducible space W_d.
   But now suppose that ι_p is an iso for some d. Then ι_x will be an iso for all $x \in X(k), i > d$. It suffices to check surjectivity; for any $\mathscr{L} \in W_{i+1}$, we have $\mathscr{L} = \mathscr{L}' \otimes \mathcal{O}(p)$ (since $\mathscr{L}$ is effective). Then $\mathscr{L}' \in W_i$, so $\mathscr{L}' = \mathscr{L}'' \otimes \mathcal{O}(x)$, then $\mathscr{L} = (\mathscr{L}'' \otimes \mathcal{O}(p)) \otimes \mathcal{O}(x)$, so ι_x is surjective.
   But we know that $\dim W_g = g$ so either equality holds, and then $\dim W_d = \dim W_g = g \geqslant d$ or equality does not hold for and $i \leqslant d$ and we win.

$\square$

Slogan 2

Cohomology groups are usually as small as possible.

For an example look up "generic vanishing theorem."

Example

$C = \mathrm{Sym}^1 C \to \mathrm{Pic}^1_{C/k}$ is a closed immersion for $g > 0$.

$C = \mathrm{Sym}^2 C \to \mathrm{Pic}^2_{C/k}$ is a closed immersion for $g \geqslant 2$ and non-hyperelliptic C. Injectivity is clear as existence of a degree 2 effective divisor with more than one global section is equivalent to being hyperelliptic.

If C is hyperelliptic then there is a unique 1 dimensional fiber so the map is birational onto its image.

$\mathrm{Sym}^{g-1} C \to \mathrm{Pic}^{g-1}_{C/k}, W_{g-1}$ is a divisor called Θ. There is lots of theory about this guy. Daniel proceeded to talk about Brill-Noether theory in generality for another few minutes.

Zeta Functions

Def: Let $X/\mathbb{F}_q$ a variety. Set

$$Z_X(t) = \exp\left(\sum_{n\geqslant 1} \frac{\#X(\mathbb{F}_{q^n})t^n}{n}\right)$$

But why should we care about this function?

Motivation

$\frac{\partial}{\partial t} \log Z_X(t) \sum_{n\geqslant 1} \#X(\mathbb{F}_{q^n})t^{n-1}$ is the normal generating functions for $\#X(\mathbb{F}_{q^n})$.

This lets us bring in ideas from analytic number theory:

Slogan

Growth of Taylor series coeffs of $\frac{\partial}{\partial z} \log f(z)$ is controlled by the zeros and poles of $f(z)$.

Let us justify this idea, at least in the only case that will matter to us: rational functions.

Example

If $f(t) = \frac{\prod_i (1-a_it)}{\prod_j (1-b_jt)}$ then

$$\begin{aligned}\frac{\partial}{\partial t}\log f(t) &= \sum_i \frac{-a_i}{1-a_it} - \sum_j \frac{-b_j}{1-b_jt} \\ &= -\sum_i a_i \sum_{n\geqslant 0}(a_it)^n - \sum_j b_j \sum_{n\geqslant 0}(b_jt)^n \\ &= \sum_{n\geqslant 0}(\sum_j b_j^{n+1} - \sum_i a_i^{n+1})\end{aligned}$$

We now state the Weil conjectures.

Weil conjectures {Now theorems :)}

1. (Dwark) $Z_X(t)$ is rational.
2. (Grothendieck) If X is smooth and proper then Z_X satisfies a functional equation (not to be stated here).
3. Riemann hypothesis: All roots and poles α of Z_X in $\bar{\mathbb{Q}}$ and for any $\bar{Q} \hookrightarrow \mathbb{C}|\alpha| = q^{-j/2}$ for j depending only on α. (Such an α is called a q-Weil number.)
4. Set $p_j(t) = \prod_{|\alpha_i|=q^{-j/2}}(1-\alpha_i^{-1}t)$. If X is smooth and proper, then

$$Z_X(t) = \prod_{j=0}^{2\dim X} p_j(t)^{(-1)^{j+1}}$$

and $\deg p_j(t) = h^j_{\text{sing}}(X(\mathbb{C}),\mathbb{Q})$ if X "lifts to a complex model".

3 & 4 are due to Deligne. We prove 1) today for curves, 2 might be a nice presentation topic, we will prove 3& 4 when we know about surfaces and intersection theory.

Example

1. $Z_{\mathrm{Spec}\,\mathbb{F}_q}(t) = \exp\left(\sum_{n\geqslant 0}\frac{t^n}{n}\right) = \exp(-\log(1-t)) = \frac{1}{1-t}$ as $\mathrm{Spec}\,\mathbb{F}_q(K)$ has a single point for any extension $K/\mathbb{F}_q$.
2. $Z_{\mathbb{A}^1}(t) = \exp\left(\sum_{n\geqslant 0}\frac{q^n t^n}{n}\right) = \frac{1}{1-qt}$ by counting polynomials.
3. $Z_{\mathbb{A}^1}(t) = \frac{1}{1-q^n t}$ in the same way.
4. $Z_{\mathbb{P}^1}(t) = Z_{\mathrm{Spec}\,\mathbb{F}_q}(t)Z_{\mathbb{A}^1}(t) = \frac{1}{1-qt}\frac{1}{1-t}$ as $\mathbb{P}^1$ is a disjoint union of a closed point and a copy of $\mathbb{A}^1$.
5. $Z_{\mathbb{P}^n}(t) = \prod_{j=0}^n Z_{\mathbb{A}^j}(t) = \prod_{j=0}^n \frac{1}{1-q^j t}$.
6. $Z_{\mathbb{F}_{q^n}}(t) = \frac{1}{1-t^k}$ as there is a map $\mathbb{F}_{q^n} \to \mathbb{F}_{q^m}$ iff $n|m$.

If $X = Y \sqcup W$ with W closed then $Z_X = Z_Y Z_W$ as $\#X(K) = \#Y(K) + \#W(K)$ and then the exponential turns the addition to multiplication.

Claim:

$Z_X(t) = \prod_{x\in X^{\mathrm{cl}}} Z_x(t) = 1 + \sum_{n\geqslant 1} \#\,\mathrm{Sym}^n(X)(\mathbb{F}_q)t^n$.

Proof. Set-theoretically, $X = \bigsqcup_{x\in X^{\mathrm{cl}}}\{x\}$ and for each $\mathbb{F}_{q^n}$ there are only finitely many closed points with F_{q^n} points. It follows that $\sum \frac{\#X(\mathbb{F}_{q^n})t^n}{n}$ can be written as a sum over all the closed points of X, therefore,

$$\begin{aligned} Z_X(t) &= \exp\left(\sum_{n\geqslant 1}\frac{\#X(\mathbb{F}_{q^n})t^n}{n}\right) \\ &= \exp\left(\sum_{x\in X^{\mathrm{cl}}}\sum_{n\geqslant 1}\frac{\#x(\mathbb{F}_{q^n})t^n}{n}\right) \\ &= \prod_{x\in X^{\mathrm{cl}}}\frac{1}{1-t^{[k(x):\mathbb{F}_q]}} \\ &= \prod_{x\in X^{\mathrm{cl}}}\left(1 + t^{[k(x):\mathbb{F}_q]} + t^{2[k(x):\mathbb{F}_q]} + \cdots\right) \end{aligned}$$

It follows that the coefficient on t^n in the final expansion will be the number of collections of points whose degrees add up to n. This is the number of effective divisors of degree n on X. This is exactly the $\mathbb{F}_q$ points of $\mathrm{Sym}^n X$. It follows that,

$$Z_X(t) = 1 + \sum_{n\geqslant 1}\#\,\mathrm{Sym}^n(X)(\mathbb{F}_q)t^n$$

□

In the following theorem we assume that $X(\mathbb{F}_q) \neq \varnothing$, but Daniel claims the reduction to that case is not that hard.

Thm

Let X be a nice curve over $\mathbb{F}_q$ with a rational point. Then $Z_X(t)$ is rational.

Proof.

$$\begin{aligned}
Z_X(t) &= \sum_{n=0}^{\infty} \# \operatorname{Sym}^n X(\mathbb{F}_q) t^n \\
&= \sum_{n=0}^{2g-2} \# \operatorname{Sym}^n X(\mathbb{F}_q) t^n + \sum_{n>2g-2} \# \operatorname{Sym}^n X(\mathbb{F}_q) t^n \\
&= p(t) + \sum_{n>2g-2} \#\mathbb{P}^{n-g}(\mathbb{F}_q) \# \operatorname{Pic}^n_{X/k}(\mathbb{F}_q) t^n && \operatorname{Sym}^n X \text{ is a } \mathbb{P}^{n-g}\text{-bundle} \\
&= p(t) + \# \operatorname{Pic}^0_{X/k}(\mathbb{F}_q) \sum_{n>2g-2} \#\mathbb{P}^{n-g}(\mathbb{F}_q) t^n && \operatorname{Pic}^n \cong \operatorname{Pic}^0 \text{ if } X \text{ has a } \mathbb{F}_q \text{ point}
\end{aligned}$$

set $\tilde{p}(t) = \sum_{n>2g-2} \#\mathbb{P}^{n-g}(\mathbb{F}_q) t^n$.

Exercise

$(1-t)(1-qt)\tilde{p}(t)$ can be seen to be polynomial using the given formulae.

□

Thm

$Z_X(t) = \frac{p(t)}{(1-t)(1-qt)}$ for some $p(t) \in \mathbb{Z}[t]$ of degree $2g$.

2 Surfaces

2.1 Lecture 12 Cont. - Intro to surfaces

Def: A ruled surface is a smooth projective variety S of dimension 2 with a rational map $S \dashrightarrow C$ with C a nice curve s.t. the generic fiber is genus 0.

Example

If C is a curve and $\mathscr{E}$ is a rank 2 vector bundle on C, then $\mathbb{P}\mathscr{E} = \operatorname{Proj}_C \operatorname{Sym} \mathscr{E}$ is a ruled surface.

Thm

Let $k = \bar{k}$ be an algebraically closed field and $S \dashrightarrow C$ be a ruled surface, then S is birational to $C \times \mathbb{P}^1$.

Example

$\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ both contain a copy of $\mathbb{A}^2$ as a dense open subset, then this yields a birational map. If $g(C) > 0$ or $g(D) > 0$, then $C \times D$ is not birational to $\mathbb{P}^2$.

2.2 Lecture 13 - Intro to intersection theory

Question: Given two curves $C, D \subseteq X$ where X is some nice surface, what is the size of $C \cap D$ and what should multiplicity of an intersection mean? We want to count something and we want it to have nice properties.

Thm 2.13.1 : intersection pairing

Let X be a nice surface. There exists a symmetric bilinear pairing $\cdot : \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \mathbb{Z}$ such that if $\mathscr{L} = \mathcal{O}_X(C)$, and $\mathscr{M} = \mathcal{O}_X(D)$ with $\dim(C \cap D) = 0$, then $\mathscr{L} \cdot \mathscr{M} = \ell(\mathcal{O}_{C \cap D})$

Corollary

There exists a pairing $\cdot : \operatorname{Div}(X) \times \operatorname{Div}(X) \to \mathbb{Z}$ bilinear such that $C \cdot D = \ell(\mathcal{O}_{C \cap D})$ and $C \cdot D$ depends only on C, D up to linear equivalence.

Proof. Idea: If $C, D \subseteq X$ are curves, then $\mathcal{O}_{C \cap D} = \mathcal{O}_C \otimes_{\mathcal{O}_X} \mathcal{O}_D$. $\mathcal{O}_C$ and $\mathcal{O}_D$ are not locally free, but we have a locally free resolution of both, so we replace them with their locally free resolution. If $C \cap D$ is 0 dimensional, then,

$$\ell(\mathcal{O}_{C \cap D}) = \chi(\mathcal{O}_{C \cap D})$$

It follows that we want to define $C \cdot D$ using the locally free resolution as,

$$\begin{aligned} C \cdot D &= \chi((\mathcal{O}_X(-C) \to \mathcal{O}_X) \otimes (\mathcal{O}_X(-D) \to \mathcal{O}_X)) \\ &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_X(-C-D)) \end{aligned}$$

For line bundles, we can then define $\mathscr{L} \cdot \mathscr{M} = \chi(\mathcal{O}_X) - \chi(\check{\mathscr{L}}) - \chi(\check{\mathscr{M}}) + \chi(\check{\mathscr{L}} \otimes \check{\mathscr{M}})$. This definition is clearly symmetric and if $\dim(C \cap D) = 0$, then it gives the length of the intersection. We want to show that it is bilinear.

Claim: If $\mathscr{L} = \mathcal{O}_X(C)$ with C nice, then $\mathscr{L} \cdot \mathscr{M} = \deg \mathscr{M}|_C$.

Pf of claim: we compute,

$$\begin{aligned} \mathscr{L} \cdot \mathscr{M} &= \chi((\mathcal{O}_X(-C) \to \mathcal{O}_X) \otimes (\check{\mathscr{M}} \xrightarrow{0} \mathcal{O}_X)) \\ &= \chi(\mathcal{O}_C \otimes (\check{\mathscr{M}} \xrightarrow{0} \mathcal{O}_X) \\ &= -\chi(\check{\mathscr{M}}|_C) + \chi(\mathcal{O}_C) \\ &= \deg(\mathscr{M}|_C) \end{aligned}$$

The last line follows from Riemann-Roch. It follows that $C \cdot - : \operatorname{Pic} X \to \mathbb{Z}$ is linear when C is a nice curve. By Bertini, we have that if $\mathscr{L}$ is ample, then for $n \gg 0$, $\mathscr{L}^n = \mathcal{O}_X(C)$ for C some nice curve.

Now for any line bundle $\mathscr{L}$ on X, $\mathcal{O}_X(1)$ is very ample, so in particular it is ample, then $\mathscr{L} \otimes \mathcal{O}_X(n)$ is globally generated and thus $\mathscr{L} \otimes \mathcal{O}_X(n)$ is . Therefore $\mathscr{L} = \mathcal{O}_X(C) \otimes \mathcal{O}_X(-n) = \mathcal{O}_X(C-D)$ with C, D nice curves. It follows that $\mathscr{L} \otimes \mathcal{O}_X(n) = \mathcal{O}_X(C)$ and if $\mathcal{O}_X(n) = \mathcal{O}(D)$, then $\mathscr{L} = \mathcal{O}_X(C-D)$. Note that the above Bertini's theorem only tells us that $(\mathscr{L} \otimes \mathcal{O}_X(n))^m \mathcal{O}_X(C)$ for some m and idk how to get it to the form I claimed.

Given the above, then for X nice, we have that $X \cdot \mathscr{L} = X \cdot C - X \cdot D$, so $- \cdot \mathscr{L}$ is linear. $\square$

Def: Let X be a nice surface. The *intersection pairing* $\cdot : \operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ is a bilinear symmetric form given by the formula,

$$\mathscr{L} \cdot \mathscr{M} = \chi(\mathcal{O}_X) - \chi(\check{\mathscr{L}}) - \chi(\check{\mathscr{M}}) + \chi(\check{\mathscr{L}} \otimes \check{\mathscr{M}})$$

Remark

$C \cdot \mathscr{L} = \deg \mathscr{L}|_C$. If $s \in \Gamma(X, \mathscr{L})$, then the degree of $\mathscr{L}|_C$ is the number of zeros of $s|_C$ counted with multiplicity.

Example

We want to compute $C^2 = C \cdot C$. We have a SES,

$$0 \to \mathscr{I}_C/\mathscr{I}_C^2 \to \Omega^1_X|_C \to \omega_C \to 0$$

where $\mathscr{I}_C$ is the ideal sheaf of C in X. Notice that $\mathcal{O}_X(C)\check{|}_C = \mathcal{O}(-C) \otimes \mathcal{O}_C = \mathscr{I}_C \otimes \mathcal{O}_X/\mathscr{I}_C = \mathscr{I}_C/\mathscr{I}_C^2$ is the dual of the normal sheaf. Therefore $\mathcal{O}_X(C)|_C = \mathscr{N}_{C/X}$, so $C^2 = \deg \mathscr{N}|_{C/X}$.

Example

For $\mathbb{P}^2$, we have that $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$, then since $\mathcal{O}(1) \cdot \mathcal{O}(1)$ is the intersection of two lines, then it is exactly 1. By bilinearity, it follows that $\cdot$ is just multiplication.

Corollary

Bézout's theorem holds in $\mathbb{P}^2$.

2.3 Lecture 14 - Curves on a Surface are Important

We begin by explaining a bit about a black box from last class:

Claim:

Let X a variety, $\mathscr{L} \in \mathrm{Pic}(X), \mathscr{M}$ ample. For $n \gg 0, \mathscr{M}^{\otimes n} \otimes \mathscr{L}$ is ample.

Proof. Let r be such that $\mathscr{M}^{\otimes r}$ is very ample. By definition of ampleness, $\mathscr{M}^{\otimes r'} \otimes \mathscr{L}$ is globally generated for all $r' \gg 0$. Therefore $\mathscr{M}^{\otimes r+r'} \otimes \mathscr{L}$ is a very ample line bundle tensored with a globally generated line bundle for $r' \gg 0$. This implies that the line bundle is very ample, as we show below. $\square$

Thm

Let $\mathscr{L}_1$ very ample, $\mathscr{L}_2$ globally generated. Then $\mathscr{L}_1 \otimes \mathscr{L}_2$ is very ample.

Proof. Consider the map

$$f_{|\mathscr{L}_1|} \times f_{|\mathscr{L}_2|} : X \hookrightarrow \mathbb{P}\Gamma(X, \mathscr{L}_1) \times \mathbb{P}\Gamma(X, \mathscr{L}_2) \hookrightarrow^{\text{Segre}} \mathbb{P}(\Gamma(X, \mathscr{L}_1) \otimes \Gamma(X, \mathscr{L}_2))$$

It is clear that this is a closed immersion, as $f_{|\mathscr{L}_1|}$ is and $f_{|\mathscr{L}_2|}$ is totally defined as $\mathscr{L}_2$ is globally generated. This map factors through $f_{|\mathscr{L}_1 \otimes \mathscr{L}_2|}$ as we have a map $\Gamma(X, \mathscr{L}_1) \otimes \Gamma(X, \mathscr{L}_2) \to \Gamma(X, \mathscr{L}_1 \otimes \mathscr{L}_2)$ and therefore $f_{|\mathscr{L}_1 \otimes \mathscr{L}_2|}$ is a closed immersion, so $\mathscr{L}_1 \otimes \mathscr{L}_2$ is very ample. $\square$

Now we compute some examples of the intersection pairing.

Example

Let $X = \mathbb{P}1 \times \mathbb{P}1$. Fact: $\text{Pic}(X) = \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2$ where ℓ_i is the fiber over any point of $\mathbb{P}1$ by the projection map π_i. Taking the fiber over distinct points gives disjoint fibers so

$$\ell_i \cdot \ell_i = 0$$

Conversely, for any points $p, q, \pi_1^{-1}(p) \cap \pi_2^{-1}(q)$ is a single reduced point so

$$\ell_1 \cdot \ell_2 = 1$$

by the result form last class.

Example

Let C be a nice curve of genus g. Set $X = C \times C$. Take points $p, q \in C$, set $\ell_1 = \pi_1^{-1}(p), \ell_2 = \pi_2^{-1}(q)$, and Δ is the diagonal.
Then by moving p, q in flat families (which preserves Euler characteristics and hence intersection numbers) we see

$$\ell_1 \cdot \ell_2 = 1, \ell_i \cdot \ell_i = 0$$

as before.

$$\ell_i \cdot \Delta = 1$$

as $\Delta \cap \ell_1 = \{(p, p)\}$ is a single reduced point. Lastly,

$$\Delta \cdot \Delta = \deg \mathcal{O}_X(\Delta)|_\Delta = \deg N_{\Delta/X} = \deg(\mathscr{I}_\Delta/\mathscr{I}_\Delta^2)^* = -\deg \omega_C = 2 - 2g$$

Thm

If $g \geqslant 1, \Delta$ is not linearly equivalent to any divisor of the form $a\ell_1 + b\ell_2$.

Proof. In that case we have that $1 = \Delta \cdot \ell_1 = a, 1 = \Delta \cdot \ell_2 = b$. But then

$$2 - 2g = \Delta \cdot \Delta = (\ell_1 + \ell_2) \cdot (\ell_1 + \ell_2) = 2$$

so $g = 0$. □

Def: The Néron Severi Group of a nice surface is defined to be

$$NS(X) = \text{Pic}(X)/\ker(\cdot)$$

Where $\ker(\cdot) = \{\mathscr{L} | \mathscr{L} \cdot - = 0\}$

Theorem 2.14.2: Theorem of the Base

Let X be a nice surface. Then $NS(X)$ is a finitely generated free abelian group.

This is quite a deep theorem and the proof is very hard in positive characteristic, so we only sketch a proof in the case where X is a surface over $\mathbb{C}$.

Sketch. By GAGA, $\mathrm{Pic}(X)$ is the same as the Cech-cohomology group $H^1(X, \mathcal{O}^*)$ in the analytic topology. Then we get a map

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*) \to^{\delta = c_1} H^2(X, \mathbb{Z}) \to H^2_{sing}(X; \mathbb{Q})$$

δ is given by the long exact sequence in cohomology from the exponential exact sequence, and the second map comes fro the fact that (on a complex manifold say) singular and sheaf cohomology of constant sheaves agree.
Then we get a diagram of maps

$$\begin{tikzcd}
\mathrm{Pic}(X) \times \mathrm{Pic}(X) \arrow[r] \arrow[d, "\cdot"] & H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, "\cup"] & H^2(X, \mathbb{Q}) \times H^2(X, \mathbb{Q}) \arrow[d, "\cup"] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} = H^4(X; \mathbb{Z}) \arrow[r] & \mathbb{Q} = H^4(X, \mathbb{Q})
\end{tikzcd}$$

with the horizontal maps being the cup products from algebraic topology. To finish the proof we would need to show the following

1. The diagram commutes
2. $H^2(X, \mathbb{Z})$ is a finitely generated abelian group, as X is projective hence compact. (To prove this you can use Morse theory to get a finite cell decomposition of X)
3. From 1, $\ker(\cdot) \subseteq \ker(c_1)$ so $NS(X)$ injects into $H^2(X, \mathbb{Z})/\text{torsion}$. Therefore $NS(X)$ is f.g. free.

$\square$

Claim:

$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$, where we define $\mathrm{Pic}^\tau(X)$ to be the preimage under the natural projection map of the torsion of the group scheme $\pi_0(\mathrm{Pic}(X)) = \mathrm{Pic}(X)/\mathrm{Pic}^e(X)$ of connected components of $\mathrm{Pic}(X)$.

The upshot of this whole discussion is that $(NS(X), \cdot)$ is a f.g. free abelian group with a non-degenerate pairing, that is we have a lattice.

Question

How does $NS(X)$ reflect the geometry of X?

Considering $NS(X) \otimes \mathbb{R}$ we get the (possibly open) cone generated by ample divisors, and larger the cone generated by effective divisors. These cones can be nasty, e.g. they need not be polyhedral or could even be spherical. However, in some circumstances they are well behaved.

Def: $\omega_X = \bigwedge^2 \Omega^1_X$. We let K_X be some divisor representing ω_X.

We now prove two important numerical results about the intersection pairing.

Thm

Let $C \subseteq X$ be a nice curve in a nice surface. Let $g = g(C)$, then $2g - 2 = C \cdot (C + K_X)$

Proof. Consider the conormal exact sequence

$$0 \to \mathscr{N}^*_{C/X} \to \Omega^1_X|_C \to \omega_C \to 0$$

By taking the second wedge power we get that

$$\omega_X|_C = \omega_C \otimes \mathscr{N}^*_{C/X} = \omega_C \otimes \mathcal{O}_X(-C)|_C$$

Then taking degrees we get that

$$C \cdot (C + K_X) = \deg(\mathcal{O}_X(C) \otimes \omega_X)|_C = \deg(\mathcal{O}_X(C)|_C \otimes \omega_X|_C) = \deg \omega_C = 2g - 2$$

as desired. □

Example

Let C be a nice plane curve of degree d. Then $2g-2 = C \cdot (C + K_{\mathbb{P}^2}) = \mathcal{O}(d) \cdot \mathcal{O}(d-3) = d(d-3)$. Therefore $g = \frac{d(d-3)}{2} + 1$ giving us another proof of the computation of the genus of a plane curve.

Example

Let $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be nice. Suppose that $C \sim a\ell_1 + b\ell_2$. From one of the SES in Hartshorne,

$$\Omega^1_X \cong \pi_1^* \omega_{\mathbb{P}^1} \oplus \pi_2^* \omega_{\mathbb{P}^1}$$

so

$$\omega_X = \bigwedge^2 \Omega^1_X = \pi_1^* \omega_{\mathbb{P}^1} \otimes \pi_2^* \omega_{\mathbb{P}^1} = -2\ell_1 - 2\ell_2.$$

Then by the adjunction theorem and our earlier computations

$$2g - 2 = a(b-2) + b(a-2) = 2ab - 2a - 2b$$

so $g = ab - a - b + 1$.

Theorem 2.14.1: Riemann Roch

For a nice surface X and any line bundle $\mathscr{L}$ on X,

$$\chi(\mathscr{L}) = \chi(\mathcal{O}_X) + \frac{1}{2} \mathscr{L} \cdot (\mathscr{L} - K_X)$$

Proof. If $\mathscr{L} = \mathcal{O}_X$ the result is clear as $\mathcal{O} \cdot \mathscr{M} = 0$ for any $\mathscr{M} \in \operatorname{Pic}(X)$.
Wlog $k = \bar{k}$ as all quantities are unaffected are unaffected by base change.

Suppose, as before, that $\mathscr{L} = \mathcal{O}(C - E)$ for C, E nice curves. BY twisting appropariate ideal sheaf exact sequences we get the two SES

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_C \to 0$$

$$0 \to \mathscr{L} \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)|_E \to 0$$

Then by the first SES

$$\begin{aligned}
\chi(\mathcal{O}_X(C)) &= \chi(\mathcal{O}) + \chi(\mathcal{O}_X(C)|_C) \\
&= \chi(\mathcal{O}) + \deg \mathcal{O}_X(C)|_C + 1 - g_C && \text{By R.R. for curves} \\
&= \chi(\mathcal{O}) + C \cdot C - \frac{1}{2} C \cdot (C + K_X) && \text{Adjunction theorem} \\
&= \chi(\mathcal{O}_X) + \frac{1}{2} C \cdot (C - K_X)
\end{aligned}$$

Doing the same for the second exact sequence gives the desired conclusion. □

Theorem 2.14.3

Let $D \in NS(X), H \in NS(X)$ is ample (i.e. some representative is ample). If $D \cdot D > 0$ and $D \cdot H > 0$, then nD is effective for $n \gg 0$.

Lemma

There exists n_0 such that if $D \cdot H > n_0$, then $H^2(X, \mathcal{O}(D)) = 0$.

Proof. By Serre duality, $H^2(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(K_X - D))$ and suppose that it is > 0. Then $K_X - D$ is that effective.

Claim: If C is effective, then $C \cdot H > 0$.

Pf: mH is very ample for some $m \gg 0$, so it is represented by a nice curve D with no components in common with C. Note that the choice of representative of an element NS doesn't matter when it comes to the intersection pairing sine NS is obtained by quotienting out degeneracy of the pairing. We now have that $m(C \cdot H) = C \cdot (mH) = C \cdot D$. But now mH is very ample, so it embeds $X \to \mathbb{P}^N$, then $D = X \cap V(\ell)$ for some linear function $\ell \in \mathcal{O}_{\mathbb{P}^N}(1)$. It follows that $C \cap D = C \cap V(\ell)$ is non-empty by Bézout's theorem, so $C \cdot D > 0$, thus $m(C \cdot H) > 0$, so $C \cdot H > 0$.

For the lemma, we now have that $K_X - D$ is effective, so $(K_X - D) \cdot H > 0$, i.e. $K_X \cdot H > D \cdot H$. It follows that if $D \cdot H > n_0 = K_X \cdot H$, then $H^2(X, \mathcal{O}(D)) = 0$. □

Proof of 2.14.3. For $n \gg 0$, $nD \cdot H > n_0$, so $H^2(\mathcal{O}(nD)) = 0$. We want to show that $h^0(\mathcal{O}(nD)) > 0$, and we know that for $n \gg 0$, $\chi(\mathcal{O}(nD)) = h^0 - h^1$ so we are done if we can show that $\chi(\mathcal{O}(nD)) > 0$. We have that,

$$\chi(\mathcal{O}(nD)) = \chi(\mathcal{O}_X) + \frac{1}{2} n^2 (D \cdot D) - \frac{1}{2} n (D \cdot K)$$

Since $D \cdot D > 0$, then this goes to infinite as $n \to \infty$. It follows that for all $n \gg 0$, nD is effective. □

The condition that $D^2 > 0$ is very important. We want to understand when $D^2 = 0$.

Theorem 2.14.4: Hodge Index Theorem

Suppose that $D \in NS(X) - \{0\}$ is such that $D \cdot H = 0$ for some ample $H \in NS(X)$, and a nice surface X. Then $D \cdot D < 0$.

Corollary

$(NS(X) \otimes \mathbb{R}, \cdot)$ has signature $(1, -1, \ldots, -1)$.

Example

This explains the claim that $\{D \cdot D = 0\}$ looks like a hyperbola. For in the $n = 2$ case then we have a basis such that $\cdot$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $D \cdot D = 0$ is equivalent in this basis to $x^2 - y^2 = 0$ which is a hyperbola.

We now prove the theorem.

Proof. Fix D so that $D \cdot H = 0$.

Case 1: $D \cdot D > 0$) As H is ample, $H' = D + nH$ is ample for $n \gg 0$. Then as $D \cdot H = 0$,

$$D \cdot H' = D \cdot D > 0$$

so mD is effective for $m \gg 0$ by theorem 2.14.3. But then by the previous corrolary we have $0 = m(D \cdot H) = (mD) \cdot H > 0$, which is a contradiction.

Case 2: $D \cdot D = 0$) Our goal is to construct a D' such that $D' \cdot H = 0$ and $D' \cdot D' > 0$ so that we win by the previous case.
As $\cdot$ is nondegenerate on $NS(X)$, there is some divisor E such that $E \cdot D \neq 0$. Set $E' = (H \cdot H)E - (E \cdot H)H$ (which is a kind of orthogonal projection). We then have

$$E' \cdot H = 0, E' \cdot D \neq 0.$$

Then set $D' = nD + E'$ for an as-of-yet undetermined n. Again it is clear that $D' \cap H = 0$ and

$$D' \cdot D' = n^2 D \cdot D + 2nD \cdot E' + E' \cdot E'$$

By assumption, $D \cdot D = 0, D \cdot E' \neq 0$ so we can pick n so that $D' \cdot D' > 0$ so we win.

□

2.4 Lecture 15 - Nakai-Moishezon & Tsen

Theorem 2.15.1: Nakai-Moishezon

Let X be a nice surface. Then $\mathscr{L} \in \operatorname{Pic}(X)$ is ample iff $\mathscr{L}^2 > 0$ and $\mathscr{L} \cdot C > 0$ for all irreducible $C \subseteq X$.

Remark

1) This does not hold for X which are not nice (note: nice $\Rightarrow$ projective).
2) If $\mathscr{L}^n$ is effective for some n, then it suffices to show that $\mathscr{L} \cdot C > 0$ for all C irreducible since $\mathscr{L}^n = \sum C_i$ implies that $n(\mathscr{L}^2) = \sum \mathscr{L} \cdot C_i > 0$, so $\mathscr{L}^2 > 0$ as well.
3) The ample cone A is the intersection of the effective cone E with the cone $\bigcup_C \{D | D \cdot C > 0\}$.

Proof. Step 1) Note that if $\mathscr{L}$ is ample, then $\mathscr{L}^n$ is very ample, so it is effective, then $\mathscr{L}^n \cdot \mathscr{L} > 0$, so $n(\mathscr{L} \cdot \mathscr{L}) > 0$ whence $\mathscr{L}^2 > 0$ and $\mathscr{L} \cdot C > 0$ for all irreducible curves C as in the proof of 2.14.3.

We now need to show the opposite direction. Suppose the statement holds for effective line bundles, then let H be an effective ample line bundle (e.g. $\mathcal{O}_X(1)$). Now $\mathscr{L}^2 > 0$ and $\mathscr{L} \cdot H > 0$, so there exists some n such that $\mathscr{L}^n$ is effective. Furthermore, $(\mathscr{L}^n)^2 = n^2 \mathscr{L}^2 > 0$ and for any irreducible curve C we have that $\mathscr{L}^n \cdot C = n(\mathscr{L} \cdot C) > 0$. Therefore $\mathscr{L}^n$ is effective and satisfies the requirements for the theorem, so $\mathscr{L}^n$ is ample and therefore $\mathscr{L}$ is ample.

Step 2) Let C be any curve in X, then we want to show that $\mathscr{L}|_C$ is ample. It is enough to show that $\mathscr{L}$ is irreducible on each component of the normalization of the reduction of C, i.e. $\overline{C}_{\text{red}}$ by [HA Ex. III.5.7]. Let $\overline{C}_{\text{red}} = \bigcup D_i$ with D_i irreducible components, then it suffices to show that $\deg \mathscr{L}|_{D_i} > 0$ since positive degree line bundles on a curve are effective. $D_i \to X$ is birational onto its image Z, so $\deg \mathscr{L}|_{D_i} = \deg \mathscr{L}|_Z = \mathscr{L} \cdot Z > 0$, therefore $\mathscr{L}|_C$ is ample.

Step 3) We want to show that $\mathscr{L}^n$ is globally generated for $n \gg 0$. On $D(s) = X \backslash V(s)$, the collection of points where s doesn't vanish with $s \in \Gamma(X, \mathscr{L})$, we have that s generates $\mathscr{L}$ on $D(s)$. Note also that s^n doesn't vanish on $D(s)$. Let $C = V(s)$, then we want to show that $\mathscr{L}$ is also globally generated on C. $\check{\mathscr{L}}$ maps into $\mathcal{O}$ by multiplication by s, with image being the ideal sheaf of C. We therefore have a short exact sequence,

$$0 \to \check{\mathscr{L}} \to \mathcal{O} \to \mathcal{O}_C \to 0$$

We can now take the tensor product with $\mathscr{L}^n$ to get a new short exact sequence,

$$0 \to \mathscr{L}^{n-1} \to \mathscr{L}^n \to \mathscr{L}^n|_C \to 0$$

Notice that on the right we have $\mathscr{L}^n|_C$ which is ample since $\mathscr{L}|_C$ is ample. We can now take a LES,

$$0 \to H^0(\mathscr{L}^{n-1}) \to H^0(\mathscr{L}^n) \to H^0(\mathscr{L}^n|_C) \to H^1(\mathscr{L}^{n-1}) \to H^1(\mathscr{L}^n) \to H^1(\mathscr{L}^n|_C)$$

Since $\mathscr{L}|_C$ is ample, then by the cohomological criterion for ampleness we have that $H^1(\mathscr{L}^n|_C) = 0$ for $n \gg 0$ [HA III.5.3]. It follows that the map $H^1(\mathscr{L}^{n-1}) \to H^1(\mathscr{L}^n)$ is surjective for all $n \gg 0$. We then get that $H^1(\mathscr{L}^n) \twoheadrightarrow H^1(\mathscr{L}^{n+1}) \twoheadrightarrow H^1(\mathscr{L}^{n+2}) \twoheadrightarrow \cdots$. Since these are all finite dimensional, then their dimension cannot go up after each surjection, thus the dimension must eventually stabilize, at which point these become isomorphisms. Thus for $n \gg 0$, the map $H^1(\mathscr{L}^{n-1}) \to H^1(\mathscr{L}^n)$ is an isomorphism, so the kernel is 0, i.e. the map $H^0(\mathscr{L}^n|_C) \to H^1(\mathscr{L}^{n-1})$ is 0, thus we have a short exact sequence,

$$0 \to H^0(\mathscr{L}^{n-1}) \to H^0(\mathscr{L}^n) \to H^0(\mathscr{L}^n|_C) \to 0$$

We know that $\mathscr{L}^n|_C$ is ample, so for $n \gg 0$, $\mathscr{L}^n|_C$ is globally generated, then for any $P \in C$ we can find some $t \in H^0(\mathscr{L}^n|_C)$ such that t does not vanish at P. Since $H^0(\mathscr{L}^n)$ surjects onto $H^0(\mathscr{L}^n|_C)$, then we can find a lift $\tilde{t}$ of t to $H^0(\mathscr{L}^n)$ but the restriction to the stalk at P will still generate $\mathscr{L}^n$, therefore $\mathscr{L}^n$ is globally generated.

Step 4) Since $\mathscr{L}^n$ is globally generated, then it defines a map $\varphi : X \xrightarrow{f|_{|\mathscr{L}^n|}} \mathbb{P}\Gamma(X, \mathscr{L}^n)$. We want to show that φ is finite. If φ is finite, then φ is a finite surjective morphism from X onto its image. The image of X lives in $\mathbb{P}\Gamma(X, \mathscr{L}^n)$ whence the pullback $\varphi^*\mathcal{O}_{\mathbb{P}\Gamma(X,\mathscr{L}^n)}(1)$ is the same as the pullback of the restriction to the image which is just $\varphi^*\mathcal{O}_{im(\varphi)}(1)$. Since $\mathcal{O}_{\mathrm{im}(\varphi)}(1)$ is ample, in fact very ample, then the pullback which is just $\mathscr{L}^n$ is also ample. To see that φ is finite, since φ is projective then to show it is finite, we need only show that it is quasi-finite (i.e. finite fibers) [HA Ex III.11.2]. Suppose that φ were not quasi-finite, then we would have some point p such that $\varphi^{-1}(p)$ contains infinitely many points and since it is closed, then it is a finite union of irreducible components, but irreducible zero dimensional components are points, and $\varphi^{-1}(p)$ is infinite, thus it must contain a higher dimensional component. It follows that $\varphi^{-1}(p)$ contains some curve C.

Let $H \subseteq \mathbb{P}\Gamma(X, \mathscr{L}^n)$ be a hyperplane missing p. Then $\mathscr{L}^n = \varphi^*\mathcal{O}(1) = \mathcal{O}(\varphi^{-1}H)$, then $\mathscr{L}^n \cdot C = \mathcal{O}(\varphi^{-1}H) \cdot C = \deg \varphi^{-1}H|_C$. Since $\varphi^{-1}H \cap C = \varnothing$, then $\varphi^{-1}H|_C = \mathcal{O}_C$ and therefore $\deg \varphi^{-1}H|_C = 0$, but $n \cdot (\mathscr{L} \cdot C) > 0$ which is a contradiction. Therefore φ is finite, so $\mathscr{L}$ is ample. $\square$

Def: A nice surface X is *geometrically ruled* if $\exists X \to C$ with C nice whose geometric fibers have genus 0.

Example

If C is a nice curve and E is a rank 2 vector bundle on $C, \mathbb{P}_C(E)$ is a ruled surface.

And in fact over an algebraically claosed field these are the only examples.

Theorem 2.15.2 : Tsen

Suppose $k = \bar{k}$, and $\pi : X \to C$ is a ruled surface. Then

1. π has a section.
2. There is a rank 2 vector bundle $\mathscr{E}$ such that X and $\mathbb{P}_C(E)$ are isomorphic over C.

Theorem 2.15.3 : Tsen 2

Let $k = \bar{k}, C/k$ a nice curve. Let $f(x_1, \ldots, x_n) \in k(C)[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $0 < d < n$. Then f has a nontrivial solution in $k(C)$.

Remark

The conclusion of Tsen 2, is called being a C_1, or quadi-algebraically closed, field. (There are a family of properties C_r where you replace d with d^r).

Remark

Over a C_1 field all genus 0 curves are isomorphic to $\mathbb{P}^1$.

Proof. C has genus 0 so C is a conic in $\mathbb{P}^2$. Thus $C = V(p)$ for p degree 2 in $k[x_1, x_2, x_3]$ so by the C_1 property, p has a soltuion in k. Thus C has a rational point and so is $\mathbb{P}^1$ □

Remark

If k is C_1, than any X/k such that $\mathbb{P}^n_{\bar{k}}$ is $\mathbb{P}^n_k$. (i.e. $Br(k) = 0$).

We now prove Tsen's theorem assuming Tsen 2.

Proof. 1.

$$\begin{tikzcd} X \arrow[d] & X_{k(C)} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ C & \text{Spec } k(C) \arrow[l] \end{tikzcd}$$

By Tsen 2, $X \times \text{Spec } k(C)$ has a rational point, giving us a section Spec $k(C) \to X \times \text{Spec } k(C)$. By definition of the function field, this lifts to a section over an open subset of C. By the valuative criterion of properness, this lifts to a section over all of C.

2. Let $D \subset X$ be the image of a section and set $\mathscr{E} = \pi_* \mathcal{O}(D)$. On fibers of $\pi, \mathscr{E}$ is $\mathcal{O}(1)$. Therefore we get a map $f : X \to \mathbb{P}\mathcal{O}(D)$ over C induced by $\pi^* \pi_* \mathcal{O}(D) \xrightarrow{ev} \mathcal{O}(D)$ which is an isomorphism as it is an isomorphism on fibers. $\mathscr{E}$ is rank 2 by cohomology and base change.

□

2.5 Lecture 16 - Picard Groups of Ruled Surfaces and Hirzebruch surfaces

Proof of Tsen 2. Lec k, C, f, d be as in the statement of Tsen 2.
Write $f(\underline{x}) = \sum_{|I|=d} a_I x^I, a_I \in l(C)$ and div $a_I = D_I$. Then there is an effective divisor D such that $D_I \leqslant D$ for any I, say $D = \sum (a_I)_0$. Then we can view a_I as being elements of $H^0(C, \mathcal{O}_C(D))$.

Then f yields a map $H^0(C, mD)^n \to H^0(C, (md+1)D)$, for any $m \in \mathbb{Z}$ (which we will determine later. f is an algebraic map from an affine space of dimension

$$n \cdot h^0(C, mD) = n(m \deg D + 1 - g)$$

to one of dimension

$$h^0(C, (md+1)D) = (md+1) \deg D + 1 - g$$

using Riemann-Roch twice and choosing $\deg D > 2g + 2$. As $d < n$ the second grows more slowly than the first so for $m \gg 0$, the domain has larger dimension than the codomain so any non empty fiber of f has positive dimensions. As $f(0) = 0, \dim f^{-1}(0) > 0$ o f has a nonzero solution. □

Remark

The same proof shows that if X/k has $\dim r, k(X)$ is C_r, once you know that $\dim H^0(X, mD) \approx m^r h^0(X, D)$ for $D \gg 0$ ample.

Thm

Let $\pi : X \to C$ be a ruled surface over $k = \bar{k}$, with section h. Then there is a natural isomorphism

$$\mathbb{Z}h \oplus \mathrm{Pic}(C) \to \mathrm{Pic}(X)$$

Proof. Let $f \in \mathrm{Pic}(X)$ be $\mathcal{O}$(fiber of π). Fix $\mathscr{L} \in \mathrm{Pic}(X)$ and set $n = \mathscr{L} \cdot f, \mathscr{L}' = \mathscr{L} - nh$. $\mathscr{L}' \cdot f = 0$ by construction as $h \cdot f$ as h is a section to π (so intersects fibers once).
Observe that $\mathscr{L}' \cdot f = 0$ is the same thing as saying that $\deg \mathscr{L}'|_{\pi^{-1}(*)} = 0$ so

$$\pi_* \mathscr{L}'$$

is a line bundle and also $\pi^* \pi_* \mathscr{L}' \to \mathscr{L}'$ is an isomorphism. Over $c \in C$ the map is the evaluation map $H^0(\mathbb{P}^1, \mathcal{O}) \otimes \mathcal{O}$, which is an isomorphism as $h^0(\mathbb{P}^1, \mathcal{O}) = 1$. As the map is an isomorphisms on fibers, by cohomology and base change is it an isomorphism (as by cohomology and base change the R^i vanish). Thus $\mathscr{L} = nh + \pi^*\pi_*\mathscr{L}$. Thus $\mathscr{L} \mapsto ((\mathscr{L} \cdot f)h, \pi_*(\mathscr{L} - nh))$ is the inverse so we have an isomorphism $\mathbb{Z}h \oplus \mathrm{Pic}(C) \to \mathrm{Pic}(X)$ as desired. □

Corollary

$\mathrm{Pic}(\mathbb{P}^1 \times C) = \mathrm{Pic}\,\mathbb{P}^1 \oplus \mathrm{Pic}\, C$

Remark

It is not true that $\mathrm{Pic}(X \times Y) = \mathrm{Pic}\, X \times \mathrm{Pic}\, Y$ in general. For example, as we have seen if C is a nice gurve of positive genus,

$$\Delta \in \mathrm{Pic}(C \times C) - \mathrm{Pic}\, C \times \mathrm{Pic}\, C$$

so $\mathrm{rk} NS(C \times C) \geqslant 3$.

Thm

$\mathrm{Pic}(C_1 \times C_2) = \mathrm{Pic}(C_1) \oplus \mathrm{Pic}(C_2) \oplus \mathrm{Hom}_{Gp-sch}(\mathrm{Jac}C_1, \mathrm{Jac}C_2)$

In order to prove this we would need spectral sequences, so we will not do the proof.

Def: (For $k = \bar{k}$) A ruled surface $X \to \mathbb{P}^1$ is called a Hirzebruch surface.

Observe that $\mathbb{P}_c(\mathscr{E}) \cong \mathbb{P}_C(\mathscr{E} \otimes \mathscr{L})$ for any $\mathscr{L} \in \mathrm{Pic}(X)$. (This is an iff for isomorphisms over C, but not necessarily as abstract schemes).

Theorem 2.16.1 : Birkhoff-Grothendieck

Any vector bundle $\mathscr{E}$ on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles in a unique way (up to reordering).

Proof sketch - Birkhoff. : $\mathscr{E}|_{\mathbb{P}^1-\{0\}} \cong \mathcal{O}^n$ and $\mathscr{E}|_{\mathbb{P}^1-\{\infty\}} \cong \mathcal{O}^n$ as $\mathbb{P}^1$ is locally a PID.
Therefore $\mathscr{E}$ is determined by an automorphism φ of $\mathcal{O}^n_{\mathbb{P}^1-\{0,\infty\}} = k[t,t^{-1}]$. Changing basis on either $\mathcal{O}^n$ corresponds to left or right multiplication by an element of $GL_n(k[t^{-1}])$ or $GL_n(k[t])$. Thus

$$\mathrm{Vec}_{n,\mathbb{P}^1} \cong GL_n(k[t^{-1}])\backslash GL_n(k[t,t^{-1}])/GL_n(k[t])$$

Birkhoff shows that any $A \in GL_n(k[t,t^{-1}])$ is equivalent to $\mathrm{diag}(t^{a_1},\dots,t^{a_n})$ under this equivalence for some unique non-increasing a_i. □

Proof for $\mathscr{E}$ rank 2. Let $\mathscr{E}$ be rank 2 vector bundle on $\mathbb{P}^1$.$S = \{\mathscr{L}|\mathrm{Hom}(\mathscr{L},\mathscr{E}) \neq 0\}$ is nonempty as $\mathscr{E}(n)$ is globally generated for $n \gg 0$ so there are nontrivial maps $\mathcal{O} \to \mathscr{E}(n)$ hence $\mathcal{O}(-n) \to \mathscr{E}$.

We then see that $\{\deg\mathscr{L}|\mathscr{L} \in S\}$ is bounded above. For $\mathrm{Hom}(\mathscr{L},\mathscr{E}) = H^0(\mathbb{P}^1,\mathscr{E}\otimes\mathscr{L}^*) = H^1(\mathbb{P}^1,\mathscr{E}^*\otimes\mathscr{L})^* = 0$ for $\deg\mathscr{L} \gg 0$ by Serre Vanishing. Thus we can take a $\mathscr{L} \in S$ of maximal degree such that there is a nonzero SES

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{E}/\mathscr{L} \to 0$$

for a nonzero map $\mathscr{L} \to \mathscr{E}$ is injective as $\mathscr{L}/\ker(\mathscr{L} \to \mathscr{E}) \cong \mathrm{im}\mathscr{L} \subseteq \mathscr{E}$ is torsion free. Therefore, $\ker(\mathscr{L} \to \mathscr{E}) = 0$ or $\mathscr{L}$ but $\mathscr{L} \to \mathscr{E} \neq 0$ so $\mathscr{L} \to \mathscr{E}$ is injective.

Claim: $\mathscr{E}/\mathscr{L}$ is a line bundle.
Suppose not. Then $(\mathscr{E}/\mathscr{L})_{tors} \neq 0$. The preimage of $(\mathscr{E}/\mathscr{L})_{tors}$ in $\mathscr{E}$ is then a line bundle strictly containing $\mathscr{L}$. Hence it has higher degree, which contradicts maximality of $\deg\mathscr{L}$.

Now recall the definition of the degree of a general vector bundle.

Def: $\deg\mathscr{E} = \deg\det\mathscr{E} = \deg\bigwedge^{top}\mathscr{E}$.

By the above SES, $\deg\mathscr{E} = \deg(\mathscr{L}\otimes\mathscr{E}/\mathscr{L}) = \deg\mathscr{L} + \deg\mathscr{E}/\mathscr{L}$. We now try to bound the degree of $\deg\mathscr{L}$ in terms of $\deg\mathscr{E}$.
We have

$$\chi(\mathscr{E}(n)) = \deg\mathscr{E}(n) + \mathrm{rank}\,\mathscr{E} = 2n + \deg\mathscr{E} + 2$$

so $\chi(\mathscr{E}(n)) > 0$ if $n > -(\frac{1}{2}\deg\mathscr{E} + 1)$, so

$$\deg\mathscr{L} \geqslant \frac{\deg\mathscr{E}}{2}$$

by maximality of $\deg\mathscr{L}$. Hence $\deg\mathscr{L} \geqslant \deg\mathscr{E}/\mathscr{L}$.
The SES $0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{E}/\mathscr{L} \to 0$ splits if $\mathrm{Ext}^1(\mathscr{E}/\mathscr{L},\mathscr{L}) = 0$ but

$$\mathrm{Ext}^1(\mathscr{E}/\mathscr{L},\mathscr{L}) = H^1(\mathbb{P}^1,\mathscr{L}\otimes(\mathscr{E}/\mathscr{L})^*) = 0$$

as $\deg\mathscr{L} \geqslant \deg\mathscr{E}/\mathscr{L}$ by the cohomology of $\mathbb{P}^1$. □

Corollary

For any Hirzebruch surface X, we have that $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$ and $\mathscr{E} \cong \mathcal{O}(a)\oplus\mathcal{O}(b)$, then we can take a tensor product with $\mathcal{O}(-b)$ and this will not change X, so $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(a)\oplus\mathcal{O})$. If $a > 0$, then we can take a tensor product with $O(-a)$ and swap the order of the terms, so $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(a))$ with $a \leqslant 0$.

Def: We define $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-n))$.

Theorem 2.16.2

$\mathbb{F}_n \cong \mathbb{F}_m$ iff $n = m$ $(n, m \geqslant 0)$.

Proof. The universal property of $\mathbb{P}(\mathscr{E}) = \operatorname{Proj}_C \operatorname{Sym} \mathscr{E}$ is that a map $X \to \mathbb{P}(\mathscr{E})$ where $f : X \to C$ corresponds to a surjective map $f^*\mathscr{E} \twoheadrightarrow \mathscr{L}$. It follows that sections of the map $\pi : \mathbb{P}(\mathscr{E}) \to C$ correspond exactly to surjective maps $\mathscr{E} \to \mathscr{L}$. Let s be the section corresponding to $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathcal{O}(-n)$ given by $(x, y) \mapsto y$. It follows that,

$$\operatorname{Pic}(\mathbb{F}_n) = \mathbb{Z}s \oplus \mathbb{Z}f$$

where f is a fiber of the map π. Then $f \cdot f = 0$ since we can move along fibers in flat family until they no longer intersect. $f \cdot s = 1$ since s is a section and f is a fiber. We want to compute $s \cdot s$.

Since $s : \mathbb{P}^1 \to \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-n))$ is given by the quotient map $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathcal{O}(-n)$, then abusing notation and writing s for the image of s in $\mathbb{F}_n$, we have that $\mathcal{O}_{\mathbb{F}_n}(s) = \mathcal{O}(1)$. Then $\pi_*\mathcal{O}(s) = \mathcal{O} \oplus \mathcal{O}(-n)$. Therefore $s \cdot s = \chi(\mathcal{O}) - 2\chi(\mathcal{O}(s)\check{)} + \chi(\mathcal{O}(2s)\check{)}$.

We want to compute $H^i(\mathbb{F}_n, \mathcal{O})$ which we can do with some spectral sequence magic. This will be given by $E_2^j = H^i(\mathbb{P}^1, R^j\pi_*\mathcal{O})$. We know that $R^1\pi_*\mathcal{O} = 0$ since the fibers are the cohomology of $\mathcal{O}$ which is 0. It follows (by black magic), that $H^i(\mathbb{F}_n, \mathcal{O}) = \begin{cases} k & i = 0 \\ 0 & i > 0 \end{cases}$ It is now relegated to some homework to compute $R^1\pi_*\mathcal{O}(-2s)$. □

Corollary

For $n > 0$, there exists a unique irreducible curve on $\mathbb{F}_n$ with negative self-intersection

Proof. Suppose that C is such a curve, i.e. $C \cdot C < 0$ and that C is not linearly equivalent to s, then $C \cdot s > 0$ since C, s are irreducible. Similarly, $C \cdot f > 0$. Let $C = as + bf$, then

$$\begin{aligned} C \cdot s &= (as + bf) \cdot s = -na + b \\ C \cdot f &= (as + bf) \cdot f = a \\ C \cdot C &= (as + bf)^2 = -na^2 + 2ab = a(2b - na) \end{aligned}$$

We know that $C \cdot s > 0$, so $b > na$ and $C \cdot f > 0$, so $a > 0$, thus $b > 0$. Therefore $C \cdot C = a(2b - na) > ab > 0$ which is a contradiction. □

2.6 Lecture 17 - Hirzebruch surface

We have $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-n))$ and $s \subseteq \mathbb{F}_n$ is the scheme theoretic image of the map corresponding to the projection $\mathcal{O} \oplus \mathcal{O}(-n) \to \mathcal{O}(-n)$ (the construction of the map is given in [HA II.7.12].

Claim : $s \cdot s = -n$. It suffices to show that $\mathscr{N}_{s/\mathbb{F}_n} = \mathcal{O}(-n)$ since $s \cdot s = \deg_s \mathscr{N}_{s/\mathbb{F}_n}$. In fact, we want to show that $\mathscr{N}_{s/\mathbb{F}_n} = \mathscr{T}_{\mathbb{F}_n/\mathbb{P}^1}|_s$. Working relative to $\mathbb{P}^1$, we have a SES:

$$0 \to \mathscr{T}_{s/\mathbb{P}^1} \to \mathscr{T}_{\mathbb{F}_n/\mathbb{P}^1}|_s \to \mathscr{N}_{s/\mathbb{F}_n} \to 0$$

This is a relative version of [HA II.8.17]. Since $s \to \mathbb{P}^1$ is an isomorphism, then $\mathcal{T}_{s/\mathbb{P}^1} = 0$, so $\mathcal{T}_{\mathbb{F}_n/\mathbb{P}^1} \cong \mathcal{N}_{s/\mathbb{F}_n}$.

Lemma

Let V be a vector space/bundle, then

$$\begin{aligned} T_{[W]}\mathbb{P}(V) &= \mathrm{Hom}(W, V/W) \\ T_{[W]}Gr(k, V) &= \mathrm{Hom}(W, V/W) \end{aligned}$$

Proof. Recall that $Gr(k, V) = GL(V)/\operatorname{Stab}_W$ given by sending an element g in this quotient to gW. We have that an element of Stab_W looks like a matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where this using a basis which extends a basis of W. We must have that $A \in GL(W)$, then the remainder of a the basis is equivalent to a basis of V/W, so $C \in GL(V/W)$ since our element must be in $GL(V)$. The bottom right must be 0 since this matrix needs to fix W. Finally, the submatrix B maps from V/W to W, so it is an element of $\mathrm{Hom}(V/W, W)$. What is left over after the quotient is the bottom left, which is $\mathrm{Hom}(W, V/W)$. We can compute the tangent space at the identity of $GL(V)/\operatorname{Stab}_W$ since $e \mapsto eW = W$, thus the tangent space is $\mathrm{Hom}(W, V/W)$. □

$\mathbb{F}_n$ has a map π to $\mathbb{P}^1$ by construction as a projective bundle. We have a surjective map:

$$0 \to K \to \pi^*(\mathcal{O} \oplus \mathcal{O}(-n)) \to \mathcal{O}_{\mathbb{F}_n}(1) \to 0$$

This surjective map is "described" in [HA II.7.11]. We know that $\mathcal{T}_{\mathbb{F}_n/\mathbb{P}^1} = \mathscr{H}om(K, \mathcal{O}_{\mathbb{F}_n}(1))$. Then $\mathcal{T}_{\mathbb{F}_n/\mathbb{P}^1}|_s = s^*\mathcal{T}_{\mathbb{F}_n/\mathbb{P}^1}$. Now $s^*\mathscr{H}om(K, \mathcal{O}_{\mathbb{F}_n}(1)) = s^*\check{K} \otimes s^*\mathcal{O}_{\mathbb{F}_n}(1) = \mathscr{H}om(s^*K, s^*\mathcal{O}_{\mathbb{F}_n}(1))$. We can do this because K will be locally free. The locus of points $p \in \mathbb{P}^1$ where K has stalk 0 is open and similarly the points where K is locally free is also open. If $K \neq 0$, then K is free at a point since $\mathbb{P}^1$ is a DVR and K is torsion-free, so then K is locally free. If $K = 0$, then the fact above sheaf hom is also true.

Since $s^*\mathcal{O}_{\mathbb{F}_n}(1) = \mathcal{O}(-n)$ is flat over $\mathbb{P}^1$, then pulling back preserves the exactness of our SES. Pulling back then yields the original SES $0 \to \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(-n) \to \mathcal{O}(-n) \to 0$. It follows that $\mathcal{T}_{\mathbb{F}_n/\mathbb{P}^1}|_s = \mathscr{H}om(\mathcal{O}, \mathcal{O}(-n)) = \mathcal{O}(-n)$ and thus we are done.

Spectral sequences

Given $E_i^{p,q}$ objects in an abelian category. Fixing i, we have a diagram:

$$\begin{tikzcd} E_i^{0,1} \arrow[drr, "d_2"] & E_i^{1,1} & E_i^{2,1} \\ E_i^{0,0} & E_i^{1,0} & E_i^{2,0} \end{tikzcd}$$

In general, $d_i^{p,q} : E_i^{p,q} \to E_i^{p+i,q-i+1}$ and $d_i \circ d_i = 0$. Furthermore, we require isomorphisms $\varphi_i^{p,q} : E_i^{p,q} \cong H^*(\cdots \xrightarrow{d_{i-1}} E_{i-1}^{p,q} \xrightarrow{d_{i-1}} \cdots)$.

Theorem 2.17.1: Leray Spectral Sequence

Let $\pi : X \to Y$ be a finite type morphism and $\mathscr{F}$ a coherent sheaf on X. There exists a spectral sequence with E_2 page given by

$$E_2^{p,q} = H^p(Y, R^q\pi_*\mathscr{F})$$

Then $E_\infty^{p,q} = E_i^{p,q}$ for $i \gg 0$. Furthermore there exists a filtration on $H^i(X, \mathscr{F})$ with the associated grading being $\oplus E_\infty^{p,q}$ where $p + q = i$.

Example

If $\dim X = 2, \dim Y = 1$ then all the differential maps on the E_2 page are 0. Since the relative dimension of π is 1, then $R^2\pi_*$ vanishes. Therefore we get:

$$H^0(X, \mathscr{F}) = H^0(Y, \pi_*\mathscr{F})$$
$$0 \to H^1(Y, \pi_*\mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(Y, R^1\pi_*\mathscr{F}) \to 0$$
$$H^2(X, \mathscr{F}) = H^1(Y, R^1\pi_*\mathscr{F})$$

We can somewhat explicitly describe the differential on the second page. We have that $R\pi_*\mathscr{F}$ is an object in the bounded derived category $D^b(Y)$. $R\pi_*\mathscr{F} = \pi_*\mathscr{I}$ where $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ is a resolution. We get,

$$0 \to \pi_*\mathscr{F} \to \pi_*\mathscr{I}^0 \to \ker d \to R^1\pi_*\mathscr{F} \to 0$$
$$0 \to R^1\pi_*\mathscr{F} \to \mathscr{I}^1/(\mathrm{im} d) \to \ker d \to R^2\pi_*\mathscr{F} \to 0$$

Then we get $\alpha \in \mathrm{Ext}^2(R^i\pi_*\mathscr{F}, R^{i-1}\pi_*\mathscr{F})$ and d_2 is given by cupping with α.

2.7 Lecture 18 - Quadrics & the Riemann Hypothesis for curves

Thm

$X \subseteq \mathbb{P}^3$ smooth quadric over $k = \bar{k}$. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. GL_3 acts on $H^0(\mathbb{P}^3, \mathcal{O}(2))$. It is a "classical fact" that GL_3 acts on smooth quadrics, and that all are isomorphic to $V(xy - wz)$, which we recognize as the segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. $\square$

Coordinate free geometry of quadrics:

Let $\operatorname{char} k \neq 2$. Then there is a correspondence between Quadratic forms and symmetric bilinear forms over k given by

$$Q(-) \mapsto B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$$

$$B(-,-) \mapsto Q(x) = B(x,x)$$

Fix a quadratic form $Q : V \to k$ with $\dim V = 4$ (if the associated billinear form B is nondegenerate, which will imply that $V(Q)$ is smooth) then

$$X = \{Q = 0\} \subseteq \mathbb{P}V \cong \mathbb{P}^3.$$

Fixing $\bar{v} \in X, Q : V \to k$ so we get $W = \{v\}^\perp \subseteq V$ with respect to the associated billinear form B. $\dim W = 3$ as B is a nondegenerate form, as X is smooth. $V(v,v) = Q(v) = 0$ since $\bar{v} \in X$ so $v \in W$, so we can choose a complement $W' \subseteq W$ to $\mathrm{span}\{v\}$.
Then $\mathbb{P}W' \subseteq \mathbb{P}V$ is a line which intersects X in two points (generically by Bezout). Then $\exists w_1, w_2 \in W'$ distinct points such that $Q(w_1) = Q(w_2) = 0$.

Claim

Then span of the lines $\mathbb{P}(\mathrm{Span}(v, w_1))$ and $\mathbb{P}(\mathrm{Span}(v, w_2))$ is contained in X.

Proof. $Q(av + bw_i) = a^2Q(v) + 2abB(v, w_i) + b^2Q(w_i) = 0$, as $v, w_i \in X$ and $w_i \in \{v\}^\perp$. □

Therefore we get 2 lines on X.

If $\mathrm{char}\, k \neq 2$ all quadrics in $\mathbb{P}^3$ are of one of the forms:

$x_1^2 = 0$	A non-reduced hyperplane
$x_1^2 + x_2^2 = 0$	Two hyperplanes meeting in a line
$x_1^2 + x_2^2 + x_3^2 = 0$	A cone over a conic in $\mathbb{P}^2$
$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$	$\mathbb{P}^1 \times \mathbb{P}^1$ are the smooth quadrics

Lemma

Let X, Y be nice curves. Then let H be the fiber over a points of X and V of a point over Y, so $NS(X \times Y) \supseteq \mathbb{Z}H \oplus \mathbb{Z}V$. Then $V + H$ is ample.

Proof. By Nakai-Moishezon, it suffices to show that $(V + H) \cdot (V + H) > 0$ and $(V + H) \cdot C > 0$ for all irreducible curves C.
Recall that $(V + H)^2 = V^2 + H^2 + 2V \cdot H = 2V \cdot H = 2$ so let an irreducible curve C be given.
$C \subseteq X \times Y$ so if C is not contained in a fiber of Y by irreducibility $C \cap V$ has dimension 0 so $C \cdot V > 0$, as $\pi_{Y,*}(C)$ dominates Y. $C \cdot H \geqslant 0$ as if $C \cdot H \leqslant 0$ then we must have $C \subseteq H$ (up to numerical equivalence of H). As C is irreduicble then $C = H$, whence $C \cdot H = 0$. Thus $(V + H) \cdot C > 0$.
The same holds for X so we win as any irreducible curve C cannot be contained in a fiber of X and a fiber of Y. □

Riemann Hypothesis

Recall the statement of the Riemann Hypothesis for curves, namely that for X a nice curve over a finite field $\mathbb{F}_q$ we have

$$Z_X(t) = \frac{f(t)}{(1-t)(1-qt)}, \qquad f \in \mathbb{Z}[t], \deg f = 2g.$$

The Riemann Hypothesis is that the zeros of $f(t)$ have absolute value $q^{-1/2}$. After some analytic number theory trickery we prove the equivalent form that

$$\#X(\mathbb{F}_q) = q^n + 1 + O(q^{n/2})$$

Proof. The idea is to use intersection theory $X \times X$.

Claim

$\#X(\mathbb{F}_q) = \Gamma_{F^n} \cdot \Delta_X$, where F^n is the q^n-Frobenius endomorphism of X.

Recall that the graph of a map $f : X \to X$ is the fiber product of either of the following diagrams (ex. use universal properties to show that they are in fact the same):

$$\begin{tikzcd} \Gamma_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ X \arrow[r, "id"] & f \end{tikzcd} \qquad \begin{tikzcd} \Gamma_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Delta_X \arrow[d] \\ X \times X \arrow[r, "(id,f)"] & X \times X \end{tikzcd}$$

Proof of claim: As $\Gamma_{F^n} \cap \Delta = X(\mathbb{F}_q)$ holds set theoretically, it suffices to show that the intersection is reduced, i.e. that Γ_{F^n} and Δ intersect transversally.

Wlog we can take $n = 1$ by base change.

Let $(x, x) \in \Gamma_F \cap \Delta, x \in X(\mathbb{F}_q)$. By dimension concerns, we want to show that $T_{(x,x)}\Delta \neq T_{(x,x)}\Gamma_F . T_{(x,x)}\Delta$ is the diagonal of T_xX in $T_{(x,x)}X \times X \cong T_xX \oplus T_xX . T_{(x,x)}\Gamma_{F^n}$ is the graph of dF in $T_xX \oplus T_xX$. But $dF = 0 \neq id$ at x as it is the Frobenius map. □

Lemma

Let $g : X \to Y$ be a finite flat map of nice surfaces, of degree d (i.e. $\mathrm{rk} g_*\mathcal{O}_X = d$). Then for $C, D \in NS(Y), g^*C \cdot g^*D = d(C \cdot D)$.

Sketch. Claim by proof of properties of $\cdot$ it is enough to show that if C, D are nice curves with $C \cap D$ 0 dimensional.
Then $g^{-1}C \cap g^{-1}D = g^{-1}(C \cap D)$ in the scheme theoretic sense, as one can check that they are the same fiber product. But then

$$g^{-1}C \cdot g^{-1}D = \mathrm{length} g^{-1}(C \cap D) = \mathrm{rk} g_*\mathcal{O}|_{C \cap D} = d\, \mathrm{length} C \cap D = d(C \cdot D)$$

□

Prop

$\Gamma_{F^n} \cdot \Delta = q^n + 1 + O(q^{n/2})$.

Proof. Define $T_n = F^n \times id$. Then $\Gamma_{F^n} = T_n^*\Delta$. Recall the definitions of H, V as vibers of the projections $X \times X \to X$. Then

$$NS(X) \otimes \mathbb{Q} = \mathrm{Span}(V, H) \oplus W$$

where $W = \mathrm{Span}(V, H)^\perp$. The sum is orthogonal as $\cdot$ is nondegenerate on $\mathrm{Span}(V, H)$. Then $\Delta = aV + bH + w$ uniquely for some $w \in W$. Then we have

$$1 = \Delta \cdot V = b = \Delta \cdot H = a,$$

as in our previous computations, so $\Delta = V + H + w$. But then

$$\Gamma_{F^n} \cdot \Delta = t_n^* \Delta \cdot \Delta = t_n^*(V + H + w) \cdot (V + H + w)$$

It follows that $T_n^* V = q^n V, T_n^* H = H$ as V, H are fibers of the projections and T_n is a product map. Then

$$\Gamma_{F^n} \cdot \Delta = (q^n V + H + T_n^* w) \cdot (V + H + w) = q^n + 1 + T_n^* w \cdot w$$

Therefore we need to bound $T_n^* w \cdot w$.

Lemma

$|T_n^* w \cdot w| = O(q^{n/2})$

Proof. W is contained in $(V + H)^\perp$ and $V + H$ is ample, so $\cdot$ is negative definite on W (by the Hodge index theorem). Then we can use Cauchy-Schwarz,

$$\begin{aligned} |T_n^* w \cdot w| &\leqslant \left|(w \cdot w) \cdot (T_n^* w \cdot T_n^* w)\right|^{1/2} \\ &\leqslant \left|(w \cdot w)^2 \deg T_n\right|^{1/2} \\ &= |w \cdot w| \left(\deg T_n\right)^{1/2} \\ &= 2g q^{n/2} \end{aligned}$$

□

□

□

"Cartoon" Digression on the standard conjectures

Let X be a variety. On $X \times X$, we have that $\#X(\mathbb{F}_q) = \Gamma_F \cdot \Delta$. Let π_1, π_2 be the projections, the we have that $\Gamma_{\mathrm{id}} = \Delta$. Let $Z \subseteq X \times X$ be a closed subscheme. This defines a map $f_Z : H^*(X) \to H^*(X)$ given by $\pi_{2*}(\pi_1^* \alpha \smile Z)$. If X/k is smooth proper, then π_{2*} is a map on $H^* X$.
Fact: $f_{\Gamma_g} = g^*$ so $f_\Delta = \mathrm{id}$.

Künneth standard conjecture says that we may write $\Delta = \sum_{i=1}^{2 \dim X} k_i$ where f_{k_i} is the projection $H^*(X) \to H^i(X)$. In the above proof of the Riemann hypothesis we had $\Delta = V + H + w$ where V, H corresponded to the projections to H^2, H^0 respectively and w corresponded to the projection to H^1.

Lefschetz standard conjecture:

$$CH^i/\text{num.} = \oplus V_j^i$$

where V_i^j have some precise definition that we did not give. They are meant to satisfy:

Hodge standard conjecture:

$$V_j^i \times V_j^i \to \mathbb{Z}$$

given by $\alpha, \beta \mapsto \alpha \cdot \beta \cdot H^{\dim X - 2i}$ is definite.

2.8 Lecture 19 - Blowups

Goal: Classification of birational invariants of surfaces.

For most invariants, we can write down the moduli determined by fixing the invariants sometimes and the remainder of the time we get a mysterious class of surfaces (general type).

We want to understand blowups.

Def: Let X be a scheme, $\mathscr{I}$ an ideal sheaf. Then we define the *blowup* of X along $\mathscr{I}$ to be $\mathrm{Bl}_{V(\mathscr{I})} X = \mathrm{Proj}_X \oplus_{n\geqslant 0} \mathscr{I}^n$ where $\oplus_{n\geqslant 0}\mathscr{I}^n$ is has $\mathcal{O}_X$ in degree 0.

We get a map $\pi : \mathrm{Bl}_{V(\mathscr{I})} X \to X$ since the blowup is a relative proj. This π is an isomorphism away from $V(\mathscr{I})$ since letting $U = X\backslash V(\mathscr{I})$ we have that $\mathscr{I}|_U \cong \mathcal{O}_U$ so $\oplus_{n\geqslant 0}\mathscr{I}^n|_U \cong \mathcal{O}_U[t]$ which becomes U when we take proj.

Def: If X/k is a surface, $x \in X$ a closed point, then $\mathrm{Bl}_x X \to X$ is called the *monoidal transformation* of X at x.

The universal property of the blowup is that

$$\mathrm{Hom}(T, \mathrm{Bl}_{V(\mathscr{I})} X) = \left\{f : T \to X \middle| f^{-1}\mathscr{I} \text{ is invertible}\right\}$$

Where we have $0 \to \mathscr{I} \to \mathcal{O}_X$ yielding $f^*\mathscr{I} \to \mathcal{O}_T$ and $f^{-1}\mathscr{I}$ is the image of $f^*\mathscr{I}$ under this map.

Example

The blowup $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$ is contained in $\mathbb{A}^2 \times \mathbb{P}^1$ and is set theoretically given by $\left\{(p,\ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \middle| p \in \ell\right\}$. Note that for any vector space V considered as affine space over its field, we have that $\mathrm{Hom}(T, \mathrm{Bl}_0 V)$ is given by a map $f : T \to V$ such that $f^{-1}\mathfrak{m}$ is invertible where $\mathfrak{m}$ is the maximal ideal corresponding to 0. A map to V is given by $\mathcal{O}_T \to V \otimes \mathcal{O}_T$, then the condition on invertibility is equivalent to asking for a map $V \otimes \mathcal{O}_T \to \mathscr{L}$, i.e. a map to $\mathbb{P}^{n-1}$ such that the composition $\mathcal{O}_T \to V \otimes \mathcal{O}_T \to \mathscr{L}$ is the 0 map.

Example

If $X = \mathrm{Spec}\, A$ a smooth affine surface and let $\mathfrak{m} \subseteq A$ a maximal ideal generated by (x, y). We have a Koszul complex:

$$0 \to A \xrightarrow{(\times x, \times(-y))} A^2 \xrightarrow{\times y + \times x} \mathfrak{m} \to 0$$

Note that since X is smooth, then the Koszul complex is exact ($A_{\mathfrak{m}}$ is regular). It follows that

$$\mathrm{Bl}_{\mathfrak{m}}\, \mathrm{Spec}\, A = \mathrm{Proj}\, A[t,u]/(ty - ux)$$

Lemma

Let X be a nice surface over $k = \bar{k}$. Let $p \in X$ a closed point. Let $E \subseteq \operatorname{Bl}_p X$ be the exceptional divisor, i.e. $\pi^{-1}(p)$. Then:

1. $E \cong \mathbb{P}^1$.

2. $E \cdot E = -1$.

3. $\operatorname{Pic}(\operatorname{Bl}_p X) \cong \mathbb{Z} \oplus \operatorname{Pic}(X)$ given by $\mathcal{O}(nE) \otimes \pi^*(\mathscr{L}) \leftrightarrow (n, \mathscr{L})$

Proof. Note that $\operatorname{Bl}_p X$ is smooth, since by the computation in the above example, we see that locally it is smooth. To see that it is projective, we note that π is projective since X is projective, i.e. it admits an ample line bundle [HA II.7.10]. Since $\pi : \operatorname{Bl}_p X$ and the structure map $X \to k$ are projective, then so is the composition, thus $\operatorname{Bl}_p X$ is a nice surface.

1) Choose $\operatorname{Spec} A \subseteq X$ such that $p = V(x, y) \in \operatorname{Spec} A$. Then $\operatorname{Bl}_p \operatorname{Spec} A = \operatorname{Proj} A[t, u]/(ty - ux)$. This is an open subscheme of $\operatorname{Bl}_p X$ containing $\pi^{-1}(p)$. We compute the fiber over p as:

$$\begin{aligned}
\pi^{-1}(p) &= \operatorname{Proj}\ A[t, u]/(ty - ux) \otimes A/(x, y) \\
&= \operatorname{Proj}\ (A/(x, y))\,[t, u]/(t\overline{y} - u\overline{x}) \\
&= \operatorname{Proj}\ (A/(x, y))\,[t, u] \\
&= \operatorname{Proj}\ k[t, u] \\
&= \mathbb{P}^1_k
\end{aligned}$$

Note that in general the fiber product with proj is more complicated than taking a tensor product however the point p is affine, so it works out.

2) We know that $E \cdot E = \deg_E \mathscr{N}_{E/\operatorname{Bl}_p X}$. We compute this to be $\mathcal{O}(-1)$ on a chart.

3) $\operatorname{Bl}_p X \backslash E = X \backslash \{p\}$, then we get a map:

$$\mathbb{Z} \to \operatorname{Pic}(\operatorname{Bl}_p X) \to \operatorname{Pic}(X \backslash \{p\}) \to 0$$

Where $1 \mapsto E$ and the map on Pics is given by intersecting a divisor on $\operatorname{Bl}_p X$ with $\operatorname{Bl}_p X \backslash E$. We have that $\operatorname{Pic}(X \backslash \{p\}) = \operatorname{Pic}(X)$ and we have a map back $\pi^* : \operatorname{Pic} X \to \operatorname{Pic}(\operatorname{Bl}_p X)$. Assuming the map is injective on the left, this gives a splitting of the exact sequence. To see that it is injective, we note that the composition $n \to nE \to nE \cdot nE = -n^2$ is injective for $n \geqslant 0$, therefore the map $\mathbb{Z} \to \operatorname{Pic}(\operatorname{Bl}_p X)$ is injective. It follows that $\operatorname{Pic}(\operatorname{Bl}_p X) = \operatorname{Pic}(X) \oplus \mathbb{Z}$. □

Example

We have that $\operatorname{Bl}_p \mathbb{P}^2 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ which then has a projection map $\pi_2 : \operatorname{Bl}_p \mathbb{P}^2 \to \mathbb{P}^1$. The fibers of this map are $\mathbb{P}^1$, so $\operatorname{Bl}_p \mathbb{P}^2$ is a ruled surface with fibers $\mathbb{P}^1$, i.e. a Hirzebruch surface. In fact since $E^2 = -1$ for the exceptional divisor, then $\operatorname{Bl}_p \mathbb{P}^2 \cong \mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$.

Theorem 2.19.1: Factorization

Any birational map of nice surfaces can be factored as:

$$\begin{tikzcd}
 & X_1 \arrow[ddl] \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & X_n \arrow[dl] \arrow[ddr] & \\
 & & Y_1 & & Y_2 & & Y_{n-1} & & \\
X \arrow[rrrrrrrr, dashed] & & & & & & & & Y
\end{tikzcd}$$

with each map being a monoidal transformation.

Def: A *simple normal crossings divisor* is a union of smooth curves C_i intersecting pairwise in a unique single point each of which is locally of the form $\widehat{\mathcal{O}}_C \cong k[\![x,y]\!]/(x,y)$.

2.9 Lecture 20 - Divisors on Blowups & Embedded resolution

Adjunction theorem II

Let X be a nice surface and C an arbitrary curve on X. Then

$$\chi(\mathcal{O}_C) = \frac{-1}{2}(C \cdot C + C \cdot K_X)$$

where we note that C is always a Cartier divisor so the intersection makes sense.

Proof. We start with the SES

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

Taking euler characteristics yields

$$\begin{aligned}
\chi(\mathcal{O}_C) &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_C(-C)) \\
&= \frac{-1}{2}(-C)\cdot(-C-K) && \text{Riemann Roch} \\
&= \frac{-1}{2}(C \cdot C + C \cdot K)
\end{aligned}$$

$\square$

Now let X be a nice surface over $k = \bar{k}$ and let $p \in X(k)$ be a rational points. Consider the blowup map $\pi : Bl_pX \to X$.
Last time we prove that for $E = \pi^{-1}(p) \subseteq Bl_pX$, we have $E \cong \mathbb{P}^1, Bl_pX$ is nice, $E \cdot E = -1$ and

$\operatorname{Pic} Bl_pX = E\mathbb{Z} \oplus \pi^*(\operatorname{Pic}(X))$ where the direct sum is an orthogonal decomposition.

Prop

$K_{Bl_pX} = \pi^*K_X + E$.

Proof. $K_{Bl_pX} = \pi^*K_X + E$ as $K_{Bl_pX-E} = \pi^*K_{X-p}$ as π is birational away from E. By the Adjunction theorem then we have

$$1 = \chi(\mathcal{O}_E) = \frac{-1}{2}(-1 + E \cdot K_{Bl_pX})$$

so $-2 = -1 - r$ hence $r = 1$, as desired. $\square$

Prop

$\pi^* : H^i(X, \mathcal{O}) \to H^i(Bl_pX, \mathcal{O})$ is an isomorphism for all i.

Proof.

1. We first show that $R^i\pi_*\mathcal{O}_{Bl_pX} \cong \begin{cases} \mathcal{O} & i = 1 \\ 0 & i > 0 \end{cases}$.
   π has relative dimension 1, so the result holds immediately for $i \geqslant 2$. But then $R^2\pi_*\mathcal{O}_{Bl_pX}$ is a line bundle so cohomology commutes with base change for $i = 1$. On fibers we get $H^1(\mathbb{P}^1, \mathcal{O}) = 0$, so $R^1\pi_*\mathcal{O}_{Bl_pX} = 0$. Thus we may again apply C&BC to conclude $\pi_*\mathcal{O}$ is a line bundle. $\pi_*\mathcal{O}$ has a nowhere vanishing section, so we conclude that $\pi_*\mathcal{O} = \mathcal{O}$, as claimed.

2. By the Lerray spectral sequence

   $$H^p(X, R^q\pi_*\mathcal{O}) \implies H^{p+q}(Bl_pX, \mathcal{O})$$

   But by part 1., most of the terms vanish, so we get isomorphisms $H^p(X, \pi_*\mathcal{O}_{Bl_pX}) \cong H^p(Bl_pX, \mathcal{O}_{Bl_pX})$. The result follows as we established in part 1 that $\pi_*\mathcal{O}_{Bl_pX} \cong \mathcal{O}_X$.

$\square$

Remark

$h^i(X, \mathcal{O})$ is thus a birational invariant of (smooth projective) surfaces, as any birational equivalence factors as a sequence of blowups.

Fix C an irreducible curve on X through $p \in X(k)$. We wish to understand what the "preimage" of C in Bl_pX is, as $\pi^{-1}C$ always contains E so is reducible.

Def: The proper (or strict) transform of C in Bl_pX is $\widetilde{C} = \pi^{-1}(\bar{C} - p)$.

Thm

Let X be a nice surface over $k = \bar{k}, p \in X(k)$, and let C be a curve through p. Let $\pi : Bl_p X \to X$ be the blowup map. Then $\pi^* = \widetilde{C} + rE$, where

$$r = \text{mult}_p C = \max\{s | f \in \mathfrak{m}_{X,P}^s, f \text{ a local equation for } C\}$$

Example

Work in $\mathbb{A}^2$. Then $\text{mult}_{(0,0)} V(y) = 1, \text{mult}_{(0,0)} V(xy) = 2, \text{mult}_{(0,0)} V(y^2 - x^3) = 2$

It is also easy to see that $\text{mult}_p C = 1$ iff C is smooth at p.

Proof. By locality, wlog $X = \text{Spec } A, \mathfrak{m}_{X,p} = (x, y)$ and $C = V(f)$,

$$f(x, y) = f_r(x, y) + g(x, y)$$

where f_r is "homogeneous" of degree r and $g \in \mathfrak{m}_{X,p}^{r+1}$. Then by our earlier computations for blowups,

$$Bl_p X = \text{Proj } A[t, u](ty - ux).$$

Computing on the chart $\{t = 1\}, y = ux$, so

$$\pi^* f = x^r (f_r(1, u) + xh(u)) \in \mathfrak{m}_{X,p}^r - m_{X,p}^{r+1}.$$

$f_r(1, u)$ is a local equation for $\widetilde{C}$ and x is a local equation for E so thus $\pi^* C = \pi^* V(f) = \widetilde{C} + rE$ as desired. □

Weak Embedded Resolution

Let X be a nice surface over $k = \bar{k}, C$ a reduced curve on X. Then there exists a sequence of monoidal transformations

$$X_n \to \cdots \to X_1 \to X_0 = X$$

such that $\widetilde{C}_n \subseteq X_n$ is smooth.

Proof. As $k = \bar{k}$ with C reduced, generic smoothness means that there are only finitely many singular points to start with. As blowups of points on X are birational, it thus suffices to show that you can resolve the singularity at any point in finitely many steps.
Claim: Suppose that $p \in X$ is a singular point of C. Let $\widetilde{C} \subseteq Bl_p X$ be the proper transform of C, then $\chi(\mathcal{O}_{\widetilde{C}}) > \chi(\mathcal{O}_C)$.

1. We first show that the claim implies the theorem.
   C and $\widetilde{C}$ are birational so therefore the normalization $\bar{C}$ of C is isomorphic to the normalization of $\widetilde{C}$. Furthermore, $\chi(\mathcal{O}_C) \leqslant \chi(\mathcal{O}_{\bar{C}})$, as there is a SES

   $$0 \to \mathcal{O}_C \to \bar{\pi}_* \mathcal{O}\bar{C} \to \text{Torsion} \to 0.$$

   Now let $\widetilde{C}_0 = C, p_{i+1}$ be a singular point of C_i, and $\widetilde{C}_{i+1}$ to be the proper transform of $\widetilde{C}_i$ under the blowup at p_{i+1}. If $\chi(\mathcal{O}_{\widetilde{C}_{i+1}}) > \chi(\mathcal{O}_{\widetilde{C}_i})$ at each step, the process eventually terminates, at these numbers are bounded above. Thus $\widetilde{C}_n$ is nonsingular.

2. We now prove the claim:
   Recall that $\widetilde{C} = \pi^*C - rE$, so

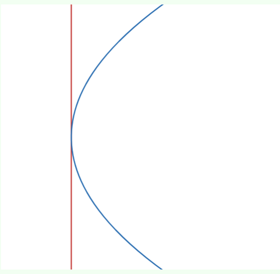
$$\begin{aligned}
\chi(\widetilde{C}) &= \frac{-1}{2}(\widetilde{C}\cdot\widetilde{C} + \widetilde{C}\cdot K_{Bl_pX}) && \text{Adjunction}\\
&= \frac{-1}{2}(\pi^*C - rE)\cdot(\pi^*C - rE + \pi^*K_X + E)\\
&= \frac{-1}{2}(C\cdot C + C\cdot K_X) + \frac{1}{2}(r^2 - r) && \deg\pi = 1\\
&= \chi(\mathcal{O}_C) + \frac{r^2 - r}{2} > \chi(\mathcal{O}_C)
\end{aligned}$$

   since $r^2 - r > 0$ for $r > 1$.

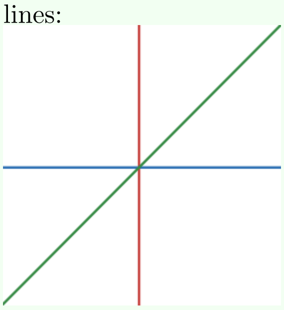
□

Example

Let $X = \mathbb{A}^2$ and $C = V(y^2 - x^3)$. Write $R = k[x,y]$, then we want to remove the singularity at 0 in C, so we blow up at 0.
$\mathrm{Bl}_0\,\mathbb{A}^2 = \mathrm{Proj}\; k[x,y][t,u]/(ty - ux)$. In the $t = 1$ chart we have $y = ux$, so $y^2 - x^3 = u^2x^2 - x^3 = x^2(u^2 - x)$. The x^2 part corresponds to the divisor $2E$ and $u^2 - x$ is smooth, however this is not a simple normal crossing divisor.



We can blow up again, now $C = V(x^2(y^2 - x)) \subseteq \mathbb{A}^2$, then we blow up and get $\mathrm{Proj}\; k[x,y][t,u]/(ty - ux)$. On the $t = 1$ chart, there is no intersection. On the $u = 1$ chart, we have that $x = ty$, then $x^2(y^2 - x) = t^2y^3(y - t)$ which is an intersection of three lines:



Blowing up one more time will then separate these three lines into four lines, one $x = 0$ line for the exceptional divisor and three lines intersecting at y-values corresponding to the slopes of the lines in this second blow up. It follows that after three blow ups we reach a simple normal crossing divisor.

Embedded Resolutions

Let X be a nice surface / $k = \bar{k}$ and C a reduced curve on X. Then there exists a sequence of monoidal transformations

$$X_n \to \cdots \to X_1 \to X_0 = X$$

such that the inverse image of C under the composite map is a snc divisor, i.e. the reduced structure on each irred components C_i is smooth, $C_i \cap C_j = \{p\}$ or $\varnothing$ for $i \neq j$, and $C_i \cap C_j \cap C_k = \varnothing$ for $i \neq j \neq k$.

The proof is similar to weak embedded resolution so we omit it.
Then Daniel talked about one of his recent twitter posts for the rest of the class.

2.10 Lecture 21 - Castelnuevo's Contraction Theorem & Minimal Models

Theorem 2.21.1: Castelnuovo's Contraction Theorem

Let X be a nice surface, $E \subseteq X$ a divisor, then there is a map $f : X \cong Bl_p Y$, where Y is a nice surface and $p \in Y$ such that $f^{-1}(p) = E$ iff $E \cong \mathbb{P}^1$ and $E \cdot E = -1$.

Def: X a nice surface is *minimal* if there are no curves $C \subseteq X$, with $C \cong \mathbb{P}^1, C \cdot C = -1$ (i.e. there do not exist -1-curves)

Proposition

If X is not birationally ruled, then X has a unique minimal model.

Theorem 2.21.2: Minimal Models Exist

Let X be nice, then $\exists X \to X'$ birational w/ X' minimal.

Proof. Set $X = X_0$. If X_i is not minimal, by Castelneuvo we obtain X_i as a blowup of some other nice surface X_{i+1} by contracting a -1-curve. It thus suffices to show that this process terminates. As we contract a -1-curve, $\dim_{\mathbb{Q}} NS(x_i) \otimes \mathbb{Q} > \dim_{\mathbb{Q}} NS(X_{i+1}) \otimes X$ (if both are finite dimensional). Thus we win by the theorem of the base, as $\dim_{\mathbb{Q}} NS(x_i) \otimes \mathbb{Q}$ are then a decreasing sequence of finite non-negative numbers. □

The theorem of the base is a big hammer, so we can instead tro to show the process terminates in another way.

Proof. 1. Define $C_i : \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X^1)$ induced by

$$d\log : \mathcal{O}_X^* \Omega_X^1, \qquad f \mapsto \frac{df}{f}$$

2. We have a symmetric cup product $\dot{\cup} : H^1(X, \Omega^1_X) \times H^1(X, \Omega^1_X) \to H^2(X, \Omega^2_X) \cong k$ which fits into the following commutative diagram (exercise: show that it commutes)

$$\begin{tikzcd} \mathrm{Pic}(X) \times \mathrm{Pic}(X) \arrow[r] \arrow[d, "C_1 \times C_1"'] & \mathbb{Z} \arrow[d] \\ H^1(X, \Omega^1_X) \times H^1(X, \Omega^1_X) \arrow[r, "\dot{\cup}"'] & k \end{tikzcd}$$

3. $H^1(X, \Omega^1_X)$ is finite dimensional by Serre. Let $V \subseteq NS(X)$ be the span of all the -1-curves. It then follows that $\dim_{\mathbb{Q}} V \otimes \mathbb{Q} = \dim_k C_1(V \otimes k)$ because $\cdot$ is nondegenerate on V, and $E \cdot F = 0$ for all distinct -1-curves E, F. Thus $\dim V < \infty$ and the same argument as before runs.

□

Proof. Idea: Find $\mathscr{L} \in \mathrm{Pic}(X)$ such that $f_{|\mathscr{L}|} : X \to \mathbb{P}H^0(X, \mathscr{L})$ factors through Y. We want $\mathscr{L}$ to separated points and tangent vectors away from E, but not on E, so we want $\mathscr{L} \cdot E = 0$.

Choose H very ample on X with $H^1(\mathcal{O}(H)) = 0$. Let $k = H \cdot E$ and $\mathscr{L} = \mathcal{O}(H + kE)$. We then have that $\mathscr{L} \cdot E = k - k = 0$ since E is a -1-curve. Since $E \cong \mathbb{P}^1$, then $\mathscr{L}|_E$ having degree 0 and being effective means that $\mathscr{L}|_E = \mathcal{O}_E$.

To show that $\mathscr{L}$ is globally generated, we proceed in a manner similar to the proof of Nakai-Moishezon. Since $\mathcal{O}(E) \cong \mathcal{O}_X$ away from E and $\mathcal{O}(H)$ is generated by global sections, then this should mean that $\mathscr{L}$ is generated by global sections away from E. More formally, let $s \in \Gamma(X, \mathcal{O}(E))$ be a nonzero section vanishing along E (this exists because E is effective). Then we have:

$$\begin{tikzcd} \Gamma(X, \mathcal{O}(H)) \otimes \mathcal{O} \arrow[r] \arrow[d] \arrow[dr, dashed] & \Gamma(X, \mathcal{O}(H + kE)) \otimes \mathcal{O} \arrow[d] \\ \mathcal{O}(H) \arrow[r] & \mathcal{O}(H + kE) = \mathscr{L} \end{tikzcd}$$

The vertical arrow on the left is surjective since $\mathcal{O}(H)$ is globally generated. The horizontal arrows are given by multiplication by s^k. Since s^k is non-vanishing away from E, then the dashed morphism is surjective away from E, and therefore the vertical morphism on the right is surjective away from E as desired.

On E, we have $\mathscr{L}|_E = \mathcal{O}_E$, so $\mathscr{L}|_E$ is globally generated and we want to lift this to X. We first show that $H^1(X, \mathcal{O}_X(H + iE) = 0$ for all $i = 0, \cdots, k$. When $i = 0$, this is by assumption that $H^1(\mathcal{O}(H)) = 0$. For $i > 0$ we have a SES:

$$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$$

Tensoring with $\mathcal{O}(H + iE)$ we get the following SES:

$$0 \to \mathcal{O}(H + (i-1)E) \to \mathcal{O}(H + iE) \to \mathcal{O}(H + iE)|_E \to 0$$

By induction the H^1 of the leftmost term vanishes, so we have that $H^1(\mathcal{O}(H + iE))$ injects into the H^1 of $\mathcal{O}(H + iE)|_E$. Since $\deg \mathcal{O}(H + iE)|_E = (H + iE) \cdot E = k - i > 0$, then $E \cong \mathbb{P}^1$ implies that the H^1 vanishes, therefore we have that $H^1(\mathcal{O}(H + iE))$ must also be 0 as desired. For $i = k$ taking global sections yields:

$$0 \to \Gamma(X, \mathcal{O}(H + (i-1)E)) \to \Gamma(X, \mathscr{L}) \to \Gamma(E, \mathscr{L}|_E) \to 0$$

Notice that $\Gamma(E, \mathcal{O}_E) = k$ since E is an irreducible curve over k. From surjectivity, it follows that we may lift $1 \in \Gamma(E, \mathcal{O}_E)$ to $\Gamma(X, \mathscr{L})$ which will give a global section of $\mathscr{L}$ not vanishing on E as desired.

To show that $\mathscr{L}$ is very ample, we need to show that $f_{|\mathscr{L}|}|_{X\setminus E}$ is an embedding. To show this, we need to show that it separates points and tangent vectors. This is an analogous argument to how we showed that $\mathscr{L}$ is globally generated away from E but on the level of local rings.

$f_{|\mathscr{L}|}$ is not injective on E, in fact $\mathscr{L}|_E = \mathcal{O}_E$, so it is constant since there is only one section. Therefore we collapse E to a point.

Finally we need to show that $\mathrm{im} f_{|\mathscr{L}|}$ is smooth. This is clear away from $f_{|\mathscr{L}|}(E) = \{p\}$ since it is isomorphic to an open subset of X and X is smooth. Let Y be the normalization of $\mathrm{im} f_{|\mathscr{L}|}$, then we have $X \xrightarrow{\tilde{f}} Y$ since X is normal, so f factors through the normalization. By Zariski's main theorem we have that $\tilde{f}_* \mathcal{O}_X = \mathcal{O}_Y$. We want to show that $\widehat{\mathcal{O}_{Y,p}} = \widehat{f_* \mathcal{O}_{X,p}} = k[\![x, y]\!]$. This follows since completion commutes with pushforward. We want $(\mathcal{O}_X|_E)\hat{}$. We have:

$$0 \to \mathscr{I}_E^n / \mathscr{I}_E^{n+1} \to \mathcal{O}_X / \mathscr{I}_E^{n+1} \to \mathcal{O}_X / \mathscr{I}_E^n \to 0$$

We know that $\deg|_E \mathscr{I}_E^n / \mathscr{I}_E^{n+1} = -(nE) \cdot E = n$, therefore it is $\mathcal{O}_{\mathbb{P}^1}(n)$. Now taking global sections sho that we indeed have $k[\![x, y]\!]$. $\square$