行列式计算拓展

行或列比例递减

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} = 0$$

(自右向左减左边相邻列)

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_1+a_2 & a_1+a_2+a_3 & a_1+a_2+a_3+a_4 \\ a_1 & 2a_1+a_2 & 3a_1+2a_2+a_3 & 4a_1+3a_2+2a_3+a_4 \\ a_1 & 3a_1+a_2 & 6a_1+3a_2+a_3 & 10a_1+6a_2+3a_3+a_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_1+a_2 & a_1+a_2+a_3 \\ 0 & a_1 & 2a_1+a_2 & 3a_1+2a_2+a_3 \\ 0 & a_1 & 3a_1+a_2 & 6a_1+3a_2+a_3 \end{vmatrix} = a_1^{\,4}$$

(自下而上减上一行)

行或列比例递减

$$
\begin{vmatrix}
n & n-1 & n-2 & \cdots & 1 \\
-1 & x & 0 & \cdots & 0 \\
0 & -1 & x & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & x
\end{vmatrix}
=
\begin{vmatrix}
n & (n-1)+nx & \sum\limits_{i=0}^{2}(n-i)x^{2-i} & \cdots & \sum\limits_{i=0}^{n-1}(n-i)x^{n-1-i} \\
-1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0
\end{vmatrix}
$$

(自左向右加左列的*x*倍)

$$
=(-1)^{1+n}\sum_{i=0}^{n-1}(n-i)x^{n-1-i}
\begin{vmatrix}
-1 & & & \\
 & -1 & & \\
 & & \ddots & \\
 & & & -1
\end{vmatrix}_{n-1}
=\sum_{k=1}^{n}kx^{k-1}
$$

$$
\begin{vmatrix}
12345 & 12245 \\
67813 & 67913
\end{vmatrix}
=
\begin{vmatrix}
100 & 12245 \\
-100 & 67913
\end{vmatrix}
=
\begin{vmatrix}
100 & 12245 \\
0 & 80158
\end{vmatrix}
=8015800
$$

(第1列减第2列)

行或列全加

$$\begin{vmatrix} a_1-b & a_2 & \cdots & a_n \\ a_1 & a_2-b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n-b \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} a_i - b & a_2 & \cdots & a_n \\ \sum_{i=1}^{n} a_i - b & a_2-b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} a_i - b & a_2 & \cdots & a_n-b \end{vmatrix} = (\sum_{i=1}^{n} a_i - b) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & -b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & -b \end{vmatrix}$$

(每一列加到第一列)

$$= (-b)^{n-1}(\sum_{i=1}^{n} a_i - b)$$

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix} = \begin{vmatrix} 2(x+y) & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$$

(每一列加到第一列)

$$= -2(x+y)(x^2 - xy + y^2) = -2(x^3 + y^3)$$

行或列全加

,

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \iddots & 1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ n-1 & n & 1 & \cdots & n-2 \\ n & 1 & 2 & \cdots & n-1 \end{vmatrix} \overset{\text{列全加}}{=} \frac{n(n+1)}{2} \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 4 & \iddots & 1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & n & 1 & \cdots & n-2 \\ 1 & 1 & 2 & \cdots & n-1 \end{vmatrix} \overset{\text{行递减}}{=} \frac{n(n+1)}{2} \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 1 & \iddots & 1-n \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 1-n & \cdots & 1 \\ 0 & 1-n & 1 & \cdots & 1 \end{vmatrix} = \frac{n(n+1)}{2} D_n$$

,

$$D_n \overset{\text{列展开}}{=} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1-n \\ 1 & 1 & 1 & \iddots & 1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & 1-n & 1 & \cdots & 1 \\ 1-n & 1 & 1 & \cdots & 1 \end{vmatrix}_{n-1} \overset{\text{列全加}}{=} \begin{vmatrix} -1 & 1 & 1 & \cdots & 1-n \\ -1 & 1 & 1 & \iddots & 1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ -1 & 1-n & 1 & \cdots & 1 \\ -1 & 1 & 1 & \cdots & 1 \end{vmatrix}_{n-1} \overset{\text{1列加到其它列}}{=} \begin{vmatrix} -1 & 0 & 0 & \cdots & -n \\ -1 & 0 & 0 & \iddots & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ -1 & -n & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \end{vmatrix}_{n-1} = (-1)^{\frac{n(n-1)}{2}} n^{n-2}$$

配套相加

$$\begin{vmatrix} a & 0 & \cdots & 0 & b \\ 0 & a & \cdots & b & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & b & \cdots & a & 0 \\ b & 0 & \cdots & 0 & a \end{vmatrix}_{n=2k}$$

(下面一半：第$2k$+1-i行加到第i行，

右面一半：第$2k$+1-i列减去第i列)

$$\overset{\substack{r_i+r_{2k+1-i} \\ i=1,2,\cdots,k}}{=} \begin{vmatrix} a+b & 0 & \cdots & 0 & a+b \\ 0 & a+b & \cdots & a+b & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & b & \cdots & a & 0 \\ b & 0 & \cdots & 0 & a \end{vmatrix} \overset{\substack{c_{2k+1-i}-c_i \\ i=1,2,\cdots,k}}{=} \begin{vmatrix} a+b & 0 & \cdots & 0 & 0 \\ 0 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & b & \cdots & a-b & 0 \\ b & 0 & \cdots & 0 & a-b \end{vmatrix}$$

$$=(a+b)^k(a-b)^k$$

化三角形消零

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ -1 & 0 & 3 & \cdots & n-1 & n \\ -1 & -2 & 0 & \cdots & n-1 & n \\ -1 & -2 & -3 & \cdots & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \cdots & -(n-1) & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 0 & 2 & 6 & \cdots & 2(n-1) & 2n \\ 0 & 0 & 3 & \cdots & 2(n-1) & 2n \\ 0 & 0 & 0 & \cdots & 2(n-1) & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n \end{vmatrix} = n!$$

(第1行加到各行)

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 2 & \cdots & n-1 \\ 3 & 2 & 1 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & \cdots & 1 \end{vmatrix} \overset{\text{行递减}}{=} \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{vmatrix} \overset{\text{最后列加到每一列}}{=} \begin{vmatrix} n+1 & n+2 & \cdots & n+n-1 & n \\ 0 & -2 & \cdots & -2 & -1 \\ 0 & 0 & \ddots & -2 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{n-1} 2^{n-2} (n+1)$$

化三角形消零

$$\begin{vmatrix} a_0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix}$$

(若所有$a_i \neq 0$，各列的-1/a_i倍加到第1列)
(若有 $a_i=0$，可按第i列原第i行展开)

$$\overset{\substack{a_i\neq 0\\ i=1,\cdots,n}}{=} \begin{vmatrix} a_0-\dfrac{1}{a_1}-\cdots-\dfrac{1}{a_n} & 1 & 1 & \cdots & 1 \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = (a_0-\frac{1}{a_1}-\cdots-\frac{1}{a_n})a_1a_2\cdots a_n$$

$$\overset{a_i=0}{=} \begin{vmatrix} a_0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & & & & & \\ 1 & & a_{i-1} & & & & 0 \\ 1 & & & 0_i & & & 0 \\ 1 & & & & a_{i+1} & & 0 \\ \vdots & & & & & \ddots & \vdots \\ 1 & \cdots & 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = (-1)^{i+2} \begin{vmatrix} 1 & a_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & a_{i-1} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & a_{i+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & a_n \end{vmatrix} = -\begin{vmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_{i-1} & & \\ & & & a_{i+1} & \\ & & & & \ddots \\ & & & & & a_n \end{vmatrix} = -\prod_{\substack{j=1,\ldots,n\\ j\neq i}} a_j$$

递推式

$$\begin{vmatrix} 1 & 1 & & \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 1 \end{vmatrix}_n \quad D_n = D_{n-1} - D_{n-2} = -D_{n-3} = \dots = \begin{cases} (-1)^k D_1 = (-1)^k, & n = 3k+1 \\ (-1)^k D_2 = 0, & n = 3k+2 \\ (-1)^{k-1} D_3 = (-1)^k, & n = 3k \end{cases}$$

$$\begin{vmatrix} 2a & a^2 & & \\ 1 & 2a & \ddots & \\ & \ddots & \ddots & a^2 \\ & & 1 & 2a \end{vmatrix}_n \quad \begin{aligned} & D_n = 2aD_{n-1} - a^2 D_{n-2}, \\ & D_n - aD_{n-1} = a(D_{n-1} - aD_{n-2}) = \dots = a^{n-2}(D_2 - aD_1) = a^n, \\ & D_n = a^n + aD_{n-1} = a^n + a^n + .. + a^n + a^{n-1} D_1 = (n+1)a^n . \end{aligned}$$

$$\begin{vmatrix} & & a^2 & 2a \\ & a^2 & 2a & 1 \\ \iddots & \iddots & \iddots & \\ 2a & 1 & & \end{vmatrix}_n \overset{\text{从右往左两两交换列}}{=} (-1)^{n-1} \begin{vmatrix} 2a & & & a^2 \\ 1 & & a^2 & 2a \\ & \iddots & \iddots & \iddots \\ 0 & 2a & 1 & \end{vmatrix} = \dots = (-1)^{(n-1)+\dots+1} \begin{vmatrix} 2a & a^2 & & \\ 1 & 2a & \ddots & \\ & \ddots & \ddots & a^2 \\ & & 1 & 2a \end{vmatrix} = (-1)^{\frac{1}{2}n(n-1)} (n+1)a^n$$

(交换成正对角线)

分裂行列式

$$\begin{vmatrix} 2a & a^2 & & \\ 1 & 2a & \ddots & \\ & \ddots & \ddots & a^2 \\ & & 1 & 2a \end{vmatrix} = \begin{vmatrix} a & a^2 & & \\ 1 & 2a & \ddots & \\ & \ddots & \ddots & a^2 \\ & & 1 & 2a \end{vmatrix} + \begin{vmatrix} a & a^2 & & \\ 0 & 2a & \ddots & \\ & \ddots & \ddots & a^2 \\ & & 1 & 2a \end{vmatrix} = \begin{vmatrix} a & 0 & & \\ 1 & a & \ddots & \\ & \ddots & \ddots & 0 \\ & & 1 & a \end{vmatrix} + aD_{n-1} = a^n + aD_{n-1}$$

$$= a^n + a^n + a^2 D_{n-2} = \cdots = (n-1)a^n + a^{n-1}D_1 = (n+1)a^n$$

$$\begin{vmatrix} x_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & x_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & x_n \end{vmatrix} = \begin{vmatrix} x_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & x_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{vmatrix} + \begin{vmatrix} x_1 & a_1b_2 & \cdots & 0 \\ a_2b_1 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & x_n - a_nb_n \end{vmatrix}$$

$$= b_n \begin{vmatrix} x_1 - a_1b_1 & 0 & \cdots & a_1 \\ 0 & x_2 - a_2b_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{vmatrix} + (x_n - a_nb_n)D_{n-1} = a_nb_n \prod_{i=1}^{n-1}(x_i - a_ib_i) + (x_n - a_nb_n)D_{n-1}$$

$$= \cdots = \left(\sum_{i=1}^{n} \frac{a_ib_i}{x_i - a_ib_i}\right)\prod_{i=1}^{n}(x_i - a_ib_i)$$

利用多项式系数

$$D=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix}$$

扩张成范德蒙行列式展开

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & y \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & \cdots & x_n^n & y^n \end{vmatrix}$$

$$=\Big(\prod_{1\le i<j\le n}(x_j-x_i)\Big)\Big(\prod_{k=1}^{n}(y-x_k)\Big)$$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & y \\ \vdots & \vdots & & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & y^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & y^{n-1} \\ x_1^n & x_2^n & \cdots & x_n^n & y^n \end{vmatrix}$$

按最后一列展开成y的多项式

$$=A_{1,n+1}+yA_{2,n+1}+\cdots+y^{n-1}A_{n,n+1}+y^nA_{n+1,n+1}=\cdots+y^{n-1}(-D)+\cdots$$

比较 y^{n-1} 项的系数： $(-\sum x_k)\prod(x_j-x_i)=-D$

故 $D=(\sum x_k)\prod(x_j-x_i)$

行列交互作用

$$D_{n+1}(x)=\begin{vmatrix} x & 1 & & & \\ n & x & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & x & n \\ & & & 1 & x \end{vmatrix} \overset{r_2+r_1}{=} \begin{vmatrix} x & 1 & & & & \\ n+x & x+1 & 2 & & & \\ & n-1 & x & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & x \end{vmatrix} \overset{c_1-c_2}{=} \begin{vmatrix} x-1 & 1 & & & & \\ n-1 & x+1 & 2 & & & \\ 1-n & n-1 & x & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & x \end{vmatrix}$$

$$\underset{c_2-c_3}{\overset{r_3+r_2}{=}} \begin{vmatrix} x-1 & 1 & & & & \\ n-1 & x-1 & 2 & & & \\ & n-2 & x+2 & 3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & x \end{vmatrix} = \cdots = \begin{vmatrix} x-1 & 1 & & & \\ n-1 & x-1 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & x-1 & n \\ & & & 0 & x+n \end{vmatrix}$$

$$=(x+n)D_n(x-1)=\cdots=\sum_{k=-n,-n+2,\cdots,n}(x+k)$$