2.3 分块矩阵

矩阵有时候是：对角、三角、O等特殊形状

$$\begin{pmatrix} 1.33 & 0 & 0 \\ 0 & -2.4 & 0 \\ 0 & 0 & -0.3 \end{pmatrix}, \quad \begin{pmatrix} 7 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

矩阵有时候：没有任何形状上的特殊性

$$\begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 & 5 \\ 1 & 3 & -5 & 7 \\ 2 & 3 & -7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

矩阵有时候是：有点对角、上三角等形状

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

有时候需要将矩阵分块来看待，如

$$\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ O_{3\times 2} & E_3 \end{pmatrix}$$

其中：$A_{11} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix}$,

$O_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

定义：对于一个$m\times n$矩阵，如果在行的方向分成s块，在列的方向分成t块，就得到A的一个$s\times t$分块矩阵. 记作

$$A = (A_{kl})_{s\times t} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{pmatrix} & \begin{matrix} m_1\text{行} \\ m_2\text{行} \\ \vdots \\ m_s\text{行} \end{matrix} \\ \begin{matrix} n_1\text{列} & n_2\text{列} & \cdots & n_t\text{列} \end{matrix} & \end{array}$$

其中 $m_1+m_2+\ldots+m_s=m$, $n_1+n_2+\ldots+n_t=n$,
而A_{kl} $(k=1,2,\ldots,s;l=1,2,\ldots,t)$称为$A$的子块.

分块矩阵的运算

(1) 分块矩阵的加法．设分块矩阵

$$A=(A_{kl})_{s\times t}\ ,\quad B=(B_{kl})_{s\times t}\ ,$$

如果A与B对应的子块 A_{kl} 和 B_{kl} 都是同型矩阵，则

$$A+B=(A_{kl}+B_{kl})_{s\times t}\ .$$

$$\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 \\ -1 & -7 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \left(\begin{array}{cc|ccc} -2 & 3 & -1 & 0 & 2 \\ 7 & 0 & 9 & 2 & 0 \\ \hline 11 & 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 3 & 6 & 0 & 0 & 3 \end{array}\right) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 \\ -1 & -7 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 \end{array}\right) + \left(\begin{array}{cc|ccc} -2 & 3 & -1 & 0 & 2 \\ 7 & 0 & 9 & 2 & 0 \\ \hline 11 & 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 & 2 \\ 3 & 6 & 0 & 0 & 3 \end{array}\right) = \left(\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & -1 \\ 8 & 3 & 10 & 0 & 0 \\ \hline 13 & 2 & 2 & 2 & 0 \\ -1 & -5 & 0 & 3 & 2 \\ 9 & 6 & 0 & 0 & 4 \end{array}\right)$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{pmatrix}$$

(2) 分块矩阵的数乘．设分块矩阵

$$A=(A_{kl})_{s\times t}\ ,$$

λ是数，则

$$\lambda A=(\lambda A_{kl})_{s\times t}\ .$$

$$\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 \\ -1 & -7 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$3\times\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 2 & 1 & 1 & 0 & 0 \\ -1 & -7 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|ccc} 9 & 3 & 6 & 0 & -9 \\ 3 & 9 & 3 & -6 & 0 \\ \hline 6 & 3 & 3 & 0 & 0 \\ -3 & -21 & 0 & 3 & 0 \\ 18 & 0 & 0 & 0 & 3 \end{array}\right)$$

$$3\times\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 3\times A_{11} & 3\times A_{12} \\ 3\times A_{21} & 3\times A_{22} \end{pmatrix}$$

(3) 分块矩阵的乘法．设分块矩阵

$$A=(a_{ij})_{m\times n}\ ,\quad B=(b_{ij})_{n\times p}\ ,$$

如果把A, B分别分块为$r\times s$和$s\times t$分块矩阵，且A的列的分块法与B的行的分块法完全相同，则

$$AB=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{pmatrix}=C=\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{pmatrix},$$

其中 C 是$r\times t$分块矩阵，且

$$C_{kl}=A_{k1}B_{1l}+A_{k2}B_{2l}+\cdots+A_{ks}B_{sl}=\sum_{i=1}^{s}A_{ki}B_{il},(k=1,2,\cdots,r;l=1,2,\cdots,t).$$

例2.3.1 设

$$A=\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},\quad B=\begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix},$$

求AB．

解 把A, B分成

$$A=\left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)=\begin{pmatrix} A_{11} & O_2 \\ A_{21} & E_2 \end{pmatrix},\quad B=\left(\begin{array}{cc} 1 & 0 \\ 1 & -2 \\ \hline 1 & 0 \\ -1 & 1 \end{array}\right)=\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix},$$

则 $$AB=\begin{pmatrix} A_{11} & O_2 \\ A_{21} & E_2 \end{pmatrix}\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix}=\begin{pmatrix} A_{11}B_{11} \\ A_{21}B_{11}+B_{21} \end{pmatrix},$$

而 $$A_{11}B_{11}=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}, A_{21}B_{11}+B_{21}=\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix},$$

于是 $$AB=\left(\begin{array}{cc} 2 & 0 \\ 1 & -2 \\ \hline 1 & -2 \\ 1 & -1 \end{array}\right).$$

(4) 分块矩阵的转置．$s\times t$分块矩阵$A=(A_{kl})_{s\times t}$ 的转置矩阵A^{T}为$t\times s$分块矩阵，如果记$A^{\mathrm{T}}=(B_{lk})_{t\times s}$，则

$$B_{lk}=A_{kl}^{\mathrm{T}},\quad (l=1,2,\ldots,t;\ k=1,2,\ldots,s).$$

例如，若 $A=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$，则． $A^{\mathrm{T}}=\begin{pmatrix} A_{11}^{\mathrm{T}} & A_{21}^{\mathrm{T}} \\ A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} \\ A_{13}^{\mathrm{T}} & A_{23}^{\mathrm{T}} \end{pmatrix}.$

$$A=\left(\begin{array}{cc|cc|c} 2 & 1 & -1 & 2 & 6 \\ \hline -3 & 4 & 1 & 0 & 4 \\ 1 & 2 & 3 & -7 & 5 \end{array}\right)=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}, \quad A^{\mathrm{T}}=\left(\begin{array}{c|cc} 2 & -3 & 1 \\ 1 & 4 & 2 \\ \hline -1 & 1 & 3 \\ 2 & 0 & -7 \\ \hline 6 & 4 & 5 \end{array}\right)=\begin{pmatrix} A_{11}^{\mathrm{T}} & A_{21}^{\mathrm{T}} \\ A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} \\ A_{13}^{\mathrm{T}} & A_{23}^{\mathrm{T}} \end{pmatrix}.$$

(5) 分块对角矩阵．设n阶矩阵A的分块矩阵只有在对角线上有非零子块，其余子块都为零矩阵，且在对角线上的子块都是方阵，即

$$A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix},$$

其中 A_i (i=1,2,…,s) 都是方阵，则称A为分块对角矩阵，也称准对角矩阵.

例如：

$$A=\left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right)=\begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{pmatrix},$$

其中：

$$A_1=\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}, A_2=(2),$$

$$A_3=\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}.$$

矩阵按行分块和按列分块是两种常见的分块法（设$A=(a_{ij})_{m\times n}$）：

(1) 按行分块 $A=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}$，其中$\alpha_i=(a_{i1},a_{i2},\cdots,a_{in}),i=1,2,\cdots,m.$

(2) 按列分块 $A=(\beta_1,\beta_2,\cdots,\beta_n)$，其中$\beta_j=\begin{pmatrix}a_{1j}\\ a_{2j}\\ \vdots\\ a_{mj}\end{pmatrix},j=1,2,\cdots,n.$

矩阵乘法可以将矩阵行列分块后再进行相乘，以例2.3.1为例。

***A*按行*B*按列：**

$$AB=\begin{pmatrix}2&0&0&0\\0&1&0&0\\-1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0\\1&-2\\1&0\\-1&1\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}\begin{pmatrix}\beta_1&\beta_2\end{pmatrix}=\begin{pmatrix}\alpha_1\beta_1&\alpha_1\beta_2\\ \alpha_2\beta_1&\alpha_2\beta_2\\ \alpha_3\beta_1&\alpha_3\beta_2\\ \alpha_4\beta_1&\alpha_4\beta_2\end{pmatrix},$$

***A*按列*B*按行：**

$$AB=\begin{pmatrix}2&0&0&0\\0&1&0&0\\-1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1&0\\1&-2\\1&0\\-1&1\end{pmatrix}=\begin{pmatrix}\tilde\alpha_1&\tilde\alpha_2&\tilde\alpha_3&\tilde\alpha_4\end{pmatrix}\begin{pmatrix}\tilde\beta_1\\ \tilde\beta_2\\ \tilde\beta_3\\ \tilde\beta_4\end{pmatrix}=\tilde\alpha_1\tilde\beta_1+\tilde\alpha_2\tilde\beta_2+\tilde\alpha_3\tilde\beta_3+\tilde\alpha_4\tilde\beta_4\ .$$

例2.3.2 设 $A=(a_{ij})_{m\times s}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}, B=(b_{ij})_{s\times n}=\begin{pmatrix}\beta_1 & \beta_2 & \cdots & \beta_n\end{pmatrix}$,

则 $$AB=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}\begin{pmatrix}\beta_1 & \beta_2 & \cdots & \beta_n\end{pmatrix}=\begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n\\ \alpha_2\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_n\\ \vdots & \vdots & & \vdots\\ \alpha_m\beta_1 & \alpha_m\beta_2 & \cdots & \alpha_m\beta_n\end{pmatrix}=(c_{ij})_{m\times n},$$

其中 $$c_{ij}=\alpha_i\beta_j=\begin{pmatrix}a_{i1} & a_{i2} & \cdots & a_{is}\end{pmatrix}\begin{pmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{sj}\end{pmatrix}=\sum_{k=1}^{s}a_{ik}b_{kj},$$

又设对角矩阵

$$\Lambda_m = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad \text{其中的}\lambda_i\text{为数,}$$

则有：

$$\Lambda_m A_{m\times n} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \lambda_1\alpha_1 \\ \lambda_2\alpha_2 \\ \vdots \\ \lambda_m\alpha_m \end{pmatrix},$$

$$A_{m\times n}\Lambda_n = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1\beta_1 & \lambda_2\beta_2 & \cdots & \lambda_n\beta_n \end{pmatrix}.$$

使用分块思想的实例：

习题一 5(2)中行列式的计算：

$$D=\left|\begin{array}{cc|cc} x & a & b & c \\ a & x & c & b \\ \hline b & c & x & a \\ c & b & a & x \end{array}\right| = \begin{vmatrix} S & T \\ T & S \end{vmatrix} = \begin{vmatrix} S+T & T+S \\ T & S \end{vmatrix} = \begin{vmatrix} S+T & O \\ T & S-T \end{vmatrix} = |S+T|\cdot|S-T|$$

例2.2.7 中行列式的计算：

$$D_{2n}=\left|\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ \hline -1 & & 0 & b_{11} & \cdots & b_{1n} \\ & \ddots & & \vdots & & \vdots \\ 0 & & -1 & b_{n1} & \cdots & b_{nn} \end{array}\right| = \begin{vmatrix} A & O \\ -E & B \end{vmatrix} = \left|\begin{pmatrix} E & A \\ O & E \end{pmatrix}\begin{pmatrix} A & O \\ -E & B \end{pmatrix}\right| = \begin{vmatrix} O & AB \\ -E & B \end{vmatrix} = \cdots$$

分块矩阵运算规律的说明

(1) 分块矩阵的加法

$$
\left(\begin{array}{c|c|c}
 & & \\ \hline
 & \underset{A_{ij}}{\begin{matrix} a_{i_1j_1} & \cdots & a_{i_1j_2} \\ \vdots & & \vdots \\ a_{i_2j_1} & \cdots & a_{i_2j_2} \end{matrix}} & \\ \hline
 & &
\end{array}\right)
+
\left(\begin{array}{c|c|c}
 & & \\ \hline
 & \underset{B_{ij}}{\begin{matrix} b_{i_1j_1} & \cdots & b_{i_1j_2} \\ \vdots & & \vdots \\ b_{i_2j_1} & \cdots & b_{i_2j_2} \end{matrix}} & \\ \hline
 & &
\end{array}\right)
=
\left(\begin{array}{c|c|c}
 & & \\ \hline
 & \underset{A_{ij}+B_{ij}}{\begin{matrix} a_{i_1j_1}+b_{i_1j_1} & \cdots & a_{i_1j_2}+b_{i_1j_2} \\ \vdots & & \vdots \\ a_{i_2j_1}+b_{i_2j_1} & \cdots & a_{i_2j_2}+b_{i_2j_2} \end{matrix}} & \\ \hline
 & &
\end{array}\right)
$$

(2) 分块矩阵的数乘

$$
k\left(\begin{array}{c|c|c}
 & & \\ \hline
 & \underset{A_{ij}}{\begin{matrix} a_{i_1j_1} & \cdots & a_{i_1j_2} \\ \vdots & & \vdots \\ a_{i_2j_1} & \cdots & a_{i_2j_2} \end{matrix}} & \\ \hline
 & &
\end{array}\right)
=
\left(\begin{array}{c|c|c}
 & & \\ \hline
 & \underset{kA_{ij}}{\begin{matrix} ka_{i_1j_1} & \cdots & ka_{i_1j_2} \\ \vdots & & \vdots \\ ka_{i_2j_1} & \cdots & ka_{i_2j_2} \end{matrix}} & \\ \hline
 & &
\end{array}\right)
$$

(3) 分块矩阵的乘法

$$
\begin{pmatrix}
 & & & & \\
\underbrace{\begin{matrix} a_{k_1 1_1} & \cdots & a_{k_1 1_2} \\ \vdots & & \vdots \\ a_{k_2 1_1} & \cdots & a_{k_2 1_2} \end{matrix}}_{A_{k1}} & \cdots\ \cdots & \underbrace{\begin{matrix} a_{k_1 i_1} & \cdots & a_{k_1 i_2} \\ \vdots & & \vdots \\ a_{k_2 i_1} & \cdots & a_{k_2 i_2} \end{matrix}}_{A_{ki}} & \cdots\ \cdots & \underbrace{\begin{matrix} a_{k_1 s_1} & \cdots & a_{k_1 s_2} \\ \vdots & & \vdots \\ a_{k_2 s_1} & \cdots & a_{k_2 s_2} \end{matrix}}_{A_{ks}} \\
 & & & &
\end{pmatrix}
\begin{pmatrix}
 & \underbrace{\begin{matrix} b_{1_1 l_1} & \cdots & b_{1_1 l_2} \\ \vdots & & \vdots \\ b_{1_2 l_1} & \cdots & b_{1_2 l_2} \end{matrix}}_{B_{1l}} & \\
 & \vdots & \\
 & \underbrace{\begin{matrix} b_{i_1 l_1} & \cdots & b_{i_1 l_2} \\ \vdots & & \vdots \\ b_{i_2 l_1} & \cdots & b_{i_2 l_2} \end{matrix}}_{B_{il}} & \\
 & \vdots & \\
 & \underbrace{\begin{matrix} b_{s_1 l_1} & \cdots & b_{s_1 l_2} \\ \vdots & & \vdots \\ b_{s_2 l_1} & \cdots & b_{s_2 l_2} \end{matrix}}_{B_{sl}} &
\end{pmatrix}
$$

$$
=\begin{pmatrix}
 & & \\
 & \underbrace{\begin{matrix} \sum_{j=1}^{n} a_{k_1 j} b_{j l_1} & \cdots & \sum_{j=1}^{n} a_{k_1 j} b_{j l_2} \\ \vdots & & \vdots \\ \sum_{j=1}^{n} a_{k_2 j} b_{j l_1} & \cdots & \sum_{j=1}^{n} a_{k_2 j} b_{j l_2} \end{matrix}}_{C_{kl}} & \\
 & &
\end{pmatrix}
$$

相关子块的关系

$$A_{k1}B_{1l}+\cdots+A_{ki}B_{il}+\cdots+A_{ks}B_{sl}$$

$$=\underbrace{\begin{pmatrix} a_{k_1 1_1} & \cdots & a_{k_1 1_2} \\ \vdots & & \vdots \\ a_{k_2 1_1} & \cdots & a_{k_2 1_2} \end{pmatrix}}_{A_{k1}}\underbrace{\begin{pmatrix} b_{1_1 l_1} & \cdots & b_{1_1 l_2} \\ \vdots & & \vdots \\ b_{1_2 l_1} & \cdots & b_{1_2 l_2} \end{pmatrix}}_{B_{1l}}+\cdots+\underbrace{\begin{pmatrix} a_{k_1 i_1} & \cdots & a_{k_1 i_2} \\ \vdots & & \vdots \\ a_{k_2 i_1} & \cdots & a_{k_2 i_2} \end{pmatrix}}_{A_{ki}}\underbrace{\begin{pmatrix} b_{i_1 l_1} & \cdots & b_{i_1 l_2} \\ \vdots & & \vdots \\ b_{i_2 l_1} & \cdots & b_{i_2 l_2} \end{pmatrix}}_{B_{il}}+\cdots+\underbrace{\begin{pmatrix} a_{k_1 s_1} & \cdots & a_{k_1 s_2} \\ \vdots & & \vdots \\ a_{k_2 s_1} & \cdots & a_{k_2 s_2} \end{pmatrix}}_{A_{ks}}\underbrace{\begin{pmatrix} b_{s_1 l_1} & \cdots & b_{s_1 l_2} \\ \vdots & & \vdots \\ b_{s_2 l_1} & \cdots & b_{s_2 l_2} \end{pmatrix}}_{B_{sl}}$$

$$=\begin{pmatrix} \sum\limits_{j=1_1}^{1_2} a_{k_1 j}b_{jl_1} & \cdots & \sum\limits_{j=1_1}^{1_2} a_{k_1 j}b_{jl_2} \\ \vdots & & \vdots \\ \sum\limits_{j=1_1}^{1_2} a_{k_2 j}b_{jl_1} & \cdots & \sum\limits_{j=1_1}^{1_2} a_{k_2 j}b_{jl_2} \end{pmatrix}+\cdots+\begin{pmatrix} \sum\limits_{j=i_1}^{i_2} a_{k_1 j}b_{jl_1} & \cdots & \sum\limits_{j=i_1}^{i_2} a_{k_1 j}b_{jl_2} \\ \vdots & & \vdots \\ \sum\limits_{j=i_1}^{i_2} a_{k_2 j}b_{jl_1} & \cdots & \sum\limits_{j=i_1}^{i_2} a_{k_2 j}b_{jl_2} \end{pmatrix}+\cdots+\begin{pmatrix} \sum\limits_{j=s_1}^{s_2} a_{k_1 j}b_{jl_1} & \cdots & \sum\limits_{j=s_1}^{s_2} a_{k_1 j}b_{jl_2} \\ \vdots & & \vdots \\ \sum\limits_{j=s_1}^{s_2} a_{k_2 j}b_{jl_1} & \cdots & \sum\limits_{j=s_1}^{s_2} a_{k_2 j}b_{jl_2} \end{pmatrix}$$

$$=\begin{pmatrix} \sum\limits_{j=1}^{n} a_{k_1 j}b_{jl_1} & \cdots & \sum\limits_{j=1}^{n} a_{k_1 j}b_{jl_2} \\ \vdots & & \vdots \\ \sum\limits_{j=1}^{n} a_{k_2 j}b_{jl_1} & \cdots & \sum\limits_{j=1}^{n} a_{k_2 j}b_{jl_2} \end{pmatrix}=\underbrace{\begin{pmatrix} c_{k_1 l_1} & \cdots & c_{k_1 l_2} \\ \vdots & & \vdots \\ c_{k_2 l_1} & \cdots & c_{k_2 l_2} \end{pmatrix}}_{C_{kl}}=C_{kl}\quad .$$

(4) 分块矩阵的转置

$$
\left(\begin{array}{c|c|c}
\quad & & \quad \\ \hline
 & \begin{matrix} a_{k_1l_1} & \cdots & a_{k_1l_2} \\ \vdots & A_{kl} & \vdots \\ a_{k_2l_1} & \cdots & a_{k_2l_2} \end{matrix} & \\ \hline
 & &
\end{array}\right)^T
=
\left(\begin{array}{c|c|c}
\quad & & \quad \\ \hline
 & \begin{matrix} a_{k_1l_1} & \cdots & a_{k_2l_1} \\ \vdots & B_{lk} & \vdots \\ a_{k_1l_2} & \cdots & a_{k_2l_2} \end{matrix} & \\ \hline
 & &
\end{array}\right)
$$

$$B_{lk} = A_{kl}^{\mathrm{T}}$$