用行列式性质简化
行列式计算

将一行或一列化为大量的0，然后展开计算

计算中行列式变换的表示：

- 第i行(列)加上j行(列)的k倍：r_i+kr_j（c_i+kc_j）
- 交换i行与j行(列)：$r_i \leftrightarrow r_j$（$c_i \leftrightarrow c_j$）
- 第i行(列)提出公因子k (k可以是0)：$\frac{1}{k}r_i$ ($\frac{1}{k}c_i$) 或 $r_i \div k$ ($c_i \div k$)

- 行用r (row)
- 列用c (column)

书上P_9例1.2.1

解：

$$\begin{vmatrix} -2 & 2 & 3 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 3 & 4 & -2 \\ -1 & 1 & 2 & 3 \end{vmatrix} \overset{c_2+c_4}{=} \begin{vmatrix} -2 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 4 & -2 \\ -1 & 4 & 2 & 3 \end{vmatrix} \overset{r_2\text{展开}}{=} 1\times \begin{vmatrix} -2 & 2 & 3 \\ 1 & 1 & 4 \\ -1 & 4 & 2 \end{vmatrix} \underset{r_3+r_2}{\overset{r_1+2r_2}{=}} \begin{vmatrix} 0 & 4 & 11 \\ 1 & 1 & 4 \\ 0 & 5 & 6 \end{vmatrix} \overset{c_1\text{展开}}{=} -1\times \begin{vmatrix} 4 & 11 \\ 5 & 6 \end{vmatrix} \overset{c_2-c_1}{=} -\begin{vmatrix} 4 & 7 \\ 5 & 1 \end{vmatrix} = -(4-35) = 31$$

1.2.3 n阶行列式的计算

例1.2.3 计算

$$A=\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{vmatrix}.$$

解:

$$A \overset{r_4-r_3}{=} \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{vmatrix} \overset{r_3-r_2}{=} \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

书上解:

$$A \overset{r_1 \leftrightarrow r_2}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{vmatrix} \overset{r_3-2r_1}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \\ 3 & 4 & 5 & 6 \end{vmatrix} = 0.$$

例1.2.4 当$x\neq a_i$ $(i=1,2,\ldots,n)$时，计算行列式

$$A=\begin{vmatrix} a_1 & x & x & \cdots & x \\ x & a_2 & x & \cdots & x \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & a_n \end{vmatrix}.$$

解 从其余各行中减去第一行得

$$A=\begin{vmatrix} a_1 & x & x & \cdots & x \\ x-a_1 & a_2-x & 0 & \cdots & 0 \\ x-a_1 & 0 & a_3-x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x-a_1 & 0 & 0 & \cdots & a_n-x \end{vmatrix} \overset{\substack{c_i\div(a_i-x)\\ i=1,2,\cdots,n}}{=} \prod_{i=1}^{n}(a_i-x)\begin{vmatrix} \frac{a_1}{a_1-x} & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

由于 $a_1/(a_1-x)=1+x/(a_1-x)$，所以把后面所有各列加到第一列得到

$$A=\prod_{i=1}^{n}(a_i-x)\begin{vmatrix} 1+\sum_{i=1}^{n}(\frac{x}{a_i-x}) & \frac{x}{a_2-x} & \frac{x}{a_3-x} & \cdots & \frac{x}{a_n-x} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = (1+\sum_{i=1}^{n}(\frac{x}{a_i-x}))\prod_{i=1}^{n}(a_i-x).$$

例1.2.5 证明
$$\begin{vmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & b_1 & b_2 & b_3 \\ -a_2 & -b_1 & 0 & c_1 & c_2 \\ -a_3 & -b_2 & -c_1 & 0 & d \\ -a_4 & -b_3 & -c_2 & -d & 0 \end{vmatrix} = 0.$$

证明 记行列式之值为D，则

$$D \overset{\text{转置}}{=} \begin{vmatrix} 0 & -a_1 & -a_2 & -a_3 & -a_4 \\ a_1 & 0 & -b_1 & -b_2 & -b_3 \\ a_2 & b_1 & 0 & -c_1 & -c_2 \\ a_3 & b_2 & c_1 & 0 & -d \\ a_4 & b_3 & c_2 & d & 0 \end{vmatrix} \overset{\substack{r_i \div (-1) \\ i=1,2,\cdots,5}}{=} (-1)^5 \begin{vmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & b_1 & b_2 & b_3 \\ -a_2 & -b_1 & 0 & c_1 & c_2 \\ -a_3 & -b_2 & -c_1 & 0 & d \\ -a_4 & -b_3 & -c_2 & -d & 0 \end{vmatrix} = (-1)^5 D = -D.$$

由此得 $D=0$.

反对称行列式：行列式满足 $a_{ij}=-a_{ji}$ $(i, j=1,2,\ldots,n)$

$$\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \\ -3 & 2 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 & 0 & -1 \\ -2 & 0 & 3 & 4 \\ 0 & -3 & 0 & -2 \\ 1 & -4 & 2 & 0 \end{vmatrix}$$

奇数阶反对称行列式值为0

*** 例1.2.6和例1.2.7的证明放在行列式性质1、2证明部分**

补充例1A 计算

$$A=\begin{vmatrix}1&3&2&-1\\-2&-1&3&1\\2&3&-2&4\\3&-1&3&6\end{vmatrix},\quad B=\begin{vmatrix}5&2&8&3\\3&-7&0&5\\-4&6&2&-3\\-2&9&7&4\end{vmatrix}.$$

解

$$A\overset{\substack{r_2+2r_1\\r_3-2r_1\\r_4-3r_1}}{=}\begin{vmatrix}1&3&2&-1\\0&5&7&-1\\0&-3&-6&6\\0&-10&-3&9\end{vmatrix}\overset{c_1\text{展开}}{=}\begin{vmatrix}5&7&-1\\-3&-6&6\\-10&-3&9\end{vmatrix}\overset{\substack{r_1\leftrightarrow r_2\\r_1\div(-3)}}{=}3\begin{vmatrix}1&2&-2\\5&7&-1\\-10&-3&9\end{vmatrix}\overset{\substack{r_2-5r_1\\r_3+10r_1}}{=}3\begin{vmatrix}1&2&-2\\0&-3&9\\0&17&-11\end{vmatrix}$$

$$\overset{c_1\text{展开}}{=}3\begin{vmatrix}-3&9\\17&-11\end{vmatrix}\overset{r_1\div(-3)}{=}-9\begin{vmatrix}1&-3\\17&-11\end{vmatrix}=-360.$$

$$B=\begin{vmatrix}5&2&8&3\\3&-7&0&5\\-4&6&2&-3\\-2&9&7&4\end{vmatrix}\overset{\substack{r_1+r_3\\r_2+r_3\\r_3-2r_4}}{=}\begin{vmatrix}1&8&10&0\\-1&-1&2&2\\0&-12&-12&-11\\-2&9&7&4\end{vmatrix}\overset{\substack{r_2+r_1\\r_4+2r_1}}{=}\begin{vmatrix}1&8&10&0\\0&7&12&2\\0&-12&-12&-11\\0&25&27&4\end{vmatrix}\overset{c_1\text{展开}}{=}\begin{vmatrix}7&12&2\\-12&-12&-11\\25&27&4\end{vmatrix}$$

$$\overset{\substack{r_1+r_2\\r_3+2r_2}}{=}\begin{vmatrix}-5&0&-9\\-12&-12&-11\\1&3&-18\end{vmatrix}\overset{r_2+4r_3}{=}\begin{vmatrix}-5&0&-9\\-8&0&-83\\1&3&-18\end{vmatrix}\overset{c_2\text{展开}}{=}-3\begin{vmatrix}-5&-9\\-8&-83\end{vmatrix}=-3(415-72)=-1029.$$

补充例1B 证明

$$D_n=\begin{vmatrix} a+b & a & 0 & \cdots & & \\ b & a+b & a & \cdots & & \\ 0 & b & a+b & \cdots & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & b & a+b & a \\ 0 & 0 & 0 & \cdots & b & a+b \end{vmatrix}=\frac{a^{n+1}-b^{n+1}}{a-b}\qquad (a\neq b).$$

证法一：

$$(a-b)D_n\overset{(a-b)\times c_1}{=}\begin{vmatrix} a^2-b^2 & a & 0 & \cdots & 0 \\ ab-b^2 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=\begin{vmatrix} a^2 & a & 0 & \cdots & 0 \\ ab & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}-\begin{vmatrix} b^2 & a & 0 & \cdots & 0 \\ b^2 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=D_a-D_b,$$

则

$$D_a=a\begin{vmatrix} a & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}\overset{c_2-c_1}{=}a\begin{vmatrix} a & 0 & 0 & \cdots & 0 \\ b & a & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}\overset{\substack{c_j-c_{j-1}\\ j=3,4,\dots,n}}{=}a\begin{vmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & a & 0 & \cdots & 0 \\ 0 & 1 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \end{vmatrix}=a^{n+1},$$

$$D_b=b^2\begin{vmatrix} 1 & a & 0 & \cdots & 0 \\ 1 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}\overset{\substack{c_j-a\times c_{j-1},\,c_j\div b\\ j=2,3,\dots,n}}{=}b^{n+1}\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}=b^{n+1}.$$

于是 $D_n=\dfrac{D_a-D_b}{a-b}=\dfrac{a^{n+1}-b^{n+1}}{a-b}.$

证法二：

$$D_n=\begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=(a+b)D_{n-1}-a\begin{vmatrix} b & a & 0 & \cdots & 0 \\ 0 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=(a+b)D_{n-1}-abD_{n-2},$$

故有 $D_n-aD_{n-1}=b(D_{n-1}-aD_{n-2})=\cdots=b^{n-2}(D_2-aD_1)=b^n.$

于是 $D_n=aD_{n-1}+b^n=a^2D_{n-2}+ab^{n-1}+b^n=\cdots=a^n+a^{n-1}b+\cdots+ab^{n-1}+b^n=\dfrac{a^{n+1}-b^{n+1}}{a-b}.$

证法三：

数学归纳法：当n=1、2时，结论显然成立

假设$n<m$时结论成立，当$n=m$时

$$D_m=\begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=(a+b)D_{m-1}-a\begin{vmatrix} b & a & 0 & \cdots & 0 \\ 0 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}=(a+b)D_{m-1}-abD_{m-2},$$

由归纳假设，有 $D_m=(a+b)D_{m-1}-abD_{m-2}=(a+b)\dfrac{a^m-b^m}{a-b}-ab\dfrac{a^{m-1}-b^{m-1}}{a-b}=\dfrac{a^{m+1}-b^{m+1}}{a-b}.$

故结论成立.

证法四：

$$D_n=\begin{vmatrix} a+b & a & 0 & \cdots & 0 \\ b+0 & a+b & a & \cdots & 0 \\ 0+0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0+0 & 0 & 0 & \cdots & a+b \end{vmatrix}=\begin{vmatrix} a & a & 0 & \cdots & 0 \\ b & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}+\begin{vmatrix} b & a & 0 & \cdots & 0 \\ 0 & a+b & a & \cdots & 0 \\ 0 & b & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

$$\overset{\substack{\text{对第一个行列式变换}\\ c_j-c_{j-1},\, j=2,3,\ldots,n}}{=}\begin{vmatrix} a & 0 & 0 & \cdots & 0 \\ b & a & 0 & \cdots & 0 \\ 0 & b & a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a \end{vmatrix}+bD_{n-1}=a^n+bD_{n-1}.$$

由于$D_n=D_n'$，利用上式得 $D_n'=b^n+aD_{n-1}'$，故有$D_n=b^n+aD_{n-1}$，于是有

$$\begin{cases} D_n=a^n+bD_{n-1}, & (1) \\ D_n=b^n+aD_{n-1}, & (2) \end{cases}$$

(1)×a-(2)×b，得 $(a-b)D_n=a^{n+1}-b^{n+1}$，即 $D_n=\dfrac{a^{n+1}-b^{n+1}}{a-b}.$

补充例1C 已知 $f(x)=\begin{vmatrix}3&2&x\\-9&4&x^2\\27&8&x^3\end{vmatrix}$，求 $f'(x)=0$ 的解.

解 行列式按第3列展开可得 $f(x)$ 是一个3次多项式(3次项非零)，因为

$$f(0)=\begin{vmatrix}3&2&0\\-9&4&0\\27&8&0\end{vmatrix}=0,\ f(-3)=\begin{vmatrix}3&2&-3\\-9&4&9\\27&8&-27\end{vmatrix}=0,\ f(2)=\begin{vmatrix}3&2&2\\-9&4&4\\27&8&8\end{vmatrix}=0.$$

故三次方程 $f(x)=0$ 有3个根0,-3,2，于是 $f(x)=a(x(x+3)(x-2))=a(x^3+x^2-6x)$,

则 $f'(x)=a(3x^2+2x-6)=0$，解得 $x=\dfrac{-1\pm\sqrt{19}}{3}$.

解法二

$$f(x)=\begin{vmatrix}3&2&x\\-9&4&x^2\\27&8&x^3\end{vmatrix}\overset{\substack{c_1\div(-3)\\c_2\div 2\\c_3\div x}}{=}(-3)\times 2x\begin{vmatrix}1&1&1\\-3&2&x\\9&4&x^2\end{vmatrix}\overset{\text{范德蒙}}{=}-6x\cdot 5(x+3)(x-2)=-30(x^3+x^2-6x).$$

则 $f'(x)=-30(3x^2+2x-6)=0$，解得 $x=\dfrac{-1\pm\sqrt{19}}{3}$.

补充例1D 计算下列行列式

$$(1)\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix},\quad (2)\begin{vmatrix} 0 & a_1 & b_1 & 0 \\ a_2 & 0 & 0 & b_2 \\ a_3 & 0 & 0 & b_3 \\ 0 & a_4 & b_4 & 0 \end{vmatrix},\quad (3)\begin{vmatrix} x & -2 & y & 3 \\ -2 & x & 3 & y \\ y & 3 & x & -2 \\ 3 & -y & -2 & x \end{vmatrix}.$$

解

$$(1)\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} \overset{c_1+c_2+c_3+c_4}{=} \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} \overset{r_i-r_1,i=2,3,4}{=} \begin{vmatrix} x+3a & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix} = (x+3a)(x-a)^3.$$

解法二

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} \overset{c_1+c_2+c_3+c_4}{=} \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} \overset{c_1\div(x+3a)}{=} (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$$\overset{c_j-a\times c_1,j=2,3,4}{=} (x+3a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x-a & 0 & 0 \\ 1 & 0 & x-a & 0 \\ 1 & 0 & 0 & x-a \end{vmatrix} = (x+3a)(x-a)^3.$$

(2) $$\begin{vmatrix} 0 & a_1 & b_1 & 0 \\ a_2 & 0 & 0 & b_2 \\ a_3 & 0 & 0 & b_3 \\ 0 & a_4 & b_4 & 0 \end{vmatrix} \overset{c_1 \leftrightarrow c_3}{=} \begin{vmatrix} b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & a_3 & b_3 \\ b_4 & a_4 & 0 & 0 \end{vmatrix} \overset{r_1 \leftrightarrow r_3}{=} \left|\begin{array}{cc:cc} b_1 & a_1 & 0 & 0 \\ b_4 & a_4 & 0 & 0 \\ \hdashline 0 & 0 & a_3 & b_3 \\ 0 & 0 & a_2 & b_2 \end{array}\right|$$

$$= \begin{vmatrix} b_1 & a_1 \\ b_4 & a_4 \end{vmatrix} \cdot \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix} = (a_4 b_1 - a_1 b_4)(a_3 b_2 - a_2 b_3).$$

(3) $$\left|\begin{array}{cc:cc} x & -2 & y & 3 \\ -2 & x & 3 & -y \\ \hdashline y & 3 & x & -2 \\ 3 & -y & -2 & x \end{array}\right| \overset{\substack{r_1 + r_3 \\ r_2 + r_4}}{=} \left|\begin{array}{cc:cc} x+y & 1 & x+y & 1 \\ 1 & x-y & 1 & x-y \\ \hdashline y & 3 & x & -2 \\ 3 & -y & -2 & x \end{array}\right| \overset{\substack{c_3 - c_1 \\ c_4 - c_2}}{=} \left|\begin{array}{cc:cc} x+y & 1 & 0 & 0 \\ 1 & x-y & 0 & 0 \\ \hdashline y & 3 & x-y & -5 \\ 3 & -y & -5 & x+y \end{array}\right|$$

$$= \begin{vmatrix} x+y & 1 \\ 1 & x-y \end{vmatrix} \cdot \begin{vmatrix} x-y & -5 \\ -5 & x+y \end{vmatrix} = (x^2 - y^2 - 1)(x^2 - y^2 - 25).$$