

## Chapter 4.1

# Specific Trees

## 4.1 Binary Search Trees

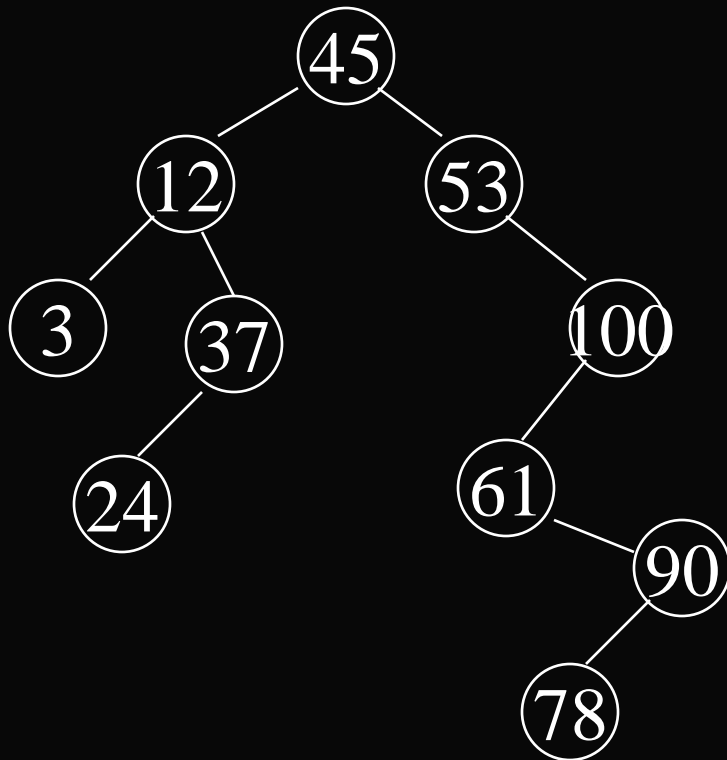
**1. Definition: A binary search tree is a binary tree that may be empty.**

**A nonempty binary search tree satisfies the following properties:**

- 1) Every element has a key and no two elements have the same key; therefore, all keys are distinct.**
- 2) The keys(if any)in the left subtree of the root are smaller than the key in the root.**
- 3) The keys(if any)in the right subtree of the root are larger than the key in the root.**
- 4) The left and right subtrees of the root are also binary search trees.**

# 4.1 Binary Search Trees

**Example:**



left	element	right
------	---------	-------

## 4.1 Binary Search Trees

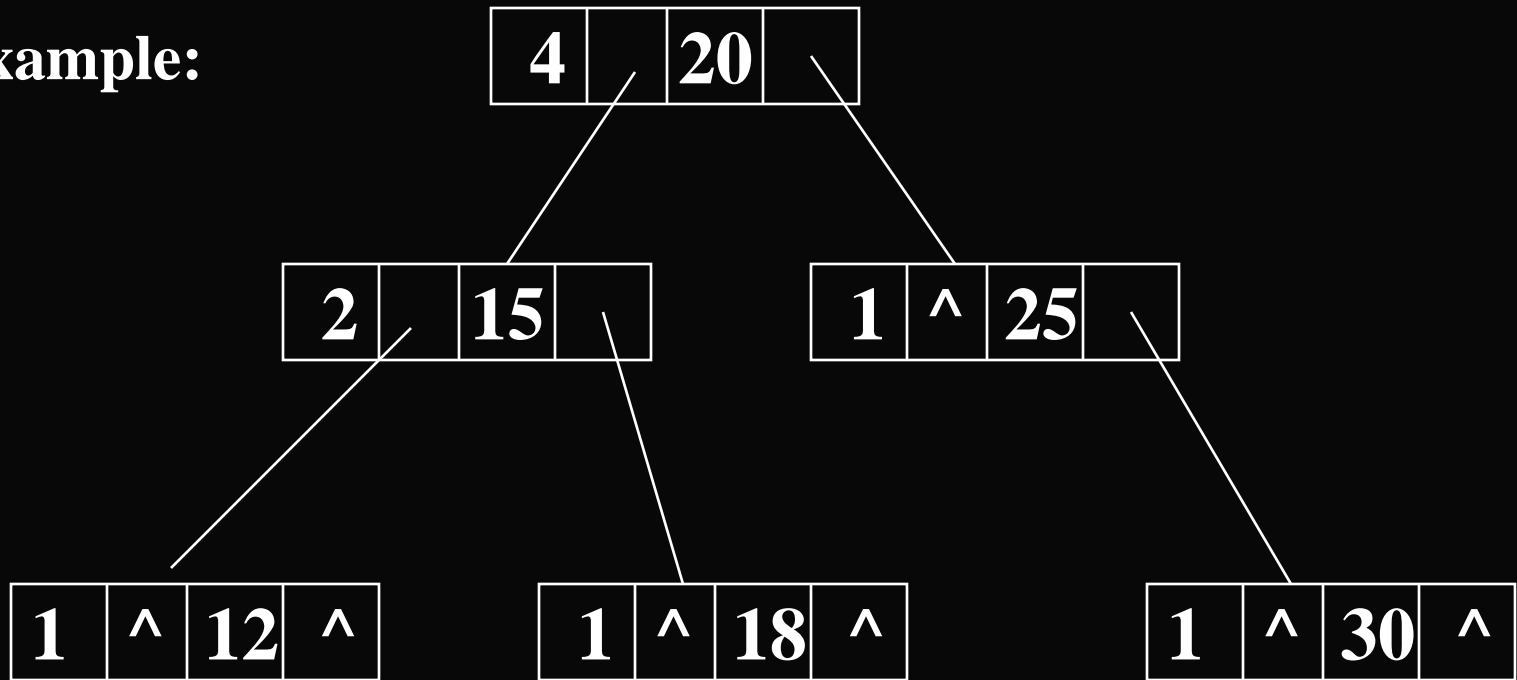
- **An indexed binary search tree is derived from an ordinary binary search tree by adding the field `leftSize` to each tree node.**
- **Value in `Leftsize` field=number of the elements in the node's left subtree +1**

<b>leftSize</b>	<b>left</b>	<b>element</b>	<b>right</b>
-----------------	-------------	----------------	--------------

# 4.1 Binary Search Trees

## Indexed binary search tree

Example:



# 4.1 Binary Search Trees

## 2. BinaryNode class

```
class BinaryNode
{ BinaryNode( Comparable theElement )
    { this( theElement, null, null ); }
  BinaryNode( Comparable theElement, BinaryNode lt,
              BinaryNode rt )
    { element = theElement; left = lt; right = rt; }

  Comparable element;
  BinaryNode left;
  BinaryNode right;
}
```

# 4.1 Binary Search Trees

## 3. Binary search tree class skeleton

```
public class BinarySearchTree  
{  public BinarySearchTree( ) { root = null; }  
    public void makeEmpty( ) { root = null; }  
    public boolean isEmpty( ) { return root == null; }  
  
    public Comparable find( Comparable x )  
        { return elementAt( find( x, root ) ); }  
    public Comparable findMin( )  
        { return elementAt( findMin( root ) ); }  
    public Comparable findMax( )  
        { return elementAt( findMax( root ) ); }  
}
```

# 4.1 Binary Search Trees

```
public void insert( Comparable x )
```

```
{ root = insert( x, root ); }
```

```
public void remove( Comparable x )
```

```
{root = remove( x, root ); }
```

```
public void printTree( )
```

```
private BinaryNode root;
```

```
private Comparable elementAt( BinaryNode t )
```

```
{ return t == null ? Null : t.element; }
```

```
private BinaryNode find( Comparable x, BinaryNode t )
```

```
private BinaryNode findMin( BinaryNode t )
```

```
private BinaryNode findMax( BinaryNode t )
```



# 4.1 Binary Search Trees

```
private BinaryNode insert( Comparable x, BinaryNode t )  
private BinaryNode remove( Comparable x, BinaryNode t )  
private BinaryNode removeMin( BinaryNode t )  
private void printTree( BinaryNode t )
```

```
}
```

# 4.1 Binary Search Trees

## 4. Find operation for binary search trees

```
private BinaryNode find( Comparable x, BinaryNode t )
{ if( t == null )
    return null;
  if( x.compareTo( t.element ) < 0 )
    return find( x, t.left );
  else if( x.compareTo( t.element ) > 0 )
    return find( x, t.right );
  else
    return t; //Match
}
```

# 4.1 Binary Search Trees

## 5. Recursive implementation of findMin for binary search trees

```
private BinaryNode findMin( BinaryNode t )
{
    if( t == null )
        return null;
    else if( t.left == null )
        return t;
    return findMin( t.left );
}
```

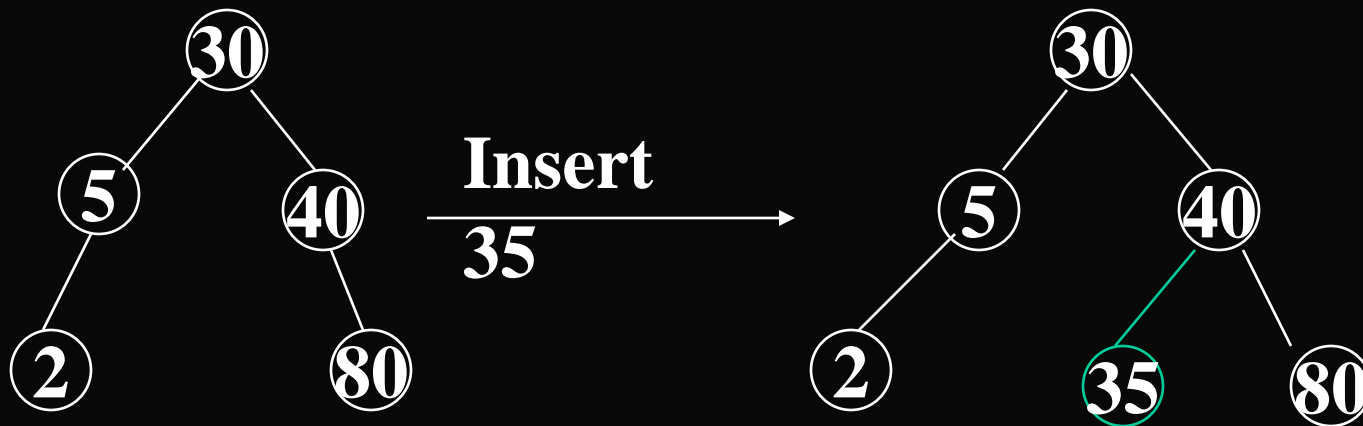
## 6. Nonrecursive implementation of findMax for binary search trees

```
private BinaryNode findMax( BinaryNode t )
{
    if( t != null )
        while( t.right != null )
            t = t.right;

    return t;
}
```

# 4.1 Binary Search Trees

## 7. Insertion into a binary search tree



## 4.1 Binary Search Trees

```
private BinaryNode insert( Comparable x, BinaryNode t )
{  if( t == null )
    t = new BinaryNode( x, null, null );
  else if( x.compareTo( t.element ) < 0 )
    t.left = insert( x, t.left );
  else if( x.compareTo( t.element ) > 0 )
    t.right = insert( x, t.right );
  else
    ; //duplicate; do nothing
  return t;
}
```

# 4.1 Binary Search Trees

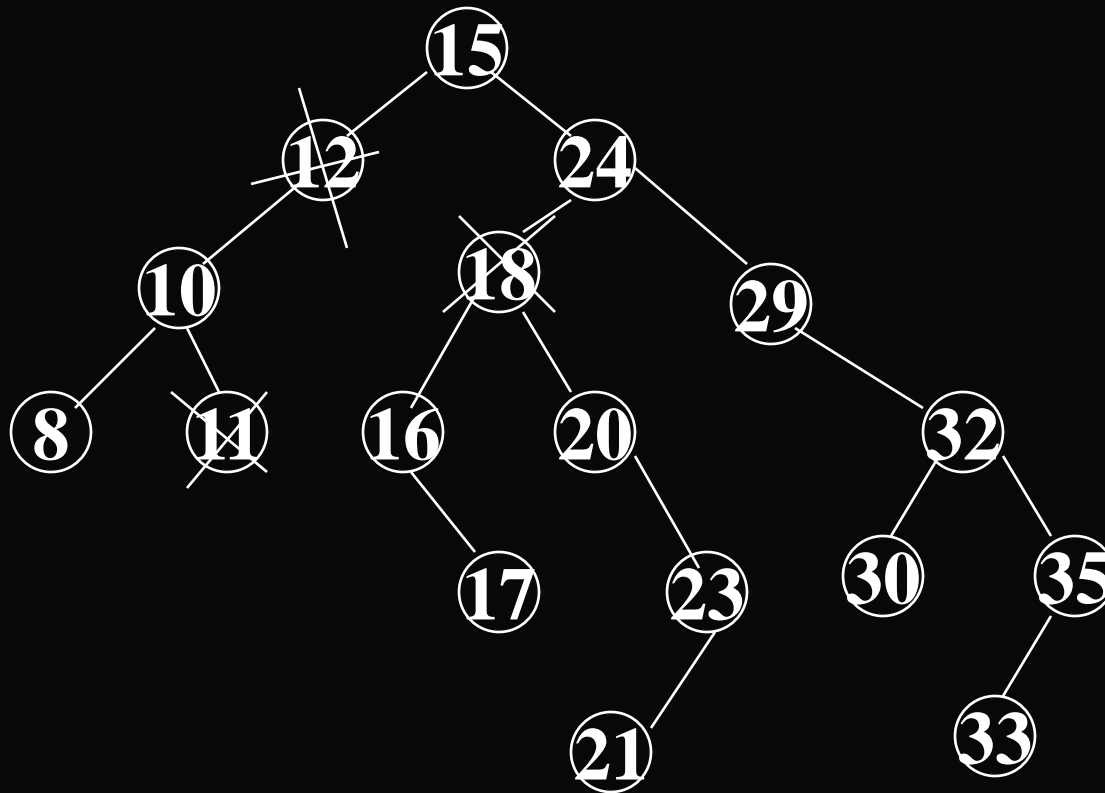
## Deletion

**It is necessary to adjust the binary search tree after deleting an element, so that the tree remained is still a binary search tree. There is three cases for deleting node  $p$  :**

- $P$  is a leaf**
- $P$  has exactly one nonempty subtree**
- $P$  has exactly two nonempty subtrees**

# 4.1 Binary Search Trees

**Example:**



# 4.1 Binary Search Trees

- **case 1: delete a leaf**
- **case 2: deleted node has exactly one nonempty subtree**
- **case 3: deleted node has exactly two nonempty subtree**

**We can replace the element to be deleted with either the largest element in the left subtree or the smallest element in the right subtree.**

**Next step is to delete the largest element in the left subtree or smallest element in the right subtree.**



# 4.1 Binary Search Trees

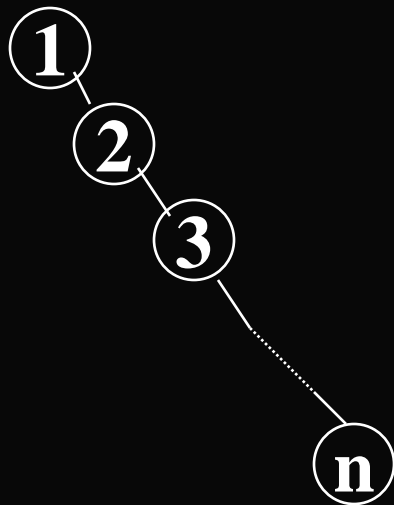
## 8. Deletion routine for binary search trees

```
private BinaryNode remove( Comparable x, BinaryNode t )
{   if( t == null )
        return t;
    if( x.compareTo( t.element ) < 0 )
        t.left = remove( x, t.left );
    else if( x.compareTo( t.element ) > 0 )
        t.right = remove( x, t.right );
    else if( t.left != null && t.right != null )
        {   t.element = findMin( t.right ).element;
            t.right = remove( t.element , t.right );
        }
    else
        t = ( t.left != null ) ? t.left : t.right;
    return t;
}
```

# 4.1 Binary Search Trees

## Height of a binary search tree

- The height of a binary search tree has influence directly on the time complexity of operations like searching, insertion and deletion.
- Worst case: add an ordered elements  $\{1,2,3\dots n\}$  into an empty binary search tree.



$O(h)$

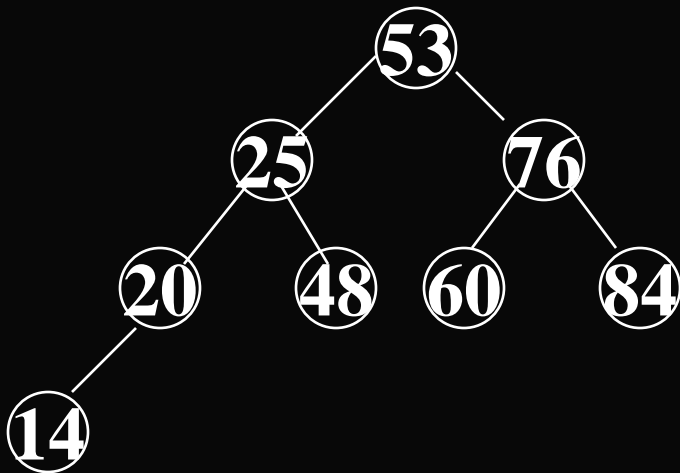
**Worst Case:**  
 $O(n)$

# 4.1 Binary Search Trees

- **Best case and average height:**

**$O(\log_2 n)$**

**Example: {53, 25, 76, 20, 48, 14, 60, 84}**



## 4.2 AVL Tree

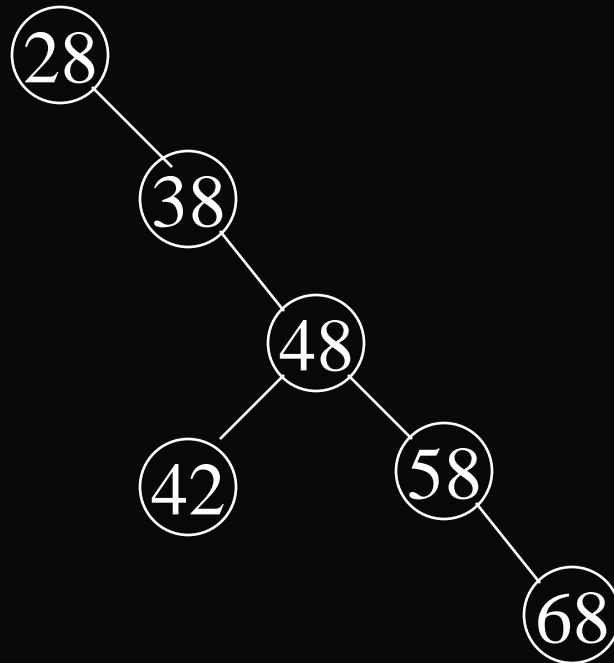
(自平衡的二叉搜索树)

**The concept of AVL tree was introduced by Russian scientists G.M.Adel'son-Vel'sky and E.M.Landis in 1962.**

**1. purpose:**

**the AVL tree was introduced to increase the efficiency of searching a binary search tree, and to decrease the average search length.**

## 4.2 AVL Tree



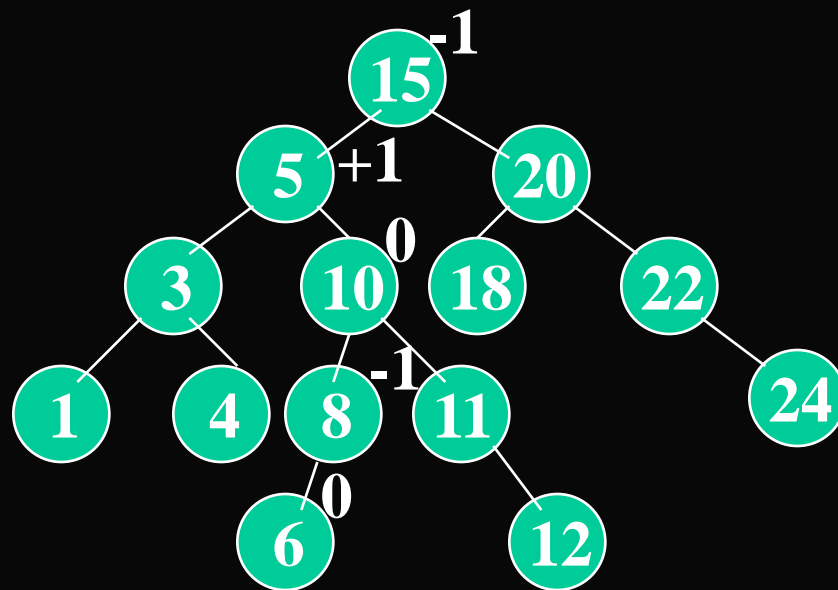
## 4.2 AVL Tree

**2 Definition of an AVL tree:**

**(1) is a binary search tree**

**(2) Every node satisfies**

**$|h_L - h_R| \leq 1$  where  $h_L$  and  $h_R$  are the heights of  $T_L$  (left subtree) and  $T_R$  (right subtree), respectively.**



## 4.2 AVL Tree

- **Height of an tree:**  
the longest path from the root to each leaf node
- **Balance factor  $bf(x)$  of a node  $x$  :**  
height of right subtree of  $x$  – height of left subtree of  $x$

Each node: 

Left	data	Right	balance(height)
------	------	-------	-----------------



## 4.2 AVL Tree

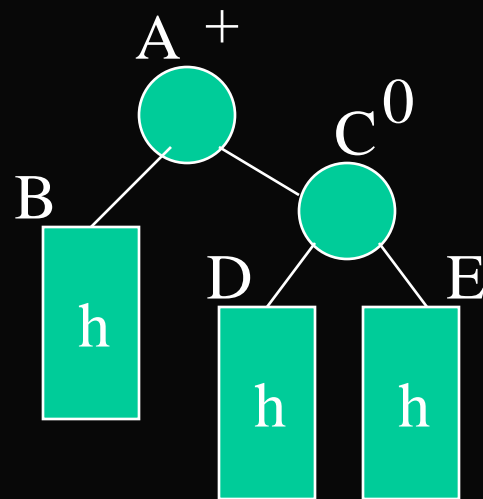
**The height of an AVL tree with  $n$  elements is  $O(\log_2 n)$ , so an  $n$ -element AVL search tree can be searched in  $O(\log_2 n)$  time.**

## 4.2 AVL Tree

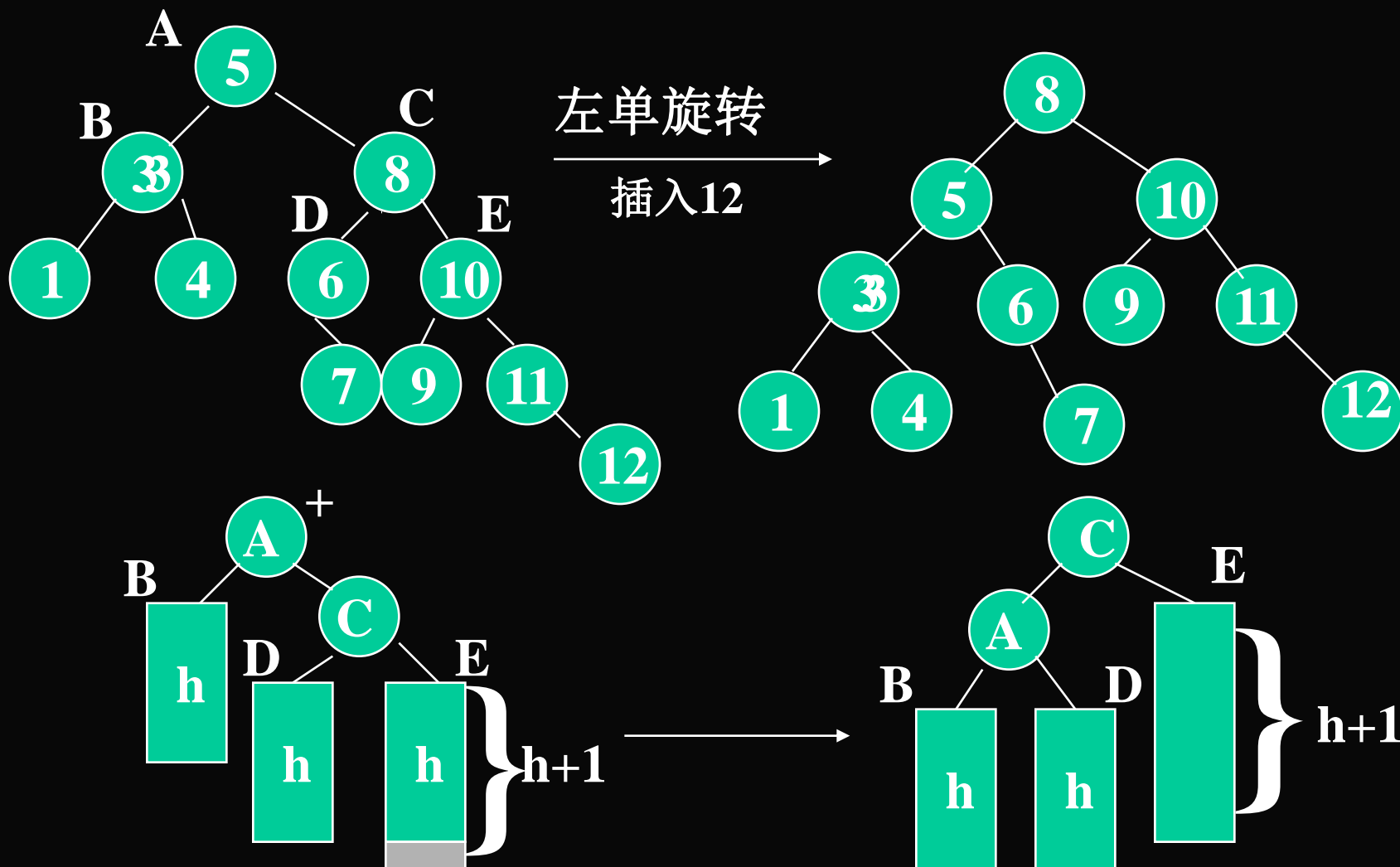
### **3.inserting into an AVL tree**

# AVL树

- 插入

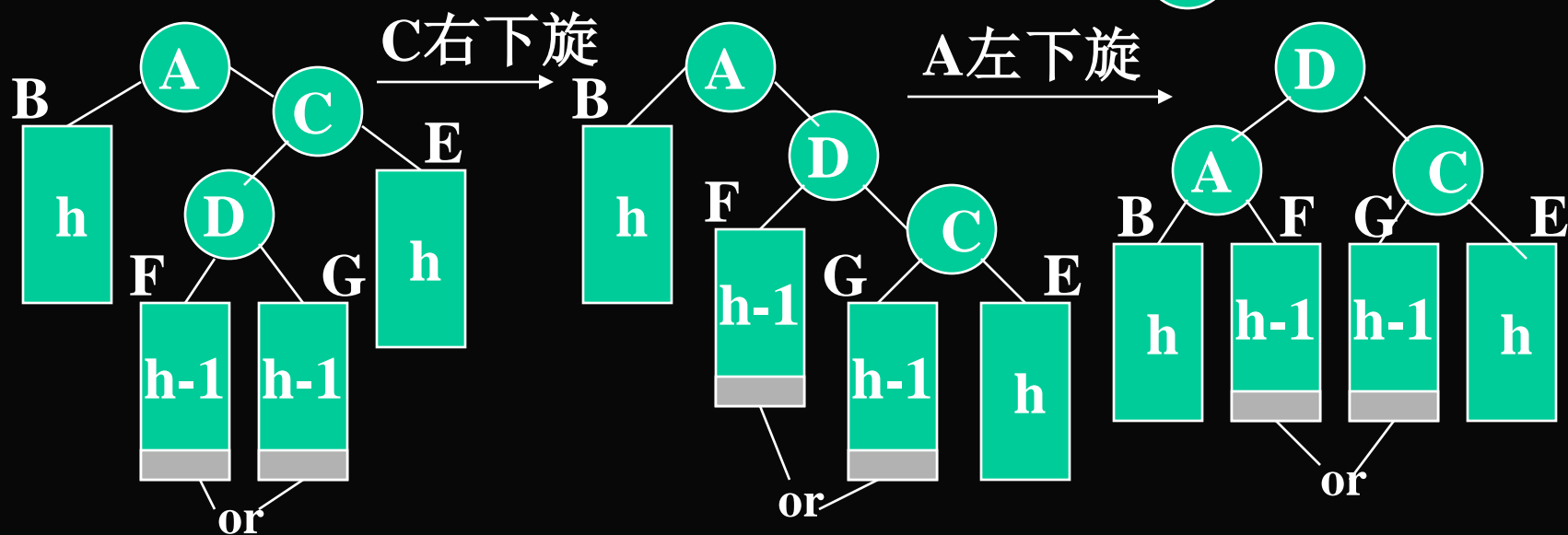
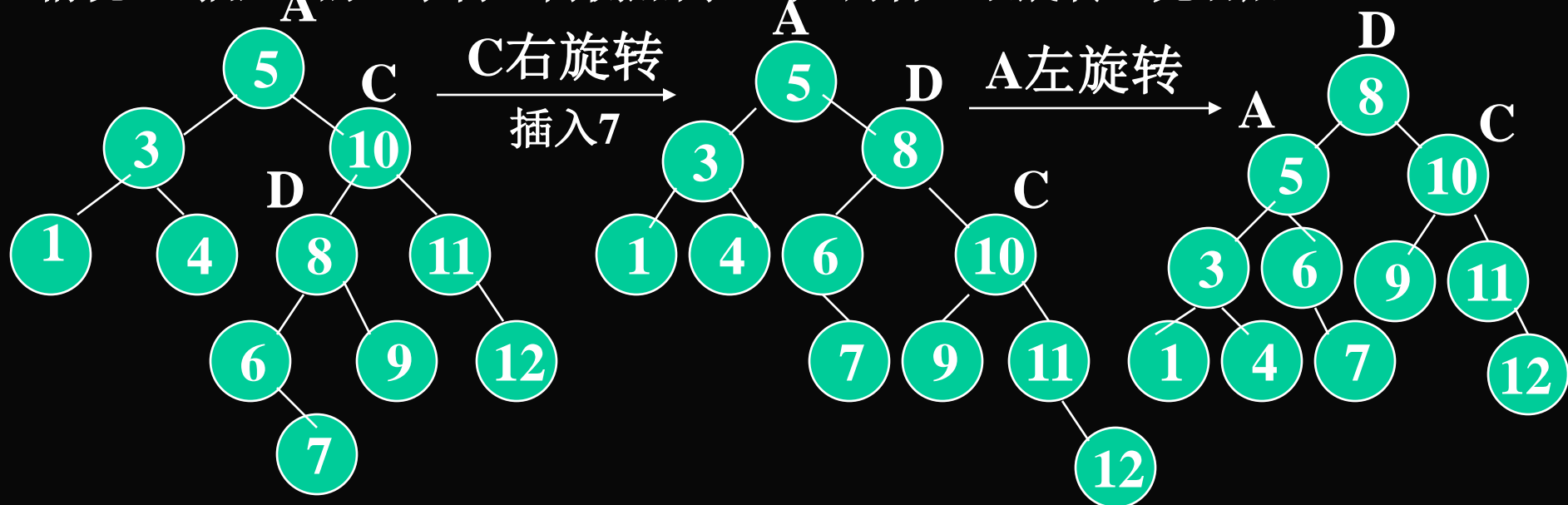


情况1:插入C的右子树\_\_外侧加高(对A而言) 单旋转(左)



调整后:树高不变.原 $h+2$ ,插入后 $h+3$ ,调整后 $h+2$ , $\therefore$ 不平衡不会向外传递.

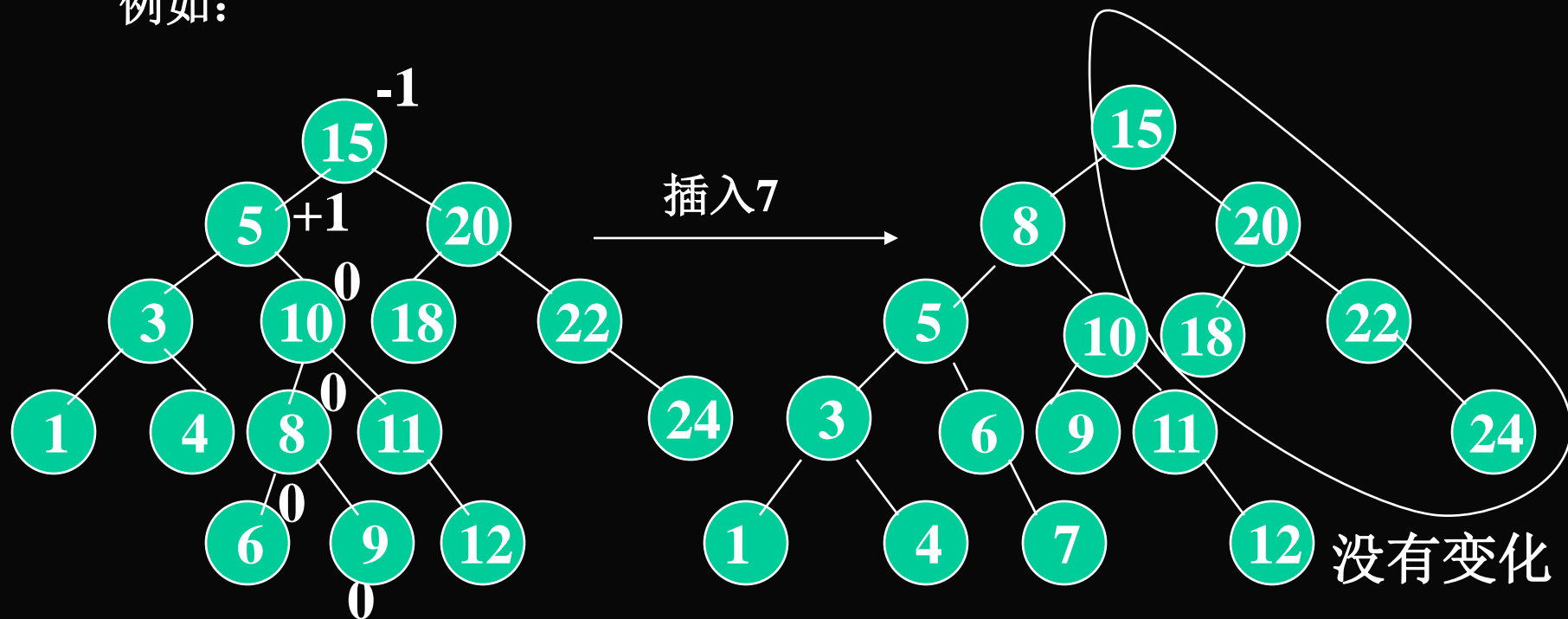
情况2: 插入C的左子树—内侧加高 (对A而言) 双旋转 (先右后左)



调整后：树高不变。原 $h+2$ ,插入后 $h+3$ ,调整后 $h+2$ .

小结一下：以A为根的子树，调整前后，其高度不变， $\therefore$ 调整不会影响到以A为根的子树以外的结点。

例如：



\*调整只要在包含插入结点的最小不平衡子树中进行,即从根到达插入结点的路径上,离插入结点最近的,并且平衡系数 $\neq 0$ 的结点为根的子树。

也可这样讲：插入一个新结点后，需要从插入位置沿通向根的路径回溯，检查各结点左右子树的高度差，如果发现某点高度不平衡则停止回溯。

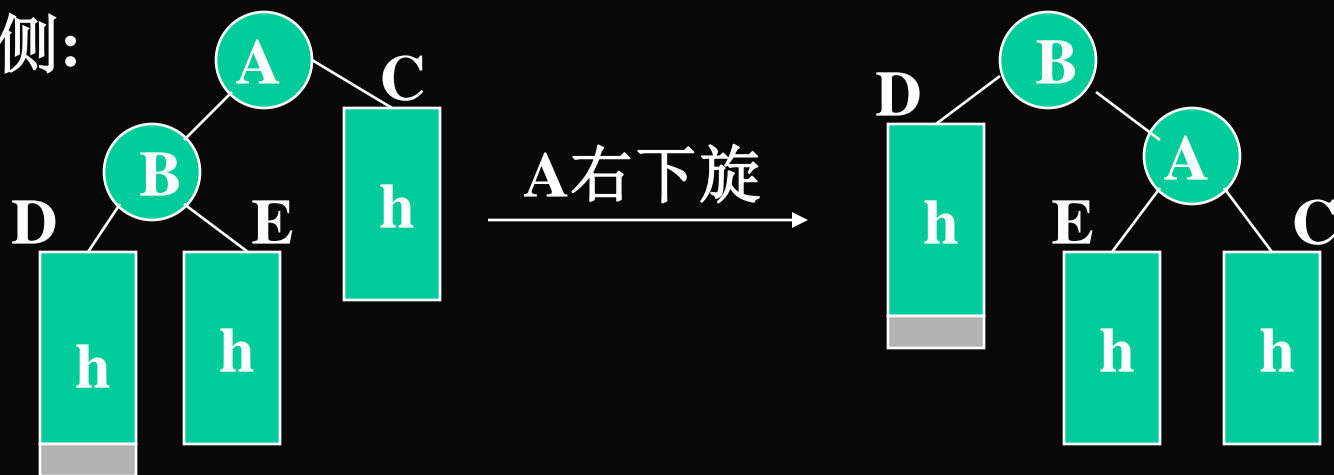
单旋转：外侧——从不平衡结点沿刚才回溯的路径取直接下两层  
如果三个结点处于一直线A, C, E

双旋转：内侧——从不平衡结点沿刚才回溯的路径取直接下两层  
如果三个结点处于一折线A, C, D

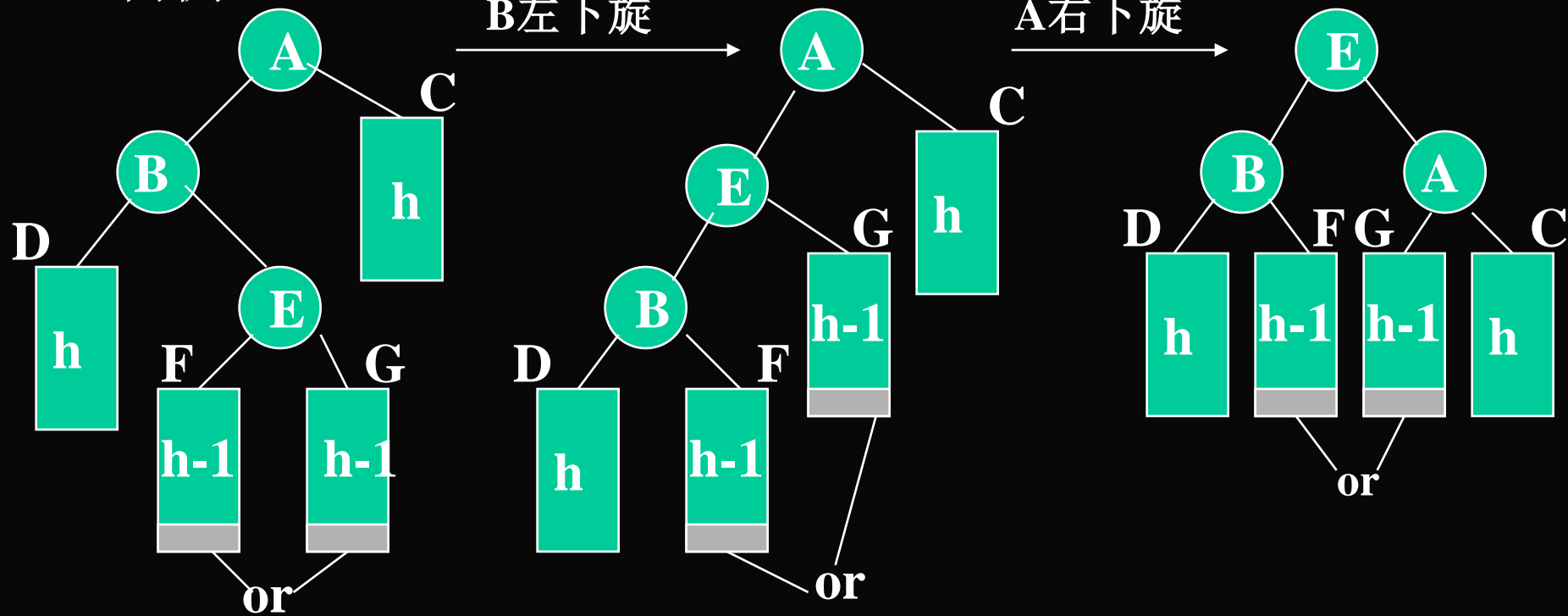
\*以上以右外侧，右内侧为例，左外侧，左内侧是对称的。

与前面对称的情况：左外侧，左内侧

左外侧：



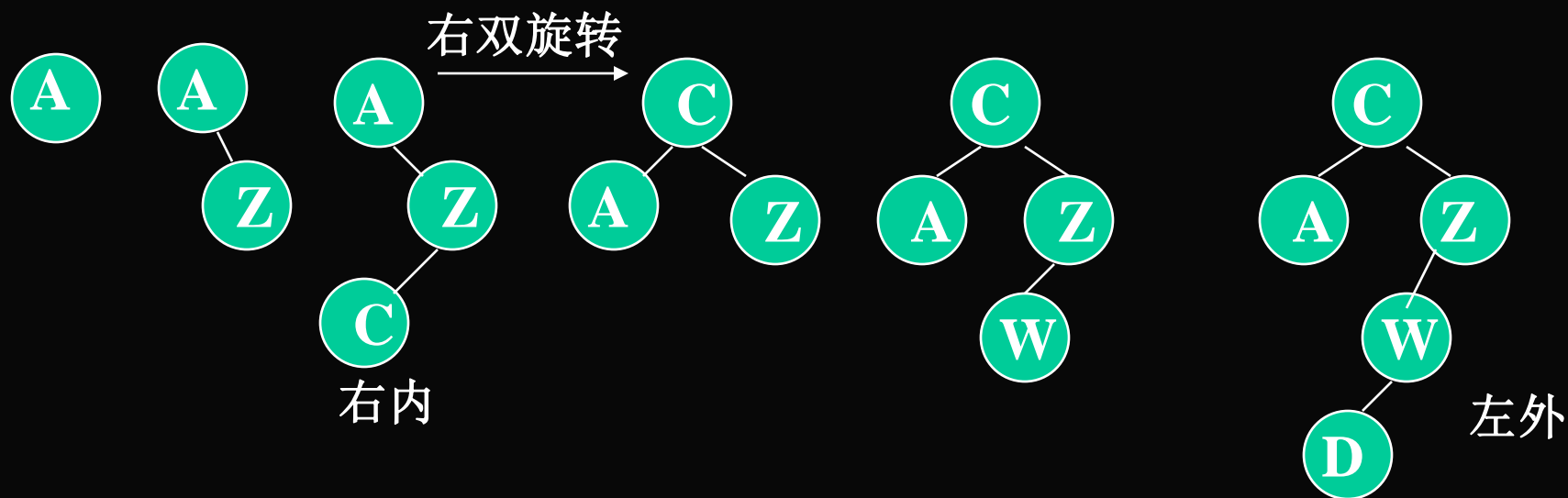
左内侧:

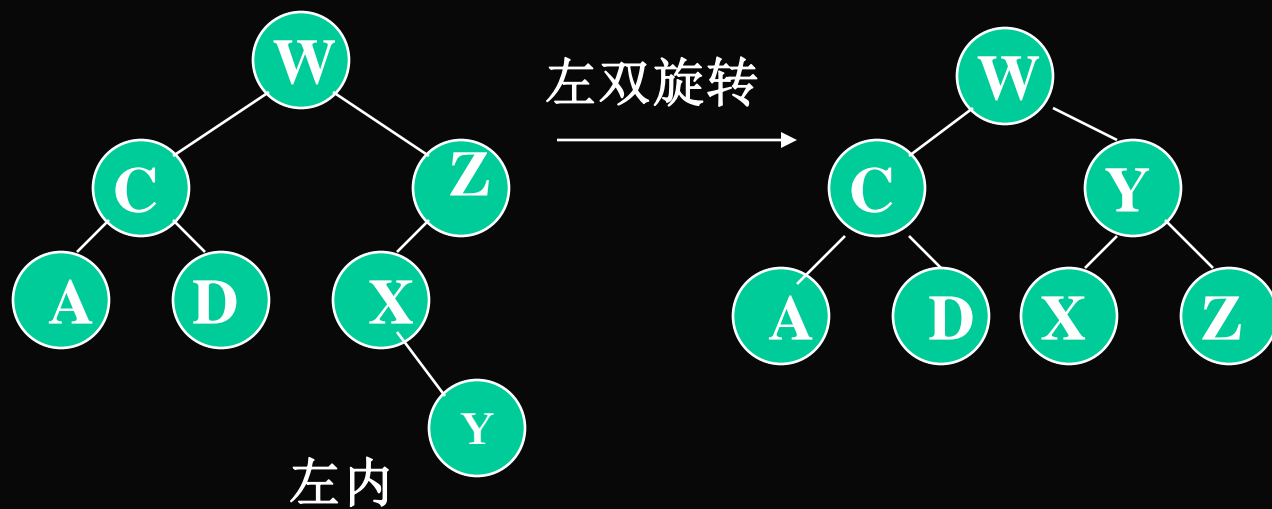
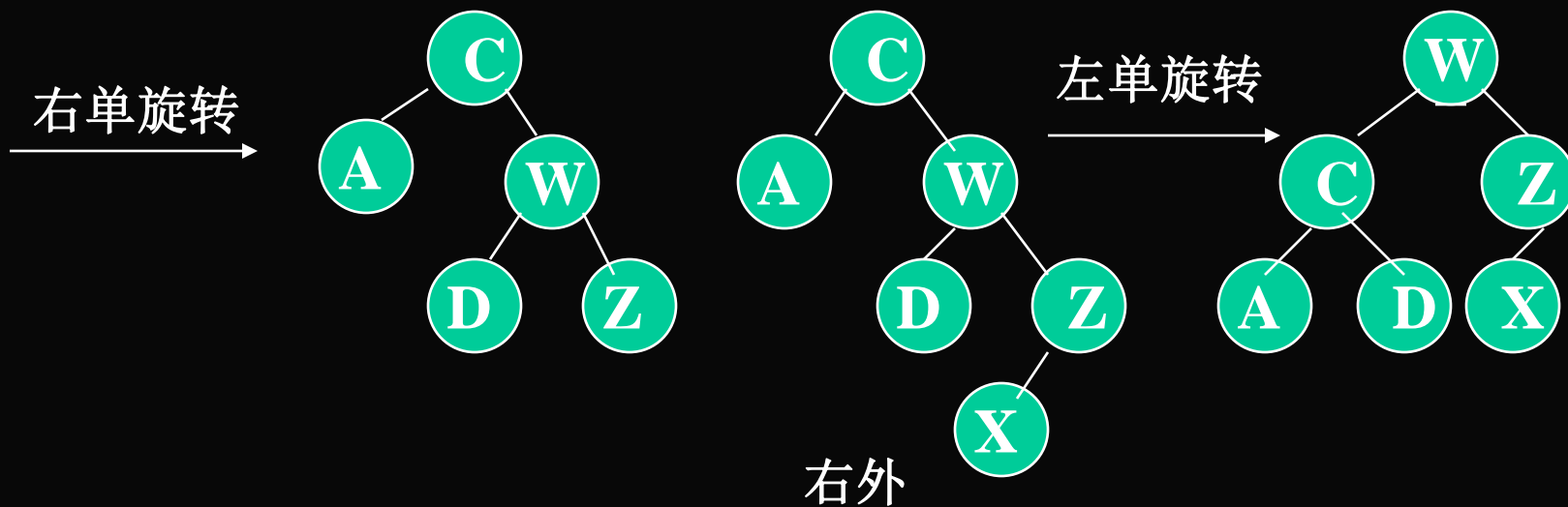




从空的AVL树建树的算法。一个例子：

7个关键码发生四种转动 A, Z, C, W, D, X, Y





# AVL Tree

```
class AVLNode
{  AVLNode( Comparable theElement )
    {  this( theElement, null, null ); }
    AVLNode( Compalable theElement, AVLNode lt, AVLNode rt )
        {  element = theElement; left = lt; right = rt; height = 0; }

    Comparable element;
    AVLNode left;
    AVLNode right;
    int height;
}

private static int height( AVLNode t )
{  return t == null ? -1 : t . height;
}
```

## AVL Tree

```
private AVLNode insert( Comparable x, AVLNode t )
{   if ( t == null )
        t = new AVLNode( x, null, null );
    else if ( x.compareTo( t.element ) < 0 )
    {   t.left = insert( x, t.left );
        if( height( t.left ) – height( t.right ) == 2 )
            if( x.compareTo( t.left.element ) < 0 )
                t = rotateWithLeftChild ( t );
            else t = doubleWithLeftChild( t );
    }
}
```

## AVL Tree

```
else if( x.compareTo( t.element ) > 0 )
{
    t.right = insert( x, t.right ) ;
    if( height( t.right ) – height( t.left ) == 2 )
        if( x.compareTo( t.right.element ) > 0 )
            t = rotateWithRightChild( t );
        else t = doubleWithRightChild( t );
}
else
    ;
t.height = max( height( t.left ), height( t.right ) ) + 1;
return t;
}
```

## AVL Tree

```
private static AVLNode rotateWithLeftChild( AVLNode k2 )
{  AVLNode k1 = k2.left;
   k2.left = k1.right;
   k1.right = k2;
   k2.height = max( height( k2.left ), height( k2.right ) ) + 1 ;
   k1.height = max( height( k1.left ), k2.height ) + 1;
   return k1;
}

private static AVLNode doubleWithLeftChild( AVLNode k3 )
{  k3.left = rotateWithRightChild( k3.left );
   return rotateWithLeftChild( k3 );
}
```

# AVL Tree

AVL树的插入:

1. 首先要正确地插入
2. 找到有可能发生的最小不平衡子树
3. 判别插入在不平衡子树的外侧还是内侧
4. 根据3的判别结果,再进行单旋还是双旋

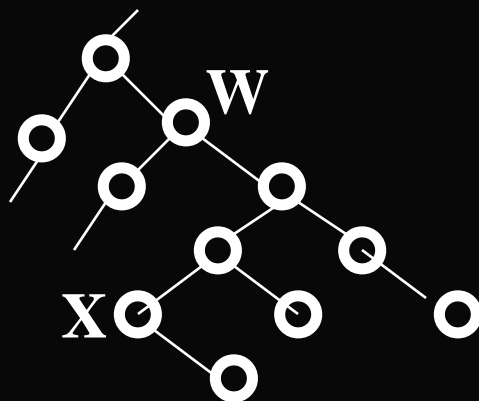
## 4.2 AVL Tree

### **4.Deletion from an AVL tree**



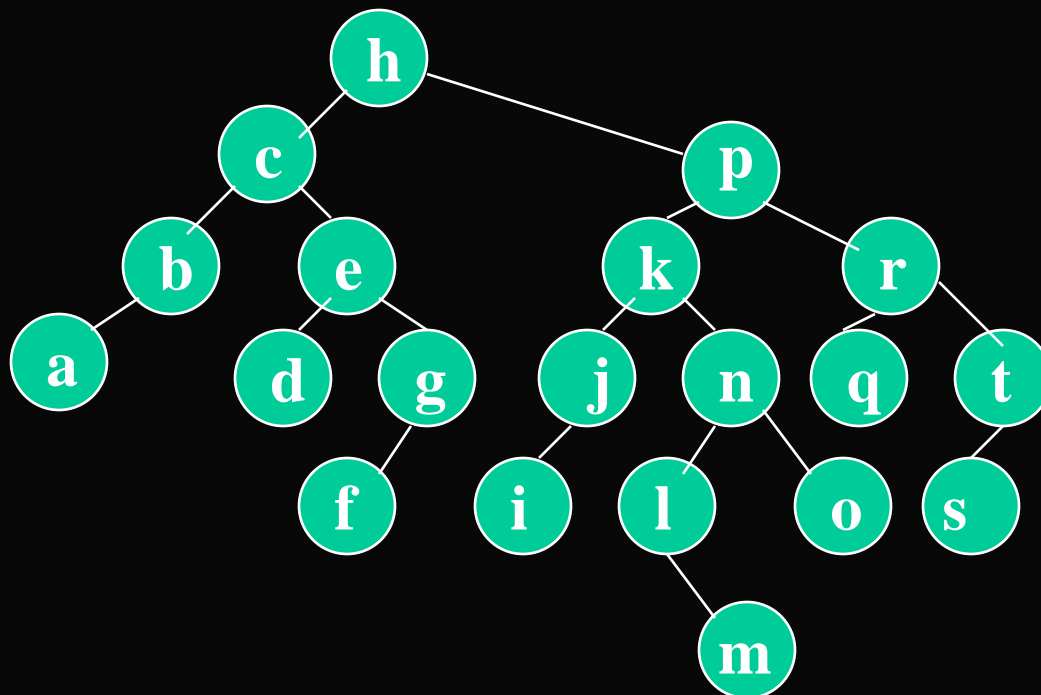
## AVL树的删除

方法：与二叉搜索树的删除方法一样。



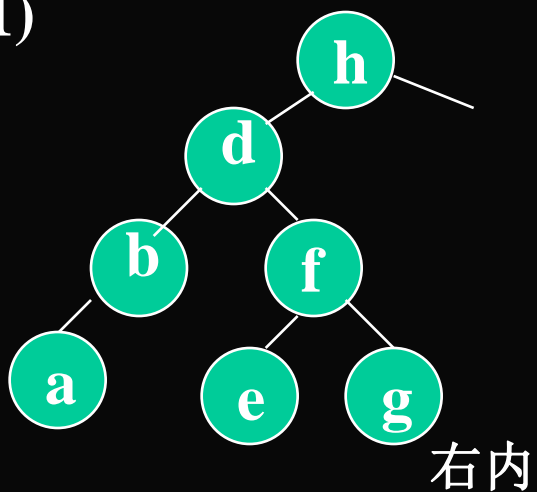
假设被删除结点为W, 它的中序后继为X, 则用X代替W, 并删除X. 所不同的是: 删除X后, 以X为根的子树高度减1, 这一高度变化可能影响到从X到根结点上每个结点的平衡因子, 因此要进行一系列调整。

例子:

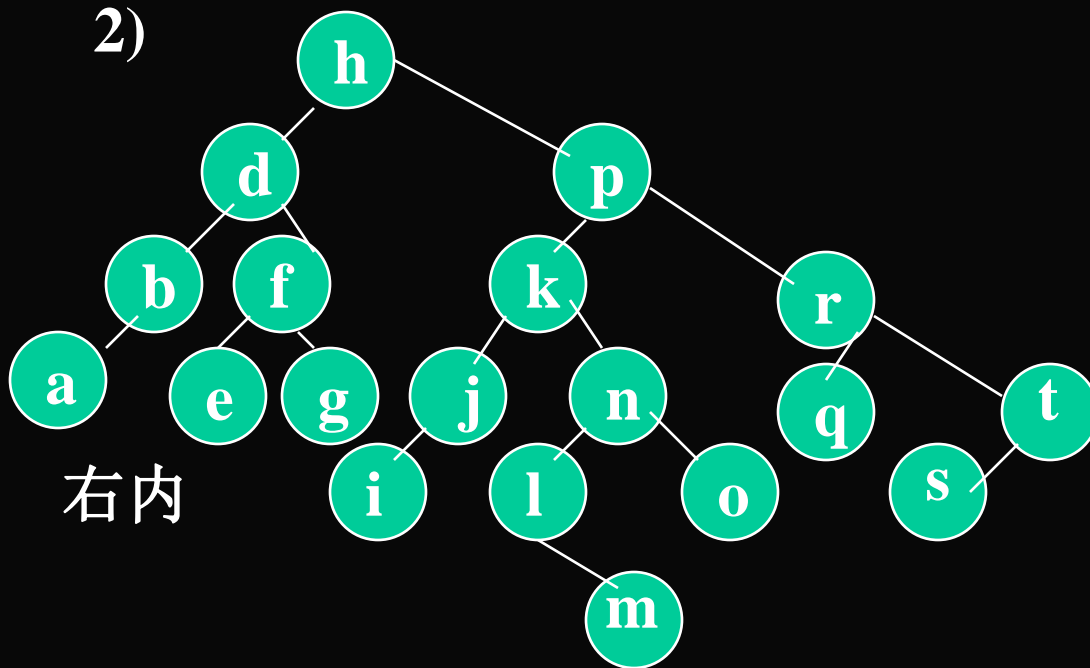


现要删除C

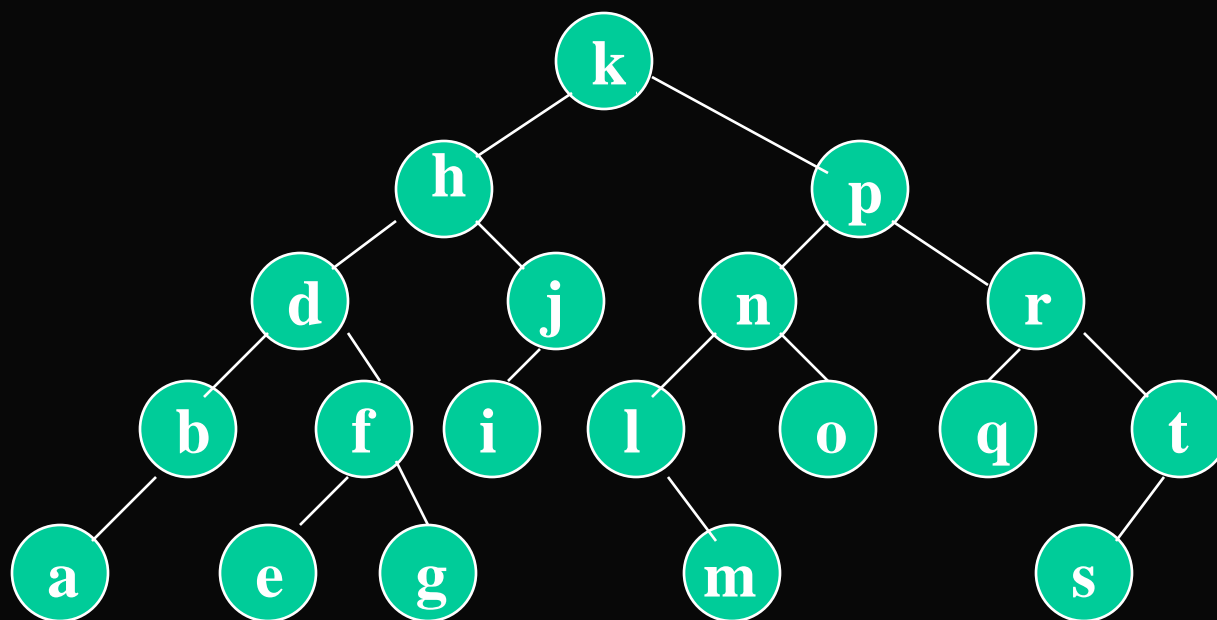
1)



2)



右内

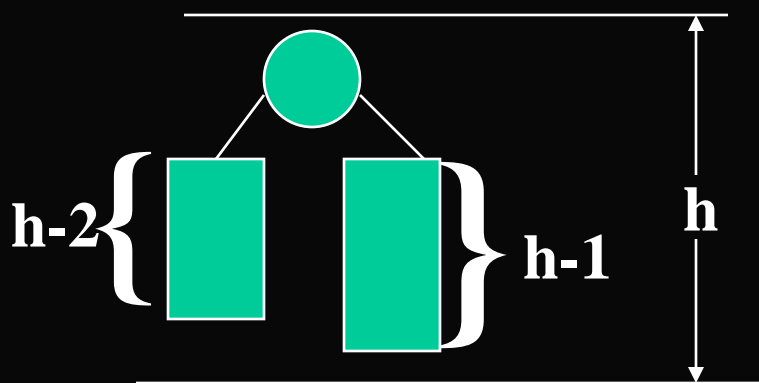


## 5.算法分析

具有 $n$ 个结点的平衡二叉树（AVL），进行一次插入或删除的时间最坏情况 $\leq O(\log_2 n)$

证明：实际上要考虑 $n$ 个结点的平衡二叉树的最大高度  
 $\leq (3/2) \log_2 (n + 1)$

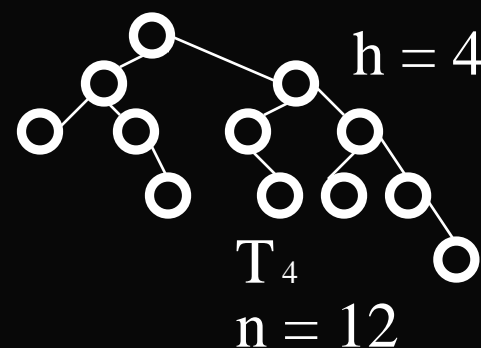
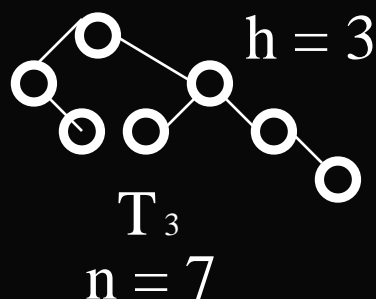
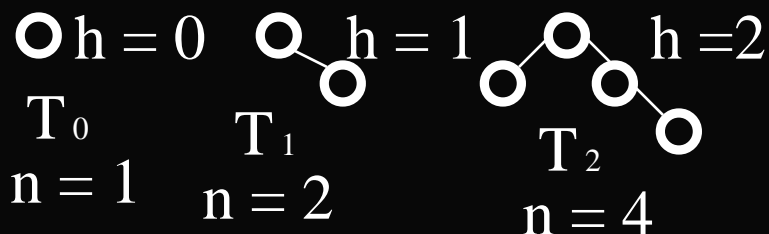
设 $T_h$ 为一棵高度为 $h$ ，且结点个数最少的平衡二叉树。



假设右子树高度为 $h-1$

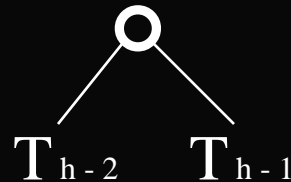
因结点个数最少,  $\therefore$  左子树高度只能是 $h-2$

这两棵左子树, 右子树高度分别为 $h-2, h-1$ , 也一定是结点数最少的:



以上五棵平衡二叉树，又称为Fibonacci树。

也可以这样说一棵高度为h的树，其右子树高度为h-1的Fibonacci树，左子树是高度为h-2的Fibonacci树，即



假设 $N_h$ 表示一棵高度为h的Fibonacci树的结点个数，则

$$N_h = N_{h-1} + N_{h-2} + 1$$

$$N_0 = 1, N_1 = 2, N_2 = 4, N_3 = 7, N_4 = 12, \dots$$

$$N_0 + 1 = 2, N_1 + 1 = 3, N_2 + 1 = 5, N_3 + 1 = 8, N_4 + 1 = 13, \dots$$

$\therefore N_h + 1$ 满足费波那契数的定义，并且 $N_h + 1 = F_{h+3}$

$$f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad \dots$$

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad \dots$$

费波那契数 $F_i$ 满足下列公式

$$F_i = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^i$$

$$\because \left| \frac{1-\sqrt{5}}{2} \right| < 1, \therefore \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^i \text{ 相当小}$$

$$N_h + 1 = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{h+3} + O(1)$$

$\therefore$  费波那契数树是具有相同高度的所有平衡二叉树中结点个  
数最少的

$$n+1 \geq N_h+1 = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{h+3} + O(1)$$

$$\therefore h \leq \frac{1}{\log_2 \frac{1+\sqrt{5}}{2}} \log_2 (n+1) + O(1) \approx \frac{3}{2} \log_2 (n+1)$$

## 4.3 B-TREES

### 1. m-way Search Trees

**Definition:** An m-way search tree may be empty. If it is not empty, it is a tree that satisfies the following properties:

- 1) In the corresponding extended search tree (obtained by replacing zero pointer with external nodes), each internal node has **up to** m children and between 1 and m-1 elements.
- 2) Every node with p elements has exactly p+1 children.
- 3) Consider any node with p elements:

$$C_0 \ k_1 \ C_1 \ k_2 \ \dots \ k_p \ C_p$$

$k_1 < k_2 < \dots < k_p$ ,  $c_0, c_1, \dots, c_p$  be the p+1 children of the node

## 4.3 B-TREES

$C_0 \ k_1 \ C_1 \ k_2 \ \dots\dots\dots \ k_p \ C_p$
---

**$C_0$ :** The elements in the subtree with root  $c_0$  have keys smaller than  $k_1$

**$C_p$ :** Elements in the subtree with root  $c_p$  have keys larger than  $k_p$

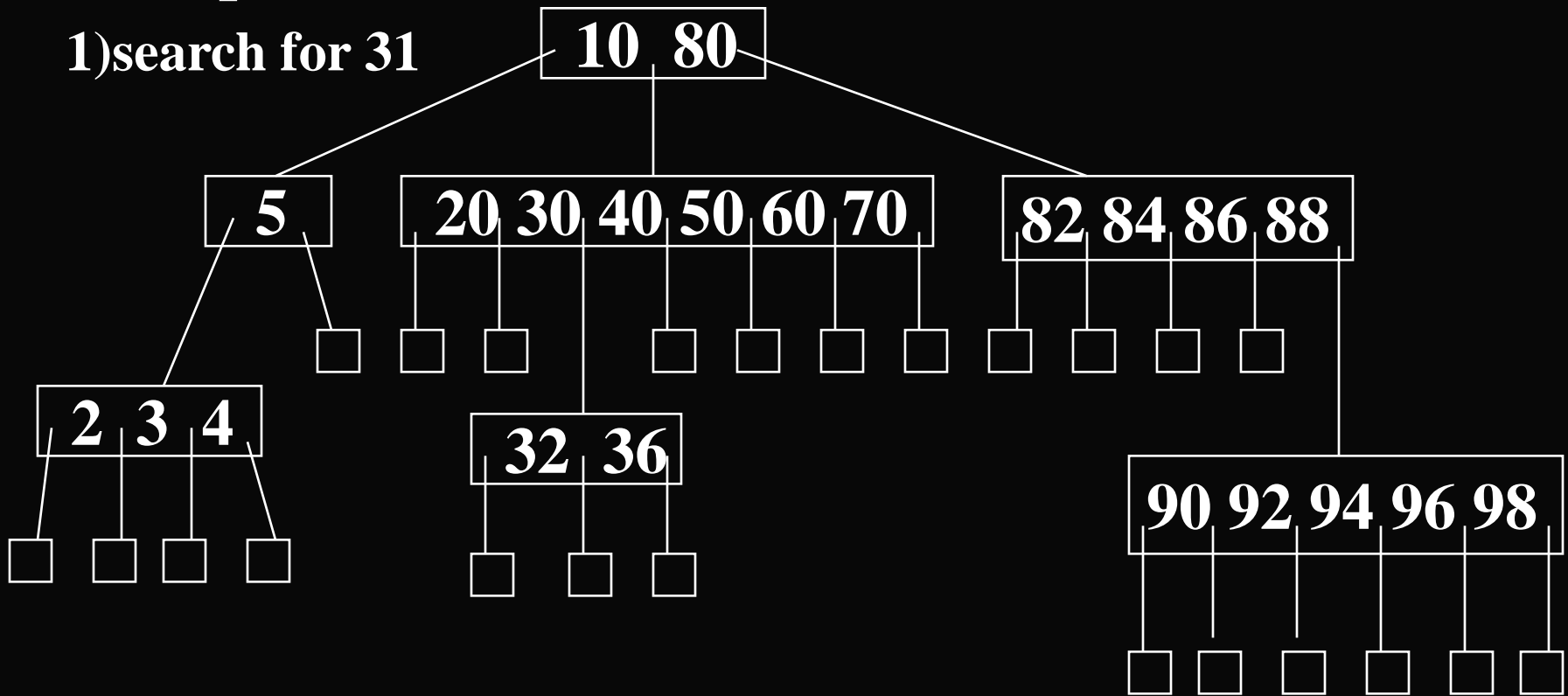
**$C_i$ :** Elements in the subtree with root  $c_i$  have keys larger than  $k_i$  but smaller than  $k_{i+1}$ ,  $1 \leq i \leq p$ .



# 4.3 B-TREES

**Example:**

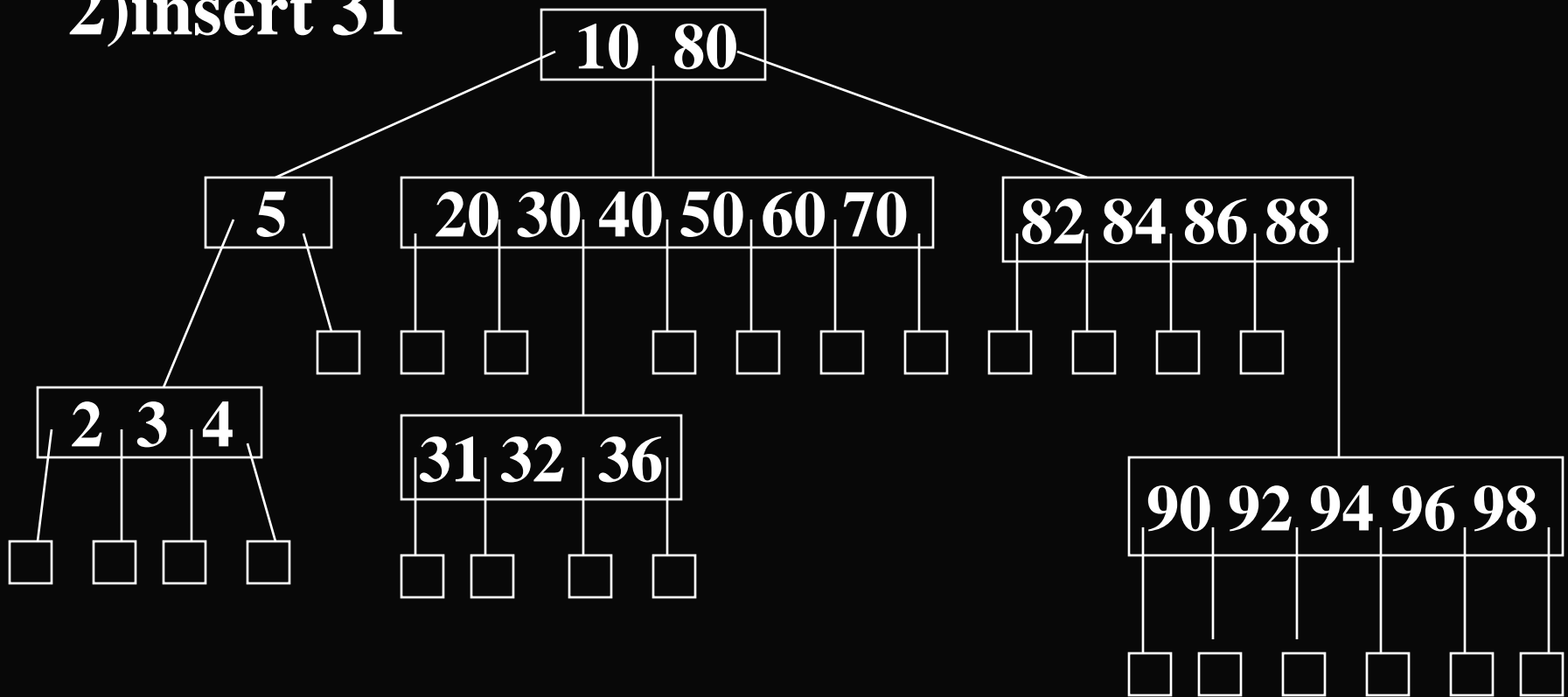
**1)search for 31**



**A seven-way search tree**

# 4.3 B-TREES

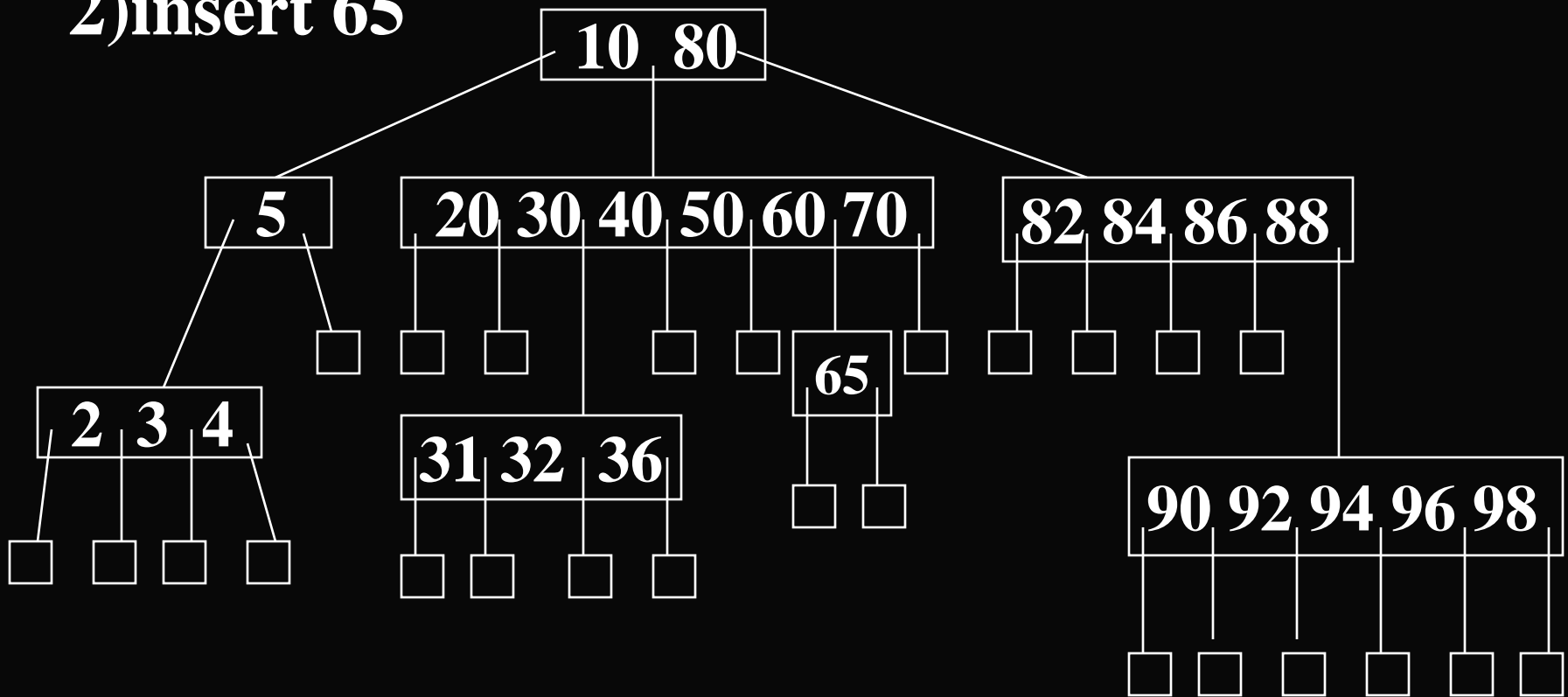
**2)insert 31**



**A seven-way search tree**

# 4.3 B-TREES

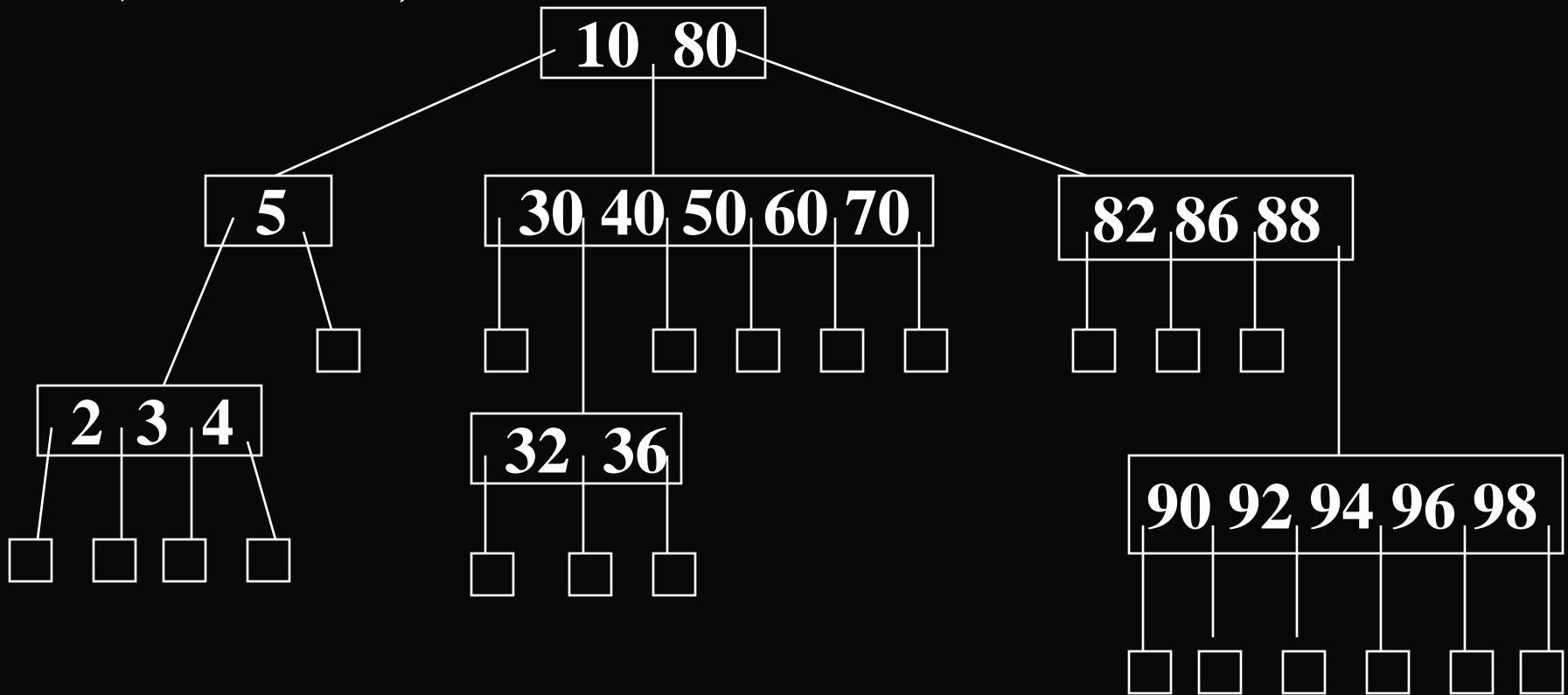
2)insert 65



A seven-way search tree

# 4.3 B-TREES

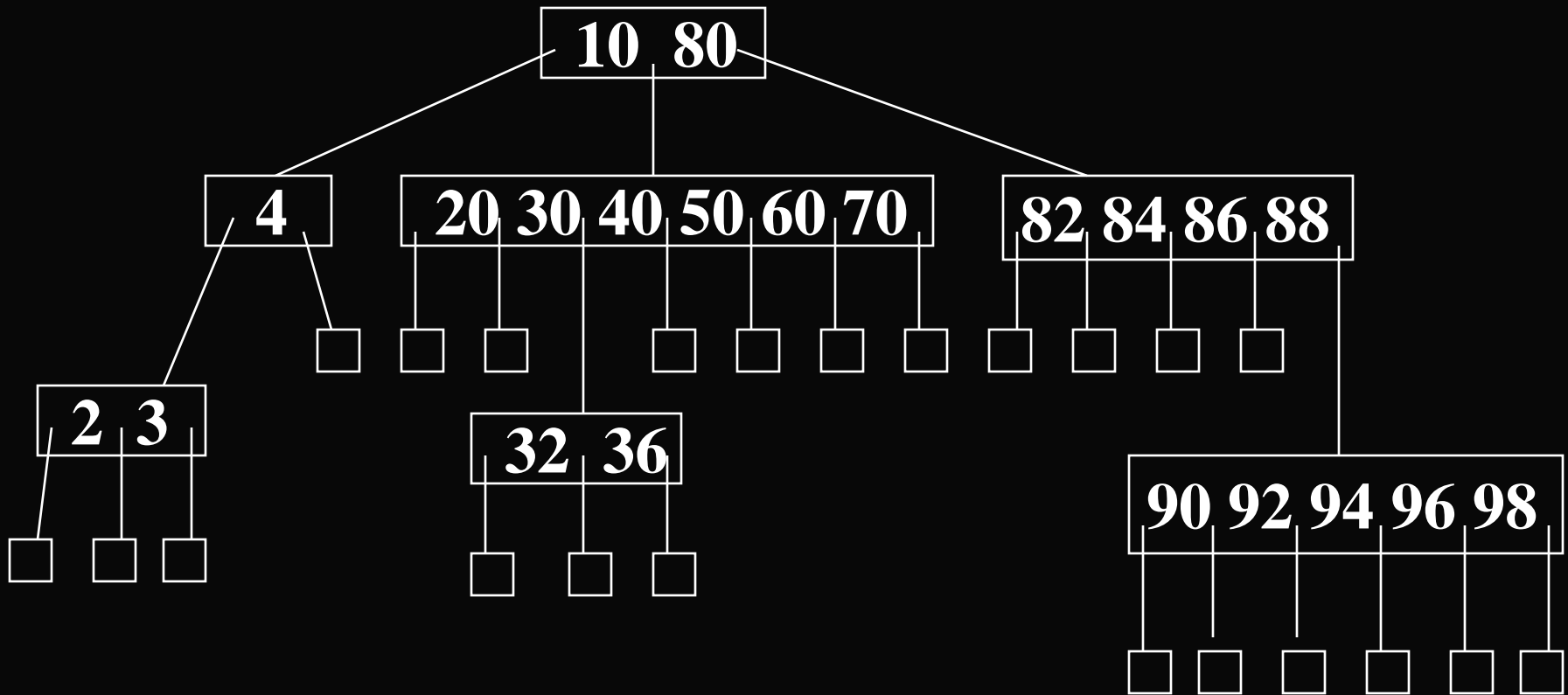
**3) Delete 20, 84**



**A seven-way search tree**

# 4.3 B-TREES

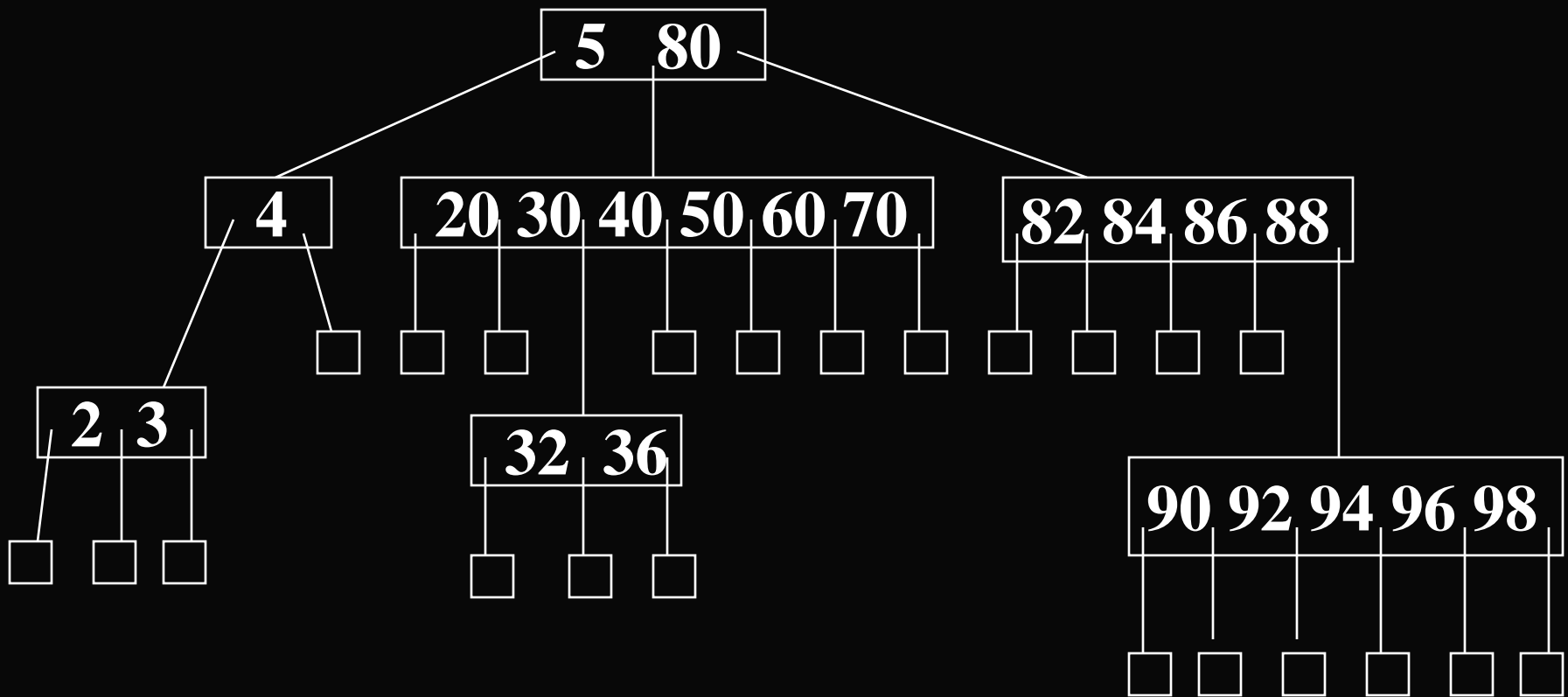
**Delete 5, move up 4**



**A seven-way search tree**

# 4.3 B-TREES

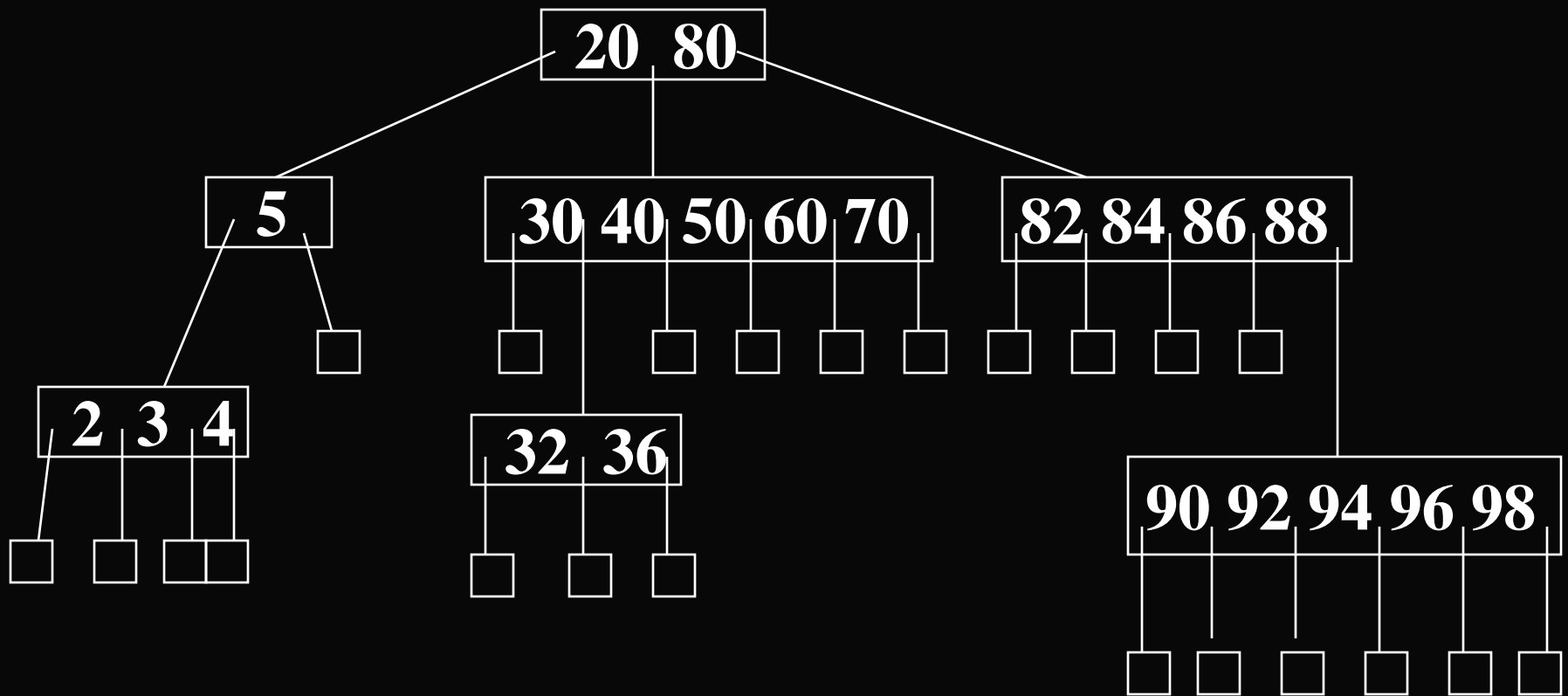
**Delete 10:** replace it with the largest element in  $c_0(5)$



**A seven-way search tree**

## 4.3 B-TREES

**Delete 10:** replace it with the smallest element in  $c_1(20)$



**A seven-way search tree**

## 4.3 B-TREES

### 4) Height of an m-way search tree

An m-way search tree of height h

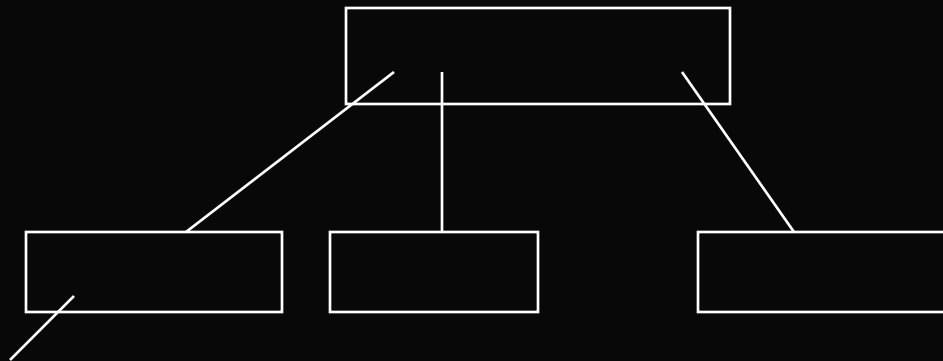
may have as few as h elements(one node per level),

as many as  $m^h-1$  elements.



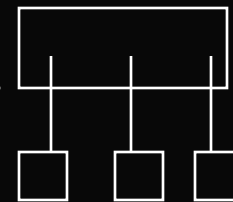
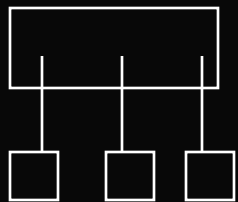
# 4.3 B-TREES

No of nodes



**0**  $m^0=1$

**1**  $m^1=m$



**h-1**  $m^{h-1}$

Sum of nodes  $\sum_{i=0}^{h-1} m^i = (m^h - 1) / (m - 1)$

## 4.3 B-TREES

- The number of elements in a  $m$ -way search tree of height  $h$  is between  $h$  and  $m^h-1$
- The height of a  $m$ -way search tree with  $n$  elements is between  $\log_m(n+1)$  and  $n$

**Example:**

height: 5

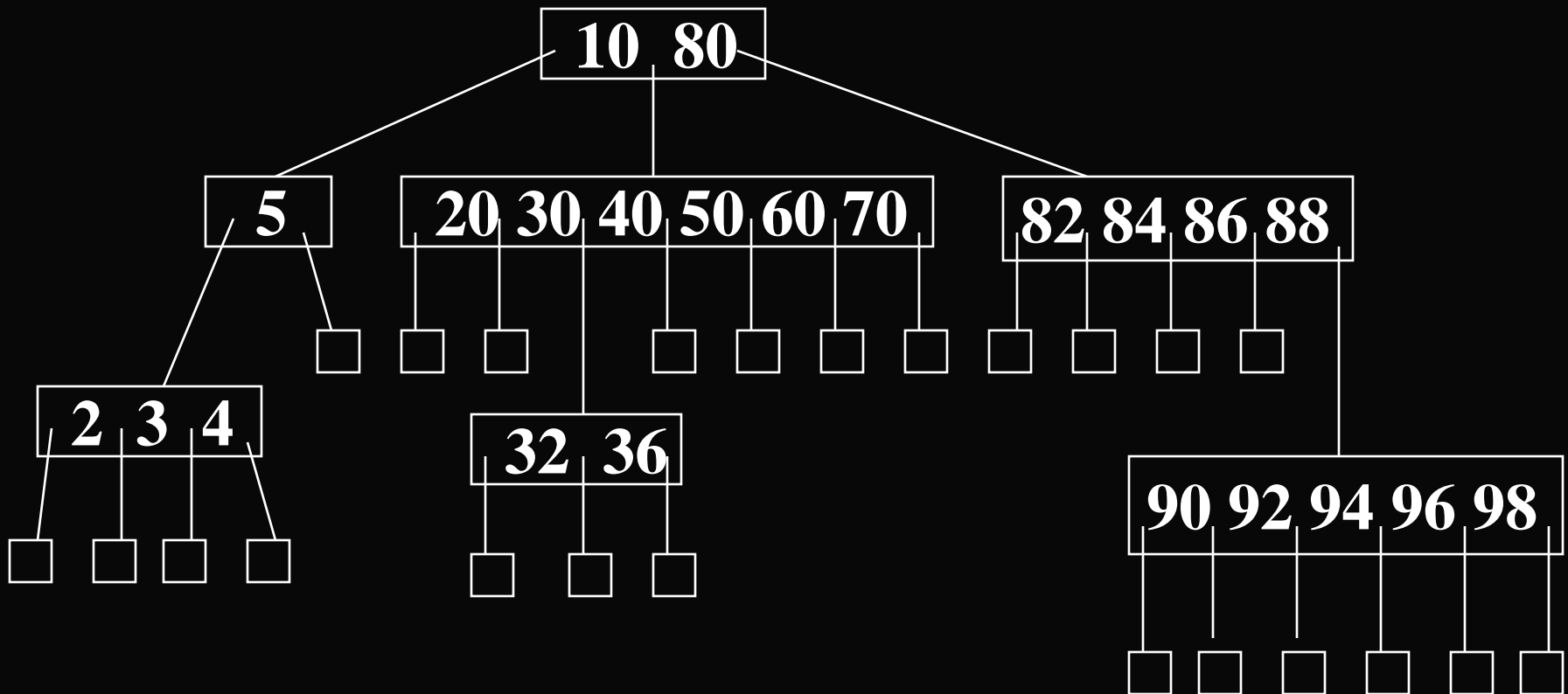
200-way search tree

$n : 200^5 - 1 = 32 * 10^{10} - 1$

# 4.3 B-TREES

二叉搜索树 ----> 平衡的二叉搜索树 (AVL树)

m路搜索树 ----> 平衡的m叉搜索树 (B-树)



A seven-way search tree

# 4.3 B-TREES

## 2.B-Trees of order m

70年 R.Bayer提出的。

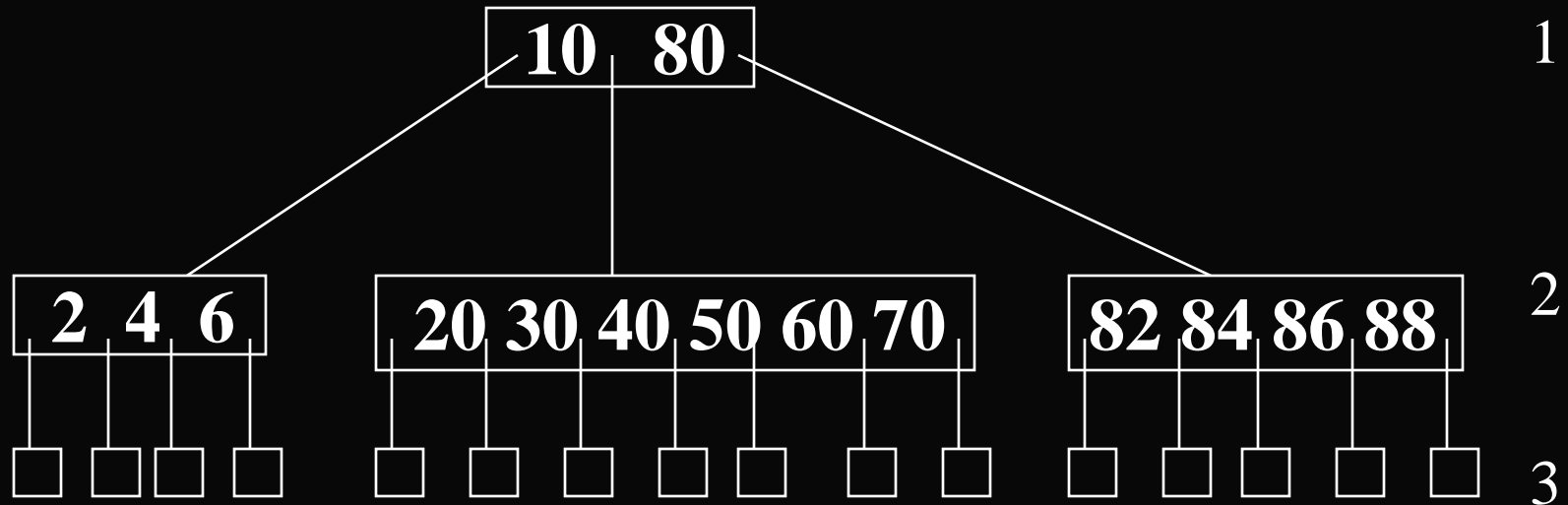
**Definition :** A B-tree of order m is an m-way search tree.

If the B-tree is not empty, the corresponding extended tree satisfies the following properties:

- 1) the root has at least two children
- 2) all internal nodes other than the root have at least  $\lceil m/2 \rceil$  children
- 3) all external nodes are at the same level

# 4.3 B-TREES

**example**



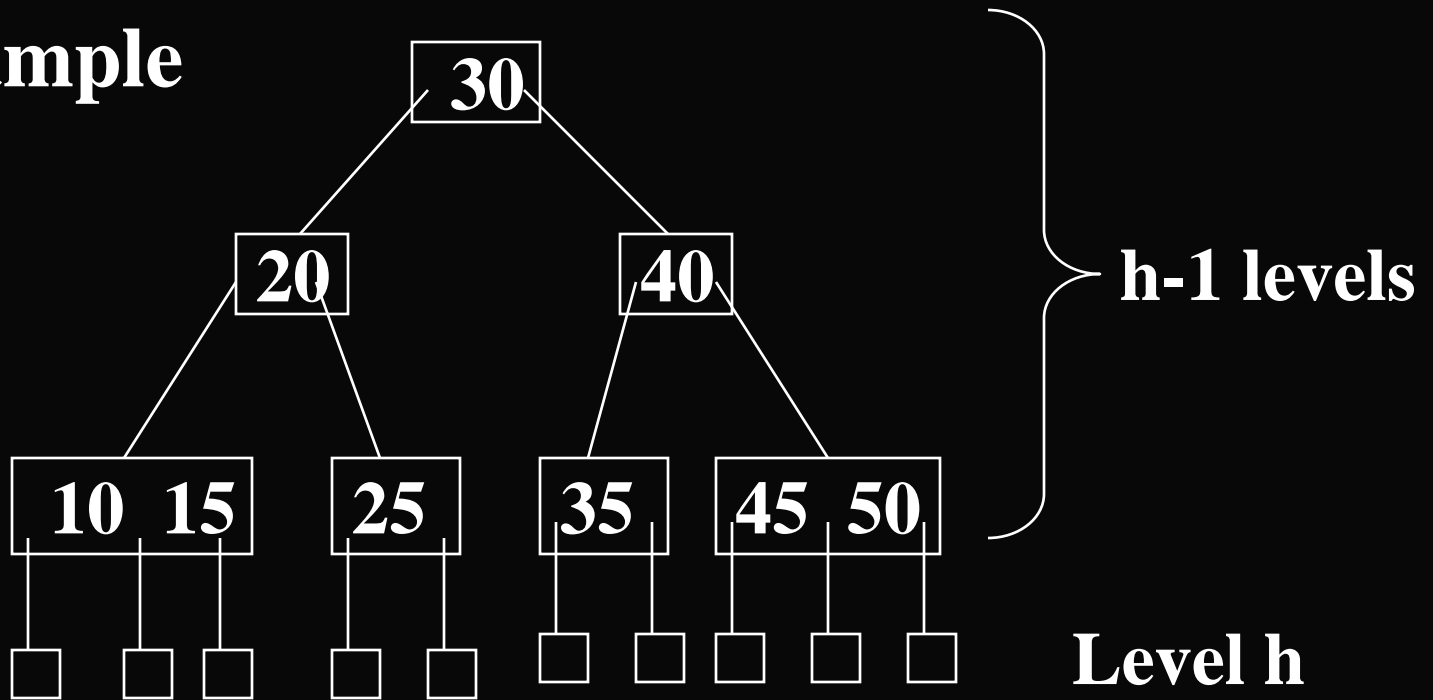
**a B-tree of order 7**

## 4.3 B-TREES

- **In a B-tree of order 2, each internal node has at least 2 children, and all external nodes must be on the same level, so a B-tree of order 2 is full binary trees**
- **In a B-tree of order 3(sometimes also called 2-3 tree) , each internal node has 2 or 3 children**

## 4.3 B-TREES

- **example**



**A B-tree of order 3**

## 4.3 B-TREES

### **B-TREES Properties:**

**1)all external nodes are on the same level**

**2)number of external nodes=number of keywords +1**

**proof:**

$$b_1=k_0+1, b_2=k_1+b_1, b_3=k_2+b_2, \dots\dots\dots,$$

$$\text{外部结点}=k_{h-1}+k_{h-2}+\dots+k_1+k_0+1=n+1$$



## 4.3 B-TREES

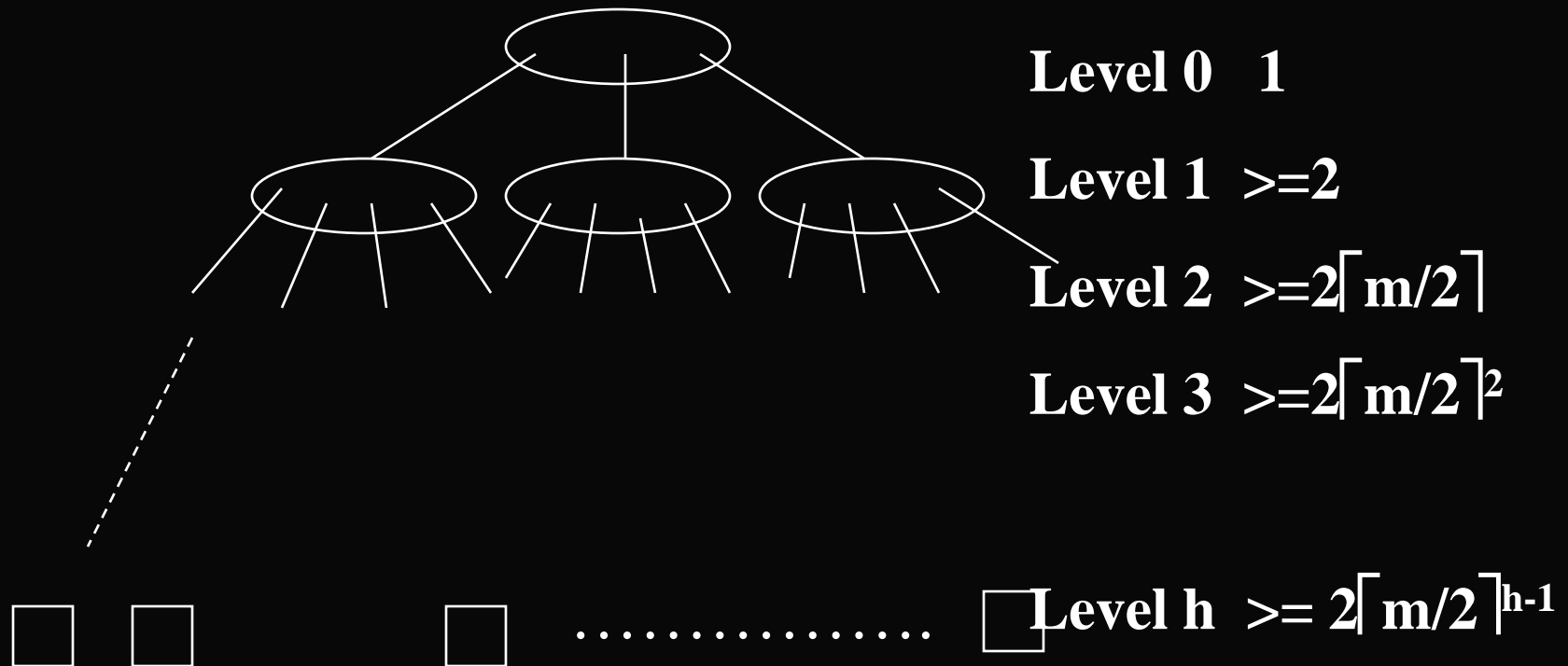
### 1) Searching a B-Tree

- A B-tree is searched using the same algorithm as used for an m-way search tree.
- Algorithm analysis: the number of disk access is at most h (h is the height of the B-Tree).

proof: T is a B-Tree of order m with height h, number of elements in T is n, each time we read a node into memory. The n+1 external nodes are on level h.

# 4.3 B-TREES

**Number of nodes on the each level of the B-Tree is:**



## 4.3 B-TREES

$$n+1 \geq 2 \lceil m/2 \rceil^{h-1}, (n+1)/2 \geq \lceil m/2 \rceil^{h-1},$$

$$h-1 \leq \log_{\lceil m/2 \rceil} (n+1)/2,$$

$$\log_m(n+1) \leq h \leq 1 + \log_{\lceil m/2 \rceil} (n+1)/2$$



In the case that each  
node has  $m$  children

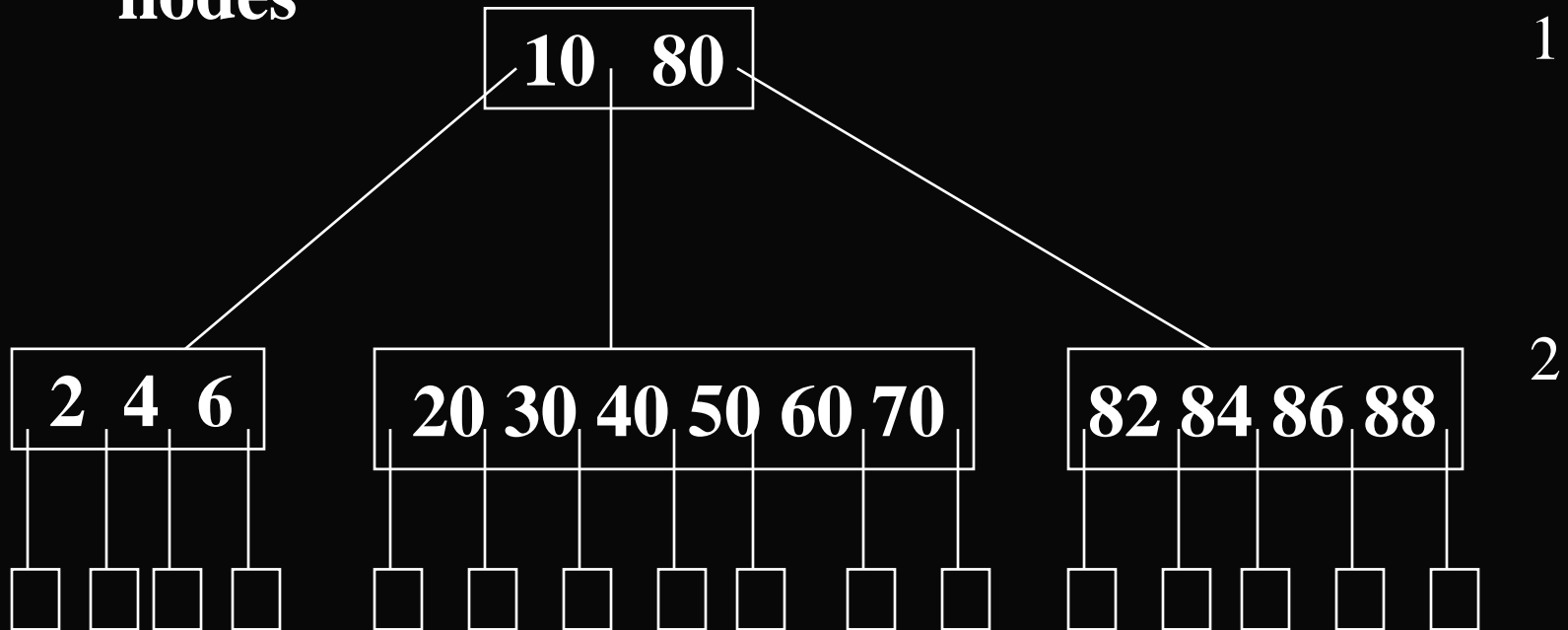
**Example:**  $n=2*10^6$ ,  $m=199$

then  $h \leq 1 + \log_{100}(10^2)^3 = 4$  search one from 200 branches

# 4.3 B-TREES

## 2) Inserting into a B-Tree

**always happen at one level above the external nodes**

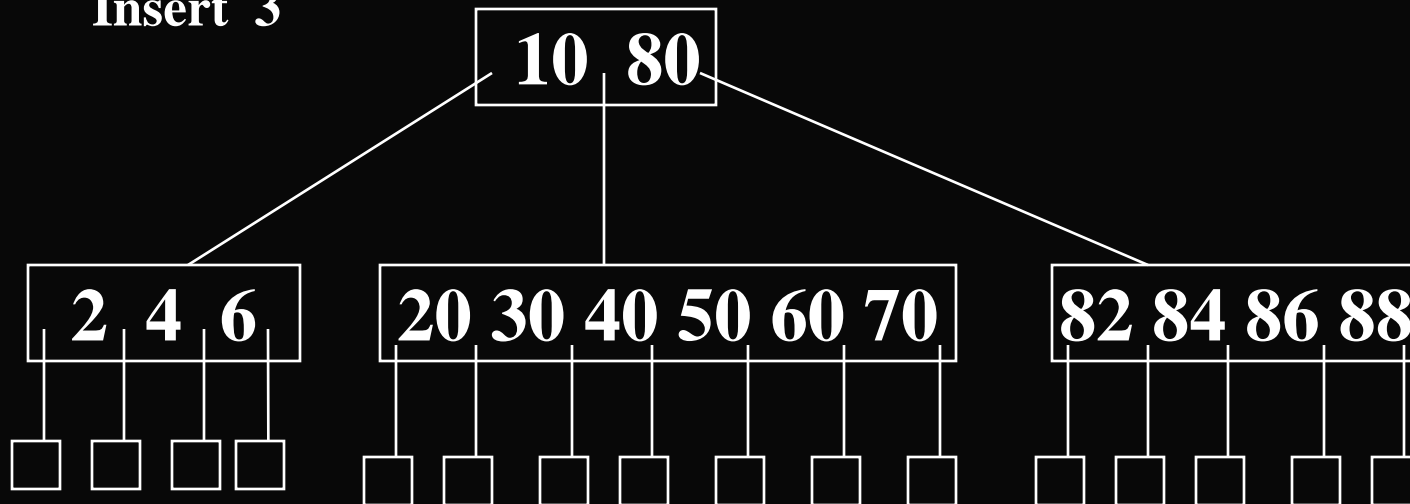


**a B-tree of order 7**

## 4.3 B-TREES

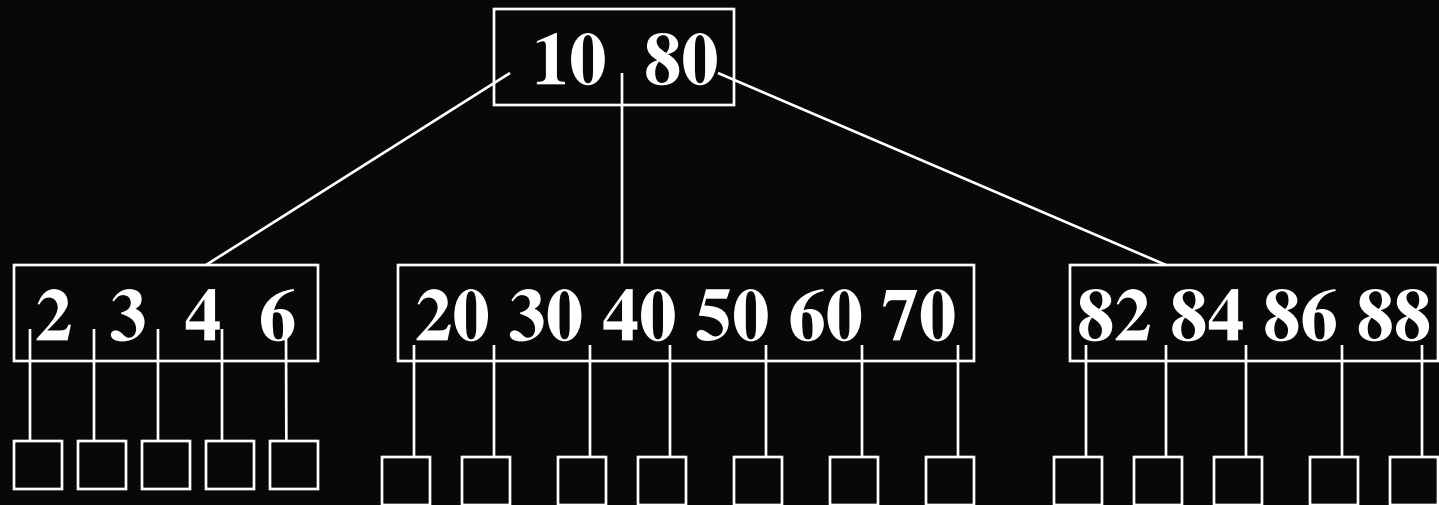
**Case 1: number of children in the node  $< m$ ,  
insert into the node as ordered**

**Insert 3**



**A B-Tree of order 7**

## 4.3 B-TREES



## 4.3 B-TREES

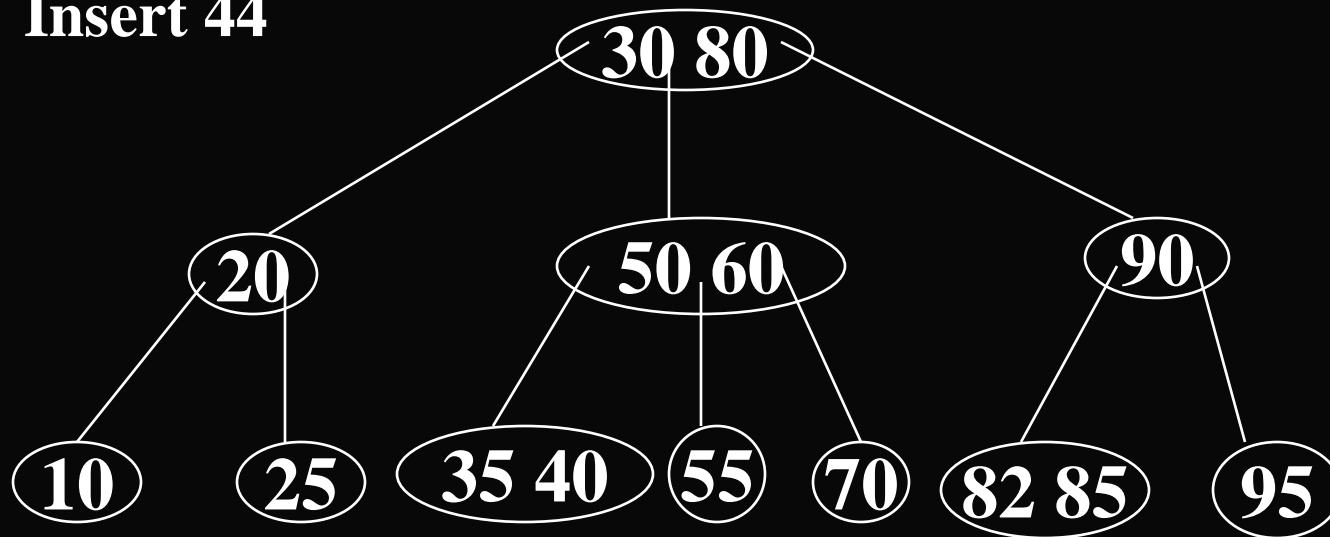
### Case 2.

- **Insert into a node with  $m$  children (also called a full node), like insert 25 into the B-Tree in the last example, the full node is split into two nodes.**
- **A new pointer will be added to the parent of the full node .**
- **Because  $k_{\lceil m/2 \rceil}$  is inserted into parent node, it may cause new split. If the root is split, the height of the tree will increased by 1.**

## 4.3 B-TREES

**Example:**

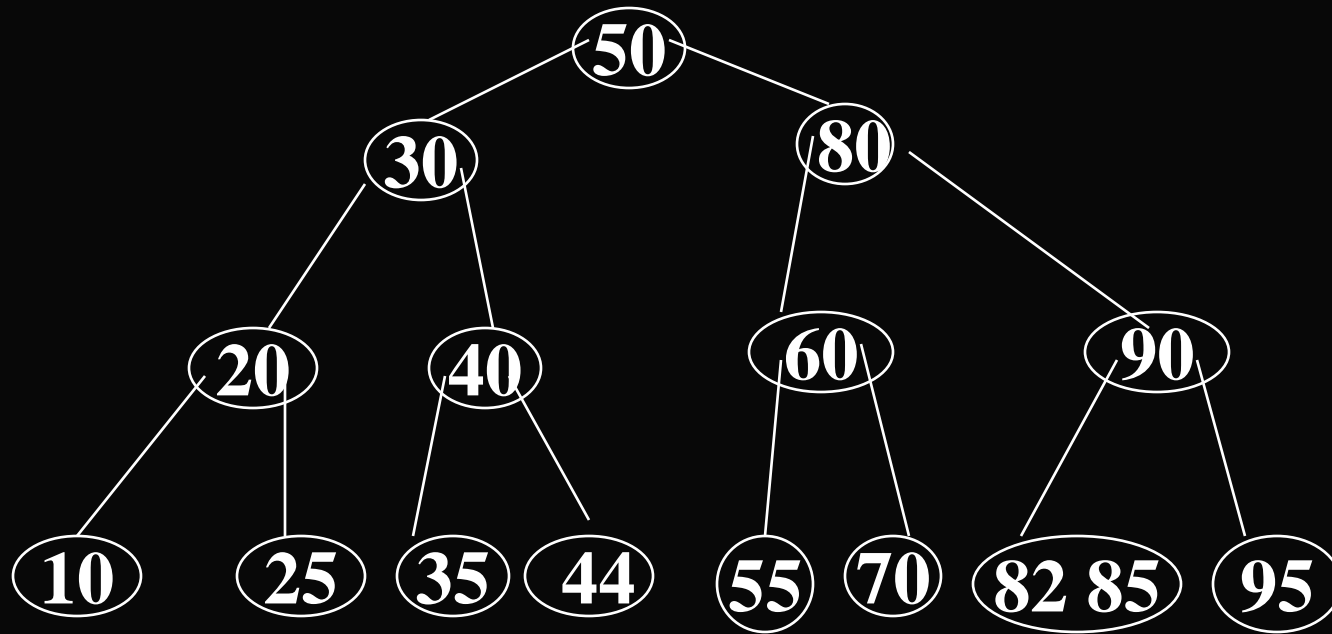
**Insert 44**



**A B-Tree of order 3**



## 4.3 B-TREES



**Algorithm analyses:**

**If the insert operation causes  $s$  node to split,  
the number of disk access is**

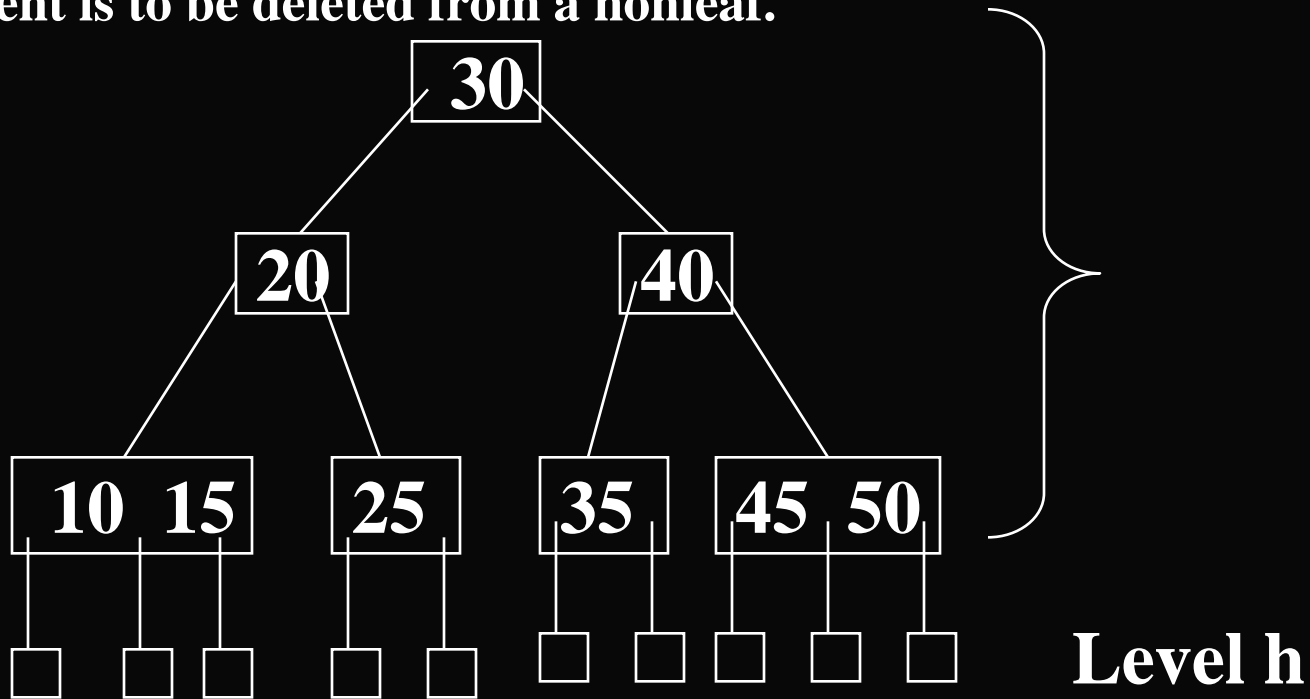
**$h$  (to read in the nodes on the search path)  
 $+2s$  (to write out the two split parts of each  
node that is split)  
 $+1$  (to write the new node).**

# 4.3 B-TREES

## 3) deletion from a B-Tree

Two cases:

- The element to be deleted is in a node whose children are external nodes(i.e. the element is in a leaf)
- The element is to be deleted from a nonleaf.



A B-tree of order 3

## 4.3 B-TREES

**a) the element to be deleted is in a leaf**

**Case1: delete it directly if it is in a node  
which has more than  $\lceil m/2 \rceil$  children**

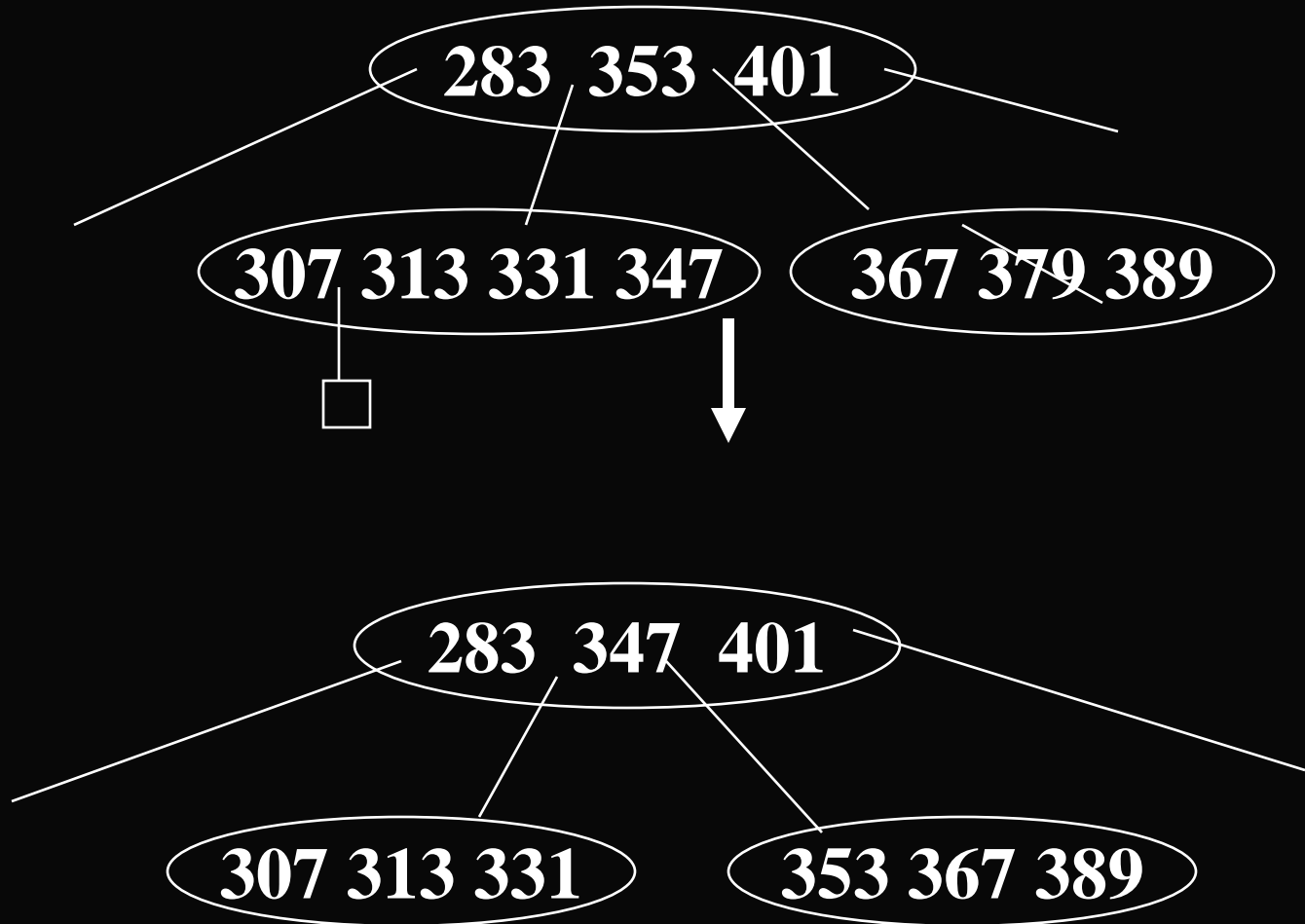
**Case2: if it is in a node which has  $\lceil m/2 \rceil$  children,  
after deletion ,the number of children( $\lceil m/2 \rceil - 1$ ) is not  
suitable for a B-Tree**

- ① borrow an element from the its nearest sibling if can,  
and do some adjusting.**

## 4.3 B-TREES

**Example: delete 379**

A B-TREE of order 7

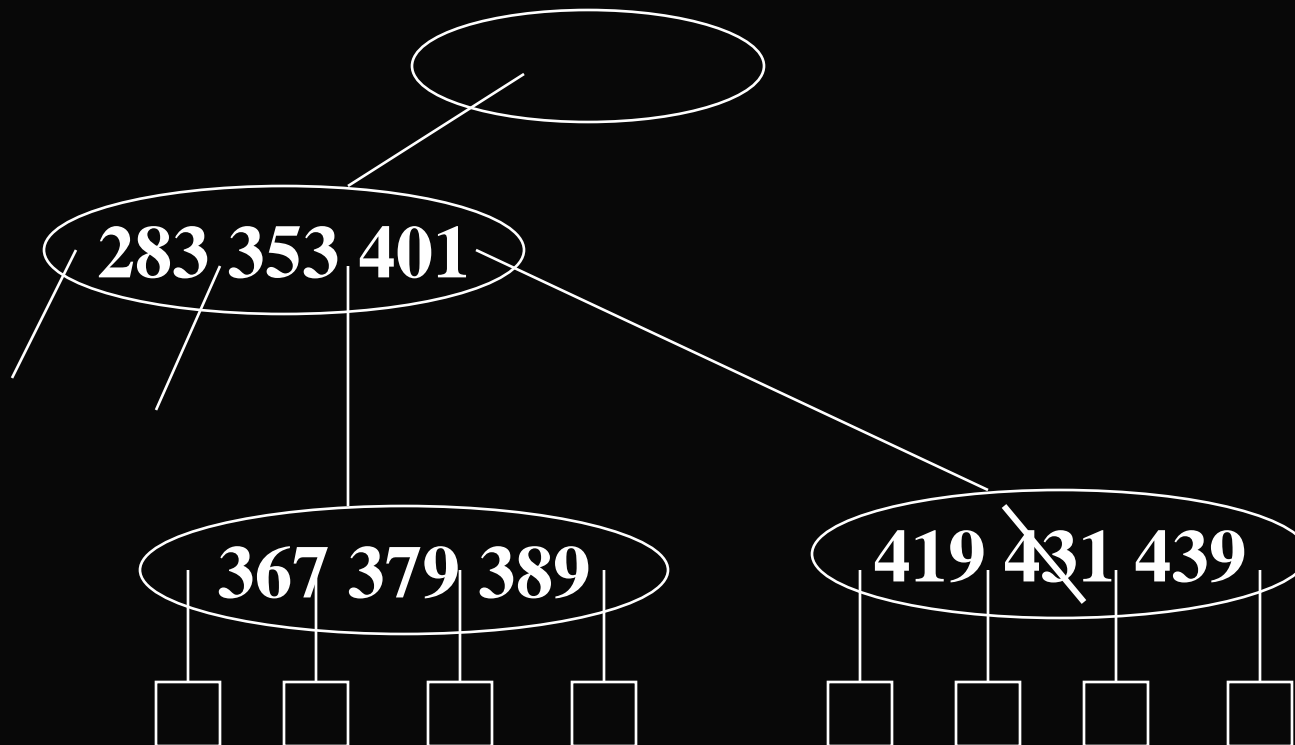


## 4.3 B-TREES

- ② If nearest left or right sibling both only has  $\lceil m/2 \rceil$  children, then merge them
  - After deletion ,merge the node and its sibling with the element between them in the parent into a single node
  - Maybe cause new merge in parent nodes
  - The height of the tree will decreased by one if root is merged.

## 4.3 B-TREES

**Example: a B-Tree of order 7 ,delete 431**





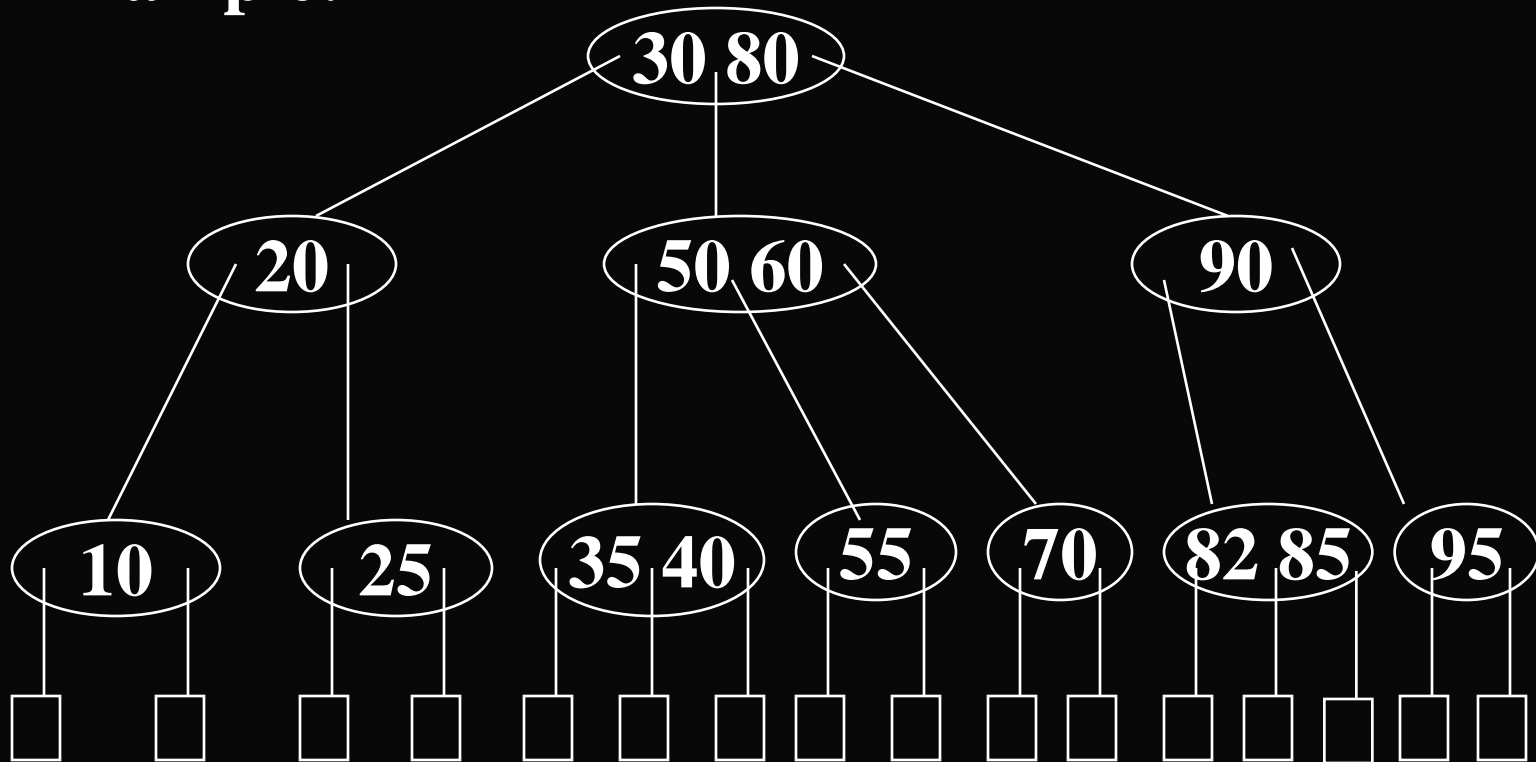
## 4.3 B-TREES

**b)delete a key in a node in the above level**

- **Delete it**
- **Replace it with the smallest key in the right subtree or the largest key in the left subtree**
- **Because delete a key in the leaf node , do the adjust mentioned in a)**

# 4.3 B-TREES

**Example:**



**A B-TREE of order 3**

**Delete 80, then replace it with 82 or 70, delete 82 or 70 at last**

## 4.3 B-TREES

### 4)Node structure

$s, c_0, (e_1, c_1), (e_2, c_2), \dots, (e_s, c_s)$
---

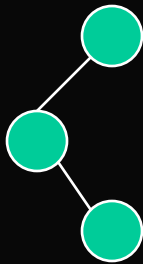
- **S is the number of elements in the node**
- **$e_i$  are the elements in ascending order of key**
- **$C_i$  are children pointers**

## Chapter 4.1

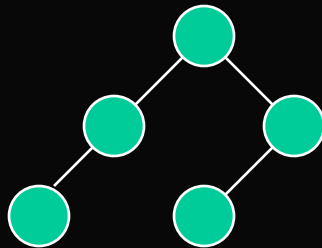
2009年统考题:

6. 下列二叉排序树中, 满足平衡二叉树定义的是

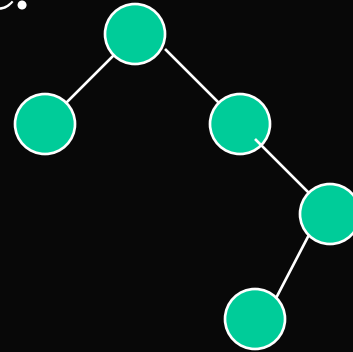
A.



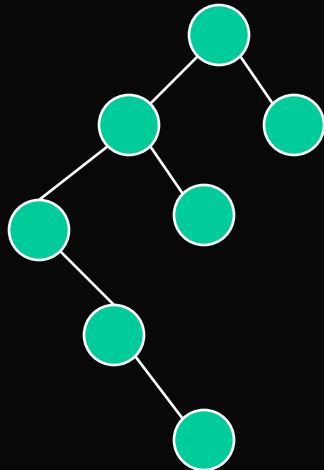
B.



C.



D.



## Chapter 4.1

2009年统考题:

7. 下列叙述中, 不符合 $m$ 阶B 树定义要求的是

- A. 根结点最多有 $m$ 棵子树
- B. 所有叶结点都在同一层上
- C. 各结点内关键字均升序或降序排列
- D. 叶结点之间通过指针链接

## Chapter 4.1

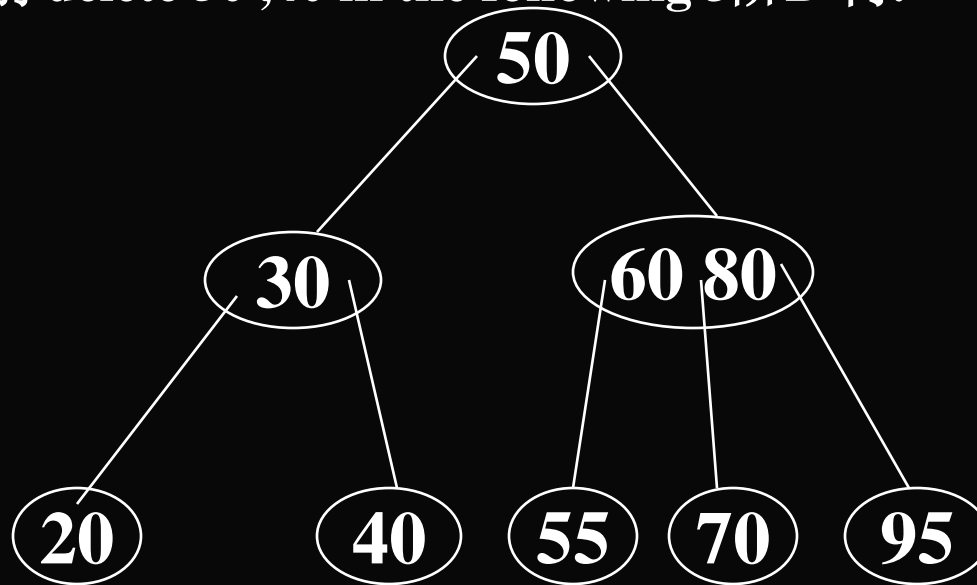
### Exercise:

1. a. Show the result of inserting 3, 1, 4, 6, 9, 2, 5, 7 into an initially empty binary search tree.
  - b. Show the result of deleting the root.
2. 写一递归函数实现在带索引的二叉搜索树 (IndexBST)中查找第k个小的元素。
3. 对一棵空的AVL树，分别画出插入关键码为{ 16, 3, 7, 11, 9, 28, 18, 14, 15} 后的AVL树。
4. 设计算法检测一个二叉树是不是一棵二叉搜索树。
5. 设有序顺序表中的元素依次为  
017,094,154,170,275,503,509,512,553,612,677,765,897,908. 试画出对其进行二分法搜索时的判定树, 并计算搜索成功的平均搜索长度。

6. 在一棵表示有序集S 的二叉搜索树中, 任意一条从根到叶结点的路径将S分为三部分: 在该结点左边结点中的元素组成集合S1; 在该路径上的结点中的元素组成集合S2; 在该路径右边结点中的元素组成集合S3,  $S=S1 \cup S2 \cup S3$ . 若对于任意的a  $\in$  S1, b  $\in$  S2, c  $\in$  S3, 是否总有 $a \leq b \leq c$ ? 为什么?
7. 将关键码DEC, FEB, NOV, OCT, JUL, SEP, AUG, APR, MAR, MAY, JUN, JAN 依次插入到一棵初始为空的AVL 树中, 画出每插入一个关键码后的AVL 树, 并标明平衡旋转的类型.
- \*8. 对于一个高度为h 的AVL 树, 其最少结点数是多少? 反之, 对于一个有n 个结点的AVL 树, 其最大高度是多少? 最小高度是多少?

## Chapter 4.1

9. 分别 delete 50 ,40 in the following 3阶B-树.



10. 分别画出插入65, 15, 40, 30后的3阶B-树。

