ECEn 672 - Laser Tag

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1 Introduction

The students of ECEn 390 design a laser tag system in which toy guns are outfitted with red LEDs to act as lasers. These lasers are modulated at different frequencies which indicate the shooter to the receiving system. In ECEn 390, the frequency of the shooter is determined by sending the received signal through a bank of bandpass filters. The shooter frequency corresponds to the filter that has the greatest output power.

Table 1: Laser Tag Player Frequencies

Frequency (Hz)									
1471	1724	2000	2273	2632	2941	3333	3571	3846	4167

For this project, we are asked to design an optimum detector using Neyman-Pearson detection principles. Our team models this project as a binary decision with a simple and a composite hypothesis shown in the block diagram of Fig. 1. So, we make two decisions. First, did we get shot, yes or no, and second, who shot us, with choices between the 10 frequencies shown in Table 1. The pull of a trigger transmits a signal 200 ms long, and with our receiver operating at 10 ksamples/s, we are able to receive 2000 samples for one hit.

This report outlines our solution to this optimal detector problem. The first test of whether or not we got shot is derived first, followed by an analysis of that test with regard to the probability of false alarm and the probability of detection. The second test is then derived to determine the frequency of the shooter. Validation of this test is shown using a simulation in MATLAB. The analysis of the second test is then provided.

2 Binary Test

The first test is a binary test of a simple hypothesis against a composite alternative. This matches the form of the uniformly most powerful (UMP) tests derived in the book and in the supplementary material for the class.

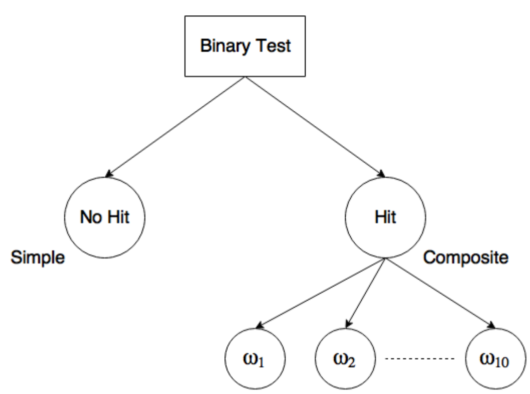


Figure 1: A block diagram of the two stages in our detector.

2.1 Derivation

Let $\mathbf{x}$ be a $N \times 1$ vector of samples of our received signal after we are hit.

$$\mathbf{x} = \mu\mathbf{s} + \mathbf{n} \tag{1}$$

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \mu \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix} + \begin{bmatrix} n_0 \\ n_1 \\ \vdots \\ n_{N-1} \end{bmatrix} \tag{2}$$

where μ is the mean of the received signal, $\mathbf{s}$ is the transmitted signal, and $\mathbf{n}$ is a realization of the random vector $\mathbf{N} \sim N[0, \sigma^2 I]$ where I is the $N \times N$ identity.

Each element, s_t, of $\mathbf{s}$ is expressed as

$$\begin{aligned} s_t &= A\cos(\omega t - \phi) \qquad\qquad t = 0, 1, \ldots, N-1, & (3)\\ &= A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) & (4) \end{aligned}$$

where ϕ is the unknown phase of our signal, ω is the frequency, and A is the amplitude. We will assume that every laser gun transmits with the same amplitude. The vector $\mathbf{s}$ can then be written as

$$\mathbf{s} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix} = \begin{bmatrix} A\cos(\phi) \\ A\cos(\omega - \phi) \\ A\cos((2)\omega - \phi) \\ \vdots \\ A\cos((N-1)\omega - \phi) \end{bmatrix}. \tag{5}$$

$\mathbf{s}$ can also be written as $\mathbf{s} = \mathbf{H}\Theta$ where

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & \ldots & 1 & 0 \\ \cos(\omega_1) & \sin(\omega_1) & \ldots & \cos(\omega_{10}) & \sin(\omega_{10}) \\ \cos(\omega_1(2)) & \sin(\omega_1(2)) & \ldots & \cos(\omega_{10}(2)) & \sin(\omega_{10}(2)) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \cos(\omega_1(N-1)) & \sin(\omega_1(N-1)) & \ldots & \cos(\omega_{10}(N-1)) & \sin(\omega_{10}(N-1)) \end{bmatrix}, \tag{6}$$

$$\Theta = \begin{bmatrix} A_0\cos(\phi_0) \\ -A_0\sin(\phi_0) \\ \vdots \\ A_{10}\cos(\phi_{N-1}) \\ -A_{10}\sin(\phi_{N-1}) \end{bmatrix}. \tag{7}$$

The matrix $\mathbf{H}$ is a $N \times 20$ matrix because we have N time samples (N rows) and ten frequencies which each correspond to two linear components, a cosine and sine, (20 columns). Θ is a 20×1 vector which is unknown because of the unknown phase and unknown amplitude of our signal.

The hypotheses in our first stage detector are

$$H_0 : \mu = 0 \tag{8}$$

$$H_1 : \mu > 0 \tag{9}$$

This is a one-sided binary test where we test a simple hypothesis against a composite alternative:

$$\text{Under} \quad H_0 : \mathbf{x} \sim N\left[0, \sigma^2\mathbf{I}\right], \tag{10}$$

$$\text{Under} \quad H_1 : \mathbf{x} \sim N\left[\mu\mathbf{H}\Theta, \sigma^2\mathbf{I}\right]. \tag{11}$$

The test statistic based on the log-likelihood ratio is

$$a' = \frac{\Theta^T\mathbf{H}^T\mathbf{x}}{\sigma^2}. \tag{12}$$

The density of the test statistic a' is

$$\text{Under} \quad H_0 : a' \sim N\left[0, \frac{E_s}{\sigma^2}\right], \tag{13}$$

$$\text{Under} \quad H_1 : a' \sim N\left[\frac{\mu E_s}{\sigma^2}, \frac{E_s}{\sigma^2}\right], \tag{14}$$

where $E_s = \Theta^T \mathbf{H}^T \mathbf{H} \Theta$.

A good statistic is one that measures the energy of $\mathbf{x}$ that lies in $\langle \mathbf{H} \rangle$. We find the energy in $\langle \mathbf{H} \rangle$ by projecting $\mathbf{x}$ into $\langle \mathbf{H} \rangle$ using the projection $\mathbf{P}_H \mathbf{x}$ where $\mathbf{P}_H = \mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T$. Then the density of $\mathbf{P}_H \mathbf{x}$ is

$$\text{Under} \quad H_0 : \mathbf{P}_H \mathbf{x} \sim N\left[0, \sigma^2 \mathbf{P}_H\right], \tag{15}$$

$$\text{Under} \quad H_1 : \mathbf{P}_H \mathbf{x} \sim N\left[\frac{\mu \mathbf{H} \Theta}{\sigma^2}, \sigma^2 \mathbf{P}_H\right]. \tag{16}$$

Since σ^2 is unknown, we have to estimate $\hat{\sigma}^2$.

First, the energy of the received signal in $\langle \mathbf{H} \rangle$ is

$$F_1 = (\mathbf{P}_H \mathbf{x})^T \mathbf{P}_H \mathbf{x} = \mathbf{x}^T \mathbf{P}_H \mathbf{x}. \tag{17}$$

Second, because of noise, $\mathbf{x} = \mathbf{H}\Theta + \mathbf{n}$ has components in both $\langle \mathbf{H} \rangle$ and $\langle \mathbf{A} \rangle$. In mathematical terms,

$$\mathbf{x} = \mathbf{P}_H \mathbf{x} + P_A \mathbf{x} = \mathbf{P}_H \mathbf{x} + (\mathbf{I} - \mathbf{P}_H)\mathbf{x}. \tag{18}$$

We are now in a position to estimate our noise variance, $\hat{\sigma}^2$, as

$$\hat{\sigma}^2 = \frac{1}{N-p}\left[(\mathbf{I} - \mathbf{P}_H)\mathbf{x}\right]^T (\mathbf{I} - \mathbf{P}_H)\mathbf{x}, \tag{19}$$

$$= \frac{1}{N-p} \mathbf{x}^T (\mathbf{I} - \mathbf{P}_H)^T (\mathbf{I} - \mathbf{P}_H)\mathbf{x}, \tag{20}$$

$$= \frac{1}{N-p} \mathbf{x}^T (\mathbf{I} - \mathbf{P}_H)\mathbf{x}, \tag{21}$$

since $(\mathbf{I} - \mathbf{P}_H)$ is both symmetric and idempotent.

Then we can establish our test statistic as

$$F' = \frac{F_1}{\hat{\sigma}^2} = \frac{\mathbf{x}^T \mathbf{P}_H \mathbf{x}}{\frac{1}{N-20} \mathbf{x}^T (\mathbf{I} - \mathbf{P}_H)\mathbf{x}}. \tag{22}$$

The numerator is a non-central chi-squared random variable with 20 degrees of freedom. And the denominator is a central chi-squared random variable with $N - 20$ degrees of freedom. We normalize each χ^2 in F' by the true variance, σ^2,

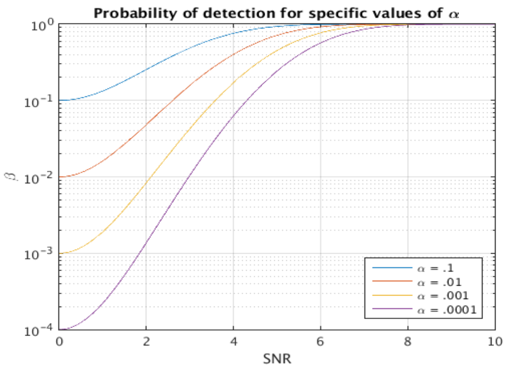
$$F = \frac{\frac{\mathbf{x}^T \mathbf{P}_H \mathbf{x}}{2\sigma^2}}{\frac{1}{\sigma^2 (N-20)} \mathbf{x}^T (\mathbf{I} - \mathbf{P}_H)\mathbf{x}}. \tag{23}$$

The ratio F is F-distributed, with non-centrality parameter $\frac{\mu E_s}{\hat{\sigma}^2}$. Our test would be

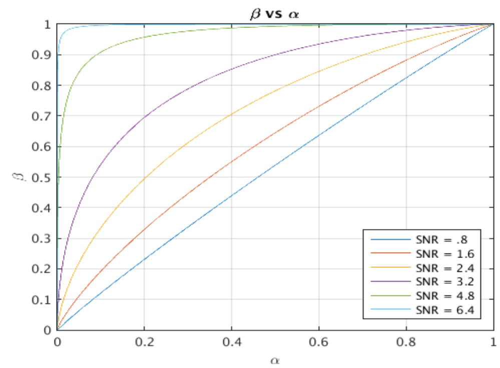
$$\phi(x) = \begin{cases} 1, & F > F_0 \\ 0, & F \leq F_0 \end{cases} \tag{24}$$

2.2 Analysis

Now that the detector is defined with an F-distributed test statistic, we can analyze the probability of false alarm, α, and the probability of detection, β, where



(a)



(b)

Figure 2: (a) Plot of β for varying α. (b) Plot of β versus α, the ROC curves.

$$\alpha = P[F > F_0 | H_0] = \int_{F_0}^{\infty} f(y|H_0) dy \tag{25}$$

$$= \int_{F_0}^{\infty} \frac{1}{B\left(\frac{p}{2}, \frac{N-p}{2}\right)} \left(\frac{p}{N-p}\right)^{p/2} y^{p/2-1} \left(1 + \frac{p}{N-p} y\right)^{-N/2} dy, \tag{26}$$

$$\beta = P[F > F_0 | H_1] = \int_{F_0}^{\infty} f(y|H_1) dy \tag{27}$$

$$= \int_{F_0}^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^k}{B\left(\frac{p}{2}, \frac{N-p}{2}\right) k!} \left(\frac{p}{N-p}\right)^{p/2+k} \left(\frac{N-p}{N-p+py}\right)^{N/2+k} y^{p/2-1+k} dy. \tag{28}$$

Ideally, we would set the value of α to be that of $1-\beta$. This selection for α and β is ideal because this would result in our threshold value F_0 to be exactly where $f(y|H_0)$ and $f(y|H_1)$ intersect. Unfortunately, we are unable to calculate this value as it requires knowledge of $\frac{\mu E_s}{\sigma^2}$ which is unknown in our problem. Fortunately, Eq. (26) shows that our calculation for α does not rely on prior knowledge of $\frac{\mu E_s}{\sigma^2}$, allowing us to choose a value for α (we chose a value of 0.001).

Analysis of α vs β for specific values of $\frac{\mu E_s}{\sigma^2}$, as well as analysis of β vs $\frac{\mu E_s}{\sigma^2}$ for specific values of alpha is shown in Fig. (2). (Note that $\frac{\mu E_s}{\sigma^2}$ is labeled as SNR in Fig. (2))

3 Composite Test

In this section, we compare the PDFs $f(x|H_m)$ and $f(x|H_{m'})$ where $m \neq m'$ and $m = 1, \ldots, 10$ to define our test for determining who shot us.

3.1 Derivation

For a given frequency m, the correct detection probability is found by integrating $f(x|H_m)$ as

$$\int_{x_1}^{x_2} f(x|H_m) dx, \tag{29}$$

where $[x_1, x_2]$ is the region where $f(x|H_m)$ is the greatest out of all ten frequencies.

We can formulate our test $\phi(x)$ as

$$\phi(x) = \Big\{ H_m, \quad f(x|H_m) > f(x|H_{m'}) \; \forall \; m' \neq m \tag{30}$$

where it is shown that our decision on who shot us is simply a comparison of the likeliness of each frequency. The inequality of Eq. (30) is expanded as

$$\frac{1}{(2\pi\sigma^2)^{N/2}} e^{\left(-\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_m]^T[\mathbf{x}-\mathbf{s}_m]\right)} > \frac{1}{(2\pi\sigma^2)^{N/2}} e^{\left(-\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_{m'}]^T[\mathbf{x}-\mathbf{s}_{m'}]\right)} \tag{31}$$

where $\mathbf{s}_m = \mathbf{H}_m \Theta_m$ and $\mathbf{H}_m$ is a $N \times 2$ matrix and Θ_m is a 2×1 vector

$$\mathbf{H}_m = \begin{bmatrix} 1 & 0 \\ \cos(\omega_m) & \sin(\omega_m) \\ \cos(\omega_m(2)) & \sin(\omega_m(2)) \\ \vdots & \vdots \\ \cos(\omega_m(N-1)) & \sin(\omega_m(N-1)) \end{bmatrix}, \qquad \Theta_m = \begin{bmatrix} \cos(\phi_m) \\ -\sin(\phi_m) \end{bmatrix}. \tag{32}$$

For a given frequency, m, each element, $s_{m,t}$, of $\mathbf{s}_m$ can be expressed as

$$s_{m,t} = \cos(\omega_m t - \phi_m) \qquad\qquad t = 0, 1, \ldots, N-1, \tag{33}$$

$$= \cos(\omega_m t)\cos(\phi_m) - \sin(\omega_m t)\sin(\phi_m) \tag{34}$$

where ϕ_m is the unknown phase of our signal, ω_m is the frequency, and A_m is the amplitude. We will assume that every laser gun transmits with the same amplitude.

$$\mathbf{s}_m = \begin{bmatrix} s_{m,0} \\ s_{m,1} \\ \vdots \\ s_{m,N-1} \end{bmatrix} = \begin{bmatrix} \cos(\phi_m) \\ \cos(\omega_m - \phi_m) \\ \cos(\omega_m(2) - \phi_m) \\ \vdots \\ \cos(\omega_m(N-1) - \phi_m) \end{bmatrix} \tag{35}$$

Eliminating like terms, Eq. (31) simplifies to

$$e^{\left(-\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_m]^T[\mathbf{x}-\mathbf{s}_m]\right)} > e^{\left(-\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_{m'}]^T[\mathbf{x}-\mathbf{s}_{m'}]\right)}. \tag{36}$$

Take the natural log of both sides and simplify,

$$-\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_m]^T[\mathbf{x}-\mathbf{s}_m] > -\frac{1}{2\sigma^2}[\mathbf{x}-\mathbf{s}_{m'}]^T[\mathbf{x}-\mathbf{s}_{m'}], \tag{37}$$

$$[\mathbf{x}-\mathbf{s}_m]^T[\mathbf{x}-\mathbf{s}_m] < [\mathbf{x}-\mathbf{s}_{m'}]^T[\mathbf{x}-\mathbf{s}_{m'}]. \tag{38}$$

Expanding each side and simplifying, we get

$$\mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{s}_m - \mathbf{s}_m^T\mathbf{x} + \mathbf{s}_m^T\mathbf{s}_m < \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{s}_{m'} - \mathbf{s}_{m'}^T\mathbf{x} + \mathbf{s}_{m'}^T\mathbf{s}_{m'}, \tag{39}$$

$$-2\mathbf{x}^T\mathbf{s}_m + \mathbf{s}_m^T\mathbf{s}_m < -2\mathbf{x}^T\mathbf{s}_{m'} + \mathbf{s}_{m'}^T\mathbf{s}_{m'}, \tag{40}$$

$$-2\mathbf{x}^T\mathbf{s}_m + E_{s,m} < -2\mathbf{x}^T\mathbf{s}_{m'} + E_{s,m'}. \tag{41}$$

We can make the approximation that $E_{s,m} \approx E_{s,m'}$ to get

$$\mathbf{x}^T\mathbf{s}_m > \mathbf{x}^T\mathbf{s}_{m'}. \tag{42}$$

The approximation $E_{s,m} \approx E_{s,m'}$ is reasonable because the quantity $\mathbf{s}_m^T\mathbf{s}_m$ for every frequency and with a random phase, is close to equal. Table 2 shows the results of computing $E_{s,m}$ numerically with ten random phase terms.

Table 2: Comparison of $E_{s,m}$ for different frequencies and random ϕ_m

	$\phi1$	$\phi2$	$\phi3$	$\phi4$	$\phi5$
$E_{s,1}$	999.98	1000.51	1000.18	999.85	1000.55
$E_{s,2}$	999.61	1000.13	999.75	1000.28	1000.21
$E_{s,3}$	1000.00	1000.00	1000.00	1000.00	1000.00
$E_{s,4}$	1000.03	1000.27	1000.12	999.89	1000.29
$E_{s,5}$	999.73	999.76	999.71	1000.29	999.79
$E_{s,6}$	999.60	1000.05	999.71	1000.31	1000.13
$E_{s,7}$	999.82	1000.16	999.92	1000.10	1000.20
$E_{s,8}$	999.41	999.83	999.48	1000.53	999.92
$E_{s,9}$	999.28	999.68	999.34	1000.68	999.78
$E_{s,10}$	999.917	999.45	999.72	1000.25	999.42

A good test statistic is the energy of $\mathbf{x}$ that lies in $\langle \mathbf{H}_m \rangle$ which is a subspace of $\langle \mathbf{H} \rangle$. The projection of $\mathbf{x}$ into $\langle \mathbf{H}_m \rangle$ is $\mathbf{P}_{H,m}\mathbf{x}$ where $\mathbf{P}_{H,m} = \mathbf{H}_m(\mathbf{H}_m^T\mathbf{H}_m)^{-1}\mathbf{H}_m^T$. The projection $\mathbf{P}_{H,m}\mathbf{x}$ is distributed as

$$\text{Under} \quad H_m : \mathbf{P}_{H,m}\mathbf{x} \sim N\left[\mu\mathbf{H}_m\Theta_m, \sigma^2\mathbf{P}_{H,m}\right] \tag{43}$$

The energy of $\mathbf{x}$ lying in $\langle \mathbf{H}_m \rangle$ is defined as $\mathbf{x}^T\mathbf{P}_{H,m}\mathbf{x}$. This is equivalent to $\mathbf{x}^T\mathbf{s}_m$ as shown

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{P}_{H,m}\mathbf{x}, \tag{44}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{P}_{H,m}(\mathbf{s}_m + \mathbf{n}), \tag{45}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T(\mathbf{P}_{H,m}\mathbf{s}_m + \underbrace{\mathbf{P}_{H,m}\mathbf{n}}_{0}), \tag{46}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{P}_{H,m}\mathbf{H}_m\Theta_m, \tag{47}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{H}_m \underbrace{(\mathbf{H}_m^T\mathbf{H}_m)^{-1}\mathbf{H}_m^T\mathbf{H}_m}_{\mathbf{I}} \Theta_m, \tag{48}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{H}_m\Theta_m, \tag{49}$$

$$\mathbf{x}^T\mathbf{s}_m = \mathbf{x}^T\mathbf{s}_m. \tag{50}$$

From Eq. (42) we calculate $\mathbf{x}^T\mathbf{P}_{H,m}\mathbf{x}$ for every frequency, m, to determine the space which contains most of the energy of $\mathbf{x}$. The block diagram in Fig. (3) shows how this statistic can be used in practice to detect the correct shooter. The frequency of the shooter is the frequency at which the corresponding space $\langle \mathbf{H}_m \rangle$ contains the most energy.

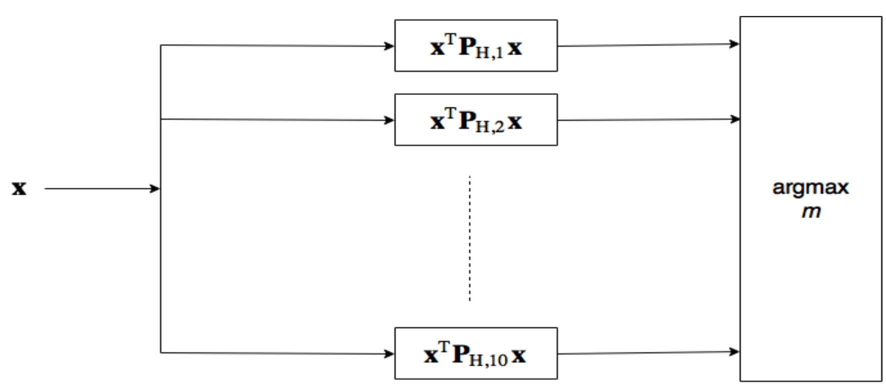


Figure 3: A block diagram of the detector for the composite hypothesis decision. Q is the non-central F-distributed random variable found in the analysis of the second stage detector by estimating the noise variance.

3.2 Analysis

The analysis of this composite hypothesis is concerned primarily with the probability of correct detection and the probability of incorrect detection since this detector is only employed when we know that we have been shot. Before we analyze these probabilities, we estimate the noise variance, $\hat{\sigma}^2$ like we did in Section 2.2 to create a F-distributed random variable. So,

$$\hat{\sigma}^2 = \frac{1}{N-p}\mathbf{x}^T(\mathbf{I} - \mathbf{P}_H)\mathbf{x}, \tag{51}$$

because we have to accept that all of the potential signals have the same noise, no matter the frequency. Then we can say,

$$\text{Under} \quad H_m : \frac{\mathbf{P}_{H,m}\mathbf{x}}{\hat{\sigma}} \sim N\left[\frac{\mu\mathbf{H}_m\Theta_m}{\hat{\sigma}}, \mathbf{P}_{H,m}\right] \tag{52}$$

After normalizing by the true noise variance, we arrive at a new test statistic, Q_m, seen in Fig. (3). Q_m is expressed as

$$Q_m = \frac{\frac{\mathbf{x}^T\mathbf{P}_{H,m}\mathbf{x}}{2\sigma^2}}{\frac{1}{\sigma^2(N-20)}\mathbf{x}^T(\mathbf{I} - \mathbf{P}_H)\mathbf{x}}, \tag{53}$$

and is F-distributed with numerator degrees of freedom 2, and denominator degrees of freedom $N - 20$. The non-centrality parameter of Q_m is the SNR:

$$\text{SNR} = \frac{\mu^2(\mathbf{H}_m\Theta_m)^T(\mathbf{H}_m\Theta_m)}{\hat{\sigma}^2} = \frac{\mu^2\mathbf{x}^T\mathbf{P}_{H,m}\mathbf{x}}{\sigma^2}. \tag{54}$$

We can now analyze the probability of error, $P_{err,m}$, by using the union bound to find an upper bound. The union bound states,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \tag{55}$$

where A_i is a set of events. In our case, the events we are concerned with are of the form

$$A_i = (\phi = i|H_m), \tag{56}$$

where ϕ is our decision on which frequency with which we were shot, and H_m is the hypothesis in force with m being the frequency of the shooter. So, our probability of error is

$$P_{err,m} = P\left(\bigcup_{i=1}^{10} A_i\right) \leq \sum_{i=1}^{10} P(A_i) \qquad i \neq m \tag{57}$$

where the probability of a single event, $P(A_i)$, is

$$P(A_i) = P(Q_m < Q_i) = P\left(\frac{Q_i}{Q_m} < 1\right). \tag{58}$$

Now, since we have defined Q as a F-distributed random variable, this ratios of Q's is undesirable to deal with. However, the degrees of freedom in the denominator of Q is quite large ($N - 20 = 1980$) and each Q can be approximated as either a chi-squared random variable. Let $\hat{Q}$ be the chi-squared approximation of Q. $\hat{Q}_m$ will be a non-central chi-squared random variable with SNR being the non-centrality parameter. Each other $\hat{Q}_i$ where $i \neq m$ will essentially be central chi-squared random variables because they are associated with other frequencies and therefore will have a very small energy in $\langle\mathbf{H}_m\rangle$.

So, the ratio of Q's in Eq. (58), with $Q_m \approx \hat{Q}_m$ and $Q_i \approx \hat{Q}_i$, is a non-central F-distributed random variable with numerator degrees of freedom of 2, denominator degrees of freedom of 2, and a non-centrality parameter SNR. The $P(A_i)$ can now be expressed as

$$P(A_i) = P\left(\frac{\hat{Q}_i}{\hat{Q}_m} < 1\right) = F_{2,2}(1, \text{SNR}). \tag{59}$$

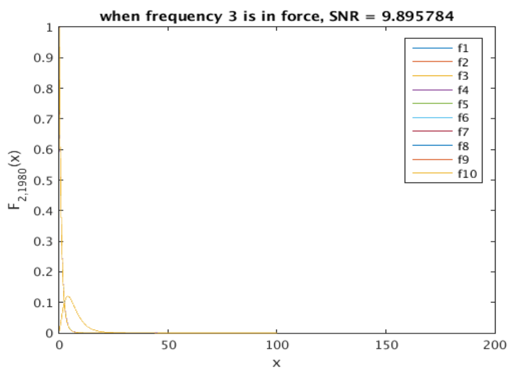
Referring back to Eq. (57), the upper bound on $P_{err,m}$ is

$$P_{err,m} \leq \sum_{i=1}^{10} F_{2,2}(1, \mathrm{SNR}) \qquad i \neq m. \tag{60}$$

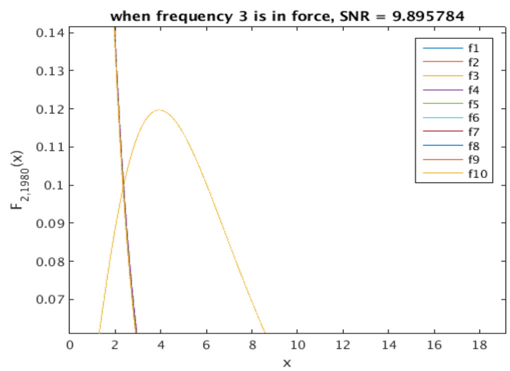
The probability of correct detection, $P_{D,m}$, is the complement of $P_{err,m}$

$$P_{D,m} = 1 - P_{err,m}. \tag{61}$$

The $F_{2,2}(1, \mathrm{SNR})$ is shown in Fig. (4). We observe that the hypothesis for the frequency that is in force has the highest non-centrality parameter. Fig. (4) is just an example given a certain value of SNR. Fig. (5) shows that for each different hypothesis, one for each frequency, the probability of correct detection versus SNR is very similar. This tells us that no shooter will be identified more easily than another. The SNR defined by Eq. (54) should be greater than 20 in order to get a good correct detection rate.



(a)



(b)

Figure 4: Plots of our F-distributed random variable Q. Here, the frequency of the shooter is $m = 3$ and the SNR $= 9.8957$. (a) Standard view. (b) Zoomed-in on intersection.

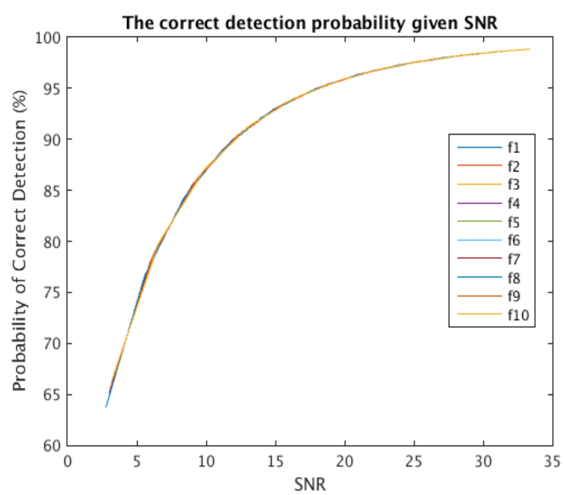


Figure 5: Plot of the probability of correct detection versus SNR.

4 Conclusion

This project demonstrated the usefulness of maixmum likelihood principles in a practical situation, at least for ECEn 390 students. A key assumption that we make throughout our development is that we are only ever being shot by one player at a time. This would make the development of our Neyman-Pearson test significantly more complex and would have turned this problem into an estimation problem as well. Another assumption we make is that we have a full set of 2000 samples from a single shooter. Depending on how this detector is implemented, it is possible that we would need to detect that a player shot us based on only a fraction of the total possible samples. The only thing that this would influence however is the mean of our received signal because the other portion of non-signal samples would be entirely noise. This would make the SNR worse and cause a greater probability of false alarm. While making the detection more difficult, the only part of our overall detection system that would be affected is the initial detection of whether or not we were hit. After that decision is made, we can proceed with the frequency detection as normal. Another issue is if the transmitter and the receiver are not aligned properly. This would simply weaken the signal.

5 Affidavit

We all worked on this project equally.

Signed: