Lecture 10

Model with serial correlation:

$$y_t = \beta_1 + \beta_2 x_{2t} + \ldots + \beta_k x_{kt} + u_t, \quad u_t = \rho u_{t-1} + e_t$$

We take the one-step-back model: $y_{t-1} = \beta_1 + \beta_2 x_{2,t-1} + \ldots + \beta_k x_{k,t-1} + u_{t-1}$;

Multiply it through by ρ : $\rho y_{t-1} = \rho\beta_1 + \rho\beta_2 x_{2,t-1} + \ldots + \rho\beta_k x_{k,t-1} + \rho u_{t-1}$;

And subtract from the original model:

$$y_t - \rho y_{t-1} = \beta_1 (1-\rho) + \beta_2 (x_{2t} - \rho x_{2,t-1}) + \ldots + \beta_k (x_{kt} - \rho x_{k,t-1}) + {\color{red} u_t - \rho u_{t-1}}$$

- The part in red equals e_t.

The model can thus be rewritten as follows:

$$y_t^* = \beta_1^* + \beta_2 x_{2t}^* + \ldots + \beta_k x_{kt}^* + e_t$$

where $y_t^* = y_t - \rho y_{t-1}$, $\beta_1^* = \beta_1(1-\rho)$ and $x_{it}^* = x_{it} - \rho x_{i,t-1}$.

This procedure is known as **quasi differencing**. More here and here.

- If knew ρ, then the transformed model would be fully efficient (and unbiased, since the original model already was).
- However, if we obtain a consistent estimate of ρ, then, as time goes to infinity, the estimated model will converge to the true (transformed) model.

Cochrane-Orcutt procedure

1. Estimate $y_t = \beta_1 + \beta_2 x_{2t} + \ldots + \beta_k x_{kt} + u_t$ with OLS. Save the residuals $\hat{u}_t,\ t = 1, 2, \ldots, T$.
   - We rely upon the fact that the estimated β_i will be unbiased albeit inefficient, so we will obtain an estimate of the residuals which is unbiased and consistent.
2. Estimate ρ with $\hat{\rho} = \dfrac{\sum_{t=2}^{T} \hat{u}_t \hat{u}_{t-1}}{\sum_{t=1}^{T-1} \hat{u}_t^2}$.
   - This is the same as a regression of $\hat{u}_t$ on $\hat{u}_{t-1} : \hat{u}_t = \rho\hat{u}_{t-1} + e_t$.
3. Transform the data, considering:
   - $y_t^* = y_t - \hat{\rho} y_{t-1}$;
   - $\beta_1^* = \beta_1(1-\hat{\rho})$;
   - $x_{it}^* = x_{it} - \hat{\rho} x_{i,t-1}$;
   - $e_t = u_t - \hat{\rho} u_{t-1}$.
4. Regress y_t^* with OLS.
   - This will produce $\hat{\beta}_1^*, \hat{\beta}_2, \ldots, \hat{\beta}_k$.
   - $\hat{\beta}_1 = \dfrac{\hat{\beta}_1^*}{1-\hat{\rho}}$.
   - The coefficients $\hat{\beta}_i$ obtained here are "less inefficient" than those of step 1.
   - The residuals of this regression are estimates of e_t (rather than u_t).
5. Use these estimates $\hat{\beta}_i$ to get a new estimate of $\hat{u}_t$.

- The new $\hat{u}_t = y_t - \hat{\beta}_1 - \hat{\beta}_2 x_{2t} + \ldots + \hat{\beta}_k x_{kt}$.

6. Repeat steps 2-5 until $\hat{\rho}$ changes by less than $.0001$ (rule of thumb).
    - By there, the coefficients will have converged to their fully efficient values.

Engle's ARCH Test

ARCH: Autoregressive Conditional Heteroskedasticity.

Most basic model: $\sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \ldots + \alpha_p \sigma_{t-p}^2$.

How do we test for ARCH?

The original regression model is of the form:

$$y_t = \beta_1 + \beta_2 x_{2t} + \ldots + \beta_k x_{kt} + u_t$$

where $\mathrm{Var}(u_t) = \sigma_t^2$.

Hypothesis test:

$$\begin{aligned} H_0 &: \alpha_1 = \alpha_2 = \ldots = \alpha_p = 0 \\ H_1 &: \exists\, \alpha_i \neq 0 \; \big(i \in \{1, 2, \ldots, p\}\big) \end{aligned}$$

Steps:

1. Run y_t on constant, $x_{2t}, \ldots, x_{kt}$. Save $\hat{u}_t$ and square it: $\hat{u}_t^2$.
2. Run $\hat{u}_t^2$ on constant (α_0), $\hat{u}_{t-1}^2, \ldots, \hat{u}_{t-p}^2$.
3. The test statistic is given by: $(T-p) \cdot R^2$.
    - $(T-p)$ is the number of observations (T) minus the number of restrictions (p).
    - R^2 is coefficient of determination.
    - The test statistic follows a chi-squared distribution with p degrees of freedom: $(T-p) \cdot R^2 \sim \chi^2(p)$.