Lecture 13

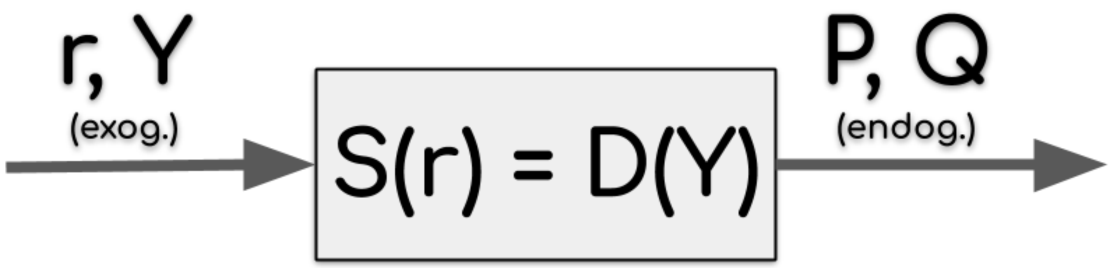
Structural Equations

Usually take the form $\text{endogenous} = f(\text{endogenous}, \text{exogenous})$.

Example 1:

$$\begin{cases} Q^D = \alpha_0 + \alpha_1 P + \alpha_2 Y + u \\ Q^S = \beta_0 + \beta_1 P + \beta_2 r + v \\ Q^D = Q^S \end{cases}$$

- The variables are:
  - Q^D: quantity demanded, Q^S: quantity supplied;
  - P: price;
  - Y: income, r: rainfall.
    - To know more about the *rainfall* variable, see the Model IV in Professor Jeffrey A. Parker's notes and Section 9.2, p. 253, of Hanushek & Jackson, 1977.
- There are 2 behavioral equations and 1 equilibrium condition.
- More on simultaneous supply and demand equations here.



- The system has 2 endogenous variables ($Q \equiv Q^D = Q^S$ and P) and 3 exogenous variables (Y, r and the constants*).
  - Didactically, the constants will be the column of ones added to the model, which multiplies α_0, β_0 etc.
  - The number of endogenous variables will be denoted as G.
    - In the supply and demand model above, $G = 2$.
  - The number of exogenous variables will be denoted as K.
    - In the supply and demand model above, $K = 3$.

Example 2:

$$\begin{cases} C_t = \alpha_0 + \alpha_1 Y_t^D + \alpha_2 Y_{t-1}^D + u_t \\ I_t = \beta_0 + \beta_1 Y_t + \beta_2 Y_{t-1} + v_t \\ Y_t = C_t + I_t + G_t \\ Y_t^D \equiv Y_t - T_t \end{cases}$$

- The variables are:
  - C_t: consumption, I_t: investment, Y_t^D: disposable income;
  - G_t: government spending, Y_t: income, T_t: taxes.
- There are 2 behavioral equations, 1 equilibrium condition and 1 identity.
- The system features:
  - 4 endogenous variables (Y_t, C_t, I_t, Y_T^D);

- 2 predetermined variables (Y_{t-1}^D, Y_{t-1}), which were endogenous in the previous step ($t-1$), but now work *as if they were* exogenous;
- 3 exogenous variables (G_t, T_t and the constants).

Another type of equation that can feature in simultaneous models is the technical equation.

- One notable example is the production function $Y = f(K, L)$.
  - E. g., $\ln Y = d_0 + d_1 \ln K + d_2 \ln L$.
- Rather than modeling a behavior, this type of equation describes the transformation of inputs into output.

Reduced-form equations

Take the form $\text{endogenous} = f(\text{exogenous})$.

Resuming the supply-demand model:

$$\begin{cases} Q = \alpha_0 + \alpha_1 P + \alpha_2 Y + u \\ Q = \beta_0 + \beta_1 P + \beta_2 r + v \end{cases}$$

- Putting the two equation together, we get: $\alpha_0 + \alpha_1 P + \alpha_2 Y + u = \beta_0 + \beta_1 P + \beta_2 r + v$.
  - Solving for $P: P = \dfrac{1}{\alpha_1 - \beta_1}\Big[(\beta_0 - \alpha_0) - \alpha_2 Y + \beta_2 r + v - u\Big]$.
  - That can be rewritten as: $P = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} - \dfrac{\alpha_2}{\alpha_1 - \beta_1} Y + \dfrac{\beta_2}{\alpha_1 - \beta_1} r + \dfrac{v - u}{\alpha_1 - \beta_1}$, which is the reduced form equation for P.
- For purposes of simplification, the equation will be rewritten as $P = \lambda_0 + \lambda_1 Y + \lambda_2 r + \varepsilon_1$:
  - $\lambda_0 = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}$;
  - $\lambda_1 = -\dfrac{\alpha_2}{\alpha_1 - \beta_1}$;
  - $\lambda_2 = \dfrac{\beta_2}{\alpha_1 - \beta_1}$;
  - $\varepsilon_1 = \dfrac{v - u}{\alpha_1 - \beta_1}$.
- Thus, to solve it for Q, we plug P back into one the original equations:
  - For the first equation, $Q = \alpha_0 + \alpha_1 \big(\lambda_0 + \lambda_1 Y + \lambda_2 r + \varepsilon_1\big) + \alpha_2 Y + u$.
  - Rewriting: $Q = (\alpha_0 + \alpha_1 \lambda_0) + (\alpha_1 \lambda_1 + \alpha_2) Y + \alpha_1 \lambda_2 r + (u + \alpha_1 \varepsilon_1)$. This is the reduced form equation for Q.
- Again, for purposes of simplification: $Q = \gamma_0 + \gamma_1 Y + \gamma_2 r + \varepsilon_2$.
  - $\gamma_0 = \alpha_0 + \alpha_1 \lambda_0$;
  - $\gamma_1 = \alpha_1 \lambda_1 + \alpha_2$;
  - $\gamma_2 = \alpha_1 \lambda_2$.
  - $\varepsilon_2 = u + \alpha_1 \varepsilon_1$.
- The reduced form of this system is, then, $\begin{cases} P = \lambda_0 + \lambda_1 Y + \lambda_2 r + \varepsilon_1 \\ Q = \gamma_0 + \gamma_1 Y + \gamma_2 r + \varepsilon_2 \end{cases}$.

Resuming the income model:

$$\begin{cases} C_t = \alpha_0 + \alpha_1 Y_t^D + \alpha_2 Y_{t-1}^D + u_t \\ I_t = \beta_0 + \beta_1 Y_t + \beta_2 Y_{t-1} + v_t \\ Y_t = C_t + I_t + G_t \\ Y_t^D \equiv Y_t - T_t \end{cases}$$

- We can solve it for Y_t by substituting the other LHS variables into its equation:
  - $\begin{aligned} Y_t &= (\alpha_0 + \alpha_1 Y_t^D + \alpha_2 Y_{t-1}^D + u_t) + (\beta_0 + \beta_1 Y_t + \beta_2 Y_{t-1} + v_t) + G_t \\ &= \alpha_0 + \alpha_1 (Y_t - T_t) + \alpha_2 (Y_{t-1} - T_{t-1}) + u_t + \beta_0 + \beta_1 Y_t + \beta_2 Y_{t-1} + v_t + G_t \end{aligned}$.
  - Isolating Y_t: $Y_t(1 - \alpha_1 - \beta_1) = (\alpha_0 + \beta_0) - \alpha_1 T_t - \alpha_2 T_{t-1} + (\alpha_2 + \beta_2) Y_{t-1} + G_t + u_t + v_t$.
  - Rewriting: $Y_t = \dfrac{\alpha_0 + \beta_0}{1 - \alpha_1 - \beta_1} - \dfrac{\alpha_1}{1 - \alpha_1 - \beta_1} T_t - \dfrac{\alpha_2}{1 - \alpha_1 - \beta_1} T_{t-1} + \dfrac{\alpha_2 + \beta_2}{1 - \alpha_1 - \beta_1} Y_{t-1} + \dfrac{1}{1 - \alpha_1 - \beta_1} G_t + \dfrac{u_t + v_t}{1 - \alpha_1 - \beta_1}$.
- Simplifying: $Y_t = \lambda_0 + \lambda_1 T_t + \lambda_2 T_{t-1} + \lambda_3 Y_{t-1} + \lambda_4 G_t + \varepsilon_t^Y$, which is the reduced form equation for Y_t.

- $\lambda_0 = \dfrac{\alpha_0 + \beta_0}{1 - \alpha_1 - \beta_1}$;
- $\lambda_1 = -\dfrac{\alpha_1}{1 - \alpha_1 - \beta_1}$;
- $\lambda_2 = -\dfrac{\alpha_2}{1 - \alpha_1 - \beta_1}$;
- $\lambda_3 = \dfrac{\alpha_2 + \beta_2}{1 - \alpha_1 - \beta_1}$;
- $\lambda_4 = \dfrac{1}{1 - \alpha_1 - \beta_1}$;
- $\varepsilon_t^Y = \dfrac{u_t + v_t}{1 - \alpha_1 - \beta_1}$.

Simultaneity bias

Consider the simple model:

$$\begin{cases} y_t = \alpha x_t + u_t \\ x_t = \beta y_t + v_t \end{cases}$$

In this case, x_t is function of y_t which, in turn, is a function of u_t.

- The terms x_t and u_t in the first equation are correlated.
  - Then, any estimate of α will be biased.
  - As seen before, $\operatorname{plim} \hat{\alpha} = \alpha + \dfrac{\operatorname{Cov}(x, u)}{\operatorname{Var}(x)}$.
- Were β equal to zero, there would not be bias.
  - There would be a correlation between x_t and u_t, given that x_t would not be a function of y_t.
  - In that case, x_t would then be an exogenous variable, since v_t is random.
- The reduced form equations for y_t and x_t are: $\begin{cases} y_t = \dfrac{1}{1 - \alpha\beta}(\alpha v_t + u_t) \\ x_t = \dfrac{1}{1 - \alpha\beta}(\beta u_t + v_t) \end{cases}$.
  - That being so, $\operatorname{Var}(x) = \left(\dfrac{1}{1 - \alpha\beta}\right)^2 \left[\beta^2 \operatorname{Var}(u) + \operatorname{Var}(v)\right]$.

TO BE CONTINUED...