Lecture 14

Simultaneity bias

Consider the model:

$$\begin{cases} y_t = \alpha x_t + u_t \\ x_t = \beta y_t + v_t \end{cases}$$

Substituting the second equation into the first one, we get the reduced-form equation:

$$y_t = \frac{1}{1-\alpha\beta}(\alpha v_t + u_t)$$

Similarly:

$$x_t = \frac{1}{1-\alpha\beta}(\beta u_t + v_t)$$

Since x_t is a function of y_t which, in turn, has u_t, there is correlation between to variable in the RHS (first equation).

- The same goes for the second equation of the system: y_t is a function of x_t which contains v_t.
- The simultaneity then introduces bias.

Is the estimate consistent if we regress y on x?

- We have already seen that $\hat{\alpha} = \dfrac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$.
- As $N \to \infty$, $\text{plim}\ \hat{\alpha} = \dfrac{\text{Cov}(x, y)}{\text{Var}(x)}$.
    - Since $y = \alpha x + u$, then: $\dfrac{\text{Cov}(x,y)}{\text{Var}(x)} = \dfrac{\text{Cov}(x, \alpha x + u)}{\text{Var}(x)} = \dfrac{\alpha\text{Cov}(x,x) + \text{Cov}(x,u)}{\text{Var}(x)} = \alpha + \dfrac{\text{Cov}(x,u)}{\text{Var}(x)}$.
    - Thus, $\text{plim}\ \hat{\alpha} = \alpha + \dfrac{\text{Cov}(x,u)}{\text{Var}(x)}$.
        - We know that $x = \dfrac{1}{1-\alpha\beta}(\beta u + v)$.
        - Then, $\text{Cov}(x,u) = \text{Cov}\left(\dfrac{1}{1-\alpha\beta}(\beta u + v), u\right)$.
        - Since we assume $u \perp v$ (uncorrelated), $\text{Cov}(x,u) = \dfrac{1}{1-\alpha\beta}\Big[\beta\text{Cov}(u,u) + \text{Cov}(v,u)\Big] = \dfrac{\beta}{1-\alpha\beta}\text{Var}(u)$.
        - Also, $\text{Var}(x) = \left(\dfrac{1}{1-\alpha\beta}\right)^2 \Big[\text{Var}(\beta u, v)\Big]$.
        - Then, $\text{Var}(x) = \left(\dfrac{1}{1-\alpha\beta}\right)^2 \Big[\text{Var}(\beta u) + \text{Var}(v) + 2\text{Cov}(\beta u, v)\Big]$.
        - Again, $u \perp v \to \text{Var}(x) = \left(\dfrac{1}{1-\alpha\beta}\right)^2 \Big[\beta^2\text{Var}(u) + \text{Var}(v)\Big]$.

- Plugging those back in and simplifying: $\text{plim}\,\hat{\alpha} = \alpha + \beta(1-\alpha\beta)\dfrac{\text{Var}(u)}{\beta\text{Var}(u) + \text{Var}(v)}$.
    1. $\hat{\alpha}$ is inconsistent;
    2. Unless $\beta = 0$, $\hat{\alpha}$ is biased.

Example:

$$\begin{cases} P = \beta_1 + \beta_2 W + u_P \\ W = \alpha_1 + \alpha_2 P + \alpha_3 \text{UN} + u_W \quad (\alpha_3 < 0) \end{cases}$$

where:

- P: price growth (inflation), W: wage growth (endogenous variables).
- UN: unemployment (exogenous variable).
- u_P and u_W: stochastic noise associated with P and W, respectively.

Substituting the second equation into the first one, we obtain the reduced-form equation for P:

$$P = \frac{1}{1-\beta_2\alpha_2}\Big[(\beta_1 + \beta_2\alpha_1) + \beta_2\alpha_3\text{UN} + (\beta_2 u_W + u_P)\Big].$$

Similarly:

$$W = \frac{1}{1-\beta_2\alpha_2}\Big[(\alpha_1 + \alpha_2\beta_1) + \alpha_3\text{UN} + (\alpha_2 u_P + u_W)\Big].$$

We can see above that the reduced form of W has u_P in it. Then, there is correlation between W and u_P and, consequently, $\hat{\beta}_2$ will be biased.

- $\text{plim}\,\hat{\beta}_2 = \beta_2 + \dfrac{\text{Cov}(W, u_p)}{\text{Var}(W)}$.
    - $\text{Cov}(W, u_P) = \dfrac{\alpha_2}{1-\beta_2\alpha_2}\text{Var}(u_P)$.
    - $\text{Var}(W) = \dfrac{1}{(1-\beta_2\alpha_2)^2}\Big[\alpha_3^2\text{Var}(\text{UN}) + \alpha_2^2\text{Var}(u_P) + \text{Var}(u_W)\Big]$.
    - Then, $\text{plim}\,\hat{\beta}_2 = \beta_2 + \alpha_2\Big(1-\beta_2\alpha_2\Big)\dfrac{\text{Var}(u_P)}{\alpha_3^2\text{Var}(\text{UN}) + \alpha_2^2\text{Var}(u_P) + \text{Var}(u_W)}$.
- If α_2 were equal to zero, there would be no simultaneity bias.
    - Note that, were α_2 equal to zero, W would not be a function of P, it would rather only be a function of the exogenous variable UN.

Identification problem

Model 1:

$$\begin{cases} Q^D = \alpha_0 + \alpha_1 P + u \\ Q^S = \beta_0 + \beta_1 P + v \\ Q^D = Q^S \equiv Q \end{cases}$$

Reduced form:

$$P = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} + \frac{1}{\alpha_1 - \beta_1}(v-u)$$

- $P = \lambda_0 + \varepsilon_P$:
    - $\lambda_0 = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}$.

- $\varepsilon_P = \dfrac{1}{\alpha_1 - \beta_1}(v - u)$.

$$Q = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} + \frac{1}{\alpha_1 - \beta_1}\Big(\alpha_1 v + \beta_1 u\Big)$$

- $Q = \gamma_0 + \varepsilon_Q$:
  - $\gamma_0 = \dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1}$.
  - $\varepsilon_Q = \dfrac{1}{\alpha_1 - \beta_1}\Big(\alpha_1 v + \beta_1 u\Big)$.

To estimate the reduced form we would regress P and Q on constants, λ_0 and γ_0, respectively.

- $\hat{\lambda}_0 = \bar{P}$, $\hat{\gamma}_0 = \bar{Q}$.

However, we have 4 structural parameters (unknowns) - $\alpha_0, \alpha_1, \beta_0, \beta_1$ -, while running the regression will produce two pieces of information ($\hat{\lambda}_0$ and $\hat{\gamma}_0$).

- That means that $\begin{cases} \hat{\lambda}_0 = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} \\ \hat{\gamma}_0 = \dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1} \end{cases}$ is an unidentified system (it has more unknowns than equations).

This model is said to be <u>under-identified</u>

Model 2:

$$\begin{cases} Q^D = \alpha_0 + \alpha_1 P + \alpha_2 Y + u \\ Q^S = \beta_0 + \beta_1 P + v \\ Q^D = Q^S \equiv Q \end{cases}$$

Reduced form:

$$P = \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} - \frac{\alpha_2}{\alpha_1 - \beta_1}Y + \frac{1}{\alpha_1 - \beta_1}(v - u).$$

- $P = \lambda_0 + \lambda_1 Y + \varepsilon_P$.
  - $\lambda_0 = \dfrac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1}$.
  - $\lambda_1 = -\dfrac{\alpha_2}{\alpha_1 - \beta_1}$.
  - $\varepsilon_P = \dfrac{1}{\alpha_1 - \beta_1}(v - u)$.
- $Q = (\beta_0 + \beta_1\lambda_0) + \beta_1\lambda_1 Y + \varepsilon_Q = \gamma_0 + \gamma_1 Y + \varepsilon_Q$.

Running the regression on the reduced-form equations we obtain four estimates: $\hat{\lambda}_0, \hat{\lambda}_1, \hat{\gamma}_0, \hat{\gamma}_1$.

- Can we thus retrieve the structural parameters $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$?
  - Generally, no, because, once again, we have more unknowns that equations!
  - But we can identify a subset of those structural parameters:
    - $\gamma_1 = \beta_1\lambda_1 \Rightarrow \hat{\beta}_1 = \dfrac{\hat{\gamma}_1}{\hat{\lambda}_1}$.
    - $\gamma_0 = \beta_0 + \beta_1\lambda_0 \Rightarrow \hat{\beta}_0 = \hat{\gamma}_0 - \hat{\beta}_1\hat{\lambda}_0$.

The system is then partially identified, due to the presence of the variable Y which discriminates the demand curve (Q^D) and allows us to identify a particular supply curve (Q^S) for a given value of Y.

TO BE CONTINUED...