Statistics II

Inferences Based on a Two Samples Confidence Intervals and Tests of Hypothesis

統計學II 假設檢定 - 兩種方法

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6/8前將遠距授課，請至ee-class查看公告

(a)求出μ_1-μ_2之95%信賴區間。
(b)檢定此藥丸是否有降低血壓的作用(α=0.05)。

Paired-Difference Experiment Small-Sample Test Quiz

某一藥劑研究員欲檢定某種藥丸是否具有預期的作用——服用者的血壓。此項研究先記錄15位大專女性學生的初始血壓，然後在她們服用此藥丸6個月後，再度測量其血壓。

	受試者							
	1	2	3	4	5	6	7	8
服藥之前 (x)	70	80	72	76	76	76	72	78
服藥之後 (y)	68	72	62	70	58	66	68	52
$d=(x-y)$	2	8	10	6	18	10	4	26

$$\bar{d}=\frac{\sum d_i}{15}=8.80,\quad S_D=\sqrt{\frac{\sum(d_i-\bar{d})^2}{14}}=10.98$$

	受試者						
	9	10	11	12	13	14	15
服藥之前 (x)	82	64	74	92	74	68	84
服藥之後 (y)	64	72	74	60	74	72	74
$d=(x-y)$	18	-8	0	32	0	-4	10

Outlines

1. Identifying the Target Parameter
2. Comparing Two Population Means: Independent Sampling
3. Comparing Two Population Means: Paired Difference Experiments
4. Comparing Two Population Proportions: Independent Sampling
5. Comparing Two Population Variances: Independent Sampling
6. Determining the Required Sample Size

Basic Concept - Example

為確定某一**新品種種子**的**產量**是否較**目前的品種大**，在適當的農作條件下，種植此二種子。

根據這種實驗，可**比較兩品種的種子之產量**，並可比較數個地段的產量(各代表著不同的氣候與土壤條件，分別影響產量的情形)。

Basic Concept - Example

在進行藥物的評估過程中，往往依據患病的動物或人類服用該藥後的效果來判斷。

例如，為了**比較二種藥物**對控制癌細胞惡化的效應，選取同品種的老鼠來進行實驗。首先，將老鼠注射癌細胞後，分別以**藥物1**與**藥物2**治療，並**觀察這兩群老鼠癌細胞惡化的情形**，做為比較藥物效力的依據。

How would you try to answer these questions?

Who gets higher grades: students majored in CS or IM?

Which program is faster to learn: Python or R?

Identifying the Target Parameter

Determining the Target Parameter

Parameter	Key Words or Phrases	Type of Data
$\mu_1 - \mu_2$	Mean difference; differences in averages	Quantitative
$p_1 - p_2$	Differences between proportions, percentages, fractions, or rates; compare proportions	Qualitative
$\dfrac{(\sigma_1)^2}{(\sigma_2)^2}$	Ratio of variances; differences in variability or spread; compare variation	Quantitative

哪種樣本?

獨立樣本 Independent Sampling

是指受測者被**隨機地分為二群**, 其中一群指定處理1, 而另一群指定處理2, 觀察此兩種處理的反應。

所得出的觀察值彼此不相關, 因為它們來自不同且不相關的受測者。

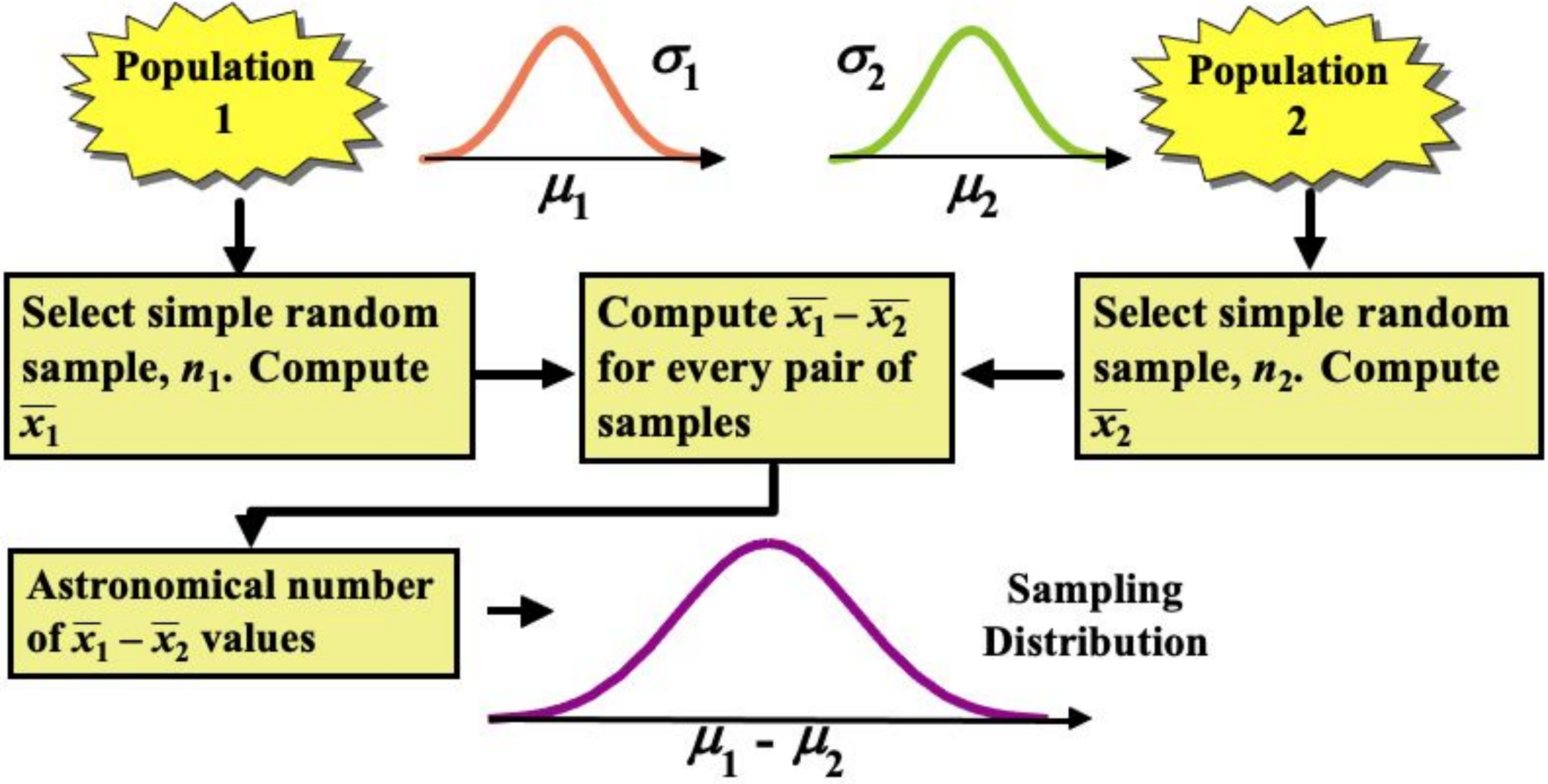
成對樣本 Paired Difference Experiments

是指受測者以**成對抽取**, 因而每對中的各元素性質相近, 而不同對的資料間性質不同。

每對的其中一個元素使用處理1, 而另一個則使用處理2。

Comparing Two Population Means: Independent Sampling

Large-sample


Population 1
σ_1
μ_1
σ_2
μ_2
Population 2
Select simple random sample, n_1. Compute $\overline{x_1}$
Compute $\overline{x_1} - \overline{x_2}$ for every pair of samples
Select simple random sample, n_2. Compute $\overline{x_2}$
Astronomical number of $\overline{x}_1 - \overline{x}_2$ values
Sampling Distribution
μ_1 - μ_2

Conditions Required for Valid Large-Sample Inferences about ($\mu_1 - \mu_2$)

1. The two samples are **randomly selected** in an **independent** manner from the **two target populations**.
2. The sample sizes, n_1 and n_2, are both **large** (i.e., ≥ 30).
   Due to the Central Limit Theorem, this condition guarantees that the sampling distribution of ($\overline{x_1} - \overline{x_2}$) will be approximately normal regardless of the shapes of the underlying probability distributions of the populations. Also, S_1^2 and S_2^2 will provide good approximations to σ_1^2 and σ_2^2 when the samples are both large.

Large-Sample Confidence Interval for ($\mu_1 - \mu_2$)

σ_1^2, σ_2^2 known:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sigma_{(\bar{x}_1 - \bar{x}_2)} = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

σ_1^2, σ_2^2 unknown:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sigma_{(\bar{x}_1 - \bar{x}_2)} = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Large-Sample Confidence Interval Example

You're a financial analyst for Charles Schwab. You want to estimate the difference in dividend yield between stocks listed on NYSE and NASDAQ.

	NYSE	NASDAQ
Number	121	125
Mean	3.27	2.53
Std Dev	1.30	1.16

What is the 95% confidence interval for the difference between the mean dividend yields (現金殖利率)?

Large-Sample Confidence Interval Solution

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$(3.27 - 2.53) \pm 1.96\sqrt{\frac{(1.3)^2}{121} + \frac{(1.16)^2}{125}}$$

$$.43 \le \mu_1 - \mu_2 \le 1.05$$

Large-Sample Confidence Interval Quiz

某項研究國人營養健康狀況調查，從東部地區隨機抽選142位男性與150位女性，測量其身高體重後計算BMI，得出分別的BMI平均數與標準差如下：

項目	女性	男性
平均數	23.6	24.4
標準差	0.3	0.3

試求出東部地區男性與女性BMI差之95%信賴區間

Large-Sample Confidence Interval Ans

令μ1代表東部地區男性BMI平均數，μ2代表女性的平均數。

$$n_1 = 142, \bar{x}_1 = 24.4, S_1 = 0.3$$
$$n_2 = 150, \bar{x}_2 = 23.6, S_2 = 0.3$$

$$\sigma_{(\bar{x}_1 - \bar{x}_2)} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{0.3^2}{142} + \frac{0.3^2}{150}} \approx 0.035$$

$$\approx \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Large-Sample Confidence Interval Ans

由於二組樣本皆為大樣本，故 $\overline{x_1}$ 為 $\overline{x_2}$ 常態分配

95%的信賴區間：

$24.4 - 23.6 \pm z_{0.025} \times 0.035$

$=0.8 \pm 1.96 \times 0.035$

$=0.8 \pm 0.0686$或$[0.7314, 0.8686]$

平均而言，東部地區男性的BMI較女性高出0.7314至0.8686間。

Large-Sample Test of Hypothesis for ($\mu_1 - \mu_2$), *z*-test

One-Tailed Test

H_0: $(\mu_1 - \mu_2) \geq D_0$ [or H_a: $(\mu_1 - \mu_2) \leq D_0$]

H_a: $(\mu_1 - \mu_2) < D_0$ [or H_a: $(\mu_1 - \mu_2) > D_0$]

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$(\mu_1 - \mu_2) < 0$ ➡️ $\mu_1 < \mu_2$

Test statistic:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sigma_{(\bar{x}_1 - \bar{x}_2)}}$$

$$\sigma_{(\bar{x}_1 - \bar{x}_2)} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \approx \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Large-Sample Test of Hypothesis for $(\mu_1 - \mu_2)$, *z*-test

Two-Tailed Test

$H_0: (\mu_1 - \mu_2) = D_0$

$H_a: (\mu_1 - \mu_2) \neq D_0$

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$(\mu_1 - \mu_2) \neq 0$ ➡️ $\mu_1 \neq \mu_2$

Test statistic:

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sigma_{(\bar{x}_1 - \bar{x}_2)}}$$

$$\sigma_{(\bar{x}_1 - \bar{x}_2)} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \approx \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Hypotheses for Means of Two Independent Populations

Hypothesis	Research Questions		
	No Difference Any Difference	Pop 1 ≥ Pop 2 Pop 1 < Pop 2	Pop 1 ≤ Pop 2 Pop 1 > Pop 2
H_0	$\mu_1 - \mu_2 = 0$	$\mu_1 - \mu_2 \geq 0$	$\mu_1 - \mu_2 \leq 0$
H_a	$\mu_1 - \mu_2 \neq 0$	$\mu_1 - \mu_2 < 0$	$\mu_1 - \mu_2 > 0$

z-test

Large-Sample Test Example

You're a financial analyst for Charles Schwab. You want to estimate the difference in dividend yield between stocks listed on NYSE and NASDAQ.

	NYSE	NASDAQ
Number	121	125
Mean	3.27	2.53
Std Dev	1.30	1.16

Is there a difference in average yield ($\alpha = .05$)?

Is there a difference => Two-tailed

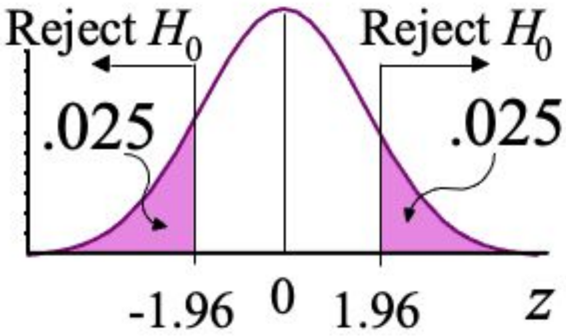
H_0: $\mu_1 - \mu_2 = 0 \; (\mu_1 = \mu_2)$

H_a: $\mu_1 - \mu_2 \neq 0 \; (\mu_1 \neq \mu_2)$

$\alpha =$ .05

$n_1 =$ 121 , $n_2 =$ 125

Critical Value(s):



Test Statistic:

$$z = \frac{(3.27 - 2.53) - 0}{\sqrt{\frac{1.69^2}{121} + \frac{1.16^2}{125}}} = +4.71$$

Decision:

Reject at $\alpha = .05$

Conclusion:

There is evidence of a difference in means

Large-Sample Test Quiz

You're an economist for the Department of Education. You want to find out if there is a difference in spending per pupil between urban and rural high schools.

	Urban	Rural
Number	35	35
Mean	$ 6,012	$ 5,832
Std Dev	$ 602	$ 497

Is there any difference in population means ($\alpha = .10$)?

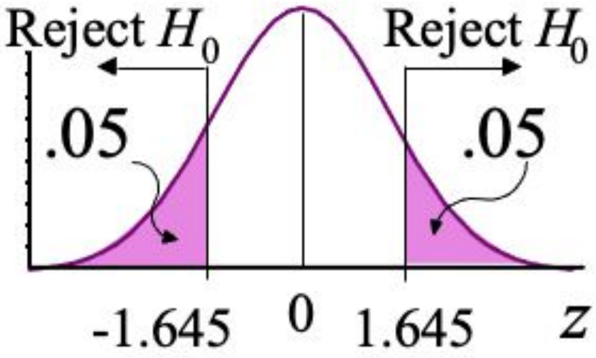
H_0: $\mu_1 - \mu_2 = 0$ ($\mu_1 = \mu_2$)

H_a: $\mu_1 - \mu_2 \neq 0$ ($\mu_1 \neq \mu_2$)

$\alpha =$ .10

$n_1 =$ 35, $n_2 =$ 35

Critical Value(s):



Test Statistic:

$$z = \frac{(6012 - 5832) - 0}{\sqrt{\frac{602^2}{35} + \frac{497^2}{35}}} = +1.36$$

Decision:

Do not reject at $\alpha = .10$

Conclusion:

There is no evidence of a difference in means

Pre-class quiz

Week 11

遠距上課，取消考試

Large-Sample Test Quiz

細懸浮微粒($PM_{2.5}$)是近年台灣用來代表空氣品質的熱門話題，一般民眾認為北部地區的空氣品質比中南部地區更糟。

以某下午2時行政院環保署發布之$PM_{2.5}$自動監測資料為例，北部地區(含桃竹苗)30個監測站算得平均數為48.8，標準差為13.57；中南部地區(含中部、雲嘉南及高屏)38個監測站算得平均數為29.9，標準差為14.29。根據此時點的監測資料，是否足以證實北部地區的空氣品質比中南部差？試以顯著水準5%檢定之。

$$H_0: \mu_1 - \mu_2 \le 0;\ H_1: \mu_1 - \mu_2 > 0$$

$$(\text{或 } H_0: \mu_1 \le \mu_2;\ H_1: \mu_1 > \mu_2)$$

μ_1為北部監測站PM2.5的濃度平均
μ_2為中南部監測站PM2.5的濃度平均

Large-Sample Test Ans

$$n_1 = 30, \bar{x}_1 = 48.8, S_1 = 13.57$$
$$n_2 = 38, \bar{x}_2 = 29.9, S_2 = 14.29$$

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{(48.8 - 29.9) - 0}{\sqrt{\frac{(13.57)^2}{30} + \frac{(14.29)^2}{38}}} \approx \frac{18.9}{3.39} \approx 5.57$$

由於$z=5.57>z_\alpha=1.645$故結論為拒絕H_0

亦即在5%顯著水準下，北部地區的空氣品質較中南部地區差。

Large-Sample Test Quiz

某公司欲瞭解兩訓練中心在教育品質方面的差異，於是對兩訓練中心的受訓者實施一項標準測驗，並以考試成績做為評估兩訓練中心之優劣的依據。此項測驗所得的資料與結果如下表所示：

訓練中心 1	訓練中心 2
$n_1 = 30$	$n_2 = 40$
$\overline{x}_1 = 82.5$	$\overline{x}_2 = 79$
$S_1 = 8$	$S_2 = 10$

試檢定此二訓練中心的教育品質是否有差異 (α=0.05)。

$$H_0 : \mu_1 - \mu_2 = 0;\ H_1 : \mu_1 - \mu_2 \neq 0$$

$$(\text{或 } H_0 : \mu_1 = \mu_2;\ H_1 : \mu_1 \neq \mu_2)$$

μ_1為訓練中心1的測驗平均
μ_2為訓練中心2的測驗平均

Large-Sample Test Ans

$$c_1 = (\mu_1 - \mu_2)_{H_0} + z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$c_2 = (\mu_1 - \mu_2)_{H_0} - z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \simeq \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

（σ_1 與 σ_2 未知，可用 S_1 與 S_2 取代）

由於82.5−79=3.5，落於接受域，
故結論為不拒絕H_0。
此二訓練中心的教育品質並無顯著差異。

$$c_1 = 0 + z_{\alpha/2} \cdot \sqrt{\frac{8^2}{30} + \frac{10^2}{40}} = 0 + 1.96 \times 2.15 = 4.21$$

$$(z_{\alpha/2} = z_{0.025} = 1.96)$$

$$c_2 = 0 - z_{\alpha/2} \cdot \sqrt{\frac{8^2}{30} + \frac{10^2}{40}} = 0 - 1.96 \times 2.15 = -4.21$$

z=3.5/2.15=1.627

Comparing Two Population Means: Independent Sampling

Small-sample

Large-sample

1. The two samples are **randomly selected** in an **independent** manner from the **two target populations**.
2. The sample sizes, n_1 and n_2, are both **large** (i.e., ≥ 30).

Conditions Required for Valid Small-Sample Inferences about $(\mu_1 - \mu_2)$

1. The two samples are **randomly selected** in an **independent** manner from the **two target populations.**
2. Both sampled **populations have distributions** that are approximately normal.
3. The populations **variances** are equal. $\sigma_1^2 = \sigma_2^2$

σ_1^2, σ_2^2 known:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sigma_{(\bar{x}_1-\bar{x}_2)} = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Large-sample

Small-Sample Confidence Interval for (μ_1 – μ_2)

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2 \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

where $s_p^2 = \dfrac{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}{n_1 + n_2 - 2}$

Pooled variance

and $t_{\alpha/2}$ is based on $(n_1 + n_2 - 2)$ degrees of freedom.

Pooled variance

$$s_p^2 = \frac{\sum_{i=1}^{k}(n_i - 1)s_i^2}{\sum_{i=1}^{k}(n_i - 1)} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2}{n_1 + n_2 + \cdots + n_k - k}$$

$$s_p^2 = \frac{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}{n_1 + n_2 - 2}$$

Small-Sample Confidence Interval Example

You're a financial analyst for Charles Schwab. You want to estimate the difference in dividend yield between stocks listed on NYSE and NASDAQ.

	NYSE	NASDAQ
Number	11	15
Mean	3.27	2.53
Std Dev	1.30	1.16

Assuming normal populations and **equal population variances**, what is the 95% confidence interval for the difference between the mean dividend yields (現金殖利率)?

$$\text{df} = n_1 + n_2 - 2 = 11 + 15 - 2 = 24 \qquad t_{.025} = 2.064$$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(11-1)(1.30)^2 + (15-1)(1.16)^2}{11+15-2} = 1.489$$

$$(3.27 - 2.53) \pm 2.064\sqrt{1.489\left(\frac{1}{11} + \frac{1}{15}\right)}$$

the difference between the mean dividend yields

$$-.26 \le \mu_1 - \mu_2 \le 1.74$$

Small-Sample Confidence Interval Quiz

為進行一項養分配給的研究，選 25頭乳牛，比較二種飼料的效果，其一為 脫水牧草，另一為枯萎的牧草。隨機地自此牛群中選出 12頭，以脫水牧草飼養，另外的 13頭乳牛則餵以枯萎的牧草。根據三個禮拜的觀察，每天平均牛奶 產量資料列示下表：

牛奶產量（磅）

枯萎牧草	44	44	56	46	47	38	58	53	49	35	46	30	41
脫水牧草	35	47	55	29	40	39	32	41	42	57	51	39	

假設食用兩種不同牧草的乳牛，其牛奶 產量的資料係分別取自平均數 μ_1與μ_2的常態母體之隨機樣本，且二個母體有 **共同的標準差 σ**。試求出 $\mu_1-\mu_2$之95%信賴區間。

Small-Sample Confidence Interval Ans

由於兩個母體均為常態，小樣本，而$\sigma_1=\sigma_2=\sigma$，但σ未知。

$$S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$
$$= \frac{(13-1)(63.97)+(12-1)(76.39)}{13+12-2}$$
$$= 69.91$$

已知$1-\alpha=0.95$，故$t_{\alpha/2(n1+n2-2)}=t_{0.025(23)}=2.069$（查表）

$$(45.15-42.25) \pm (2.069)\left(\sqrt{69.91}\sqrt{\frac{1}{13}+\frac{1}{12}}\right)$$

即2.90±6.92或[-4.02, 9.82]，我們有95%的信心認為食用枯萎牧草的乳牛其產量較食用脫水牧草乳牛的產量，多出-4.02至9.82磅。

t-test

Small-Sample Test of Hypothesis for $(\mu_1 - \mu_2)$

One-Tailed Test

$H_0: (\mu_1 - \mu_2) \geq D_0$ [or $H_a: (\mu_1 - \mu_2) \leq D_0$]

$H_a: (\mu_1 - \mu_2) < D_0$ [or $H_a: (\mu_1 - \mu_2) > D_0$]

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$(\mu_1 - \mu_2) < 0$ ➡️ $\mu_1 < \mu_2$

Test statistic (df=$(n_1 + n_2 - 2)$):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$s_p^2 = \frac{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}{n_1 + n_2 - 2}$$

Small-Sample Test of Hypothesis for $(\mu_1 - \mu_2)$

Two-Tailed Test

$H_0: (\mu_1 - \mu_2) = D_0$

$H_a: (\mu_1 - \mu_2) \neq D_0$

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$(\mu_1 - \mu_2) \neq 0$ ➡️ $\mu_1 \neq \mu_2$

Test statistic (df=(n_1 + n_2 – 2)):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$s_p^2 = \frac{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}{n_1 + n_2 - 2}$$

Small-Sample Test Example

You're a financial analyst for Charles Schwab. You want to estimate the difference in dividend yield between stocks listed on NYSE and NASDAQ.

	NYSE	NASDAQ
Number	11	15
Mean	3.27	2.53
Std Dev	1.30	1.16

Assuming normal populations and equal population variances, is there a difference in average yield ($\alpha = .05$)?

μ_1為the mean of dividend yield of stocks listed on NYSE
μ_2為the mean of dividend yield of stocks listed on NASDAQ

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(11-1)(1.30)^2 + (15-1)(1.16)^2}{11+15-2} = 1.489$$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{(3.27 - 2.53) - (0)}{\sqrt{1.489\left(\frac{1}{11} + \frac{1}{15}\right)}} = +1.53$$

μ_1為the mean of dividend yield of stocks listed on NYSE
μ_2為the mean of dividend yield of stocks listed on NASDAQ

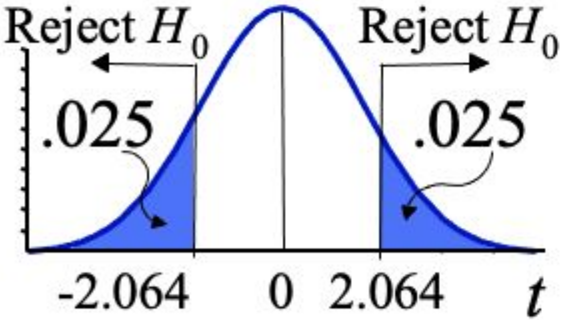
H_0: $\mu_1 - \mu_2 = 0 \ (\mu_1 = \mu_2)$

H_a: $\mu_1 - \mu_2 \neq 0 \ (\mu_1 \neq \mu_2)$

$\alpha =$ .05

df = $11 + 15 - 2 = 24$

Critical Value(s):



Test Statistic:

$$t = +1.53$$

Decision:

Do not reject at $\alpha = .05$

Conclusion:

There is no evidence of a difference in means

Small-Sample Test Quiz

You're a research analyst for General Motors. Assuming equal variances, is there a difference in the average miles per gallon (mpg) of two car models (α = .05)?

	Sedan	Van
Number	15	11
Mean	22.00	20.27
Std Dev	4.77	3.64

μ_1為the mean of mpg of Sedan
μ_2為the mean of mpg of Van

$$s_p^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}$$

$$= \frac{(15-1)(4.77)^2+(11-1)(3.64)^2}{15+11-2} = 18.793$$

$$t = \frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sqrt{s_p^2\cdot\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{(22.00-20.27)-(0)}{\sqrt{18.793\cdot\left(\frac{1}{15}+\frac{1}{11}\right)}} = +1.00$$

μ_1為the mean of mpg of Sedan
μ_2為the mean of mpg of Van

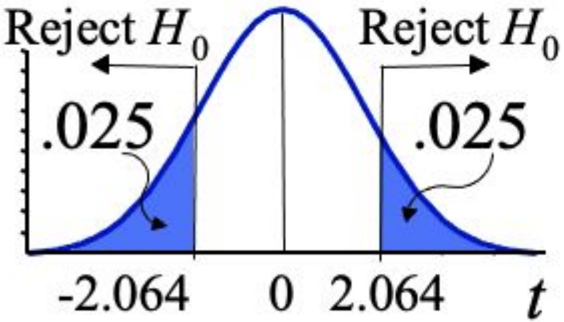
H_0: $\mu_1 - \mu_2 = 0 \ (\mu_1 = \mu_2)$

H_a: $\mu_1 - \mu_2 \neq 0 \ (\mu_1 \neq \mu_2)$

$\alpha =$ .05

df = $15 + 11 - 2 = 24$

Critical Value(s):



Test Statistic:

$$t = +1.00$$

Decision:

Do not reject at $\alpha = .05$

Conclusion:

There is no evidence of a difference in means

食用枯萎牧草：$\bar{x}_1 = 45.15$

食用脫水牧草：$\bar{x}_2 = 42.25$

$S_p^2 = 69.91$

***t*-test**

Small-Sample Test Quiz

為進行一項養分配給的研究，選 25頭乳牛，比較二種飼料的效果，其一為 脫水牧草，另一為枯萎的牧草。隨機地自此牛群中選出 12頭，以脫水牧草飼養，另外的 13頭乳牛則餵以枯萎的牧草。根據三個禮拜的觀察，每天平均牛奶 產量資料列示下表：

牛奶產量（磅）

枯萎牧草	44	44	56	46	47	38	58	53	49	35	46	30	41
脫水牧草	35	47	55	29	40	39	32	41	42	57	51	39	

假設食用兩種不同牧草的乳牛，其牛奶 產量的資料係分別取自平均數 μ1與μ2的常態母體之隨機樣本，且二個母體有 **共同的標準差σ**。這些資料是否足以證明食用枯萎牧草之乳牛的平均牛乳 產量大於食用脫水牧草之乳牛的平均 產量？試以α=0.05檢定之。

μ_1為食用枯萎牧草的乳牛平均 產乳量(母體平均)
μ_2為食用脫水牧草的乳牛平均 產乳量(母體平均)

食用枯萎牧草：$\overline{x}_1 = 45.15$

食用脫水牧草：$\overline{x}_2 = 42.25$

$$S_p^2 = 69.91$$

Small-Sample Test Ans

$$H_0 : \mu_1 - \mu_2 \leq 0; \quad H_1 : \mu_1 - \mu_2 > 0$$

（或 $H_0 : \mu_1 \leq \mu_2; \quad H_1 : \mu_1 > \mu_2$）

由於兩個母體均為常態，小樣本，
而$\sigma_1=\sigma_2=\sigma$，但σ未知。

$$t = \frac{45.15 - 42.25}{8.36\sqrt{\frac{1}{13} + \frac{1}{12}}} = \frac{2.90}{3.35} = 0.87$$

$$t = \frac{(\overline{X}_1 - \overline{X}_2) - \mu_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \; df = n_1 + n_2 - 2$$

就df=23而言，查表知$t_{0.05}$=1.714。因此，在α=0.05之下，不能拒絕虛無假設。也就是說，沒有足夠證據證明食用枯萎牧草之乳牛的平均牛乳產量大於食用脫水牧草之乳牛的平均產量

σ_1^2, σ_2^2 known:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sigma_{(\bar{x}_1-\bar{x}_2)} = (\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Large-sample

Approximate Small-Sample Procedures with Unequal Population Variance

Confidence interval: $(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}$

Test statistic H_0: $(\mu_1 - \mu_2) = 0$

$$t = (\bar{x}_1 - \bar{x}_2) \Big/ \sqrt{(s_1^2/n_1) + (s_2^2/n_2)}$$

Welch's t-test

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

df = round(v)

Equal Population Variance

Test statistic (df=(n_1 + n_2 – 2)):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$s_p^2 = \frac{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}{n_1 + n_2 - 2}$$

Pooled degrees of freedom, v

$$v = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$$

Welch–Satterthwaite equation:

approximation to the effective degrees of freedom of a linear combination of independent sample variances

$$\nu = \left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2 \Bigg/ \left(\frac{s_1^4}{n_1^2(n_1 - 1)} + \frac{s_2^4}{n_2^2(n_2 - 1)}\right).$$

Proof: https://math.stackexchange.com/questions/1746329/proof-and-precise-formulation-of-welch-satterthwaite-equation

- 母體分配
  - 常態母體
    - 大樣本 $\begin{pmatrix} n_1 \geq 30 \\ n_2 \geq 30 \end{pmatrix}$ $\longrightarrow$ $\overline{X}_1 - \overline{X}_2$ 為常態分配，且 $\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2, \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)$ 註：若 σ_1 或 σ_2 未知，則以 S_1 或 S_2 取代 (Ⅰ)
    - 小樣本 $\begin{pmatrix} n_1 < 30 \\ n_2 < 30 \end{pmatrix}$
      - 母體標準差已知（σ_1 與 σ_2 皆已知）$\rightarrow$ $\overline{X}_1 - \overline{X}_2$ 為常態分配，且 $\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2, \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)$ (Ⅱ)
      - 母體標準差未知（σ_1 與 σ_2 皆未知）
        - $\sigma_1 = \sigma_2 = \sigma \rightarrow \dfrac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1 + n_2 - 2)$ (Ⅲ)

          $$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$
        - $\sigma_1 \neq \sigma_2 \rightarrow \dfrac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \sim t(df)$ (Ⅳ)

          $$df = \frac{(S_1^2/n_1 + S_2^2/n_2)^2}{\left[(S_1^2/n_1)^2/(n_1 - 1)\right] + \left[(S_2^2/n_2)^2/(n_2 - 1)\right]}$$
  - 非常態母體
    - 大樣本 $\begin{pmatrix} n_1 \geq 30 \\ n_2 \geq 30 \end{pmatrix}$ $\longrightarrow$ $\overline{X}_1 - \overline{X}_2$ 為常態分配，且 $\overline{X}_1 - \overline{X}_2 \sim N\left(\mu_1 - \mu_2, \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}\right)$
    - 小樣本 $\longrightarrow$ 視母體分配而定 註：若 σ_1 或 σ_2 未知，則以 S_1 或 S_2 取代 (Ⅴ)

Advanced Example - Ages of self-employed immigrants (1/5)

Is self-employment for immigrant workers a faster route to economic advancement in the country?

One aspect of the study involved **comparing the ages of self-employed and wage-earning immigrants.** The researcher found that in Sweden, native wage earners tend to be younger than self employed natives. However, immigrant wage earners tend to be older than self-employed immigrants. This inference was based on summary statistics for male Swedish immigrants shown in the table.

Advanced Example - Ages of self-employed immigrants (2/5)

	Self-Employed Immigrants	Wage-Earning Immigrants
Sample Size	870	84,875
Mean Age (years)	44.88	46.79

Source: Based on L. Andersson, "Occupational Choice and Returns to Self-Employment Among Immigrants," *International Journal of Manpower,* Vol. 32, No. 8, 2011 (Table I).

Advanced Example - Ages of self-employed immigrants (3/5)

1. Based on the information given, why is it impossible to provide a measure of reliability for the inference "Self Employed immigrants are younger, on average, than wage-earning immigrants in Sweden"?
2. What information do you need to obtain a measure of reliability for the inference, part a?

Advanced Example - Ages of self-employed immigrants - Solution part 1

1. We cannot provide a measure of reliability because we have no measure of the **variability or variance** of the data.
2. We would need the variances of the two samples.

$$\sigma_{(\bar{x}_1-\bar{x}_2)} = \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

$$z = \frac{(\bar{x}_1-\bar{x}_2)-D_0}{\sigma_{(\bar{x}_1-\bar{x}_2)}} \qquad \approx \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$$

Advanced Example - Ages of self-employed immigrants (4/5)

3. Give a value of the test statistic that would lead you to conclude that **the true mean age of self-employed immigrants is less than the true mean age of wage-earning immigrants** if you are willing to risk a Type I error rate of 0.01.

Advanced Example - Ages of self-employed immigrants - Solution part 2

μ_1=mean age for self-employed immigrants
μ_2=mean age for the wage-earning immigrants
To determine if **the mean age for self-employed immigrants is less than the mean age for wage-earning immigrants**, we test:

$$H_0: \mu_1 - \mu_2 >= 0$$
$$H_1: \mu_1 - \mu_2 < 0$$

The rejection region requires in the lower tail of the z-distribution.

$z_{0.01}$=2.33. The rejection region is $z < -2.33$.

Advanced Example - Ages of self-employed immigrants (5/5)

4. Assume that σ, the standard deviation of the ages, **is the same for both self-employed and wage-earning immigrants. Give an estimate of σ** that would lead you to conclude that the true mean age of self-employed immigrants is **less than** the true mean age of wage-earning immigrants using **α = 0.01**.
5. Is the true value of σ likely to be larger or smaller than the one you calculated in part d?

Advanced Example - Ages of self-employed immigrants - Solution part 3

$z_{0.01}=2.33$

$$z=\frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}=\frac{(44.88-46.79)-0}{\sqrt{\frac{\sigma^2}{870}+\frac{\sigma^2}{84,875}}}\le -2.33$$

$$\Rightarrow -1.91\le \sigma\sqrt{\frac{1}{870}+\frac{1}{84,875}}(-2.33)\Rightarrow \sigma\le 24.056$$

The true value of **σ is likely to be smaller than 24.056**.

This standard deviation would be too large for the ages of people.

Comparing Two Population Means: Paired Difference Experiments

Paired-Difference Experiment Data Collection Table

Observation	Group 1	Group 2	Difference
1	x_{11}	x_{21}	$d_1 = x_{11} - x_{21}$
2	x_{12}	x_{22}	$d_2 = x_{12} - x_{22}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	x_{1i}	x_{2i}	$d_i = x_{1i} - x_{2i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	x_{1n}	x_{2n}	$d_n = x_{1n} - x_{2n}$

Conditions Required for Valid Inferences about μ_d

Large-Sample

1. A **random sample of differences** is selected from the target population of differences.
2. The sample sizes, n_d, are **large** (i.e., ≥ 30)

Small-Sample

1. A **random sample of differences** is selected from the target population of differences.
2. The **population** of differences has a distribution that is approximately **normal**.

Paired-Difference Confidence Interval for $\mu_d = \mu_1 - \mu_2$

Large Sample

$$\bar{d} \pm z_{\alpha/2} \frac{\sigma_d}{\sqrt{n_d}} \approx \bar{d} \pm z_{\alpha/2} \frac{s_d}{\sqrt{n_d}}$$

Small Sample

$$\bar{d} \pm t_{\alpha/2} \frac{s_d}{\sqrt{n_d}}$$

where $t_{\alpha/2}$ is based on $(n_d - 1)$ degrees of freedom.

Paired-Difference Experiment Confidence Interval Example

You work in Human Resources. You want to see if there is a difference in test scores after a training program. You collect the following test score data:

Name	Before (1)	After (2)
Sam	85	94
Tamika	94	87
Brian	78	79
Mike	87	88

Find a 90% confidence interval for the mean difference in test scores.

Computation Table

Observation	Before	After	Difference
Sam	85	94	–9
Tamika	94	87	7
Brian	78	79	–1
Mike	87	88	–1
Total			– 4

$\overline{d} = -1 \qquad s_d = 6.53$

Paired-Difference Experiment Confidence Interval Solution

$$\text{df} = n_d - 1 = 4 - 1 = 3 \qquad t_{.05} = 2.353$$

$$\bar{d} \pm t_{\alpha/2} \frac{S_d}{\sqrt{n_d}}$$

$$-1 \pm 2.353 \frac{6.53}{\sqrt{4}}$$

$$-8.68 \le \mu_d \le 6.68$$

paired *z(t)*-test

Paired-Difference Test of Hypothesis for $\mu_d = \mu_1 - \mu_2$

Test statistic:

One-Tailed Test

$H_0: \mu_d \geq D_0$ [or $H_a: \mu_d \leq D_0$]

$H_a: \mu_d < D_0$ [or $H_a: \mu_d > D_0$]

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$\mu_d < 0$ ➡️ $\mu_1 < \mu_2$

$$z = \frac{\bar{d} - D_0}{\sigma_d / \sqrt{n_d}} \approx \frac{\bar{d} - D_0}{s_d / \sqrt{n_d}}$$

$$t = \frac{\bar{d} - D_0}{s_d / \sqrt{n_d}}$$

$df = (n_d - 1)$

paired *z(t)*-test

Paired-Difference Test of Hypothesis for $\mu_d = \mu_1 - \mu_2$

Two-Tailed Test

$H_0: \mu_d = D_0$

$H_a: \mu_d \neq D_0$

D_0 = Hypothesized difference between the means

$D_0 = 0$ ➡️ 沒差的意思

$\mu_d \neq 0$ ➡️ $\mu_1 \neq \mu_2$

Test statistic:

$$z = \frac{\bar{d} - D_0}{\sigma_d / \sqrt{n_d}} \approx \frac{\bar{d} - D_0}{s_d / \sqrt{n_d}}$$

$$t = \frac{\bar{d} - D_0}{s_d / \sqrt{n_d}}$$

$df = (n_d - 1)$

Hypotheses for Paired-Difference Experiment

Hypothesis	Research Questions		
	No Difference Any Difference	Pop 1 ≥ Pop 2 Pop 1 < Pop 2	Pop 1 ≤ Pop 2 Pop 1 > Pop 2
H_0	$\mu_d = 0$	$\mu_d \geq 0$	$\mu_d \leq 0$
H_a	$\mu_d \neq 0$	$\mu_d < 0$	$\mu_d > 0$

Note: $d_i = x_{1i} - x_{2i}$ for i^{th} observation

Paired-Difference Experiment Small-Sample Test Example

You work in Human Resources. You want to see if a training program is effective. You collect the following test score data:

Name	Before	After
Sam	85	94
Tamika	94	87
Brian	78	79
Mike	87	88

At the 0.10 level of significance, **was the training effective?**

Null Hypothesis Solution

1. Was the training effective?
2. Effective means ‘Before’ < ‘After’.
3. Statistically, this means $\mu_{Before} < \mu_{After}$.
4. Rearranging terms gives $\mu_{Before} - \mu_{After} < 0$.
5. Defining $\mu_d = \mu_{Before} - \mu_{After}$ and substituting into (4) gives $\mu_d<0$.
6. The alternative hypothesis is H_a: $\mu_d<0$.

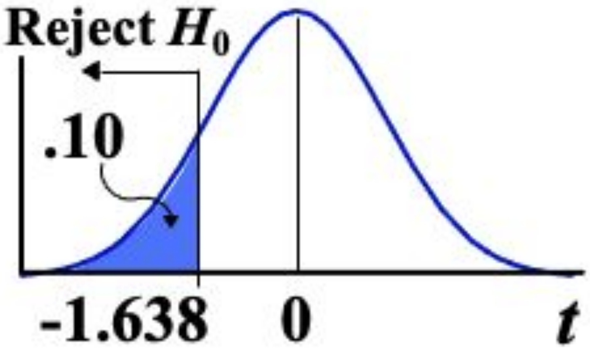
H_0: $\mu_d \geq 0$ ($\mu_d = \mu_B - \mu_A$)

H_a: $\mu_d < 0$

$\alpha =$ .10

df = $4 - 1 = 3$

Critical Value(s):



Test Statistic:

$$t = \frac{\bar{d} - D_0}{\frac{S_d}{\sqrt{n_d}}} = \frac{-1-0}{\frac{6.53}{\sqrt{4}}} = -.306$$

Decision:

Do not reject at $\alpha = .10$

Conclusion:

There is no evidence training was effective

Paired-Difference Experiment Small-Sample Test Quiz

You're a marketing research analyst. You want to compare a client's calculator to a competitor's. You sample 8 retail stores. At the 0.01 level of significance, does your client's calculator sell for less than their competitor's?

Store	(1) Client	(2) Competitor
1	$ 10	$ 11
2	8	11
3	7	10
4	9	12
5	11	11
6	10	13
7	9	12
8	8	10

Paired-Difference Experiment Small-Sample Test Ans

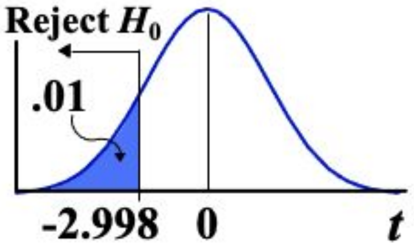
H_0: $\mu_d \geq 0$ ($\mu_d = \mu_1 - \mu_2$)

H_a: $\mu_d < 0$

$\alpha =$ .01

df = $8 - 1 = 7$

Critical Value(s):



Test Statistic:

$$t = \frac{\bar{d} - d_0}{\frac{s_d}{\sqrt{n_d}}} = \frac{-2.25 - 0}{\frac{1.16}{\sqrt{8}}} = -5.486$$

Decision:

Reject at $\alpha = .01$

Conclusion:

There is evidence client's brand (1) sells for less

Pre-class quiz

Week 12

遠距上課，取消考試

(a)求出μ_1-μ_2之95%信賴區間。
(b)檢定此藥丸是否有降低血壓的作用(α=0.05)。

Paired-Difference Experiment Small-Sample Test Quiz

某一藥劑研究員欲檢定某種藥丸是否具有預期的作用——服用者的血壓。此項研究先記錄15位大專女性學生的初始血壓，然後在她們服用此藥丸6個月後，再度測量其血壓。

	受試者							
	1	2	3	4	5	6	7	8
服藥之前 (x)	70	80	72	76	76	76	72	78
服藥之後 (y)	68	72	62	70	58	66	68	52
$d=(x-y)$	2	8	10	6	18	10	4	26

$$\bar{d}=\frac{\sum d_i}{15}=8.80,\quad S_D=\sqrt{\frac{\sum(d_i-\bar{d})^2}{14}}=10.98$$

	受試者						
	9	10	11	12	13	14	15
服藥之前 (x)	82	64	74	92	74	68	84
服藥之後 (y)	64	72	74	60	74	72	74
$d=(x-y)$	18	−8	0	32	0	−4	10

Advanced Example - Taking “power naps” during work breaks (1/2)

Lack of sleep costs companies about $18 billion a year in lost productivity, according to the National Sleep Foundation.

In recognition of this problem, many companies now make quiet rooms available for employees to take “power naps” during work breaks. Consider a major airline that encourages reservation agents to nap during their breaks. The accompanying table lists **the number of complaints received about each of a sample of 10 reservation agents during the 6 months before naps were encouraged and during the 6 months after the policy change**.

Advanced Example - Taking “power naps” during work breaks (2/2)

1. Do the data present sufficient evidence to conclude that the **new napping policy reduced the mean number of customer complaints about reservation agents**? Test using a = 0.05
2. What **assumptions** must hold to ensure the validity of the test?
3. What variables, not controlled in the study, **could lead to an invalid conclusion**?

Operator	Before Policy	After Policy
1	10	5
2	3	0
3	16	7
4	11	4
5	8	6
6	2	4
7	1	2
8	14	3
9	5	5
10	6	1

Advanced Example - Taking "power naps" during work breaks - Solution part 1

Operator	Difference (Before - After)
1	5
2	3
3	9
4	7
5	2
6	−2
7	−1
8	11
9	0
10	5

$H_0 : \mu_d \leq 0$

$H_a : \mu_d > 0$

$$\bar{d} = \frac{\sum d}{n_d} = \frac{39}{10} = 3.9$$

$$s_d^2 = \frac{\sum d^2 - \frac{\left(\sum d\right)^2}{n_E}}{n_d - 1} = \frac{319 - \frac{39^2}{10}}{10 - 1} = 18.5444$$

$$s_d = \sqrt{18.5444} = 4.3063$$

The test statistic is $t = \frac{\bar{d} - 0}{\frac{s_d}{\sqrt{n_d}}} = \frac{3.9 - 0}{\frac{4.3063}{\sqrt{10}}} = 2.864$

The rejection region requires $\alpha = .05$ in the upper tail of the t-distribution with $df = n_d - 1 = 10 - 1 = 9$. From Table III, Appendix D, $t_{.05} = 1.833$. The rejection region is $t > 1.833$.

Since the observed value of the test statistic falls in the rejection region $(t = 2.864 > 1.833)$, H_0 is rejected. There is sufficient evidence to indicate the new napping policy reduced the mean number of customer complaints at $\alpha = .05$.

Advanced Example - Taking “power naps” during work breaks - Solution part 2

2. In order for the above test to be valid, we must assume that
    a. The population of differences is **normal**
    b. The differences are **randomly selected**

3. Variables that were not controlled that could lead to an invalid conclusion include **time of day agents worked**, **day of the week agents worked**, and **how much sleep the agents got before working**, among others.

哪種樣本?

	獨立樣本 $(n_1 = n_2 = n)$	成對樣本 (n 對)
估計的標準誤	$S_p\sqrt{\frac{1}{n}+\frac{1}{n}}$	$\frac{S_D}{\sqrt{n}}$
t 的自由度	$2n-2$	$n-1$

Comparing Two Population Proportions: Independent Sampling

Properties of the Sampling Distribution of $(p_1 - p_2)$

1. The mean of the sampling distribution of $(\overline{p_1} - \overline{p_2})$ is $(p_1 - p_2)$; that is, $E(\overline{p_1} - \overline{p_2}) = p_1 - p_2$
2. The standard deviation of the sampling distribution of $(\overline{p_1} - \overline{p_2})$ is $\sigma_{(\overline{p_1} - \overline{p_2})} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$
3. If the sample sizes n_1 and n_2 are large, the sampling distribution of $(\overline{p_1} - \overline{p_2})$ is approximately normal.

Conditions Required for Valid Large-Sample Inferences about ($p_1 - p_2$)

1. The two samples are randomly selected in an independent manner from the two target populations.
2. The sample sizes, n_1 and n_2, are both large so that the sampling distribution of $(\overline{p_1} - \overline{p_2})$ will be approximately normal. (This condition will be satisfied if both

$$n_1\overline{p_1} \geq 15,\ n_1\overline{q_1} \geq 15,\ and\ n_2\overline{p_2} \geq 15,\ n_2\overline{q_2} \geq 15$$

Large-Sample Confidence Interval for ($p_1 - p_2$)

$$
\begin{aligned}
(\overline{p_1} - \overline{p_2}) \pm z_{\alpha/2} \times \sigma_{(\overline{p_1}-\overline{p_2})} &= (\overline{p_1} - \overline{p_2}) \pm z_{\alpha/2} \times \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}} \\
&\approx (\overline{p_1} - \overline{p_2}) \pm z_{\alpha/2} \times \sqrt{\frac{\overline{p_1}\,\overline{q_1}}{n_1} + \frac{\overline{p_2}\,\overline{q_2}}{n_2}}
\end{aligned}
$$

Confidence Interval for $p_1 – p_2$ Example

As personnel director, you want to test the perception of fairness of two methods of performance evaluation.

63 of 78 employees rated Method 1 as fair.

49 of 82 rated Method 2 as fair.

Find a 99% confidence interval for the difference in perceptions.

Confidence Interval for $p_1 - p_2$ Solution

$$\hat{p}_1 = \frac{63}{78} = .808 \quad \hat{q}_1 = 1 - .808 = .192$$

$$\hat{p}_2 = \frac{49}{82} = .598 \quad \hat{q}_2 = 1 - .598 = .402$$

$$(.808 - .598) \pm 2.58\sqrt{\frac{.808(.192)}{78} + \frac{.598(.402)}{82}}$$

$$.029 \le p_1 - p_2 \le .391$$

Large-Sample Test of Hypothesis about $(p_1 - p_2)$

Test statistic:

$$z = \frac{(\overline{p_1} - \overline{p_2}) - (p_1 - p_2)}{\sigma_{(\overline{p_1} - \overline{p_2})}}$$

One-Tailed Test

$H_0: (p_1 - p_2) \geq 0$ [or $H_a: (p_1 - p_2) \leq 0$]

$H_a: (p_1 - p_2) < 0$ [or $H_a: (p_1 - p_2) > 0$]

比例p的混合估計值 (pooled estimate of the proportion)

$$\sigma_{(\overline{p_1} - \overline{p_2})} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$$

$$\approx \sqrt{\overline{pq}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \ \overline{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Note: H_0假設$p_1 = p_2$，因此需用**混和估計值**來替代p_1 與 p_2

Large-Sample Test of Hypothesis about $(p_1 - p_2)$

Two-Tailed Test

$H_0: (p_1 - p_2) = 0$

$H_a: (p_1 - p_2) \neq 0$

Test statistic:

$$z = \frac{(\overline{p_1} - \overline{p_2}) - (p_1 - p_2)}{\sigma_{(\overline{p_1} - \overline{p_2})}}$$

$$\sigma_{(\overline{p_1} - \overline{p_2})} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$$

$$\approx \sqrt{\overline{pq}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}, \ \overline{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

比例p的混合估計值 (pooled estimate of the proportion)

Note: H_0假設$p_1 = p_2$，因此需用混和估計值來替代p_1與p_2

Hypotheses for Two Proportions

Hypothesis	Research Questions		
	No Difference Any Difference	Pop 1 ≥ Pop 2 Pop 1 < Pop 2	Pop 1 ≤ Pop 2 Pop 1 > Pop 2
H_0	$p_1 - p_2 = 0$	$p_1 - p_2 \geq 0$	$p_1 - p_2 \leq 0$
H_a	$p_1 - p_2 \neq 0$	$p_1 - p_2 < 0$	$p_1 - p_2 > 0$

Test for Two Proportions Example

As personnel director, you want to test the perception of fairness of two methods of performance evaluation.

63 of 78 employees rated Method 1 as fair.

49 of 82 rated Method 2 as fair.

At the 0.01 level of significance, is there a difference in perceptions?

p_1為覺得Method1公平的比例
p_2為覺得Method2公平的比例

Test for Two Proportions Solution

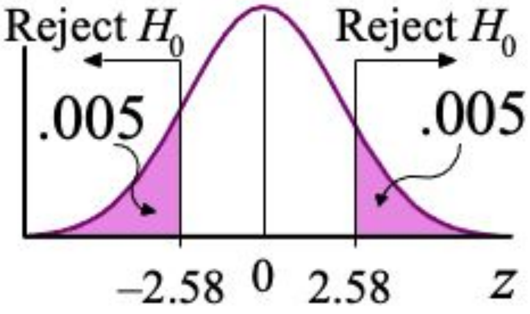
H_0: $p_1 - p_2 = 0$

H_a: $p_1 - p_2 \neq 0$

$\alpha =$ .01

$n_1 =$ 78 $n_2 =$ 82

Critical Value(s):



Test Statistic:

$z = +2.90$

Decision:

Reject at $\alpha = .01$

Conclusion:

There is evidence of a difference in proportions

Test for Two Proportions - Quiz

You're an economist for the Department of Labor. You're studying unemployment rates.

In MA, 74 of 1500 people surveyed were unemployed.

In CA, 129 of 1500 were unemployed.

At the 0.05 level of significance, does MA have a lower unemployment rate than CA?

Ans

$$\hat{p}_{MA} = \frac{x_{MA}}{n_{MA}} = \frac{74}{1500} = .0493 \qquad \hat{p}_{CA} = \frac{x_{CA}}{n_{CA}} = \frac{129}{1500} = .0860$$

$$\hat{p} = \frac{x_{MA} + x_{CA}}{n_{MA} + n_{CA}} = \frac{74 + 129}{1500 + 1500} = .0677$$

$$z \approx \frac{(.0493 - .0860) - (0)}{\sqrt{(.0677) \cdot (1 - .0677) \cdot \left(\frac{1}{1500} + \frac{1}{1500}\right)}}$$

$$= -4.00$$

Ans

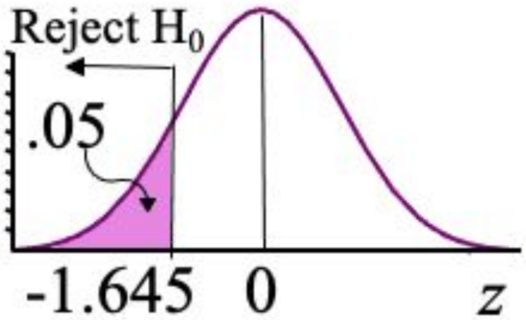
H_0: $p_{MA} - p_{CA} >= 0$

H_a: $p_{MA} - p_{CA} < 0$

$\alpha =$ .05

$n_{MA} =$ 1500 $n_{CA} =$ 1500

Critical Value(s):



Test Statistic:

$z = -4.00$

Decision:

Reject at $\alpha = .05$

Conclusion:

There is evidence MA is less than CA

Test for Two Proportions - Quiz

某公司考慮以兩種不同的電台廣告來促銷一種新產品。根據過去的經驗，銷售經理**相信A廣告較B廣告有效**。茲將A廣告與B廣告分別在不同的市場進行測試；其中一組60位收聽A廣告的消費者之隨機樣本中有18位試用此產品，而另一組100位收聽B廣告的消費者之隨機樣本中，有22位試用此產品。若顯著水準為α=0.05，試問此抽樣結果是否保證A廣告較B廣告有效？並請求出P值。

Test for Two Proportions - Ans

p_1、p_2分別代表收聽A廣告與B廣告後將會試用此產品的母體比例

$$H_0: p_1 - p_2 \le 0;\ H_1: p_1 - p_2 > 0$$

$$(\text{或} H_0: p_1 \le p_2;\ H_1: p_1 > p_2)$$

$$\overline{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{18 + 22}{60 + 100} = 0.25$$

$$\sigma_{\overline{p}_1 - \overline{p}_2} = \sqrt{\frac{\overline{p}(1-\overline{p})}{n_1} + \frac{\overline{p}(1-\overline{p})}{n_2}}$$

$$= \sqrt{\frac{(0.25)(0.75)}{60} + \frac{(0.25)(0.75)}{100}} = 0.071$$

$$z = \frac{(\overline{p}_1 - \overline{p}_2) - p_0}{\sigma_{\overline{p}_1 - \overline{p}_2}} = \frac{0.08 - 0}{0.071} = 1.13$$

因z=1.13小於z_α，因此結論為不拒絕H_0

亦即銷售經理無充分的證據相信A廣告的效果優於B廣告

Advanced Example - Religious symbolism in TV commercials

Gonzaga University professors conducted a study of television commercials and published their results in the Journal of Sociology, Social Work and Social Welfare.

The key research question was as follows: "**Do television advertisers use religious symbolism to sell goods and services?**" In a sample of **797** TV commercials collected ten years earlier, only **16** commercials used religious symbolism. Of the sample of **1,499** TV commercials examined in the more recent study, **51** commercials used religious symbolism. Conduct an analysis to **determine if the percentage of TV commercials that use religious symbolism has changed over time**. If you detect a change, estimate the **magnitude of the difference** and **attach a measure of reliability** to the estimate.

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{16}{797} = .020 \qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{51}{1,499} = .034 \qquad \hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{16+51}{797+1,499} = \frac{67}{2,296} = .029$$

Advanced Example - Religious symbolism in TV commercials - Solution part 1

p_1=proportion of TV commercials ten years earlier that used religious symbolism

p_2=proportion of TV commercials in a recent study that used religious symbolism

$$H_0 : p_1 - p_2 = 0$$
$$H_a : p_1 - p_2 \neq 0$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{.020 - .034}{\sqrt{.029(.971)\left(\frac{1}{797} + \frac{1}{1,499}\right)}} = -1.90$$

Since no α was given, we will use $\alpha = .05$. The rejection region requires $\alpha/2 = .05/2 = .025$ in each tail of the z-distribution. $z_{.025} = 1.96$. The rejection region is $z < -1.96$ and $z > 1.96$.

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{16}{797} = .020 \qquad \hat{p}_2 = \frac{x_2}{n_2} = \frac{51}{1,499} = .034 \qquad \hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{16+51}{797+1,499} = \frac{67}{2,296} = .029$$

Advanced Example - Religious symbolism in TV commercials - Solution part 2

$$H_0 : p_1 - p_2 = 0$$
$$H_a : p_1 - p_2 \neq 0$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{.020 - .034}{\sqrt{.029(.971)\left(\frac{1}{797} + \frac{1}{1,499}\right)}} = -1.90$$

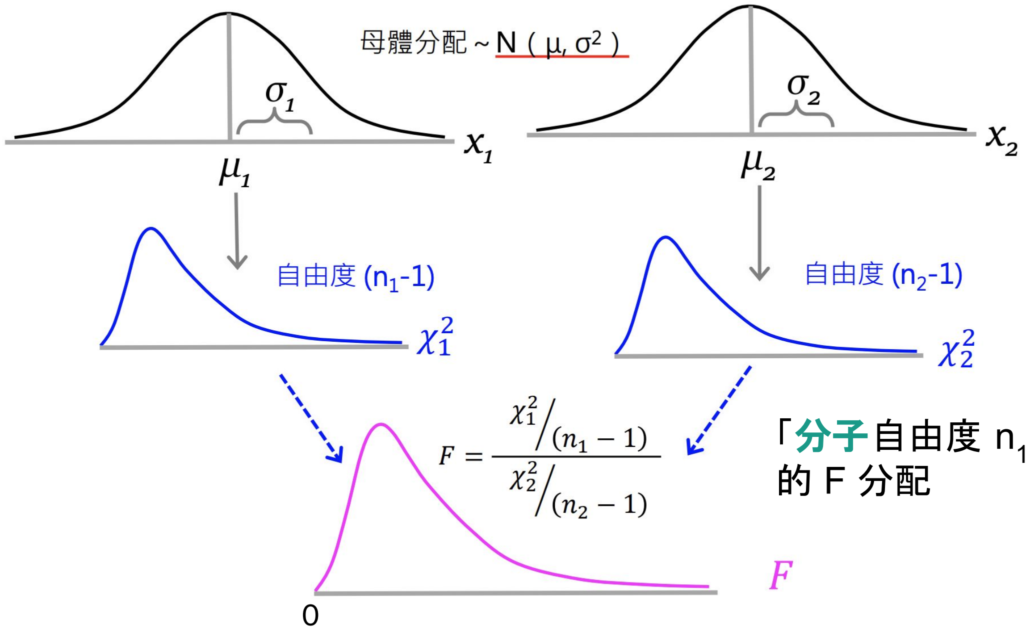
Since the observed value of the test statistic does not fall in the rejection region $Z = -1.90 > -1.96$, H_0 is not rejected.

There is **insufficient evidence** to indicate the percentage of **TV commercials that use religious symbolism has changed** in the previous ten years at $\alpha=0.05$.

Comparing Two Population Variances: Independent Sampling

F-distribution

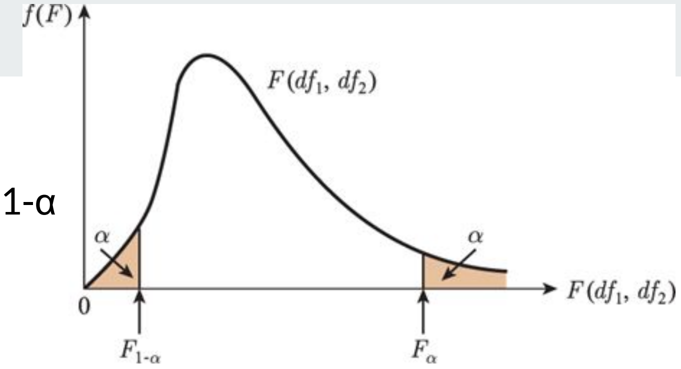
$$F = \frac{\chi_1^2/df_1}{\chi_2^2/df_2}$$



「**分子**自由度 n_1 -1、**分母**自由度n_2 -1 」的 F 分配

http://si.secda.info/kevinho_stat/wp-content/uploads/2014/10/3-1_5_2Sample-Distributions.pdf

F_α = F分配下某點的右尾發生機率為 α
$F_{1-\alpha}$ = F分配下某點的右尾發生機率為 1-α



F-distribution

自由度交換，交換前分配與交換後分配的倒數為同一 F分配

$$f(v_1, v_2) = \frac{\frac{\chi_1^2}{\nu_1}}{\frac{\chi_2^2}{\nu_2}} = \frac{1}{\frac{\frac{\chi_2^2}{\nu_2}}{\frac{\chi_1^2}{\nu_1}}} = \frac{1}{f(\nu_2,\nu_1)}$$

$$Right\,tail: f_\alpha(\nu_1, \nu_2)$$

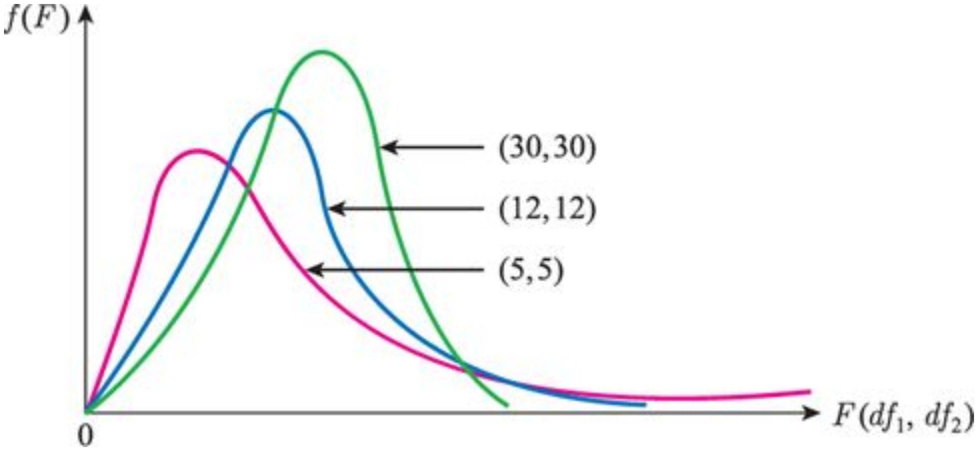
$$Left\,tail: f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_\alpha(\nu_2, \nu_1)}$$

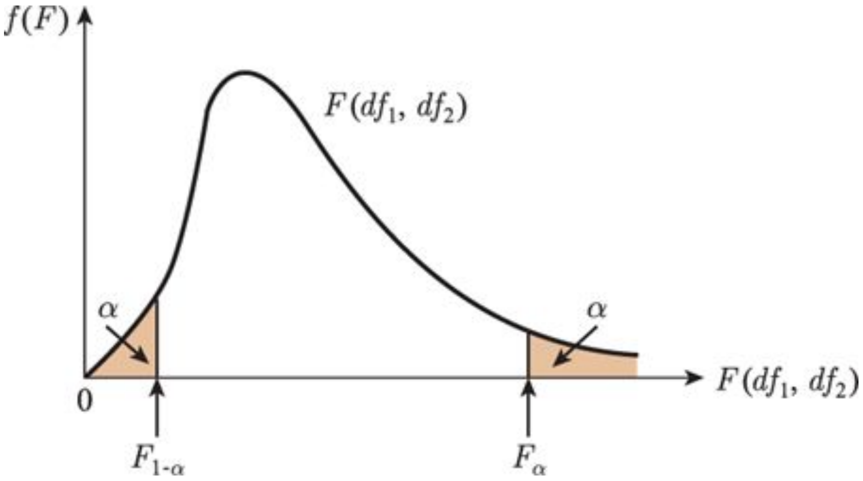
$$P[f(\nu_1, \nu_2) > f_\alpha(\nu_1, \nu_2)] = \alpha$$
$$P[f(\nu_1, \nu_2) < f_\alpha(\nu_1, \nu_2)] = 1 - \alpha$$
$$P\left[\frac{1}{f(\nu_2, \nu_1)} < f_\alpha(\nu_1, \nu_2)\right] = 1 - \alpha$$
$$P\left[\frac{1}{f_\alpha(\nu_1, \nu_2)} < f(\nu_2, \nu_1)\right] = 1 - \alpha$$
$$P\left[f(\nu_2, \nu_1) > \frac{1}{f_\alpha(\nu_1, \nu_2)}\right] = 1 - \alpha$$

$$P[f(\nu_2, \nu_1) > f_{1-\alpha}(\nu_2, \nu_1)] = 1 - \alpha$$

F-distribution

$$F = \frac{\chi_1^2 / df_1}{\chi_2^2 / df_2}$$





$$Right\,tail: \; f_\alpha(\nu_1, \nu_2)$$

$$Left\,tail: \; f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_\alpha(\nu_2, \nu_1)}$$

F-distribution

$$F(v_1, v_2) = \frac{\chi_1^2/df_1}{\chi_2^2/df_2} = \frac{\left[\frac{(n_1-1)S_1^2}{\sigma_1^2}\right]/(n_1-1)}{\left[\frac{(n_2-1)S_2^2}{\sigma_2^2}\right]/(n_2-1)}$$

$$= \frac{\sigma_2^2}{\sigma_1^2} \cdot \frac{S_1^2}{S_2^2}$$

F-Test for Equal Population Variances

One-Tailed Test

$$H_0 : \sigma_1^2 \geq \sigma_2^2 \quad or \ \sigma_1^2 \leq \sigma_2^2$$
$$H_a : \sigma_1^2 < \sigma_2^2 \quad or \ \sigma_1^2 > \sigma_2^2$$

Rejection region: $F > F_\alpha$

where F_α is based on v_1 = numerator degrees of freedom and v_2 = denominator degrees of freedom; v_1 and v_2 are the degrees of freedom for the numerator and denominator sample variances, respectively.

Test statistic:

$$F = \frac{s_2^2}{s_1^2} \quad or \quad F = \frac{s_1^2}{s_2^2}$$

Note: H_0假設 $\frac{\sigma_1^2}{\sigma_2^2} = 1$

F-Test for Equal Population Variances

Two-Tailed Test

$$H_0 : \sigma_1^2 = \sigma_2^2$$
$$H_a : \sigma_1^2 \neq \sigma_2^2$$

Rejection region: $F > F_{\alpha/2}$

where $F_{\alpha/2}$ is based on v_1 = numerator degrees of freedom and v_2 = denominator degrees of freedom; v_1 and v_2 are the degrees of freedom for the numerator and denominator sample variances, respectively.

Test statistic:

$$F = \frac{s_2^2}{s_1^2} \quad when\, s_2^2 > s_1^2$$

$$F = \frac{s_1^2}{s_2^2} \quad when\, s_1^2 > s_2^2$$

Conditions Required for a Valid *F*-Test for Equal Variances

1. Both sampled **populations** are **normally distributed**.
2. The samples are **random** and independent.

F-Test for Equal Variances Example

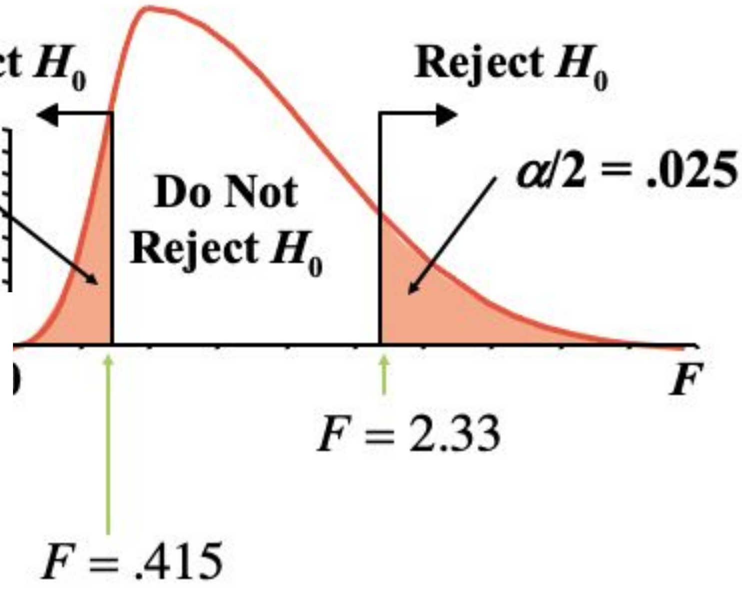
You're a financial analyst for Charles Schwab. You want to compare dividend yields between stocks listed on the NYSE & NASDAQ. You collect the following data:

	NYSE	NASDAQ
Number	21	25
Mean	3.27	2.53
Std Dev	1.30	1.16

Is there a difference in variances between the NYSE & NASDAQ at the .05 level of significance?

σ1為the sd of dividend yield of stocks listed on NYSE
σ2為the sd an of dividend yield of stocks listed on NASDAQ

F-Test for Equal Variances Solution



H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

$\alpha =$ .05

$\nu_1 =$. 20 $\nu_2 =$ 24

Table VII *(continued)*

ν_2 \ ν_1 Numerator Degr	10	12	15	20	24
1	968.6	976.7	984.9	993.1	997.2
2	39.40	39.41	39.43	39.45	39.46
3	14.42	14.34	14.25	14.17	14.12
4	8.84	8.75	8.66	8.56	8.51
5	6.62	6.52	6.43	6.33	6.28
6	5.46	5.37	5.27	5.17	5.12
7	4.76	4.67	4.57	4.47	4.42
8	4.30	4.20	4.10	4.00	3.95
9	3.96	3.87	3.77	3.67	3.61
10	3.72	3.62	3.52	3.42	3.37
11	3.53	3.43	3.33	3.23	3.17
12	3.37	3.28	3.18	3.07	3.02
13	3.25	3.15	3.05	2.95	2.89
14	3.15	3.05	2.95	2.84	2.79
15	3.06	2.96	2.86	2.76	2.70
16	2.99	2.89	2.79	2.68	2.63
17	2.92	2.82	2.72	2.62	2.56
18	2.87	2.77	2.67	2.56	2.50
19	2.82	2.72	2.62	2.51	2.45
20	2.77	2.68	2.57	2.46	2.41
21	2.73	2.64	2.53	2.42	2.37
22	2.70	2.60	2.50	2.39	2.33
23	2.67	2.57	2.47	2.36	2.30
24	2.64	2.54	2.44	2.33	2.27
25	2.61	2.51	2.41	2.30	2.24

Denominator Degrees of Freedom

1/2.41=0.415

$$Right\,tail: f_{\alpha}(\nu_1, \nu_2)$$

$$Left\,tail: f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_{\alpha}(\nu_2, \nu_1)}$$

F-Test for Equal Variances Solution

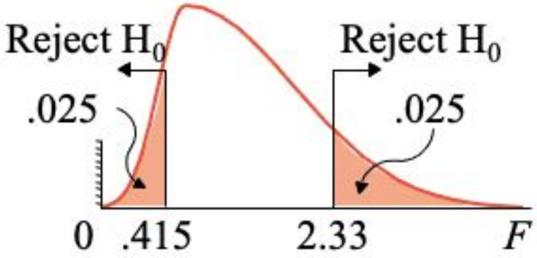
H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

$\alpha = .05$

$\nu_1 = 20 \quad \nu_2 = 24$

Critical Value(s):



Test Statistic:

$$F = \frac{s_1^2}{s_2^2} = \frac{1.30^2}{1.16^2} = 1.25$$

Decision:

Do not reject at $\alpha = .05$

Conclusion:

There is no evidence of a difference in variances

Table VII *(continued)*

ν_2 \ ν_1 (Numerator Degr)	10	12	15	20	24
1	968.6	976.7	984.9	993.1	997.2
2	39.40	39.41	39.43	39.45	39.46
3	14.42	14.34	14.25	14.17	14.12
4	8.84	8.75	8.66	8.56	8.51
5	6.62	6.52	6.43	6.33	6.28
6	5.46	5.37	5.27	5.17	5.12
7	4.76	4.67	4.57	4.47	4.42
8	4.30	4.20	4.10	4.00	3.95
9	3.96	3.87	3.77	3.67	3.61
10	3.72	3.62	3.52	3.42	3.37
11	3.53	3.43	3.33	3.23	3.17
12	3.37	3.28	3.18	3.07	3.02
13	3.25	3.15	3.05	2.95	2.89
14	3.15	3.05	2.95	2.84	2.79
15	3.06	2.96	2.86	2.76	2.70
16	2.99	2.89	2.79	2.68	2.63
17	2.92	2.82	2.72	2.62	2.56
18	2.87	2.77	2.67	2.56	2.50
19	2.82	2.72	2.62	2.51	2.45
20	2.77	2.68	2.57	2.46	2.41
21	2.73	2.64	2.53	2.42	2.37
22	2.70	2.60	2.50	2.39	2.33
23	2.67	2.57	2.47	2.36	2.30
24	2.64	2.54	2.44	$\boxed{2.33}$	2.27
25	2.61	2.51	2.41	2.30	2.24

1/2.41=0.415

$$Right\,tail:\ f_\alpha(\nu_1,\nu_2)$$

$$Left\,tail:\ f_{1-\alpha}(\nu_1,\nu_2) = \frac{1}{f_\alpha(\nu_2,\nu_1)}$$

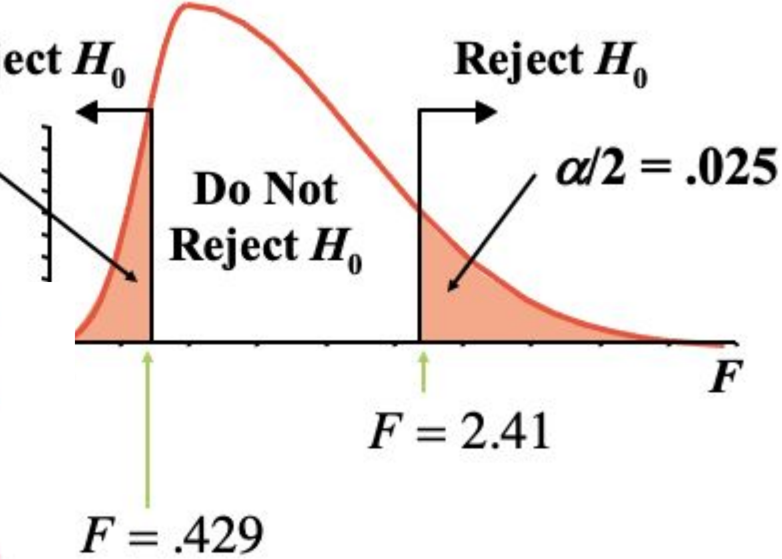
F-Test for Equal Variances Quiz

You're an analyst for the Light & Power Company. You want to compare the electricity consumption of single-family homes in two towns. You compute the following from a sample of homes:

	Town 1	Town 2
Number	25	21
Mean	$ 85	$ 68
Std Dev	$ 30	$ 18

At the .05 level of significance, is there evidence of a difference in variances between the two towns?

F-Test for Equal Variances Ans



H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

$\alpha = .05$

$\nu_1 = 24$ $\nu_2 = 20$

Table VII *(continued)*

Denominator Degrees of Freedom ν_2 \ Numerator Degr[ees of Freedom] ν_1	10	12	15	20	24
1	968.6	976.7	984.9	993.1	997.2
2	39.40	39.41	39.43	39.45	39.46
3	14.42	14.34	14.25	14.17	14.12
4	8.84	8.75	8.66	8.56	8.51
5	6.62	6.52	6.43	6.33	6.28
6	5.46	5.37	5.27	5.17	5.12
7	4.76	4.67	4.57	4.47	4.42
8	4.30	4.20	4.10	4.00	3.95
9	3.96	3.87	3.77	3.67	3.61
10	3.72	3.62	3.52	3.42	3.37
11	3.53	3.43	3.33	3.23	3.17
12	3.37	3.28	3.18	3.07	3.02
13	3.25	3.15	3.05	2.95	2.89
14	3.15	3.05	2.95	2.84	2.79
15	3.06	2.96	2.86	2.76	2.70
16	2.99	2.89	2.79	2.68	2.63
17	2.92	2.82	2.72	2.62	2.56
18	2.87	2.77	2.67	2.56	2.50
19	2.82	2.72	2.62	2.51	2.45
20	2.77	2.68	2.57	2.46	2.41
21	2.73	2.64	2.53	2.42	2.37
22	2.70	2.60	2.50	2.39	2.33
23	2.67	2.57	2.47	2.36	2.30
24	2.64	2.54	2.44	$\boxed{2.33}$	2.27
25	2.61	2.51	2.41	2.30	2.24

1/2.33=0.429

$$Right\,tail:\ f_{\alpha}(\nu_1,\nu_2)$$

$$Left\,tail:\ f_{1-\alpha}(\nu_1,\nu_2) = \frac{1}{f_{\alpha}(\nu_2,\nu_1)}$$

σ_1為the sd of electricity consumption of single-family homes in town 1
σ_2為the sd of electricity consumption of single-family homes in town 2

F-Test for Equal Variances Ans

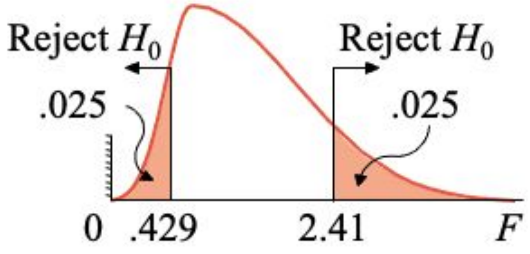
H_0: $\sigma_1^2 = \sigma_2^2$

H_a: $\sigma_1^2 \neq \sigma_2^2$

$\alpha = .05$

$\nu_1 = 24 \quad \nu_2 = 20$

Critical Value(s):



Test Statistic:

$$F = \frac{s_1^2}{s_2^2} = \frac{30^2}{18^2} = 2.778$$

Decision:

Reject at $\alpha = .05$

Conclusion:

There is evidence of a difference in variances

Table VII *(continued)*

ν_2 \ ν_1 (Numerator Degr… ; Denominator Degrees of Freedom)	10	12	15	20	24
1	968.6	976.7	984.9	993.1	997.2
2	39.40	39.41	39.43	39.45	39.46
3	14.42	14.34	14.25	14.17	14.12
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9	3.96	3.87	3.77	3.67	3.61
10	3.72	3.62	3.52	3.42	3.37
11	3.53	3.43	3.33	3.23	3.17
12	3.37	3.28	3.18	3.07	3.02
13	3.25	3.15	3.05	2.95	2.89
14	3.15	3.05	2.95	2.84	2.79
15	3.06	2.96	2.86	2.76	2.70
16	2.99	2.89	2.79	2.68	2.63
17	2.92	2.82	2.72	2.62	2.56
18	2.87	2.77	2.67	2.56	2.50
19	2.82	2.72	2.62	2.51	2.45
20	2.77	2.68	2.57	2.46	2.41
21	2.73	2.64	2.53	2.42	2.37
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23	2.67	2.57	2.47	2.36	2.30
24	2.64	2.54	2.44	2.33	2.27
25	2.61	2.51	2.41	2.30	2.24

1/2.33=0.429

$$Right\,tail:\ f_{\alpha}(\nu_1,\nu_2)$$

$$Left\,tail:\ f_{1-\alpha}(\nu_1,\nu_2) = \frac{1}{f_{\alpha}(\nu_2,\nu_1)}$$

F-test + *t*-test

F-test before *t*-test

$(\overline{X}_1 - \overline{X}_2)$ 的抽樣分配在**常態母體**且**小樣本**的假設下，需判斷此二組隨機樣本的**變異數是否相等**(即 與σ_1^2 是σ_2^2相等)，以決定採用何種性質的t分配。

因此，這類的問題往往必須先檢定二個母體變異數是否相等

F-test before *t*-test Example

在甲、乙二種品牌燈泡中各抽10支檢查，得其統計資料如下：

	平均壽命（小時）	標準差（小時）
甲品牌	1,600	120
乙品牌	1,200	100

在顯著水準α=0.1下，**比較甲、乙二品牌燈泡之平均壽命是否有差異**？(假設燈泡之壽命服從**常態分配**)

F-test before *t*-test Solution

由於是小樣本，t分配，需先檢定二個母體的變異數是否相等

(a)先檢定母體變異數是否相等

(b)再檢定母體平均數是否相等

$$(a)\ H_0 : \sigma_1^2 = \sigma_2^2;\ H_1 = \sigma_1^2 \neq \sigma_2^2$$

F-test

$$F = \frac{S_1^2}{S_2^2} = \frac{(120)^2}{(100)^2} = 1.44$$

在顯著水準α=0.1下,

$F_{0.95}(9, 9)=0.314<1.44<3.18= F_{0.05}(9,9)$(查表)

因此, 不能拒絕H_0, 亦即沒有證據顯示此二母體的變異數不相等。

(b) 假設此二母體的變異數相等

$$H_0 : \mu_1 = \mu_2; \quad H_1 : \mu_1 \neq \mu_2$$

檢定統計量 $t = \dfrac{(\overline{x}_1 - \overline{x}_2)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

t-test

$$\overline{x}_1 = 1,600, \ \overline{x}_2 = 1,200$$

$$S_p^2 = \frac{(10-1)S_1^2 + (10-1)S_2^2}{10+10-2} = 12,200, \ S_p = 110.45$$

$$t = \frac{(1,600 - 1,200)}{(110.45)\sqrt{\dfrac{1}{10} + \dfrac{1}{10}}} = 8.098 > 1.736 = t_{0.05}(18)$$

在α=0.05的條件下，拒絕H_0，亦即甲、乙二品牌燈泡之平均壽命有顯著差異

Advanced Example - Shopping vehicle and judgment

Recall that design engineers want to know **whether the mean choice of vice-over-virtue score is higher** when a consumer's arm is flexed (as when carrying a shopping basket) than when the consumer's arm is extended (as when pushing a shopping cart).

The average choice score for the $n_1 = 11$ consumers with a flexed arm was $x_1 = 59$, while the average for the $n_2 = 11$ consumers with an extended arm was $x_2 = 43$.

In which scenario is the assumption required for a t-test to compare means more likely to be violated, $s_1 = 4$ and $s_2 = 2$, or, $s_1 = 10$ and $s_2 = 15$? Explain.

Advanced Example - Shopping vehicle and judgment - Solution

For the first scenario, with $s_1 = 4$ and $s_2 = 2$, the test statistic is

$$F = \frac{\text{Larger sample variance}}{\text{Smaller sample variance}} = \frac{s_1^2}{s_2^2} = \frac{4^2}{2^2} = 4.$$

For the second scenario, with $s_1 = 10$ and $s_2 = 15$, the test statistic is

$$F = \frac{\text{Larger sample variance}}{\text{Smaller sample variance}} = \frac{s_2^2}{s_1^2} = \frac{15^2}{10^2} = 2.25.$$

In both cases, the degrees of freedom for the tests are the same.

Thus, the assumption required for the t-test for the first scenario, with S_1=4 and S_2=2, would be the most likely be violated because the value of the test statistic is larger.

Pre-class quiz

Week 13

遠距上課, Zuvio考試

要寫計算過程

Determining the Required Sample Size

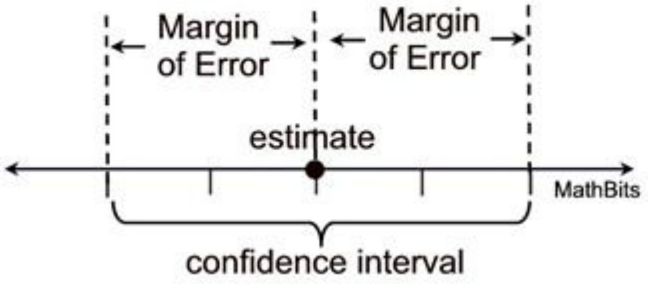
$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sigma_{(\bar{x}_1-\bar{x}_2)} = (\bar{x}_1 - \bar{x}_2) \pm \underbrace{z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}_{\text{Margin of Error}}$$

Determination of Sample Size for Estimating $\mu_1 - \mu_2$

To estimate $(\mu_1 - \mu_2)$ with a given margin of error **margin of error** (ME) and with confidence level $(1 - \alpha)$, use the following formula to solve for equal sample sizes that will achieve the desired reliability:

$$n_1 = n_2 = \frac{(z_{\alpha/2})^2(\sigma_1^2 + \sigma_2^2)}{(\text{SE})^2}$$

SE= Sampling Error=ME

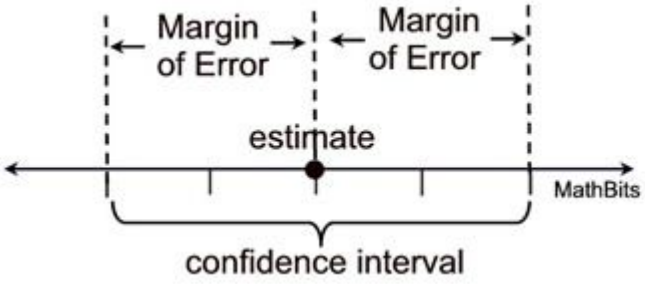


$$(\overline{p_1} - \overline{p_2}) \pm z_{\alpha/2} \times \sigma_{(\overline{p_1}-\overline{p_2})} = (\overline{p_1} - \overline{p_2}) \pm \underbrace{z_{\alpha/2} \times \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}_{\text{Margin of Error}}$$

Determination of Sample Size for Estimating $p_1 - p_2$

To estimate $(p_1 - p_2)$ with a given margin of error ME and with confidence level $(1 - \alpha)$, use the following formula to solve for equal sample sizes that will achieve the desired reliability:

$$n_1 = n_2 = \frac{\left(z_{\alpha/2}\right)^2 \left(p_1 q_1 + p_2 q_2\right)}{(\mathrm{SE})^2}$$



Sample Size Example

What sample size is needed to estimate $\mu_1 - \mu_2$ with 95% confidence and a **margin of error** of 5.8?

Assume prior experience tells us $\sigma_1 = 12$ and $\sigma_2 = 18$.

$$n_1 = n_2 = \frac{(1.96)^2 \left(12^2 + 18^2\right)}{(5.8)^2} = 53.44 \approx 54$$

Sample Size Quiz

What sample size is needed to estimate $p_1 - p_2$ with 90% confidence and a width of .05?

$\hat{p}_1 = .184$ and $\hat{p}_2 = .177$.

Advanced Example - Angioplasty's benefits challenged (1/2)

Half the patients were randomly assigned to get an angioplasty (血管成形術) and half were not.

The researchers compared the proportion of patients with **subsequent heart attacks** for the two groups and reported **no significant difference between the two proportions**.

Although the study involved over **2,000 patients**, the sample size may have been too small to detect a difference in heart attack rates.

Advanced Example - Angioplasty's benefits challenged (2/2)

1. **How many patients must be sampled in each group** to estimate the difference in heart attack rates to **within ±0.015 with 95% confidence**?
2. Comment on the practicality of carrying out the study with the sample sizes determined in part 1.
3. Comment on the practical significance of the difference detected in the confidence interval for the study, part 1.

Advanced Example - Religious symbolism in TV commercials - Solution

$$n_1 = n_2 = \frac{(z_{\alpha/2})^2 (p_1 q_1 + p_2 q_2)}{(ME)^2} = \frac{1.96^2 (.184(.816) + .177(.823))}{.015^2} = 5{,}050.7 \approx 5{,}051$$

- The study would involve 5051*2=10102 patients. A study this large would be extremely time consuming and expensive.
- Since a difference of 0.015 is so small, the practical significance detecting a 0.015 difference may not be very worthwhile. A difference of 0.015 is so close to 0, that it might not make any difference.

Key Ideas

Key Words for Identifying the Target Parameter

Parameter	Key Words or Phrases	Type of Data
$\mu_1 - \mu_2$	Mean difference; differences in averages	Quantitative
$p_1 - p_2$	Differences between proportions, percentages, fractions, or rates; compare proportions	Qualitative
$\dfrac{(\sigma_1)^2}{(\sigma_2)^2}$	Ratio of variances; differences in variability or spread; compare variation	Quantitative

Conditions Required for Inferences about $\mu_1 - \mu_2$

Large Samples:

1. Independent random samples
2. $n_1 \geq 30, \; n_2 \geq 30$

Small Samples:

1. Independent random samples
2. Both populations normal
3. $\sigma_1^2 = \sigma_2^2$

Conditions Required for Inferences about $p_1 - p_2$

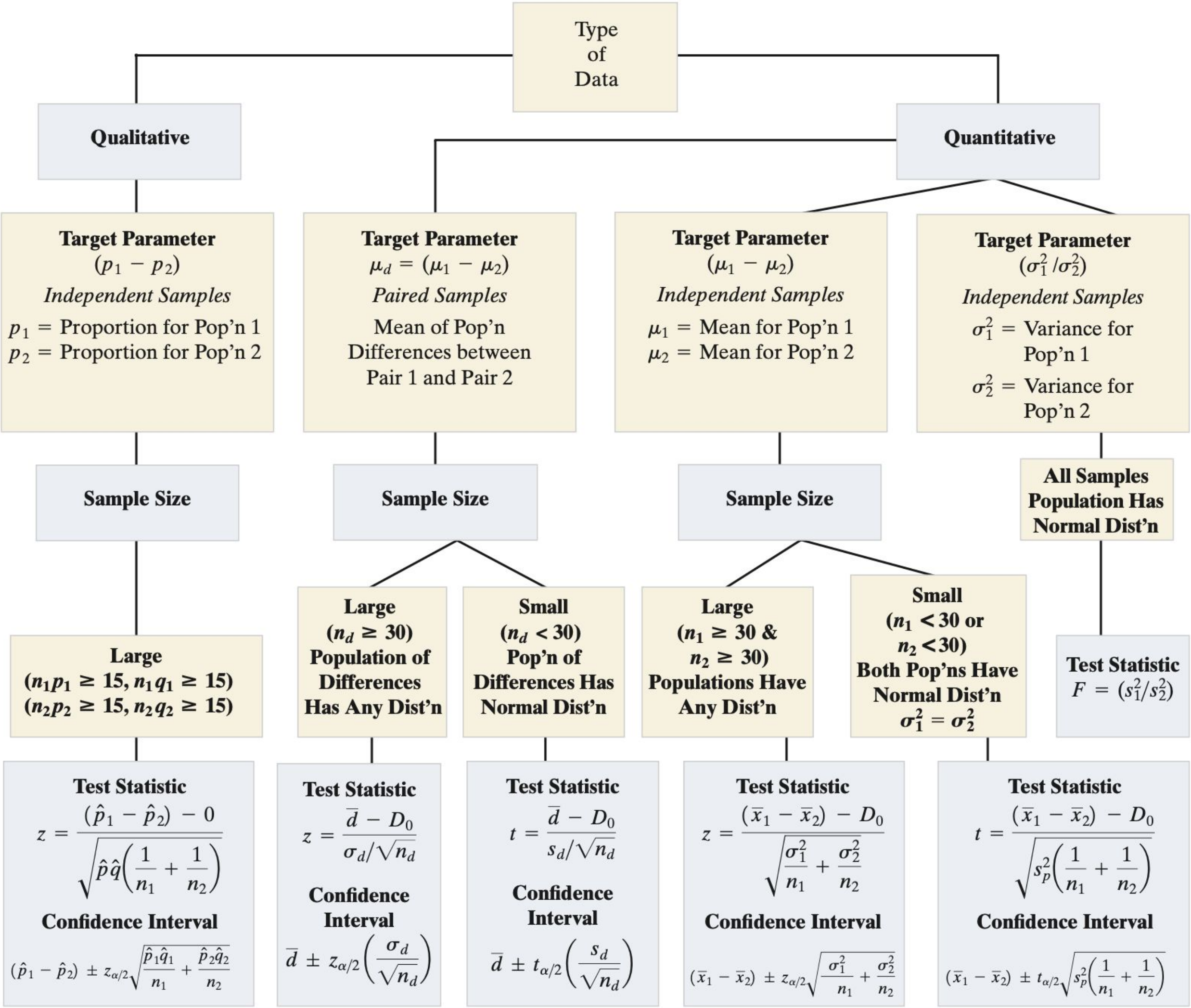
Large Samples:

1. Independent random samples
2. $n_1 p_1 \geq 15$, $n_1 q_1 \geq 15$
3. $n_2 p_2 \geq 15$, $n_2 q_2 \geq 15$

Conditions Required for Inferences about *F*-Test for Equal Variances

Large or small Samples:

1. Independent random samples
2. Both populations normal



Type of Data
Qualitative
Quantitative
Target Parameter $(p_1 - p_2)$
Independent Samples
p_1 = Proportion for Pop'n 1
p_2 = Proportion for Pop'n 2
Target Parameter $\mu_d = (\mu_1 - \mu_2)$
Paired Samples
Mean of Pop'n Differences between Pair 1 and Pair 2
Target Parameter $(\mu_1 - \mu_2)$
Independent Samples
μ_1 = Mean for Pop'n 1
μ_2 = Mean for Pop'n 2
Target Parameter $(\sigma_1^2 / \sigma_2^2)$
Independent Samples
σ_1^2 = Variance for Pop'n 1
σ_2^2 = Variance for Pop'n 2
Sample Size
Sample Size
Sample Size
All Samples Population Has Normal Dist'n
Large $(n_1 p_1 \geq 15, n_1 q_1 \geq 15)$ $(n_2 p_2 \geq 15, n_2 q_2 \geq 15)$
Large $(n_d \geq 30)$ Population of Differences Has Any Dist'n
Small $(n_d < 30)$ Pop'n of Differences Has Normal Dist'n
Large $(n_1 \geq 30$ & $n_2 \geq 30)$ Populations Have Any Dist'n
Small $(n_1 < 30$ or $n_2 < 30)$ Both Pop'ns Have Normal Dist'n $\sigma_1^2 = \sigma_2^2$
Test Statistic $F = (s_1^2 / s_2^2)$
Test Statistic $z = \dfrac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}\hat{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$
Confidence Interval $(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}_1\hat{q}_1}{n_1} + \dfrac{\hat{p}_2\hat{q}_2}{n_2}}$
Test Statistic $z = \dfrac{\bar{d} - D_0}{\sigma_d / \sqrt{n_d}}$
Confidence Interval $\bar{d} \pm z_{\alpha/2}\left(\dfrac{\sigma_d}{\sqrt{n_d}}\right)$
Test Statistic $t = \dfrac{\bar{d} - D_0}{s_d / \sqrt{n_d}}$
Confidence Interval $\bar{d} \pm t_{\alpha/2}\left(\dfrac{s_d}{\sqrt{n_d}}\right)$
Test Statistic $z = \dfrac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$
Confidence Interval $(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$
Test Statistic $t = \dfrac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s_p^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$
Confidence Interval $(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2}\sqrt{s_p^2\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}$

[統計學2概念整理]

Determining the Sample Size

Estimating $\mu_1 - \mu_2$: $n_1 = n_2 = \dfrac{\left(z_{\alpha/2}\right)^2\left(\sigma_1^2 + \sigma_2^2\right)}{(\mathrm{SE})^2}$

Estimating $p_1 - p_2$: $n_1 = n_2 = \dfrac{\left(z_{\alpha/2}\right)^2\left(p_1 q_1 + p_2 q_2\right)}{(\mathrm{SE})^2}$