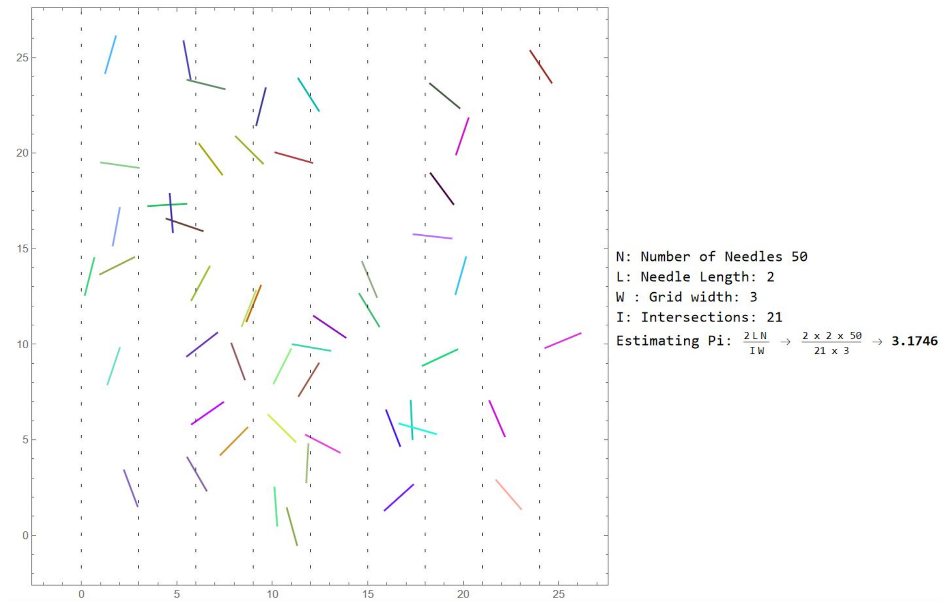
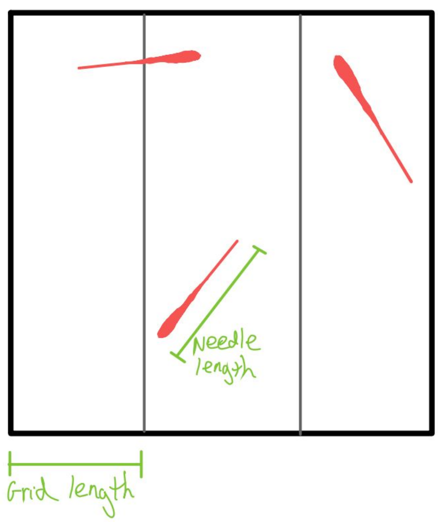
Buffon’s needle Problem

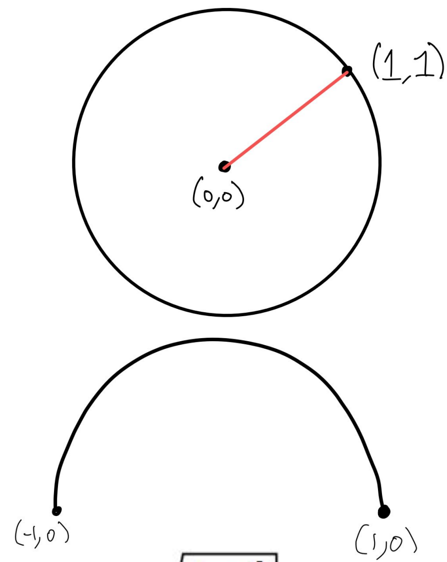


Ethan zhang

The buffon’s needle is a geometric probability problem that can be used to estimate pi with monte carlo methods, which is a field of computational mathematics that rely on repeated random sampling to obtain numerical solutions, in this case the value of pi.

What is it?



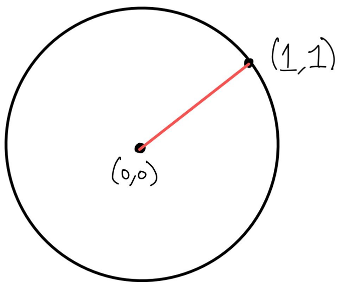


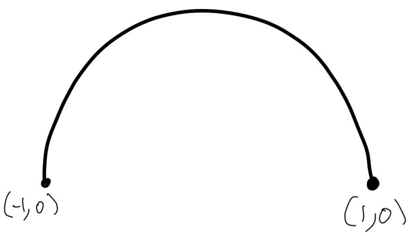
$$\sqrt{1 - x^2}$$

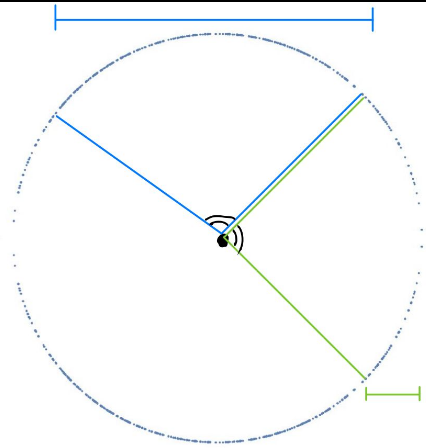
$$-1 \leq x \leq 1$$

The basic procedure of recreating buffon's needle is by getting a flat surface and constructing a set of grids in one axis parallel and similarly apart. Next, a number of needles of a size less than the width between each grid such as a grid separation of 3 units and needle size of 2. Now you throw these needles randomly on the grid, ensuring all needles are within the bounds of the table. Count the number of needles that intersect the grid, and using this formula, the value of pi can be roughly calculated.

Simulation







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Solve[x^2 + y^2 == 1, x, Reals]
Solve[x^2 + y^2 == 1, y, Reals]
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$$\left\{\left\{x \to -\sqrt{1-y^2} \text{ if } -1 < y < 1\right\}, \left\{x \to \sqrt{1-y^2} \text{ if } -1 < y < 1\right\}\right\}$$

$$\left\{\left\{y \to -\sqrt{1-x^2} \text{ if } -1 < x < 1\right\}, \left\{y \to \sqrt{1-x^2} \text{ if } -1 < x < 1\right\}\right\}$$

To simulate the Buffon's needle, we can use program mathematica to throw N needles of length L onto a table of a certain size, separated by gridlines.

From here we can run the function. First it randomly plots N points, each which will act as the one of the endpoints of our needles. For each pair of x and y coordinate we can model a loci of points or circle of a radius predetermined by the length of the needle.

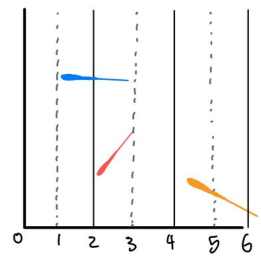
At this point we have one of endpoints defined as the center of the circle and another endpoint which can be any point on the circle ultimately forming a needle. To find the second endpoint we will need to solve each equation for a variable. However it should be noted that choosing strictly one variable will result in a uneven distribution of random locus points. This is illustrated by this diagram that clearly shows points will tend to distribute around the center of the circle. The reason for this is that for the same angles that encompass the same set of loci points, the x domain of the green portion is much smaller than the blue.

The result of this evaluation will provide us with two solutions as the value of square roots can be either positive or negative. Specifically they will draw either a top or bottom semicircle if solved in terms of x and left or right when solved for y. Because of this we will randomly select one of the equations as only one other endpoint is required.

Now we have semicircle functions in terms of either the vertical or horizontal axis, and we know by subbing in for the corresponding variable with a random value, we will get the other axis coordinate. However, unfortunately doing this will return with non real numbers if the substituted point is not within the domain of the circle, so we must restrict the random number to its respective semicircle domain, which are also its endpoints.

With the two endpoints for each needle we can now simulate it with listlineplot in mathematica.

Calculating Number of Intersections



Gridlines: {2,4,6}

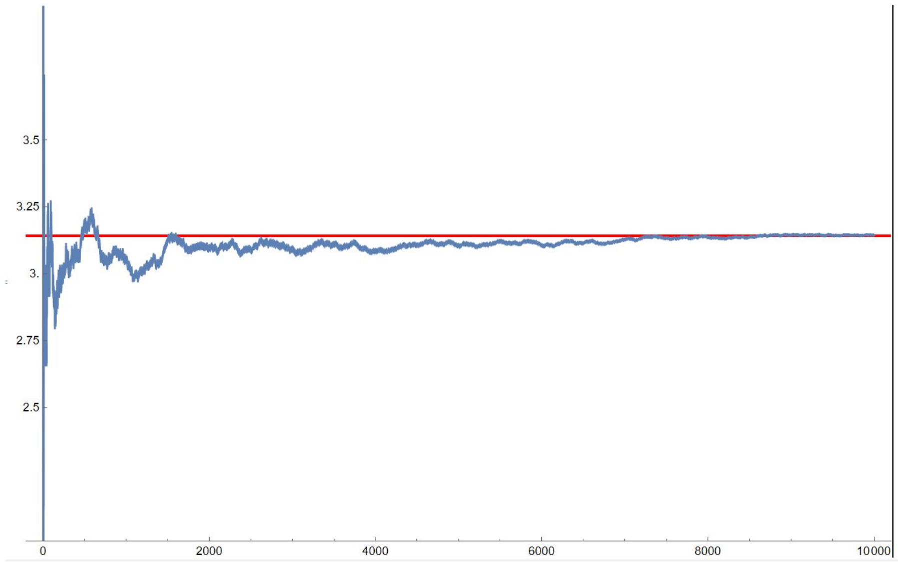
Intervals

[1,3]	2 True	4 False	6 False	1
[2.1,3]	2 False	4 False	6 False	0
[4.2,6.1]	2 False	4 False	6 True	1

Total Intersections: 2

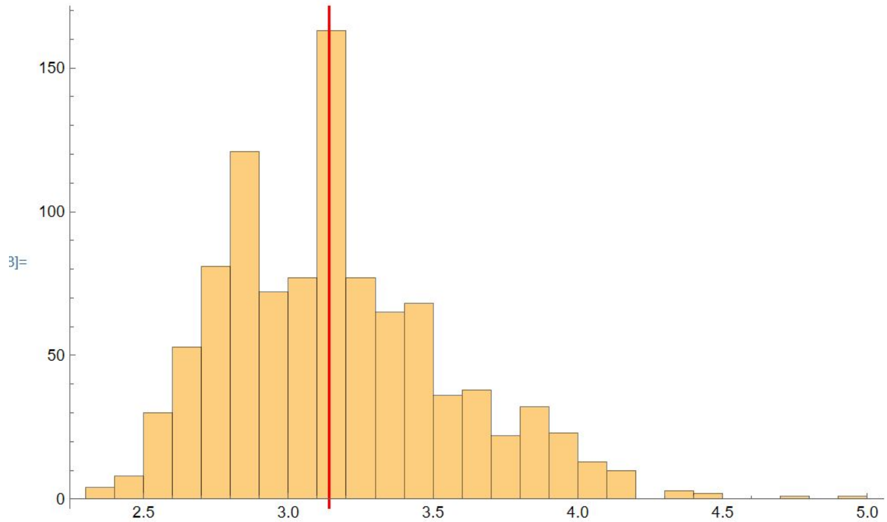
Now that the buffon’s needle can be visually demonstrated, we now need to find the number of needle intersection with the grid. To do this we will use intervals. The grid lines in my simulations are vertical, so they have the equation x = some number. We can gather all these gridline equations into a gridline set for example {1,2,3}. We also know the coordinates of the two endpoints of each needle. For each needle we extract the two x coordinates of each side which gives us the interval for which the set of points for that line lies on. We now test whether each element of the gridline set is a member of a needle interval. If this is true for any element, then we know there is a intersection. We can iterate this test for every needle interval, sowing the number of positive results. This provides us with the total number of intersections.

Cumulative needle pi Estimate



Process of estimation of pi with 10000 needles graphically.
X: needles thrown
Y: pi estimate

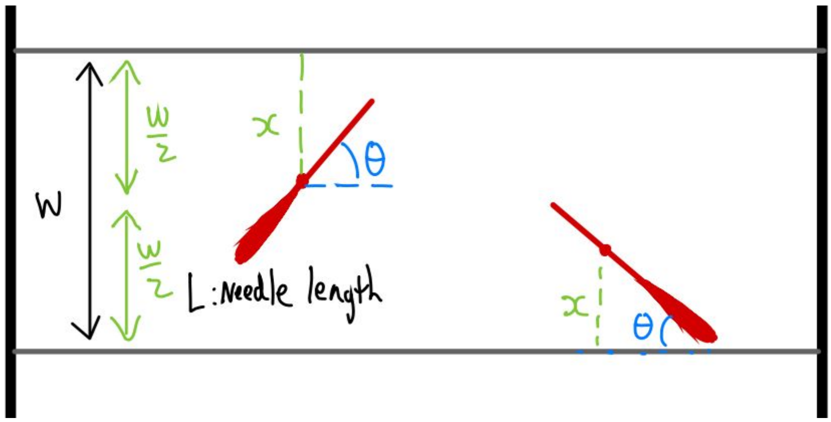
Histogram of pi estimates



Histogram of 1000 samples.
Threw 100 needles per sample
X: pi estimation
Y: number of samples that fall within particular column

Derivation of Formula

N: Number of Needles 20
L: Needle Length: 2
W : Grid width: 3
I: Intersections: 8
Estimating Pi: $\frac{2LN}{IW} \rightarrow \frac{2 \times 2 \times 20}{8 \times 3} \rightarrow$ **3.33333**



$$0 \leq x \leq \frac{W}{2}$$

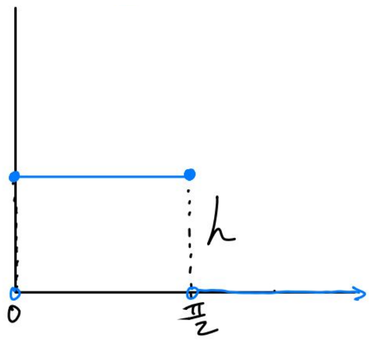
$$0 \leq \theta \leq \frac{\pi}{2}$$

To derive the formula we have to tackle the Buffon's needle problem which is to calculate the probability of a needle falling on a gridline.

Being a probability problem, we must define the random variables. The two factors that effect the probability of the needle intersecting is the angle the needle creates with respect to the horizontal axis and the distance from the midpoint of the needle to the closest grid line.

As the position that the needle lands is completely random, we can say that the random variables follow uniform distributions. We can set domains for each of these probabilities

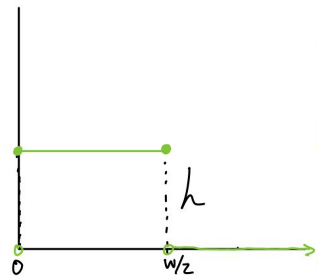
Uniform Distribution Probability



$$\frac{\pi}{2} \times h_\theta = 1$$

$$h_\theta = \frac{2}{\pi}$$

$$f_\theta[\theta] = \frac{2}{\pi}$$



$$\frac{w}{2} \times h_x = 1$$

$$h_x = \frac{2}{w}$$

$$f_\theta[\theta] = \frac{2}{\pi}$$

As the angle and position of the needle is completely randomised, the probability of these two variables are equally distributed, so we can say that the random variables follow uniform distributions. To construct the PDF's of these random variables, we know that the integral or area under the curve within the specified domain should always total to 1. As these two probabilities are constant over the domain, we can calculate the area under the line as length times height of the rectangle. Solving for the variable h, we can find the PDF.

Bivariate Joint PDF

$$f_\theta[\theta] = \frac{2}{\pi}$$

$$f_x[x] = \frac{2}{w}$$

$$P(A \cap B) = P(A) \times P(B)$$

$$P(f_x \cap f_\theta) = P(f_x) \times P(f_\theta)$$

$$f_{x\,\theta}[x, \theta] = \frac{4}{w\,\pi}$$

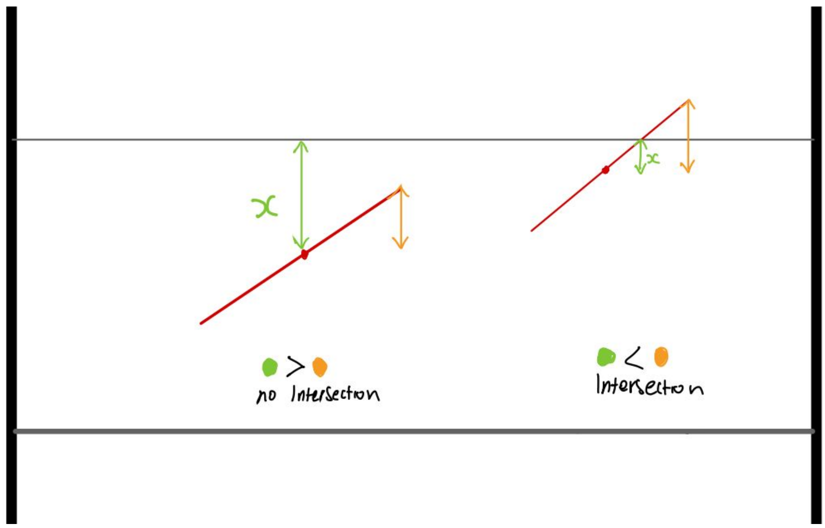
$$0 \le x \le \frac{w}{2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \int_0^{w/2} \frac{4}{w\,\pi}\, dx\, d\theta = 1$$

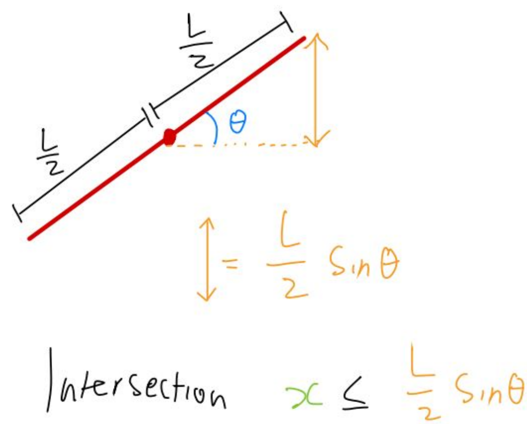
To find the probability of two events occurring simultaneously, and knowing the fact that these two events are independent of each other we can multiply the two probabilities by the independent rule. This gives us a new joint PDF with two variables x and theta. To double check we can try integrating within the bounds to ensure it evaluates to 1.

Conditions for needle intersection



Now that we have found our sample space, we must find the conditions that there will be a intersection. We can observe in this diagram that if x is larger than the vertical hieight of the needle from the midpoint, there will be no intersection. On the other hand, if this vertical distance is greater, there will be a intersection.

Condition for needle intersection



To mathematically represent this we can define the vertical distance as L/2 sin theta and say that for a intersection to occur, the value of x has to be smaller than this.

Integral Evaluation

$$\Pr(\text{Intersection With Gridline})$$

$$= \Pr\left(x \le \frac{L}{2}\operatorname{Sin}[\theta]\right)$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{L}{2}\operatorname{Sin}[\theta]} \frac{4}{W\pi}\, dx\, d\theta$$

$$= \frac{2L}{\pi W}$$

$\hat{p}$: Sample proportion of intersections

$$\hat{p} = \frac{\text{Intersections}}{\text{Needles thrown}} = \frac{I}{N}$$

$$\frac{2L}{\pi W} \approx \frac{I}{N}$$

$$\pi \approx \frac{2LN}{IW}$$

Now similar to other PDF questions, to find this probability, we will integrate within the required bounds. Doing this gives us 2 length/pi width. However if we didn't know the value of pi, and were trying to figure out this constant, we can approximately represent this equation as a sample proportion of intersections to the number of needles thrown. Rearranging this new equation for pi, we can now estimate pi's value through simulations which is also known as a monte carlo method to find pi.

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