

## Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #9 (18 November 2002):

Linear Algebra V — Tensors, more eigenstuff, and a bit on inner products

*The terms “proper value”, “characteristic value”, “secular value”, and “latent-value” or “latent root” are sometimes used [for “eigenvalue”] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are “latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf.” We will not adhere to his terminology.*

— N. Dunford, J.T. Schwartz: *Linear Operators, Part I*, pages 606–7.

We begin with some basic problems on tensors and tensor products. Recall that the *rank* of a linear transformation  $T : U \rightarrow V$  is the dimension of its image  $T(U)$ . The *rank* of a matrix is the rank of the linear transformation it represents.

1. Let  $\{u_i\}_{i=1}^m$  and  $\{v_j\}_{j=1}^n$  be bases of the  $F$ -vector spaces  $U$  and  $V$ , and consider the general element  $w = \sum_i \sum_j w_{ij}(u_i \otimes v_j)$  of  $U \otimes V$ . Prove that  $w$  is the sum of  $r$  pure tensors if and only if the matrix  $(w_{ij})$  has rank at most  $r$ .
2. Let  $V$  be a vector space of finite dimension  $n$  over a field  $F$ . We constructed a linear map, the trace, from  $\mathcal{L}(V)$  to  $F$ . Hence the map from  $\mathcal{L}(V) \times \mathcal{L}(V)$  to  $F$  taking  $(S, T)$  to the trace of  $ST$  is bilinear. Prove that it is symmetric. For what  $n$  can there exist  $S, T \in \mathcal{L}(V)$  such that  $ST - TS$  is the identity map? (By comparison, we observed that the operators  $d/dz$  and  $z$  on the infinite-dimensional space  $\mathcal{P} = F[z]$  satisfy  $ST - TS = I$ .)

Tensors and eigenstuff:

3. Fix  $a \in \mathbf{C}$ , and let  $T : \mathbf{C} \rightarrow \mathbf{C}$  be the map  $z \mapsto az$ . This is an  $\mathbf{R}$ -linear operator, so we may consider the linear operator  $T' = T \otimes 1$  on the complex vector space  $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ . What are the eigenvalues and eigenvectors of  $T'$ ? (Warning: The answer depends on whether  $a \in \mathbf{R}$ .)
4. Let  $U, V$  be vector spaces over a field  $F$ , equipped with linear operators  $S \in \mathcal{L}(U)$ ,  $T \in \mathcal{L}(V)$ . Consider  $S \otimes T \in \mathcal{L}(U \otimes V)$ .
  - i) If  $\lambda \in F$  is an eigenvalue of  $S$ , and  $\mu \in F$  is an eigenvalue of  $T$ , prove that  $\lambda\mu$  is an eigenvalue of  $S \otimes T$ .
  - ii) If  $U, V$  are finite dimensional and  $F$  is algebraically closed, prove that every eigenvalue of  $S \otimes T$  is the product of an eigenvalue of  $S$  with an eigenvalue of  $T$ .
  - iii) Show, by constructing a counterexample with finite-dimensional vector spaces  $S, T$  over  $\mathbf{R}$ , that (ii) no longer holds when the hypothesis on  $F$  is dropped.

Apropos eigenstuff... The next result generalizes what we proved in class about involutions (which are the special case  $m = 2$ ,  $\lambda_i = \pm 1$ ).

5. Suppose  $V$  is a vector space over a field  $F$  and  $T$  is a linear operator on  $V$  such that  $\prod_{i=1}^m (T - \lambda_i I) = 0$  for some *distinct*  $\lambda_i \in F$ . Prove that  $V$  is the direct sum of the  $\lambda_i$ -eigenspaces of  $T$ . [NB:  $V$  may *not* be assumed finite-dimensional.]

**Tensor products of  $A$ -modules.** Like direct sums, quotient spaces, and duals, tensor products can be defined in the same way for modules over rings  $A$  that need not be fields. Basic properties such as  $M \otimes (N \oplus N') \cong (M \otimes N) \oplus (M \otimes N')$  hold in this more general setting, and for much the same reason; but some new phenomena emerge, as in parts (ii) and (iii) of the next problem:

6. i) Show that if  $A$  is a commutative ring with unit, and  $I \subseteq A$  is an ideal (an additive subgroup such that  $aI \subseteq I$  for all  $a \in A$ , or equivalently a submodule of the  $A$ -module  $A$ ), then  $(A/I) \otimes_A (A/I)$ , the tensor product of the quotient  $A$ -module  $A/I$  with itself, is isomorphic with  $A/I$ .
- ii) On the other hand, show that  $(\mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/3\mathbf{Z})$  is the trivial  $\mathbf{Z}$ -module  $\{0\}$ .
- iii) For positive integers  $m, n$ , what is the  $\mathbf{Z}$ -module  $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z})$ ?

Finally, a bit about inner products:

7. Solve Exercises 7 and 13 on pages 122, 123 of Axler. For #13,  $V$  is either a real or complex inner-product space, which need not be finite dimensional.
8. Is the symmetric bilinear pairing constructed in Problem 2 nondegenerate? When  $F = \mathbf{R}$ , is it positive definite?

Axler's exercise #7, as well as the more familiar #6, is often referred to as the "polarization identity". This shows that a linear transformation preserves the norm if and only if it preserves the inner product [more precisely, it shows the harder, "only if" part of this result]. These are basically also the identities used to prove Propositions 2 and 4 in the next chapter (pages 129, 130).

This problem set is due Wednesday [sic], 27 November, at the beginning of class.