

Math 55a: Honors Abstract Algebra

Homework Assignment #9 (1 November 2017):

Linear Algebra IX:

Trace, determinant, and more exterior algebra

Quotes? We don't need no stinkin' quotes!

—adapted or misquoted from *Blazing Saddles* (1974), in turn adapted or misquoted from *The Treasure of the Sierra Madre*¹

A few basic facts about trace and determinant:

1. Solve problem 9 in Axler 10.A (p.305) for any finite-dimensional vector space V over any field F . Note the corollary that the trace of P is a nonnegative integer. [As I may have noted in lecture, operators satisfying $P^2 = P$ (and thus $P^k = P$ for all $k = 1, 2, 3, \dots$, so the same as each power of P) are called “idempotent”.]
2. Solve problem 6 in Axler 10.B (p.331). This should be easy to do with the \bigwedge^n definition of the determinant, even though Axler didn't intend that. Again the ground field is arbitrary.

More about traces and such:

3. Let V be an inner-product space with orthonormal basis (e_1, \dots, e_n) . For operators $S, T \in \mathcal{L}(V)$, define

$$\langle S, T \rangle := \sum_{j=1}^n \langle Se_j, Te_j \rangle.$$

Prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{L}(V)$, that it satisfies the identity $\langle S, T \rangle = \langle T^*, S^* \rangle$, and that the inner product does not depend on the choice of orthonormal basis for V . [Thus an inner product on finite-dimensional space V yields canonically an inner product on $\mathcal{L}(V)$. Cf. also Problem 2 on Problem Set 6.]

- 4–5. Solve Exercises 18 and 21 from Chapter 10 of the textbook (10.A, pages 305 and 306).

Next, another use of the sign homomorphism (which we introduced to study exterior algebra) to prove familiar(?) facts about Rubik's Cube, which is the 20th (and now 21st) century counterpart to the 19th century's Fifteen Puzzle. Some terminology first (which I may have used casually in lecture without formally defining):

A permutation is called *even* or *odd* according as its sign is $+1$ or -1 respectively. If i_1, i_2, \dots, i_m are m distinct integers in $\{1, 2, \dots, n\}$, the permutation of $\{1, 2, \dots, n\}$ that takes i_1 to i_2 , i_2 to i_3 , \dots , i_r to i_{r+1} , \dots , i_{m-1} to i_m , and i_m back to i_1 , while leaving the rest of $\{1, 2, \dots, n\}$ fixed, is called an *m-cycle*. (In particular, the identity permutation is a 1-cycle.)

6. i) Prove that an m -cycle has sign $(-1)^{m+1}$, i.e., is even iff m is odd.
ii) Prove that no sequence of turns of Rubik's Cube can have the effect of flipping one of its edge pieces while leaving the rest unchanged.
iii) Prove that no sequence of turns of Rubik's Cube can have the effect of switching two of its edge pieces while leaving the rest unchanged. Does this approach work for the $4 \times 4 \times 4$ Cube?

¹According to Wikipedia's “Stinking badges” page. Yes, Wikipedia has a page on “Stinking badges”!

Exterior algebra, continued:

7. Fix a nonnegative integer k . Let $F = \mathbf{R}$ or \mathbf{C} , and let V/F be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. For $v_1, \dots, v_k, w_1, \dots, w_k \in V$, define

$$\langle \langle v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k \rangle \rangle$$

to be the determinant of the $k \times k$ matrix whose (i, j) entry is $\langle v_i, w_j \rangle$. Prove that $\langle \langle \cdot, \cdot \rangle \rangle$ extends to an inner product on the exterior power $\bigwedge^k V$.

8. Let V be a finite-dimensional vector space over a field F . For $\omega \in \bigwedge^2 V$, define the *rank* of ω to be the rank of the associated alternating pairing on V^* , which in turn is the rank of the corresponding map $V^* \rightarrow V$. [If $\omega = \sum_i a_i (v_i \wedge w_i)$, the pairing is given by

$$\langle v^*, w^* \rangle = \sum_i a_i (v^*(v_i) w^*(w_i) - v^*(w_i) w^*(v_i)).]$$

In Problem 7 of the last problem set we saw in effect that this rank is an even integer, say $2k$. Use the results of that problem to give the following characterizations of k :

- i) For each $m = 0, 1, 2, \dots$, there exist $u_i, v_i \in V$ ($i = 1, \dots, m$) such that

$$\omega = \sum_{i=1}^m u_i \wedge v_i$$

if and only if $m \geq k$. Moreover, if $m = k$ then the $2k$ vectors u_i, v_i are linearly independent.

- ii) In characteristic 0 or $p > 2k$, the k -th exterior power of ω (that is, the element $\omega \wedge \omega \wedge \dots \wedge \omega$ of $\bigwedge^{2k} V$, with k factors of ω) is nonzero, but the $(k+1)$ -st exterior power vanishes.

And finally: A square matrix A with entries a_{ij} in a field F is said to be *skew-symmetric* if its entries satisfy $a_{ij} = -a_{ji}$ for all i, j and the diagonal entries a_{ii} all vanish. We'll show in class that $\det A = 0$ if A has odd order; here we study the even case.

9. If A has even order $2n$, and $n!$ is invertible in F , the *Pfaffian* $\text{Pf}(A)$ can be defined thus: let $\omega \in \bigwedge^2(F^{2n})$ be defined by $\omega = \sum_{1 \leq i < j \leq 2n} a_{ij} e_i \wedge e_j$; then $\text{Pf}(A) \in F$ is the scalar such that

$$\omega^n = n! \text{Pf}(A) (e_1 \wedge e_2 \wedge \dots \wedge e_{2n})$$

in $\bigwedge^{2n}(F^{2n})$. (Of course ω^n means $\omega \wedge \omega \wedge \dots \wedge \omega$ with n factors.) Give an explicit formula for $\text{Pf}(A)$ in terms of the a_{ij} , analogous to the formula for the determinant as a sum of $n!$ monomials. Prove that

$$\det(A) = (\text{Pf}(A))^2.$$

This problem set is due *Friday*, 10 November, at the beginning of class.