Solutions to Homework 3

Math 55b

1. *Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ is not continuous.*

   An example is the function $f(x) := x^2 \sin(1/x)$ for $x \neq 0$, and 0, for $x = 0$. That this function is differentiable at 0 must be justified; it follows from $|f(x)| = O(x^2)$ as $x \to 0$ that f is differentiable at 0 with $f'(0) = 0$. At $x \neq 0$, the derivative is $f'(x) = 2x \sin(1/x) - \cos(1/x)$. Thus $f'(x)$ is not continuous at 0: $\cos(1/x)$ does not have a limit as $x \to 0$. ∎

2. *Let $\mathcal{F}$ be the smallest collection of functions $f : [0,1] \to \mathbb{R}$ that contains $C[0,1]$ and is closed under pointwise limits. Prove that the characteristic function the set $[0,1] \cap \mathbb{Q}$ is in $\mathcal{F}$.*

   Solution 1. The question asks to write the function defined by $g(x) = 1$ for $x \in [0,1] \cap \mathbb{Q}$ and $g(x) = 0$ for $x \in [0,1] - \mathbb{Q}$ as a pointwise limit of a sequence of continuous functions. Noting that $q \in \mathbb{Q}$ if and only if $n!q \in \mathbb{Z}$ for $n \gg 0$, and that $\lim_{m\to\infty} \cos^{2m}(\pi x) = 1$, if $x \in \mathbb{Z}$, and 0, if $x \notin \mathbb{Z}$, it follows that, pointwise,

   $$g(x) = \lim_{n\to\infty} \lim_{m\to\infty} \cos^{2m}(n!x).$$

   ∎

   Solution 2. (Alex) For each $q \in [0,1]$, the characteristic function χ_q of $\{q\}$ is the pointwise limit $\lim_{k\to\infty} \left(1 - (x-q)^2\right)^k$, and is therefore in $\mathcal{F}$. Thus the characteristic function of every countable subset $(q_n) \subset [0,1]$, being the pointwise limit $\lim_{n\to\infty} \sum_{i=1}^n \chi_{q_i}$, is in $\mathcal{F}$. ∎

3. *Let (X,d) be a metric space, and let S denote the set of Cauchy sequences $s = (x_i)$ in X. Prove that the limit $\bar{d}(s,s') := \lim_{i\to\infty} d(x_i, x_i')$ exists for all $s, s' \in S$, and defines a pseudometric on S. Let $\overline{X}$ be the quotient of S by the equivalence relation $s \sim s'$ iff $\bar{d}(s,s') = 0$; then $\bar{d}$ descends to a metric on $\overline{X}$. Prove that $(\overline{X}, \bar{d})$ is a complete metric space, and define a natural isometry $\pi : X \to \overline{X}$ whose image $\pi(X)$ is dense.*

   To say that the limit $\lim_{i\to\infty} d(x_i, x_i')$ exists is to say that the sequence $\left(d(x_i, x_i')\right)_i$ of real numbers is Cauchy. As s, s' are Cauchy, for any $\varepsilon > 0$

there exists an integer $N(\varepsilon)$ such that $i,j > N(\varepsilon)$ implies $d(x_i,x_j) < \varepsilon/2$ and $d(x'_i,x'_j) < \varepsilon/2$. Then the triangle inequality gives $d(x_i,x'_i) - d(x_j,x'_j) \le d(x_i,x_j) + d(x'_i,x'_j) < \varepsilon$ for $i,j > N(\varepsilon)$, and this shows that $\big(d(x_i,x'_i)\big)_i$ is Cauchy, as required. By definition, the $\bar{d}$ is a symmetric function $S \times S \to \mathbb{R}_{\ge 0}$; the triangle inequality for $\bar{d}$ follows from the triangle inequality for d upon noting that limits preserve nonstrict inequalities: for any convergent sequence (A_i) of nonnegative real numbers, $\lim_i A_i \ge 0$. Thus $\bar{d}$ descends to a metric on $\overline{X}$.

To show that $(\overline{X},\bar{d})$ is complete, note that any subsequence of a Cauchy sequence $s \in S$ is equivalent to s, i.e. descends to the same element of $\overline{X}$. Call a Cauchy sequence $s = (x_i)$ *fast* if $d(x_i,x_j) < 2^{-\min(i,j)}$ for all i,j. Every Cauchy sequence has a fast Cauchy subsequence, and a Cauchy sequence has a limit iff some Cauchy subsequence has a limit; hence, in showing completeness of $(\overline{X},\bar{d})$, it suffices to show that every fast Cauchy sequence $(s_i)_i$ of fast Cachy sequences $s_i := (x_{i,j})$ on X has a limit. We simply check that the diagonal sequence $x_i := x_{i,i}$ is a limit; this means verifying that the assumptions $d(x_{i,j},x_{i,j'}) < 2^{-\min(j,j')}$ and $\lim_{j\to\infty} d(x_{i,j},x_{i',j}) < 2^{-\min(i,i')}$ imply the conclusion $\lim_{i,j\to\infty} d(x_{i,j},x_{j,j}) = 0$. Fix i,j; we are given that there exists a $k > \max(i,j)$ such that $d(x_{i,k},x_{j,k}) < 2^{-\min(i,j)}$. By the triangle inequality, $d(x_{i,j},x_{j,j}) \le d(x_{i,j},x_{i,k}) + d(x_{i,k},x_{j,k}) + d(x_{j,k},x_{j,j}) < 2^{-j} + 2^{-\min(i,j)} + 2^{-j} \le 3\cdot 2^{-\min(i,j)}$; since this bound approaches 0 as $i,j\to\infty$, the conclusion follows: the diagonal sequence $(x_{j,j})_j$ is a limit of the Cauchy sequence $\big((x_{i,j})_j\big)_i$. This proves completeness of $(\overline{X},\bar{d})$.

Finally, there is the natural map $\pi : X \to \overline{X}$ sending $x \in X$ to the class of $(x,x,\ldots,x,\ldots) \in S$; this map clearly satisfies $\bar{d}(\pi(x),\pi(y)) = d(x,y)$, meaning that it is an isometry. The image $\pi(X)$ is dense, since, by the definition of a Cauchy sequence $s = (x_i)$, $\bar{d}(s,\pi(x_i)) \to 0$ as $i \to \infty$. ∎

4. *Let $X := \ell^1(\mathbb{N})$ be the vector space of all sequences $a : \mathbb{N} \to \mathbb{R}$ such that $\|a\|_1 := \sum_i |a_i| < \infty$. Prove that the metric $d(a,b) := \|a-b\|_1$ makes X into a complete metric space. Prove that the closed unit ball $\overline{B}(0,1)$ in X is not compact. Prove that for every $b \in X$, the set $K(b) := \{a \in \ell^1(\mathbb{N}) \mid |a_i| \le |b_i|$ for all $i\}$ is compact.*

Note that the topology induced by the norm $\|\cdot\|_1$ on $\ell^1(\mathbb{N})$ has the following characterizing property: a sequence $(a^{(i)})_i$ in $\ell^1(\mathbb{N})$ converges to $a \in \ell^1(\mathbb{N})$ if and only if $\sum_{j\ge N} |a^{(i)}_j| \longrightarrow_{i,N\to\infty} 0$ and $a^{(i)}_j \to_{i\to\infty} a_j$ for each j. Since $\ell^1(\mathbb{N})$ is a vector space, this suffices to be verified for $a = 0$; and the forward implication is obvious, so what needs to be verified is that the assumptions $\sum_{j\geq N}|a_j^{(i)}| \longrightarrow_{i,N\to\infty} 0$ and $a_j^{(i)} \to_{i\to\infty} 0$ for each j imply the conclusion $\|a^{(i)}\|_1 \to_{i\to\infty} 0$. The required conclusion $\|a^{(i)}\|_1 \to 0$ is simply equivalent to $\sum_{j=1}^{N}|a_j^{(i)}| \to 0$ for each N; by passing to a subsequence of the indexing ordered set $(i)=\mathbb{N}$, we may assume that $|a_j^{(i)}| < 2^{-i-j}$ for $j=1,\ldots,N$ and all i, which in turn yields $\sum_{j=1}^{N}|a_j^{(i)}| < \sum_{j=1}^{N}2^{-i-j} < 2^{-i}\sum_{j=1}^{\infty}2^{-j} = 2^{-i}$, hence the conclusion.

Completeness of $\ell^1(\mathbb{N})$ is a consequence of this and of the completeness of $\mathbb{R}$, as follows. For any Cauchy sequence $(a^{(i)})_i$ in $\ell^1(\mathbb{N})$, each coordinate sequence $(a_j^{(i)})_i$ is Cauchy in $\mathbb{R}$, and thus has a limit a_j; moreover, since $(a^{(i)})_i$ is Cauchy, the condition $\sum_{j\geq N}|a_j^{(i)}| \longrightarrow_{i,N\to\infty} 0$ is fulfilled: it follows from the estimates $\sum_{j\geq N}|a_j^{(i)}| \longrightarrow_{N\to\infty} 0$, $\|a^{(i)}-a^{(i')}\|_1 \longrightarrow_{i,i'\to\infty} 0$, and the triangle inequality. It remains to show that the sequence $a := (a_j)_j$ is in $\ell^1(\mathbb{N})$, i.e. that $\sum_j |a_j| < \infty$. If not, there exists an N such that $\sum_{j=1}^{N}|a_j| > 2\sup_i \|a^{(i)}\|_1$ (notice that the supremum is finite, because $(a^{(i)})_i$ is Cauchy and hence $\left(\|a^{(i)}\|_1\right)_i$ is Cauchy). Let $M := \sup_i \|a^{(i)}\|_i$, and consider a large enough $i_0 = i_0(N)$ for which $|a_j - a_j^{(i_0)}| < M/N$ for $1 \leq j \leq N$. Combining these N inequalities with $\sum_{j=1}^{n}|a_j| > 2M$, the triangle inequality implies $\|a^{(i_0)}\| \geq \sum_{j=1}^{N}|a_j^{(i_0)}| > M$, which is absurd. Hence, $a \in \ell^1(\mathbb{N})$, proving completeness.

That the closed unit ball $\overline{B}(0,1)$ is not compact follows from the existence of the infinite discrete set $\{(\delta_{ij})_i \mid i\} \subset \overline{B}(0,1)$ consisting of the vertices of the unit cube.

Finally, the compactness of each cube $K(b) = \prod_{i=1}^{\infty}[-b_i, b_i]$ follows from the compactness of the closed intervals $[-b_i, b_i]$ by a usual diagonalization argument. Note that, by the first paragraph above, the induced topology on $K(b)$ is simply the topology of pointwise convergence (for a sequence $(a^{(i)})_i$ in $K(b)$, the condition $\sum_{j\geq N}|a_j^{(i)}| \longrightarrow_{i,N\to\infty} 0$ is automatic, since each $\sum_{j\geq N}|a_j^{(i)}|$ is dominated by $\sum_{j\geq N}|b_j|$). We need to show that every countably infinite subset $\{(a^{(i)})_i \mid i \in \mathbb{N}\} \subset K(b)$ has a limit point in $K(b)$. The sequence $(a_1^{(i)})_i$ of points of the compact interval $[-b_1, b_1]$ has a convergent subsequence, say indexed by $I_1 \subset \mathbb{N}$; let a_1 be the limit of this subsequence. The sequence $(a_2^{(i)})_{i\in I_1}$ of points on the compact interval $[-b_2, b_2]$ has a convergent subsequence, say indexed by $I_2 \subset I_1$; let a_2 be the limit of this subsequence. Continuing, we obtain a sequence

$a = (a_j)_j \in K(b)$ and a nested sequence $\mathbb{N} \supset I_1 \supset I_2 \supset \cdots$ of infinite sets such that, for each $j \in \mathbb{N}$, the sequence $(a_j^{(i)})_{i \in I_j}$ is convergent to a_j. Then $a \in K(b)$ is a limit point of $\{a^{(i)} \mid i \in \mathbb{N}\}$. ■

Remark. For showing compactness of $K(b)$, some of you verified instead that $K(b)$ is closed and totally bounded; using the sequential definition of compactness, though, is more technically convenient here, and the simple diagonalization argument of this proof is very standard and conceptual. Also, noting that $K(b) = \prod_{i=1}^{\infty}[-b_i, b_i]$ with the topology of pointwise convergence, the compactness part of this question hints at a very important theorem in point set topology, due to **Tychonoff**. In a special case, the above proof works *verbatim*: let (X_i) be a countable collection of compact metric spaces, and suppose that the set $\prod_i X_i$ is given a metric for which the convergent sequences are exactly the pointwise convergent ones (such a metric always exists; for example, if B_i is a bound for the metric d_i on X_i, such is the metric $d(x, y) := \sum_i B_i^{-1}|x_i - y_i|2^{-i}$). Then the metric space $(\prod_i X_i, d)$ is compact. ■

5. *Let $X = B[0,1]$ denote the vector space of bounded functions $f : [0,1] \to \mathbb{R}$. Is there a metric d on X such that $d(f_n, f) \to 0$ iff $f_n \to f$ pointwise.*

There is no such metric; one reason is that the indexing set $[0,1]$ (note that its topology is not used!) is *too large*: let us show that, for an arbitrary set S, the topology of pointwise convergence on the set $\mathbb{R}_{\mathrm{b}}^S$ of bounded maps $S \to \mathbb{R}$ is metrizable if and only if S is at most countable; in particular, it is not metrizable for $S = [0,1]$. Suppose there exists a metric d on this set that induces the topology of pointwise convergence. For $q \in S$, let $\chi_q \in \mathbb{R}_{\mathrm{b}}^S$ be the characteristic function of $\{q\}$, or the elementary function supported at q: this is the function taking value 1 at $x = q$ and 0 at all $x \neq q$. Every sequence of distinct elements of the set $\{\chi_q \mid q \in S\}$ converges pointwise to the zero function $0 \in \mathbb{R}_{\mathrm{b}}^S$; hence, for each $n = 1, 2, 3, \ldots$, the set $S_n := \{q \in S \mid d(\chi_q, 0) > 1/n\}$ is finite. Since $d(\chi_q, 0) > 0$ for all $q \in S$ and hence $S = \bigcup_n S_n$ is a countable union of finite sets, it follows that S is at most countable, as claimed. Finally, let us note that when S is at most countable, the topology of pointwise convergence is metrizable: this follows upon noting that the metric $d(x, y) := \sum_i |x_i - y_i|2^{-i}$ on the set $\mathbb{R}_{\mathrm{b}}^{\mathbb{N}}$ induces the topology of pointwise convergence (on $\mathbb{R}^{\mathbb{N}}$, it is also know as the **product topology** induced from the topologies of each factor of $\mathbb{R}$). ■

6. *Does the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = \sin(nx)$ contain a uniformly convergent subsequence?*

No. Fixing m and letting $n \gg_m 0$ large and $x_0 := \pi/2n$, it suffices to note that $\sin(nx_0) - \sin(mx_0) = 1 - \sin(mx_0)$ can be arbitrarily close to 1, since $mx_0 \to_{n\to\infty} 0$. ■

Remark. It can be shown that $\big(\sin(nx)\big)$ does not even have a *pointwise* covergent subsequence; but this is more difficult. ■

7. *Let $\alpha > 0$ be rational. Without appeal to calculus, determine*

$$\lim_{n\to\infty} (n+1)^\alpha - n^\alpha.$$

The limit is ∞, for $\alpha > 1$; 1, for $\alpha = 1$; and 0, for $\alpha < 1$. To really prove this from first principles, we will have to go back to an "obvious" statement we proved in the beginning of the course as a consequence of the completeness axiom of the real numbers $\mathbb{R}$: for every $C \in \mathbb{R}$, there exists an $n \in \mathbb{Z}$ with $n > C$. (Recall the **proof**: this states that the subset $\mathbb{Z} \subset \mathbb{R}$ is unbounded. Assuming the contrary, the completeness of $\mathbb{R}$ implies the existence of $M := \sup \mathbb{Z}$. But then $n \le M$ for every $n \in \mathbb{Z}$ implies $n+1 \le M$ for every $n \in \mathbb{Z}$, implying $n \le M-1$ for every $n \in \mathbb{Z}$, giving the contradiction $M - 1 \ge \sup \mathbb{Z} = M$). This basic property implies something that most of you assumed for obvious, namely that $\lim_{n\to\infty} n^\beta = \infty$ for $\beta \in \mathbb{Q}_{>0}$. Indeed, write $\beta = p/q$ with $p, q \in \mathbb{N}$ and let $B \in \mathbb{R}_{>0}$ be arbitrary. By the unboundedness of $\mathbb{Z}$ in $\mathbb{R}$, there exists some $n_0 \in \mathbb{N}$ with $n_0 > B^q$. For $n \ge n_0$, multiplying the inequality $n \ge n_0$ with $p-1$ copies of the inequality $n \ge 1$, we conclude that $n^p > B^q$; this, for $n, B > 0$, is precisely equivalent to $n^{p/q} > B$; and since $B > 0$ was arbitrary, this proves $n^{p/q} \to_{n\to\infty} \infty$.

Now, we may formally deduce the required limit as follows. We have $(n+1)^\alpha - n^\alpha = n^\alpha\Big(\big(1+\frac{1}{n}\big)^\alpha - 1\Big)$; if $\alpha \ge 1$, then $(1+n^{-1})^\alpha = (1+n^{-1})^{\alpha-1} \cdot (1+n^{-1}) \ge 1 + n^{-1}$ (we have used the inequality $x^\beta > 1$ for $x > 1$ and $\beta \in \mathbb{Q}_{>0}$, which follows for $\beta \in \mathbb{N}$ by β times multiplying $x > 1$, and in general from the equivalence "$x > 1$ iff $x^m > 1$" for $m \in \mathbb{N}$) the factor in the brackets is $\ge n^{-1}$; hence, for $\alpha \ge 1$, $(n+1)^\alpha - n^\alpha \ge n^{\alpha-1}$, which is 1 if $\alpha = 1$ and diverges to ∞ if $\alpha > 1$, but the preceding paragraph. Likewise, for $\alpha < 1$, $\alpha - 1 < 0$ and $1 + n^{-1} > 1$ imply

$(1+n^{-1})^\alpha = (1+n^{-1})^{\alpha-1}\cdot(1+n^{-1}) < 1+n^{-1}$, and the factor in the brackets is $< n^{-1}$; in that case, $(n+1)^\alpha - n^\alpha < n^{\alpha-1}$, which converges to 0 since $n^{1-\alpha} \to_{n\to\infty} \infty$, again by the preceding paragraph. ■

Remark. Alternatively, the second paragraph can be replaced by applying the identity $x^q - y^q = (x-y)(x^{q-1}+\cdots+y^{q-1})$ to $x=(n+1)^\alpha, y = n^\alpha$, $p/q=\alpha$ to obtain the more precise estimate $(n+1)^\alpha - n^\alpha = \alpha n^{\alpha-1} + o(n^{\alpha-1})$. ■