Homework 10

Math 55b
Due Tuesday, 14 April 2009.

Notation: $\Delta = \{z : |z| < 1\}$.

1. Let $p(z) = z^3 + z^n$ with $n \geq 3$. Prove that $p(z) = 1$ for some z with $\operatorname{Re} z < 0$.

2. Give an expression for $\sin(x + iy)$ in terms of real-valued spherical and hyperbolic sines and cosines. Where are the zeros of the function $\sin(z)$ on $\mathbb{C}$?

3. Suppose $f(z) = \sum a_n z^n$ is analytic for $|z| < 1$. Prove that for any $r < 1$, we have
$$\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum |a_n|^2 r^{2n}.$$

4. Given $J \in \mathrm{M}_2(\mathbb{R})$, let $\mathbb{R}[J] \subset \mathrm{M}_2(\mathbb{R})$ denote the set of matrices of the form $aI + bJ$, $a, b \in \mathbb{R}$. (i) Prove that $\mathbb{R}[J]$ is closed under addition and (matrix) multiplication. (ii) When do these two operations make $\mathbb{R}[J]$ into a field?

5. Prove that for any polynomial $p(z)$ there exists a $z \in S^1$ such that $|\overline{z} - p(z)| \geq 1$.

6. Let $u \in C(\overline{\Delta})$ be a real-valued continuous function on the unit disk which is harmonic on its interior. Prove that for any $p \in \Delta$, we have
$$u(p) = \frac{1}{2\pi} \int_{S^1} \frac{1 - |p|^2}{|z - p|^2} u(z) \, |dz|.$$
(Hint: for $p = 0$ this is just the mean value theorem; reduce to this case using a Möbius transformation $f : \Delta \to \Delta$ such that $f(p) = 0$.)

7. Prove Hadamard's 3-circles theorem: if $f(z)$ is analytic on the annulus $R_1 < |z| < R_2$, and $M(r) = \sup_{S^1(r)} |f(z)|$, then $\log M(e^s)$ is a convex function of $s \in (\log R_1, \log R_2)$. (Hint: apply the maximum principle to the function $z^\alpha f(z)$ for suitable $\alpha \in \mathbb{R}$.)

8. Let u and v be smooth functions on $\overline{\Delta}$ such that $u|\Delta$ is harmonic and $u|S^1 = v|S^1$. Show that
$$\int_\Delta |\nabla v|^2 \geq \int_\Delta |\nabla u|^2.$$

9. Let $\sum a_n z^n$ be the Laurent series for $f(z) = 1/(e^z - 1)$ near $z = 0$. Find a_n for all $n \leq 3$. What is the radius of convergence of this series?

10. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying $f(a + b) = f(a)f(b)$ for all $a, b \in \mathbb{C}$. Prove that either $f(z) = 0$ or $f(z) = \exp(\alpha z)$ for some $\alpha \in \mathbb{C}$.