**Homework 9**

Math 55b
Due Tuesday, 7 April 2009.

Notation: $S^1(r)$ denotes the circle of radius $r$ about the origin in $\mathbb{C}$, oriented counterclockwise as usual; $S^1 = S^1(1)$; $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

1. Show, directly from the definition of $\int_\gamma f(z)\,dz$ as a limit of Riemann sums, that $\int_\gamma z\,dz = 0$ for any closed loop $\gamma$ in the plane.

2. What is the most general form of a rational function $f(z)$ which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles of $f$ related to each other?

3. Let $f : U \to \mathbb{C}$ be an analytic function on a connected domain and suppose $|f(z)|$ is constant. Prove that $f(z)$ is constant.

   *More generally, prove that if $f_i : U \to \mathbb{C}$ are analytic and $\sum_1^n |f_i(z)|^2$ is constant, then all the functions $f_i(z)$ are constant.

4. Show that
$$\prod_1^\infty (1 + a_n) := \lim_{N\to\infty} \prod_1^N (1 + a_n) \neq 0$$
   provided $a_n \in \mathbb{C}$ satisfies $a_n \neq -1$ and $\sum |a_n| < \infty$.

5. Let $p(n)$ be the *partition function*; that is, the number of ways to write $n$ as the sum of an increasing sequence of positive integers. (For example $p(5) = 7$ because $5 = 1+1+1+1+1 = 1+1+1+2 = 1+1+3 = 1+4 = 1+2+2 = 2+3$.) Show that
$$1 + \sum_{n=1}^\infty p(n) z^n \quad = \quad \prod_{n=1}^\infty \frac{1}{1 - z^n} \neq 0$$
   for all complex $z$ with $|z| < 1$.

6. Let $G$ denote the group of rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree one (called Möbius transformations), with composition as the group operation. Prove that the map $\phi : \mathrm{SL}_2(\mathbb{C}) \to G$ given by
$$\phi : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) = \frac{az + b}{cz + d}$$
   is a surjective homomorphism, and compute its kernel. Explain this homomorphism geometrically in terms of the 'slope map' $s : \mathbb{C}^2 - \{(0,0)\} \to \widehat{\mathbb{C}}$ given by $s(z_1, z_2) = z_1/z_2$. Which vectors in $\mathbb{C}^2$ correspond to the fixed-points of $f$?

7. Prove that every $f \in G$ is conjugate to either $f(z) = \lambda z$ (for some $\lambda \in \mathbb{C}^*$) or $f(z) = z + 1$. Show that the value of $\lambda$ can be determined from $\mathrm{tr}(A)$ if $f = \phi(A)$. Is it unique? What value(s) of $\mathrm{tr}(A)$ correspond to $f(z) = z + 1$?

8. Let $H \subset G$ be the subgroup generated by a single map of th form $f(z) = \alpha z$, $\alpha \neq 0$. What is the centralizer of $H$ in $G$? What is the normalizer? Answer the same question where $H$ is generated by $f(z) = z + 1$, and where $H$ is the group of *all* translations $f(z) = z + t$, $t \in \mathbb{C}$.

9. Prove that the image of a circle or a line under a Möbius transformation is a circle or a line.