Homework 6

Math 55b
Due Tuesday, 10 Mar 2009.

1. Let $K_n(x) = c_n(1-x^4)^n$ for $|x| < 1$ and 0 otherwise, where c_n is chosen so that $\int_{\mathbb{R}} K_n = 1$. Prove that $\langle K_n \rangle$ is an approximate identity. (You need to show that $\int_{-r}^{r} K_n(x)\,dx \to 1$ for every $r > 0$.)

2. Determine which of the following sequences of functions are equicontinuous, and give a uniform modulus of continuity $h(r)$ for the ones that are.[1]

   (a) $f_n(x) = \exp(nx)$, $x \in (-\infty, 1]$;

   (b) $f_n(x) = \sin(\sin(\cdots(\sin x)))$ (n times), $x \in [0, 2\pi]$;

   (c) $f_n(x) = n + x^n$, $x \in [0, 1/2]$.

   (d) $f_n(x) = (1 + x/n)^n$, $x \in [0, \infty)$.

3. Let X, Y be a pair of compact metric spaces. Show that the continuous functions of the form $f(x)g(y)$ span a dense subspace of $C(X \times Y)$.

4. Compute the Fourier series of the function $f(x) = |x|$ on $[-\pi, \pi]$ (extended periodically to the whole line). Does the series converge absolutely?

5. Let $S \subset C[a, b]$ be a finite-dimensional subspace. Prove that if $f_n \in S$, and $g : [a, b] \to \mathbb{R}$, then $f_n \to g$ pointwise iff $g \in S$ and $f_n \to g$ uniformly.

6. Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$, and suppose df/dx and df/dy exist for every (x, y). Prove or disprove each of the following assertions.

   (a) f is continuous.

   (b) If, in addition, $|df/dx| \le M$ and $|df/dy| \le M$, then f is continuous.

   (c) If, in addition, $|df/dx| \le M$ and $|df/dy| \le M$, then f is differentiable.

   Hint: consider $xy/(x^2 + y^2)$ and $x^3/(x^2 + y^2)$.

7. Give an example of a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such $\det Df(x) \neq 0$ for all x, but f is not one-to-one.

[1] This means $h(r) \to 0$ as $r \to 0$ and $|x - y| < r \implies |f_n(x) - f_n(y)| < h(r)$ for all n.