

## Math 55b: Honors Real and Complex Analysis

Homework Assignment #6 (2 March 2018):  
Univariate integral calculus

Fejér discovered his theorem<sup>1</sup> at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.<sup>2</sup>

As happened a few times before (see for example problem 6 of the 7th problem set in Math 55a), several of the problems here rely on properties of elementary transcendental functions ( $e^x$  for  $x$  real or pure imaginary,  $\log$ , and  $\tan^{-1}$ ) that we have yet to officially introduce and prove. We'll finally make good on this a week or two after Spring Break.

First, a typical use of Stieltjes integration by parts with discontinuous  $\alpha$ :

1. Recall that if  $f, g$  are real-valued functions on  $[a, \infty)$  or  $(a, \infty)$  then we write  $f \sim g$  (read “ $f$  is asymptotic to  $g$ ”) if  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow \infty$  (which presumes that there exists  $x_0 \geq a$  such that  $g(x) \neq 0$  for all  $x > x_0$ ). For example, the Prime Number Theorem (PNT)<sup>3</sup> states that  $\pi(x) \sim x/\log x$ , where  $\pi(x)$  is the number of primes  $p \leq x$ . It turns out that for technical reasons it is easier to prove that  $L(x) := \sum_{p \leq x} \log p$  is asymptotic to  $x$ . When  $x$  is not itself prime we can write

$$L(x) = \int_1^x \log(y) d(\pi(y)).$$

Use this and integration by parts to prove that PNT implies  $L(x) \sim x$ . Similarly prove that “ $L(x) \sim x$ ” implies PNT by writing  $\pi(x)$  in terms of  $L$ .

Next, some of the few cases where it's feasible to evaluate an integral via Riemann sums:

2. [Bernoulli polynomials]<sup>4</sup> Prove that for each positive integer  $m$  there exists a polynomial  $B_m \in \mathbf{Q}[X]$  such that  $\sum_{i=1}^n i^{m-1} = B_m(n)$  for all positive integers  $n$ . [Hint: What must the polynomial  $B_m(x) - B_m(x-1)$  be? The map taking any polynomial  $P(x)$  to the polynomial  $Q(x) := P(x) - P(x-1)$  is linear.] Determine the leading coefficient of  $B_m$ , and deduce the value of  $\int_0^b x^{m-1} dx$  for any  $b > 0$  (and thus also of  $\int_a^b x^{m-1} dx$ ) without using the Fundamental Theorem of Calculus. Beyond the leading term, what further patterns can you detect in the coefficients of  $B_m$ ? Can you prove any of these patterns? (You may need to go at least to  $m = 6$  or  $m = 7$  to see what's going on; a computer algebra system could help to handle the linear algebra manipulations.)
3. [Fermat] Prove *using the Riemann-sum definition of the integral* that  $\int_a^b x^{r-1} dx = (b^r - a^r)/r$  for every nonzero rational number  $r$  and all real  $a, b$  such that  $0 < a < b$ . [Note: since Fermat (1607–1665) predated Newton (1642–1726), the solution cannot

<sup>1</sup>That the uniform closure of trigonometric polynomials is the full space of continuous functions on  $\mathbf{R}/2\pi\mathbf{Z}$ ; see Rudin, pages 199–200.

<sup>2</sup>Körner, *Fourier Analysis*, p.294 (end of Chapter 59: “Weierstrass’s proof of Weierstrass’s theorem”).

<sup>3</sup>Proved by Hadamard and (independently) de la Vallée-Poussin in 1896.

<sup>4</sup>Not that it matters for our purposes, but the “Bernoulli polynomials” usually seen in the literature differ from our  $B_m$  by an additive constant.

use the Fundamental Theorem of Calculus. Besides the special case that  $r$  is a positive integer, addressed in the previous problem, you might also find a solution for the special case that  $1/(r-1)$  is a positive integer — but this will not directly lead you to a solution of the general case.]

4. Prove that if  $c$  is a complex number such that  $|c| < 1$  then  $\int_0^{2\pi} \log |1 - ce^{ix}| dx = 0$ . What is the value of this integral for  $|c| > 1$ ? (To be continued when we reach complex analysis.)

More about  $\int f d\mu$  when  $\mu$  need not be monotone:

5. For a bounded function  $f : [a, b] \rightarrow \mathbf{R}$  let

$$M_f := \{\alpha - \beta : f \in \mathcal{R}(\alpha) \cap \mathcal{R}(\beta)\}$$

(with a condition that  $\alpha$  and  $\beta$  be nondecreasing functions  $[a, b] \rightarrow \mathbf{R}$ , which is implicit in the notations  $\mathcal{R}(\alpha)$ ,  $\mathcal{R}(\beta)$ ). Show that  $M_f$  is a vector space, and that

$$I_f : \alpha - \beta \mapsto \int_a^b f d\alpha - \int_a^b f d\beta$$

yields a well-defined linear map on  $M_f$ . Naturally we write this map as  $I_f(\mu) = \int_a^b f d\mu$ . Find an intrinsic definition of  $M_f$ , i.e., a definition that lets us recognize functions  $\mu \in M_f$  directly and define  $I_f(\mu)$  without finding  $\alpha, \beta$ .<sup>5</sup>

Finally, (indefinite) integration of arbitrary rational functions:<sup>6</sup>

6. [Partial fractions] Let  $k$  be an algebraically closed field. Let  $K = k(x)$ , the field of rational functions in one variable  $x$  with coefficients in  $k$ . Show that the following elements of  $K$  constitute a basis for  $K$  as a vector space over  $k$ :  $x^n$  for  $n = 0, 1, 2, 3, \dots$ , and  $1/(x - x_0)^n$  for  $x_0 \in k$  and  $n = 1, 2, 3, \dots$ . (Linear independence is easy. To prove that the span is all of  $K$ , consider for any polynomial  $Q \in k[x]$  the subspace  $V_Q := \{P/Q : P \in k[x], \deg(P) < \deg(Q)\}$  of  $K$ , and compare its dimension with the number of basis vectors in  $V_Q$ .)
7. Prove that the integral of any  $f \in \mathbf{R}(x)$  is a rational function plus a linear combination of functions of the form  $\log |x - x_0|$ ,  $\log((x - x_0)^2 + c)$ , and  $\tan^{-1}(ax + b)$  ( $x_0, a, b, c \in \mathbf{R}$ ,  $c > 0$ ).

This problem set due Friday, 9 March, at the beginning of class. You may (without penalty) postpone any one or two of these problems until the Monday the 19th, but you probably don't want to take advantage of this unless you really have to, even though there will be no problem set due over Spring Break.

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<sup>5</sup>It is possible, but not required here, to do this in a way that generalizes smoothly to vector-valued  $f$  or  $\mu$ , and makes the integration-by-parts formula  $f(b)g(b) - f(a)g(a) = \int_a^b f dg + \int_a^b g df$  work even when  $f$  is vector valued. In that case we must of course start from the definition of  $\int_a^b f d\alpha$  in the `vint` handout. You might also get a sense of what to do from Rudin's discussion of "rectifiable curves" at the end of Chapter 6.)

<sup>6</sup>Along the way we again encounter a natural example of a vector space with an uncountable algebraic basis (assuming  $k$  is uncountable, e.g.  $k = \mathbf{C}$ ).