Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #4 (21 February 2003):
Multivariate differential calculus

> Mathematicians have long regarded it as demeaning to work on problems
> related to elementary geometry in two or three dimensions, in spite of the
> fact that it is precisely this sort of mathmetics which is of practical value.
> — B. Grünbaum and G.C. Shephard, in *Handbook of Applied Mathematics*

[But then there isn't much new to be done in 2D or 3D geometry, and higher-dimensional geometry can have practical uses too — see the last two problems.]

1. Solve Problems 7 and 10 on page 239.

2. Solve Problems 17 and 18 on page 241 (and see #5 below).

3. [Multivariate Taylor expansions] solve Problem 30 on pages 243–4. (NB What Rudin calls "$\mathcal{C}^{(m)}(E)$" is our "$\mathcal{C}^m(E, \mathbf{R})$"; as usual this works with $\mathbf{R}$ replaced by an arbitrary normed vector space.)

4. The *Laplacian* Δf of a $\mathcal{C}^2$ function f on $E \subset \mathbf{R}^n$ is defined by

$$\Delta f := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2} = \sum_{j=1}^{n} D_{jj} f.$$

Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Show that $\Delta(f \circ A) = (\Delta f) \circ A$ for all $f \in \mathcal{C}^2$ if and only if A is orthogonal. In particular, the "harmonic functions" (those in the kernel of Δ, i.e. $\mathcal{C}^2$ functions f with $\Delta f = 0$) are preserved by orthogonal changes of variable.

5. [Cauchy-Riemann equtions] The usual identification of $\mathbf{C}$ with $\mathbf{R}^2$ (identify the complex number $z = x + iy$ with the vector (x, y)) lets us regard any map $w : \mathbf{C} \to \mathbf{C}$ as a map of $\mathbf{R}^2$ to $\mathbf{R}^2$, or equivalently as a pair of real-valued functions $u(x, y) = \mathrm{Re}\, f(x + iy)$, $v(x, y) = \mathrm{Im}\, f(x + iy)$ on $\mathbf{R}^2$. Prove the following criterion for a $\mathcal{C}^1$ function w to be differentiable *as a map from $\mathbf{C}$ to $\mathbf{C}$*, i.e., for there to exist a function $w' : \mathbf{C} \to \mathbf{C}$ such that $w(z+h) = w(z) + hw'(z) + o(|h|)$ as $h \to 0$: the functions u, v must satisfy the *Cauchy-Riemann equations*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show that this is the case if w is any polynomial in z with complex coefficients, or the exponential function $w(z) = e^z = e^x(\cos y + i \sin y)$. Prove that such u, v, w are necessarily harmonic functions on $\mathbf{C}$. [It is known that conversely every harmonic function on $\mathbf{C}$ is the real part of a differentiable function of a complex variable, but we shall not go in that direction in Math 55b!]

6. [A foray into algebraic geometry] Recall that as an example of an implicit function we used the "folium of Descartes" consisting of all $(x, y) \in \mathbf{R}^2$ such that $x^3 + y^3 = 3xy$. For $y \neq 0$ let $t = x/y$ and solve for x, y as functions of t. Use this and the Inverse Function Theorem to obtain another proof of the differentiability of y with respect to x at all but two points of the "folium". Explain what happens at those two points in terms of these functions $x(t), y(t)$. [Careful around $(0, 0)$!]

7. Let E be the subspace of the $n+1$-dimensional real vector space $\mathcal{P}_n$ consisting of the polynomials of degree n, i.e. $E = \{\sum_{j=0}^{n} a_j T^j : a_n \neq 0\}$. Fix $P_0 \in E$ and a root t_0 of P_0. Give necessary and sufficient conditions on P_0, t_0 for there to exist a $\mathcal{C}^1$ real-valued function t on a neighborhood of P_0 such that $t(P_0) = t_0$ and $t(P)$ is a root of P for each P in the neighborhood. What is the derivative $t'(P_0)$?

8. Let M_0 be an $n \times n$ matrix, and $\lambda_0 \in \mathbf{R}$ an eigenvalue of M_0 that is a simple root of the characteristic polynomial of M_0. Let v_0 be a λ_0-eigenvector of M_0. Regard M_0 as an element of the n^2-dimensional vector space $\mathcal{M}_n$ of $n \times n$ matrices. Prove that there is a neighborhood U of M_0 in $\mathcal{M}_n$ and $\mathcal{C}^1$ functions $\lambda : U \to \mathbf{R}$ and $v : U \to \mathbf{R}^n$ such that $\lambda(M_0) = \lambda_0$, $v(M_0) = v_0$, and, for all $M \in U$, $\lambda(M)$ is an eigenvalue of M with eigenvector v_0. What is the derivative $\lambda'(M_0)$? What can you say about $v'(M_0)$?

This problem set due Friday, 28 February, at the beginning of class.