Solutions to Homework 12

Math 55b

1. *Evaluate* $\sum_1^\infty 1/n^6$ *using the Laurent series for* $\pi \cot \pi z$ *around* $z = 0$.

The meromorphic 1-form $\omega := \pi \cot \pi z \, dz/z^6$ on $\mathbb{C}$ has poles at the integers, and simple poles at the nonzero integers $z = n$, with residue $1/n^6$, $n \in \mathbb{Z} \setminus \{0\}$. As for the residue at $z = 0$, it equals the coefficient of z^5 in the Laurent expansion of $\pi \cot \pi z$ near $z = 0$; found to be $-2\pi^6/945$. Apply the residue theorem to the integral of ω over a big circle $|z| = N + 1/2$, $N \in \mathbb{N}$. Since $\pi \cot \pi z$ is bounded below on the circle $|z| = N + 1/2$, this integral approaches 0 as $N \to \infty$. The residue theorem then gives the information that $\sum_{n\in\mathbb{Z}} \operatorname{Res}(\omega, n) = 0$, which translates into $\sum_{n\geq 1} 1/n^6 = \pi^6/945$. ■

2. *Evaluate* $\int_0^{2\pi} d\theta/(2 - \sin\theta)$.

We have, by the simplest type of applications of the residue theorem in evaluating real integrals, the reparametrization and evaluation as follows:

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta} = \int_{S^1} \frac{1}{2 - (z - z^{-1})/2i} \frac{dz}{iz}$$

$$= -2 \int_{S^1} \frac{dz}{z^2 - 4iz - 1} = -4\pi i \operatorname{Res}\Big(\frac{dz}{z^2 - 4iz - 1}, i(2 - \sqrt{3})\Big)$$

$$= -4\pi i \cdot \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}},$$

by a simple calculation. ■

3. *Evaluate* $\int_0^\infty dx/(1 + x^2)^2$.

We integrate the 1-form $dz/(1 + z^2)^2$ on the half-circular path $[-R, R] \cup \{|z| = R, \operatorname{Im}(z) > 0\}$, in the clockwise direction. Since the integrand is $o(R^{-1})$ as $R \to \infty$ while the arclength is proportional to R, the limit of this integral as $R \to \infty$ equals the real limit integral $\int_{-\infty}^{+\infty} dx/(1 + x^2)^2 = 2\int_0^\infty dx/(1 + x^2)^2$. On the other hand, the meromorphic 1-form $dz/(1 + z^2)^2$ has the unique simple pole $z = i$, with residue $-i/4$, in the region enclosed by the integration path; and the residue theorem shows the integral to be equal to $2\pi i \cdot (-i/4) = \pi/2$, for every $R > i$. The conclusion, upon taking the limit $R \to \infty$, is that $2\int_0^\infty dx/(1+x^2)^2 = \pi/2$, or $\int_0^\infty dx/(1+x^2)^2 = \pi/4$. ■

4. *Evaluate* $\int_0^\infty x^{-a}/(x+1)\,dx$ *for* $0 < a < 1$.

Consider the following path of integration: join the point $\infty \in \widehat{\mathbb{C}}$ to the point 0 along the real positive axis, then traverse a circular loop based at 0 and perpendicular to the real axis, of large radius R, and then return to ∞ along the real positive axis. Integrate the 1-form $z^{-a}\,dz/(z+1)$ along this path. The integral over the first leg of the journey (the real positive axis) is the negative $-I(a)$ of the desired answer; the integral along the circular part approaches 0 as $R \to \infty$, because the integrand is $O(R^{-1-a})$ and the arclength of the circle is proportional to R. In the approach back to ∞ along the positive real axis, the integral gets twisted by $1^{-a} := e^{-2a\pi i}$ from the analytic continuation of x^{-a}, and the total integral along over loop, as $R \to \infty$, approaches $(e^{-2a\pi i} - 1)I(a)$. On the other hand, the integral (for $R > 1$) is equal, by the residue theorem, to $2\pi i \operatorname{Res}(z^{-a}\,dz/(z+1), -1) = 2\pi i(-1)^{-a} := 2\pi i e^{-a\pi i}$; letting $R \to \infty$, we conclude the evaluation $I(a) = 2\pi i \frac{e^{-a\pi i}}{e^{-2a\pi i}-1} = \pi \cdot \frac{2i}{e^{-a\pi i}-e^{a\pi i}} = \pi/\sin a\pi$. ■

5. *Given* $p \in \mathbb{C}$, *construct explicitly a sequence* $z_n \to 0$ *with* $\exp(1/z_n) \to p$. *Can you, in fact construct such a sequence with* $\exp(1/z_n) = p$*?*

If $p \neq 0$, we may as well construct such a sequence with $\exp(1/z_n) = p$: choose any logarithm a of p, and define $z_n := (a+2\pi i n)^{-1}$, which certainly approaches 0 as $n \to \infty$. For $p = 0$, do this construction for a sequence $p_1, p_2, \ldots$ approaching 0, and diagonalize. ■

6. *Let* $p_t(z) = z^n + a_1(t)z^{n-1} + \cdots + a_0(t)$ *be a polynomial whose coefficients are analytic functions near* $t = 0$. *Suppose* $p_0(z)$ *has only simple zeros. Prove that there are analytic functions* $b_i(t)$ *defined near* $t = 0$ *such that* $p_t(z) = \prod_1^n (z - b_i(t))$.

Let $q_1, \ldots, q_n$ be the zeros of $p_0(z)$. Choose $r > 0$ sufficiently small so that the disks $|z - q_i| \leq r$ are disjoint, and let $s > 0$ be sufficiently small so that $p_t(z)$ does not vanish on the regions $|z - q_i| = r, |t| < s$ (such an s exists, simply by continuity of $p_t(z)$). Then, for $|t| < s$, the polynomial $p_t(z)$ has precisely one zero, call it $b_i(t)$, in each disks $|z - q_i| < r$: this, by the usual **Rouché** argument, is because the function

$n_i(t) = \frac{1}{2\pi i}\int_{|z-q_i|=r} \frac{(d/dz)p_t(z)}{p_t(z)}\,dz$ counting the zeros inside $|z-q_i|<r$ is integer-valued and continuous, hence identically $n_i(0)=1$. Given this, using the general formula $\operatorname{Res}(g\,df/f, p) = g(p)\mathrm{mult}_p(f(z))$ for analytic functions f, g, the **residue theorem** provides an integral formula for $b_i(t)$, ensuring its analyticity:

$$b_i(t) = \frac{1}{2\pi i}\int_{|z-q_i|=r} z\frac{(d/dz)p_t(z)}{p_t(z)}\,dz, \quad i=1,\dots,n.$$

■

7. *Prove or disprove: if $f:\Delta\to\mathbb{C}$ is an analytic function with n zeros, then $f'(z)$ has at least $n-1$ zeros in Δ. What happens if "at least" is replaced by "at most?"*

   Both propositions are false. The function $e^{nz}-1$ has many zeros on Δ, but its derivative ne^{nz} has none. The function $e^{z^{n+1}}$ has no zeros, but its derivative $(n+1)z^n e^{z^{n+1}}$ vanishes to order n at $z=0$. ■

8. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function satisfying $f(z+1)=f(z)$. Show that there exists an analytic function $g(z)$ on the punctured plane $\mathbb{C}^\times$ such that $f(z)=g(e^{2\pi i z})$. What is $g(z)$ for $f(z)=\tan\pi z$? Show that if $f(z_n)\to 0$ whenever $\mathrm{Im}(z_n)\to\infty$, then $f(z)=0$.*

   We may define $g(z) := f(\log(z)/(2\pi i))$ without ambiguity, because of the $\mathbb{Z}$-periodicity of $f(z)$. Then $g(z)$ is analytic on $\mathbb{C}-\mathbb{R}^{\le 0}$, because $\log z$ has an analytic branch on this region; and likewise, $g(z)$ is analytic on $\mathbb{C}-\mathbb{R}^{\ge 0}$. Analyticity being a local property, it follows that the function $g(z)$, which we defined set-theoretically on $\mathbb{C}^\times$, is in fact analytic on $(\mathbb{C}-\mathbb{R}^{\le 0})\cup(\mathbb{C}-\mathbb{R}^{\ge 0}) = \mathbb{C}^\times$, as required.

   For the example with the function $\tan\pi z$, note that $\tan\pi z = i(e^{-i\pi z}-e^{i\pi z})/(e^{i\pi z}+e^{-i\pi z})$.

   Finally, if $f(z)\to 0$ as $|\mathrm{Im}(z)|\to\infty$, $\mathbb{Z}$-periodicity implies $f(z)$ is bounded, therefore constant, therefore 0. ■

9. *Show that $f(z) := \sum_{n\in\mathbb{Z}}(z-n)^{-2}$ converges locally uniformly to an analytic function on $\mathbb{C}-\mathbb{Z}$, and conclude that $f(z)=\pi^2/\sin^2(\pi z)$.*

   Uniform convergence on $\{|z|\le R\}\cap\{|z-n|\ge r,\text{ all } n\in\mathbb{Z}\}$ is by comparison with $\sum n^{-2}$: on this compact region we have the uniform domination

$|f(z)| \le \sum_{n\in\mathbb{Z}} |z-n|^{-2} = \sum_{|n|\le R+1} |z-n|^{-2} + \sum_{|n|>R+1} |z-n|^2 \le (2R+3)/r + 2\sum_{k\ge 1} k^{-2} < (2R+3)/r + 4$, showing in particular uniform convergence on this region. Uniform limits of analytic functions are analytic, and the compact regions $\{|z| \le R\} \cap \{|z-n| \ge r, \text{ all } n \in \mathbb{Z}\}$ cover $\mathbb{C} - \mathbb{Z}$, hence we obtain the first clause: the series for $f(z)$ is absolutely and locally uniformly convergent away from $\mathbb{Z}$, and defines an analytic function on $\mathbb{C} - \mathbb{Z}$. This function is $\mathbb{Z}$-periodic, has double poles at the integers, with principal part $1/(z-n)^2$ of the pole $z = n$. We verify that $\pi^2/\sin^2(\pi z)$ has these same properties: it is a $\mathbb{Z}$-periodic, meromorphic function on $\mathbb{C}$ whose poles are located at the integers $\mathbb{Z}$; and to find the principal part of the pole $z = n$, it suffices by periodicity to consider $n = 0$, in which case the expansion $\pi^2/\sin^2(\pi z) = \pi^2/(\pi^2 z^2 + o(z^3)) = z^{-2} + o(1)$ shows the principal part to be z^{-2}, as required. Thus, the principal parts of the difference $G(z) := f(z) - \pi^2/\sin^2(\pi z)$ cancel, and $G(z)$ is a $\mathbb{Z}$-periodic entire function; to show it is 0, it suffices by 8 above to check that $|G(z)| \to 0$ as $|\mathrm{Im}(z)| \to \infty$, and this in turn suffices to be checked separately for $|f(z)|$ and $|\pi^2/\sin^2(\pi z)|$. For the latter case, it suffices to note that $|\sin(x+iy)|^2 = |\sin x \cosh y + i \cos x \cosh y| = \cosh^2 y + \sinh^2 y$ approaches ∞ exponentially fast as $y \to \infty$, as both sinh, cosh do: $\sinh y = (e^y - e^{-y})/2$, $\cosh y = (e^y + e^{-y})/2$. And for the former, note $|f(x+iy)| \le \sum_{n\in\mathbb{Z}} |(x-n)+iy|^{-2} = \sum_{n\in\mathbb{Z}} |(x-n)^2+y^2|^{-1} \le 2\sum_{k\ge 1} 1/(k^2+y^2) \to_{y\to\infty} 0$. ■

10. *Find A, B, C such that $\sum_{n\in\mathbb{Z}}(z-n)^{-4} = \frac{A}{\sin^4(\pi z)} + \frac{B}{\sin^2(\pi z)} + C$, and use this result to evaluate $\sum 1/n^4$.*

We twice differentiate the result in 9. to obtain $6\sum_{n\in\mathbb{Z}}(z-n)^{-4} = (\pi^2/\sin^2 \pi z)'' = (-2\pi \cot \pi z \cdot \pi^2/\sin^2(\pi z))' = 2\pi^2/\sin^2 \pi z \cdot \pi^2/\sin^2 \pi z - 4\pi^2 \cot^2 \pi z \cdot \pi^2/\sin^2 \pi z = 2\pi^4/\sin^4 \pi z - 4\pi^4(1-\sin^2 \pi z)/\sin^2 \pi z = \pi^4/\sin^4 \pi z - 4\pi^4/\sin^2 \pi z$, getting the required evaluation with $A = \pi^4, B = -2\pi^4/3, C = 0$. Subtracting $1/z^4$ on both sides and evaluating at $z = 0$ (that is, comparing the constant terms of the Laurent expansions): using $\pi^2/\sin^2 \pi z = \frac{1}{z^2}\frac{1}{1-\pi^2 z^2/3+2\pi^4 z^4/45+O(z^6)} = \frac{1}{z^2}\cdot(1+\pi^2 z^2/3+\pi^4 z^4/15+O(z^6))$ and hence $\pi^4/\sin^2 \pi z = \frac{1}{z^4}\cdot(1+2\pi^2 z^2/3+11\pi^4 z^4/45+O(z^6))$, we find this coefficient to be $\frac{11}{45}\pi^4 - \frac{2\pi^4}{3}\frac{1}{3} = \frac{\pi^4}{45}$. Thus, $\sum_{n\ne 0} 1/n^4 = \pi^4/45$, or $\sum_{n\ge 1} = \pi^4/90$. ■