Homework 5

Math 55b

Due Tuesday, 3 Mar 2009.

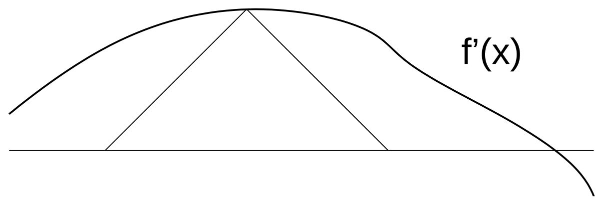

Figure 1. Littlewood's proof.

1. Let $\alpha > 1$. Find all functions $f : [0,1] \to \mathbb{R}$ satisfying
$$|f(x) - f(y)| \le C|x-y|^\alpha$$
for some $C > 0$.

2. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable, $f(a) = 0$ and $|f'(x)| \le f(x)$ for all x. Prove that f is constant.

3. Given $p \ge 1$ and $f \in C[a,b]$, let
$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}.$$
Prove that $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

4. Recall that $\|f\|_\infty = \sup |f(x)|$. Prove that for $f \in C[a,b]$, we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

5. Suppose $f \in C^2(\mathbb{R})$, $|f''(x)| \le 1$ and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$. (Hint: Littlewood suggests that Figure 1 is already a proof.)

6. Let $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and let $f(0) = 0$. Prove that all derivatives of f vanish at $x = 0$.

7. Let $L(n) = n(\log n)(\log\log n)(\log\log\log n)\cdots$, where the product is continued until you reach a term $< e$. Is $\sum 1/L(n)$ convergent or divergent?

8. Prove or disprove the following:
   (a) If $f : [a,b] \to \mathbb{R}$ is differentiable, then $f(b) - f(a) = \int_a^b f'(x)\,dx$.
   (b) If $f : [a,b] \to \mathbb{R}$ is monotone and differentiable, then $f'(x)$ is continuous.
   (c) If $f_n : [0,1] \to \mathbb{R}$ are differentiable functions, $f_n \to g$ uniformly, and g is differentiable, then $f_n' \to g'$ pointwise.