Math 556 Lecture 14 – March 3 – More about covering spaces ①

(Munkres §60 and §79 = §74 in international edition)

Recall: $p:(E,e_0)\to(B,b_0)$ covering map
ie. $\forall b\in B$ has an evenly covered nbd. U st

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\simeq"', "\text{homeo}"] \arrow[dr] & U\times A \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

$\leftarrow$ (discrete set indexing sheets of the cover)

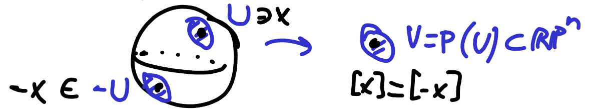
$\Rightarrow$ paths or loops in B lift to paths in E, uniquely if we choose lift of the starting point.
+ path homotopies lift to path homotopies.

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ f:(I,0) \arrow[r] \arrow[ur, "\exists!\,\tilde f"] & (B,b_0) \end{tikzcd}$$

Considering lifts of loops in (B,b_0) & considering end point of lifted path, get lifting correspondence $\varphi:\pi_1(B,b_0)\to p^{-1}(b_0)$.

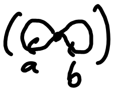
We've seen:
- if E is path connected then φ is surjective (can go from e_0 to any point of $p^{-1}(b_0)$!)
- if E is simply connected then φ bijection

Ex: recall from HW4: the quotient of S^n by $x\sim -x$, $p:S^n\to S^n/\sim\ \simeq \mathbb{RP}^n$ is a degree 2 covering map.



$p^{-1}(V)=U\amalg(-U)$ ✓

+ last time: for $n\geq 2$, S^n is simply connected. Hence: $\pi_1(\mathbb{RP}^n,b_0)\to p^{-1}(b_0)=\{2\text{ points}\}$ is bijective for $n\geq2$, hence $\pi_1(\mathbb{RP}^n;b_0)\simeq\mathbb{Z}/2$ (the only group of order 2!)
(but $\pi_1(\mathbb{RP}^1)\simeq\mathbb{Z}$)

Ex: can use covering maps to show $\pi_1($  $)$ non-abelian – $a*b\neq b*a$.
cf. Munkres 60.5 (on HW6)

Q: Let $p:(E,e_0)\to(B,b_0)$ covering map. How are $\pi_1(E)$ and $\pi_1(B)$ related?
(Always assume E and B are path-connected).

Thm: $p_*:\pi_1(E,e_0)\to\pi_1(B,b_0)$ is an injective homomorphism.

Pf: if $\tilde h$ is a loop at e_0 and $p_*([\tilde h])=id$, then $\exists$ path-homotopy $H:I\times I\to B$ from $p\circ\tilde h$ to the constant loop at b_0. Its lift $\tilde H:I\times I\to E$ starting at e_0 is then a path-homotopy from $\tilde h$ to the constant loop, so $[\tilde h]=id$. □

Hence, the covering $p:E\to B$ gives a subgroup $H=\operatorname{Im}(p_*)\subset\pi_1(B,b_0)$, with $\pi_1(E,e_0)\overset{\text{iso.}}{\underset{p_*}{\xrightarrow{\sim}}}H$

It turns out that:

(1) The subgroup $H\subset\pi_1(B,b_0)$ determines the covering p. (Munkres §79)

(2) Assuming B is path-connected and "sufficiently nice" ("semi-locally simply connected"). For each subgroup H of $\pi_1(B,b_0)$ $\exists$ covering $p:E\to B$ st. $p_*(\pi_1(E))=H$. (§82, won't do)

Equivalence of covering spaces:

Def: $p: E \to B$, $p': E' \to B$ coverings. p and p' are <u>equivalent</u> if $\exists$ homeomorphism $h: E \to E'$ st. $p = p' \circ h$:

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

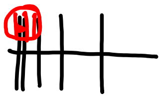
. Say h is an <u>equivalence of coverings</u>

(NB: $\forall b \in B$, h gives a bijection $p^{-1}(b) \xrightarrow{\sim} p'^{-1}(b)$ between the sheets of p and p'. By continuity, over a connected evenly covered subset $U \subset B$ this looks like $p^{-1}(U) \simeq U \times A \xrightarrow[id \times \sigma]{} U \times A' \simeq p'^{-1}(U)$. $\sigma: A \to A'$ bijection between sets of sheets).

- Goal: if two coverings have same corresponding subgroup of $\pi_1(B)$ then they are equivalent. For this we need a general lifting lemma.

Def: A space X is <u>locally path-connected</u> if $\forall x \in X$, $\forall U \ni x$, $\exists V \subset U$ path-connected neighborhood of x.

Counterexample:



$(\{\frac{1}{n}, n \geq 1\} \cup \{0\}) \times \mathbb{R} \cup \mathbb{R} \times 0$ in $\mathbb{R}^2$ is path-connected but not locally path-connected.

* From now on, assume $p: E \to B$ covering, E and B path-connected and locally path-connected.

Lifting lemma for loops:

Thm: A loop f in (B, b_0) lifts to a loop in (E, e_0) iff $[f] \in p_*(\pi_1(E, e_0)) \subset \pi_1(B, b_0)$

Pf:
- if the lift $\tilde{f}$ of f at e_0 is a loop in E, then $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}]) \in p_*(\pi_1(E))$.
- if $[f] = p_*([\tilde{g}])$ for some loop $\tilde{g}$ in (E, e_0), then $p \circ \tilde{g}$ is path-homotopic to f. Lifting this path-homotopy to E, we get a path-homotopy in E between $\tilde{g}$ and the lift $\tilde{f}$ of f. Since $\tilde{g}$ is a loop, so is $\tilde{f}$. $\square$

General lifting lemma:

Thm: Let $p: E \to B$ covering map, $p(e_0) = b_0$. Let Y be path-connected & loc. path-connected, and $f: Y \to B$ continuous map st. $f(y_0) = b_0$. Then f can be lifted to $\tilde{f}: Y \to E$ with $\tilde{f}(y_0) = e_0$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$. If it exists, the lift is unique

Pf:
- if f can be lifted to $\tilde{f}$, then $f = p \circ \tilde{f}$, so

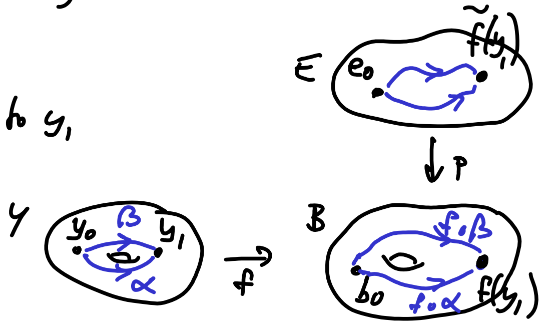
$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (B, b_0) \end{tikzcd}$$

$$f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0))) \subset p_*(\pi_1(E, e_0)) \checkmark$$

- Conversely, assume the condition holds, and let $y_1 \in Y$. Choose a path α from y_0 to y_1 in Y. Lift $f \circ \alpha: I \to B$ to a path in E starting at e_0. Define $\tilde{f}(y_1)$ = the end point of this path.

(this is the only possibility for $\tilde{f}(y_1)$ if a continuous lift $\tilde{f}$ exists, since the unique lift of $f\circ\alpha$ will then be $\tilde{f}\circ\alpha$.)

Need to check $\tilde{f}$ is well-defined and continuous!



- Well-defined? Let β be a different path in Y from y_0 to y_1.

  Then $\alpha*\bar{\beta}$ is a loop in (Y, y_0)

  $f\circ(\alpha*\bar{\beta})$ loop in (B,b_0), representing

  $f_*([\alpha*\bar{\beta}]) \in \mathrm{Im}\, f_* \subset p_*(\pi_1(E,e_0))$

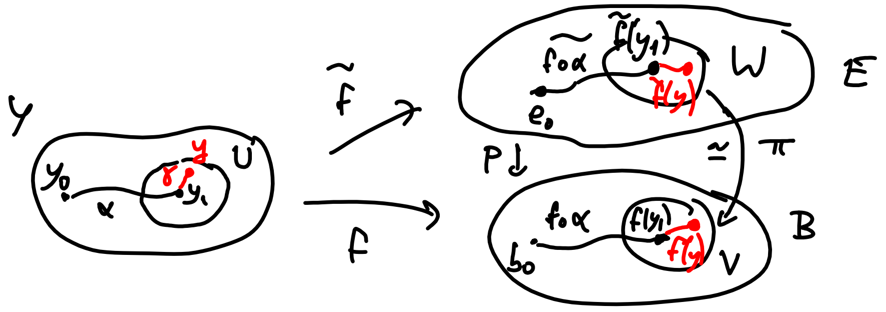
  so it lifts to a loop in E (by previous theorem).

  So: $f\circ\alpha$ lifts to a path from e_0 to $\tilde{f}(y_1)$ as defined above, and $\overline{f\circ\beta}$ lifts to a path from $\tilde{f}(y_1)$ back to e_0, hence $f\circ\beta$ lifts to a path from e_0 to $\tilde{f}(y_1)$. Thus $\tilde{f}(y_1)$ is independent of the choice of path $y_0 \to y_1$.

- Continuity of $\tilde{f}$: enough to check on a neighborhood of y_1.

  Let $V\subset B$ be an evenly covered nbd. of $f(y_1)$, and using local path-connectedness of Y, can find $U\subset f^{-1}(V)$ path-connected neighborhood of y_1 in Y.

  Let $W\subset p^{-1}(V)\subset E$ be the slice containing $\tilde{f}(y_1)$; $p_{|W}=\pi : W \xrightarrow{\sim} V$ homeo.



For $y\in U$, $\exists$ path γ in U from y_1 to y, and $\pi^{-1}\circ f\circ\gamma$ is a lift of $f\circ\gamma$ to $W\subset E$ starting at $\tilde{f}(y_1)$. And so the lift of $f\circ(\alpha*\gamma)$ to E starting at e_0 is the composition of $\widetilde{f\circ\alpha}$ (from e_0 to $\tilde{f}(y_1)$) and $\pi^{-1}\circ f\circ\gamma$ from $\tilde{f}(y_1)=\pi^{-1}(f(y_1))$ to $\pi^{-1}(f(y))$. Hence $\tilde{f}(y)=\pi^{-1}(f(y))$.

So $\tilde{f}_{|U}=\pi^{-1}\circ f_{|U}$ is continuous, and hence $\tilde{f}$ is continuous. □

★ Now we can tell when two coverings are equivalent, as long as all maps preserve base points!

Thm: Let $p: E\to B$, $p': E'\to B$ covering maps with $p(e_0)=p'(e_0')=b_0$.
There is an equivalence $h: E\xrightarrow{\sim}E'$ s.t. $h(e_0)=e_0'$ if and only if the subgroups $H=p_*(\pi_1(E,e_0))$ and $H'=p'_*(\pi_1(E',e_0'))$ are equal (the same subgroup of $\pi_1(B,b_0)$).
Moreover, if h exists it is unique.

Pf: ⇒ if $h: E \to E'$ is an equivalence with $h(e_0) = e_0'$, then $h_*(\pi_1(E,e_0)) = \pi_1(E',e_0')$.
The conclusion then follows from $p'_* \circ h_* = p_*$.

⇐ assume $H = H'$. Then by the lifting lemma, ∃ unique base point preserving lifts

$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E \arrow[ur, dashed, "h"] \arrow[r, "p"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E \arrow[d, "p"] \\ E' \arrow[ur, dashed, "h'"] \arrow[r, "p'"'] & B \end{tikzcd}.$$

So $p' \circ h = p$ and $p \circ h' = p'$.

Now, $p \circ h' \circ h = p' \circ h = p$, so $h' \circ h : E \to E$ is a lifting $\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, dashed, "h'\circ h"] \arrow[r, "p"'] & B \end{tikzcd}$

But so is id_E. By uniqueness of lifting, we get $h' \circ h = id_E$.
Similarly $h \circ h' = id_{E'}$. So h is a homeomorphism s.t. $p' \circ h = p$, hence an equivalence of coverings. □

Ex: $p_k : S^1 \to S^1$, $z \mapsto z^k$ $\quad (p_k)_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ (both $\cong \mathbb{Z}$) mult. by $k \Rightarrow H = k\mathbb{Z} \subset \mathbb{Z}$

$p_0 : \mathbb{R} \to S^1$, $x \mapsto (\cos x, \sin x)$ $\quad (p_0)_*(\pi_1(\mathbb{R})) = \{0\}$

[these are all the subgroups of $\mathbb{Z}$, so every connected covering of S^1 is equivalent to exactly one of these!

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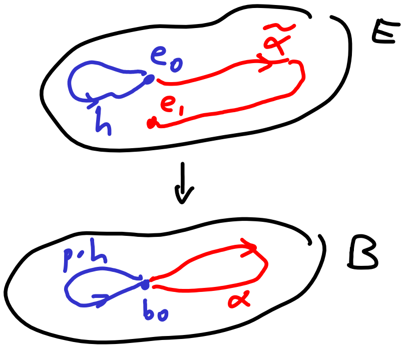
* What if we consider equivalences $h: E \to E'$ that don't map e_0 to e_0'?
Then the corresponding subgroups of $\pi_1(B, b_0)$ are conjugate.

- Indeed, if we change the base point in a (path-connected) covering space $p: E \to B$...
if $e_0, e_1 \in p^{-1}(b_0)$, and $\tilde{\alpha}$ is a path from e_0 to e_1, recall

$$\pi_1(E, e_0) \xrightarrow{\sim} \pi_1(E, e_1)$$
$$[h] \longmapsto [\tilde{\alpha}^{-1} * h * \tilde{\alpha}]$$



Then $\alpha = p \circ \tilde{\alpha}$ is a loop in (B, b_0), so whenever

$$[p \circ h] = p_*([h]) \in H_0 = p_*(\pi_1(E, e_0))$$
$$\Rightarrow [\alpha]^{-1} * [p \circ h] * [\alpha] \in H_1 = p_*(\pi_1(E, e_1))$$

So: $[\alpha]^{-1} H_0 [\alpha] \subset H_1$, and similarly in the reverse direction $[\alpha] H_1 [\alpha]^{-1} \subset H_0$, hence $=$

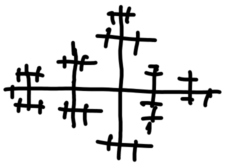
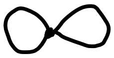
- Conversely, if H_0, H_1 are conjugate subgroups of $\pi_1(B, b_0)$, ie. $\exists [\alpha]$ st. $H_1 = [\alpha]^{-1} H_0 [\alpha]$ and $H_0 = p_*(\pi_1(E, e_0))$, then let $\tilde{\alpha}$ = lift of α to a path in E starting at e_0, and let $e_1 = \tilde{\alpha}(1)$, then $H_1 = p_*(\pi_1(E, e_1))$.

⇒ Theorem: $p: E \to B$, $p': E' \to B$ covering maps, $p(e_0) = p'(e_0') = b_0$. Then p and p' are equivalent ⇔ the subgroups $H = p_*(\pi_1(E, e_0))$, $H' = p'_*(\pi_1(E', e_0'))$ of $\pi_1(B, b_0)$ are conjugate.

Universal covering space:

Def: If $p_0: E_0 \to B$ covering and E_0 is simply connected, say E_0 is a universal covering of B.

Note: this corresponds to the trivial subgroup $p_{0*}(\pi_1(E_0)) = \{1\} \subset \pi_1(B)$; unique up to equiv by the above.

Ex:
- $p: \mathbb{R} \to S^1$
  $p \times p: \mathbb{R}^2 \to S^1 \times S^1 =$ torus
-  infinite tree $\longrightarrow$ 
  (horizontal edges $\to a$
  vertical $\to b$).

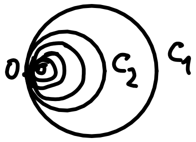
- **Thm:** $p_0: E_0 \to B$ universal covering, $p': E' \to B$ any path-connected covering, then $\exists$ covering map $q_0: E_0 \to E'$ st. $p' \circ q_0 = p_0$; and q_0 is univ. covering of E'.

q_0 is constructed by lifting:
$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E_0 \arrow[ur, "q_0"] \arrow[r, "p_0"'] & B \end{tikzcd}$$
($\exists$ since $p_{0*}(\pi_1(E_0)) = \{1\} \subset p'_*(\pi_1(E'))$.
& can show it's a covering map as well.

So, in fact, if B has a universal covering, all other coverings can then be obtained as quotients!

- Some spaces have no universal covering!

Ex: "Hawaiian earrings" $= \bigcup_{n \geq 1} C_n$ circles of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ inside $\mathbb{R}^2$



Any covering space must evenly cover a neighborhood of the origin, which prevents it from being simply connected. (for n suff'tly large, loop around C_n lifts to a loop).

- If one avoids such pathological examples - assuming B is (semi) locally simply connected, can build univ. cover as space of pairs (b, γ) where $\begin{cases} b \in B \\ \gamma = \text{homotopy class of path } b_0 \to b \end{cases}$

This has a preferred topology for which any simply conn'd nbd $U \ni b$ is evenly covered: if $b' \in U$, adding a path $b \rightsquigarrow b'$ inside U or its inverse gives a preferred bijection $\{\text{htpy classes of paths } b_0 \to b\} \longleftrightarrow \{\text{htpy classes of paths } b_0 \to b'\}$ independent of choice of path $b \rightsquigarrow b'$ inside U since U simply connected).