Zoom logistics:

- please turn on your video*, mute yourself except to ask/answer questions
  use your real name* (* if possible)
- Lectures are recorded. (speaker view = mostly me) (remind me to start recording when class starts)
- Ask questions either verbally or in Zoom chat. (I usually don't watch for raised hands in participant window)
- Internet issues:
  - short freezes will happen (if I don't seem to have noticed, a CA should tell me)
  - outage on my end: CAs lead Q&A for 1-2 minutes while I reconnect
  - major outage: check e-mail.

- Outside of lecture:
  → Canvas (notes, assignments, ...)
  → Slack (please join + introduce yourself in #general)
  → e-mail
  → discussions + office hours

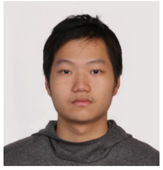
Course staff:

Prof. Denis AUROUX office hours Mondays 12-1 & Wednesdays 9-10 + 12-1.
auroux@math.harvard.edu ↳ not Sept 7 (holiday)

TF: Dr. Mark Shusterman

CAs: Avery Parr

Alfian Tjandra



Richard Xu

Cheng Zhou

Gaurav Goel (volunteer)

- Office hours & sections: to be announced on Canvas.

- See course information & syllabus on Canvas (more logistics, policies, exams)
- Homework due Wednesdays on Canvas. HW 1 (due Sept 9) is posted.
  Handwritten submissions are fine, or try LaTeX / Overleaf (+ Richard Xu's tutorial video)
  Collaboration encouraged (but write your own solution!). Ask CAs for hints if needed!
  Use slack (#studygroups, #homework). List your collaborators.
- Feedback survey to be completed this weekend (after lecture 2, before lecture 3)

Course Content:

1. Group theory (~ Artin chapter 2)
2. Fields and vector spaces, linear + multilinear algebra (Axler)
3. More group theory (Artin)
4. Intro to Representation theory (Artin + other texts)

You should have: { Artin, "Algebra" (2nd edition)
Axler, "Linear Algebra Done Right"

Groups = abstract structure that models the common features of concrete objects such as
- numbers
- permutations
- linear transformations
- symmetries

Definition: A group G consists of a set S together with a law of composition, ie. a map $m: S\times S \to S$, $(a,b)\mapsto a\cdot b$ (sometimes $a*b, \dots$) satisfying the following axioms:

1) there exists an identity element $e\in S$ st. $\forall a\in S$, $ae = ea = a$.
"for all"

[note: e is unique! if e, e' both act as identity then $e = ee' = e'$].

2) inverses exist: $\forall a \in S$, $\exists b\in S$ st. $ab = ba = e$. Write $b = a^{-1}$.
"for all" "there exists"

3) associativity: $\forall a,b,c\in S$, $(ab)c = a(bc)$.
[so we can write just: abc].

Rmk:
- associativity implies the cancellation law: $\forall a,b,c \in S$, $ab = ac \Rightarrow b = c$.
  (Pf: $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac) \underset{\text{associativity + inverse}}{\Rightarrow} eb = ec \Rightarrow b = c$.)
- technically the group is the pair (S, m), but in real life we'll just write G for the set and talk of elements of G.

Variants:
- ★ if we omit the second axiom (inverses), we have a semigroup.
- ★ if we have a group whose law is commutative, ie. $ab = ba$ $\forall a,b$ we say that G is abelian (and may denote the operation $+$ instead)

Examples: 0) the trivial group $G = \{e\}$, $e\cdot e = e$.
(usually not an interesting example. Don't give this as answer to a HW problem asking for an example.)

1) number systems: $(\mathbb{Z}, +)$ or $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with addition. Identity: 0
integers; rationals, reals, complex. Inverse: $-x$.
but natural numbers $(\mathbb{N}, +)$ only form a semigroup!

2) a group with two elements? if $|G| = 2$, let e = identity, x = the other element, necessarily $e\cdot e = e$, $e\cdot x = x$, $x \cdot e = x$. What about $x\cdot x$?
Can think of
- $\{0,1\}$ or $\{\text{even}, \text{odd}\}$, with addition mod 2 $(1+1=0)$
- $\{+1, -1\}$ with multiplication.

Breakout room challenge
- Come up with an example of a group with 8 elements + convince yourselves that it is a group.
- If this is too easy, try to find several different groups!
- When done, leave room & have one of your team report on your example in the main zoom chat.

Ex's continued:

3.) $\mathbb{Z}/n = \{0, 1, \dots, n-1\}$ with group law given by addition mod n:

$$(a,b) \mapsto \begin{cases} a+b & \text{if } a+b \leq n-1 \\ a+b-n & \text{otherwise} \end{cases}$$

(denote this by $+$) (finite group w/ n elements)

Similarly, $\mathbb{R}/\mathbb{Z}$: $S = [0,1) \subset \mathbb{R}$ with addition $(a,b) \mapsto \begin{cases} a+b & \text{if } a+b < 1 \\ a+b-1 & \text{otherwise.} \end{cases}$

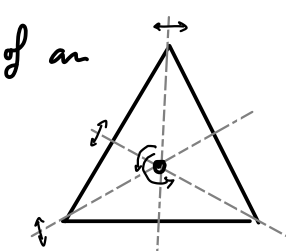
4) nonzero numbers $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^*$, $\mathbb{C}^*$ with multiplication. Identity: 1, inverse: $1/x$.
Inside $\mathbb{C}^*$, the unit circle $S^1 = \{z \in \mathbb{C} \,/\, |z| = 1\}$ is also a group for multiplication.
These are still abelian (aside: nonzero quaternions form a nonabelian mult. group)

5) symmetries and permutations:

Recall $f: A \to B$ is
- injective (1-to-1) if $\forall x, y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$
- surjective (onto) if $\forall b \in B$ $\exists x \in A$ st. $f(x) = b$.
- bijective if injective and surjective.

A permutation of a set A is a bijection $f: A \to A$. The set of permutations of A, with operation = composition, is a group, $\mathrm{Perm}(A)$. (Why?)

The symmetric group on n elements: $S_n = \mathrm{Perm}(\{1, \dots, n\})$



- S_3 has a geometric interpretation if we think of symmetries of an equilateral triangle = rotations which preserve it (3 incl. identity) and reflections (3 of those).

Symmetries permute the vertices, and every permutation of the set of vertices arises from exactly one symmetry (+ composition laws agree).
So: S_3 also occurs as the group of symmetries of Δ.
(Other groups arise from symmetries of other geometric figures in $\mathbb{R}^2$ and $\mathbb{R}^3$).

6) Groups of matrices: $GL_n(\mathbb{R}) = \{$invertible $n \times n$ matrices with real coefficients$\}$
"general linear group" (with matrix multiplication)

also $SL_n(\mathbb{R}) = \{n \times n$ real matrices with determinant $1\}$
"special linear group".

also $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$ for matrices with complex coefficients... or $\mathbb{Q}$ or $\mathbb{Z}/n$ coeff's!

Products of groups:

- Given two groups G, H, the product group is $G \times H = \{(g,h) \mid g \in G, h \in H\}$
  with composition law $(g,h)\cdot(g',h') = (gg', hh')$.
- If G, H are finite, of <u>order</u> $m = |G|$ and $n = |H|$, then $G \times H$ is a finite group of order mn.
- Similarly for product of n groups:
  <u>Ex:</u> $\mathbb{Z}^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{Z}\}$, $(a_1,\dots,a_n) + (b_1,\dots,b_n) = (a_1+b_1, \dots, a_n+b_n)$
  (similarly $\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$ with componentwise addition)
- Given infinitely many groups $G_1, G_2, G_3, \dots$ there are two different notions:
  → the direct product $\prod_{i=1}^{\infty} G_i = \{(a_1, a_2, a_3, \dots) \mid a_i \in G_i\}$
  → the direct sum $\bigoplus_{i=1}^{\infty} G_i = \{(a_1, a_2, a_3, \dots) \mid a_i \in G_i, \text{ all but finitely many are identity}\}$

  <u>Ex:</u> consider $G_0 = G_1 = \dots = (\mathbb{R}, +)$, denote $(a_0, a_1, a_2, \dots)$ by $\sum a_i x^i$.
  Then $\prod_{i=0}^{\infty} \mathbb{R} = \mathbb{R}[[x]]$ formal power series $\sum_{i=0}^{\infty} a_i x^i$ (w/ addition)
  $\bigoplus_{i=0}^{\infty} \mathbb{R} = \mathbb{R}[x]$ polynomials $\sum_{\text{finite}} a_i x^i$.

★ Subgroups & homomorphisms:

<u>Def:</u> A <u>subgroup</u> H of a group G is a non-empty! subset $H \subset G$ which is closed under composition ($a, b \in H \Rightarrow ab \in H$) and inversion ($a \in H \Rightarrow a^{-1} \in H$).
Since $H \neq \emptyset$, these 2 conditions imply $e \in H$. So H (with same operation) is a group in its own right.

- say H is a <u>proper subgroup</u> if $H \subsetneq G$.

<u>Def:</u> Given two groups G, H, a <u>homomorphism</u> $\varphi: G \to H$ is a map which respects the composition law: $\forall a, b \in G$, $\varphi(ab) = \varphi(a)\varphi(b)$.
(This implies $\varphi(e_G) = e_H$, and $\varphi(a^{-1}) = \varphi(a)^{-1}$).

- an <u>isomorphism</u> is a bijective homomorphism
  (if G and H are isomorphic, then they are secretly the "same" group even if elements and law may have different names).

<u>Breakout room challenge:</u> among examples seen so far, which groups are isomorphic to each other? or to subgroups of other groups?