Homework 4

Math 55b

Due Tuesday, 24 Feb 2009.

1. Let $U_n \subset \mathbb{R}$ be a sequence of open sets containing the rational numbers. Prove that $\bigcap U_n$ contains an irrational number.

2. Let $f : [0,1] \to \mathbb{R}$ be a function, and let $\Gamma(f) \subset [0,1] \times \mathbb{R}$ be its graph. Prove or disprove each of the 4 implications below:

   f is continuous $\iff$ $\Gamma(f)$ is compact $\iff$ $\Gamma(f)$ is connected.

3. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then so does $\sum \sqrt{a_n}/n$.

4. Let $a_1 = 1$ and let $a_{n+1} = (a_n + 2/a_n)/2$. Let $\epsilon_n = a_n - \sqrt{2}$. (i) Show that $\epsilon_n \to 0$. (ii) Compute $\lim\,(\log\log(1/\epsilon_n))/n$.

5. Let $p(t) = t^d + a_1 t^{d-1} + \cdots + a_d$ be a polynomial with integral coefficients, and suppose $x \in \mathbb{R}$ satisfies $p(x) = 0$. Prove there is a $C > 0$ such that
$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^d}$$
   for all $p/q \in \mathbb{Q} - \{x\}$. Use this to show that $x = \sum_0^\infty 1/10^{n!}$ is transcendental.

6. Prove that the function $S(t) = \sup_N \sum_0^N \sin(nt)$ is finite for all $t \in \mathbb{R}$. Is it bounded?

7. Prove that if $\sum a_n = S$, then $\lim_{r \nearrow 1} \sum a_n r^n = S$.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f'(x)$ is continuous. Prove that $f_n(x) = n(f(x+1/n) - f(x))$ converges to $f'(x)$ uniformly on any fixed interval $[a,b]$. Give an example where the convergence is not uniform on $\mathbb{R}$.