Math 55b: Honors Real and Complex Analysis

Homework Assignment #7 (19 March 2018):
Fourier series via Stone-Weierstrass;
Power series and some classical integrals and product formulas

Sketch of a proof *n*. I couldn't verify all the details, so I'll break it down into the parts I couldn't prove.[1]

Please avoid merely "sketching" (as defined in the above quote) a proof. In all problem sets, you may use the result in one problem (or problem part) to solve another, even if you have not proved the first one, unless this becomes circular [EXCEPTION: when problem B is clearly a generalization of A, don't use B to solve A unless you've solved B!]. As in Math 55b, the problems are generally <u>not</u> in order of difficulty.

Connections between Stone-Weierstrass and Fourier analysis:

1. Let K be the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. Prove that the functions $K \to \mathbf{C}$ of the form[2] $\sum_{n=-N}^{N} a_n z^n$ (for any integer $N > 0$ and any $a_n \in \mathbf{C}$) constitute a subalgebra of $\mathcal{C}(K, \mathbf{C})$ that satisfies the hypothesis of the complex Stone-Weierstrass theorem, and is thus dense in $\mathcal{C}(K, \mathbf{C})$. [This proves the theorem of Fejér mentioned at the start of the previous problem set, though it is not the original proof, which constructed an explicit sequence of functions $\sum_{n=-N}^{N} a_n z^n$ converging uniformly to a given $f \in \mathcal{C}(K, \mathbf{C})$.]

2. Deduce Parseval's theorem for continuous functions $f : K \to \mathbf{C}$: let

$$a_n := \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) \, d\theta$$

for $n \in \mathbf{Z}$; then

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 \, d\theta.$$

(Remember that the functions $e^{in\theta}$ are orthonormal with respect to the inner product $\langle a, b \rangle = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) \, \overline{b(e^{i\theta})} \, d\theta$. You'll have to combine this inner-product structure on $\mathcal{C}(K, \mathbf{C})$, and the corresponding norm, with the sup norm relevant to Problem 1.)

3. State and prove generalizations of problems 1 and 2 to continuous functions on the torus K^d $(d = 1, 2, 3, \ldots)$.

In the first problem set of Math 55a we introduced in effect the algebra of formal power series, with no concern for convergence. Now we can show that the power series convergent up to a given radius constitute a subalgebra. Closure under the vector space operations is easy, so we consider only multiplication.

4. i) Suppose a_n and b_n $(n \geq 0)$ are complex numbers such that the power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ converge absolutely (and thus uniformly) in $|z| \leq R$. Prove that the same is true of $\sum_{n=0}^{\infty} (a * b)_n z^n$ where $a * b$ is the convolution

$$(a * b)_n := \sum_{m=0}^{n} a_m b_{n-m},$$

and that this series converges to $A(z)B(z)$ for all $z \in \mathbf{C}$ such that $|z| \leq R$. (See Rudin's Theorem 3.55 for one ingredient.)

   ii) Suppose in addition that $b_0 \neq 0$. Show that there exists some positive r such that $A(z)/B(z)$ is analytic in $|z| \leq r$. (Construct a geometric series converging to A/B on some circle.)

A preview of complex-analytic functions:

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[1] *Definitions of Terms Commonly Used in Higher Math*, R. Glover et al.

[2] Equivalently, these are the "trigonometric polynomials" = finite linear combinations of the functions $e^{in\theta}$ $(n \in \mathbf{Z})$ on $\mathbf{R}/2\pi\mathbf{Z}$, or of $e^{2\pi in\theta}$ $(n \in \mathbf{Z})$ on $\mathbf{R}/\mathbf{Z}$, under the identification of this group with K via the $n = 1$ function.

5. i) Suppose the complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely (and thus uniformly) in $|z| \le R$. Prove that for each n

$$a_n = R^{-n} \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) e^{-in\theta} \, d\theta.$$

(This works also for $n < 0$, giving $a_n = 0$ in that case, which should be familiar from Rudin's discussion of Stone-Weierstrass.) Use this to prove an integral formula of the form $f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) K_w(\theta) \, d\theta$ for all $w \in \mathbf{C}$ with $|w| < R$. (For example, $K_0(\theta) = 1$ works by taking $n = 0$ in our integral formula for a_n.)

The remaining problems give a path to several classical product formulas and integrals that is more elementary than usual but requires finesse in several places. Be careful about justifying all steps!

6. [Wallis' integrals] Prove that $\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx$ for all $n \ge 2$. Deduce that

$$\int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{2}{3} \frac{4}{5} \frac{6}{7} \cdots \frac{n-1}{n}, & \text{if } n \text{ is odd;} \\[2mm] \frac{\pi}{2} \frac{1}{2} \frac{3}{4} \frac{5}{6} \cdots \frac{n-1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

7. [Wallis' product (1655)] It follows that

$$\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \cdot \frac{\int_0^{\pi/2} \cos^{2m} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx}.$$

Show that

$$1 < \frac{\int_0^{\pi/2} \cos^{2m} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx} < \frac{\int_0^{\pi/2} \cos^{2m-1} x \, dx}{\int_0^{\pi/2} \cos^{2m+1} x \, dx} = 1 + \frac{1}{2m},$$

and therefore

$$\frac{\pi}{2} = \lim_{m \to \infty} \left(\frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \right).$$

[This is usually written as the "infinite product"

$$\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \cdots .]$$

8. Use the formulas of the previous problem to prove that

$$\lim_{n \to \infty} \int_0^{\sqrt{n} \frac{\pi}{2}} \cos^n \frac{x}{\sqrt{n}} \, dx = \sqrt{\pi/2}.$$

Now show that $\lim_{n \to \infty} \cos^n(x/\sqrt{n}) = \exp(-x^2/2)$ for any $x \ge 0$, and use this to prove that[3]

$$\int_0^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi/2}.$$

9. Define $I_n(\lambda)$ for $0 < \lambda < 1$ by

$$I_n(\lambda) = \int_0^{\pi/2} \cos^n x \cos(\lambda x) \, dx \qquad (n = 0, 1, 2, \dots).$$

Integrate by parts twice to prove that $(n^2 - \lambda^2) I_n(\lambda) = (n^2 - n) I_{n-2}(\lambda)$ for $n \ge 2$. Then evaluate $I_0(\lambda)$ and $I_1(\lambda)$ to obtain a formula for $I_n(\lambda)$ for all n. Deduce a product formula for $\tan(\pi\lambda/2)$, and verify that Wallis' product can be recovered from your formula by taking the limit as $\lambda \to 0$. Can you obtain any further formulas by investigating the behavior of $I_n(\lambda)$ as $n \to \infty$?

This problem set is due Monday, 26 March, at the beginning of class.

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[3] As I may have noted some times ago in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral $\int \exp(cx^2) \, dx$ cannot be simplified. We shall give another proof of $\int_0^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi/2}$ when we come to the change of variable formula for multiple integrals.