Math 55a Homework 2

Due Wednesday September 16, 2020.

- You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. Also, please list at the end of the problem set the sources you consulted and people you worked with on this assignment.
- Questions marked * may be on the harder side.

Material covered: Subgroups, normal subgroups, quotients; modular arithmetic, permutations, etc. (most of Artin chapter 2).

0. Sometime over the weekend of September 12-13, please complete the week 2 feedback survey (in Canvas). This is important to help us assess how well the course structure, pacing, and our efforts at getting students to know each other are working. (There will be more surveys).

1. Describe a polygon in $\mathbb{R}^2$ whose symmetry group is $\mathbb{Z}/3$. What is the fewest number of vertices that such a polygon can have?

2. Let G be the set of *affine transformations* of the real line, i.e. maps $f_{a,b} : \mathbb{R} \to \mathbb{R}$ of the form $f_{a,b} : x \mapsto ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$.

(a) Show that G is a group, with group law given by composition.

(b) What is the center of G? (Recall the center of a group G is the subgroup of elements which commute with all other elements, $Z(G) = \{a \in G \,|\, ax = xa \ \forall x \in G\}$.)

(c) Show that the subsets $H = \{f_{a,b} \,|\, a = 1\}$ and $K = \{f_{a,b} \,|\, b = 0\}$ are subgroups of G. Which one is normal, and what is the quotient of G by that subgroup?

3. Let $G = S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ be the group of complex numbers of modulus 1, with multiplication. Let $H \subset G$ be the subgroup $\{\pm 1\}$. Show that the quotient G/H is isomorphic to G.

4. Let G be the group of rotations preserving a cube in $\mathbb{R}^3$. Show that G is isomorphic to the symmetric group S_4 (permutations of a four-element set). (Hint: which four-element set?)

5. Show that, if G is a finite group and $H \subset G$ is a subgroup of index 2 (i.e., $|G|/|H| = 2$), then H is a normal subgroup of G.

6*. The dihedral group D_n is the group of order $2n$ consisting of all symmetries (rotations and reflections) of the regular n-gon in the plane.

(a) Find the center of D_n.

(b) Find all normal subgroups of D_n.

7. Prove or provide a counterexample: if A is a normal subgroup of B and B is a normal subgroup of C, then A is a normal subgroup of C.

8. Find the order of the group $GL_2(\mathbb{Z}/2)$ of 2×2 matrices with entries in $\mathbb{Z}/2$ and nonzero determinant (mod 2). Is $GL_2(\mathbb{Z}/2)$ isomorphic to another group we've encountered?

9*. (a) Let p be a prime number, and suppose $H \subset S_p$ is a subgroup of the group of permutations of $\{1, 2, \dots, p\}$. Show that if H contains the p-cycle $(123\dots p)$ and a transposition (a permutation that exchanges two elements and fixes all others), then $H = S_p$. (In other words: S_p is generated by $(123\dots p)$ and by *any* transposition).

(b) Show that the conclusion of part (a) may be false if we don't assume p is prime (even though it is still true that S_p is generated by $(123\dots p)$ and a *suitably chosen* transposition).

10*. Let G be a finitely generated group, and $H \subset G$ a subgroup of finite index. Show that H is finitely generated.

(Hint: Choose a finite subset $S \subset G$ containing one representative of each coset of H, so every element of G is the product of an element of S and an element of H. Given a word in the generators of G and their inverses, how do you rewrite it as the product of an element of S and an element of H?)

11*. (Optional, extra credit): Let G be a non-abelian finite group, and consider its center $Z(G) = \{a \in G \,|\, ax = xa \ \forall x \in G\}$.

(a) Show that $G/Z(G)$ is not a cyclic group.

(b) Show that at most $5/8^{\text{ths}}$ of the pairs of elements of G commute, i.e. the set $C = \{(a, b) \in G \times G \,|\, ab = ba\}$ satisfies $|C| \le \frac{5}{8}|G|^2$.

(c) Show that this bound is optimal, i.e. there exists a non-abelian finite group for which $|C| = \frac{5}{8}|G|^2$.

12*. (Optional, extra credit): Given a set S, let $\mathcal{E} \subset \mathcal{P}(S)$ be such that (1) $S \in \mathcal{E}$, (2) if $A \in \mathcal{E}$ then $S - A \in \mathcal{E}$, (3) if $A, B \in \mathcal{E}$ then $A \cup B \in \mathcal{E}$ and $A \cap B \in \mathcal{E}$.

Prove that if S is finite then there is a set T and a surjective map $f : S \to T$ such that $\mathcal{E} = \{f^{-1}(A),\ A \subset T\}$. What happens if S is infinite?

13. How long did this assignment take you? How hard was it? What resources did you use, and how much help did you need? (Remember to list the students you collaborated with on this assignment.) Did you have any prior experience with this material?