

## Math 55a: Honors Abstract Algebra

### Tensor products

**Slogan.** Tensor products of vector spaces are to Cartesian products of sets as direct sums of vector spaces are to disjoint unions of sets.

**Description.** For any two vector spaces  $U, V$  over the same field  $F$ , we will construct a *tensor product*  $U \otimes V$  (occasionally still known also as the “Kronecker product” of  $U$  and  $V$ ), which is also an  $F$ -vector space. If  $U, V$  are finite dimensional then so is  $U \otimes V$ , with  $\dim(U \otimes V) = \dim U \cdot \dim V$ . If  $U$  has basis  $\{u_i : i \in I\}$  and  $V$  has basis  $\{v_j : j \in J\}$ , then  $U \otimes V$  has basis  $\{u_i \otimes v_j : (i, j) \in I \times J\}$ .

This notation  $u_i \otimes v_j$  is a special case of a map  $\otimes : U \times V \rightarrow U \otimes V$ , which is bilinear: for each  $u_0 \in U$ , the map  $v \mapsto u_0 \otimes v$  is a linear map from  $V$  to  $U \otimes V$ , and for each  $v_0 \in V$ , the map  $u \mapsto u \otimes v_0$  is a linear map from  $U$  to  $U \otimes V$ . So, for instance,

$$(2u_1 + 3u_2) \otimes (4v_1 - 5v_2) = 8u_1 \otimes v_1 - 10u_1 \otimes v_2 + 12u_2 \otimes v_1 - 15u_2 \otimes v_2.$$

The element  $u \otimes v$  of  $U \otimes V$  is called the “tensor product of  $u$  and  $v$ ”.

**Definitions.** Such an element  $u \otimes v$  is called a “pure tensor” in  $U \otimes V$ . The general element of  $U \otimes V$  is not a pure tensor; for instance you can check that if  $\{u_1, u_2\}$  is a basis for  $U$  and  $\{v_1, v_2\}$  is a basis for  $V$  then

$$a_{11}u_1 \otimes v_1 + a_{12}u_1 \otimes v_2 + a_{21}u_2 \otimes v_1 + a_{22}u_2 \otimes v_2$$

is a pure tensor if and only if  $a_{11}a_{22} = a_{12}a_{21}$ . But any element of  $U \otimes V$  is a linear combination of pure tensors. The basis-free construction of  $U \otimes V$  is obtained in effect by declaring that  $U \otimes V$  consists of linear combinations of pure tensors subject to the condition of bilinearity. More formally, we define  $U \otimes V$  as a quotient space:

$$U \otimes V := Z/Z_0,$$

where  $Z$  is the (huge) vector space with one basis element  $u \otimes v$  for every  $u \in U$  and  $v \in V$  (that is,  $Z$  is the space of formal (finite) linear combinations of the symbols  $u \otimes v$ ), and  $Z_0 \subseteq Z$  is the subspace generated by the linear combinations of the form

$$(u + u') \otimes v - u \otimes v - u' \otimes v, \quad u \otimes (v + v') - u \otimes v - u \otimes v',$$

$$(au) \otimes v - a(u \otimes v), \quad u \otimes (av) - a(u \otimes v)$$

for all  $u, u' \in U$ ,  $v, v' \in V$ ,  $a \in F$ .

**Properties.** To see this definition in action and verify that it does what we want, let us prove our claim above concerning bases: If  $\{u_i\}_{i \in I}$  and  $\{v_j\}_{j \in J}$  are bases for  $U$  and  $V$  then  $\{u_i \otimes v_j\}_{(i,j) \in I \times J}$  is a basis for  $U \otimes V$ . Naturally, for any vectors  $u \in U$ ,  $v \in V$ , we write “ $u \otimes v$ ” for the image of  $u \otimes v \in Z$  under the quotient map  $Z \rightarrow Z/Z_0 = U \otimes V$ .

Let  $W$  be a vector space with basis  $\{w_{ij}\}$  indexed by  $I \times J$ . We construct linear maps

$$\alpha : W \rightarrow U \otimes V, \quad \beta : U \otimes V \rightarrow W,$$

with  $\alpha(w_{ij}) = u_i \otimes v_j$  and  $\beta(u_i \otimes v_j) = w_{ij}$ . We prove that  $\alpha$  and  $\beta$  are each other's inverse. This will show that  $\alpha, \beta$  are isomorphisms that identify  $w_{ij}$  with  $u_i \otimes v_j$ , thus proving our claim. In each case we use the fact that choosing a linear map on a vector space is equivalent to choosing an image of each basis vector. The map  $\alpha$  is easy: we must take  $w_{ij}$  to  $u_i \otimes v_j$ . As to  $\beta$ , we don't yet have a basis for  $U \otimes V$ , so we first define a map  $\tilde{\beta} : Z \rightarrow W$ , and show that  $Z_0 \subseteq \ker \tilde{\beta}$ , so  $\tilde{\beta}$  "factors through  $Z_0$ ", i.e., descends to a well-defined map from  $Z/Z_0 = U \otimes V$ . Recall that  $\{u \otimes v : u \in U, v \in V\}$  is a basis for  $Z$ . For all  $u = \sum_i a_i u_i \in U$  and  $v = \sum_j b_j v_j \in V$ , we define

$$\tilde{\beta}(u \otimes v) = \sum_i \sum_j a_i b_j (u_i \otimes v_j).$$

Note that this sum is actually finite because the sums for  $u$  and  $v$  are finite, so the sum represents a legitimate element of  $W$ . We then readily see that  $\ker \tilde{\beta}$  contains  $Z_0$ , because each generator of  $Z_0$  maps to zero. We check that  $\tilde{\beta} \circ \alpha$  and  $\alpha \circ \beta$  are the identity maps on our generators of  $W$  and  $U \otimes V$ . The former check is immediate:  $\tilde{\beta}(u_i \otimes v_j) = w_{ij}$ . The latter takes just a bit more work: it comes down to showing that

$$u \otimes v - \sum_i \sum_j a_i b_j (u_i \otimes v_j) \in Z_0.$$

But this is straightforward, since the choice of  $\tilde{\beta}(u \otimes v)$  was forced on us by the requirement of bilinearity. This exercise completes the proof of our claim.

Our initial Slogan, and/or the symbol  $\otimes$  for tensor product, and/or the formula for  $\dim(U \otimes V)$  in the finite-dimensional case, lead us to expect identities such as

$$V_1 \otimes V_2 \cong V_2 \otimes V_1, \quad (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3),$$

and

$$U \otimes (V_1 \oplus V_2) \cong (U \otimes V_1) \oplus (U \otimes V_2).$$

These are true, and in fact are established by canonical isomorphisms taking  $v_1 \otimes v_2$  to  $v_2 \otimes v_1$ ,  $(v_1 \otimes v_2) \otimes v_3$  to  $v_1 \otimes (v_2 \otimes v_3)$ , and  $u \otimes (v_1, v_2)$  to  $(u \otimes v_1, u \otimes v_2)$ . In each case this is demonstrated by first defining the linear maps and their inverses on the level of the  $Z$  spaces and then showing that they descend to the tensor products which are the quotients of those  $Z$ 's. Even more simply we show that

$$V \otimes F \cong V, \quad V \otimes \{0\} = \{0\}$$

for any  $F$ -vector space  $V$ .

**A universal property.** Suppose now that we have a linear map

$$F : U \otimes V \rightarrow X$$

for some  $F$ -vector space  $X$ . Define a function  $f : U \times V \rightarrow X$  by

$$f(u, v) := F(u \otimes v).$$

Then this map is bilinear, in the sense described above. Conversely, for any function  $f : U \times V \rightarrow X$  we may define  $\tilde{F} : Z \rightarrow X$  by setting  $\tilde{F}(u \otimes v) = f(u, v)$ , and  $\tilde{F}$  descends

to  $Z/Z_0 = U \otimes V$  if and only if  $f$  is bilinear. Thus a bilinear map on  $U \times V$  is tantamount to a linear map on  $U \otimes V$ ; more precisely, there is a canonical isomorphism between the vector space of bilinear maps:  $U \times V \rightarrow X$  and the space  $\text{Hom}(U \otimes V, X)$  that takes  $f$  to the map  $u \otimes v \mapsto f(u, v)$ . Stated yet another way, every bilinear function on  $U \times V$  factors through the bilinear map  $(u, v) \mapsto u \otimes v$  from  $U \times V$  to  $U \otimes V$ . This “universal property” of  $U \otimes V$  could even be taken as a definition of the tensor product (once one shows that it determines  $U \otimes V$  up to canonical isomorphism).

For example, a bilinear form on  $V$  is a bilinear map from  $V \times V$  to  $F$ , which is now seen to be a linear map from  $V \otimes V$  to  $F$ , that is, an element of the dual space  $(V \otimes V)^*$ . We shall come back to this important example later.

**Tensor products of linear maps.** Here is another key example. For any linear maps  $S : U \rightarrow U'$  and  $T : V \rightarrow V'$  we get a bilinear map  $U \times V \rightarrow U' \otimes V'$  taking  $(u, v)$  to  $S(u) \otimes T(v)$ . Thus we have a linear map from  $U \otimes V$  to  $U' \otimes V'$ . We call this map  $S \otimes T$ . The map  $(S, T) \mapsto S \otimes T$  is itself a bilinear map from  $\text{Hom}(U, U') \times \text{Hom}(V, V')$  to  $\text{Hom}(U \otimes V, U' \otimes V')$ , which yields a canonical map  $\text{Hom}(U, U') \otimes \text{Hom}(V, V') \rightarrow \text{Hom}(U \otimes V, U' \otimes V')$ . At least if  $U, U', V, V'$  are all finite dimensional, this map is an isomorphism. This can be seen by choosing bases for  $U, V, U', V'$ . This yields bases for  $U \otimes V$  and  $U' \otimes V'$  (the  $u_i \otimes v_j$  construction above), for  $\text{Hom}(U, U')$  and  $\text{Hom}(V, V')$  (the matrix entries), and thus for  $\text{Hom}(U \otimes V, U' \otimes V')$  and  $\text{Hom}(U, U') \otimes \text{Hom}(V, V')$ ; and our map takes the  $(i, j, i', j')$  element of the first basis to the  $(i, j, i', j')$  element of the second. If we represent  $S, T, S \otimes T$  by matrices, we get a bilinear map

$$\text{Mat}(m, n) \times \text{Mat}(m', n') \rightarrow \text{Mat}(mm', nn')$$

called the Kronecker product of matrices; the entries of  $\mathcal{M}(S \otimes T)$  are the products of each entry of  $\mathcal{M}(S)$  with every entry of  $\mathcal{M}(T)$ .

**Tensor products and duality.** If the above seems hopelessly abstract, consider some special cases. Suppose  $U' = V = F$ . We then map  $U^* \otimes V'$  to the familiar space  $\text{Hom}(U, V')$ , and the map is an isomorphism if  $U, V'$  are finite dimensional. Thus if  $V_i$  are finite dimensional then we have identified  $\text{Hom}(V_1, V_2)$  with  $V_1^* \otimes V_2$ . If instead we take  $U' = V' = F$  then we get a map  $U^* \otimes V^* \rightarrow (U \otimes V)^*$ , which is an isomorphism if  $U, V$  are finite dimensional. In particular, if  $U = V$  we find that a bilinear form on a finite-dimensional vector space  $V$  is tantamount to an element of  $V^* \otimes V^*$ .

**Changing the ground field.** In another direction, suppose  $F'$  is a field containing  $F$ , and let  $V' = V \otimes_F F'$ . (When more than one field is present, we'll use the subscript to indicate the intended ground field for the tensor product. A larger ground field gives more generators for  $Z_0$  and thus may yield a smaller tensor product  $Z/Z_0$ . In most of the applications we'll have  $F = \mathbf{R}$ ,  $F' = \mathbf{C}$ .) We claim that  $V'$  is in fact a vector space over  $F'$ . For each  $a \in F'$ , consider multiplication by  $a$  as an  $F$ -linear map on  $F'$ . Then  $1_V \otimes a$  is a linear map from  $V'$  to itself, which we use as the multiplication-by- $a$  map on  $V'$ . The fact that multiplication by  $ab$  is the same as multiplication by  $b$  followed by multiplication by  $a$  is then a special case of the fact that composition of linear maps is consistent with tensor products:

$$(S_1 \circ S_2) \otimes (T_1 \circ T_2) = (S_1 \otimes T_1) \circ (S_2 \otimes T_2).$$

This in turn is true because it holds on pure tensors  $u \otimes v$ . We usually think of  $V'$  as  $V$  with scalars extended from  $F$  to  $F'$ .

If  $V$  has dimension  $n < \infty$  with basis  $\{v_i\}_{i=1}^n$  then  $\{v_i \otimes 1\}_{i=1}^n$  is a basis for  $V'$ . (To see this, begin by using  $\{v_i\}$  to identify  $V$  with  $F \oplus F \oplus \cdots \oplus F$ , and tensor the direct sum with  $F'$ . If  $\{v_i\}_i \in I$  is a basis of arbitrary cardinality for  $V$ , is it still true that  $\{v_i \otimes 1\}_{i \in I}$  is a basis for  $V'$ ?) If  $T : U \rightarrow V$  is a linear map between  $F$ -vector spaces then  $T \otimes 1$  is an  $F'$ -linear map from  $U \otimes F = U'$  to  $V'$ ; when  $U, V$  are finite dimensional, this map has the same matrix as  $T$  as long as we use the bases  $\{u_i \otimes 1\}, \{v_j \otimes 1\}$  for  $U', V'$ . We usually think of  $T \otimes 1$  as  $T$  with scalars extended from  $F$  to  $F'$ .

**Symmetric and alternating tensor squares.** The *tensor square*  $V^{\otimes 2}$  of  $V$  is defined by

$$V^{\otimes 2} := V \otimes V.$$

Likewise we can define tensor cubes and higher tensor powers. (Of course  $V^{\otimes 1}$  is  $V$  itself; what should  $V^{\otimes 0}$  be?) Our isomorphism  $V_1 \otimes V_2 \cong V_2 \otimes V_1$  then becomes an isomorphism  $s$  from  $V \otimes V$  to itself. This map is not the identity (unless  $V$  has dimension 0 or 1), but it is always an involution; that is,  $s^2$  is the identity. The subspace of  $V \otimes V$  fixed under  $s$  is the *symmetric square* of  $V$ , denoted  $\text{Sym}^2 V$ . It can also be defined as a quotient  $Z/Z_1$ , with  $Z$  as in the definition of  $V \otimes V$ , and  $Z_1$  generated by  $Z_0$  and combinations of the form  $v_1 \otimes v_2 - v_2 \otimes v_1$ . Likewise we may define the symmetric cube and higher symmetric powers of  $V$  by declaring  $\text{Sym}^k V$  to be the subspace of  $V^{\otimes k}$  invariant under arbitrary permutations of the  $k$  indices. If  $V$  has finite dimension  $n$  then  $\text{Sym}^2 V$  has dimension  $(n^2 + n)/2$ ; do you see why? What does this correspond to in terms of our motivating Slogan? Can you determine the dimension of  $\text{Sym}^k V$  for  $k = 3, 4, \dots$ ?

We can also regard  $\text{Sym}^2 V$  as the  $+1$ -eigenspace of  $s$ . Since  $s^2 = 1$ , we know that the only possible eigenvalues are  $\pm 1$ . What then of the  $-1$  eigenspace? Usually this is called the *alternating square* of  $V$ , denoted by  $\wedge^2 V$ , and can be obtained as the quotient of  $Z$  by the subspace generated by  $Z_0$  and combinations of the form  $v_1 \otimes v_2 + v_2 \otimes v_1$ ; the image of  $v_1 \otimes v_2$  in  $\wedge^2 V$  is denoted by  $v_1 \wedge v_2$ . The caveat “usually” is necessary because in characteristic 2 one cannot distinguish between  $-1$  and  $+1$ ! Note however that if  $2 \neq 0$  then the identity  $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$  entails  $v \wedge v = 0$  for all  $v$ . Conversely, in any characteristic the identity  $v \wedge v = 0$  entails  $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$  for all  $v_1, v_2 \in V$ . In other words, the subspace  $Z_2$  of  $Z$  generated by  $Z_0$  and all elements of the form  $v \otimes v$  contains all combinations  $v_1 \otimes v_2 + v_2 \otimes v_1$ . Proof:

$$v_1 \otimes v_2 + v_2 \otimes v_1 = (v_1 + v_2) \otimes (v_1 + v_2) - (v_1 \otimes v_1) - (v_2 \otimes v_2) - B,$$

where  $B \in Z_0$  (why?). So, we actually define  $\wedge^2 V$  to be  $Z/Z_2$ ; this is identical with the  $-1$  eigenspace of  $s$  when  $2 \neq 0$ , and does what we want it to even when  $2 = 0$ . If  $V$  has finite dimension  $n$  then  $\wedge^2 V$  has dimension  $(n^2 - n)/2$ , and if  $V$  has basis  $\{v_i\}_{i \in I}$  for some totally ordered index set  $I$  then  $\wedge^2 V$  has basis  $\{v_i \wedge v_j : i < j\}$ . We will later define higher alternating powers  $\wedge^k V$  of dimension  $\binom{n}{k}$  (so  $\wedge^k$  will correspond to unordered  $k$ -tuples under our Slogan). The key ingredient is the existence of the sign homomorphism from the group of permutations of  $\{1, 2, \dots, k\}$  to the two-element group  $\{\pm 1\}$ .

If we apply the  $\text{Sym}^k$  construction to the space  $V^*$  of linear functionals on  $V$ , we obtain the space of homogeneous polynomial functions of degree  $k$  from  $V$  to  $F$ . For instance, a *symmetric* bilinear form on  $V$  is an element of  $\text{Sym}^2 V^*$ . Likewise  $\wedge^2 V^*$  consists of the alternating (a.k.a. antisymmetric) bilinear forms on  $V$ .