Math 55b Midterm Exam

Feb. 16–19, 2021 – due by end of day (US Eastern) on Friday February 19.

No collaboration allowed; no materials other than lecture notes and Munkres.

Directions

1. This take-home exam is **due by Friday February 19** (end of the day), online on Canvas. You are welcome to write your answers on paper and upload a scan or photos; LaTeX strictly optional. The exam is intended to take less than 3 hours, but your experience may vary.

2. In your solutions, you may refer to any of the results seen in class or in the relevant parts of the book. (You do not need to cite the exact source unless it's an obscure fact). Please show your reasoning (enough so that I can tell how you arrived at the answer). The material covered is what we've seen in class up to Lecture 7 (February 10) included; equivalently: Munkres sections 12–29 (except 22).

3. **You may not use any external sources on this exam other than the course materials (class notes and Munkres).** You may not use any other textbooks or external websites. You may not discuss the contents of this midterm with anyone, including in office hours (exception: check with Prof. Auroux via e-mail if a clarification is necessary) until after the due date, *even if you have turned it in.*

4. **On your exam, please include the following statement, with your signature.** (If you are typing the exam, no need to print and sign: copy the statement and type your name.)

"I affirm my awareness of the standards of the Harvard College Honor Code. While completing this exam, I have not consulted any external sources other than class notes and the textbook (Munkres). I have not discussed the problems or solutions of this exam with anyone, and will not discuss them until after the due date."

Signed: __________________

There are three problems. For multi-part problems, each question can be attempted independently.

Problem 1. (12 points)

Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) If $A_i \subset X_i$ are closed subsets for all $i \in I$, then $\prod_{i\in I} A_i$ is a closed subset of $\prod_{i\in I} X_i$ with the product topology.

(b) If $x_1, x_2, \dots \in X$ are limit points of a subset $A \subset X$, and if the sequence x_n converges to a limit $x \in X$, then x is a limit point of A.

(c) If X is Hausdorff, and $A \subset X$ is connected, then its boundary $\partial A = \overline{A} - \mathrm{int}(A)$ is connected.

(d) If X is Hausdorff, and $A \subset X$ is compact, then its boundary $\partial A = \overline{A} - \mathrm{int}(A)$ is compact.

(e) $[0,1] \subset \mathbb{R}_\ell$ with the lower limit topology (generated by the basis $\{[a,b),\ a<b\}$) is compact.

(f) The addition map $f : \mathbb{R}_\ell \times \mathbb{R}_\ell \to \mathbb{R}_\ell$ defined by $f(x,y) = x+y$ is continuous (equipping $\mathbb{R}_\ell$ with the lower limit topology and $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with the product topology).

Problem 2. (6 points)

Let X be a topological space. In this problem (and only in this problem), we consider $Y = X \times X$ equipped with the topology defined by the basis $\mathcal{B} = \{U \times U \,|\, U \subset X \text{ open}\}$.

(a) Show that $\mathcal{B}$ is a basis, and that the topology on Y is coarser than the product topology.

(b) Show that the diagonal map $\Delta : X \to Y$ defined by $\Delta(x) = (x, x)$ is continuous.

(c) If X is Hausdorff, does it follow that Y is Hausdorff? If X is connected, does it follow that Y is connected? (for each statement, give a proof or a counterexample)

Problem 3. (7 points)

Let X, Y be topological spaces. The graph of $f : X \to Y$ is the subset $G_f = \{(x, f(x)) \,|\, x \in X\}$ of $X \times Y$.

(a) Show that if Y is Hausdorff and $f : X \to Y$ is continuous then its graph G_f is a closed subset of $X \times Y$ (with the product topology).

(b) Show that if Y is compact and Hausdorff, then the converse is true: if the graph G_f is closed in $X \times Y$ then f is continuous.

(Hint: given an open $V \subset Y$ and $x \in f^{-1}(V)$, show that the subset $\{x\} \times (Y - V)$ of $X \times Y$ can be covered by open subsets $U_i \times V_i$ which are disjoint from G_f, and use this to find a neighborhood U of x such that $U \times (Y - V)$ is disjoint from G_f.)

(c) Give an example showing that the result of (b) need not hold if Y is not compact.