Important announcements (made in class on Wednesday)

Course evaluations: please take a moment to fill course evaluations, they are the most important feedback you can give on your classes!

Final exam info: → exam will be posted on Canvas Monday 5/3, due by Wed. 5/12. It shouldn't take 10 days to complete.

→ allowed: lecture notes, Munkres + Rudin + Ahlfors, McMullen's notes. No other materials, no collaboration.

→ goal = test understanding of semester's material at large + basic problem-solving / proof writing skills — like the more straightforward homework problems, not the extra-hard ones.

(scope & length broader than the midterm, and perhaps slightly harder, but not meant to be insanely challenging.)

Review: → will post 3 videos / notes reviewing topology, real analysis, complex analysis
→ will hold office hours + available by Slack and email.
→ CAs' (esp. Richard's) office hours / review sessions.

Please try to catch up with any late homeworks etc. so the CAs can finish their grading jobs before the end of the semester! Psets not submitted by the time the CAs grade them won't necessarily get graded, and unless I am told otherwise by your Resident Dean I will assign final grades soon after the May 12 final due date.

Today's topic: special functions - Γ and ζ especially

This is another application of infinite sums and products: build new functions!

Warm-up: the partition generating function

Let $p(n)$ = number of partitions = # ways of expressing n as an (unordered) sum of positive integers. (by convention $p(0)=1$).

1 — $p(1)=1$

$2 = 1+1$ — $p(2)=2$

$3 = 2+1 = 1+1+1$ — $p(3)=3$

$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$ — $p(4)=5$ etc.

This has many remarkable properties, eg. arithmetic (Ramanujan: $p(5k+4)\equiv 0 \mod 5$ (!?!)) but our point here is rather to study the growth rate of $p(n)$: polynomial? exponential?

* One way to approach this is to introduce the generating function $P(z) = \sum_{n=0}^{\infty} p(n) z^n$ and ask about its properties (radius of convergence, etc.). The key formula for this is a

product expansion $P(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1-z^n}$. (Euler 1753)

To see this, write the product as $(1+z+z^2+\dots)(1+z^2+z^4+\dots)(1+z^3+z^6+\dots)\dots$

A partition of n as a sum of a_1 1's, a_2 2's, etc. corresponds to the contribution to the coeff^t of z^n that comes from multiplying z^{a_1} in the first factor, z^{2a_2} in the second, and so on. So the total coeff^t of z^n is indeed $p(n)$.

* This infinite product expansion, and comparison between $\sum \log(1-z^n)$ and $\sum z^n$, shows that $P(z)$ is well-defined and analytic in the unit disc $D = \{|z|<1\}$.

But we also see that, since the factors have poles at all roots of unity = a dense subset of the unit circle ($e^{2\pi i\alpha}, \alpha \in \mathbb{Q}$), there is no way to extend $P(z)$ beyond D.

This tells us the radius of convergence is 1, but in fact a <u>much</u> more detailed analysis of $P(z)$ yields more info ... $p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ (Hardy-Ramanujan! 1918)

<u>The Gamma function:</u>

<u>Q.1:</u> does there exist a meromorphic function that generalizes $n!$ beyond non-negative integers?

Since $n! = n \cdot (n-1)!$, the functional identity we'd hope for is $F(z) = z F(z-1)$.

This can't be a polynomial, though – comparing the zeros on both sides of this identity, we get that the zeros of $F(z)$ are those of $F(z-1)$ (ie. those of F shifted by 1) + one more at 0.

$\Rightarrow$ if F is an entire function, it must have zeros at all non-negative integers.

This isn't really consistent with wanting to generalize $n!$, though.

Better idea: we'll want a meromorphic function with poles at the negative integers (& no zeroes).

<u>Q.2:</u> is there an entire function whose zeros are exactly the negative integers?

Yes, we've seen how to do this! $G(z) = \prod_{n=1}^{\infty} \left(\left(1+\frac{z}{n}\right) e^{-z/n}\right)$ ← to achieve convergence of $\sum_{n\geq 1} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)$

Observe: $zG(z)G(-z) = \frac{1}{\pi}\sin(\pi z)$ by comparing to last time.

What functional equation does G satisfy? $G(z-1)$ has zeros at $z = 0, -1, -2, \dots$, same as $zG(z)$. Hence $\frac{G(z-1)}{zG(z)}$ is an entire function without poles or zeros $\Rightarrow$ it's $e^{\gamma(z)}$ for some entire function $\gamma(z)$.

So: $G(z-1) = zG(z)e^{\gamma(z)}$. What's $\gamma(z)$?

Take logarithmic derivative on both sides: $\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right)$

$\frac{G'(z-1)}{G(z-1)} = \frac{1}{z} + \frac{G'(z)}{G(z)} + \gamma'(z) \Rightarrow \gamma'(z) = 0 \Rightarrow \gamma(z) = \gamma$ = constant - <u>Euler's constant</u>.

$G(0) = 1 = G(1)e^{\gamma} \Rightarrow \gamma = -\log G(1) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right) = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n} - \log(n)\right) \simeq 0.57722\dots$

* To get rid of the e^{γ}, let $H(z)=e^{\gamma z}G(z)$, so $H(z-1)=e^{\gamma z}e^{-\gamma}G(z-1)=e^{\gamma z}zG(z)=zH(z)$.

(<u>Note:</u> $H(z)=e^{\gamma z}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)e^{-z/n}\right)=\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\left(1+\frac{1}{n}\right)^{-z}$)

* Finally, let $\Gamma(z)=\dfrac{1}{zH(z)}=\dfrac{1}{H(z-1)}$ - <u>Euler's Gamma function</u>.

<u>Properties</u>:
- Γ is a meromorphic function, with simple poles at $0,-1,-2,\dots$ and no zeroes.
- $\Gamma(z)=\dfrac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}=\frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}$
- $\Gamma(z+1)=z\Gamma(z)$ (since both $=\frac{1}{H(z)}$).
- since $\Gamma(1)=1$ from product expansion, this yields $\Gamma(n)=(n-1)!\ \forall n\in\mathbb{Z}_{>0}$.
- from $\pi z\,G(z)G(-z)=\sin(\pi z)$ we get $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$

and so on.

- Stirling's formula: $\Gamma(z)\sim\sqrt{2\pi}\,z^{z-\frac{1}{2}}e^{-z}$ (ie. ratio $\to 1$) for $Re(z)\to\infty$.
  ($\Rightarrow n!\sim\sqrt{2\pi n}\,n^{n}e^{-(n+1)}$, cf. HW7) (pf is painful, see Ahlfors §5.2.5)

- <u>the other formula</u>: $\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt$ for $Re(z)>0$

  Integration by parts shows (for $Re(z)>0$) $\int_0^{\infty}t^{z}e^{-t}\,dt=\big[\underbrace{-t^{z}e^{-t}}_{0}\big]_0^{\infty}+z\int_0^{\infty}t^{z-1}e^{-t}\,dt$

  so the integral satisfies the same identity as $\Gamma(z)$.

  The ratio of the two is 1-periodic, entire function; Stirling's formula allows one to show it's bounded, hence constant ($=1$ by comparing values at positive integers)

- there are many other wonderful formulas, eg Legendre duplication formula
  $\sqrt{\pi}\,\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ - see Ahlfors.

<u>The Riemann zeta function</u>: we've seen how to encode a sequence of numbers a_n into a generating function = power series $\sum a_n z^n$, but one can also try something different: the Dirichlet series $f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ (for traditional reasons the variable is denoted s not z).

Simplest of these: the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges absolutely for $Re(s)>1$, uniformly on $\{Re(s)\geq 1+\varepsilon\}\ \forall\varepsilon>0$

so this is an analytic function on $\{Re\ s>1\}$.

<u>Fact</u>: even though the series doesn't converge for $Re\ s<1$, the function $\zeta(s)$ can be extended to a meromorphic f^{n} on whole plane, with a pole at $s=1$.

The key questions about ζ concern its behavior in regions of the plane where the series diverges.

Number theoretic significance: $\zeta(s)=\prod_{p\text{ prime}}\dfrac{1}{1-p^{-s}}$ $\left(\frac{1}{1-p^{-s}}=1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right.$ + prime factorization$\left.\right)$.

Because of this representation, the behavior of $\zeta(s)$ as complex analytic function reflects properties of the primes.

Eg. the fact that $\sum \frac{1}{n}$ diverges $\longleftrightarrow$ pole of ζ at $s=1$

$\longleftrightarrow$ there are infinitely many primes p, and the series $\sum \log\left(\frac{1}{1-p^{-1}}\right) \sim \sum \frac{1}{p}$ diverges.

(cf. exercise on HW7 !)

But there are much deeper facts – the location of the zeros of ζ implies estimates on the error term in the classical approximation $\pi(x) = \#\{\text{primes } p \le x\} \sim \frac{x}{\log x} + \dots$ (lookup "Prime number theorem"). This is the subject of the Riemann hypothesis.

- Back to complex analysis: ζ is intricately related to Γ, because:

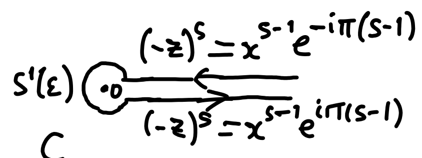
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt \quad \rightsquigarrow \quad n^{-s}\Gamma(s) = \int_0^\infty t^{s-1} e^{-nt}\,dt$$

change of variables to nt
$t^{z-1}dt \to n^z t^{z-1}\,dt$

Summing over $n \ge 1$, we get for $Re(s) > 1$: $\quad \zeta(s)\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt$

since $\sum_{n=1}^\infty e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}$

This allows us to re-express $\zeta(s)$ as a path integral: $\frac{(-z)^{s-1}}{e^z - 1}$ has branching behavior at $z=0$ (and poles at $2\pi i n$, $n \in \mathbb{Z}$)



$$\int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz = -\int_0^\infty \frac{x^{s-1}e^{-i\pi(s-1)}}{e^x - 1}\,dx + \int_0^\infty \frac{x^{s-1}e^{i\pi(s-1)}}{e^x - 1}\,dx$$

(Cauchy $\Rightarrow \int_C$ indep. of $\varepsilon \in (0, 2\pi)$, and $Re(s) > 1 \Rightarrow \lim_{\varepsilon \to 0} \int_{S^1(\varepsilon)} = 0$)

So $\int_C \frac{(-z)^{s-1}dz}{e^z - 1} = 2i \sin(\pi(s-1))\,\zeta(s)\Gamma(s)$. Since $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$,

this yields: $$\boxed{\zeta(s) = -\frac{\Gamma(1-s)}{2i\pi}\int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz} \quad (*)$$

The point is: the right-hand side is defined and meromorphic $\forall s \in \mathbb{C}$!
($\int$: convergence at ∞ ok because e^z in denominator $\gg |z|^{s-1}$; analytic dependence on s follows from our usual tricks for integral formulas – "differentiation under $\int$").

Since $\Gamma(1-s)$ has poles at $1-s \in \{0,-1,-2,\dots\}$ ie. $s = \{1,2,3,\dots\}$, the only possible poles of $\zeta(s)$ are at $s = 1,2,3,\dots$ but for $s \ge 2$ manifestly $\zeta(s) = \sum \frac{1}{n^s}$ converges, so the pole of $\Gamma(1-s)$ is cancelled by the vanishing of $\int_C$ (no branching behavior for $s \in \mathbb{Z}$!).

$\Rightarrow$ <u>Corollary</u>: $\zeta(s)$ extends to an entire meromorphic function, whose only pole is a simple pole at $s=1$.

Further consideration of the integral formula (*) yields the "functional equation" for $\zeta(s)$: ⑤

Thm. $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$.

(This is proved by further manipulation of the integral (*), and closing the path in $\mathbb{C}$. see Ahlfors §5.4.3)

This is important: we know that $\zeta(s)$ has no zeroes in the halfplane $Re(s)>1$ (as seen from the product expansion $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$, which converges for $Re(s)>1$), so this equation determines the behavior of ζ in the halfplane $Re(s)<0$: namely it has simple zeroes at $s=-2,-4,-6,\dots$ and no other zeroes.

The remaining zeroes are in the "critical strip" $0<Re\, s<1$; the Riemann hypothesis states that these all lie on the line $Re(s)=\frac{1}{2}$. This has been verified experimentally for the first few million zeroes (starting with $\frac{1}{2} \pm 14.134725\dots i$, $\frac{1}{2} \pm 21.022039\dots i$, etc.), and is widely believed to be true (which has implications for the distribution of prime numbers), but a proof remains out of reach. (The Clay Math Institute offers \$1M for a proof or disproof.)