Math 55a Review – Part 1: Linear algebra

- A **field** $(k,+,\times)$ is a set with two operations: $(k,+)$ abelian group with identity 0, $(k^* = k - \{0\}, \times)$ abelian group with identity 1; distributive law $a(b+c) = ab+ac$.

Lec. 6 Axler ch. 1 — Examples: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (characteristic 0: $\underbrace{1+\dots+1}_{n} = n.1 \neq 0$), $\mathbb{F}_p = \mathbb{Z}/p$ p prime (char. $= p$).

- A **vector space** over k is a set V with addition $+: V\times V \to V$ $(V,+)$ abelian group, $0 \in V$; scalar mult. $k \times V \to V$ associative, distributive.

Ex: k^n, $k[x]$, ... **Subspace**: $W \subset V$ closed under $+, \times$.

- $\mathrm{span}(v_1,\dots,v_n) = \{\sum a_i v_i \,/\, a_i \in k\} \subset V$, say (v_i) span V if $\mathrm{span}(v_i) = V$.

Axler ch. 2 — Say (v_i) are linearly independent if $a_1 v_1 + \dots + a_n v_n = 0 \Rightarrow a_i = 0\ \forall i$.

Basis = linearly independent vectors which span V: $k^n \to V$ isomorphism. $(a_i) \mapsto \sum a_i v_i$

- All bases of V have same cardinality $= \underline{\dim V}$

Lec. 7 — Any linearly independent set can be completed to a basis.

- $\mathrm{Hom}(V,W)$ = linear maps $\varphi: V \to W$, $\varphi(u+v) = \varphi(u) + \varphi(v)$, $\varphi(\lambda u) = \lambda \varphi(u)$.

This is a vector space.

- Axler ch. 3 — given bases $(v_j)_{i=1..n}$ of V, $(w_i)_{i=1..m}$ of W, represent $v = \sum x_j v_j \in V$ by column $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\varphi \in \mathrm{Hom}(V,W)$ by matrix $A = (a_{ij})$ whose columns represent $\varphi(v_j)$ in basis (w_i), $\varphi(v_j) = \sum_i a_{ij} w_i$.

Then $\varphi(v)$ is represented in basis (w_i) by column vector $Y = AX$.

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \text{basis} \uparrow \simeq & & \uparrow \simeq \text{basis} \\ k^n & \xrightarrow{A} & k^m \end{array}$$

Lec. 8 — Change of basis: $P = (p_{ij}) = \mathcal{M}(id, (v_i'), (v_i))$ ie. $v_j' = \sum p_{ij} v_i$, then for $\varphi: V \to V$, $\mathcal{M}(\varphi, (v_i')) = A' = P^{-1}AP$

- $V \simeq W_1 \oplus \dots \oplus W_n$ direct sum decomp.^{n} if $\begin{cases} W_i \text{ span } V: \forall v \in V\ \exists w_i \in W_i \text{ st. } v = w_1 + \dots + w_n \\ W_i \text{ independent}: w_1 + \dots + w_n = 0,\ w_i \in W_i \Rightarrow w_i = 0\ \forall i. \end{cases}$

ie. $\varphi: \oplus W_i \to V$ is an isomorphism.
$(w_i) \mapsto \sum w_i$

- V finite dim. $\Rightarrow$ $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = \{0\}$ and $\dim W_1 + \dim W_2 = \dim V$.
- **dim/rank formula**: V, W finite dim., $\varphi \in \mathrm{Hom}(V,W) \Rightarrow \dim V = \dim \ker \varphi + \underbrace{\dim \mathrm{Im}\, \varphi}_{= \mathrm{rank}(\varphi)}$.
- $\exists$ bases (v_i) of V, (w_j) of W st. $\mathcal{M}(\varphi) = \left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)$ $\}\ \mathrm{Im}\,\varphi$, $\ker\varphi$

- **Dual**: $V^* = \mathrm{Hom}(V,k)$.

(e_i) basis of V (finite dim) $\Rightarrow$ dual basis (e_i^*) of V^* st. $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else.} \end{cases}$

$V \to V^{**}$, $v \mapsto ev_v : \begin{pmatrix} V^* \to k \\ \ell \mapsto \ell(v) \end{pmatrix}$ is an isomorphism if $\dim V < \infty$ (injective if $\dim V = \infty$).

The annihilator of $U \subset V$ is $\mathrm{Ann}(U) = \{\ell \in V^* / \ell(u) = 0\ \forall u \in U\}$; $\dim \mathrm{Ann}(U) = n - \dim U$.

Lec. 9 — The **transpose** of $\varphi \in \mathrm{Hom}(V,W)$ is $\varphi^t : W^* \to V^*$, $\varphi^t(\ell) = \ell \circ \varphi$

$\ker \varphi^t = \mathrm{Ann}(\mathrm{Im}\,\varphi)$, $\mathrm{Im}\,\varphi^t = \mathrm{Ann}(\ker \varphi)$ if $\dim < \infty$, $\mathcal{M}(\varphi^t, (f_j^*), (e_i^*)) = \mathcal{M}(\varphi)^T$

- <u>Quotient</u>: $U \subset V$ subspace $\Rightarrow V/U = \{\text{cosets } v+U\}$ is a vector space.
 $V \xrightarrow{q} V/U$, $v \mapsto v+U$ is surjective with kernel $= U$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "q"'] & W \\ V/U \arrow[ur, dotted, "\exists \bar\varphi"'] & \end{tikzcd}$$

factors through V/U iff $U \subset \ker\varphi$.

Axler ch. 5

Lec. 10

- $W \subset V$ is an <u>invariant subspace</u> for $\varphi \in \mathrm{Hom}(V,V)$ if $\varphi(W) \subset W$.
 Ex. $\ker(\varphi)$, $\mathrm{Im}(\varphi)$; <u>eigenspaces</u> $\mathrm{Ker}(\varphi - \lambda I)$.
- if $V = \oplus V_i$, V_i invariant for $\varphi \Rightarrow \exists$ basis where $\mathcal{M}(\varphi) =$ block diagonal $\left(\begin{array}{c|c} \varphi_{|V_1} & 0 \\ \hline 0 & \varphi_{|V_2} \end{array}\right)$
 A basis of <u>eigenvectors</u> $v_i \in V$, $v_i \neq 0$, $\varphi(v_i) = \lambda_i v_i \iff \varphi$ <u>diagonalizable</u>
 $\mathcal{M}(\varphi,(v_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
- Eigenvectors of φ for distinct eigenvalues are linearly indep^{t}.
- If k is algebraically closed (eg. $\mathbb{C}$) then any linear op. $\varphi \in \mathrm{Hom}(V,V)$ has an eigenvector.
 Corollary: $\exists$ basis in which $\mathcal{M}(\varphi)$ is upper triangular $\begin{pmatrix} * & & /// \\ & \ddots & \\ 0 & & * \end{pmatrix}$
 $\lambda \in k$ is an eigenvalue of $\varphi \iff (\varphi - \lambda)$ not invertible $\iff \lambda$ appears on diagonal in a triangular matrix representing φ.

Axler ch. 8

Lec. 11

- The generalized kernel $\mathrm{gKer}(\varphi) = \mathrm{Ker}(\varphi^N)$ $\forall N$ large (eg. $\geq \dim V$).
 φ is <u>nilpotent</u> if $\varphi^N = 0$; $\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(\varphi^2) \subset \dots$ $\exists$ basis st. $\mathcal{M}(\varphi)$ block diagonal with blocks $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$
- generalized eigenspaces $V_\lambda = \mathrm{gKer}(\varphi - \lambda) = \mathrm{Ker}(\varphi-\lambda)^N$ are linearly independent invariant subspaces.
- if k is alg. closed then $V =$ direct sum $\oplus V_\lambda$ of the gen^{d} eigenspaces of φ.
 This gives the <u>Jordan normal form</u> : $\mathcal{M}(\varphi)$ block diagonal with blocks $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$
 (φ diagonalizable $\iff$ all blocks have size 1).
- characteristic polynomial of φ: $\chi_\varphi(x) = \det(xI - \varphi) = \prod_{\lambda_i} (x-\lambda_i)^{n_i}$, $n_i = \mathrm{mult}(\lambda_i) = \dim V_{\lambda_i}$.
 minimal polynomial: $\mu_\varphi(x) = \prod (x-\lambda_i)^{m_i}$, $m_i = \min\{m \,/\, V_{\lambda_i} = \mathrm{Ker}(\varphi-\lambda_i)^m\} =$ size of largest Jordan block in V_{λ_i}.
 - $p(\varphi) = 0$ iff $\mu_\varphi \mid p(x)$. In particular $\mu_\varphi \mid \chi_\varphi$.
 φ diagonalizable $\iff m_i = 1$ $\forall i$.

Lec. 12

Axler ch. 9A

- Over $\mathbb{R}$, $\varphi: V \to V$ need not have eigenvectors, but by considering $V_{\mathbb{C}} = V \times V = \{v + iw \,/\, v,w \in V\}$ and $\varphi_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}}$, $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w)$ $\Rightarrow$ any real operator has an invariant subspace of dimension 1 (eigenvector!) or 2.

Handout

- <u>Categories</u> have objects, and morphisms $\mathrm{Mor}(A,B)$ $\forall A,B \in \mathrm{ob}\,\mathcal{C}$, with operation = composition.
 Axioms: $\forall A \in \mathrm{ob}\,\mathcal{C}$, $\exists\, id_A \in \mathrm{Mor}(A,A)$, $f \circ id_A = id_B \circ f = f$; associativity $(f\circ g)\circ h = f \circ (g \circ h)$.
 <u>Ex</u>: sets, groups, vector spaces/k

Lec. 13

- A <u>functor</u> $F: \mathcal{C} \to \mathcal{D}$ assigns
 - to each $X \in \mathrm{ob}\,\mathcal{C}$, $F(X) \in \mathrm{ob}\,\mathcal{D}$
 - to $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$, $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X),F(Y))$

 st. $F(id_X) = id_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. (contravariant functors = reverse dir^{n} of morphisms)
- Natural transformation t between functors $F, G: \mathcal{C} \to \mathcal{D}$
 = for each $X \in \mathrm{ob}\,\mathcal{C}$, $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$ st. $\forall \begin{smallmatrix} X \\ \downarrow f \\ Y \end{smallmatrix}$, $\begin{tikzcd} F(X) \arrow[r, "t_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "t_Y"] & G(Y) \end{tikzcd}$ commutes.

Axler ch.6
Lec. 13

- A bilinear form on V is $b: V\times V \to k$, linear in each input $b(u+v,w) = b(u,w)+b(v,w)$, $b(\lambda u, v) = \lambda b(u,v)$ etc.
  b is symmetric if $b(u,v) = b(v,u)$, skew-symmetric if $b(u,v) = -b(v,u)$.
- $B(V) = \{\text{bilinear } b: V\times V \to k\} \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$ (isom. of vector spaces)
  $b \longmapsto \varphi_b : v \longmapsto (b(v,\cdot): V\to k)$
  $\mathrm{rank}(b) = \mathrm{rank}(\varphi_b)$, b is nondegenerate if $\varphi_b : V \xrightarrow{\sim} V^*$ isomorphism.
- in a basis (e_i) of V, b is represented by a matrix $B = (b_{ij}) = (b(e_i, e_j))$.
  if $u = \sum x_i e_i$, $v = \sum y_i e_i$ are represented by column vectors X, Y, $b(u,v) = X^T B Y$.
- the orthogonal of $S \subset V$ for b is $S^\perp = \{v\in V / b(v,w) = 0 \ \forall w\in S\} = \mathrm{Ker}(V \to S^*)$, $v \mapsto \varphi_b(v)|_S$
  If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$
  If b is an inner product then $S \cap S^\perp = \{0\}$ and $V = S \oplus S^\perp$.

Lec. 14

- A real inner product $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$ is a symmetric definite positive bilinear form.
  $\hookrightarrow \langle u,u\rangle = \|u\|^2 > 0 \ \forall u \neq 0$.
  Cauchy-Schwarz ineq: $\langle u,v\rangle \le \|u\|\,\|v\|$.

Over $\mathbb{C}$, we consider Hermitian inner products $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{C}$, not quite bilinear: $\langle \lambda u, v\rangle = \bar\lambda \langle u,v\rangle$
require Hermitian-symmetric $\langle v,u\rangle = \overline{\langle u,v\rangle}$, and definite positive $\langle u,u\rangle = \|u\|^2 > 0 \ \forall u\neq 0$.
The map $V \to V^*$ induced by such $\langle\cdot,\cdot\rangle$ is $\mathbb{C}$-antilinear: $\varphi(\lambda u) = \bar\lambda \varphi(u)$.

- Every finite dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) has an orthonormal basis $(e_1, \dots, e_n)$ st. $\langle e_i, e_j\rangle = \delta_{ij}$. (build by induction eg. using Gram-Schmidt).

Axler ch. 7

- Let V, $\langle\cdot,\cdot\rangle$ inner product space (over $\mathbb{R}$ or $\mathbb{C}$), $T: V\to V$ linear operator.
  The adjoint operator $T^*: V\to V$ satisfies $\langle v, Tw\rangle = \langle T^* v, w\rangle \ \forall v,w\in V$.
  (corresponds to the transpose of T via $V \underset{\varphi}{\xrightarrow{\sim}} V^*$; over $\mathbb{C}$: complex conjugate of T^t).
  In an orthonormal basis, $\mathcal{M}(T^*) = \mathcal{M}(T)^t$ (real case) or $\overline{\mathcal{M}(T)}^t$ (complex Hermitian case)
  $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^\perp$ and vice-versa.
- $T: V\to V$ is self-adjoint if $T^* = T$
  T is orthogonal (unitary over $\mathbb{C}$) if $T^* = T^{-1}$, ie. $\langle Tu, Tv\rangle = \langle u,v\rangle \ \forall u,v\in V$.
  ($\Leftrightarrow$ T maps orthonormal basis to orthonormal basis)
- If $S\subset V$ is invariant under a self-adjoint/orthogonal/unitary operator then so is $S^\perp$.
  $\Rightarrow$ spectral theorem (real and complex versions):

Lec. 15
Lec. 16

- If $T: V\to V$ is self adjoint then T is diagonalizable, with real eigenvalues, and can be diagonalized in an orthonormal basis.
- If $T: V\to V$ is orthogonal for a real inner product, then V is a direct sum of orthogonal invariant subspaces of dim. 1 or 2, with T acting by ± 1 on the 1-dim! pieces, rotations on 2-dim! pieces.
- If $T: V\to V$ is unitary for a Hermitian inner product, then T is diagonalizable in an orthonormal basis, with eigenvalues $|\lambda_i| = 1$.
  More generally if $T: V\to V$ (Hermitian) is normal ie. $TT^* = T^*T \Rightarrow T$ is diagonalizable in an orthonormal basis.

- Besides inner products, one can also consider arbitrary nondegenerate symmetric bilinear forms (without assuming positivity); eg. over $\mathbb{R}$ (resp. $\mathbb{C}$), $\exists$ orthogonal basis st.
$$b(e_i,e_j)=\begin{cases}\pm 1 & i=j\\ 0 & i\neq j\end{cases}\quad (\text{resp. } b(e_i,e_j)=\delta_{ij});$$ or skew-symmetric bilinear forms.

Handout Lec. 17

- <u>Tensor product</u>: $V\otimes W$ vector space, with a bilinear map $\begin{array}{l}V\times W\to V\otimes W\\ (v,w)\mapsto v\otimes w\end{array}$, st.
bilinear maps $V\times W\xrightarrow[b]{} U$ correspond to linear maps $V\otimes W\xrightarrow[\varphi]{} U$ $(\varphi(v\otimes w)=b(v,w))$
Elements of $V\otimes W$ are finite linear combinations $\sum v_i\otimes w_i$
If (e_i) basis of V and (f_j) basis of W, then $(e_i\otimes f_j)$ basis of $V\otimes W$.
- $V^*\otimes W\cong \mathrm{Hom}(V,W)$, by mapping $\ell\otimes w\in V^*\otimes W$ to $(v\mapsto \ell(v)w)\in\mathrm{Hom}(V,W)$.
- the trace $tr(T:V\to V)=\sum\lambda_i\in k$ can be defined by $\mathrm{Hom}(V,V)\xrightarrow{\sim} V^*\otimes V\to k$, $\ell\otimes v\mapsto \ell(v)$

Lec. 18
- multilinear maps $V_1\times\ldots\times V_n\to U$ $\Leftrightarrow$ linear maps $V_1\otimes\ldots\otimes V_n\to U$.
- $V^{\otimes n}=V\otimes\ldots\otimes V$ contains subspaces
$\mathrm{Sym}^n(V)$ = symmetric tensors ($\leftrightarrow$ symmetric multilinear maps) $v_{\sigma(1)}\cdots v_{\sigma(n)}=v_1\cdots v_n$
$\Lambda^n(V)$ = exterior powers: alternating tensors $v_{\sigma(1)}\wedge\ldots\wedge v_{\sigma(n)}=(-1)^\sigma v_1\wedge\ldots\wedge v_n$.

Lec. 19
- if $\dim V=n$ then $\Lambda^n V$ has dim. 1; for $T:V\to V$, $\Lambda^n T:\Lambda^n V\to\Lambda^n V$ is multiplication by a scalar, the <u>determinant</u> $\det(T)\in k$.

- The theory of modules over a ring $(R,+,\times)$ (elements need not have multiplicative inverses) is more complicated than that of vector spaces.
Finitely generated modules need not have a basis; those that do are called <u>free</u>.
- $\mathbb{Z}$-modules $\Leftrightarrow$ abelian groups.

Lec. 20 (parts of Artin ch. 14)

Every finitely generated $\mathbb{Z}$-module M with generators $(e_1\ldots e_n)$ is a quotient of $\mathbb{Z}^n$ $\left(\varphi:\begin{array}{l}\mathbb{Z}^n\twoheadrightarrow M\\ (a_i)\mapsto\sum a_ie_i\end{array}\right)$ and $\ker(\varphi)\subset\mathbb{Z}^n$ is itself a free module, ie. $\exists\ T:\mathbb{Z}^m\to\mathbb{Z}^n$ st. $M\simeq\mathbb{Z}^n/\mathrm{Im}\,T$

$\rightsquigarrow$ via linear algebra over $\mathbb{Z}$, one finds:
Every finitely generated abelian group is $\simeq\mathbb{Z}^r\times\mathbb{Z}/n_1\times\ldots\times\mathbb{Z}/n_k$ for some $r, n_1\ldots n_k$.