Solutions to Homework 1

Math 55b

1. *Compute the indefinite integrals of* $\int \tan^{-1}(x)\,dx$.

The answer $\int \tan^{-1}(x)\,dx = x\,\tan^{-1}(x) - \frac{1}{2}\log(1+x^2) + C$ follows by integration by parts. For this, we first need to know the derivative of $\tan^{-1}(x)$; by differentiating $\tan \circ \tan^{-1} = \mathrm{id}$, it is found to be $1/(1+x^2)$. Integrating by parts then gives $\int \tan^{-1}(x)\,dx = x\,\tan^{-1}(x) - \int x\,d(\tan^{-1}(x)) = x\,\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx = x\,\tan^{-1}(x) - \frac{1}{2}\log(1+x^2) + C$. Everyone got this question. ∎

2. *Construct a continuous bijection* $f : (0,1] \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$.

The basic idea is to represent both rational intervals $(0,1] \cap \mathbb{Q}$ and $(0,1) \cap \mathbb{Q}$ as disjoint unions of an open rational interval and a semi-open rational interval, and to exhibit separate continuous bijections between the corresponding pairs of intervals; there are many ways to proceed with this construction. Specifically, choose an irrational number $a \in (0,1)$ and a rational number $b \in (0,1)$, and note that $(0,1] \cap \mathbb{Q} = (0,a) \cap \mathbb{Q} \bigcup (a,b] \cap \mathbb{Q}$ and $(0,1) \cap \mathbb{Q} = (0,b) \cap \mathbb{Q} \bigcup [0,1) \cap \mathbb{Q}$. It suffices then to construct continuous bijections $(0,a) \cap \mathbb{Q} \to (0,b) \cap \mathbb{Q}$ and $(a,1] \cap \mathbb{Q} \to [b,1) \cap \mathbb{Q}$; since $a \notin \mathbb{Q}$, these will jointly give a *continuous* bijection $(0,1] \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$.

One way to construct the required continuous bijections $(0,a) \cap \mathbb{Q} \to (0,b) \cap \mathbb{Q}$ and $(a,1] \cap \mathbb{Q} \to [b,1)$ is by noting that every monotonous bijection between two (rational) intervals is a *homeomorphism*, i.e. both it and its inverse are continuous; given this, it suffices to simply construct an increasing bijection $(0,a) \cap \mathbb{Q} \to (0,b) \cap \mathbb{Q}$ and a decreasing bijection $(a,1] \cap \mathbb{Q} \to [b,1) \cap \mathbb{Q}$. The following inductive argument provides such a construction. ∎

Cantor's Lemma. *Every totally ordered set* $(X,<)$ *which is countable, order-dense, and has no maximal or minimal element is isomorphic to* $\mathbb{Q}$ *with the usual ordering.*

Proof. Label $x_1, x_2, \ldots$ the elements of X, and label $q_1, q_2, \ldots$ the rational numbers. We need to construct a bijection $f : \mathbb{Q} \to X$ with the respects the orderings of X and $\mathbb{Q}$, which is to say that two rational numbers satisfy $q < r$ if and only if the corresponding elements of X

satisfy $f(q) < f(r)$. For the inductive construction, suppose f has already been defined on $\{q_1, \ldots, q_n\}$, and define $f(q_{n+1})$ to be the element $x_n \in X \setminus \{f(q_1), \ldots, f(q_n)\}$ *with the minimal available index* n and having the property that, for $k \leq n$, $q_k < q_{n+1}$ if and only if $f(q_k) < f(q_{n+1})$. First we need to verify that the construction is possible at every stage; this follows from the assumption that X is order-dense and has no maximal or minimal element. Second, we need to show that f is surjective; this is ensured by our systematic choice of element with x_n with minimal available index. Indeed, suppose to the contrary that some $x_k \in X$ is not in the image of the inductively constructed map f. For $n \geq n_0$ sufficiently large, all $f(q_n)$ have index $> k$, and x_k will lie on a unique minimal segment $(f(q_i), f(q_j))$ with $i, j \in \{1, \ldots, n_0\}$; considering the minimal $r > n_0$ for which q_r lies between q_i and q_j gives a contradiction with the inductive construction. Hence the surjectivity of f. ■

Remark. The general fact is that any two countable metric spaces with no isolated points are homeomorphic; this result goes by the name of Waclaw Sierpinski. ■

3. *Given distinct points $a, b \in \mathbb{R}^k$, show that the locus $S := \{x \mid |x - a| = 2|x - b|\}$ is a sphere. What is its center? Its radius?*

   For $x, a, b \in \mathbb{R}$, note the identity $4(x-b)^2 - (x-a)^2 = 3x^2 + 2(a-4b)x + 4b^2 - a^2 = 3 \cdot (x^2 + 2(a-4b)x/3 + (a-4b)^2/9) - (4a^2/3 - 8ab/3 + 4b^2/3) = 3\big(x - \frac{4b-a}{3}\big)^2 - \big(2a-2b\big)^2/3$.

   On to our problem in $\mathbb{R}^k$, note that the condition $|x - a| = 2|x - b|$ is equivalent to $\big(4|x-b|^2 - |x-a|^2\big)/3 = 0$; using the identity from the preceding paragraph, we may rewrite the latter condition equivalently as $\big|x - \frac{4b-a}{3}\big|^2 = \big|\frac{2a-2b}{3}\big|^2$, showing that S is the sphere with center $\frac{4b-a}{3} \in \mathbb{R}^k$ and radius $\big|\frac{2a-2b}{3}\big|$. ■

4. *Given $z, w \in \mathbb{C}$ with $z \neq 0$, give a definition of the multivalued function z^w. What are the possible values of $|i^i|$, were $i := \sqrt{-1}$?*

   The multivalued **logarithm function** log on the punctured plane $\mathbb{C} \setminus \{0\}$ is defined by $\log(z) := \log|z| + i\mathrm{Arg}(z) + 2\pi i\mathbb{Z}$, where $\mathrm{Arg}(z) := \tan^{-1}(\mathrm{Im}(z)/\mathrm{Re}(z))$ is the **complex argument function**, the angle that the abscess $y = 0$ forms with the ray joining the origin with z. The **exponential function** is defined on the entire complex plane $\mathbb{C}$ by the normally convergent power series $e^w := \sum_{n \geq 0} w^n/n!$. Given this, the multivalued function z^w is defined, for $z \neq 0$, to be $e^{w \log z}$.

In particular, since $\log(i) = \log(1) + i\text{Arg}(i) + 2\pi i\mathbb{Z} = \pi i/2 + 2\pi i\mathbb{Z}$, the possible values for i^i are $e^{-\pi/2+2\pi k}$, $k \in \mathbb{Z}$ (these are all positive real numbers). ∎