Homework 7

Math 55b
Due Tuesday, 17 Mar 2009.

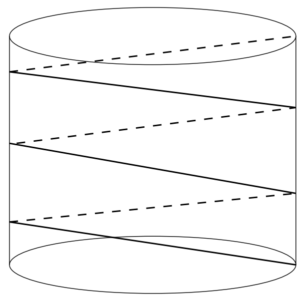

Figure 1. A path on a cylinder in $\mathbb{R}^3$.

1. Let X be a nonempty compact metric space, and suppose $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. (i) Prove that f has a unique fixed point. (ii) Give an example of such a map which is not a *strict* contraction (there is no $\lambda < 1$ such that $d(f(x), f(y)) < \lambda d(x, y)$).

2. Let $U = B(0, 1) \subset \mathbb{R}^n$ be the open unit ball, and let $f : U \to \mathbb{R}^n$ be the identity map. (i) Show that if $\|f - g\|_{C^1(U)}$ is small enough, then $g : U \to \mathbb{R}^n$ is injective. (ii) Give an example of a different open set U such that (i) fails.

3. Prove that $d(d(\omega)) = 0$ for any smooth k-form on $\mathbb{R}^n$.

4. (i) Find an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which sends the unit square S to the parallelogram P with vertices $(1, 1), (3, 2), (4, 5), (2, 4)$. What is $\det Df$?

   (ii) Let $\omega = \exp(x - y)\, dx\, dy$ on P. Compute $f^*\omega$.

   (iii) Compute $\int_P \omega$ using an integral over S.

5. Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$. Prove directly (without using Stokes' theorem) that if $\omega = f\, dx + g\, dy$ satisfies $dg/dx = df/dy$, then $\int_{\partial R} \omega = 0$.

6. (i) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth loop enclosing a region U (so that $\gamma(a) = \gamma(b)$). Using Stokes' theorem, prove that the area of U is given by $(1/2) \int_a^b \det(\gamma(t), \gamma'(t))\, dt$.

   (ii) Parameterize the curve given in polar coordinates by $r^2 = \cos 2\theta$, compute the area it encloses, and explain the answer.

7. Give a formula relating Euclidean (x, y, z) coordinates to spherical (r, θ, ϕ) coordinates on $\mathbb{R}^3$. (Here r is the length of the vector (x, y, z), ϕ measures its angle with the z-axis, and θ measure the angle between the vector $(x, y, 0)$ and the x-axis.) Use your formula to compute the Euclidean volume element $dx\, dy\, dz$ in spherical coordinates.

8. Find a 2-form ω on $\mathbb{R}^3$ such that $\int_{S^2} \omega$ computes the area of the unit sphere, and $\int_{B^3} d\omega$ computes the volume of the unit ball (up to a constant). Use Stokes' theorem to relate the two integrals and show $\text{area}(S^2) = 3\,\text{vol}(B^3)$. Generalize your proof to $\mathbb{R}^n$. Explain geometrically why this relationship should hold.

9. Let γ be the oriented path in $\mathbb{R}^3$ that connects $(1,0,0)$ to $(1,0,1)$ by spiraling three times around the surface of the cylinder $x^2 + y^2 = 1$ at a constant slope. (see Figure 1). Let $\omega = y \sin z\, dx + x \sin z\, dy + xy \cos z\, dz$. Compute $\int_\gamma \omega$.

10. Let $S \subset \mathbb{R}^3$ be the part of the hypersurface $x^4 + y^4 + z^4 = 1$ with $z \geq 0$. Give S a well-defined orientation, and then compute $\int_S \omega$ where

$$\omega = e^y z^2\, dx\, dy - e^y 2xz\, dy\, dz + \cos(z)\, dx\, dz.$$