THE INTEGRAL $\int_{0}^{\infty} \frac{\sin x}{x} dx.$

No one can possibly welcome more cordially than I do the widening of the school curriculum in mathematics that has taken place during the last twenty years. Even I can remember the days when 'Conics' and 'Mechanics' were the privilege of a select few, and only a prodigy was initiated into the mysteries of the Differential (much less the Integral) Calculus. All this is changed now, and it is fairly safe (at least that is my experience) to assume that a boy who has won a scholarship has learnt something about the Integral Calculus at school.

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I do not believe for a moment that the curriculum has been made harder. These happy results are not due to any striking development in the schoolboy himself, either in intelligence, or in capacity or willingness for work. They are merely the result of improvement in the methods of teaching and a wider range of knowledge in the teacher, which enables him to discriminate, in a way his predecessor could not, between what is really difficult and what is not. No one now supposes that it is easy to resolve sin x into factors and hard to differentiate it!—and even in my time it was the former of these two problems to which one was set first.

Still there are difficulties, even in the Integral Calculus; it is not every schoolboy who can do right everything that Todhunter and Williamson did wrong; and, when I find my pupils handling definite integrals with the assurance of Dr. Hobson or Mr. Bromwich, I must confess I sometimes wonder whether even the school curriculum cannot be made too wide. Seriously, I do not think that schoolboys ought to be taught anything about definite integrals. They really are difficult, almost as difficult as the factors of sin x: and it is to illustrate this point that I have written this note.

I have taken the particular integral

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{1}{2}\pi$$

as my text for two reasons. One is that it is about as simple an example as one can find of the definite integral proper—that is to say the integral whose value can be expressed in finite terms, although the indefinite integral of the subject of integration cannot be so determined. The other is that two interesting notes by Mr. Berry and Prof. Nanson have been published recently,* in which they discuss a considerable number of different ways of evaluating this particular integral. I propose now to apply a system of marking to the different proofs of the equation (1) to which they allude, with a view to rendering our estimate of their relative difficulty more

Practically all methods of evaluating any definite integral depend ultimately upon the inversion of two or more operations of procedure to a limit+—e.g. upon the integration of an infinite series or a differentiation under the integral sign: that is to say they involve 'double-limit problems.' The only exceptions to this statement of which I am aware (apart, of course, from cases in which the indefinite integral can be found) are certain cases in which the integral can be calculated directly from its definition as a sum, as can the integrals

$$\int_0^{\frac{1}{2}\pi} \log \sin \theta \, d\theta, \quad \int_0^{\pi} \log (1-2p\cos \theta+p^2) \, d\theta.$$

Such inversions, then, constitute what we may call the standard difficulty of the problem, and I shall base my system of marking primarily upon them. For every such inversion involved in the proof I shall assign 10 marks. Besides these marks I shall add, in a more capricious way, marks for artificiality, complexity, etc. The proof obtaining least marks is to be regarded as the simplest and best.

^{*} A. Berry, Messenger of Mathematics, vol. 37, p. 61; E. J. Nanson, ibid. p. 113.

⁺ Whether this is true of a proof by 'contour integration' depends logically on what proof of Cauchy's Theorem has been used (see my remarks under 3 below).

[‡] For an explanation of the general nature of a 'double-limit problem' see my Course of Pure Mathematics (App. 2, p. 420); for examples of the working out of such problems see Mr. Bromwich's Infinite Series (passim, but especially App. 3, p. 414 et seq.).

1. Mr. Berry's first proof.* This is that expressed by the series of equations

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \lim_{a \to 0} \left(e^{-ax} \frac{\sin x}{x} \right) dx = \lim_{a \to 0} \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$$

$$= \lim_{a \to 0} \int_0^\infty e^{-ax} dx \int_0^1 \cos tx dt = \lim_{a \to 0} \int_0^1 dt \int_0^\infty e^{-ax} \cos tx dx$$

$$= \lim_{a \to 0} \int_0^1 \frac{a dt}{a^2 + t^2} = \lim_{a \to 0} \arctan\left(\frac{1}{a}\right) = \frac{1}{2}\pi.$$

This is a difficult proof, theoretically. Each of the first two lines involves, on the face of it, an inversion of limits. In reality each involves more: for integration, with an infinite limit, is in itself an operation of repeated procedure to a limit; and we ought really to write

$$\int_0^\infty \lim_{a \to 0} = \lim_{X \to \infty} \int_0^X \lim_{a \to 0} = \lim_{X \to \infty} \lim_{a \to 0} \int_0^X = \lim_{a \to 0} \lim_{X \to \infty} \int_0^X = \lim_{a \to 0} \int_0^\infty.$$

Thus two inversions are involved in the first line; and so also in the second. This involves 40 marks. On the other hand this proof is, apart from theoretical difficulties, simple and natural: I do not think it is necessary to add more marks. Thus our total is 40.

2. Mr. Berry's second proof. This is expressed by the equations

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \sum_{i=-\infty}^{\infty} \int_{i\pi}^{(i+1)\pi} \frac{\sin x}{x} dx$$

$$= \frac{1}{2} \sum_{-\infty}^{\infty} (-1)^{i} \int_{0}^{\pi} \frac{\sin x}{x - i\pi} dx = \frac{1}{2} \int_{0}^{\pi} \sin x \sum_{-\infty}^{\infty} \frac{(-1)^{i}}{x + i\pi} dx$$

$$= \frac{1}{2} \int_{0}^{\pi} \sin x \csc x dx = \frac{1}{2} \pi.$$

It is obvious that this has a great advantage over the first proof in that only one inversion is involved. On the other hand the uniform convergence of the series to be integrated is, as Mr. Berry remarks, 'not quite obvious.' Moreover the fact that the cosecant series contains the terms

$$\frac{1}{x} - \frac{1}{x - \pi}$$

which become infinite at the limits, although it does not really add to the difficulty of the proof, does involve a slight amount of additional care in its statement. I am inclined to assign 15 marks, therefore, instead of 10 only, for this inversion. For the first three steps in the proof I assign 3 marks each; for the use of the cosecant series 4. Thus the total mark may be estimated at 9+15+4=28.

3. Mr. Berry's third proof (by contour integration). It is naturally a little harder to estimate the difficulties of a proof which depends upon the theory of analytic functions. It seems to me not unreasonable to assign 10 marks for the use of Cauchy's theorem in its simplest form, when we remember that the proof of the theorem which depends upon the formula †

$$\iint \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx dy = \iint (p dx + q dy)$$

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$$\int_0^\infty \frac{\sin x}{x} dx$$
.

involves the reduction of a double integral partly by integration with respect to x and partly by integration with respect to y. The actual theoretical difficulty of this proof of Cauchy's theorem I should be disposed to estimate at about 20; but half that amount seems sufficient for the mere use of a well known standard theorem.*

The particular problem of contour integration with which we are concerned is, however, by no means a really simple one. The range of integration is infinite, the pole occurs on the contour; we have to use the fact that

$$\lim_{R=\infty} \lim_{\epsilon=0} \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right) \frac{e^{ix}}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx.$$

For this I assign 8 marks; for the evaluation of the limit of the integral round the 'small semicircle' another 8. There remains the proof that the integral round the 'large semicircle' tends to zero. As Mr. Berry points out, the ordinary proof of this is really rather difficult; I estimate its difficulty at 16. Mr. Berry has suggested a simplification of this part of the argument; the difficulty of his proof I estimate at 8. Thus we have 10+8+8+16=42 marks for the ordinary proof, and 10+8+8+8=34 for Mr. Berry's modification of it.

4. Prof. Nanson's proof. This proof, I must confess, does not appeal to me. Starting from the integral

$$u = \int_0^\infty \frac{a \cos cx}{a^2 + x^2} dx,$$

where α and c are positive, we show, by a repeated integration by parts, and a repeated differentiation under the integral sign, that u satisfies the equation

$$\frac{d^2u}{da^2} = c^2u.$$

As the upper limit is infinite we ought to assign 40 marks for this, following the principles we adopted in marking the first proof; but as the two operations of differentiation involve inversions of the same character I reduce this to 30, adding 6, however, for the repeated integration by parts.

The transformation x=ay shows that u is a function of ac, so that $u=Ae^{ac}+Be^{-ac}$, where A and B are independent of both a or c. This step in the proof I assess at 4.

That A=0 is proved by making c tend to ∞ and observing that

$$|u| < \int_0^\infty \frac{a dx}{\alpha^2 + x^2} = \frac{1}{2}\pi.$$

For this I assign 6 marks.

That $c=\frac{1}{2}\pi$ is proved by making c=0. Here we assume the continuity of the integral, and this should involve 20 marks. But as the proof is simple I reduce this to 12. Thus by the time that we have proved that

$$u = \frac{1}{2}\pi c^{-ca},$$

we have incurred 30+6+4+6+12=58 marks.

We have then

$$\int_{0}^{\infty} \frac{a \sin mx \, dx}{x(a^{2} + x^{2})} = \int_{0}^{\infty} \frac{a \, dx}{a^{2} + x^{2}} \int_{0}^{m} \cos cx \, dc$$
$$= \int_{0}^{m} u \, dc = \frac{\pi}{2a} (1 - e^{-am}).$$

^{*}I mean, of course, the first proof mentioned by Mr. Berry—the proof itself is

⁺ Forsyth, Theory of Functions, p. 23.

^{*}Goursat's proof is better and more general, and does not involve my inversion of limit-operations, but its difficulties are far too delicate for beginners.

This involves 20 marks. We then obtain

$$\int_0^\infty \frac{x \sin mx}{a^2 + x^2} \, dx = \frac{1}{2} \pi e^{-ax}$$

by two differentiations with respect to m. This should strictly involve 40, and I cannot reduce the number to less than 30, as the second differentiation is none too easy to justify. The result then follows by multiplying the last formula but one by α and adding it to the second.

I do not think that the 58+20+30=108 marks which we have assigned are more than the difficulties of the proof deserve. But of course it is hardly fair to contrast this heavy mark with the 40, 28, 42, 36 which we have obtained for the others; for Prof. Nanson has evaluated the integrals

$$\int_0^\infty \frac{\sin mx}{x(\alpha^2+x^2)} dx, \int_0^\infty \frac{\cos mx}{\alpha^2+x^2} dx, \int_0^\infty \frac{x\sin mx}{\alpha^2+x^2} dx$$

as well as the integral (1).

5. Mr. Michell's proof.* This depends on the equations

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \lim_{h \to 0, H \to \infty} \int_{h}^{H} \frac{\sin x}{x} dx$$

$$= -\lim_{h \to 0, H \to \infty} \int_{h}^{H} dx \int_{0}^{\frac{1}{2}\pi} \frac{d}{dx} \left\{ e^{-x \sin \theta} \cos(x \cos \theta) \right\} d\theta$$

$$= -\lim_{h \to 0, H \to \infty} \int_{0}^{\frac{1}{2}\pi} d\theta \int_{h}^{H} \frac{d}{dx} \left\{ e^{-x \sin \theta} \cos(x \cos \theta) \right\} dx$$

$$= -\lim_{h \to 0, H \to \infty} \int_{0}^{\frac{1}{2}\pi} \left\{ e^{-H \sin \theta} \cos(H \cos \theta) - e^{-h \sin \theta} \cos(h \cos \theta) \right\} d\theta$$

$$= -(0 - \frac{1}{2}\pi) = \frac{1}{2}\pi.$$

I have myself for several years used a proof, in teaching, which is in principle substantially the same as the above, though slightly more simple in details and arrangement, viz.:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2i} \int_0^\infty dx \int_0^\pi e^{i(t+xe^{it})} dt$$

$$= \frac{1}{2i} \int_0^\pi e^{it} dt \int_0^\infty e^{ixe^{it}} dx$$

$$= \frac{1}{2} \int_0^\pi dt = \frac{1}{2}\pi.$$

The successive steps of the proof can of course be stated in a form free from *i* by merely taking the real part of the integrand. To justify the inversion we have to observe that

$$\int_0^{\pi} e^{it} dt \int_0^X e^{ixe^{it}} dx = \int_0^X dx \int_0^{\pi} e^{it+ixe^{it}} dt$$

for any positive value of X, in virtue of the continuity of the subject of integration, and that

$$\left| \int_{0}^{\pi} e^{it} dt \int_{X}^{\infty} e^{ixe^{it}} dx \right| = \left| \int_{0}^{\pi} e^{iXe^{it}} dt \right| < \int_{0}^{\pi} e^{-X\sin t} dt$$

$$< 2 \int_{0}^{\frac{1}{2}\pi} e^{-X\sin t} dt < 2 \int_{0}^{\frac{1}{2}\pi} e^{-2Xt/\pi} dt$$

$$< \pi (1 - e^{-X})/X,$$

the closing piece of argument being essentially the same as that at the end of Prof. Nanson's paper. Hence

$$\int_0^{\pi} \int_0^{\infty} = \lim_{X \to \infty} \int_0^{\pi} \int_0^X = \lim_{X \to \infty} \int_0^X \int_0^{\pi} = \int_0^{\infty} \int_0^{\pi}.$$

This proof involves an inversion of a repeated integral, one limit being infinite. This involves 20 marks. To this I add 15 on account of the artificiality of the initial transformation, deducting 3 on account of the extreme shortness and elegance of the subsequent work. Thus we obtain 20+15-3=32 marks. To Mr. Michell's proof, as stated by Prof. Nanson, I should assign a slightly higher mark, say 40.

We may therefore arrange the proofs according to marks, thus:

1.	Mr. Berry's second proof,		-	28
2.	Mr. Michell's proof (my form), -	-	-	32
	Mr. Berry's third proof (his form), -	-	-	34
	Mr. Berry's first proof,	-	-	4()
	Mr. Michell's proof (his form), -	-		40
6.	Mr. Berry's third proof (ordinary form),	·. •••	-	42
	Prof. Nanson's proof		•	108

And this fairly represents my opinion of their respective merits: possibly,* however, I have penalized Nos. 2 and 5 too little on the score of artificiality and 4 too much on the score of theoretical difficulty. I conclude, however, with some confidence that Mr. Berry's second proof is distinctly the best. But whether this be so or not, one thing at any rate should be clear from this discussion: whatever method be chosen, the evaluation of the integral (1) is a problem of very considerable difficulty. This integral (and all the other standard definite integrals) lie quite outside the legitimate range of school mathematics.

G. H. HARDY.

*This I know to be Mr. Berry's opinion. He would put 2 below 3 and 4, and 5 below 6. To compare 3 and 4 is very difficult, and they might fairly be bracketed.

Since writing this note I have recollected another proof which I had forgotten, and this is given in \$172 (Fig. 1) of Mr. Browwich's Infinite Series. Mr. Browwich there

which is given in § 173 (Ex. 1) of Mr. Bromwich's *Infinite Series*. Mr. Bromwich there proves that, provided the real parts of a and b are positive or zero,

$$\int_0^\infty (e^{-ax}-e^{-bx})\frac{dx}{x} = \int_0^\infty \left(\frac{1}{a+y}-\frac{1}{b+y}\right)dy.$$

Putting a=1, b=i, we obtain

$$\int_0^{\infty} (e^{-x} - \cos x) \, \frac{dx}{x} = 0, \quad \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{1}{2}\pi.$$

This proof, as presented by Mr. Bromwich, should be marked at about 45. It has the advantage of giving the values of several other interesting integrals as well, so that (as in the case of Prof. Nanson's proof) it is hardly fair to contrast this mark with the considerably lower marks obtained by some of the other proofs.

CORRECTIONS

p. 100, line 16. For $x+i\pi$ read $x-i\pi$.

p. 101, line 5 up. For c^{-ca} read e^{-ca} .

p. 102, line 1 up. For < read =.

COMMENT

Some additional remarks on this topic appear in 1916, 11.

^{*}This proof is also referred to by Prof. Nanson. The only proofs which are 'classical' are 1, 2, 3, viz. the three discussed by Mr. Berry.