Math 556 Lecture 34 – Wed April 21 – Product expansions

After infinite sum formulas in the spirit of partial fractions, we now look at infinite products.
The convention/definition we want to use for products is:

Def: $\prod_{i=1}^{\infty} p_i$ <u>converges</u> if 1) at most finitely many terms p_i are zero, and
2) the products of the nonzero terms $\prod_{\substack{1\le i\le n\\ p_i\neq 0}} p_i$ converge to a nonzero limit as $n\to\infty$.

This feels awkward, and less natural than the obvious idea ($\prod_{i=1}^{n} p_i$ converges to a limit which may be zero), but is more suitable for expressing analytic functions as products.

The requirements ensure:
- → adding/removing finitely many factors to the product doesn't affect convergence
- → when a convergent product of analytic functions vanishes, it does so to a finite order ($=\sum$ orders of the factors that equal zero) & we can factor out the zeros. (a convergent product of nonzero factors is nonzero by definition of convergence!)
- → for nonzero products, the convergence of $\prod p_i$ is equivalent to that of $\sum \log p_i$.

Since convergence forces $\log(p_i)\to 0$ ie. $p_i\to 1$, it's customary to write infinite products in the form $\prod_{n=1}^{\infty}(1+a_n)$, and convergence $\Leftrightarrow$ $\sum \log(1+a_n)$ converges
(with necessarily $a_n\to 0$, so we take our <u>preferred choice of log, with $|\mathrm{Im}(\log)|<\pi$.</u>)

Moreover: $\sum \log(1+a_n)$ converges absolutely iff $\sum a_n$ converges absolutely
(using comparison: since either implies $a_n\to 0$, for suff. large n we have $\frac{|a_n|}{2}\le |\log(1+a_n)|\le 2|a_n|$)
When this happens we say the product <u>converges absolutely</u>. However, non-absolute convergence may involve more subtle cancellations, and cannot be reduced to that of $\sum a_n$.

Goal: given an entire analytic function $f(z)$, express it as a product that makes the zeroes of f immediately apparent, just as we write a polynomial in the form $c\prod(z-b_i)^{m_i}$.
Since an infinite product of $(z-b_i)$'s isn't going to converge, instead we aim for a product of factors of the form $\prod_{i=1}^{\infty}\left(1-\frac{z}{b_i}\right)^{m_i}$. (For $b_i\neq 0$. If f has a zero at $z=0$, we keep that factor as z^{m_0}).

If the infinite product converges $\forall z$, and if the convergence is <u>uniform</u> on compact subsets of $\mathbb{C}\setminus\{b_i\}$ (which, by definition, means $\sum m_i \log(1-\frac{z}{b_i})$ converges uniformly), then it defines an analytic function with the same zeros as f. So the ratio of $f(z)$ and this function is an entire function without zeros, hence can be written as $e^{g(z)}$ for some entire analytic function $g(z)$ (cf. homework! eg: lifting lemma for
$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{C} \arrow[ur, dashed, "g"] \arrow[r] & \mathbb{C}^* \end{tikzcd}$$
)

In summary: our hope is to arrive at $f(z) = z^{m_0} e^{g(z)} \prod_{i=1}^{\infty} \left(1 - \frac{z}{b_i}\right)^{m_i}$.

Just like the case of sums, the questions that come up are:

→ can we represent given functions in this way?

→ when do these expressions converge?

→ given $b_i \in \mathbb{C}$ without limit points (ie. $b_i \to \infty$), can we find an entire function with zeros of prescribed orders at b_i?

The answers to these questions parallel what we did with partial fractions.
Just like last time, we start with an example: the function $\sin(\pi z)$.

Expressing $\sin(\pi z)$ as an infinite product:

Since $\sin(\pi z)$ has zeros exactly at the integers, our naive guess is $z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right)$.
Unfortunately the series $\sum \log\left(1 - \frac{z}{n}\right)$ diverges (just like $\sum \frac{1}{n}$).

Just like we did for partial fractions, we cancel the divergence by subtracting from each term the beginning of its Taylor series.
Here: $\log\left(1 - \frac{z}{n}\right) = -\frac{z}{n} - \frac{z^2}{2n^2} - \cdots$ so we can consider $\sum \left(\log\left(1 - \frac{z}{n}\right) + \frac{z}{n}\right)$, which converges (comparison: $\sum \frac{z^2}{n^2}$ converges). This yields the product

$$z \prod_{n \neq 0} \left(\left(1 - \frac{z}{n}\right) e^{z/n}\right)$$

, which does converge (by convergence of $\sum \log(\ldots)$).

Now we can write $\sin(\pi z) = z e^{g(z)} \prod_{n \neq 0} \left(\left(1 - \frac{z}{n}\right) e^{z/n}\right)$ for some analytic $g(z)$.

How do we find $g(z)$? Answer: compare logarithmic derivatives $\left(\frac{f'(z)}{f(z)}\right)$ for both sides

Note: uniform convergence of the series $\sum \left(\log\left(1 - \frac{z}{n}\right) + \frac{z}{n}\right)$ over compact subsets of $\mathbb{C} - \mathbb{Z}$ implies that we can differentiate term by term $\left(\left(\sum f_n\right)' = \sum (f_n')\right)$
(remember, this is for analytic f^ns. In real analysis we need to assume the uniform convergence of $\sum (f_n')$, not just that of $\sum f_n$.)
so that the logarithmic derivative of a product is the sum of those of the factors.

Logarithmic derivatives. $\quad \sin \pi z \rightsquigarrow \frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot(\pi z)$

$$z \rightsquigarrow 1/z$$

$$\prod_{n \neq 0} \left(\left(1 - \frac{z}{n}\right) e^{z/n}\right) \rightsquigarrow \sum_{n \neq 0} \left(\frac{-1/n}{1 - z/n} + \frac{1}{n}\right) = \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

$$e^{g(z)} \rightsquigarrow g'(z).$$

So: $\pi \cot(\pi z) = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right)$.

$\Rightarrow$ using formula seen last time, $g'(z)=0$, so $e^{g(z)}=$ constant.

To find the constant c s.t. $\sin(\pi z) = c\, z \prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$,

divide both sides by z and evaluate at $z=0$ $\Rightarrow$ $\lim_{z\to 0} \frac{\sin \pi z}{z} = c$.

So $c=\pi$ and $\sin(\pi z) = \pi z \prod_{n\neq 0}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)$.

Or combining the terms corresponding to $+n$ and $-n$, $\sin(\pi z) = \pi z \prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$.

<u>Remark</u>: Last time & today, the series $\sum_{n\neq 0} \frac{1}{z-n}$ and $\sum_{n\neq 0}\log\left(1-\frac{z}{n}\right)$ are considered divergent because one's supposed to think of $\sum_{n\neq 0} = \sum_{n>0} + \sum_{n<0}$, and the latter two are divergent.

The simpler rewriting by grouping $\pm n$ together amounts to the observation that, for these specific divergent series, there is a convergent rearrangement:

$\exists \lim_{N\to\infty}\left(\sum_{\substack{n=-N\\ n\neq 0}}^{N} a_n\right) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}(a_n+a_{-n})\right)$. The series $a_1+a_{-1}+a_2+a_{-2}+\dots$ converges non-absolutely.

However, rearrangement of a non absolutely convergent series is not a benign operation, it can change the value of the sum — in fact for series of real numbers you can make it take any value you'd like (!!) (Rudin Thm 3.54)

<u>The general existence theorem</u>: analogous to what we've seen for sums,

<u>Thm.</u>: Given a subset $\{b_1, b_2, \dots\}\subset\mathbb{C}$ with $|b_j|\to\infty$ ($\Leftrightarrow$ no limit points), and multiplicities $m_j\geq 1$, there exists an entire analytic function $f(z)$ with zeroes exactly at the points b_j, with order m_j at each.

The proof is the same as for partial fractions: we want to modify the sum $\sum m_j \log\left(1-\frac{z}{b_j}\right)$ to achieve convergence. As before we do this by subtracting part of the Taylor series (*) $\log\left(1-\frac{z}{b_j}\right) = -\frac{z}{b_j} - \frac{z^2}{2b_j^2} - \dots$ stopping at some degree d_j.

$\Rightarrow$ we consider the infinite product $z^{m_0} \prod_j \left[\left(1-\frac{z}{b_j}\right) e^{\frac{z}{b_j}+\frac{1}{2}\left(\frac{z}{b_j}\right)^2+\dots+\frac{1}{d_j}\left(\frac{z}{b_j}\right)^{d_j}}\right]^{m_j}$

if $\exists\, b_0 = 0$.

The same sort of argument as for partial fractions shows that, for a suitable choice of d_j's the remainders $r_j(z)$ in (*) form a series s.t. $\sum m_j r_j(z)$ converges uniformly on compact subsets; the infinite product is then (uniformly) convergent. $\square$

<u>Corollary</u>: Any meromorphic function on $\mathbb{C}$ is the quotient of two analytic entire functions.

<u>Proof</u>: Suppose f has poles at $\{b_j\}$ with orders m_j: the above thm gives the existence of an entire function $g(z)$ with zeroes precisely at b_j with order m_j. So $h(z) = g(z)f(z)$ is everywhere analytic (zeroes of g cancel poles of f), and we have $f(z) = \dfrac{h(z)}{g(z)}$. □

<u>Next topic</u>: special functions – Γ and ζ especially

This is another application of infinite sums and products, besides abstract existence questions + explicit formulas for known functions such as $\sin(\pi z)$.

<u>Warm-up</u>: the <u>partition generating function</u>

Let $p(n)$ = number of partitions = # ways of expressing n as an (unordered) sum of positive integers. (by convention $p(0)=1$).

1	$p(1) = 1$
$2 = 1+1$	$p(2) = 2$
$3 = 2+1 = 1+1+1$	$p(3) = 3$
$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$	$p(4) = 5$ etc.

This has many remarkable properties, eg. arithmetic (Ramanujan: $p(5k+4) \equiv 0 \mod 5$ (!?!)) but our point here is rather to study the growth rate of $p(n)$: polynomial? exponential?

* One way to approach this is to introduce the generating function $P(z) = \sum_{n=0}^{\infty} p(n) z^n$ and ask about its properties (radius of convergence, etc.). The key formula for this is a product expansion
$$P(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1-z^n}. \qquad \text{(Euler 1753)}$$

To see this, write the product as $(1+z+z^2+\dots)(1+z^2+z^4+\dots)(1+z^3+z^6+\dots)\dots$

A partition of n as a sum of a_1 1's, a_2 2's, etc. corresponds to the contribution to the coeff. of z^n that comes from multiplying z^{a_1} in the first factor, z^{2a_2} in the second, and so on. So the total coeff. of z^n is indeed $p(n)$.

* This infinite product expansion, and comparison between $\sum \log(1-z^n)$ and $\sum z^n$, shows that $P(z)$ is well-defined and analytic in the unit disc $D = \{|z|<1\}$.

But we also see that, since the factors have poles at all roots of unity = a dense subset of the unit circle ($e^{2\pi i\alpha}, \alpha \in \mathbb{Q}$), there is no way to extend $P(z)$ beyond D.

This tells us the radius of convergence is 1, but in fact a <u>much</u> more detailed analysis of $P(z)$ yields more info ... $p(n) \sim \dfrac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ (Hardy-Ramanujan! 1918)