Math 55a Take-Home Final

Due by 5 pm, Thursday, 8 January 2009
Hand in to McMullen's mailbox, outside 325 Science Center

Instructions. Write your answers neatly on separate paper, stapled together, with your name on the first page. All work should be your own. Refer only to class notes (your own and those online) and the course texts. Problems continue on the back of the page.

Part I.

1. Let V an n-dimensional vector space and let $T_1, \ldots, T_n$ be pairwise commuting nilpotent operators on V.

   (i) Show that $T_1 T_2 \cdots T_n = 0$.

   (ii) Does this continue to hold if the T_i are no longer assumed pairwise commuting?

2. Let V be a finite-dimensional real vector space equipped with a symplectic form $B(x, y)$. A *Lagrangian subspace* $L \subset V$ is a maximal subspace such that
$$L \subset L^\perp = \{x \in V \;:\; B(x, y) = 0 \;\; \forall y \in L\}.$$
   (i) Show that every Lagrangian subspace satisfies $\dim L = (\dim V)/2$.

   (ii) Show for every Lagrangian subspace L there is a second Langrangian subspace L' such that $V = L \oplus L'$.

   (iii) Give an example of such a Lagrangian splitting in the case $V = \mathbb{C}^n$, considered as a real vector space, with $B(z, w) = \operatorname{Im} \sum_1^n z_i \overline{w}_i$.

3. Prove that any simple group of order 60 is isomorphic to A_5.

4. Let $C \subset S_n$ be a conjugacy class in the symmetric group.

   (i) Show that $C \cap A_n$ is the union of 0, 1 or 2 different conjugacy classes in the alternating group A_n.

   (ii) Determine whether the number is 0, 1 or 2 in terms of the lengths $a_1, \ldots, a_r$ of the cycles of an element $g \in C$ (which satisfy $\sum a_i = n$).

5. (i) Let G be a group and let p^e be a prime power which divides $|G|$. Show that G has a subgroup of order p^e.

   (ii) Give an example of a group G and an integer $n > 0$ such that n divides $|G|$ but G has no subgroup of order n.

6. In the character table below, one row and one column is missing.

G	(1)	(1)	(2)	(2)	(3)
	a	b	c	d	e
χ_1	1	1	1	1	1
χ_2	1	1	1	1	-1
χ_3	1	-1	1	-1	i
χ_4	1	-1	1	-1	$-i$
χ_5	2	2	-1	-1	0

(i) Complete the table.

(ii) Find the orders of elements in each conjugacy class.

(iii) Show that c generates a normal subgroup.

(iv) Describe the group.

7. Let $G \subset \mathrm{O}_n(\mathbb{R})$ be a finite group of orthogonal transformations acting irreducibly on $\mathbb{R}^n$. Let $B(x, y)$ be a symmetric bilinear form preserved by G. Show that $B(x, y) = \lambda\langle x, y\rangle$ for some $\lambda \in \mathbb{R}$.

Part II. Mark each of the following assertions True (T) or False (F).

1. The sets $\mathbb{R}$ and $\mathbb{R}^{\mathbb{R}}$ have the same cardinality.

2. The function $q(A) = \det(A)$ on $\mathrm{M}_2(\mathbb{R})$ is a quadratic form of signature $(3, 1)$.

3. For any two nonzero matrices $A, B \in \mathrm{M}_n(\mathbb{C})$, there exists a $\lambda \in \mathbb{C}$ such that $\det(A + \lambda B) = 0$.

4. Every linear map $T : \mathbb{R}^9 \to \mathbb{R}^9$ has an eigenvector.

5. Every linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ can be expressed in the form $T = DS$, where D is diagonal and $S \in \mathrm{O}_n(\mathbb{R})$.

6. Let $G \subset \mathrm{Isom}(\mathbb{R}^2)$ be a discrete group such that only the identity element has a fixed-point. Then G is a group of translations.

7. If $A, B \in \mathrm{GL}_n(\mathbb{R})$ are invertible matrices such that $\mathrm{tr}(A^n) = \mathrm{tr}(B^n)$ for all $n \in \mathbb{Z}$, then A is similar to B.

8. Every trilinear form is the sum of a symmetric and an antisymmetric form (i.e. $\otimes^3 V \cong \mathrm{Sym}^3 V \oplus \wedge^3 V$).

9. Any subgroup of index 5 in a group of order 5005 must be normal.

10. The group of transformations $G \subset \mathrm{Isom}(\mathbb{R}^3)$ that send a dodecahedron into itself has a subgroup of order 2.