Solutions to Homework 10

Math 55b

1. *Let $p(z) = z^3 + z^n$ with $n \geq 3$. Prove that $p(z) = 1$ for some z with* $\mathrm{Re}(z) < 0$.

   Solution 1. This is also true for $n = 2$. By the **Gauss-Lucas theorem** proved in class (the critical points of a complex polynomial lie in the convex hull of its zeros), $(z^3 + z^n - 1)' = 3z^2 + nz^{n-1}$ shows that $p(z)$ has a zero in $\mathrm{Re}(z) \leq 0$. It remains to remark that, for $n \geq 2$, $z^3 + z^n - 1$ has no zero on the imaginary axis $\mathrm{Re}(z) = 0$: if $z = it$ is a purely imaginary zero, equating the imaginary part in $-it^3 + (it)^n = 1$ shows that n is odd (since $t \neq 0$), but then $-it^3 + (it)^n$ is purely imaginary, contradiction. ■

   Solution 2. The sum of the zeros of z^3+z^n-1 is equal to the negative of the coefficient of the z^{n-1}-term, which is 0, if $n \neq 2, 4$, and -1 if $n = 2, 4$; in either case, it is a real number ≤ 0, showing the sum of the real parts of the zeros of $z^3 + z^n - 1$ is ≤ 0. The conclusion follows from the same remark that $z^3 + z^n - 1$ has no zero on the imaginary axis. ■

2. *Give an expression for* $\sin(x+iy)$ *in terms of the real-valued spherical and hyperbolical sines and cosines. Where are the zeros of the function* $\sin(z)$ *on* $\mathbb{C}$*?*

   The identity $\sin(z) = (e^{iz} - e^{-iz})/2i = e^{-iz}/2i \cdot (e^{2iz} - 1)$ makes it manifest that the zeros of $\sin(z)$ are the same as the zeros of $e^{2iz} = 1$, which are the integral multiples of π.

   The required expression $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$ is a special case of the addition formula $\sin x + y = \sin x \cos y + \cos x \sin y$, upon noting that $\cos iy = \cosh y$ and $\sin iy = i \sinh y$. Since $\cosh y = (e^y + e^{-y})/2$ does not vanish (for $y \in \mathbb{R}$) and $\sinh y = (e^y - e^{-y})/2$ vanishes only at 0, this expression also shows that $\sin(x + iy) = 0$ iff $y = 0$ and $\sin x = 0$; that is, $\sin z = 0$ iff $z \in \pi\mathbb{Z}$. ■

3. *Suppose $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in $|z| < 1$. Prove that for any $|r| < 1$, we have $\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = 2\pi \sum_{n\geq 0} |a_n|^2 r^{2n}$.*

   By the uniform convergence of $\sum_{n\geq 0} a_n z^n$ in $|z| \leq r$ and the identity $\bar{z} = r^2/z$ on $|z| = r$, we have $|f(z)|^2 = f(z)\overline{f(z)} = \Big(\sum_{n\geq 0} a_n z^n \Big) \Big(\sum_{n\geq 0} \overline{a_n} \bar{z}^n \Big) =$

$\sum_k c_k z^k$ on $|z| = r$, where $c_0 = \sum_{n\geq 0} |a_n|^2 r^{2n}$ and the c_k are certain coefficients. Integrating term by term on $|z| = r$ (again justified by uniform convergence), and using $\int_0^{2\pi} z^k = 0$, for $k \neq 0$, and 2π, for $k = 0$, proves the required identity. ∎

4. *Given $J \in \mathrm{M}_2(\mathbb{R})$, let $\mathbb{R}[J] \subset \mathrm{M}_2(\mathbb{R})$ denote the set of matrices of the form $aI + bJ$, $a, b \in \mathbb{R}$. (i) Prove that $\mathbb{R}[J]$ is closed under addition and multiplication. (ii) When do these operations make $\mathbb{R}[J]$ into a field?*

   That $\mathbb{R}[J]$ is closed under addition is obvious, and its closedness under multiplication follows upon noting that J satisfies its characteristic polynomial, which has degree 2, so that $J^2 \in \mathbb{R}[J]$; this proves (i). For (ii), $\mathbb{R}[J]$ is a field if and only if the characteristic polynomial of J is irreducible over $\mathbb{R}$, which is the case if and only if either $J = \lambda I$ with $\lambda \in \mathbb{R}$, or J has no real eigenvalue. ∎

5. *Prove that for any polynomial $p(z)$ there exists a $z \in S^1$ such that $|\overline{z} - p(z)| \geq 1$.*

   Proof 1. This follows from the maximum principle, applied to the polynomial $1 - zp(z)$. ∎

   Proof 2. Assuming otherwise, integrating $1/z - p(z)$ over the unit circle leads to the contradictory inequality $2\pi = \left| \int_{S^1} (1/z - p(z))\, dz \right| \leq \int_{S^1} |\overline{z} - p(z)|\, dz < 2\pi$. ∎

6. *Let $u \in C(\overline{\Delta})$ be a real-valued continuous function which is harmonic in Δ. Prove that for every $p \in \Delta$, $u(p) = \frac{1}{2\pi} \int_{S^1} \frac{1-|p|^2}{|z-p|^2} u(z)\, |dz|$.*

   For $p = 0$, this is just the mean value property $u(0) = (1/2\pi) \int_{S^1} u(z)\, |dz|$ of harmonic functions. To reduce to this case, compose with a Möbius transformation moving 0 to p and preserving Δ; such is given by $q(z) := (p - z)/(1 - \overline{p}z)$. The new function $u \circ q^{-1} \in C(\overline{\Delta})$ is harmonic in Δ and takes the value $u(p)$ at $z = 0$. It remains to compute that $|q^* \, dz| = \frac{1-|p|^2}{|z-p|^2}\, |dz|$. ∎

7. *Prove Hadamard's* 3 circles theorem*: if $f(z)$ is analytic on the annulus $R_1 < |z| < R_2$, and $M(r) := \sup_{|z|=r} |f(z)|$, then $\log M(e^s)$ is a convex function of $s \in (\log R_1, \log R_2)$.*

For any $\alpha \in \mathbb{R}$, the function $\alpha \log|z| + \log|f(z)|$ is harmonic on the annulus $A := \{R_1 < |z| < R_2\}$, away from the zeros of f. Since $\alpha \log|z| + \log|f(z)| \ll 0$ near a zero of f, the maximum principle shows that the function $\alpha \log|z| + \log|f(z)| = \log|z^\alpha f(z)|$ attains its maximum on the boundary of A, which is the union of the circles $|z| = R_1$ and $|z| = R_2$. Choose $\alpha := \frac{\log M(R_1) - \log M(R_2)}{\log(R_2) - \log(R_1)}$, so that $\alpha \log R_1 + \log M(R_1) = \alpha \log R_2 + \log M(R_2)$. Then the preceding observation gives, for all $R_1 < R < R_2$, the inequality $\alpha \log R + \log M(R) \leq \alpha \log R_1 + \log M(R_1)$. By the choice of α, a rearrangement gives $\log M(R) \leq \frac{\log R_2 - \log R}{\log R_2 - \log R_1} \log M(R) + \frac{\log R - \log R_1}{\log R_2 - \log R_1} \log M(R_2)$. The conclusion follows: $\log M(R)$ is a convex function of $\log R$. ■

8. *Let u, v be smooth functions on $\overline{\Delta}$ with $u|\Delta$ harmonic and $u\big|_{S^1} = v|_{S^1}$. Show that $\int_\Delta |\nabla v|^2 \geq \int_\Delta |\nabla u|^2$.*

More precisely, the following identity takes place:

$$\int_\Delta |\nabla v|^2 = \int_\Delta |\nabla u|^2 + \int_\Delta |\nabla(v-u)|^2.$$

To prove it, we need to show $\int_\Delta \nabla v \cdot \nabla(v-u) = 0$. Its proof reduces to the divergence theorem. Since $\nabla(\nabla u) = \Delta u = 0$, we have $\nabla((v-u)\nabla u) = \nabla(v-u) \cdot \nabla u$, and the divergence theorem gives $\int_\Delta \nabla v \cdot \nabla(v-u) = \int_\Delta \nabla((v-u)\nabla u) = \int_{S^1} (v-u)\nabla u\, |dz| = 0$, the last equality holding by the assumption $v - u|_{S^1} \equiv 0$. ■

Remark. Conversely, if u minimizes the **Dirichlet energy integral** $E(u) := \frac{1}{2}\int_\Delta |\nabla u|^2$ among the functions on the disk with a given boundary condition, then u is harmonic. This is proven in exactly the same way, with the following **calculus of variations** twist. For a test function φ vanishing on S^1, analyze the function $\mathbb{R} \to \mathbb{R}$, $\quad t \mapsto E(u + t\varphi)$. If $E(u)$ is a minimum among the functions restricting to $u|_{S^1}$ on S^1, this (smooth) function attains a minimum at $t = 0$, showing $\frac{d}{dt}E(u + t\varphi) = 0$. Differentiating under the integral and applying the divergence theorem in exactly the same way, this yields $\int_\Delta \varphi \cdot \Delta u = 0$. Since this is to hold for all functions φ vanishing on S^1, it follows that $\Delta u = 0$, i.e. u is harmonic. ■

9. *Let $\sum a_n z^n$ be the Laurent series for $1/(e^z - 1)$ near $z = 0$. Find a_n for $n \leq 3$. What is the radius of convergence of this series?*

Expand in geometric series $1/(e^z-1) = \frac{1}{z}\frac{z}{e^z-1} = \frac{1}{z}\frac{1}{1+(z/2+z^2/6+z^3/24+z^4/120+o(z^4))} = \frac{1}{z}\sum_{n\geq 0}(-z/2-z^2/6-z^3/24-z^4/120)^n+o(z^4) = \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+o(z^3)$. The radius of convergence if the series is 2π, because the singularity of the meromorphic function $1/(e^z-1)$ of minimal absolute value is the pole $2\pi\sqrt{-1}$.

The coefficients a_n in the expansion of $1/(e^z-1)$ near 0 are called the **Bernoulli numbers**, and have important arithmetical properties. They also arise in a formula expressing $\sum_{j\leq n} j^k$ as a polynomial in n of degree $k+1$: it is obtained by comparing coefficients in $(e^{(k+1)z}-e)/(e^z-1) = 1+e^z+e^{2z}+\cdots+e^{kz}$. But note that $1/(e^z-1)$ cannot be expanded directly in geometric series (even formally), because e^z-1 has a zero at $z=0$. ■

10. *Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function satisfying $f(x+y)=f(x)f(y)$. Prove that either $f(z)=0$, or $f(z)=\exp(\alpha z)$ for some $\alpha\in\mathbb{C}$.*

The functional equation $f(x+y)=f(x)f(y)$ implies $f'(x)=f'(0)f(x)$, hence $f^{(k)}(x)=\alpha^k f(x)$, where $\alpha := f'(0)$. Also $f(0)^2=f(0)$, showing $f(0)$ is either 1 or 0. In the latter case, $f(z)=0$. In the former case, $f^{(k)}(0)=\alpha^k f(0)=\alpha^k$, showing that the Taylor expansion near 0 of the entire function $f(z)$ is $\sum_{k\geq 0}\alpha^k z^k/k! = \exp(\alpha z)$, as required. ■