Closed sets: **Def:** a subset A of a topological space X is **closed** if $X \setminus A$ is open.

⚠ subsets can be both closed & open, e.g. $\emptyset$ and X, or neither (e.g. $[0,1)$ or $\mathbb{Q}$ in $\mathbb{R}$)

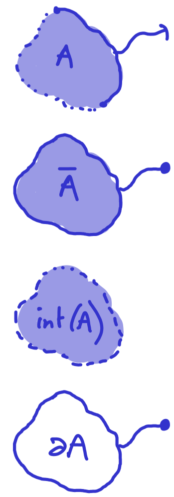
Axioms of open sets imply:
- $\emptyset, X$ are closed
- arbitrary intersections of closed sets are closed
- finite unions of closed sets are closed.

Def: $A \subset X$ any subset $\Rightarrow$ we define

1) the **closure** of A: $\bar{A}$ = smallest closed set containing A = $\bigcap_{\substack{F \supset A \\ F \text{ closed}}} F$
($\bar{A} \supset A$, $\bar{A}$ closed since it's $\cap$ of closed)

2) the **interior** of A, $\operatorname{int}(A)$ = largest open set contained in A
$= \bigcup_{U \subset A,\, U \text{ open}} U$ (open).

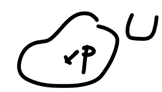
3) the **boundary** of A is $\partial A = \bar{A} - \operatorname{int}(A)$



Ex: $A = [0,1) \subset \mathbb{R}$, usual top. $\Rightarrow$ $\bar{A} = [0,1]$, $\operatorname{int}(A) = (0,1)$, $\partial A = \{0,1\}$

Rmk:
- A is closed iff $\bar{A} = A$, open iff $\operatorname{int}(A) = A$.
- $\overline{X \setminus A} = X \setminus \operatorname{int}(A)$, $\operatorname{int}(X - A) = X - \bar{A}$. $(*)$ ← since open sets $\subset X-A$ are exactly complements of closed sets $\supset A$!

Def: Say $U \subset X$ is a **neighborhood** of $p \in X$ if U is open and $p \in U$.



→ **Prop:**
(1) $p \in \operatorname{int}(A)$ iff A contains a neighborhood of p.
(2) $p \in \bar{A}$ iff every neighborhood of p intersects A nontrivially.

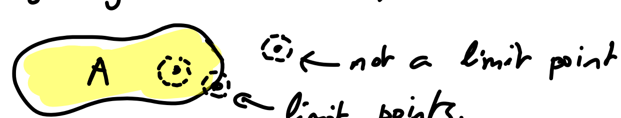
(check this! (1) follows from def$^{\text{ns}}$: $p \in \operatorname{int}(A) \Leftrightarrow \exists U$ open s.t. $p \in U \subset A$.
(2) follows from (1) + $(*)$: $p \in \bar{A} \Leftrightarrow p \notin \operatorname{int}(X - A) \Leftrightarrow \forall U \ni p$ open, $A \cap U \neq \emptyset$).

Def: say A is **dense** if $\bar{A} = X$. (i.e. every nonempty open subset of X intersects A nontrivially).

Ex: $\mathbb{Q}$ is dense in $\mathbb{R}$ (for usual topology).

Closed sets & limit points:

Def: $x \in X$ is a **limit point** of $A \subset X$ if, for every neighborhood $U \ni x$, $U \cap (A - \{x\}) \neq \emptyset$



Ex: in $\mathbb{R}_{std}$, 1 is a limit point of $(0,1)$ and of $[0,1]$.
1 is not a limit point of $\{\frac{1}{n}, n \geq 1\} \cup \{0\}$, but 0 is.

Prop: $\overline{A} = A \cup \{\text{limit points of } A\}$.

Pf: $A \subset \overline{A}$ by def^{n}, so enough to consider points not in A.

if $x \notin A$, $\forall U \ni x$ neighborhood, $U \cap (A - \{x\}) = U \cap A$ so $x \in \overline{A}$ iff x limit pt.

x is a limit point if these always $\neq \phi$ $\quad$ $x \in \overline{A}$ if these always $\neq \phi$. □

Corollary: A is closed iff A contains all of its limit points.

- ⚠ What is the connection between limit points and limits of sequences?

Recall: $\{p_n\}$ sequence in X converges to $p \in X$ if $\forall U$ neighborhood of p, $\exists N$ st. $n \geq N \Rightarrow p_n \in U$.

Fact: $p \in X$, if $\exists \{p_n\}$ sequence in $A \subset X$ st. $p_n \to p$ then $p \in \overline{A}$

if $\exists \{p_n\}$ seq. in A, $p_n \neq p$ for ∞ many n, $p_n \to p$, then p is a limit pt of A.

Pf: any neighborhood $U \ni p$ contains p_n for all large n, hence contains points of A. (distinct from p in 2nd case) □

The converse is true in metric spaces: if $p \in \overline{A}$ (resp. a limit point of A) then $\forall n > 0$ $\exists p_n \in B_{1/n}(p) \cap A$ (resp. with $p_n \neq p$), so $\exists$ sequence in A st. $p_n \to p$.

This holds more generally in spaces whose points have countable bases of neighborhoods $U_1 \supset U_2 \supset \dots$ (ie. $\forall p$ $\exists$ nbds $U_1, U_2, \dots$ st. $\forall$ nbd $U \ni p$, $\exists n$ st. $p \in U_n \subset U$), but not in arbitrary topological spaces!

Hausdorff spaces: In a metric space, a sequence converges to at most one limit.

This is not true in an arbitrary topological space!

Ex: $X = \mathbb{R}$ with finite complement topology: open subsets $= \phi$ and $\mathbb{R} - \{\text{finite sets}\}$

let $a_1, a_2, \dots$ be a sequence in X with all a_i distinct.

Then $\forall x \in X$, every neighborhood $U \ni x$ contains all but finitely many of the a_i, hence $\exists N$ st. $a_n \in U$ $\forall n \geq N$. Thus the sequence converges to every point of X!

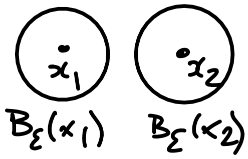
To avoid such pathological behavior:

Def: A top-space is Hausdorff (or T_2) if $\forall x_1 \neq x_2 \in X$, $\exists$ neighborhoods $U_1 \ni x_1$, $U_2 \ni x_2$ st. $U_1 \cap U_2 = \phi$.

Ex: 1) any metric space is Hausdorff:

given $x_1 \neq x_2$, choose $0 < \varepsilon < \frac{1}{2} d(x_1, x_2)$

then $U_i = B_\varepsilon(x_i)$ disjoint neighborhoods of x_i.



2) the finite complement topology on $\mathbb{R}$ is not Hausdorff, since any two non-empty open sets intersect (in infinitely many points).

3) the discrete topology is always Hausdorff ($U_i = \{x_i\}$ disjoint neighborhoods of x_i)

4) One can show: X Hausdorff, $Y \subset X \Rightarrow$ the subspace top. is Hausdorff.
X, Y Hausdorff $\Rightarrow X \times Y$ Hausdorff. (Homework!)

<u>Thm:</u> if X is Hausdorff then every sequence in X converges to at most one limit.

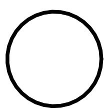
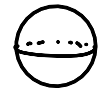
<u>Proof.</u> assume $x_1, x_2, \dots$ converges to $x \in X$, and let $y \neq x$. Choose $U_x \ni x$, $U_y \ni y$ disjoint neighborhoods. Since $x_n \to x$, $\exists N$ st. $\forall n \geq N$ $x_n \in U_x$.
Hence $x_n \notin U_y$ for $n \geq N$, so the sequence doesn't converge to y. $\square$

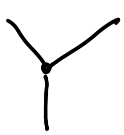
<u>Rmk:</u>
- There's in fact a whole hierarchy of "separation axioms": eg. a weaker one is:
  A top. space is T_1 if $\forall x \neq y \in X$, $\exists U_y \ni y$ neighborhood st. $x \notin U_y$.
  equivalently: X is $T_1 \Leftrightarrow \{x\}$ is closed in X $\forall x \in X$. (exercise!)
  Hausdorff (T_2) $\Rightarrow T_1$, but eg. ($\mathbb{R}$, finite complement top.) is T_1 but not Hausdorff.
- Hausdorff spaces are fairly nice to work with, and we will generally be working with this assumption. There are more subtle reasons why not every Hausdorff topology comes from a metric, but one can give pretty good criteria for a topology to be metrizable involving further separation conditions ("normal" or T_4). (+ a countability condition). We'll see the Urysohn metrization theorem.

<u>Manifolds & CW complexes:</u>

Metric spaces are nice, but they can still be pretty nasty. (We'll see conditions such as local connectedness, local compactness etc. come up). Algebraic topologists like to focus on even nicer spaces. For example:

<u>Def:</u> An n-dimensional topological manifold is a top. space X st. every point $p \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$ (or equivalently, an open ball in $\mathbb{R}^n$).

<u>Example:</u>  $S^1 \subset \mathbb{R}^2$ is a 1-d. top. manifold;   $\subset \mathbb{R}^3$ 2d top. manifolds.

<u>Example:</u>  isn't a top. manifold (vertex looks wrong) - but it is part of a more general class of spaces called CW complexes, built by attaching "cells" (closed balls of dim. 0, 1, ...) onto each other inductively.

We'll see more on this later when we get to alg. top. In decreasing order of generality:
{top spaces} $\supset$ {Hausdorff} $\supset$ {metrizable} $\supset$ {CW-complex} $\supset$ {manifold}.

Topologies on infinite products: given topological spaces X_i, $i \in I$ index set:
What is the natural topology on $X = \prod_{i\in I} X_i = \{(p_i)_{i\in I} \mid p_i \in X_i \ \forall i \in I\}$?

First idea: Def: the box topology on $\prod_{i\in I} X_i$ has basis $\{\prod_{i\in I} U_i \mid U_i \subset X_i \text{ open } \forall i\}$

(this is a basis: box $\cap$ box $=$ box, since $(\prod U_i) \cap (\prod V_i) = \prod (U_i \cap V_i)$)

This is actually too fine for most purposes.

Example: consider the diagonal map $\Delta : \mathbb{R} \to \mathbb{R}^\omega = \mathbb{R}^{\mathbb{N}}$ $(= \mathbb{R}_0 \times \mathbb{R}_1 \times \mathbb{R}_2 \times \dots)$
$$\Delta(x) = (x, x, x, \dots)$$
giving $\mathbb{R}^{\mathbb{N}}$ the box topology, Δ is not continuous!! (unlike case of finite products)

Indeed, let $U = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$ open in box topology.
$$\Delta^{-1}(U) = \bigcap_{n \geq 1} (-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\} \text{ not open in } \mathbb{R}.$$

Better: Def: the product topology on $X = \prod X_i$ has basis
$\{\prod_{i\in I} U_i \mid U_i \subset X_i \text{ open, and } U_i = X_i \text{ for all but finitely many } i\}$

(This is the same as the box topology if I is finite; for infinite I this is coarser)

Unless otherwise specified, the product topology is the one we'll use on $\prod X_i$.

Theorem: $f : Z \to X = \prod X_i$ (product top) is continuous $\iff$ each component $f_i : Z \to X_i$ is continuous.
$z \mapsto (f_i(z))_{i\in I}$

Ex: this now implies the diagonal map $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is continuous, since each $\Delta_i =$ identity.

Pf:
- the projection $p_i : X \to X_i$ to the i^{th} factor is continuous ($\forall U \subset X_i$ open, $p_i^{-1}(U)$ is open in product top.). Hence, if f is continuous, so is $f_i = p_i \circ f$.
- conversely, assume all f_i are continuous, and consider basis element $\prod U_i \subset X$ where $U_i = X_i$ for all but finitely many i,
  then $f^{-1}(\prod U_i) = \{z \in Z \mid (f_i(z))_{i\in I} \in \prod U_i\} = \bigcap_{i\in I} f_i^{-1}(U_i)$
  Each $f_i^{-1}(U_i) \subset Z$ is open, and all but finitely many are $= f_i^{-1}(X_i) = Z$, so can be omitted from the intersection. So $f^{-1}(\prod U_i)$ is the intersection of finitely many open sets in Z, hence open. □

Ex: given a set X & top. space Y, let $\mathcal{F} = \{\text{functions } X \to Y\} = Y^X$ with product top.
Then a sequence $f_n \in \mathcal{F}$ converges to $f \in \mathcal{F}$ iff $\forall x \in X$, $f_n(x) \to f(x)$ in Y.
(check this!) So: the product topology is the topology of pointwise convergence.