Homework 12

Math 55b
Due Tuesday, 28 April 2009.

1. Evaluate $\sum_1^\infty 1/n^6$ using the Laurent series for $\pi/\cot(\pi z)$ around $z = 0$.

2. Evaluate $\int_0^{2\pi} d\theta/(2 - \sin\theta)$.

3. Evaluate $\int_0^\infty dx/(1 + x^2)^2$.

4. Evaluate $\int_0^\infty x^{-a}/(x + 1)\,dx$, where $0 < a < 1$.

5. Given $p \in \mathbb{C}$, construct explicitly a sequence $z_n \to 0$ such that $\exp(1/z_n) \to p$. Can you, in fact, construct a sequence with $\exp(1/z_n) = p$?

6. Let $p_t(z) = z^d + a_1(t)z^{d-1} + \cdots + a_d(t)$ be a polynomial whose coefficients are analytic functions near $t = 0$. Suppose $p_0(z)$ has only simple zeros. Prove there are analytic functions $b_i(t)$ defined near $t = 0$ such that $p_t(z) = \prod_1^d (z - b_i(t))$.

7. Prove or disprove: if $f : \Delta \to \mathbb{C}$ is an analytic function with n zeros (counted with multiplicities), then $f'(z)$ has at least $n - 1$ zeros in Δ. What happens if 'at least' is replaced with 'at most'?

8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(z + 1) = f(z)$. Show there is an entire function $g(z)$ such that $f(z) = g(e^{2\pi i z})$. What is $g(z)$ if $f(z) = \tan \pi z$? Show that if $f(z_n) \to 0$ whenever $|\operatorname{Im} z_n| \to \infty$, then $f(z) = 0$.

9. Show that the sum $f(z) = \sum_{-\infty}^\infty (z - n)^{-2}$ converges locally uniformly to an analytic function on $\mathbb{C} - \mathbb{Z}$. Then show $f(z) = \pi^2/\sin^2(\pi z)$. (Hint: apply the preceding result to the difference.)

10. Find A, B, C such that

$$\sum_{-\infty}^{\infty} \frac{1}{(z-n)^4} = \frac{A}{\sin^4(\pi z)} + \frac{B}{\sin^2(\pi z)} + C.$$

Then, by considering the Laurent series of both sides around $z = 0$, evaluate $\sum 1/n^4$.