Math 55b Take-Home Final

Due by 5 pm, Thursday, 7 May 2009
Hand in to McMullen's mailbox, outside 325 Science Center

Instructions. Write your answers neatly on separate paper, stapled together, with your name on the first page. All work should be your own. Refer only to class notes (your own and those online) and the course texts. The exam is 3 pages long.

Part I.

1. Given $1 \le p < \infty$, let $E_p \subset C^1[0,1]$ denote the space of functions such that $f(0) = 0$ and $\int_0^1 |f'(x)|^p \, dx < 1$. Show that the closure of E_p in $C[0,1]$ is compact iff $p > 1$.

2. Let $f(x) \ge 0$ be a smooth, compactly supported function on $\mathbb{R}^3$. The gravitional force at $p \in \mathbb{R}^3$ coming from the mass distribution $f(x)\,|dx|$ is given by the vector

$$F(p) = \int_{\mathbb{R}^3} \frac{x-p}{|x-p|^3} f(x)\,|dx|.$$

(a) Show that $F(x) = \nabla\phi$ for a suitable function $\phi(x)$.

(b) Show that if $f(x)$ vanishes on a neighborhood of p, then ϕ is harmonic near p.

(c) More precisely, show that $\Delta\phi = Cf$ for some constant C.

(d) Show that the gravitational force is zero inside a hollow, spherically symmetric planet.

3. Let $f \in C^1(\mathbb{R})$ be a function such that $\|f\|_\infty$ and $\|f'\|_\infty$ are both bounded. Define $I : C[0,1] \to C[0,1]$ by $I(u) = v$ where

$$v(x) = \int_0^x f(u(t))\,dt.$$

Show that the differential equation $u'(x) = f(u(x))$ has a unique solution on $[0,1]$ with $u(0) = 0$, by showing:

(a) $I^n(u)$ converges uniformly, for any $u \in C[0,1]$, to a function g satisfying $I(g) = g$;

(b) The fixed point g of I is unique; and

(c) The fixed points of I correspond bijectively to solutions to the given differential equation.

4. Suppose $f(z)$ is analytic on the unit disk $\Delta \subset \mathbb{C}$, $f(0) = 0$ and $\operatorname{Re} f(z) \leq 1$ for all z.

   (a) What is the largest possible value $M(r)$ for $|\operatorname{Im} f(z)|$ on the circle $|z| = r < 1$?

   (b) Let $f_n : \Delta \to \mathbb{C}$ be analytic functions with $\operatorname{Re} f_n \leq 1$ and $f_n(0) = 0$, and suppose $\operatorname{Re} f_n$ converges uniformly on the unit disk. Prove that $\operatorname{Im} f_n$ converges uniformly on the disk $|z| \leq r$ for each $r < 1$.

   (c) Give an example where $\operatorname{Re} f_n$ converges uniformly on Δ but $\operatorname{Im} f_n$ does not.

5. Let $f(x) = \int_0^x dt/\sqrt{t(1-t^2)}$ for $x \in [0,1]$.

   (a) Show that there is a unique analytic function $F(z)$ defined on $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ such that $F(z_n) \to f(x)$ whenever $z_n \to x \in [0,1]$.

   (b) Show that $S = F(\mathbb{H})$ is an open square in $\mathbb{C}$, and that $F : \mathbb{H} \to S$ is a homeomorphism.

Part II. Mark each of the following assertions True (T) or False (F).

1. A smooth map $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$, is analytic iff for all 1-forms α on $\mathbb{C}$, $f^*(*\alpha) = *f^*(\alpha)$.

2. If α, β are k-forms on $\mathbb{R}^n$, $k > 0$, then $\alpha\beta = -\beta\alpha$.

3. If $f_n(z)$ are analytic functions and $f_n \to f$ uniformly on a domain U, then f is analytic and $f_n' \to f'$ uniformly on U.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous map such that the zeros of $f(z) - a$ are isolated for every $a \in \mathbb{C}$. Then f is an open map ($f(U)$ is open whenever U is open).

5. If E is any subset of $\mathbb{R}^n$, then the boundary of the boundary of the interior of E is empty.

6. If $f_n \in C[0,1]$ converges pointwise to 0, and $|f_n(x)| \leq 1$ for all n, x, then $\int_0^1 f_n(x)\, dx \to 0$.

7. Suppose $f_n \in C[0,1]$ converges uniformly to 0, and $\alpha_n \in C[0,1]$ are monotone increasing functions with $\alpha_n(1) - \alpha_n(0) = 1$. Then $\int_0^1 f_n d\alpha_n \to 0$.

8. Suppose v is a smooth vector field on $\mathbb{R}^3$. Then $\nabla \times \nabla \times v = 0$.

9. There exists a sequence of nonempty, disjoint, closed intervals $I_i \subset [0,1]$ such that $\bigcup I_i = [0,1]$.

10. The analytic function defined on the unit disk by $f(z) = \sum n^5 z^n$ extends to a rational function on the Riemann sphere.