

Math 55b: Honors Real and Complex Analysis

Homework Assignment #11 (24 April 2017):

Definite integrals, and other (mostly) real uses of complex analysis

The shortest path to any truth involving real quantities often passes through the complex plane. —*J. Hadamard* [attributed by several online sources]

You’ve probably seen some instances of this already (e.g. using complex numbers to describe rotations in the Euclidean plane), and we’ve done more last term (such as using the Fundamental Theorem of Algebra to describe factorization in $\mathbf{R}[X]$), but complex analysis is the most prominent source of examples. The usual illustration is the evaluation of real definite integrals via contour integration. This and some other examples are included in the problems below.

1. [From our graduate Qualifying Exam, Spring 1997] For $b > 0$, compute $\int_0^\infty \log x \, dx/(x^2 + b^2)$.
2. Recall that for $z \in \mathbf{C}$ the *hyperbolic cosine* $\cosh z$ is defined as $\cos(iz) = (e^z + e^{-z})/2$.¹ Prove that

$$\int_0^\infty \frac{\cos(mx)}{\cosh(\pi x)} e^{-mx^2} dx = \frac{1}{2} e^{-m/4}$$

for all $m > 0$. [From the Fall 1998 Qualifying Exam.] Can you evaluate any other such integrals this way (other than those obtained trivially from this formula by linear change of variable etc.)?

More about residues, contour integrals, and the complex Gamma function:

3. Determine for any entire function f the residue at $z = 0$ of the differential $f(\cot(z)) dz$. In particular, what is the residue at the origin of $\sin(\cot(z)) dz$?

Note that in general $f(\cot(z))$ has an essential singularity at $z = 0$. As far as I know, the other odd-order coefficients of the Laurent expansion of $\sin(\cot(z))$ about $z = 0$ are not known in closed form.

4. Use contour integration to find, for any $c \in \mathbf{C}$, the coefficients a_n of the analytic function $w(z) = \sum_{n=1}^\infty a_n z^n$ on a neighborhood of $z = 0$ such that $w(z)(1-z)^c = z$. [The expression $(1-z)^c$ can be unambiguously defined near $z = 0$ by

$$(1-z)^c = \exp(c \log(1-z)) = \exp(-c \sum_{m=1}^\infty z^m/m).$$

Check that your answer agrees with the elementary formulas for $c = 0, \pm 1, -2, -1/2$.]

5. Prove that for $c, x > 0$ the integral $\int_{-\infty}^\infty \Gamma(c + iy) x^{-iy} dy$ converges to $2\pi x^c e^{-x}$. [While we did not prove Stirling’s approximation for the complex Gamma function, here it will be enough to use judiciously the elementary inequality $|\Gamma(z)| \leq |\Gamma(\operatorname{Re}(z))|$ for $\operatorname{Re}(z) > 0$.]

¹The abbreviation “cosh” is from the Latin *cosinus hyperbolicus*, but is still often pronounced to rhyme with “mosh” or even expanded to “coshine”...

The next problems concern functions to and from the Riemann sphere, which can be defined as the “projective line” $\mathbf{P}^1(\mathbf{C}) = (\mathbf{C}^2 - \{(0,0)\})/\mathbf{C}^*$. An analytic map f from some open $E \subset \mathbf{C}$ to $\mathbf{P}^1(\mathbf{C})$ is a pair (f_1, f_2) of analytic functions on E that do not have a common zero in E ; another such pair (g_1, g_2) gives the same function if the vectors $(f_1(z), f_2(z))$ and $(g_1(z), g_2(z))$ are proportional for all $z \in E$. You should check that every z has a neighborhood in which f has a representative such that either f_1 or f_2 is the constant function 1. We usually identify f with f_1/f_2 , with the understanding that this may take the value ∞ (when $f_2 = 0$). The idea is that $\mathbf{P}^1(\mathbf{C})$ is made up of two copies of \mathbf{C} , one represented by vectors $(z_1, 1)$ and the other by $(1, z_2)$, and intersecting on \mathbf{C}^* with $z_2 = z_1^{-1}$. Since the map $z_1 \mapsto z_1^{-1}$ is analytic on \mathbf{C}^* it makes sense to say that an analytic or meromorphic map F from $\mathbf{P}^1(\mathbf{C})$ to \mathbf{C} or $\mathbf{P}^1(\mathbf{C})$ is one whose restriction to each of these two copies of \mathbf{C} is analytic or meromorphic respectively.²

6. Show that an analytic maps $E \rightarrow \mathbf{P}^1(\mathbf{C})$ other than the constant map ∞ is a meromorphic functions on E . Show that the meromorphic functions on $\mathbf{P}^1(\mathbf{C})$ are precisely the rational functions (note that the rational functions of z_1 are the same as the rational functions of $z_2 = z_1^{-1}$ so this is well-defined). [Recall our generalization of Liouville’s theorem to entire functions bounded by $C|z|^n$ as $|z| \rightarrow \infty$ — which indeed is the special case of a meromorphic function with no poles except possibly at ∞ .]
7. Let $A, B \in \mathbf{C}[z]$ be polynomials such that B has distinct roots z_1, \dots, z_n . Let ω be the differential $(A(z)/B(z)) dz$ on $\mathbf{C} - \{z_1, \dots, z_n\}$. Show that the residue of ω at each z_j is $A(z_j)/B'(z_j)$. Conclude that if $\deg(A) \leq \deg(B) - 2$ then $\sum_{j=1}^n A(z_j)/B'(z_j) = 0$. What happens if $\deg(A) = \deg(B) - 1$?

Since these identities are purely algebraic results, they must hold for polynomials over any algebraically closed field; but — as with invariance of the residue under coordinate change — a direct algebraic proof, though possible, is harder and less revealing.

8. Let E be the open right half-plane $\{z \in \mathbf{C} : \operatorname{Re}(z) > 0\}$, and $f : E \rightarrow \mathbf{C}$ a bounded analytic function. Suppose $f(x_k) = 0$ for some real numbers $x_k > 0$. Prove that

$$|f(1)| \leq \left(\prod_{k=1}^n \frac{1 - x_k}{1 + x_k} \right) B$$

where $B = \sup_{z \in E} |f(z)|$, and find all f for which equality is attained. Deduce that if f is not identically zero but vanishes at $x_k > 0$ for each $k = 1, 2, 3, \dots$ then $\sum_{k=1}^{\infty} 1/x_k < \infty$.

This underlies one approach to the proof of the Müntz-Szász theorem: if the topological span of $\{t^{x_k}\}_{k=1}^{\infty}$ is not dense in $\mathcal{C}(0,1)$ then its L^2 closure is a proper subspace of $L^2(0,1)$, and then there’s a nonzero $\phi \in L^2(0,1)$ orthogonal to each t^{x_k} ; then $f(x) := \int_0^1 \phi(t) t^x dt$ is holomorphic and bounded on E and vanishes at $x = x_k$ for each k , “etc.”. We didn’t officially define $L^2(0,1)$ in this year’s Math 55: it’s the completion of the inner product space $\mathcal{C}([0,1])$ with respect to its norm $\|f\| = \left(\int_0^1 f(x)^2 dx \right)^{1/2}$.

This problem set is due Monday, April 31 [that is, May 1], at the beginning of class.

²This is basically saying that $\mathbf{P}^1(\mathbf{C})$ is a 1-dimensional complex manifold.