Math 55b: Honors Real and Complex Analysis

Homework Assignment #11 (8 April 2011):
Convexity; change of variable

> Old MacDonald had a form: $e_i \wedge e_i = 0$.
> —*Mike Stay (October 2009), in a* `mathoverflow` *thread "Do good math jokes exist?".*

Some basic facts about convex sets and functions. Recall that a subset E of a real vector space V is said to be *convex* if $x, y \in E \Rightarrow px + qy \in E$ for all $p, q \in [0, 1]$ such that $p + q = 1$. If E is convex, an (upward) *convex function* on E is a function $f : E \to \mathbf{R}$ such that $f(px + qy) \le pf(x) + qf(y)$ for all $x, y \in E$, $p, q \in [0, 1]$ with $p + q = 1$; equivalently, f is convex if $\{(x, t) \in V \oplus \mathbf{R} : t > f(x)\}$ is a convex subset of $V \oplus \mathbf{R}$.

1. Solve Exercise 7 on page 289. (The simplex Q^k is defined in Example 10.4 on page 247.)

2. i) Show that any convex function on a convex open set in $\mathbf{R}^k$ is continuous.
   ii) Let U be a convex open set in $\mathbf{R}^k$, and fix $B \in (0, \infty)$ and a compact subset $K \subset U$. Let $\mathcal{C}$ be the set of all convex functions $f : U \to [-B, B]$. Prove that the restriction of $\mathcal{C}$ to the space of continuous functions on K is equicontinuous.

3. Let E be a convex subset of $\mathbf{R}^n$, and $f : E \to \mathbf{R}$ a $\mathcal{C}^2$ function. Prove that if f is convex then the $n \times n$ matrix $(D_i D_j f(x))_{i,j=1}^n$ of second partial derivatives is positive semidefinite for all $x \in E$. Is the converse true?

4. [Jensen's inequalities] Let f be a convex function on a convex set E in some real vector space.
   i) If $x_i \in E$, $p_i \ge 0$, and $\sum_{i=1}^n p_i = 1$, prove that $x := \sum_{i=1}^n p_i x_i$ is in E and $f(x) \le \sum_{i=1}^n p_i f(x_i)$. (This contains many classical inequalities as special cases; e.g., the inequality on the arithmetic and geometric means is obtained by taking $E = (0, \infty)$, $f(x) = -\log x$, and $p_i = 1/n$.)
   ii) If $\phi : [a, b] \to E$ is a continuous function and $\alpha : [a, b] \to \mathbf{R}$ is an increasing function such that $\alpha(b) - \alpha(a) = 1$, prove that $x := \int_a^b \phi(t) \, d\alpha(t)$ is in E and $f(x) \le \int_a^b f(\phi(t)) \, d\alpha(t)$.

5. [Cf. PS8, problem 7] For real ν, λ with and $|\lambda| < 1$ and $\nu > |\lambda|$, prove that

$$\int_0^{\pi/2} \cos^\nu x \cos(\lambda x) \, dx = \frac{\pi}{2^{\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma\left(1 + \frac{\nu-\lambda}{2}\right) \Gamma\left(1 + \frac{\nu+\lambda}{2}\right)} \, .$$

   [The hypotheses on ν and λ can be relaxed considerably, as we'll see when we develop complex analysis.]

6. The logarithmic convexity of $\Gamma(x)$, or more generally of any function of the form $f(x) = \int (\alpha(t))^x \beta(t) \, dt$, can be interpreted as the nonnegativity of the determinant of a symmetric 2×2 matrix. Generalize this to larger determinants. For instance, prove that for any positive reals $a_1, \ldots, a_n$ the determinant of the $n \times n$ matrix with entries $\Gamma(a_i + a_j)$ is nonnegative, as is the determinant with entries $(a_i + a_j)^{-k}$ for any $k > 0$. [Hint for this last part: remember $\int_0^\infty t^{x-1} e^{-ct} \, dt$?]

Change of variable and related ideas:

7. [Calabi] Prove that

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

(Note that this is an improper integral (why?) so some care will be needed here.) Now let Δ be the triangle $\{(u,v) \in \mathbf{R}^2 | u, v > 0, u + v < \pi/2\}$. Prove that the map $T : \Delta \to \mathbf{R}^2$ defined by $T(u,v) = (\sin u/\cos v, \sin v/\cos u)$ is a $\mathcal{C}^1$ and $\mathcal{C}^1$-invertible map of Δ to the open unit square $0 < x, y < 1$, and compute its Jacobian determinant. Conclude that

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \iint_\Delta 1\,du\,dv = \frac{\pi^2}{8}$$

as desired.

[Can you generalize this to evaluate $\sum_{n=1}^{\infty} 1/n^4$, or show more generally that $\sum_{n=1}^{\infty} 1/n^s$ is a rational multiple of π^s for all positive even s? Can you prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + - \cdots = \frac{\pi^3}{32}$$

using this method?]

8. [Newton] Let V be the vector space $\mathbf{R}^3$ with the usual Euclidean norm, and let $\omega = dx_1 \wedge dx_2 \wedge dx_3$ be the standard volume form on V. Define a vector-valued function g from $V - \{\mathbf{0}\}$ to V by $g(v) = v/|v|^3$. Fix $r_1, r_2 \in \mathbf{R}$ with $0 < r_1 < r_2$, and let E be the spherical shell $\{x \in V : r_1 < |x| < r_2\}$.

i) Prove that if $v_0 \in V$ with $|v_0| < r_1$ then $\int_E g(x - v_0)\,\omega = 0$.

ii) Prove that if $v_0 \in V$ with $|v_0| > r_2$ then $\int_E g(x - v_0)\,\omega = Ig(-v_0)$, where $I = \int_E \omega = 4\pi(r_2^3 - r_1^3)/3$ is the volume of E.

Note: The manipulations required, while straightforward, may be somewhat lengthy. I don't assign many such problems, but this one has special significance, both theoretical and historical: Newton had observed that the force of gravity near the earth's surface and the acceleration of the moon in its orbit around the earth are consistent with a universal inverse-square law of gravitation, provided the gravitational force of a spherical body was equivalent to that of an equal point mass; but it was only some twenty years later that he succeeded in proving this result and thus clinching the inverse-square law. That's the significance of part (ii); part (i) also figures in the physics of electrostatic forces: a uniformly charged sphere exerts no force on its interior. (For the effect on the exterior of the sphere, see again part (ii).) It is now known that this can be proved in a more "conceptual" way, albeit at the cost of introducing more machinery (surface integrals, etc.), from the fact that the inverse-square force exerted by a point mass is the gradient of a potential function $G(x) = C/|x - x_0|$ satisfying the Laplace equation $\Delta G(x) = 0$ for all $x \neq x_0$.

This problem set due Friday, April 15, at the beginning of class.