Stone-Weierstrass theorem:

Thm (Weierstrass): Polynomials are dense in $C^0([a,b])$, ie. $\forall f\in C^0([a,b])$ $\exists P_n$ polynomials st. $P_n\to f$ uniformly on $[a,b]$.

Pf. uses convolution and its use to approximate / smooth functions.

Def: convolution: $(f*g)(x) = \int_{s+t=x} f(s)\,g(t)\,dt = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt = \int_{-\infty}^{\infty} f(s)\,g(x-s)\,ds.$

well-def'd if e.g. f and g are (piecewise) C^0 + one of them is compactly supported (ie. 0 outside some $[-M,M]$). (to avoid improper integrals).

Principle: "$f*g$ inherits the best properties of f and g".

This is because $\|f*g\|_\infty \le \|f\|_{L^1}\|g\|_\infty$.

here $(*)$ $(f*g)(x+h)-(f*g)(x) = \int f(x-t)\big(g(t+h)-g(t)\big)\,dt \le \|f\|_{L^1}\|g_h-g\|_\infty$ where $g_h(t):=g(t+h)$ (over relevant intervals, per compact support)

→ If g is C^0 then uniform continuity (over a compact interval) gives $\lim_{h\to 0}\|g_h-g\|_\infty = 0$.
($|g(t+h)-g(t)|<\varepsilon$ $\forall t$ when $|h|<\delta$). $\Rightarrow$ continuity of $f*g$.

→ If g is C^1 (continuously differentiable) then dividing $(*)$ by h and using mean value thm + unif. continuity of g' on a compact interval $\Rightarrow$ $f*g$ is continuously differentiable and $(f*g)'=f*g'$.

Hence if g is C^∞ then $f*g$ is C^∞!! (even if f isn't even continuous)

→ and... if g is a polynomial of degree d then so is $f*g$!
eg. because $g^{(d+1)}=0$ so $(f*g)^{(d+1)} = f*g^{(d+1)} = 0$, or more directly: $g(x)=\sum_{k=0}^d a_k x^k$

$$\Rightarrow (f*g)(x) = \sum_{k=0}^d a_k \int f(t)\,(x-t)^k\,dt = \sum_{k=0}^d\sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} a_k x^{k-\ell} \underbrace{\int f(t)\,t^\ell\,dt}_{\text{constant}}$$

manifestly a polynomial in x.

* Approximate identities:

Def: A sequence of functions $K_n\ge 0$ approximates identity if $\begin{cases} \bullet \int K_n\,dx = 1 \\ \bullet\ \forall\delta>0,\ \int_{|x|\ge\delta} K_n\,dx \to 0\end{cases}$

Thm: $\left.\begin{array}{l} f \text{ compactly supported \& continuous} \\ K_n \text{ approximate identity}\end{array}\right\} \Rightarrow f*K_n\to f$ uniformly.

Pf: $(f*K_n)(x)-f(x) = \int\big(f(x-t)-f(x)\big)\,K_n(t)\,dt = \int_{|t|\le\delta} + \int_{|t|\ge\delta}$. Estimate each term as follows.

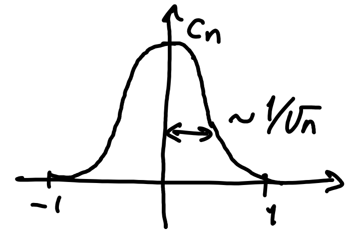
Given $\varepsilon > 0$, uniform continuity of f on its support $\Rightarrow \exists\, \delta$ (indep. of x) st.
$|t| < \delta \Rightarrow |f(x-t) - f(x)| < \varepsilon/2$

Then $\left| \int_{-\delta}^{\delta} (f(x-t) - f(x))\, K_n(t)\, dt \right| \leq \frac{\varepsilon}{2} \int_{-\delta}^{\delta} K_n(t)\, dt \leq \frac{\varepsilon}{2}$. (indeptly of x).

while $\left| \int_{|t| \geq \delta} (f(x-t) - f(x))\, K_n(t)\, dt \right| \leq 2\, \|f\|_\infty \int_{|x| \geq \delta} K_n\, dt \longrightarrow 0$ as $n \to \infty$
becomes $< \frac{\varepsilon}{2}$ for n suffly large.
$\rightarrow \exists N$ st. $\forall x$, $|(f * K_n)(x) - f(x)| < \varepsilon \quad \forall n \geq N$. $n \geq N$ (indept of x!)

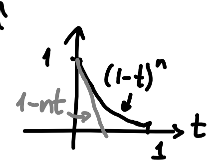
□

<u>Ex:</u> $K_n(x) = c_n (1-x^2)^n$ for $|x| \leq 1$, 0 elsewhere
where $c_n > 0$ is chosen so that $\int_{-1}^{1} K_n\, dx = 1$.



Claim: K_n approximate identity.

<u>Pf:</u> • for $|x| \leq \frac{1}{\sqrt{2n}}$, $(1-x^2)^n \geq 1 - nx^2 \geq \frac{1}{2}$, so $\int_{-1}^{1} (1-x^2)^n\, dx \geq \int_{-\frac{1}{\sqrt{2n}}}^{1/\sqrt{2n}} \geq \frac{1}{\sqrt{2n}}$
convexity



$\Rightarrow c_n \leq \sqrt{2n}$

• for $|x| \geq \delta$, $(1-x^2)^n \leq (1-\delta^2)^n$ so $\int_{|x| \geq \delta} K_n\, dx \leq 2 c_n (1-\delta^2)^n$
$\leq 2\sqrt{2n}\,(1-\delta^2)^n \underset{n \to \infty}{\longrightarrow} 0$

$\Rightarrow$ <u>Thm:</u> (Weierstrass):
‖ $\forall f \in C^0([a,b])$ $\exists\, P_n$ polynomials st. $P_n \to f$ uniformly.

<u>Pf:</u> • by linear change of variables, we can assume $[a,b] = [0,1]$.
• subtracting a degree 1 polynomial from f we can assume $f(0) = f(1) = 0$.
Then extend f to $\mathbb{R}$ by $f(x) = 0$ if $x \notin [0,1]$.
• now let $K_n(x) =$ as above, and $p_n = f * K_n$.
Then K_n approx. identity, f C^0 compactly supported $\Rightarrow p_n \to f$ uniformly.
• p_n is a polynomial of degree $2n$ on $[0,1]$ because, given that $f = 0$ outside $[0,1]$, the formula $(f * K_n)(x) = \int f(x-t)\, K_n(t)\, dt$ for $x \in [0,1]$ doesn't involve the values of K_n outside $[-1,1]$, and $K_n|_{[-1,1]}$ is polynomial.

□

• Stone's theorem generalizes this to other families of functions:

<u>Def:</u> | $\mathcal{A} \subset C^0(K)$ is an <u>algebra</u> if $f, g \in \mathcal{A} \Rightarrow f+g \in \mathcal{A}$, $cf \in \mathcal{A}$, $fg \in \mathcal{A}$.
| $\mathcal{A}$ <u>separates points</u> if $\forall a \neq b \in K$, $\exists f, g \in \mathcal{A}$ st. $f(a) = 1$, $f(b) = 0$, $g(a) = 0$, $g(b) = 1$

(0, 1 are arbitrary – this is equivt to $\mathcal{A} \to \mathbb{R}^2$, $f \mapsto (f(a), f(b))$ is surjective $\forall a \neq b$).

* For complex-valued functions, further assume $\mathcal{A}$ is conjugation-invariant, ie. $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$ (equivalently: $\mathrm{Re}\, f \in \mathcal{A}$, $\mathrm{Im}\, f \in \mathcal{A}$).

Thm (Stone): K compact metric space, $\mathcal{A} \subset C^0(K)$ algebra which separates points (+ conjugation invariant in $\mathbb{C}$ case), then $\mathcal{A}$ is dense is $(C^0(K), \|\cdot\|_\infty)$

(Weierstrass = special case $K = [a,b]$, $\mathcal{A}$ = polynomials).

Pf: • $\bar{\mathcal{A}}$ (uniform closure of $\mathcal{A}$) is an algebra ($f_n \to f$, $g_n \to g \Rightarrow f+g = \lim(f_n+g_n)$, $fg = \lim(f_n g_n)$)
so enough to show assumptions + $\mathcal{A}$ closed $\Rightarrow \mathcal{A} = C^0(K)$

• given $f \in \mathcal{A}$, $\mathcal{A}$ algebra & closed $\Rightarrow P(f) \in \mathcal{A}$ $\forall P$ polynomial st. $P(0)=0$
By Weierstrass, $|x|$ is a uniform limit of polynomials on $[-M,M]$, so $|f| \in \bar{\mathcal{A}} = \mathcal{A}$.
Hence: $f, g \in \mathcal{A} \Rightarrow \max(f,g) = \frac{f+g+|f-g|}{2} \in \mathcal{A}$, same for $\min(f,g)$.

• Now: given $f \in C^0(K)$, $\varepsilon > 0$, want to show $\exists h \in \mathcal{A}$ st. $\sup |h-f| \leq \varepsilon$. ($\Rightarrow f \in \bar{\mathcal{A}} = \mathcal{A}$).
given $x \in K$ $\forall y \neq x$ $\exists g_y \in \mathcal{A}$ st. $\begin{cases} g_y(x) = f(x) \\ g_y(y) = f(y). \end{cases}$ ($\mathcal{A}$ separates points).

$\exists U_y \ni y$ open st. $g_y > f - \varepsilon$ on U_y; and K compact $\Rightarrow \exists y_1 \dots y_n$ st. $U_{y_1} \cup \dots \cup U_{y_n} = K$.

Then $h_x := \max(g_{y_1}, \dots, g_{y_n}) \in \mathcal{A}$ satisfies $\begin{cases} h_x > f - \varepsilon \text{ everywhere} \\ h_x(x) = f(x). \end{cases}$

By the same argument, $\exists x_1, \dots x_n$ st. $k = \min(h_{x_1}, \dots h_{x_k})$ satisfies $|k-f| < \varepsilon$ everywhere.
($\exists V_x \ni x$ open st. $h_x < f + \varepsilon$ on U_x; K compact $\Rightarrow \exists x_1 \dots x_n$ st. $V_{x_1} \cup \dots \cup V_{x_n} = K$) □

Fourier series: we consider continuous 2π-periodic functions $f: \mathbb{R} \to \mathbb{C}$ with complex values, or equivalently functions on $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, with L^2 inner product $\langle f,g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \bar{f}(x) g(x)\, dx$

The complex exponentials $e_n(x) = e^{inx}$, $n \in \mathbb{Z}$ satisfy $\langle e_i, e_j \rangle = \delta_{ij}$ - orthonormality.

Def: The Fourier coefficients of f are $c_n(f) = \langle e_n, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx$.

→ the Fourier series of f is $\sum_{n \in \mathbb{Z}} c_n e_n = \sum_{-\infty}^{\infty} c_n(f) e^{inx}$

Q: (Fourier, Dirichlet, Féjer, ...) does the Fourier series accurately represent f? (eg. does it converge? to f?).

Def: Trigonometric polynomials = the vector space of finite linear combinations of e_n.

* Clearly this is an algebra, complex conj. invariant, and separates points of S^1, which is compact: hence by Stone-Weierstrass, trig. polynomials are dense in $(C^0(S^1), \|\cdot\|_\infty)$
... hence also in L^2-norm $\left(\|f\|_{L^2} = \left(\frac{1}{2\pi}\int |f|^2 dx\right)^{1/2} \leq \sup |f|\right)$.

* The n^{th} Fourier sum $f_n = s_n(f) = \sum_{-n}^{n} c_k e^{ikx} = \sum_{-n}^{n} \langle e_k, f\rangle e_k$ is the orthogonal projection of f onto $V_n = \mathrm{span}(e_{-n},\dots,e_n)$ for $\langle\cdot,\cdot\rangle$.

Indeed: $\langle e_j, f_n\rangle = \sum_{k=-n}^{n} c_k \langle e_j, e_k\rangle = c_j = \langle e_j, f\rangle$, so $\langle e_j, f-f_n\rangle = 0 \quad \forall -n\le j\le n$.

Thus: $\forall g\in V_n$, $\|f-f_n\|_{L^2} \le \|f-g\|_{L^2}$ – the point of V_n closest to f for $\|\cdot\|_{L^2}$

(This follows from $(f-f_n)\perp V_n$: $(f-g) = \underbrace{(f-f_n)}_{\perp V_n} + \underbrace{(f_n-g)}_{\in V_n} \Rightarrow \|f-g\|^2 = \|f-f_n\|^2 + \|f_n-g\|^2 \ge \|f-f_n\|^2$.)

$\Rightarrow$ Theorem (Parseval): Let $f\in C^0(S^1)$, $c_n = \langle e_n, f\rangle$ Fourier coeffts, $f_n = \sum_{-n}^{n} c_k e_k$ partial sums:

(1) $f_n \to f$ in L^2, ie. $\|f_n - f\|_{L^2}^2 = \frac{1}{2\pi}\int |f(x)-f_n(x)|^2 dx \to 0$ as $n\to\infty$.

(2) $\sum_{n\in\mathbb{Z}} |c_n|^2 = \|f\|_{L^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx$ (in particular $\sum |c_n|^2$ converges so $c_n\to 0$ as $|n|\to\infty$)

Pf: (1) Since trig.polynomials $= \bigcup_n V_n$ are dense in $(C^0(S^1), \|\cdot\|_{L^2})$,
$\forall \varepsilon>0\ \exists N$ st. $\exists g\in V_N$ with $\|f-g\|_{L^2}<\varepsilon$.
Now for $n\ge N$, $g\in V_N\subset V_n$ and $f_n =$ closest point to f, so
$\|f-f_n\|_{L^2} \le \|f-g\|_{L^2} < \varepsilon$. This shows $f_n\to f$ in L^2.

(2) since $f_n\in V_n$ and $f-f_n\in V_n^\perp$, $\|f\|_{L^2}^2 = \|f_n\|_{L^2}^2 + \|f-f_n\|_{L^2}^2$
where $\|f_n\|_{L^2}^2 = \|\sum_{-n}^{n} c_k e_k\|^2 = \sum_{-n}^{n} |c_k|^2$ by orthonormality, and
$\|f-f_n\|_{L^2}^2 \to 0$ by the first part. $\square$

Corollary: if $f,g\in C^0(S^1)$ have same Fourier series then $\frac{1}{2\pi}\int |f-g|^2 dx = \sum |c_n(f)-c_n(g)|^2 = 0$, hence $f=g$.

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* The fact that $f_n\to f$ in L^2 is the best approximation (in L^2 norm) of f by trig. polynomials, and that trig. polynomials are dense in $\|\cdot\|_\infty$ (so $\exists$ trig. polynomials $\to f$ uniformly) makes one hope that $f_n\to f$ uniformly or at least pointwise... alas not!

Fact: $\exists f\in C^0(S^1)$ st. the Fourier series of f does not converge ($s_n(f)(0)$ unbounded!) (but the example is hard to construct).

Thm (Dirichlet): if f is C^1 then $f_n = s_n(f) \to f$ uniformly.

The proof uses convolution – redefine, for periodic functions, $(f*g)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)g(x-t)\,dt$.
& note $c_n e_n(x) = \frac{1}{2\pi}\left(\int f(t)e^{-int}dt\right)e^{inx} = (f*e_n)(x)$.

So: $s_n(f) = \sum_{-n}^{n} c_k e_k = f*\left(\sum_{-n}^{n} e_k\right) = f*D_n$ where

$$D_n(x) = \sum_{-n}^{n} e^{ikx} = \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})}$$ <u>Dirichlet kernel</u> 5

Dirichlet's proof studies this convolution for $f \in C^1$ to prove unif. convergence.

The fact that convergence can sometimes fail makes it remarkable that $\forall f \in C^0$, f can be recovered from the partial sums $s_n(f) = f_n = \sum_{-n}^{n} c_k e^{ikx}$ ...

<u>Thm</u> (Féjer): If $f \in C^0(S^1)$ then $\dfrac{s_0(f) + \ldots + s_{n-1}(f)}{n}$ converges uniformly to f.

The reason is that this process amounts to convolution with the Féjer kernel $F_n = \dfrac{D_0 + \ldots + D_{n-1}}{n}$, which actually approximates identity (in the sense seen above) unlike D_n.