Solutions to Homework 9

Math 55b

1. *Show, directly from the definition of $\int_\gamma f(z)\,dz$ as a limit of Riemann sums, that $\int_\gamma z\,dz = 0$ for any closed loop γ in the plane.*

   It suffices to show that, for γ any rectifiable path from a to b, $\int_\gamma z\,dz = \frac{1}{2}b^2 - \frac{1}{2}a^2$. Given $\varepsilon > 0$, uniform continuity of γ implies that there exists a $\delta > 0$ such that, for z, w points on γ with $|z-w| < \delta$, the segment of γ with endpoints z and w has diameter $< \varepsilon$. Thus, for any subdivision $z_0 = a, \ldots, z_n = b$ of γ with diameter $< \delta$, the Riemann sum $\sum_1^n w_i(z_{i+1} - z_i)$ is, up to an error of $\varepsilon \sum_1^n |z_{i+1} - z_i| \le \varepsilon L(\gamma)$, independent of the points w_i between z_i and z_{i+1} are chosen; letting $\varepsilon \to 0$, we justify the existence of the integral. Taking the average of the Riemann sums with $w_i = z_i$ and $w_i = z_{i+1}$, we may as well take $w_i = \frac{z_i + z_{i+1}}{2}$ in computing the limit, and in this case the Riemann sum telescopes to $\sum_1^n \frac{z_i+z_{i+1}}{2}(z_{i+1} - z_i) = \sum_1^n \left(\frac{1}{2}z_{i+1}^2 - \frac{1}{2}z_i^2\right) = \frac{1}{2}b^2 - \frac{1}{2}a^2$, hence the conclusion. ∎

   Remark. This computation, together with the Goursat argument given in class, can be used to give a proof of the Cauchy theorem that is completely independent of Stokes' theorem. Specifically, the Goursat argument works just as well to prove Cauchy's theorem $\int_\gamma f(z)\,dz = 0$ for the case that γ is the boundary of a *triangle* in whose interior $f(z)$ is analytic (divide the triangle with its median lines into four congruent triangles, then into 16 smaller congruent triangles, etc. – the proof literally carries through, using only the basic integrals $\int_\gamma dz = \int_\gamma z\,dz = 0$ and *not* Stokes' theorem). Then, given an arbitrary (connected) region U, approximate ∂U by a piecewise linear loop γ lying in U, so that $\int_{\partial U} f(z)\,dz$ is uniformly approximated by $\int_\gamma f(z)\,dz$, reducing the assertion to the case of regions bounded by piecewise linear loops. But for these, choosing a point $p \in U$ and joining it to all vertices of the piecewise linear part ∂U to form (counterclockwise oriented) triangles $\gamma_1, \ldots, \gamma_n$, we have $\int_{\partial U} = \int_{\gamma_1} + \cdots + \int_{\gamma_n}$, because the *inner* paths of integration, those entering and leaving p, get mutually canceled out. Since Goursat's argument proves each $\int_{\gamma_i} f(z)\,dz = 0$, the Cauchy theorem follows. ∎

2. *Find those rational functions $f(z)$ that perserve the unit circle $|z| = 1$.*

Answer: $f(z) = e^{i\theta} \prod \frac{z-a}{1-\bar{a}z}$, where $a = \infty$ is also allowed, with the meaning of $1/z$. Those either preserve the unit *disk* Δ (if no factors of $1/z$ occur; those are called **finite Blaschke products**), or *turn the disk* Δ *inside out*, i.e. invert it with its complement $|z| > 1$.

That each such product preserves the circle S^1 follows from the calculation $|z-a| = |\overline{z-a}| = |1/z - \bar{a}| = |1 - \bar{a}z|$, valid on $|z| = 1$. In general, given $f \in \mathbb{C}(z)$ a rational function that preserves the circle $|z| = 1$, we can multiply f by a suitable Blaschke product to clear out all zeros and poles of f inside Δ, *without introducing any new zeros or poles in* Δ; this is because $|a| < 1$ is equivalent to $|\bar{a}^{-1}| > 1$, so that the zero and pole of a factor $\frac{z-a}{1-\bar{a}z}$ are never simultaneously inside Δ. Thus the problem reduces to showing: *if $f \in \mathcal{O}(\overline{\Delta})$ is an analytic function on $\overline{\Delta}$ without zeros or poles in Δ and preserving the circle $|z| = 1$, then f is constant.* For such a function, f and $1/f$ are simultaneously analytic in $\overline{\Delta}$ and satisfy $|f| = |1/f| = 1$ on $\partial\Delta = S^1$, so that the **maximum principle** implies $\sup_{\overline{\Delta}} |f| = 1 = \inf_{\overline{\Delta}} |f|$, implying that $|f| \equiv 1$, and hence (see ex. 3 below) f, is constant. To complete the proof, I include a discussion (and proof) of the maximum principle below. ∎

Maximum principle. A real-valued function f on a region U in $\mathbb{C}$ is called **subharmonic** if it satisfies the following **mean value inequality**: for any $p \in U$ and disk $|z-p| \le r$ contained in U,

$$f(p) \le \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta.$$

Equivalently, if f is smooth, it can be shown (and not needed here) that this subharmonicity condition is equivalent to the nonnegativity $\Delta f \ge 0$ of the Laplacian $\Delta = 4\partial\bar{\partial} = \partial_x^2 + \partial_y^2$ of f (recall that f is said to be *harmonic* if it satisfies $\Delta f = 0$; subharmonicity is the condition $\Delta f \ge 0$). For example, for f an analytic function on U, the real-valued function $|f|$ is subharmonic on U: this follows upon combining the Cauchy integral formula $f(p) = \frac{1}{2\pi\sqrt{-1}} \int_{|z-p|=r} f(z)\, \frac{dz}{z-p} = \frac{1}{2\pi\sqrt{-1}} \int_0^{2\pi} f(p + re^{i\theta})\, \sqrt{-1}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta$ with the triangle inequality. The **maximum principle** states that *a nonconstant subharmonic function on U cannot attain a maximum at an interior point.* This is immediate: if f attains a maximum at $p \in U$, then the mean value inequality shows that f takes this maximum value on the boundary of any closed disk centered at p and contained in U, hence on any disk centered at p and contained in U, hence f is locally constant, implying by connectedness of U that f is constant on U. As a consequence, since $|f|$ is subharmonic whenever f is analytic on U, it follows in conjunction with ex. 3 below:

The maximum principle for analytic functions. If $f \in \mathcal{O}(U)$ is a nonconstant analytic function on the connected region $U \subset \mathbb{C}$, then $\max|f|$ is never attained at an interior point of U. ■

Second proof. Alternatively, applying to the analytic functions $g = f$ and $g = 1/f$ the mean-value inequality $|g(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|\, d\theta$, which for analytic functions g is just a combination of the Cauchy integral formula and the triangle inequality as above, and using $|g| = 1$ on S^1 in both cases, we obtain $|f(0)| = 1$. Expanding the rational function $f(z) = \sum_{n\geq 0} a_n z^n$ in an absolutely and uniformly (in fact, geometrically) convergent power series in $\overline{\Delta}$ (which is possible by the assumption that f has no poles in $\overline{\Delta}$, by the geometric series $1/(z-a) = -a^{-1}/(1-a^{-1}z) = -a^{-1}\sum_{n\geq 0}(a^{-1}z)^n$ that converge absolutely and uniformly in $\overline{\Delta}$ for $|a| > 1$), we thus have $|a_0| = |f(0)| = 1$. On the other hand, on $|z| = 1$, we have $1 = |f(z)|^2 = f(z)\overline{f(z)} = \sum_{n\geq 0} |a_n|^2 + \sum_{k\neq 0} c_k z^k$ for certain coefficients c_k. Multiplying by $1/z$ and integrating over S^1 using $\int_{S^1} z^k\, dz = \int_{S^1} d(z^{k+1}/(k+1)) = 0$ for $k \neq -1$, we obtain $\sum_{n\geq 0} |a_n|^2 = 1$; together with $|a_0| = 1$, this shows all $a_n = 0$ for $n > 0$, and hence that $f(z) = a_0$ is constant, as required. ■

3. *Let $f_1, \ldots, f_n$ be analytic functions on a (connected) region $U \subset \mathbb{C}$ such that $\sum_1^n |f_i|^2$ is constant. Prove that all f_i are constant.*

Proof 1. Denote $\partial = d/dz$, $\bar\partial = d/d\bar z$, so that the analyticity condition becomes $\bar\partial f = 0$. Note that the Laplacian $\Delta = d^2/dx^2 + d^2/dy^2 = \partial_x^2 + \partial_y^2$ becomes $\Delta = 4\partial\bar\partial$. Applying it to $|f|^2$ for f an analytic function, we obtain $\Delta f = 4\partial\bar\partial|f^2| = 4\partial\bar\partial(f\bar f) = 4\partial(f\bar f') = 4f'\bar f' = 4|f'|^2$, where we have used that $\bar\partial f = 0, \partial f = f', \bar\partial \bar f = \bar f', \partial \bar f' = 0$ for f analytic. By this calculation, $\sum_1^n |f_i|^2 = \text{const}$ implies $0 = \Delta \sum_1^n |f_i|^2 = \sum_1^n \Delta\, |f_i|^2 = 4\sum_1^n |f_i'|$, showing all $f_i' = 0$ and hence all $f_i = \text{const}$. ■

Proof 2. We may alternatively use the Cauchy integral formula as our point of departure, proving something stronger: if $\sum_1^n |f_i| = \text{const}$, then all $f_i = \text{const}$ (this is stronger upon replacing each f_i with f_i^2). The Cauchy integral formula for f_i gives $f_i(p) = \frac{1}{2\pi\sqrt{-1}}\int_{|z-p|=r} f(z)\frac{dz}{z-p} = \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi} f(p+re^{i\theta})\sqrt{-1}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\, d\theta$ whenever the disk

$|z-p| \leq r$ is contained in U. Taking absolute values and summing, we obtain

$$\sum_1^n |f_i(p)| = \sum_1^n \frac{1}{2\pi}\Big|\int_0^{2\pi} f_i(p+re^{i\theta})\,d\theta\Big|$$

On the other hand, the constancy of $\sum_1^n |f_i|$ implies trivially the equality

$$\sum_1^n |f_i(p)| = \sum_1^n \frac{1}{2\pi}\int_0^{2\pi} |f_i(p+re^{i\theta})|\,d\theta.$$

Combining the two equalities, and using that, for a continuous function $g : S \to \mathbb{C}$ the triangle inequality $\left|\int_S g\right| \leq \int_S |g|$ can only turn into equality when $\arg(g)$ is constant on S, we conclude (again by connectedness of U) that each $\arg(f_i)$ is constant on U. This reduces the proof of our assertion to showing that a *real-valued* analytic function $f \in \mathcal{O}(U)$ is constant. This, by pre-composing in turn with the exponential $\exp(iz)$ (note that f analytic trivially implies $g(z) := e^{if(z)}$ analytic), reduces the problem to the case $n = 1$: if $g \in \mathcal{O}(U)$ is analytic, then $|g| \equiv 1$ implies $g = \text{const}$. Indeed, the Cauchy formula (or mean value property) for the analytic function $g(z) = e^{if(z)}$ implies $e^{if(p)} = \frac{1}{2\pi\sqrt{-1}}\int_{|z-p|=r} e^{if(z)}\frac{dz}{z-p} = \frac{1}{2\pi}\int_0^{2\pi} e^{if(p+re^{i\theta})}\,d\theta$, and hence, upon dividing by $e^{if(p)}/2\pi$ and taking real parts, $2\pi = \int_0^{2\pi}\cos\big(f(p+re^{i\theta}) - f(p)\big)\,d\theta$, were we have used that the argument $f(p+re^{i\theta}) - f(p)$ is real-valued. Since $\cos t \leq 1$ for $t \in \mathbb{R}$, a real-valued continuous function $g : [0, 2\pi] \to \mathbb{R}$ with $\int_0^{2\pi}\cos g(\theta)\,d\theta = 2\pi$ is a constant multiple of 2π, hence the conclusion: $f = f(p)$ on the boundary of every closed disk $|z-p| \leq r$ contained in U and centered at p, hence $f = \text{const}$ on every disk contained in U and centered at p, hence (by connectedness of U), f is constant on U. The alternative proof is complete. ■

4. *Show that* $\prod_1^\infty(1+a_n) := \lim_{N\to\infty}\prod_1^N(1+a_n)$ *exists and is nonzero, provided* $a_n \neq -1$ *and* $\sum |a_n| < \infty$.

We need to show that $\prod_N^M(1+a_n) \to 1$ as $N \to \infty$; the key is the easily verified inequality $-2|x| \leq \log(1+x) \leq |x|$ for $|x| < 1/2$. For N large enough so that $|a_n| < 1/2$ for $n \geq N$, these inequalities imply $-2\sum_N^M |a_n| \leq \sum_N^M \log(1+a_n) \leq \sum_N^M |a_n|$, proving the claim by letting $N \to \infty$. ■

5. *Let $p(n)$ be the partition function. Show that $\sum_{n\geq 0} p(n)z^n = \prod_{n\geq 1} \frac{1}{1-z^n}$ for all complex z with $|z| < 1$.*

That the coefficients of the formal expansion $\prod_{n\geq 1} \frac{1}{1-z^n} = \prod_{n\geq 1}(1 + z^n + z^{2n} + \cdots)$ are given by the partition function is easy combinatorics; the point is to justify the formal product expansion for $|z| < 1$. This follows from ex. 3 above, by the geometric convergence of $\sum \left|\frac{z^n}{1-z^n}\right|$ for $|z| < 1$, since the latter implies $\prod_N^M \frac{1}{1-z^n} \to 1$ as $N \to \infty$, while the partial product expansion satisfies $\prod_1^N \frac{1}{1-z^n} = \sum_{n\leq N} p(n)z^n \mod z^{N+1}$. ■

6. *Let G denote the group of Möbius transformations $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Prove that the map $\phi : \mathrm{SL}_2(\mathbb{C}) \to G$ given by*

$$\phi :\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) = \frac{az+b}{cz+d}$$

is a surjective homomorphism, and compute its kernel. Explain the homomorphism geometrically in terms of the 'slope' map $s : \mathbb{C}^2 - \{(0,0)\} \to \widehat{\mathbb{C}}$ given by $s(z_1, z_2) = z_1/z_2$. Which vectors of $\mathbb{C}^2$ correspond to the fixed points of f?

The surjectivity of the map ϕ follows upon noting that every expression $\frac{az+b}{cz+d}$ of a Möbius transformation can be normalized upon dividing the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\sqrt{ad-bc} \neq 0$, yielding a coefficient matrix of determinant 1. The pre-image of the identity of G is $\{\pm I\}$: the identity $z = \frac{az+b}{cz+d}$ of rational functions is equivalent to $b = c = 0, a = d$, which for $ad = 1$ amounts to the two possibilities $a = d = 1$ and $a = d = -1$. That ϕ is a group homomorphism can be verified by a direct ('brute force') computation; but one may simply (as Alex did) note that both groups $\mathrm{SL}_2(\mathbb{R})$ and G act transitively on the set $\mathbb{CP}^1$ of 1-dimensional subspaces of $\mathbb{C}^2$, with $A \in \mathrm{SL}_2(\mathbb{C})$ acting in the same way as $\phi(A) \in G$, *proving* (from the axioms of a transitive group action!) that ϕ is a group homomorphism! Thus $\phi : \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbb{C}}$ is a surjective homomorphism with kernel $\{\pm I\}$, yielding an isomorphism $\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\{\pm I\} \cong G$. Identifying $\widehat{\mathbb{C}}$ with $\mathbb{CP}^1$ by identifying $z \in \mathbb{C}$ with the 1-dimensional subspace $\mathbb{C} \cdot (z, 1) \in \mathbb{CP}^1$ of $\mathbb{C}^2$, and $\infty \in \widehat{\mathbb{C}}$ with $\mathbb{C} \cdot (1, 0) \in \mathbb{CP}^1$, the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{CP}^1$ becomes identified with the action of G on $\widehat{\mathbb{C}}$; this explains the action in terms of the slope map on $\mathbb{C}^2 - \{(0,0)\}$. The fixed points of $\phi(A) \in G$ correspond to the images in $\mathbb{CP}^1$ of the eigenvectors of $A \in \mathrm{SL}_2(\mathbb{C})$. ■

7. *Prove that every $f \in G$ is conjugate to either $f(z) = \lambda z$ (for some $\lambda \in \mathbb{C}^*$) or $f(z) = z + 1$. Show that the value of λ can be determined from* $\mathrm{tr}(A)$ *if $f = \phi(A)$. Is it unique? What value(s) of* $\mathrm{tr}(A)$ *correspond to $f(z) = z+1$?*

Every $f = (az + b)/(cz + d) \in G$ has two fixed points, the solutions to the quadric $az + b = cz^2 + dz$, with double zeros counted twice.

Case 1. *f has two distinct fixed points.*

If these are $p \neq q$, we may use the Möbius transformation $g(z) := (z - p)/(z-q)$ to conjugate f to a Möbius transformation $g \circ f \circ g^{-1}$ that fixes 0 and ∞. Clearly, the Möbius transformations that fix 0 and ∞ are those of the form λz, $\lambda \in \mathbb{C}^*$; thus f is conjugate to such Möbius transformation.

Case 2. *f has a unique (double) fixed point.*

If p is this point, then conjugating, similarly, f by $1/(z - p)$ to move p to ∞, we see that f is conjugate to a Möbius transformation that has ∞ for its unique fixed point; the Möbius transformation having this property clearly being the translations $z \mapsto z + c$, $c \neq 0$, which are all conjugate to $z \mapsto z + 1$ via $z \mapsto z/c$, it follows that, in this case, f is conjugate to $z \mapsto z + 1$.

If $f(z) = \lambda z$ equals $\phi(A)$, then $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is diagonal with $ad = 1$ and $a/d = \lambda$, and then λ is recovered from tr(A) (non-uniquely) by the quadratic relation $\lambda + \frac{1}{\lambda} = \frac{a}{d} + \frac{d}{a} = \frac{a^2+d^2}{ad} = a^2 + d^2 = (a + d)^2 - 2ad = \mathrm{tr}(A)^2 - 2$. For $f(z) = z + 1$, the corresponding values of $\mathrm{tr}(A)$ are ± 2, corresponding to $A = \pm I$. ∎

8. *Let $H \subset G$ be the subgroup generated by a single map of the form αz, $\alpha \neq 0$. What is the centralizer of H? What is the normalizer? Answer the same question with $H = \langle z \mapsto z + 1 \rangle$, as well as with $H = \langle z \mapsto z + t \mid t \in \mathbb{C} \rangle$.*

Note that $f(z)$ commutes with $z \mapsto \alpha z$ iff $f(\alpha z) = \alpha f(z)$, and normalizes $\langle z \mapsto \alpha z \rangle$ iff $f(\alpha z) = \alpha^k f(z)$ for some $k \in \mathbb{Z}$. The first condition requires that $f(0) = \alpha f(0)$ and $f(\infty) = \alpha f(\infty)$, which subsequently require either $\alpha = 1$, or $f(0), f(\infty) \in \{0, \infty\}$. If $f(0) = 0$, then also $f(\infty) = \infty$ (since α is injective), and hence $f(z) = \beta z$ for some $\beta \neq 0$. If $f(0) = \infty$, then $f(\infty) = 0$, and hence $f(z) = \beta/z$ for some $\beta \neq 0$ (those are the only Möbius transformations that swap 0 and ∞), and this case requires

$\alpha^2 = 1$ and hence $\alpha = \pm 1$ for $f(\alpha z) = \alpha f(z)$ to hold. Thus the centralizer of H is: $\langle z \mapsto \beta z \mid \beta \in \mathbb{C}^* \rangle$, if $\alpha \neq \pm 1$; $\langle \beta z, \beta/z \mid \beta \in \mathbb{C}^* \rangle$, if $\alpha = -1$; and G, if $\alpha = 1$. (**NB**: Note that $z \mapsto -z$ and $z \mapsto 1/z$ commute, which many of you missed; in this question, as was explicitly clarified, H is generated by a *single* transformation $z \mapsto \alpha z$, and not by *all* such transformations!). The normalizer of $\langle z \mapsto \alpha z \rangle$ (for $\alpha \neq 1$) is always $\langle \beta z, \beta/z \mid \beta \in \mathbb{C}^* \rangle$; this is because $f(z) = \beta/z$ satisfy $f(\alpha z) = \alpha^{-1} f(z)$.

The centralizer of $\langle z \mapsto z + 1 \rangle$ consists of those f satisfying $f(z+1) = f(z) + 1$; namely, of the translations $z \mapsto z + c$. The normalizer consists of the f satisfying $f(z+1) = f(z) + k$ for some $k \in \mathbb{Z} \setminus \{0\}$; namely, it is $\langle z \mapsto kz + c \mid k \in \mathbb{Z} \setminus \{0\}, c \in \mathbb{C} \rangle$.

Finally, the centralizer of $\langle z \mapsto z + t \mid c \in \mathbb{C} \rangle$ consists of those f satisfying $f(z+t) = f(z) + t$ for all t; namely, again the translations $f(z) = z + c$; while the normalizer consists of the f satisfying $f(z+t) = f(z) + kt$ for some $k = k(t) \in \mathbb{Z} \setminus \{0\}$; namely, it is, once again, $\langle z \mapsto kz + c \mid k \in \mathbb{Z} \setminus \{0\}, c \in \mathbb{C} \rangle$. ∎

9. *Prove that the image of a circle or a line under a Möbius transformation is a circle or a line.*

Note that the group G of Möbius transformations is generated by the subgroup $\{z \mapsto az+b\}$ of affine transformations, together with the transformation $z \mapsto \frac{z+i}{z-i}$; since the affine transformations act transitively on lines and circles and preserve *both* lines and circles, it suffices to verify that $z \mapsto i\frac{z+1}{z-1}$ transforms the unit circle $|z| = 1$ onto the imaginary axis $\mathrm{Re}(z) = 0$. To this end, simply note that $\frac{z+i}{z-i} = \frac{x+i(y+1)}{x+i(y-1)} = \frac{(x+i(y+1))(x-i(y-1))}{x^2+(y-1)^2} = \frac{x^2+y^2-1}{x^2+(y-1)^2} + i\frac{2x}{x^2+(y-1)^2}$ has real part 0 iff $x^2 + y^2 = 1$, iff $|z| = 1$. ∎