Homework 11

Math 55b
Due Tuesday, 21 April 2009.

Notation: $\Delta(r) = \{z : |z| < r\}$; $\Delta = \Delta(1)$.

1. Let $f_n(z)$ be a sequence of analytic functions on Δ converging uniformly to $f(z)$. (i) Show that for each $r < 1$, $f_n'(z) \to f'(z)$ uniformly on $\Delta(r)$. (ii) Show the same statement for $r = 1$ is false.

2. Let $u : S^1 - \{\pm 1\} \to \mathbb{R}$ be the function which is 1 if $\operatorname{Im} z > 0$ and 0 if $\operatorname{Im} z < 0$. Find a continuous, harmonic extension of u to the unit disk. (The extension will be defined on $\overline{\Delta} - \{\pm 1\}$.) Then find the harmonic conjugate of u.

3. What is the residue at $z = 0$ of $\sin^3(1/z)$? What are the residues of $z/(1 - e^{z^2})$ at its singularities?

4. Compute the first 3 nonzero terms in the Taylor series $\sum a_n z^n = \sin^{-1}(z)$, by formally inverting $\sin(z) = z - z^3/3! + z^5/5! - \cdots$. What is the radius of convergence of the series for $\sin^{-1}(z)$?

5. Prove that a positive harmonic function on $\mathbb{C}$ must be constant.

6. Prove that if $f : \Delta \to \mathbb{C}$ satisfies $f(0) = 0$ and $\operatorname{Re} f(z) \leq 1$ for all $z \in \Delta$, then $|f'(0)| \leq 2$. For what $f(z)$ does equality hold?

7. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and there exist $A, B > 0$ such that $\operatorname{Re} f(z) \leq A|z|^n + B$, then $f(z)$ is a polynomial.

8. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that (i) f is never 0 and (ii) there exists a C such that $|f(z)| \leq \exp(C|z|^2)$ for all $z \in \mathbb{C}$.

9. Show that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(z_n) \to \infty$ whenever $z_n \to \infty$, then $f(z)$ is a polynomial.

10. Find the Laurent series for $f(z) = 1/(z(z-1)(z-2))$ valid (i) in the region $1 < |z| < 2$, (ii) in the region $|z| > 2$.