Zoom logistics:

- please turn on your video*, mute yourself except to ask/answer questions use your real name* (* if possible)
- Lectures are recorded. (speaker view = mostly me) (remind me to start recording when class starts)
- Ask questions either verbally or in Zoom chat. (I usually don't watch for raised hands in participant window)
- Internet issues:
  - short freezes will happen (if I don't seem to have noticed, a CA should tell me)
  - outage on my end: CAs lead Q&A for 1-2 minutes while I reconnect
  - major outage: check e-mail.

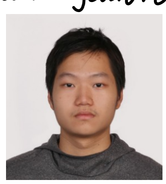
- Outside of lecture:
  - → Canvas (notes, assignments, ...)
  - → Slack (please join + introduce yourself in #general)
  - → e-mail
  - → discussions + office hours

Course staff:

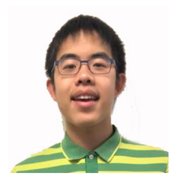
Prof. Denis AUROUX office hours Mondays 12-1 & Wednesdays 9-10 + 12-1.
auroux@math.harvard.edu

CAs: Avery Parr

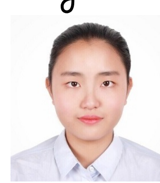
Alfian Tjandra



Richard Xu



Cheng Zhou



Gaurav Goel (volunteer)

- Office hours & sections: to be announced on Canvas.

- See course information & syllabus on Canvas (more logistics, policies, exams)
- Homework due Wednesdays on Canvas. HW 1 (due Feb. 3) is posted.
  Handwritten submissions are fine, or try LaTeX / Overleaf
  Collaboration encouraged (but write your own solution!). Ask CAs for hints if needed!
  Use slack (#studygroups, #homework). List your collaborators.
- Feedback survey to be completed this weekend.

- Please be civil & respectful of each other, and the rest of the math/Harvard community, at all times.

Course Content: first half = topology + real analysis.

1. Point-set topology: topological spaces (incl. some pieces of analysis)
2. Intro to algebraic topology: fundamental groups.
3. A bit more real analysis.

Then move on to complex analysis.

Books you should have:
- → Munkres, Topology, 2nd ed.
- → Ahlfors, Complex analysis, 3rd ed.
- → recommended: Rudin, Principles of Mathematical Analysis

<u>What is topology?</u> Unlike geometry, which concerns quantitative information about spaces (distances, volumes, ...), topology concerns itself with qualitative properties that are invariant under continuous deformation.

<u>Eg:</u> is it connected? (a single piece) simply connected?  vs. 

Point-set topology also gives a <u>language</u> (topological spaces, open & closed sets, compactness ...) both for algebraic topology (associate alg. invariants to spaces, eg. fundamental group) and for analysis.

<u>Ex:</u> extreme value theorem says: $f:[a,b]\to\mathbb{R}$ continuous $\Rightarrow$ f achieves its max and min at some points of $[a,b]$.

This is in fact true for any continuous $f: X\to\mathbb{R}$ whenever X is a <u>compact</u> topological space, and is a special instance of:

<u>Theorem:</u> If $f: X\to Y$ continuous mapping between topological spaces, & X compact, then $f(X)$ is compact.

Since the general notion of topological space is quite abstract, let's start with a more familiar class of examples: <u>METRIC SPACES</u>

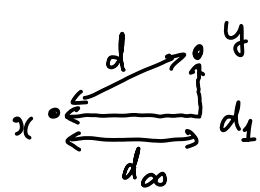
<u>Def:</u> A metric space (X,d) is a set X together with a <u>distance</u> function $d: X\times X\to\mathbb{R}_{\geq 0}$ s.t.

1) For $p,q\in X$, $d(p,q)=0 \Leftrightarrow p=q$
2) —— " ——, $d(p,q)=d(q,p)$
3) For $p,q,r\in X$, $d(p,r)\leq d(p,q)+d(q,r)$ (triangle inequality)

<u>Ex:</u> $X=\mathbb{R}^n$ with Euclidean distance $d(x,y)=\left(\sum_{i=1}^n (y_i-x_i)^2\right)^{1/2}$.

<u>Ex.</u> If $Y\subset X$ then $(Y, d_{|Y})$ is a metric space. ("induced metric").

<u>Ex:</u> different metrics on $\mathbb{R}^n$:
$$d_1(x,y)=\sum_{i=1}^n |y_i-x_i|$$
$$d_\infty(x,y)=\max(|y_i-x_i|)$$

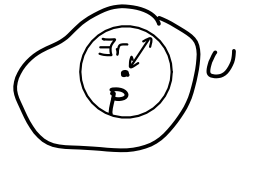


(Exercise: check $(\mathbb{R}^n,d_1)$ & $(\mathbb{R}^n,d_\infty)$ are metric spaces. What do balls look like?)

<u>Def:</u> • (X,d) metric space, $p\in X$, $r>0$: the <u>open ball</u> of radius r around p is (or <u>neighborhood</u>)
$$B_r(p)=\{q\in X \mid d(p,q)<r\}.$$

Here's a more general notion:

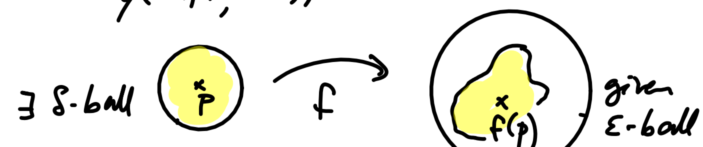
Def: • $U \subset X$ is <u>open</u> if $\forall p \in U$, $\exists r > 0$ st. $B_r(p) \subset U$.



Prop: (HW!) open balls are open; so are arbitrary unions & <u>finite</u> intersections of open sets.

- In fact: open sets are unions of open balls! $\left(U = \bigcup_{p \in U} B_{r(p)}(p)\right)$
- This is useful to a general discussion of <u>continuity</u>:

Def: (X, d_X), (Y, d_Y) metric spaces. $f: X \to Y$ is <u>continuous</u> if $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ st. $d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon$.



Theorem: $f: X \to Y$ is continuous iff $\forall U \subset Y$ open, $f^{-1}(U) \subset X$ is open.

Pf: • assume f continuous, let $U \subset Y$ open, let $p \in f^{-1}(U)$, ie. $f(p) \in U$..
Since U is open, $\exists \varepsilon > 0$ st. $B_\varepsilon(f(p)) \subset U$.
By continuity, $\exists \delta > 0$ st. $d_X(p, x) < \delta \Rightarrow f(x) \in B_\varepsilon(f(p)) \subset U$.
Hence $B_\delta(p) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open.

• conversely, assume U open $\Rightarrow f^{-1}(U)$ open.
Fix $p \in X$, $\varepsilon > 0$. $B_\varepsilon(f(p))$ is open in Y, so $f^{-1}(B_\varepsilon(f(p))) \ni p$ is open in X.
Hence $\exists \delta > 0$ st. $B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$.
This means $d_X(p, x) < \delta \Rightarrow x \in f^{-1}(B_\varepsilon(f(p))) \Rightarrow f(x) \in B_\varepsilon(f(p))$ ✓ $\square$

(Our first ε-δ proof, but not our last!).

We can also talk about sequences and their limits:

Def: A sequence $p_1, p_2, \ldots$ in (X, d) converges to a <u>limit</u> $p \in X$ (write $p_n \to p$ or $\lim_{n \to \infty} p_n = p$) if $\forall \varepsilon > 0$ $\exists N$ st. $\forall n \geq N$, $d(p_n, p) < \varepsilon$.

(unique if it exists).

p_n get closer to p!
vs. get closer to each other

Def: A sequence $p_1, p_2, \ldots$ in X is <u>Cauchy</u> if $\forall \varepsilon > 0$ $\exists N$ st. $\forall m, n \geq N$, $d(p_n, p_m) < \varepsilon$.

Exercise: if a sequence converges then it is Cauchy, but not necessarily vice-versa.

A metric space is <u>complete</u> if every Cauchy sequence converges.

Ex: $\mathbb{R}$ is complete, but $\mathbb{Q}$ (with induced metric) isn't complete.

* The notion of Cauchy seq. is specific to metric spaces, but really useful for real analysis.

Ex: $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ - if we take this to be the def^n of e, we can't prove directly that $x_n = \sum_{k=0}^{n} \frac{1}{k!}$ converges to e, instead use Cauchy criterion to show that the limit exists. $\square$

* Interlude: What is $\mathbb{R}$?

Ans: it's an ordered field (ie: $+ - \times /$, and $<$ compatible with usual rules) with the least upper bound property: every nonempty subset $E \subset \mathbb{R}$ that admits an upper bound ($\exists M \in \mathbb{R}$ st. $\forall x \in E$, $x \leq M$) has a least upper bound $\sup(E) \in \mathbb{R}$. (ie. $\sup(E)$ is an upper bound, and every upper bound for E is $\geq \sup(E)$).

The l.u.b. property is equivalent to completeness of $\mathbb{R}$; any ordered field with this property is isomorphic to $(\mathbb{R}, +, \times, <)$. Constructions of $\mathbb{R}$ from $\mathbb{Q}$ involve adding the missing elements (irrationals) so that l.u.b. property / completeness holds; the elements of $\mathbb{R}$ end up being either the sups of certain subsets of $\mathbb{Q}$ or the limits of Cauchy seq's in $\mathbb{Q}$

↳ see eg. Rudin ↳ see HW.

Returning to limits of sequences...

• Prop: if $p_n \to p$, then every open subset $U \ni p$ contains p_n for all but finitely many n.
This will be the definition of limit outside the metric case.

(Pf: $U \ni p$, U open $\Rightarrow \exists \varepsilon > 0$ st. $B_\varepsilon(p) \subset U$. So $\exists N$ st. $n \geq N \Rightarrow p_n \in B_\varepsilon(p) \subset U$).

• Def: $Z \subset X$ is closed if its complement $X \setminus Z$ is open.

⚠ Most subsets of X are neither open nor closed !! and... $\emptyset$ and X are both!

Prop: If $Z \subset X$ is closed, then
$\forall$ sequence $\{p_n\}$ in Z which converges to a limit $p \in X$, then $p \in Z$.

(the converse is true in metric spaces and in nice enough top. spaces - "first countable")

Pf: Assume $\exists \{p_n\} \in Z$, $p \in X - Z$, $p_n \to p$: $\forall U \ni p$ open, U contains p_n for all but finitely many n, but $p_n \in Z$, so $U \not\subset X - Z$.
If Z is closed then $U = X - Z$ is open and we get a contradiction. $\square$.

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* Our goal will be to reformulate / generalize all this in the context of topological spaces, ie. sets equipped with a topology which may or may not come from a metric.

Def: A *topology* $\mathcal{T}$ on a set X = collection of subsets of X, which we'll declare to be the *open sets* in X. Needs to satisfy axioms:

- $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$
- any union of elements of $\mathcal{T}$ is in $\mathcal{T}$
- the intersection of finitely many elements of $\mathcal{T}$ is in $\mathcal{T}$.

Why bother? One answer: many natural topologies do not come from a metric!

Eg, in analysis:

- on the space of (bounded) functions $f: X \to \mathbb{R}$, uniform convergence topology ($f_n \to f$ iff $\sup_x |f_n(x) - f(x)| \to 0$) comes from a metric $\left(d(f,g) = \sup_x |f(x) - g(x)|\right)$

  but pointwise convergence ($f_n \to f$ iff $\forall x \in X$ $f_n(x) \to f(x)$) doesn't. ("product topology")
- C^∞ topology on smooth functions $\mathbb{R} \to \mathbb{R}$ doesn't come from a metric either.

And on the other hand, a metric contains extraneous information for topology

Eg. $(\mathbb{R}^n, d)$, $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_\infty)$ have the same open sets $\Rightarrow$ same top.