Solutions to Homework 7

Math 55b

1. *Let X be a nonempty compact metric space, and suppose $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. (i) Prove that f has a unique fixed point. (ii) Give an example of such a map which is not a strict contraction.*

   (i) The map $X \to \mathbb{R}^{\geq 0}, \quad x \mapsto d(x, f(x))$ is manifestly continuous, being the composite of the continuous maps $X \to X \times X, \quad x \mapsto (x, f(x))$ and $d : X \times X \to \mathbb{R}^{\geq 0}$; by compactness of X, it attains a minimum. This minimum cannot be strictly positive, because $d(f(x), f(f(x))) < d(x, f(x))$ for $x \neq f(x)$, by assumption; hence the minimum is 0. A point with $d(x, f(x)) = 0$ is a fixed point of f; and conversely, the condition $d(f(x), f(y)) < d(x, y)$ for $x \neq y$ precludes the existence of more than one fixed points. Thus, f has a unique fixed point.

   (ii) The simplest example is to take $X := [0, 1]$ and $f(x) := x^2/2$: this map is a contraction because it satisfies $|f(x) - f(y)| = \frac{x+y}{2} \cdot |x - y| < |x - y|$ for $x \neq y$, and is not a strict contraction since $(x + y)/2$ can be arbitrarily close to 1 for $x, y \in [0, 1]$, $x \neq y$. ∎

2. *Let $U = B(0, 1) \subset \mathbb{R}^n$ be the open unit ball, and let* id $: U \to \mathbb{R}^n$ *be the identity map. (i) Show that if $\|\mathrm{id} - g\|_{C^1(U)}$ is small enough, then $g : U \to \mathbb{R}^n$ is injective. (ii) Give an example of a different open set U such that (i) fails.*

   (i) The relevant property of the unit ball is its convexity. We have shown, as an ingredient of the proof of the inverse function theorem, and an immediate consequence of the mean value theorem: if $U \subset \mathbb{R}^n$ is convex and $F : U \to \mathbb{R}^n$ is differentiable, then $|F(x) - F(y)| \leq |x - y| \cdot \|DF\|$ for all $x, y \in U$. Using this, the first idea from the proof of the inverse function theorem works without change to show: *if $U \subset \mathbb{R}^n$ is convex and $g : U \to \mathbb{R}^n$ satisfies $\|I - Dg\| < 1$, then g is injective.* (Note that the condition $\|I - Dg\| < 1$ is implied by, and is weaker than, the condition $\|\mathrm{id} - g\|_{C^1(U)} < 1$, which implies additionally $\sup_U |\mathrm{id} - g| < 1$). In detail, pick any $y_0 \in \mathbb{R}^n$, and consider the map $F : U \to \mathbb{R}^n, \quad x \mapsto y_0 + x - g(x)$, whose differential is $DF = I - Dg$. Convexity of U and the assumption $\|I - Dg\| < 1$ imply $|F(x) - F(y)| < |x - y|$ for $x \neq y$. In particular, F

has at most one fixed point, and this means that the equation $g(x) = y_0$ has at most one solution. Thus g is injective, as claimed.

(ii) Consider for U the union of two open balls touching (externally). Translating these balls slightly towards each other can be arbitrarily close to the identity in the C^1-norm, but is never injective. ■

3. *Prove that $d(d(\omega)) = 0$ for any smooth form ω on $\mathbb{R}^n$.*

The reason lies in the *commutativity of partial differentiation*, or the symmetry of the Hessian $\left(d^2 f/dx_i\, dx_j\right)_{i,j}$ of a smooth function f on $\mathbb{R}^n$. Indeed, in the case of a 0-form, i.e. a function f, the statement is, by virtue of $d(d(f)) = \sum_{i,j} \frac{d^2 f}{dx_i\, dx_j}\, dx_i\, dx_j$ and the anticommutativity $dx_i\, dx_j = -dx_j\, dx_i$, simply a restatement of the symmetry of the Hessian matrix. In the case of an arbitrary smooth form $\omega = \sum_I f_I\, dx_I$, the assertion then follows from this special case by $d(d\omega) = d\left(\sum(df_I\, dx_I + f_I\, d(dx_I))\right) = d\left(\sum df_I\, dx_I\right) = \sum(d(df_I)\, dx_I + df_I\, d(dx_I)) = 0$, which holds by virtue of $d(dx_I) = 0$. ■

4. *(i) Find an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which sends the unit square S to the parallelogram P with vertices $(1,1), (3,2), (4,5), (2,4)$. What is $\det Df$? (ii) Let $\omega := \exp(x-y)\, dx\, dy$ on P. Compute $f^*\omega$. (iii) Compute $\int_P \omega$ using an integral over S.*

(i) The required affine map is $(x, y) \mapsto (2x + y + 1, x + 3y + 1)$, because it maps the vertices of S to the vertices of P, and the affine image of a square is a parallelogram. The differential of f is identified with the Jacobian matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, which has determinant 5.

(ii) The pull-back of $\omega = \exp(x - y)\, dx\, dy$ under f is $f^*\omega = \exp(x - 2y) \det(Df)\, dx\, dy = 5 \exp(x - 2y)\, dx\, dy$.

(iii) By (i), the affine map f perserves the orientation of $\mathbb{R}^2$. Hence, by (ii), we have $\int_P \omega = \int_S f^*\omega = 5 \int_S \exp(x - 2y)\, dx\, dy = 5 \int_0^1 \left(\int_0^1 \exp(x - 2y)\, dx \right) dy = 5 \int_0^1 (e^{1-2y} - e^{-2y})\, dy = \frac{-5}{2}(e^{-1} - e^1) + \frac{5}{2}(e^{-2} - e^0) = \frac{5}{2}(e - 1 - e^{-1} - e^{-2})$. ■

5. *Let $R := [a, b] \times [c, d] \subset \mathbb{R}^2$. Prove directly (without using Stokes' theorem) that if $\omega = f\, dx + g\, dy$ satisfies $dg/dx = df/dy$, then $\int_{\partial R} \omega = 0$.*

Integrating the condition $dg/dx = df/dy$ using the Fubini theorem and the fundamental theorem of calculus, we have $\int_c^d \big(g(a,y) - g(b,y)\big)\,dy = \int_c^d \Big(\int_a^b \frac{dg}{dx}\,dx\Big)dy = \int_a^b \Big(\int_c^d \frac{df}{dy}\,dy\Big)dx = \int_a^b \big(f(x,c) - f(x,d)\big)\,dx$, which translates into $\int_{\partial R} \omega = 0$. ∎

6. *(i) Let $\gamma : [a,b] \to \mathbb{R}^2$ be a smooth loop enclosing a region U. Using Stokes' theorem, prove that the area of U equals $\frac{1}{2}\int_a^b \det(\gamma(t), \gamma'(t))\,dt$. (ii) Parametrize the curve $r^2 = \cos 2\theta$, and compute the area it encloses.*

   (i) Follows from Stokes' theorem by the identity $2\,dx\,dy = x\,dy - y\,dx$ and by $\gamma^*(x\,dy - y\,dx) = \det(\gamma(t), \gamma'(t))\,dt$.

   (ii) The loop is based at the origin, with two components parametrized by $\gamma_\pm : [-\pi/4, \pi/4] \to \mathbb{R}^2$, $\big(\pm\sqrt{\cos(2t)}\cos t, \sqrt{\cos(2t)}\sin t\big)$. Since $\gamma'_+(t) = \big(-\frac{\sin 2t \cos t}{\sqrt{\cos 2t}} - \sqrt{\cos 2t}\sin t, -\frac{\sin 2t \sin t}{\sqrt{\cos 2t}} + \sqrt{\cos 2t}\cos t\big) = \big(\frac{-\sin 3t}{\sqrt{\cos 2t}}, \frac{\cos 3t}{\sqrt{\cos 2t}}\big)$ and hence $\det(\gamma(t), \gamma'(t)) = \cos 2t$, (i) shows that the area enclosed by γ_+ is $\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos 2t\,dt = 1/2$. The area enclosed by γ_- is the same by symmetry, and the total area enclosed is 1. ∎

7. *Give a formula relating Euclidean (x,y,z) coordinates to spherical (r,θ,ϕ) coordinates on R^3. Use it to compute the Euclidean volume element $dx\,dy\,dz$ in spherical coordinates.*

   The formula relating Euclidean to spherical coordinates is $(x,y,z) = \big(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi\big)$. The Jacobian matrix of the *Euclidean-to-spherical* map is then easily computed to be

$$\begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\cos\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix},$$

   and its determinant is easily found to be $-r^2\sin\phi$, showing that the Euclidean volume form $dx\,dy\,dz$ pulls back to the form $-r^2\sin\phi\,dr\,d\theta, d\phi = r^2\sin\phi\,dr\,d\phi\,d\theta$. ∎

8. *Find a 2-form ω on $\mathbb{R}^3$ such that $\int_{S^2}\omega$ computes the area of the unit sphere, and $\int_{B^3} d\omega$ computes the volume of the unit ball. Relating the two integrals by Stokes' theorem, show that* $\text{area}(S^2) = 3\text{vol}(B^3)$. *Generalize to $\mathbb{R}^n$, explaining geometrically this relationship.*

The area element dA of the unit sphere is given by the restriction to S^2 of the 2-form $\omega := *dr = *d\,(x^2+y^2+z^2)^{1/2}$ on $\mathbb{R}^3$, or, which is the same, by the restriction to S^2 of the 2-form $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ on $\mathbb{R}^2$: this is because $dr * dr = dx\,dy\,dz$ on S^2 (we could take for ω any 2-form on $\mathbb{R}^3$ satisfying $dr\,\omega = dx\,dy\,dz$ on S^2, with the same result). As $d\omega = 3\,dx\,dy\,dz = 3\,dV$, Stokes' theorem shows $\text{area}(S^2) = \int_{S^2}\omega = \int_{B^3} d\omega = 3\int_{B^3} dx\,dy\,dz = 3\text{vol}(B^3)$. This generalizes on $\mathbb{R}^n$: the $(n-1)$-form $*dr = *d(x_1^2+\cdots+x_n^2)^{1/2}$ restricts to S^{n-1} as $\sum_i x_i * dx_i$ and satisfies $dr * dr = \sum_{i,j} x_i x_j\,dx_i * dx_j = (x_1^2+\cdots+x_n^2)\,dx_1\cdots dx_n = dx_1\cdots dx_n$ on S^{n-1}, and hence gives the $(n-1)$-dimensional volume element on S^{n-1}. Thus we may take the $(n-1)$-form $\omega := \sum_i x_i * dx_i$. Its exterior derivative is $d\omega = \sum_i dx_i * dx_i = n\,dx_1\cdots dx_n$, which is n times the volume form on $\mathbb{R}^n$, and Stokes' theorem again gives $\text{vol}_{n-1}(S^{n-1}) = \int_{S^{n-1}}\omega = \int_{B^n} d\omega = n\int_{B^n} dx_1\cdots dx_n = n\,\text{vol}_n(B^n)$. ■

9, 10. *(9) Let γ be the oriented path in $\mathbb{R}^3$ that connects $(1,0,0)$ to $(1,0,1)$ by spiraling three times around the surface of the cylinder $x^2+y^2=1$ at a constant slope. Let $\omega = y\sin z\,dx + x\sin z\,dy + xy\cos z\,dz$. Compute $\int_\gamma \omega$; (10) Let $S \subset \mathbb{R}^3$ be the part of the surface $x^4+y^4+z^4=1$ with $z \geq 0$. Give S a well-defined orientation, and then compute $\int_S \omega$, where $\omega = e^y z^2\,dx\,dy - 2e^y xz\,dy\,dz + \cos z\,dx\,dz$.*

In the case of (10), we first need to give S an orientation. It inherits an orientation from the planar region $x^4+y^4 \leq 1$ via projection from the point $(0,0,-1)$, which maps S diffeomorphically onto $\{x^4+y^4 \leq 1\} \subset \mathbb{R}^2$, inheriting the anti-clockwise orientation of the latter region given by the 2-form $dx\,dy$. Alternatively, we could orient the entire hypersurface $x^4+y^4+z^4=1$ by just pointing out a 2-form on $\mathbb{R}^3$ that does not vanish on the hypersurface; such a 2-form is $\omega := *d(x^4+y^4+z^4-1) = 4(x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy)$: it does not vanish along S, since $\omega * \omega = 16(x^6+y^6+z^6)\,dx\,dy\,dz$ does not vanish outside the origin (and in particular, does not vanish along S). More generally, the formula $df * (df) = |\nabla f|^2\,dx_1\cdots dx_n$ shows that the hypersurface $S = \{f=0\}$ in $\mathbb{R}^n$ is oriented by the $(n-1)$-form $*df|_S$, provided that the gradient ∇f does not vanish along the level set $S = \{f=0\}$; that is, provided the hypersurface is **nonsingular**.

Once this is done (either way in the case of (10), and regardless of which of the two possible orientations is chosen), both integrals are equal to 0, because in both cases, ω is of the form $d\eta$ with η vanishing along the boundary: in the case of (9) we have $\omega = d(xy\sin z)$ with $xy\sin z$ vanishing along the boundary $\{(1,0,0),(1,0,1)\}$, and in the case of (10) we have $\omega = d(xe^y z^2\, dy - x\cos z\, dz)$ with $xe^y z^2\, dy - x\sin z\, dz$ vanishing along the boundary $\partial S = \{z = 0\} \cap S$. ∎