Solutions to Homework 5

Math 55b

1. *Let $\alpha > 1$. Find all functions $f : [0,1] \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq C|x-y|^\alpha$ for some $C > 0$.*

   The answer is, as everyone shows, $f \equiv$ const. The difference quotient satisfies $\left|\frac{f(x)-f(y)}{x-y}\right| \leq C|x-y|^{\alpha-1}$, and since $\alpha - 1 > 0$, this implies that f is differentiable, with $f' \equiv 0$ identically. By the **fundamental theorem of calculus** (and this is the main thing to note!), this implies $f(y) - f(x) = \int_x^y 0\,dx = 0$ for all $x, y \in [0,1]$; that is, $f \equiv$ const. ∎

   Remark. Alternatively (and equivalently), some of you quoted the mean value theorem instead: for arbitrary $x < y$, it implies the existence of $c \in (x,y)$ with $f(y) - f(x) = (y-x)f'(c)$; which by $f' \equiv 0$ again yields $f \equiv$ const. Finally, of course, one may give direct proofs that even avoid the mean value theorem. ∎

2. *Suppose $f : [a,b] \to \mathbb{R}$ differentiable, $f(a) = 0$, and $|f'(x)| \leq f(x)$ for all x. Prove that $f \equiv 0$ identically.*

   Upon dividing the interval $[a,b]$ into closed subintervals of lengths < 1, we lose no generality in assuming $b - a < 1$. By compactness of $[a,b]$, the supremum $M := \sup_{[a,b]} f$ is attained, say at $c \in (a,b)$. The mean value theorem gives $M = f(c) = f(c) - f(a) = (c-a)f'(\xi)$ for some $\xi \in (a,c)$, which by $c - a < b - a$ and the assumption $|f'(\xi)| \leq f(\xi) \leq M$ implies $M \leq (b-a)M$; as $b - a < 1$ and $M \geq 0$, this last inequality yields $M = 0$, showing that $f \equiv 0$. ∎

3. *Given $p \geq 1$ and $f, g \in C[a,b]$, prove the* **Minkowski inequality**

   $$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

   One proof is by reduction to the Hölder inequality shown in class, namely $\left|\int_a^b f(x)g(x)\,dx\right| \leq \|f\|_p\,\|g\|_q$ for $1/p + 1/q = 1$. Observe to this end that $|f+g|^p \leq |f| \cdot |f+g|^{p-1} + |g| \cdot |f+g|^{p-1}$ by the triangle inequality, and that Hölder's inequality bounds the terms on the right hand side by $\|f\|_p \cdot \||f+g|^{p-1}\|_q$ and $\|g\|_p \cdot \||f+g|^{p-1}\|_q$, respectively. The result, upon noting $(p-1)q = p$, is precisely the Minkowski inequality. ∎

4. *Prove that, for $f \in C[a,b]$, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty := \sup_{[a,b]} f(x)$.*

   The inequality $\|f\|_p \leq \|f\|_\infty$ is obvious since $|f(x)| \leq \|f\|_\infty$ for all $x \in [a,b]$; it remains to show, for a given $\varepsilon > 0$, the inequality $\|f\|_p > \|f\|_\infty - \varepsilon$ for $p \gg_\varepsilon 0$. By continuity of f and compactness of $[a,b]$, the supremum $\|f\|_\infty$ is attained at some point $c \in [a,b]$; let $\delta = \delta(\varepsilon) > 0$ be such that $|x-c| < \delta$ implies $|f(x)| > \|f\|_\infty - \varepsilon$. Then $\|f\|_p^p \geq \int_{|x-c|<\delta} |f(x)|^p \, dx \geq \int_{|x-c|<\delta} (\|f\|_\infty - \varepsilon)^p \, dx = 2\delta \cdot (\|f\|_\infty - \varepsilon)^p$ for all p; equivalently, $\|f\|_p \geq (2\delta)^{1/p} \cdot \big(\|f\|_\infty - \varepsilon\big)$ for all $p \geq 1$, and the proof is completed upon noting that $\lim_{p\to\infty} (2\delta)^{1/p} = 1$. ∎

5. *Suppose $f \in C^2(\mathbb{R})$ with $|f''(x)| \leq 1$ and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.*

   The picture shown in this assignment relates both f and f'' to the graph of f': it interprets f as the area underneath the graph, and f'' as the slope of the graph; it is already an intuitive proof, as it simply says that if the area below the graph is finite and the slope of the graph is everywhere bounded (so that no "spikes" occur), then the graph converges to the x-axis as $x \to \infty$.

   Rigorous proofs, with the precise estimate $\sup |f'|^2 \leq 4 \sup |f| \cdot \sup |f''|$, are immediate from this interpretation. Alternatively, some of you argued as follows. Fix $\varepsilon > 0$, and note that Taylor's theorem implies the existence of a map $\xi : \mathbb{R} \to \mathbb{R}$, $x \mapsto \xi_x$ such that $x < \xi_x < x+\varepsilon$ and $f(x+\varepsilon) = f(x) + \varepsilon f'(x) + \varepsilon^2/2 \cdot f''(\xi_x)$ for all $x \in \mathbb{R}$; since $\sup |f''| \leq 1$ and $\lim_{x\to\infty} f(x) = 0$ by assumption, this implies, for fixed $\varepsilon > 0$, that $|f'(x)| \leq \varepsilon$ for $x \gg_\varepsilon 0$, completing the proof by letting $\varepsilon \to 0$. ∎

6. *Let $f(x) := e^{-1/x^2}$, for $x \neq 0$, and 0, for $x = 0$. Prove that f is infinitely differentiable, and all its derivatives vanish at $x = 0$.*

   Note the basic inequality $e^t > 1 + t$ for $t > 0$; it implies $e^{-t} < t^{-1}$ for $t > 0$, proving $e^{-1/x^2} = o(x)$ as $x \to 0$, proving differentiability of f at 0 and $f'(0) = 0$. In general, $f(x)$ is clearly infinitely differentiable away from 0, being the composition of two infinitely differentiable functions on $\mathbb{R} \setminus \{0\}$; a straightforward induction shows that $f^{(n)}(x)e^{1/x^2} \in \mathbb{R}(x)$ is a rational function for each n; and since a rational function is $O(x^k)$ as $x \to 0$ for some $k \in \mathbb{Z}$, it suffices to show that $\lim_{x\to 0} x^k e^{-1/x^2} = 0$ for each $k \in \mathbb{Z}$: this will prove that f is infinitely differentiable, with $f^{(n)}(0) = 0$ for all n. Since $(x^k)' = kx^{k-1}$ and $(e^{-1/x^2})' = 2x^{-3}e^{-1/x^2}$, **L'Hôspital's rule** shows that $\lim_{x\to 0} x^k e^{-1/x^2} = 0$ is equivalent to $\lim_{x\to 0} x^{k-4} e^{-1/x^2} = 0$. Since the assertion is true for $k = 0$ and since its truth for k evidently implies its truth for $k+1$, it follows that it holds for all $k \in \mathbb{Z}$. ■

7. *Let $L(n) := n \log(n) \log(\log(n)) \log(\log(\log(n))) \cdots$, where the product is continued until you reach a term $< e$. Is $\sum_n 1/L(n)$ convergent or divergent?*

   It is divergent. The proof is by comparing the sum to an integral, which can be done in the following more general situation. Suppose we have a decreasing sequence (a_n) of positive real numbers, and that we are interested in the convergence or divergence of the series $\sum_n a_n$. Suppose $f(x) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is any continuous decreasing function interpolating between the terms of this sequence: $f(n) = a_n$ for $n = 1, 2, \ldots$. *Then $\sum_n a_n = \sum_n f(n)$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent.* This follows upon noting the majorizations $\sum_{n=2}^N a_n \leq \int_1^N f(x)\,dx \leq \sum_{n=1}^{N-1} a_n$, which are valid since, f being decreasing, the inequality $a_{n+1} \leq \int_n^{n+1} f(x)\,dx \leq a_n$ for each n (this is known as **area comparison**, or the **integral test**).

   In our situation, there is an obvious continuous decreasing function that interpolates our sequence: this is the function $f(x) := 1/L(x)$, where $L(x) := x \log(x) \log(\log(x)) \cdots$, with the product continued until a term $< e$ is reached. By the preceding paragraph, it suffices to show that the divergence of the integral $\int_1^\infty \frac{dx}{L(x)}$. The key to this is that, by the chain rule, the function $\log^{[k]}(x) := \log(\log(\cdots \log(x)))$ (k times iterated log) is an antiderivative of $\frac{1}{x \log(x) \log^{[2]}(x) \cdots \log^{[k-1]}(x)}$, which coincides with $1/L(x)$ on the interval $[\exp^{[k-1]}(1), \exp^{[k]}(1)]$ (likewise, $\exp^{[k]}$ denotes k-times iterated exp). By the fundamental theorem of calculus, $\int_1^\infty \frac{dx}{L(x)} = \sum_{k\geq 1} \int_{\exp^{[k-1]}(1)}^{\exp^{[k]}(1)} \frac{dx}{L(x)} = \sum_k \left(\log^{[k]}(\exp^{[k]}(1)) - \log^{[k]}(\exp^{[k-1]}(1)) \right) = \sum_k 1 = \infty$, as claimed. ■

8. *Prove or disprove each of the following: (a) If $f : [a, b] \to \mathbb{R}$ is differentiable, then $f(b) - f(a) = \int_a^b f'(x)\,dx$. (b) If $f : [a, b] \to \mathbb{R}$ is monotone and differentiable, then $f'(x)$ is continuous. (c) If $f_n : [0, 1] \to \mathbb{R}$ are*

differentiable and $f_n \to g$ converge uniformly to a differentiable function g, then $f_n' \to g'$ pointwise.

All three propositions are false. To disprove (a), simply note the existence of differentiable $f : [a, b] \to \mathbb{R}$ with f' unbounded, hence not Riemann integrable; an example is $f(x) := x^2 \sin(1/x^2)$, $x \neq 0$, and 0, $x = 0$, on the interval $[-1, 1]$. Differentiability at 0, with $f'(0) = 0$, follows from $f(x) = O(x^2)$ as $x \to 0$; at $x \neq 0$, we have $f'(x) = 2x \sin(1/x^2) - 2x^{-1} \cos(1/x^2)$, which is unbounded as $x \to 0$.

To disprove (b), we may modify the example of $f(x) = x^2 \sin(1/x)$, which was our first example of a differentiable function whose derivative is discontinuous at 0, by adding a sufficiently increasing continuous function; for example, $3x$ suffices. Thus, we take $f(x) := x^2 \sin(1/x) + 3x$, at $x \neq 0$, and 0, at $x = 0$; consider this function on $[-1, 1]$. It is differentiable, whose derivative is discontinuous at 0 and everywhere nonnegative: away from 0 it equals $2x \sin(1/x) - \cos(1/x) + 3$, and this is ≥ 0 for $x \in [-1, 1]$.

To disprove (c), take $f_n(x) := \sin(n^2 x)/n$. This sequence converges uniformly to the zero function; but the sequence $f_n'(x) = n \cos(n^2 x)$ does not converge pointwise. ∎

Remark. Part (a) raises the question of the existence of differentiable functions $f : [a, b] \to \mathbb{R}$ with f' bounded yet not Riemann integrable. There exist such functions (due to Volterra and Pompeiu), but their construction is more difficult. ∎