Solutions to Homework 6

Math 55b

1. *Let $K_n(x) := c_n(1-x^4)^n$ for $|x|<1$ and 0 otherwise, where c_n is chosen to make $\int_{\mathbb{R}} K_n = 1$. Prove that $\langle K_n \rangle$ is an approximate identity.*

   We need to show that $\int_{-r}^{r} K_n(x)\,dx \to_{n\to\infty} 1$ for each $r>0$; for this we need to estimate c_n from above. Most of you gave the (rough) estimate $c_n < n^{-1/4}$, obtained upon noting the majorization $(1-x^4)^n \geq 1-nx^4$ on $[0,1]$: to justify this inequality, it suffices to note that the function $f(x) := (1-x^4)^n - (1-nx^4)$ has $f(0)=0$ and derivative $f'(x) = 4nx^3(1-(1-x^4)^{n-1})$, which is nonnegative on $[0,1]$. This majorization yielding $\int_{-1}^{1} K_n = 2\int_0^1 (1-x^4)^n\,dx > 2\int_0^{n^{-1/4}} (1-nx^4)\,dx = 8n^{-1/4} > n^{-1/4}$, hence the claimed estimate $c_n < n^{-1/4}$. The estimate implies

   $$K_n \leq n^{-1/4}(1-r^4)^n \quad \text{in the region } r \leq |x| \leq 1,$$

   which approaches 0 with $n \to \infty$, for each fixed $r>0$. Integrating this inequality gives the required $\int_{-r}^{r} K_n(x)\,dx \to 0$. ∎

2. *Determine which of the following sequences of functions are equicontinuous, and give a uniform modulus of continuity $h(r)$ for the ones that are.*

   (a) $f_n(x) := \exp(nx), \quad x \in (-\infty, 1]$.

   This sequence is not equicontinuous, by the unboundedness of $f_n(1) - f_n(0) = e^n - 1$.

   (b) $f_n(x) := \sin(\sin(\cdots(\sin(x))))$ (n times), $\quad x \in [0, 2\pi]$; and

   (c) $f_n(x) := n + x^n, \quad x \in [0, 1/2]$.

   Both these sequences are equicontinuous, with uniform modulus of continuity $h(r) = r$. This is an instance of the following general principle, useful to be aware of, which is an immediate consequence of the mean value theorem: if F is the family of differentiable functions on $[a,b]$ such that $\sup_{f\in F} \|f'\| \leq M < \infty$, then F is equicontinuous with uniform modulus $h(r) = Mr$.

   (d) $f_n(x) := (1 + x/n)^n, \quad x \in [0, \infty)$

Since $f_2(x) = x^2/4 + x + 1$ is not even uniformly continuous on $[0, \infty)$, this family is *a fortiori* not equicontinuous. ∎

Remark. The sequence in (d) *is* equicontinuous on any bounded interval, since $f_n' = f_{n-1}$ is uniformly bounded by $2e^t$ on $[0, t]$. ∎

3. *Let X, Y be compact metric spaces. Show that the continuous functions of the form $f(x)g(y)$ span a dense subset of $C(X \times Y)$.*

This is, of course, an application of the Stone-Weierstrass approximation theorem. Note (and N.B.!) that the span of continuous functions of the form fg is $A := \{f_1 g_1 + \cdots + f_n g_n \mid f_i \in C(X), g_i \in C(Y)\}$, and these clearly form and algebra. The product of compact metric spaces $(X, d_X), (Y, d_Y)$, defined to be the metric space $(X \times Y, \sup(d_X, d_Y))$, is compact (its topology being the topology of pointwise convergence, or the **product topology**), and to apply the Stone-Weierstrass approximation theorem, we need to show that the algebra A separates points of $X \times Y$. For $(x_1, y_1), (x_2, y_2) \in C(X \times Y)$ arbitrary distinct points, the function $(x, y) \mapsto \frac{d_X(x,x_1)+d_Y(y,y_1)}{d_X(x_1,y_1)+d_Y(x_2,y_2)}$, which is manifestly in A, evaluates to 0 at (x_1, y_1), and 1 at (x_2, y_2). ∎

4. *Compute the Fourier series of the function $f(x) = |x|$ on $[-\pi, \pi]$ (extended periodically to the whole real line). Does the series converge absolutely?*

The Fourier coefficients are given by $c_n = \langle f(x), e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, e^{-inx} \, dx = \frac{1}{\pi} \int_0^{\pi} x \cos(nx) \, dx = \frac{1}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi} = \frac{(-1)^n - 1}{\pi n^2}$, for $n \geq 1$; and $c_0 = \pi/2$. The Fourier series thus converges absolutely. ∎

5. *Let $S \subset C[a, b]$ be a finite-dimension subspace. Prove that if $f_n \in S$ and $g : [a, b] \to \mathbb{R}$, then $f_n \to g$ pointwise iff $g \in S$ and $f_n \to g$ uniformly.*

The nontrivial implication is to show that the pointwise convergence $f_n \to g$ yields $g \in S$ and $f_n \to g$ uniformly. Fix a basis $g_1, \ldots, g_m$ of S. The key is to note that the linear independence of $g_1, \ldots, g_m$ yields the existence of m points $x_1, \ldots, x_m \in [a, b]$ and linear forms $L_1, \ldots, L_m \in \mathbb{R}[X_1, \ldots, X_m]_{(1)}$ such that every $f \in S$ has the expansion $f(x) = \sum_{i=1}^{m} L_i\big(f(x_1), \ldots, f(x_m)\big) g_i(x)$ in the basis $g_1, \ldots, g_m$; the conclusion is then immediate. To prove the claim, simply reformulate the linear independence of $g_1, \ldots, g_m$ as the injectivity of the linear transformation $\mathbb{R}^m \to \mathbb{R}^{[a,b]}$, $(y_1, \ldots, y_m) \mapsto$

$\left(\sum_{i=1}^m g_i(x)y_i\right)_{x\in[a,b]}$; by linear algebra, the latter is equivalent to the existence of an invertible $m\times m$-minor $\left(g_i(x_j)\right)_{i,j=1}^m$, and it remains to note that invertibility of this minor is equivalent to the linear independence of the m linear forms $T_i := \sum_{i=1}^m g_i(x_j)X_i$, which in turn amounts to the existence of the linear forms $L_1,\dots,L_m$ with $X_i = L_i(T_1,\dots,T_m)$—as claimed. ■

Second solution (Alex). Enumerate the rationals in $[a,b]$ as (r_k), and let $A_i := \{r_1,\dots,r_i\}$. Let $S_i\subset S$ be the subspace $\{f\in S \mid f|_{A_i}=0\}$, and note that $S\supset S_1\supset S_2\supset\cdots$ is a descending nested sequence of vector spaces. Since $\dim S<\infty$, this sequence stabilizes to $\bigcap_i S_i$; but the latter space is $\{0\}$, since the rational are dense: $f(x)=0$ for $x\in[a,b]\cap\mathbb{Q}$ implies $f=0$. Thus there exists some k with $S_k=\{0\}$, and this means that the evaluation map $S\to\mathbb{R}^{A_k}$ of normed vector spaces is injective and hence a homeomorphism onto its image; where S is given the sup-norm (inducing the topology of uniform convergence), and $\mathbb{R}^{A_k}$ the Euclidean norm (inducing, by definition, the topology of coordinate-wise convergence); note, indeed, that any injective linear transformation from a finite-dimensional normed vector space to a normed vector space is a homeomorphism onto its image. This means, for $f_n\in S$, that the uniform limit $\lim_n f_n$ exists (in S) iff each limit $\lim_n f_n(r_i)$ exists in $\mathbb{R}$, for $1\le i\le k$. The conclusion follows. ■

Comment. Apparently, this question caused some headache: most of you did not solve it. This I found surprising, especially since nobody asked for a discussion on this question during section: you should have — you would have received the hint! I hope you had looked at this question *before* the Monday section. Please look at every individual question as soon as it becomes available! ■

6. *Let $f(x,y)$ be a real-valued function on $\mathbb{R}^2$, and suppose df/dx and df/dy exist for every (x,y). Prove or disprove each of the following assertions: (a) f is continuous; (b) if, additionally, $|df/dx|,|df/dy|\le M$, the f is continuous; (c) if, additionally, $|df/dx|,|df/dy|\le M$, then f is differentiable.*

   (a) This is false, f need not be continuous. Consider the example $f(x,y) := xy/(x^2+y^2)$, for $(x,y)\neq(0,0)$, and 0, for $(x,y)=(0,0)$. This function is discontinuous at $(0,0)$; yet, as f vanishes along either coordinate axis, both partial derivatives of f at $(0,0)$ exist and equal 0.

(b) This is true: if the partial derivatives exist *and are bounded*, then f is continuous; this is a consequence of the mean value theorem, or equivalently, the main theorem of calculus for Riemann integration. Indeed, $|(x-x', y-y')| < \varepsilon$ implies $|f(x,y) - f(x',y')| \leq |f(x,y) - f(x,y')| + |f(x,y') - f(x',y')| \leq M|y-y'| + M|x-x'| < 2M\varepsilon$, hence the continuity.

(c) This is false, f need not be differentiable even if its partial derivatives exist and are bounded. Consider the example $f(x,y) := x^3/(x^2+y^2)$, whose partial derivatives $\partial f/\partial x = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$ and $\partial f/\partial y = -\frac{2x^3y}{(x^2+y^2)^2}$ are bounded, but whose differential does not exist: if it did, $f(\varepsilon,\varepsilon) = \varepsilon/2$ and $\partial_x f(0,0) = 1, \partial_y f(0,0) = 0$ would contradict the identity $Df(z) = f(z) + \partial_x f(z) + \partial_y f(z) + o(|z|), \quad z := (x,y)$. ■

7. *Give an example of a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that Df is invertible at every point, but f is not one-to-one.*

Of the many examples, the simplest is to identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ and take $f(z) := e^z$, whose derivative is invertible and which is has period $2\pi\sqrt{-1}$ and is hence not one-to-one. In real coordinates $x + \sqrt{-1}y = z$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $(x,y) \mapsto (e^x \cos y, e^x \sin y)$. ■