Homework 3

Math 55b
Due Tuesday, 17 Feb 2009.

1. Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ is not continuous.

2. Let $\mathcal{F}$ be the smallest collection of functions $f : [0,1] \to \mathbb{R}$ that contains $C[0,1]$ and is closed under pointwise limits: whenever $f_n \in \mathcal{F}$ and $g(x) = \lim f_n(x)$ exists for each $x \in [0,1]$, then $g \in \mathcal{F}$.

   Prove that $g \in \mathcal{F}$, where $g(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$ and $g(x) = 0$ otherwise.

3. Let (X,d) be a metric space, let S denote the set of Cauchy sequences $s = (x_i)$ in X. Prove that the function

   $$\overline{d}(s,s') = \lim_{i\to\infty} d(x_i, x_i')$$

   exists for all pairs $s, s' \in S$, and satisfies the triangle inequality.

   Let $\overline{X}$ be the quotient of S by the equivalence relation $s \sim s'$ if $\overline{d}(s,s') = 0$. Observe that $\overline{d}$ is naturally a function on $\overline{X}$ as well, and prove $(\overline{X}, \overline{d})$ is a complete metric space. Finally define a natural isometric map $\pi : X \to \overline{X}$, and prove that $\pi(X)$ is dense.

4. Let $X = \ell^1(\mathbb{N})$ be the vector space of all sequences $a : \mathbb{N} \to \mathbb{R}$ such that $\|a\|_1 = \sum |a_i| < \infty$. Prove that the metric $d(a,b) = \|a - b\|$ makes X into a complete metric space. Prove that the closed unit ball $\overline{B}(0,1)$ in X is not compact. Finally prove that for any $b \in X$ the set

   $$K(b) = \{a \in \ell^1(\mathbb{N}) \ : \ |a_i| \le |b_i| \ \forall i\}$$

   is compact.

5. Let $X = B[0,1]$ denote the vector space of bounded functions $f : [0,1] \to \mathbb{R}$. Is there a metric on X such that $d(f_n, g) \to 0$ if and only if $f_n(x) \to g(x)$ for all $x \in [0,1]$?

6. Does the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = \sin(nx)$ have a uniformly convergent subsequence?

7. Let $\alpha > 0$ be rational. Without appeal to calculus, determine

   $$\lim_{n\to\infty} (n+1)^\alpha - n^\alpha.$$