References: - Rudin "Principles of mathematical analysis" review ch.4 = continuity; (today ch.3, beginning of ch.7&8)
- McMullen's 55b notes sequences/series in $\mathbb{R}$ seq./series of functions) power series
review sec.3, today sec.4, start 5 (+ start ch.5: differentiation)

Basic object of real analysis = functions $f:\mathbb{R}\to\mathbb{R}$ (or a subset of $\mathbb{R}$, the domain of f)
+ their continuity, differentiability, integrals, ...
+ sequences and series of functions.

Review: real functions

* Continuity at x ($\forall\varepsilon>0\ \exists\delta$ st. $\forall y$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$) $\iff \lim_{t\to x} f(t)=f(x)$.
more general limits: $\lim_{x\to\infty} f(x)$, $\lim_{t\to x, t<x} f(t)$, ... ; infinite limits and limits at infinity can be understood as taking place in compactification $\mathbb{R}\cup\{\pm\infty\}$, or explicitly, eg.
$\lim_{x\to 0, x>0} f(x)=\infty$ means $\forall M>0\ \exists\delta$ st. $\forall x$, $0<x<\delta\Rightarrow f(x)>M$.

* Things we've already seen, using compactness & connectedness of $[a,b]\subset\mathbb{R}$:
  - continuous functions $f:[a,b]\to\mathbb{R}$ are uniformly continuous (same δ works at all x).
    ↳ ($\forall\varepsilon\ \forall x\ \exists\delta/\forall y$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$) ↳ ($\forall\varepsilon\ \exists\delta/\forall x,y$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$)
  - intermediate value theorem $f([a,b])$ is connected $\Rightarrow$ contains all reals between $f(a)$ & $f(b)$.
  - extreme value theorem $f([a,b])$ is compact $\Rightarrow$ bounded and contains its inf & sup.

* Two topologies on spaces of functions so far (here $\mathbb{R}\to\mathbb{R}$, but similarly for $\mathbb{R}^n\to\mathbb{R}^m$).
  - $f_n\to f$ pointwise if $\forall x$ $f_n(x)\to f(x)$ in $\mathbb{R}$ (= product topology)
  - $f_n\to f$ uniformly if $\|f_n-f\|_\infty := \sup_x |f_n(x)-f(x)| \to 0$ (= uniform topology).

  We've seen: f_n continuous + $f_n\to f$ uniformly $\Rightarrow f$ is continuous. (Lecture 4)
  + spaces of functions $\mathbb{R}\to\mathbb{R}$, $[a,b]\to\mathbb{R}$ ($\mathbb{R}^n\to\mathbb{R}^m$) with uniform topology are complete metric spaces, C^0 (continuous functions) are a closed subspace hence complete as well (but... unless we restrict to bounded functions, $\sup|f-g|$ isn't quite a metric).

  Analysis considers many different spaces of functions (bounded, integrable, continuous, differentiable) and topologies (often but not always metrics) on them.

* Beyond polynomials & a few other explicit examples, many functions are defined using limits of sequences or series. A key example (also for complex analysis!):
  Power series = expressions of the form $f(x)=\sum_{n=0}^{\infty} a_n x^n$ for some coeff's $a_n\in\mathbb{R}$.
  (Writing this expression doesn't in itself guarantee the series converges for any $x\neq 0$!)
  We'll want to understand convergence (pointwise, uniformly over certain subsets of $\mathbb{R}$, ...)
  → basic facts about real sequences & series in $\mathbb{R}$ come in handy.

More review: sequences & series in $\mathbb{R}$

* $\mathbb{R}$ is complete $\Rightarrow$ a sequence in $\mathbb{R}$ converges iff it is Cauchy
* compactness of $[-M,M]$ $\Rightarrow$ every bounded sequence in $\mathbb{R}$ has convergent subsequences.
* a monotonic sequence (eg. $a_n \le a_{n+1}$) converges iff it is bounded ($\Rightarrow \lim_{n\to\infty} a_n = \sup\{a_n\}$)
* sometimes write $a_n \to +\infty$ or $-\infty$; can interpret as convergence in compactification $\mathbb{R}\cup\{\pm\infty\}$. Such a sequence is still said to diverge.
* if (a_n) bounded then $M_n = \sup\{a_k, k\ge n\} \searrow$,
  $\limsup a_n := \lim M_n$ = largest limit of a convergent subsequence of (a_n). Similarly liminf.
  Ex: $a_n = \sin(\sqrt{n}\,\pi)$ or $(-1)^n\left(1+\frac{1}{n}\right)$ both have $\limsup = 1$, $\liminf = -1$.

* Recall a series $\sum a_n$ converges iff its partial sums $s_n = \sum_{k=0}^{n} a_k$ form a convergent sequence, and then we write $\sum_{n=0}^{\infty} a_n$ for the limit.
  * If $\sum a_n$ converges then $a_n \to 0$ (by Cauchy criterion for (s_n): $|s_n - s_{n-1}| \to 0$), but not vice versa ($\sum 1/n$ diverges even though $1/n \to 0$)
  * For $a_n \ge 0$, $\sum a_n$ converges iff partial sums are bounded. (since $s_n \nearrow$)
  * Hence: $0 \le a_n \le b_n$, $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent
    $\sum a_n$ divergent $\Rightarrow \sum b_n$ divergent (comparison criterion)
  * The geometric series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$ (does not converge if $|x|\ge 1$, in fact terms $\not\to 0$!).
  * $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges iff $\alpha > 1$. (Proof is a variant of comparison argument:
    $$\frac{2^k}{(2^{k+1})^\alpha} \le \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n^\alpha} \le \frac{2^k}{(2^k)^\alpha} \quad \text{so} \quad 2^{-\alpha}\sum_{k=0}^{m} 2^{(1-\alpha)k} \le \sum_{n=2}^{2^{m+1}} \frac{1}{n^\alpha} \le \sum_{k=0}^{m} 2^{(1-\alpha)k} \quad \text{geometric series!}$$
    $\Rightarrow$ partial sums are bounded iff $2^{1-\alpha} < 1$).
  * $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty} \frac{1}{k!}$ (equality comes from applying binomial theorem to $\left(1+\frac{1}{n}\right)^n$ and showing for fixed k, $\binom{n}{k}\left(\frac{1}{n}\right)^k \nearrow$ with n and $\to \frac{1}{k!}$ as $n\to\infty$).
    e is irrational: because denoting the partial sum $\sum_{k=0}^{n} \frac{1}{k!} = \frac{p_n}{n!}$, $e - \frac{p_n}{n!} \in \left(0, \frac{1}{n!}\right)$
    $\Rightarrow$ e isn't an integer multiple of $\frac{1}{n!}$ $\forall n$, so not rational.

* A series is <u>absolutely convergent</u> if $\sum |a_n|$ converges. $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges
  (using Cauchy: $|s_n - s_m| = \left|\sum_{m+1}^{n} a_k\right| \le \sum_{m+1}^{n} |a_k|$)

  but not viceversa, eg. <u>alternating series</u>:

  if $\begin{cases} a_n \text{ has the same sign as } (-1)^n \\ |a_n| \text{ decreasing with } n, \\ a_n \to 0 \end{cases}$ then $\sum a_n$ converges!
  (Proof: odd/even partial sums $\nearrow$ and $\searrow$ towards common limit)

Ex: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots = \log 2$, $\quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots = \arctan(1)=\frac{\pi}{4}$.

Absolutely convergent series can be safely rearranged ($\sum a_{\varphi(n)} = \sum a_n$), multiplied, etc.; others, not always.

* Root test: if $\limsup |a_n|^{1/n} < 1$ then $\sum a_n$ converges (absolutely) (comparison w/ geom. series)
$\quad > 1 \quad$ diverges $(a_n \not\to 0)$

This is used to great effect for power series!

Def: The radius of convergence of $\sum a_n x^n$ is $R = \frac{1}{\limsup(|a_n|^{1/n})} \in [0,\infty]$.

Thm:
- $\sum a_n x^n$ converges pointwise $\forall x \in \mathbb{C}$ st. $|x|<R$
- convergence is uniform on $\overline{B_r(0)} = \{x \,/\, |x| \le r\}$ $\forall r<R$ (but not necess. on $B_R(0)$)
- thus $f(x) = \sum a_n x^n$ is continuous over $B_R(0) = \{|x|<R\}$.
- the series diverges whenever $|x|>R$; at $|x|=R$ it may converge or diverge.

Pf:
- root test: $\limsup |a_n x^n|^{1/n} = |x| \limsup |a_n|^{1/n} = \frac{|x|}{R}$.
  $\Rightarrow$ converges for $|x|<R$, diverges for $|x|>R$.
- uniform convergence: if $|x|\le r$ then $\left|f(x) - \sum_0^n a_k x^k\right| = \left|\sum_{n+1}^\infty a_k x^k\right| \le \underbrace{\sum_{n+1}^\infty |a_k| r^k}_{\varepsilon_n}$.
  The series $\sum |a_n| r^n$ converges by root test, so $\varepsilon_n \to 0$,
  $\sup\left\{\left|f(x) - \sum_0^n a_k x^k\right|, \ |x|\le r\right\} \le \varepsilon_n \to 0 \quad \Rightarrow$ uniform convergence.
- partial sums are continuous, so f is continuous on $\{|x|\le r\}$ by unif convergence.
  hence continuous on $\bigcup_{r<R} \overline{B}_r = B_R(0)$ $\square$

Ex:
- $\sum_{n=1}^\infty (-1)^{n+1} \frac{x^n}{n} = \log(1+x)$ for $|x|<1$ $\quad (n^{1/n} \to 1$ so $R=1)$
  converges at $x=1$ (alternating series), diverges at -1.
- $\sum_{n=0}^\infty \frac{x^n}{n!} = \exp(x)$ converges everywhere (uniformly over bounded subsets)
  ($R=\infty$: indeed $n! > \left(\frac{n}{2}\right)^{n/2}$ so $(n!)^{1/n} > \left(\frac{n}{2}\right)^{1/2} \to \infty$).

* Power series form a ring (can add & multiply);
facts about sums & products of numerical series $\Rightarrow$ where convergent, sum/product as series are indeed equal to the sum/product as functions.

Differentiation in one variable (Rudin ch 5 = McMullen §5)

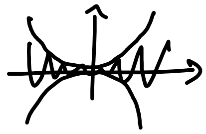
Def: $f: [a,b] \to \mathbb{R}$ is differentiable at x if $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} =: f'(x)$ exists.
(ie. $\forall \varepsilon\ \exists \delta$ st. $0<|t-x|<\delta \Rightarrow \left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \varepsilon$).

- Prop: f differentiable at $x \Rightarrow f$ continuous at x. (The converse is false, eg. $|x|$ at 0).

  Pf: $f(t)-f(x) = \underbrace{\frac{f(t)-f(x)}{t-x}}_{\to f'(x) \text{ as } t\to x} \cdot \underbrace{(t-x)}_{\to 0 \text{ as } t\to x}$ + multiplication is continuous $\Rightarrow f(t)-f(x) \to f'(x)\,0 = 0$.

- Usual rules of calculation hold: derivatives of $f+g$, fg, ...; $\underbrace{(f\circ g)'(x) = f'(g(x)) \cdot g'(x)}_{\text{chain rule}}$.
  (see Rudin p 104-105).

- **Ex:**
  - $\begin{cases} f(x) = x \sin \frac{1}{x} & (x \neq 0) \\ f(0) = 0 \end{cases}$  For $x \neq 0$, $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$

    continuous but not differentiable at 0 $\left(\nexists \lim_{x\to 0} \frac{f(x)}{x}\right)$.
  - $\begin{cases} g(x) = x^2 \sin \frac{1}{x} \\ g(0) = 0 \end{cases} \Rightarrow$  differentiable $(g'(0)=0)$ but g' not continuous at 0.
  - $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n!\,x)$ continuous (series converges uniformly, since $\sum \frac{1}{n^2}$ conv.), nowhere differentiable!

    (See also Rudin 7.18 for a related example).

* <u>Mean value theorem</u>: ‖ $f : [a,b] \to \mathbb{R}$ differentiable $\Rightarrow \exists c \in (a,b)$ st. $f(b) - f(a) = f'(c).(b-a)$.

Follows logically from easier results:

(1) | if $f : [a,b] \to \mathbb{R}$ has a local max (or min) at $x \in (a,b)$ (ie. max of $f_{|(x-\delta,\, x+\delta)}$) and f is differentiable at x, then $f'(x) = 0$.

(because $\frac{f(t)-f(x)}{t-x}$ is ≥ 0 for $t \in (x-\delta, x)$, ≤ 0 $t \in (x, x+\delta)$ $\Rightarrow$ take lim. as $t \to x$ from left and from right.)

(2) | if $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$ then $\exists c \in (a,b)$ st. $f'(c) = 0$.

clear if f is constant; otherwise look at max or min of f over $[a,b]$ & apply (1)

(3) mean val. thm = apply (2) to $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}\,x$.

<u>Corollary</u>: mean value inequality: $m \leq f'(x) \leq M$ $\forall x \in (a,b) \Rightarrow m(b-a) \leq f(b) - f(a) \leq M(b-a)$.

* Generalization: <u>Taylor's theorem</u>:

‖ $f : [a,b] \to \mathbb{R}$ n times differentiable. The deg. $(n-1)$ Taylor polynomial of f at a is:

$$P(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$ Then $\exists c \in (a,b)$ st. $f(b) = P(b) + \frac{f^{(n)}(c)}{n!}(b-a)^n$.

<u>Pf</u>:
- subtracting $P(x)$ from both sides, we can reduce to the case $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$. (and $P = 0$).
- let $g(x) = f(x) - f(b)\frac{(x-a)^n}{(b-a)^n} \Rightarrow g(b) = g(a) = 0$ + still have $g'(a) = \dots = g^{(n-1)}(a) = 0$.
- now: the mean value thm for g ; $g(a) = g(b) = 0 \Rightarrow \exists x_1 \in (a,b)$ st. $g'(x_1) = 0$.

  —— " —— g' : $g'(a) = g'(x_1) = 0 \Rightarrow \exists x_2 \in (a, x_1)$ st. $g''(x_2) = 0$

and so on until $\exists c = x_n \in (a, x_{n-1})$ st. $g^{(n)}(c) = 0$. Ie: $f^{(n)}(c) - \frac{n!\,f(b)}{(b-a)^n} = 0$. □

<u>Rmk</u>:
- can compare $f(x)$ to $P(x)$ by applying thm. to $[a,x]$ instead!
- as with mean value inequality: a bound $|f^{(n)}| \leq M$ gives a bound $|f(x) - P(x)| \leq \frac{M(x-a)^n}{n!}$ over $[a,b]$.

<u>Rmk</u>: there exist nonzero functions whose Taylor polynomials are all zero!

eg. $f(x) = \exp\left(-\frac{1}{x^2}\right)$, $f(0) = 0$: $f \in C^\infty$ (all derivatives exist), $f^{(k)}(0) = 0$ $\forall k$

so the Taylor polynomials are all zero! The Taylor series of f at 0 converges but $\neq f$! (in other examples, it can also have $R = 0$ ie. never convergent for $x \neq a$).