Math 55b Midterm Exam – Solutions

Problem 1. (12 points)

Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) If $A_i \subset X_i$ are closed subsets for all $i \in I$, then $\prod_{i\in I} A_i$ is a closed subset of $\prod_{i\in I} X_i$ with the product topology.

True: the complement of $\prod_{i\in I} A_i$ is the union $\bigcup_{i\in I} U_i$, where U_i is the set of points whose i-th component is not in A_i, i.e. the product of $X_i - A_i$ (open in X_i) and the whole space X_j for all $j \neq i$. Since U_i is open in the product topology, $\bigcup U_i$ is open, and hence $\prod A_i$ is closed. (Or: the result follows from Theorem 19.5 in Munkres).

(b) If $x_1, x_2, \dots \in X$ are limit points of a subset $A \subset X$, and if the sequence x_n converges to a limit $x \in X$, then x is a limit point of A.

False: let $X = \{a, b\}$ with the non-Hausdorff topology $\{\emptyset, X\}$, and let $A = \{a\}$. Then b is a limit point of A (every neighborhood of b contains $a \in A$, $a \neq b$), and the sequence $b, b, b, \dots$ converges to a (since every neighborhood of a contains b), but a is not a limit point of A (because the intersection of A with a neighborhood of a consists of only a, and does not contain any other point).

If we assume X is Hausdorff (or just T1: points are closed) then the result is true. Proof: If $x_n = x$ for some n then the conclusion obviously holds (since x_n is a limit point of A). Otherwise: let U be any neighborhood of x in X. Since $x_n \to x$, there exists N such that $x_n \in U$ $\forall n \geq N$, in particular $x_N \in U$. By assumption x_N is a limit point of A; and since $x \neq x_N$, $U - \{x\}$ is a neighborhood of x_N, hence contains some point $a \in A$. Hence U contains $a \in A$, with $a \neq x$. Since his holds for every neighborhood of x, we conclude that x is a limit point of A.

(c) If X is Hausdorff, and $A \subset X$ is connected, then its boundary $\partial A = \overline{A} - \operatorname{int}(A)$ is connected.

False: for example $[0, 1] \subset \mathbb{R}$ is connected, but its boundary $\{0, 1\}$ isn't.

(d) If X is Hausdorff, and $A \subset X$ is compact, then its boundary $\partial A = \overline{A} - \operatorname{int}(A)$ is compact.

True: since X is Hausdorff, the compact subset A must be closed, and $\overline{A} = A$. Meanwhile, $\operatorname{int}(A)$ is open in X and hence in A, so its complement $\partial A = A - \operatorname{int}(A)$ is closed in A; a closed subset of a compact set is compact, so ∂A is compact.

(e) $[0, 1] \subset \mathbb{R}_\ell$ with the lower limit topology (generated by the basis $\{[a, b),\ a < b\}$) is compact.

False: the open sets $[0, 1-\frac{1}{n})$ for $n \geq 2$, and $\{1\}$ (=$[1, 2) \cap [0, 1]$ hence open in the subspace topology of $\mathbb{R}_\ell$), form an open cover $[0, 1] = \{1\} \cup \bigcup_{n\geq 2} [0, 1 - \frac{1}{n})$ which has no finite subcover.

(f) The addition map $f : \mathbb{R}_\ell \times \mathbb{R}_\ell \to \mathbb{R}_\ell$ defined by $f(x, y) = x + y$ is continuous (equipping $\mathbb{R}_\ell$ with the lower limit topology and $\mathbb{R}_\ell \times \mathbb{R}_\ell$ with the product topology).

True: if $U \subset \mathbb{R}_\ell$ is open and $(x, y) \in f^{-1}(U)$, then $x + y \in U$ so there exists $\varepsilon > 0$ such that $[x + y, x + y + \varepsilon) \subset U$. Then $[x, x + \frac{\varepsilon}{2}) \times [y, y + \frac{\varepsilon}{2}) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Problem 2. (6 points)

Let X be a topological space. In this problem (and only in this problem), we consider $Y = X \times X$ equipped with the topology defined by the basis $\mathcal{B} = \{U \times U \,|\, U \subset X \text{ open}\}$.

(a) Show that $\mathcal{B}$ is a basis, and that the topology on Y is coarser than the product topology.

We check the axioms of a basis: $X \times X \in \mathcal{B}$ so the elements of $\mathcal{B}$ cover $X \times X$; and the second requirement follows from the observation that, if $U \times U, V \times V$ are in $\mathcal{B}$, then $(U \times U) \cap (V \times V) = (U \cap V) \times (U \cap V) \in \mathcal{B}$. So $\mathcal{B}$ is a basis. Since every element of $\mathcal{B}$ is open in the product topology, the topology generated by $\mathcal{B}$ is coarser than the product topology (strictly coarser in most cases).

(b) Show that the diagonal map $\Delta : X \to Y$ defined by $\Delta(x) = (x, x)$ is continuous.

It is enough to show that the inverse images of elements of $\mathcal{B}$ are open in X (the result will then follow for arbitrary open subsets of Y by taking unions). And indeed, given any open $U \subset X$, $\Delta^{-1}(U \times U) = \{x \in X \mid (x, x) \in U \times U\} = U$ is open in X.

(Or: Δ is continuous if we equip $X \times X$ with the product topology, which implies the result since the topology on Y is coarser).

(c) If X is Hausdorff, does it follow that Y is Hausdorff? If X is connected, does it follow that Y is connected? (for each statement, give a proof or a counterexample)

Hausdorff: false (as soon as X contains more than one point). If $x_1 \neq x_2 \in X$, then every neighborhood of (x_1, x_2) in Y contains a basis element $U \times U$ with $(x_1, x_2) \in U \times U$, i.e. $\{x_1, x_2\} \subset U \subset X$, U open in X. So every neighborhood of (x_1, x_2) also contains (x_1, x_1) (as well as (x_2, x_1) and (x_2, x_2)). So we can't separate (x_1, x_1) and (x_1, x_2) by open subsets of Y.

Connected: true: if X is connected then $X \times X$ is connected in the product topology; since the topology on Y is coarser, a separation of Y would give a separation of $X \times X$ in the product topology, so can't exist. Hence Y is connected if X is.

Problem 3. (7 points)

Let X, Y be topological spaces. The graph of $f : X \to Y$ is the subset $G_f = \{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(a) Show that if Y is Hausdorff and $f : X \to Y$ is continuous then its graph G_f is a closed subset of $X \times Y$ (with the product topology).

(a) Let $(x, y) \in (X \times Y) - G_f$. Then $y \neq f(x)$, and since Y is Hausdorff there exist disjoint neighborhoods $V \ni y$ and $W \ni f(x)$ in Y. Since f is continuous, $U = f^{-1}(W)$ is open in X, and contains x. Now $U \times V$ is disjoint from G_f: given any point $(x', y') \in U \times V$, $f(x') \in W$ which is disjoint from $V \ni y'$, so $y' \neq f(x')$ i.e. $(x', y') \notin G_f$. So every point of $(X \times Y) - G_f$ has a neighborhood which is disjoint from G_f, i.e. the complement of G_f is open in $X \times Y$.

(b) Show that if Y is compact and Hausdorff, then the converse is true: if the graph G_f is closed in $X \times Y$ then f is continuous.

(Hint: given an open $V \subset Y$ and $x \in f^{-1}(V)$, show that the subset $\{x\} \times (Y - V)$ of $X \times Y$ can be covered by open subsets $U_i \times V_i$ which are disjoint from G_f, and use this to find a neighborhood U of x such that $U \times (Y - V)$ is disjoint from G_f.)

As per the hint: let $V \subset Y$ be an open subset, and let $x \in f^{-1}(V)$. For every $y \in Y - V$, $y \neq f(x)$ so $(x, y) \notin G_f$; since G_f is closed, there exists a product neighborhood $U_y \times V_y$ of (x, y) which is disjoint from G_f. The collection of open subsets $\{U_y \times V_y\}_{y \in Y - V}$ gives an open cover of $\{x\} \times (Y - V)$. Since $Y - V$ is closed in Y which is compact, it is also compact, hence there exists a finite subcover, i.e. there exist $y_1, \dots, y_n \in Y - V$ such that $\{x\} \times (Y - V) \subset \bigcup_{i=1}^n U_{y_i} \times V_{y_i} \subset (X \times Y) - G_f$.

Now, the finite intersection $U = U_{y_1} \cap \dots \cap U_{y_n}$ is a neighborhood of x in X, and by construction $U \times (Y - V) \subset U \times (\bigcup V_{y_i}) \subset (X \times Y) - G_f$. (Note: one could also find such a U directly by applying the tube lemma to the open set $(X \times (Y - V)) - G_f$ in the product $X \times (Y - V)$.)

Finally: the statement that $U \times (Y - V)$ is disjoint from G_f means that $\forall x' \in U$, $f(x') \notin Y - V$ (else $(x, f(x'))$ would be in $(U \times (Y - V)) \cap G_f$), hence $f(x') \in V$: in other terms, $U \subset f^{-1}(V)$, so $f^{-1}(V)$ contains a neighborhood of x. This completes the proof that $f^{-1}(V)$ is open in X for every open $V \subset Y$, i.e. f is continuous.

(c) Give an example showing that the result of (b) need not hold if Y is not compact.

Take $X = Y = \mathbb{R}$, and consider the function defined by $f(x) = 1/x$ if $x > 0$, $f(x) = 0$ if $x \leq 0$. This function is not continuous at 0, but its graph is a closed subset of $\mathbb{R}^2$, since it is the union of the two closed subsets $(-\infty, 0] \times \{0\}$ and $\{(x, y) \mid xy = 1,\ x, y \geq 0\}$.