Solutions to Homework 11

Math 55b

1. *Let f_n be a sequence of analytic functions on Δ converging uniformly to $f(z)$. (i) Show that for each $r < 1$, $f_n' \to f'$ uniformly on $\Delta(r)$; (ii) Show that convergence need not be uniform on Δ.*

   Uniform convergence on $\Delta(r)$ follows from the **Cauchy derivative estimate**, which is conveniently stated in the form: *for any analytic map h sending a disk of radius r centered at p into a disk of radius s centered at $f(p)$, $|h'(p)| \le s/r$; in particular, if $h(\Delta_p(r))$ is contained in some disk of radius s, then $|h'(p)| \le 2s/r$.* Applying this to $h := f - f_n$ with r replaced by $1 - r$, we conclude the bound $|f'(z) - f_n'(z)| \le 2M/(1-r)$, where $M := \sup_n |f - f_n|$, which is uniform on $\Delta(r)$.

   Alternatively (and equivalently), one may simply write the Cauchy formula $g'(p) = \frac{1}{2\pi\sqrt{-1}} \int_{S_1} g(z)\, dz/(z-p)^2$, and note that $1/(z-p)^2$ is uniformly bounded by $1/(1-r)^2$ on S^1 for $p \in \Delta(r)$.

   For an example where the convergence $f_n' \to f'$ is not uniform on Δ, take $f_n(z) := z^n/n$, which converges uniformly to 0 on Δ, but the convergence $z^n \to 0$ is not uniform on Δ. ∎

2. *Let $u : S^1 - \{\pm 1\}$ be the function which is 1 if $\operatorname{Im} z > 0$ and 0 if $\operatorname{Im} z < 0$. Find a continuous, harmonic extension of u on Δ. Then find the conjugate of u.*

   The Möbius transformation $\frac{1-z}{1+z}$ transforms Δ into the right half-plane $\operatorname{Re} z > 0$ (because $0 \mapsto 0$, $-1 \mapsto \infty$, $i \mapsto -i$, so that the boundaries agree; and $0 \mapsto 1$). In this, the upper-arc $\operatorname{Im} z > 0$ of S^1 is mapped to the positive imaginary axis $i\mathbb{R}^{>0}$, and the lower arc is mapped to the negative imaginary axis $-i\mathbb{R}^{<0}$. There is an obvious harmonic function on $\mathbb{C}^- := \mathbb{C} \setminus \mathbb{R}^{\le 0}$ which is equal to $\pi/2$ on the positive imaginary axis and to $-\pi/2$ on the negative imaginary axis: the argument function $\operatorname{Arg}(z)$ (which is the imaginary part of the analytic function $\log z := \log|z| + i\operatorname{Arg}(z)$ on $\mathbb{C}^-$). Then $\operatorname{Arg}\left(\frac{1-z}{1+z}\right)$ is a harmonic function on Δ equal to $\pi/2$ on the upper arc and to $-\pi/2$; accordingly, $\frac{1}{2} + \frac{1}{\pi}\operatorname{Arg}\left(\frac{1-z}{1+z}\right)$ is the required harmonic extension. It is the imaginary part of $\frac{1}{2} + \frac{1}{\pi}\log\left(\frac{1-z}{1+z}\right)$, and so its harmonic conjugate is $\frac{1}{2} + \frac{1}{\pi}\log\left|\frac{1-z}{1+z}\right|$. ∎

3. *What is the residue at $z = 0$ of $\sin^3(1/z)$? What are the residues of $z/(e^{z^2} - 1)$ at its singularities?*

Since $\sin(z)$ has a simple zero at $z = 0$, $\sin^3(1/z)$ has a triple pole at $z = 0$, therefore no residue.

In general, the residue at the simple pole $z = p$ of a meromorphic function f/g is equal to $f(p)/g'(p)$ (here, f, g are assumed to have no zeros in common). In the case of $z/(e^{z^2} - 1)$, all poles $\sqrt{2\pi i\mathbb{Z}}$ are simple, and $(e^{z^2} - 1)' = 2ze^{z^2}$. For the poles other than 0, we conclude that the residue is $1/2$. For $z = 0$, note that $(e^{z^2} - 1)/z = z + O(z^3)$, showing that the residue is 1 at $z = 0$.

Thus: the residue of $z/(e^{z^2} - 1)$ at each of its nonzero poles $\sqrt{2\pi i\mathbb{Z}} \setminus \{0\}$ is equal to $1/2$; and is equal to 1 at the pole 0. ■

4. *Compute the first three nonzero terms in the Taylor series of $\sum a_n z^n = \sin^{-1}(z)$ by formally inverting $\sin z = z - z^3/3! + z^5/5! - \cdots$. What is the radius of convergence of this series?*

The function $\sin^{-1}(z)$ is odd, so its expansion begins $\sin^{-1}(z) = a_1 z + a_3 z^3 + a_5 a^5 + \cdots$. These coefficients are determined by the requirement $P(z) - P(z)^3/6 + P(z)^5/120 = z$ for $P(z) := a_1 z + a_3 z^3 + a_5 z^5$. The left-hand side expands as $a_1 z + (a_3 - a_1^2/6)z^3 + (a_5 - a_1^2 a_3/2 + a_1^5/120)z^5$; equating these to $1, 0, 0$ respectively, we find successively $a_1 = 1, a_3 = 1/6, a_5 = 3/40$. Thus the expansion begins: $\sin^{-1}(z) = z + z^3/6 + 3z^5/40 + \cdots$.

But note, alternatively, that, since $\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$ (as determined from $\sin(\sin^{-1}(z)) = z$ and the chain rule), we may write $\sin^{-1}(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}} = \int_0^z (1 - t^2/2 - t^4/8 + \cdots)\,dt = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} z^{2n+1}/(2n+1)$, which (in this explicit form) is easily seen to have radius of convergence 1.

There are, of course, many direct, alternative ways to show that the radius of convergence is 1, one of then being to derive the formula $\sin^{-1}(z) = -i\log(iz + \sqrt{z^2 - 1})$, noting that the singularities appear at $z = \pm 1$ (and the right-hand side has an analytic, single-valued branch on the unit disk Δ). But the simplest of all is this: the derivative $\frac{d}{dz}\sin^{-1}(z) = \frac{1}{\sqrt{1-z^2}}$ has radius of convergence 1, since $\frac{1}{\sqrt{1-z^2}}$ is analytic on Δ and has poles $z = \pm 1$ on the boundary. Since a series $f(z)$ has, in general, the same radius of convergence as its derivative $f'(z)$, the conclusion follows. ■

5. *Prove that a positive harmonic function on $\mathbb{C}$ is constant.*

This is **Liouville's theorem**. If u is a positive harmonic function, then $v := e^{-u}$ is a bounded harmonic function (it satisfies $0 < |e^{-u}| \leq 1$), hence constant, by the **derivative estimate** $|v'(z)| = \frac{1}{2\pi} \int_{S(R)} \left|\frac{v(z)}{z}\right| dz \leq 1/R$, for arbitrarily large $R > 0$. ∎

Remark. This theorem has an interesting discrete analog:

Challenge problem. *A positive discrete harmonic function on $\mathbb{Z}^2$ is constant. In other words, let $f : \mathbb{Z}^2 \to \mathbb{R}^{\geq 0}$ be a positive function having the mean value property that*

$$f(x, y) = \frac{f(x, y+1) + f(x+1, y) + f(x-1, y) + f(x, y-1)}{4}$$

Then f is constant.

Hint. A random walk on $\mathbb{Z}^2$ starting from the origin revisits the origin with probability 1. In fact, a radom walk on $\mathbb{Z}^2$ visits *any* given point with probability 1!

The result is also true on $\mathbb{Z}^n$ (a positive discrete harmonic function on $\mathbb{Z}^n$ is constant), but the proof is more difficult: random walks are of no use already for $\mathbb{Z}^3$; a three-dimensional random walk starting at the origin revisits the origin with positive probability $0.3405\ldots$, which, being *less* than 1, is not helpful to the problem!

On the other hand, the weaker statement with f bounded (rather than just nonnegative) is easily proven, in $\mathbb{Z}^n$, by a perfect analogy with the *Cauchy estimate* argument given for the continuous case. What is special on $\mathbb{R}^2$ is the argument of replacing the nonnegative harmonic function f with the bounded harmonic function e^{-f}; this has no discrete analog, and no analog in $\mathbb{R}^3$. The proof that a nonnegative harmonic function on $\mathbb{R}^3$ is constant is also slightly more complicated, due to the absence of this trick. ∎

6. *Prove that if $f : \Delta \to \mathbb{C}$ satisfies $f(0) = 0$ and $\mathrm{Re}(f) \leq 1$, then $|f'(0)| \leq 2$. Find the cases of equality.*

The key is the Möbius transformation $q(z) := z/(2 - z)$, which transforms the half-plane $\mathrm{Re}\, f(z) \leq 1$ to the disk $|z| \leq 1$.

First solution. We can use, as in 1., the following consequence of the **Cauchy derivative estimates**: *if h maps a disk of radius r centered at*

z into a disk of radius s centered at $f(z)$, then $|h'(z)| \leq s/r$. Our situation is this statement for $h = q \circ f = f/(2-f)$, $s = r = 1$ and $z = 0$, where we get the claim from $1 \geq |(q \circ f)'(0)| = |f'(0)| \cdot |q'(f(0))| = |f'(0)| \cdot |q'(0)| = |f'(0)|/2$. More generally, if h maps a disk of radius r centered at z into *any* disk of radius s (not necessarily centered at $f(z)$), then $|f'(z)| \leq 2s/r$: this is since the disk of radius $2s$ centered at $f(z)$ still covers the image of h. Taking this statement with $s = 1$ and $r = 1 - |z|$, we additionally conclude the estimate $|f'(z)| \leq 4/(1 - |z|)$, under the same conditions $f(0) = 0$ and $\mathrm{Re}(f) \leq 1$. ∎

Second (phrasing of the) solution. Equivalently, we can use the Schwarz lemma towards the same conclusion. Thus, $q \circ f = f/(2-f)$ is an analytic function $\Delta \to \Delta$ which vanishes at the origin. By the Schwarz lemma, $1 \geq |(q \circ f)'(0)| = |q'(f(0))| \cdot |f'(0)| = |q'(0)| \cdot |f'(0)| = |f'(0)|/2$, which is the required inequality. Equality holds if and only if $f(z)/(2 - f(z)) = cz$ for some constant with $|c| = 2$, or equivalently, for $f(z) = 2cz/(1 + cz)$, $|c| = 2$. ∎

7. *Prove that if $f(z)$ is entire and $\mathrm{Re}\, f(z) \ll |z|^k$, then $f(z)$ is a polynomial.*

First solution. This combines two separate ideas. The first is the weaker statement that $|f(z)| \ll |z|^n$ implies $f(z)$ is a polynomial; the second is that $M(f;R) := \sup_{|z|=R} |f|$ and $N(f;R) := \sup_{|z|=R} \mathrm{Re}(f)$ are comparable: $N(f';R) \leq M(f';R) \leq 8N(f;2R) + |f(0)|$. The two statements together clearly imply the required conclusion. The first step is an easy application of the Cauchy derivative estimates, and was done in class. The second statement is sufficient to be shown for $R = 1/2$ and for $f(0) = 0$, and is hence a special case of the following variation of 6. above: *if $f : \Delta \to \mathbb{C}$ is an analytic function such that $f(0) = 0$ and $\mathrm{Re}\, f(z) \leq 1$ for all $z \in \Delta$, then $|f'(z)| \leq 8$ for $|z| \leq 1/2$.* This is proved in the first solution of 6. above; and also follows from the second solution. The proof is complete. ∎

Second solution. (In this concise form, by Alex) We need to show that if $u(z) = \sum_{n \geq 0} (a_n z^n + \overline{a_n} z^n)$ is a harmonic function on $\mathbb{C}$ bounded by $A|z|^k + B$, then $a_n = 0$ for $n \gg 0$: $u(z)$ is the real part of a polynomial. On the unit circle S^1, $AR^n + B \geq u(Rz) = \sum_{n \geq 0} (a_n R^n z^n + \overline{a_n} R^n z^{-n})$ for $R > 0$. Multiplying by z^{-1-n} and integrating over S^1 gives $2\pi\sqrt{-1} a_n R^n$, and the bound $R^n |a_n| \leq AR^k + B$, for all $R > 0$. For $n > k$, this shows $a_n = 0$, concluding the proof. ∎

Comment. In a nutshell, the first proof has two ideas: a Möbius transformation $z/(2M - z)$, which is a conformal isomorphism from the half-plane $\operatorname{Re} z \leq M$ onto the unit disk Δ; and the use of Schwarz's lemma on Δ to translate bounds for $\operatorname{Re} f(z)$ into bounds for $|f'(z)|$. ∎

8. *Find all entire functions f such that (i) f is never 0; and (ii) $|f(z)| \leq \exp(|z|^2)$ for all $z \in \mathbb{C}$.*

An nonvanishing analytic function $f \in \mathcal{O}(U)^\times$ on a simply connected domain U has an analytic logarithm: $f = e^g$ for some $g \in \mathcal{O}(U)$. To recall the proof, fix a point $p \in U$, and note that Cauchy's integral theorem implies that $g(z) := \int_p^z df/f$ is independent of the chosen path of integration, because the function f'/f is analytic in the interior of any closed loop (simple connectedness of U). This defines an analytic function on U; it satisfies $(e^g/f)' = (g'e^g f - f'e^g)/f^2 = e^g(g' - f'/f)/f = 0$, hence $e^g/f = c$ is a (nonzero) constant, and then we may write $c = e^a$, and replace g by $g - a$ to $f = e^g$, as claimed.

In particular, a nonvanishing entire function is of the form $f = e^g$, with g entire. (Alternatively, this also follows from the Poincaré lemma: $\log|f(z)|$ is a real harmonic function on $\mathbb{C}$, and is therefore the real part of an entire function $g(z)$). The condition $|f(z)| \leq \exp(|z|^2)$ then translates to $\operatorname{Re} g(z) \leq |z|^2$, which by 7. above implies g is a polynomial. It is then clearly of degree at most 2, for the real part of $g(x)$ is a real polynomial of degree $\deg g$ bounded by x^2. In fact, if this (real) polynomial is ax^2+bx+c, then $(1-a)x^2-bx-c \geq 0$ means $a \leq 1, c \geq 0$, $b^2 \leq 4c(1-a)$. Similarly, the real part of $g(y)$ is a real polynomial $\alpha y^2 + \beta y + c$ bounded by y^2, meaning again $\alpha \leq 1, \beta^2 \leq 4c(1-\alpha)$. Then $\operatorname{Re} g(x+iy) = ax^2+bx+\alpha y^2+\beta y+qxy+c$ for some coefficient $q \in \mathbb{R}$; and note that an obvious constraint on q is that the quadratic form $(a-1)x^2 + qxy + \alpha y^2$ be nonnegative-definite (as is seen by taking $y = rx$ and letting $x \to \infty$); in other words, we arrive at the constraint $q^2 \leq 4\alpha(a-1)$. Those three constraints are also sufficient; but since this was not an intended part of the problem, and grades are not deducted for it, we omit the routine but boring proof: the answer is $f(z) = e^{g(z)}$, where $g(z)$ is a quadratic polynomial whose real part $ax^2 + bx + \alpha y^2 + \beta y + qxy + c$ is subject to the conditions $a \leq 1, c \geq 0, b^2 \leq 4(1-a)c, \beta^2 \leq 4(1-\alpha)c, q^2 \leq 4(a-1)\alpha$. ∎

Claim. *A real quadratic function $ax^2 + \alpha y^2 + qxy + bx + \beta y + c$ is nonnegative if and only if $a, \alpha, c \geq 0, b^2, \leq 4ac, \beta^2 \leq 4\alpha c$, and $q^2 \leq 4a\alpha$.* ∎

9. *Show that if $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(z_n) \to \infty$ whenever $z_n \to \infty$, then $f(z)$ is a polynomial.*

This follows from the classification of isolated singularities (more precisely, from the **Casoratti-Weierstrass theorem**): $z = 0$ is an isolated singularity of the analytic function $g(z) := 1/f(1/z)$, defined on $\Delta(r)^\times$ for some $r > 0$, and the condition $g(z) \to 0$ as $z \to 0$ means that this singularity is removable: the function $G(z) := \frac{1}{2\pi\sqrt{-1}} \int_{\Delta(r)} \frac{g(w)\,dw}{w-z}$ is defined at $z = 0$ with value 0 (because of $g(z) \to_{z\to 0} 0$ and the Cauchy integral formula), and is an analytic extension of $g(z)$ on $\Delta(r)$. On the other hand, if $f(z)$ is not a polynomial, then $z = 0$ is an essential singularity of $f(1/z)$, and hence of $1/f(1/z) = g(z)$.

Alternatively, the analyticity of $1/f(1/z)$ (which is a consequence of the **Riemann removable singularity theorem**, whose sketch of proof was recalled above) shows that $f(z)$ is a meromorphic function on the Riemann sphere $\widehat{\mathbb{C}}$, hence a rational function; being also entire on $\mathbb{C}$, $f(z)$ is therefore a polynomial. ■

10. *Find the Laurent series for $f(z) = 1/(z(z-1)(z-2))$ valid (i) in the region $1 < |z| < 2$; (ii) in the region $|z| > 2$.*

The continued fraction expansion is $f(z) = 2/z - 1/(z-1) + 2/(z-2)$. The Laurent expansions of these functions in $1 < |z| < 2$ are, respectively, $2/z$, $-1/(z-1) = -z^{-1}/(1-z^{-1}) = -\sum_{n\geq 1} 1/z^n$, and $2/(z-2) = -1/(1-z/2) = -\sum_{n\geq 0} z^n/2^n$; therefore, the Laurent expansion of $f(z)$ in the annulus is

$$f(z) = 1/z - \sum_{n\leq -2} z^n - \sum_{n\geq 0} z^n/2^n, \quad 1 < |z| < 2.$$

For the region $|z| > 2$, we use instead the expansion $2/(z-2) = 2z^{-1}/(1-2z^{-1}) = \sum_{n\geq 1} 2^n/z^n$, arriving at the Laurent expansion

$$f(z) = -1/z - \sum_{n\leq -2} (-1-2^n) z^n, \quad |z| > 2.$$

■