Solutions to Homework 8

Math 55b

1. *Let f be a compactly supported smooth function on $\mathbb{R}^2$. Are the relations $\int f(x,y)\,dx\,dy = \int f(y,x)\,dy\,dx$ and $dx\,dy = -dy\,dx$ both true? How can they be reconciled?*

Answer: It depends on the orientations of $\mathbb{R}^2$ in which the two integrals are taken, but it is **false** if we mean implicitly the *anticlockwise* orientation of $\mathbb{R}^2$, given by the form $dx\,dy$ (which is what we did in the course: we *defined* $\int f(x,y)\,dx\,dy := \int f(x,y)\,|dx\,dy| := \int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} f(x,y)\,dx\Big)\,dy = \int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} f(x,y)\,dy\Big)\,dx$ as an iterated integral, once we declared the standard ordered basis $\{(0,1),(1,0)\}$ to be *positive*). Since the differential of the diffeomorphism $(x,y)\mapsto(y,x)$ has constant determinant -1, the change of variables formula for iterated integration gives $\int f(x,y)\,|dx\,dy| = \int f(y,x)\,|-1|\,|dx\,dy| = \int f(y,x)\,|dx\,dy| = \int f(y,x)\,dx\,dy = -\int f(y,x)\,dy\,dx$, the third equality holding under the understanding that $\mathbb{R}^2$ is anticlockwise oriented (by $dx\,dy$) throughout. ∎

Further comment. The general **change of variables formula** takes the following form (where you may think of M, N as regions in $\mathbb{R}^n$ with given orientations): for $\phi : N \to M$ a diffeomorphism of oriented manifolds and ω a top-degree differential form on M (i.e. a $\dim M$-form), then $\int_M \omega = \pm\int_N \phi^*\omega$, where the sign is "+" if ϕ preserves orientation, and "$-$" if ϕ reverses orientation. Note in this that a diffeomorphism ϕ either preserves or reverses an orientation, because the invertibility at every point of the differential $D\phi$ implies that the (continuous) function $\det D\phi$ is nonvanishing and hence has constant sign.

Since the diffeomorphism $\phi : (x,y)\mapsto(y,x)$ of $\mathbb{R}^2$ is orientation-reversing (it pulls back the area form $dx\,dy$ to its negative $dy\,dx = -dx\,dy$; or equivalently, its Jacobian is identically $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of constant negative determinant -1), while $f(y,x)\,dy\,dx = \phi^*\big(f(x,y)\,dx\,dy\big)$, it follows that $\int f(x,y)\,dx\,dy = \int f(y,x)\,dy\,dx$, if the two integrals are taken in opposite orientations of $\mathbb{R}^2$, and $\int f(x,y)\,dx\,dy = -\int f(y,x)\,dy\,dx$ if they are taken in the same orientation of $\mathbb{R}^2$. ∎

2. *Prove that $\nabla \cdot v$ on $\mathbb{R}^3$ is the limit, as the size of a cube Q goes to zero, of the flux of v through ∂Q divided by the volume of Q.*

This is a consequence (upon taking the limiting statement as $Q \to 0$) of the **divergence theorem**, which implies $\int_Q (\nabla \cdot v)\,|dV| = \int_{\partial Q} (n \cdot v)\,|dA|$. In detail, fix $\varepsilon > 0$ and a point $p \in \mathbb{R}^3$. By continuity of $\nabla \cdot v$, there exists a $\delta > 0$ such that $|(\nabla \cdot v)(x) - (\nabla \cdot v)(p)| < \varepsilon$ whenever x belongs to a cube Q containing p of volume $< \delta$. By the divergence theorem, the flux of v through ∂Q equals $\int_Q (\nabla \cdot v)\,|dV|$, which satisfies $\big((\nabla \cdot v)(p) - \varepsilon\big)\mathrm{vol}(Q) \leq \int_Q (\nabla \cdot v)\,|dV| \leq \big((\nabla \cdot v)(p) + \varepsilon\big)\mathrm{vol}(Q)$. The conclusion follows upon letting $\varepsilon \to 0$. ■

3. *State and prove a similar theorem for the three components of $\nabla \times v$ on $\mathbb{R}^3$.*

The statement is the following: the component $(\nabla \times v) \cdot n$ of the curl of v in the direction of the unit vector n is given by the limit

$$\lim_{S \to 0} \frac{\int_{\partial S} (v \cdot s)\,|ds|}{\mathrm{area}(S)},$$

as the size of a square S in a plane orthogonal to n goes to 0, of the circulation of v around ∂S divided by the area of S. (Apply this to the three basis unit vectors). This follows from Green's theorem in exactly the same way as the preceding statement follows from the divergence theorem. ■

4. *Prove directly that $\nabla \cdot \nabla \times v = 0$ on $\mathbb{R}^3$, and explain how this is a consequence of $d^2 = 0$.*

We compute:

$$\nabla \cdot \nabla \times v = \nabla \cdot \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{d}{dx_1} & \frac{d}{dx_2} & \frac{d}{dx_3} \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \nabla \cdot \Big((\frac{dv_3}{dx_2} - \frac{dv_2}{dx_3})e_1 - \frac{dv_3}{dx_1} - \frac{dv_1}{dx_3})e_2 + \frac{dv_2}{dx_1} - \frac{dv_1}{dx_2})e_3 \Big)$$

$$= \Big(\frac{d^2 v_3}{dx_1\,dx_2} - \frac{d^2 v_2}{dx_1\,dx_3} \Big) - \Big(\frac{d^2 v_3}{dx_2\,dx_1} - \frac{d^2 v_1}{dx_2\,dx_3} \Big) + \Big(\frac{d^2 v_2}{dx_3\,dx_1} - \frac{d^2 v_1}{dx_3\,dx_2} \Big)$$

$$= 0,$$

by the commutativity of mixed partials (the symmetry of the Hessian of a smooth function).

The consequence from $d^2 = 0$ is immediate by the identifications $\nabla \cdot w \Leftrightarrow *d*\eta$ and $\nabla \times v \Leftrightarrow *d\omega$ of div and curl in terms of the operators d and $*$ for differential 1-forms ω, η identified with the vector fields v, w, respectively. Indeed, the identifications show $\nabla \cdot \nabla \times v = *d*(*d\omega) = *dd\omega = 0$. ∎

5. *How does the Hodge star on $\mathbb{R}^2$ operate on the differentials dr and $d\theta$ coming from polar coordinates? Use your answer to compute the Laplacian of a function $f(r, \theta)$ in polar coordinates. Then, find all radially symmetric functions $f(r)$ on $\mathbb{R}^2 - \{(0,0)\}$.*

The Hodge star on $\mathbb{R}^2$ is defined by $*dx = dy$, $*dy = -dx$, $*1 = dx\,dy$, $*dx\,dy = 1$, and linearity in functions: $*(\omega + \eta) = *\omega + *\eta$, $*(f\omega) = f * \omega$. To compute $*dr$ and $*d\theta$, we first need to express dr, $d\theta$ in terms of dx, dy; this is immediate from $(x, y) = (r\cos\theta, r\sin\theta)$, which shows $dx = \cos\theta\,dr - r\sin\theta\,d\theta$, $dy = \sin\theta\,dr + r\cos\theta\,d\theta$, hence $dr = \cos\theta\,dx + \sin\theta\,dy$, $d\theta = \frac{\cos\theta}{r}dy - \frac{\sin\theta}{r}dx$, and ultimately, $*dr = rd\theta$, $*d\theta = -dr/r$.

The Laplacian is given by $*d*d$, and we compute:

$$\begin{aligned}
\Delta(f) = (*d*d)(f) = (*d*)\Big(\frac{df}{dr}dr + \frac{df}{d\theta}d\theta\Big) \\
= (*d)\Big(r\frac{df}{dr}d\theta + \frac{df}{d\theta}d\theta\Big) \\
= *\Big[\Big(\frac{df}{dr} + r\frac{d^2f}{dr^2} + \frac{1}{r}\frac{d^2f}{d\theta^2}\Big)dr\,d\theta\Big] \\
= \frac{1}{r}\frac{df}{dr} + \frac{d^2f}{dr^2} + \frac{1}{r^2}\frac{d^2}{d\theta^2},
\end{aligned}$$

because $dr\,d\theta = \frac{1}{r}dx\,dy$.

Thus, the condition on a radially symmetric function $f(r)$ to be harmonic is that it satisfy the second-order differential equation $\frac{1}{r}\frac{df}{dr} + \frac{d^2f}{dr^2} = 0$, or equivalently, $\frac{d}{dr}\big(r\frac{df}{dr}\big) = 0$. Since the solution to $r\frac{df}{dr} = A = \text{const}$ is $f(r) = \log(Ar) + B$ for $B = \text{const}$, it follows that the radially symmetric harmonic functions on the punctured plane are exactly the *logarithmic* functions $\log(Ar) + B$. ∎

6. *Suppose α, β are forms of degree k, ℓ on $\mathbb{R}^n$. Prove a formula relating $\alpha\beta$ to $\beta\alpha$, and establish a* product formula *for $d(\alpha\beta)$.*

The required formulas are $\beta\alpha = (-1)^{k\ell}\alpha\beta$ and $d(\alpha\beta) = (d\alpha)\cdot\beta + (-1)^k\alpha\cdot d\beta$. The former follows upon noting that the permutation $(k+1,\dots,k+\ell,1,\dots,k)$ of $(1,\dots,k+\ell)$ has sign $(-1)^{k\ell}$, so that $dx_I\,dx_J = (-1)^{k\ell}dx_J\,dx_I$ for disjoint index sets I,J with $|I|=k, |J|=\ell$, and linearity of both sides in α,β. The latter ultimately reduces to the Leibnitz rule: by linearity of both sides in α,β, it suffices to establish the identity for the cases $\alpha = f\,dx_I$, $\beta = g\,dx_J$ with I,J disjoint index sets with $|I|=k, |J|=\ell$, and for these, $d(\alpha\beta) = d\big(fg\,dx_I\,dx_J\big) = (df)\,g\,dx_I\,dx_J + f\,(dg)\,dx_I\,dx_J = (d\alpha)\beta + (-1)^k\alpha\cdot(d\beta)$, as claimed. ∎

7. *Give an example of a differentiable map $f:\mathbb{R}\to\mathbb{R}$ which is a homeomorphism but not a diffeomorphism.*

The standard example is $f(x) = x^3$, whose inverse $\sqrt[3]{x}$ is continuous, but not differentiable at 0. ∎

8. *For any smooth function $f:U\to\mathbb{C}$, where $U\subset\mathbb{C}$, let*

$$\frac{df}{dz} := \frac{1}{2}\Big(\frac{df}{dx} - \sqrt{-1}\frac{df}{dy}\Big), \quad \frac{df}{d\bar z} := \frac{1}{2}\Big(\frac{df}{dx} + \sqrt{-1}\frac{df}{dy}\Big).$$

(i) Prove that $df = \frac{df}{dz}dz + \frac{df}{d\bar z}d\bar z$; (ii) Prove that $(d/dz)(z^n\bar z^m) = nz^{n-1}\bar z^m$; (iii) Prove that if $\sum_{i,j=0}^N a_{ij}z^i\bar z^j = \sum_{i,j=0}^N b_{i,j}z^i\bar z^j$ for all $z\in\mathbb{C}$, then all $a_{ij} = b_{ij}$; (iv) Prove that a smooth function $f(z)$ is analytic if and only if $df/d\bar z = 0$, in which case $f'(z) = df/dz$.

For (i), we have the immediate computation

$$df = \frac{df}{dx}dx + \frac{df}{dy}dy$$
$$= \frac{1}{2}\Big(\frac{df}{dx} - \sqrt{-1}\frac{df}{dy}\Big)(dx + \sqrt{-1}dy) + \frac{1}{2}\Big(\frac{df}{dy} + \sqrt{-1}\frac{df}{dy}\Big)(dx - \sqrt{-1}dy)$$
$$= \frac{df}{dz}dz + \frac{df}{d\bar z}d\bar z.$$

For (ii), noting that $(d/dx)z^n = nz^{n-1}$, $(d/dy)z^n = \sqrt{-1}\,nz^{n-1}$, $(d/dx)\bar z^m = m\bar z^{m-1}$, $(d/dy)\bar z^m = -\sqrt{-1}\,m\bar z^{m-1}$, we compute:

$$(d/dz)z^n = \frac{1}{2}\big(nz^{n-1} - \sqrt{-1}(\sqrt{-1}\,nz^{n-1})\big) = nz^{n-1},$$
$$(d/dz)\bar z^m = \frac{1}{2}\big(m\bar z^{m-1} + \sqrt{-1}(\sqrt{-1}\,mz^{m-1})\big) = 0.$$

By the Liebnitz rule $(d/dz)(fg) = f(dg/dz) + g(df/dz)$ (which holds for the operators d/dx and d/dy, and hence for any linear combination of these), we compute $(d/dz)(z^n \bar{z}^m) = z^n (d/dz)(\bar{z}^m) + \bar{z}^m (d/dz) z^n = n z^{n-1} \bar{z}^m$, as required.

For (iii), we are asked to show that a polynomial identity $\sum_{i,j=0}^{N} a_{ij} z^i \bar{z}^j = 0$ for all $z \in \mathbb{C}$ implies all coefficients $a_{ij} = 0$. If not, then there exists some $a_{i_0 j_0} \neq 0$ such that (i_0, j_0) is maximal in the lexicographic partial ordering of $\mathbb{Z} \times \mathbb{Z}$. Applying the differential operator $d^{i_0+j_0} \big/ (dz^{i_0}\, d\bar{z}^{j_0})$ to the given polynomial identity and using part (ii), we obtain $i_0! j_0! a_{i_0 j_0} = 0$ (all other terms in the polynomial are annihilated, by the lexicographic maximality of (i_0, j_0)), contradictory to the assumption.

To establish (iv), write $f(x + \sqrt{-1} y) = u(x,y) + \sqrt{-1}\, v(x,y)$, where u, v are smooth real-valued functions on $\mathbb{R}^2$; then $df/d\bar{z} = 0$ is the system of **Cauchy-Riemann equations** $du/dx = dv/dy$, $dv/dx = -du/dy$. These follow from analyticity (which means, complex differentiability of $f(z)$) upon comparing the real and imaginary parts of the limit difference quotients

$$\lim_{r \to 0^+} \frac{\big(\alpha(x+r,y) - \alpha(x,y)\big) + \sqrt{-1}\, \big(\beta(x+r,y) - \beta(x,y)\big)}{r} = f'(x + \sqrt{-1}\, y)$$
$$= \lim_{r \to 0^+} \frac{\big(\alpha(x,y+r) - \alpha(x,y)\big) + \sqrt{-1}\, \big(\beta(x,y+r) - \beta(x,y)\big)}{\sqrt{-1}\, r}$$

along the real and the imaginary axes.

Conversely, suppose the Cauchy-Riemann equations $df/d\bar{z} = 0$ hold, and consider f as the map $\mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto \big(u(x,y), v(x,y)\big)$. The Cauchy-Riemann equations are precisely the assertion that the (usual) differential $Df(z) : \mathbb{C} \to \mathbb{C}$ is in $\mathbb{R} \cdot \mathrm{SO}_2(\mathbb{R})$, or equivalently, acts on $\mathbb{C}$ as multiplication by a complex number (the composition of a rotation and a homothecy). Since $f(z+h) = f(z) + Df(z)h + o(|h|)$, it follows that f is complex differentiable, and $Df(z) : \mathbb{C} \to \mathbb{C}$ is multiplication by the complex number $f'(z)$. Since the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{R} \cdot \mathrm{SO}_2(\mathbb{R})$ acts as multiplication by the complex number $a - \sqrt{-1}\, b$, it follows that $f'(z) = du/dx - \sqrt{-1}\, dv/dy = \frac{1}{2}(df/dx - \sqrt{-1}\, df/dy) = df/dz$, as required. ■