Solutions to Homework 2

Math 55b

1. *Give an example of a subset $A \subset \mathbb{R}$ such that, by repeatedly taking closures and complements.*

Take $A := \big((-5,-4)-\mathbb{Q}\big) \cup [-4,-3] \cup \{2\} \cup [-1,0] \cup (1,2) \cup (2,3) \cup \big((4,5) \cap \mathbb{Q}\big)$. (The number of intervals in this sets is unnecessarily large, but, as **Kevin** suggested, taking a set that is complement-symmetric about 0 cuts down the verification in half. Note that this set is the union of $\{0\} \cup (1,2) \cup (2,3) \cup ([4,5] \cap \mathbb{Q})$ and the complement of the reflection of this set about the the origin). Because of this symmetry, we only need to verify that the *closure-complement-closure* sequence of the set $\{0\} \cup (1,2) \cup (2,3) \cup \big((4,5) \cap \mathbb{Q}\big)$ consists of seven distinct sets. Here are the (first) seven members of this sequence; from the eighth member the sets start repeating periodically: $\{0\} \cup (1,2) \cup (2,3) \cup \big((4,5) \cap \mathbb{Q}\big)$; $\{0\} \cup [1,3] \cup [4,5]$; $(-\infty,0) \cup (0,1) \cup (3,4) \cup (5,\infty)$; $(-\infty,1] \cup [3,4] \cup [5,\infty)$; $(1,3) \cup (4,5)$; $[1,3] \cup [4,5]$; $(-\infty,1) \cup (3,4) \cup (5,\infty)$. Thus, by symmetry, both the *closure-complement-closure* and *complement-closure-complement* sequences of A consist of seven distinct sets, and hence there is a total of 14 different sets obtained from A by taking closures and complements. ■

Remark. That 14 is the maximal possible number of sets obtainable for any metric space is the **Kuratowski complement-closure problem**. This is proved by noting that, denoting respectively the closure and complement operators by a and b and by e the identity operator, the relations $a^2 = a, b^2 = e$, and $aba = abababa$ take place, and then one can simply list all 14 elements of the **monoid** $\langle a, b \mid a^2 = a, b^2 = e, aba = abababa \rangle$. ■

2. *Prove that for any subset E of a metric space (X,d), we have $\partial E = \overline{E} - \operatorname{int}(E)$.*

The boundary ∂E consists, by definition, of all $x \in X$ having the property that every neighborhood of x meets both E and $X - E$. The condition that every neighborhood of x meets E is precisely the condition that $x \in \overline{E}$, while the condition that every neighborhood of x meets $X - E$ is precisely the condition $x \notin \operatorname{int}(E)$. The conclusion follows. ■

3. *Let E_1 denote the set of limit points of E, and E_{n+1} the set of limit points of E_n. For each $n > 0$, give an example of a set $E \subset [0,1]$ such that $E_n \neq \emptyset$, but $E_{n+1} = \emptyset$. Prove or disprove: for any $E \subset [0,1]$, the set $F := \bigcap_n E_n$ is perfect.*

The answer to the question is negative: for example, F may consist of the single point $\{0\}$. Two possible constructions are as follows.

First example. For the first part of the problem, where n is fixed, take $E := \{\sum_{i=1}^n 2^{-r_i} \mid r_1 < \cdots < r_n\}$; this is the set of dyadic rationals in $[0,1]$ having n digits 1. For the second part of the problem, modify this construction by taking $E := \{\sum_{i=1}^n 2^{-r_i} \mid n < r_1 < \cdots < r_n\}$. Then $E_n \subset [0, 2^{-n})$ for all n and 0 is in every E_n, so that the intersection $F = \bigcup_n E_n = \{0\}$.

Second example. For the first part of the problem, take E_n to be the set of all continued fractions

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}}$$

of length n. For the second part of the problem, modify this example by taking E to be the set of all continued fractions all of whose terms a_i are larger than their length n. Then each E_n contains 0 and, otherwise consisting entirely of continued fractions of length (and hence terms) $> n$, is contained in $[0, 1/n)$. It follows that $\bigcup_n E_n = \{0\}$, which is not perfect. ∎

4. *Prove that any open subset $U \subset \mathbb{R}$ can be expressed as a union $U = \bigcup_{i \in U}(a_i, b_i)$ of disjoint open intervals (allowing $\pm\infty$ as endpoints). Prove that the number $|I|$ of intervals appearing in this union is at most countable. If U is bounded, does $\partial U = \bigcup_{i \in I}\{a_i, b_i\}$?*

The completeness of the real numbers means that, on the extended real line $\mathbb{R} \cup \{\pm\infty\}$, every subset has a supremum and an infimum. For each $x \in U$, let $a_x := \inf\{z < x \mid (z, x) \subset U\}$ and $b_x := \inf\{z > x \mid (x, z) \subset U\}$; then $I_x := (a_x, b_x)$ is a maximal open interval contained in U. It is key to note that, for $x, y \in U$, the intervals I_x and I_y are either equal or disjoint; taking $I \subset U$ to be a set of representatives for the equivalence relation "$x \sim y$ iff $I_x = I_y$," we have $U = \bigcup_{x \in I} I_x$ expressed as a disjoint union of disjoint open intervals. That the number of these intervals is at most countable follows upon noting that $\mathbb{R}$ contains the countable dense subset $\mathbb{Q}$, allowing to define a map $I \to \mathbb{Q}$ that sends $x \in I$ to a choice of a rational number in the interval I_x: as the I_x are pairwise disjoint, this map is necessarily injective, hence the countability of I.

It is generally not true that ∂U is the union of the endpoints of the intervals I_x appearing in the union. For example, consider $U := \bigcup_n \left(\frac{1}{n+1}, \frac{1}{n}\right)$, which has 0 as a boundary point. For another example which shows that ∂U can even be uncountable (whereas the union of the endpoints of the at most countable set of intervals I_x is at most countable), consider U to be the complement in $[0,1]$ of the Cantor set: then $\partial U = C$, as a reflection of the total disconnectedness of C. ∎

5. *(i) Prove that $\mathbb{R}$ has a countable base. (ii) Prove that if X has a countable base, then any open cover of X has a countable subcover. (iii) Suppose every infinite subset of X has a limit point. Prove directly that X has a countable base.*

A countable base of $\mathbb{R}$ is given by the collection $\{(\alpha, \beta)\}_{\alpha,\beta\in\mathbb{Q}}$ of intervals with rational endpoints: the key was that $\mathbb{R}$ contained a countable dense subset, such as the rational numbers $\mathbb{Q}$. This settles (i).

If X has a countable base $(B_i)_{i\in I}$ and if $(U_\alpha)_{\alpha\in A}$ is any open cover, then having the B_i form a base implies that, for each $i \in I$, the set $\{\alpha \in A \mid B_i \subset U_\alpha\}$ is nonempty; thus[1], there exists a choice function $f : I \to A$ such that $B_i \subset U_{f(i)}$ for all $i \in I$; and then $\bigcup_{i\in I} B_i = X$ implies $\bigcup_{i\in I} U_{f(i)} = X$, so $(U_{f(i)})_{i\in I}$ is a countable subcover of $(U_\alpha)_{\alpha\in A}$. This settles (ii).

For (iii), we assume that every infinite subset of X has a limit point, and first show that X is **totally bounded**: for every $\varepsilon > 0$, X can be covered with a finite number of ε-balls. Pick a point $x_0 \in X$; if $X = B_\varepsilon(x_0)$, then stop; else, pick a point $x_1 \in X \setminus B_\varepsilon(x_0)$. If $X = B_\varepsilon(x_0) \cup B_\varepsilon(x_1)$, then stop; else, pick a point $x_2 \in X \setminus \bigcup_{i=0}^{1} B_\varepsilon(x_i)$. This process either terminates—in which case we have covered X by a finite union of ε-balls; or it produces an infinite sequence $x_0, x_1, x_2, \ldots$ with all $d(x_i, x_j) \geq \varepsilon$ for $i \neq j$. But in the latter case, the infinite set $\{x_0, x_1, x_2, \ldots\}$ has no limit point, contradicting our hypothesis; this proves that X is totally bounded.

Thus, for every $n = 1, 2, 3, \ldots$, X can be expressed as a finite union $\bigcup_{i=1}^{k_n} B_i(1/n)$ of balls of radii $1/n$. Then the countable collection $\mathcal{B} := \big(B_i(1/n)\big)_{\substack{n\in\mathbb{N} \\ 1\leq i\leq k_n}}$ of open balls forms a base for X: if B is any ball of radius $\rho > 0$, then (for any chosen n) the center of B lies in one of the balls

[1] Since I is countable, the Axiom of Choice is not needed in this argument.

$B_i(1/n)$, $1 \le i \le k_n$; for $n > 2/\rho$, this implies that the entire ball $B_i(1/n)$ is contained in B, showing that every open ball contains a member of $\mathcal{B}$, which therefore forms a base for X. ∎

6. *Prove that* $[0,1]$ *is connected. Is the locus*

$$X := (\{0\} \times [-1,1]) \cap \{(x,y) \mid x \neq 0, y = \sin 1/x\}$$

in $\mathbb{R}^2$ *also connected?*

Suppose to the contrary that $[0,1]$ is disconnected; this means that it can be partitioned into the disjoint union of two nonempty open subsets A, B. Assume that $0 \in A$, and let $c := \inf B$, which exists by the completeness of the real numbers and is in the interval $(0,1)$ since B is nonempty. By the definition of c we have that $c \in \partial B$ is a boundary point, and since by assumption $A = [0,1] - B$, this means that every neighborhood of c meets both A and B. As c lies in either A or B, this is a contradiction with the assumption that A, B are open and disjoint, proving the connectedness of $[0,1]$.

Next, recall (as seen in class and in section) that if $f : X \to Y$ is any continuous surjective map from a connected metric space X to a metric space Y, then Y is connected (this fact is referred to as *the image of a connected set is connected*): if $Y = U \cap V$ with U, V nonempty, open, and disjoint, then surjectivity and continuity of f implies that $X = f^{-1}(Y) = f^{-1}(U) \cup f^{-1}(V)$ is a disjoint union of nonempty open sets, contradicting the connectedness of X. By what was shown, every compact interval on the real line is connected; this also implies connectedness of $(0,1)$, since $(0,1) = \bigcup_n \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ is an increasing union of compact intervals. It follows that the graphs of the functions $\mathbb{R}_{>0} \to \mathbb{R}, \quad x \mapsto \sin(1/x)$ and $\mathbb{R}_{<0} \to \mathbb{R}, \quad x \mapsto \sin(1/x)$, which are continuous limits of an open interval, are connected.

The set X in this question is connected. First note that X is is the union of the compact segment $\{0\} \times [-1,1]$ and the latter two connected graphs; it is partitioned disjointly into these three connected sets. Noting that the boundaries of the two graphs intersect at the line segment, it suffices to establish the following general claim: *if* $A, B \subset Y$ *are two connected subsets of a metric space* Y*, then* $A \cap \overline{B} \neq \emptyset$ *implies that* $A \cap B$ *is connected.* Suppose to the contrary that $A \cup B = U \cap V$ with $U, V \subset A \cup B$ disjoint nonempty open subsets. Then $A = (A \cap U) \cup (A \cap V)$ with $A \cap U, A \cap V$ open and disjoint, and it follows from the connectedness of A that one of these empty; say, $A \subset U$ and $A \cap V = \emptyset$. Likewise, connectedness of B implies, together with the assumption that V is nonempty, that $B \subset V, B \cap U = \emptyset$. But U, V open and disjoint implies $U \cap \overline{V} = \emptyset$; as $A \subset U, \overline{B} \subset \overline{V}$, this is in contradiction with $A \cap \overline{B} \neq \emptyset$. Thus $A \cup B$ is connected, establishing the claim and completing the solution. ∎