McMullen §10, Ahlfors ch. 2, Lec. 24

(1) Analytic functions:

- $f: U \subset \mathbb{C} \to \mathbb{C}$ is analytic (= holomorphic) if $\forall z \in U\ \exists f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$
- f is analytic $\Leftrightarrow$ f differentiable in real sense and $\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$
  and then $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = f'(z)$, and $df = f'(z)\,dz$ Cauchy-Riemann eq^n
  ($\Leftrightarrow Df: \mathbb{R}^2 \to \mathbb{R}^2$ is complex-linear).
  ($\Leftrightarrow$ conformal transformation: Df preserves angles between vectors (& orientation))
- Ex: polynomials, rational functions $\frac{P(z)}{Q(z)}$
  Rational functions extend to the Riemann sphere $S = \mathbb{C}\cup\{\infty\}$: $f: S \to S$
  $\deg(f) = \max(\deg P, \deg Q)$ (after simplifying any common zeroes)
  $= \#\text{poles} = \#\text{zeros} = \# f^{-1}(c)\ \forall c$ (with multiplicities)
  deg. 1 case = fractional linear transformations $\mathrm{Aut}(S) = \left\{ z \mapsto \frac{az+b}{cz+d} \right\} \simeq PGL(2,\mathbb{C})$.

Lec. 25

- Power series $\sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R = \frac{1}{\limsup |a_n|^{1/n}}$, uniformly on $\{|z|\le r\}\ \forall r<R$.
  $f(z)$ is analytic on D_R, $f'(z) = \sum n a_n z^{n-1}$.
- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^x e^{iy}$ ($|\cdot| = e^{\mathrm{Re}\, z}$, $\mathrm{Arg} = \mathrm{Im}\, z$). $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, ...
  $\log(z)$ only defined up to $+2\pi i\mathbb{Z}$, $z^a = e^{a\log z}$ also multivalued if $a \notin \mathbb{Z}$
  $\log'(z) = \frac{1}{z}$, $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots\ (R=1)$ $\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \dots\ (R=1)$.
- key facts: f analytic $\Rightarrow$ f has derivatives to all orders
  f analytic on $D(z_0,r) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ on $D(z_0,r)$, where $a_n = \frac{1}{n!} f^{(n)}(z_0)$

McMullen §10, Ahlfors 4.1–4.2

(2) Complex integration – Cauchy's integral formula (and applications)

- $\omega = f(z)\,dz$ complex 1-form $\rightsquigarrow \int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$
  - Setting for Cauchy's thm. and applications: $D \subset \mathbb{C}$ bounded region with piecewise smooth boundary $\gamma = \partial D$, $f(z)$ analytic on $U \supset \bar D$ (or on $U \setminus \{z_i\}$, $z_i \in \mathrm{int}(D)$ isolated)
  - Cauchy's thm: $f(z)$ analytic on $U \supset \bar D$, $\partial D = \gamma \Rightarrow \int_\gamma f(z)\,dz = 0$.
    (= follows from Stokes, since f analytic $\Rightarrow f(z)\,dz$ is a closed 1-form).
    Still holds if f analytic in $U \setminus \{z_0\}$ and $\lim_{z\to z_0} (z-z_0) f(z) = 0$.

Lec. 26

- $\int_{S^1(z_0,r)} (z-z_0)^n\,dz = 0$ if $n \ne -1$, $\int_{S^1(z_0,r)} \frac{dz}{z-z_0} = 2\pi i$. (or any γ around z_0).
- Cauchy's integral formula: f analytic on $U \supset \bar D \Rightarrow f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}\ \forall z \in \mathrm{int}(D)$.
- for derivatives: $\frac{1}{n!} f^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z)^{n+1}}\ \forall z \in \mathrm{int}(D)$.
- in fact: φ any C^0 function on $\gamma = \partial D \Rightarrow g(z) = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(w)\,dw}{w-z}$ is analytic on $\mathrm{int}(D)$.

Lec. 27

- Cauchy's bound: f analytic in $U \supset \overline{B(z_0,R)} \Rightarrow \left|\frac{f^{(n)}(z_0)}{n!}\right| \le \frac{1}{R^n} \sup_{w\in S^1(z_0,R)} |f(w)|$
- this implies: a bounded entire analytic function is constant
  a nonconstant entire function has dense image $\overline{f(\mathbb{C})} = \mathbb{C}$.

Ahlfors 4.3

- Taylor: f analytic on $B(z_0,R) \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ on $B(z_0,R)$, where $a_n = \frac{1}{n!} f^{(n)}(z_0)$
  (radius of conv. $\ge R$; if $=R$ then $\exists$ nonremovable singularity on $S^1(z_0,R)$)
- if $f^{(n)}(z_0) = 0\ \forall n$, $z_0 \in U$ connected $\Rightarrow f(z) = 0\ \forall z \in U$
  $f^{(n)}(z_0) = g^{(n)}(z_0)\ \forall n$ $\qquad f(z) = g(z)$
- $f: U \to \mathbb{C}$ analytic, not $\equiv 0$ $\Rightarrow$ the zeros of f are isolated (no limit pts in U) and $\exists$ finite order at each
- uniqueness: f, g analytic on U connected open, $f = g$ on a non-isolated subset of U $\Rightarrow f = g$ on U.
- if $f_n(z)$ analytic on U, $f_n \to f$ uniformly on compact subsets of U, then f is analytic on U; and f_n' converge (unif. on compact subsets) to f'
  (⚠ this doesn't hold for real functions)

Lec. 28

- uniformly bounded sequences of analytic functions on U are equicontinuous on compact subsets (by Cauchy's bound) $\Rightarrow \exists$ subsequence which converges uniformly on compact subsets.
- if $f(z)$ is analytic on $U \subset \mathbb{C}$ simply connected open, then $\exists$ analytic function $F: U \to \mathbb{C}$ st. $F'(z) = f(z)$ (take $F(z) = \int_{z_0}^{z} f(z)\,dz$): antiderivative
  (ex: • apply to $1/z$ to define log over a simply conn. subset of $\mathbb{C}^* = \mathbb{C} - \{0\}$
  • if f has no zeroes on U simply conn^d, applying to $\frac{f'}{f}$ gives $g(z)$ st. $f = e^g$.)
- inverse function: f analytic, $f(a) = b$, $f'(a) \neq 0 \Rightarrow \exists$ analytic inverse function g on a nbd. of b, $g'(b) = \frac{1}{f'(a)}$

Ahlfors 4.3 (also 5.1)
McMullen §12

③ Poles and singularities

- Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ converges for $R_1 = \limsup_{n\to-\infty} |a_n|^{1/|n|} < |z| < R_2 = \frac{1}{\limsup_{n\to+\infty} |a_n|^{1/n}}$
- if $f(z)$ is analytic in $A_{R_1,R_2} = \{R_1 < |z| < R_2\}$ then it can be expressed as a Laurent series $\sum_{-\infty}^{\infty} a_n z^n$ $\left(a_n = \frac{1}{2\pi i}\int_{S^1(r)} \frac{f(w)\,dw}{w^{n+1}}\right)$ which converges on A_{R_1,R_2}.

Lec. 29

- if f is analytic in $D^*(R) = D(R) - \{0\}$, isolated singularity at 0, then one of these holds:
  → f has removable singularity at 0, ie. has analytic extension on $D(R)$
  $\Leftrightarrow$ Laurent series of f has no negative part. $\Leftrightarrow$ $f(z)$ is bounded in a nbd. of 0
  → f has a pole at 0 (of order $m \ge 1$), ie. $\exists g(z)$ analytic on $D(R)$ st. $f(z) = \frac{g(z)}{z^m}$
  $\Leftrightarrow$ Laurent series of f is $\sum_{-m}^{\infty} a_n z^n$ (finite negative part) $\Leftrightarrow |f(z)| \to \infty$ as $z \to 0$
  → f has essential singularity (ie. neither removable nor pole)
  $\Leftrightarrow$ Laurent series has infinite negative part $\Leftrightarrow \forall \varepsilon > 0$, $f(D^*(\varepsilon))$ is dense in $\mathbb{C}$.
- f is meromorphic if f is analytic in $U \setminus \{p_i\}$ (isolated) poles at p_i, no essential singularity.

- f meromorphic function on U extends to $\hat{f}: U \to S = \mathbb{C} \cup \{\infty\}$ (set $\hat{f} = \infty$ at poles)
  $\hat{f}$ is analytic $U \to S$, ie. f analytic away from its poles
  $1/f$ analytic away from its poles = the zeros of f.
- meromorphic functions = quotients of analytic functions $\frac{f(z)}{g(z)}$.
- f entire function, $|f(z)| \le M|z|^n$ for $|z| \to \infty \Rightarrow f$ is a polynomial of degree $\le n$.
  $f: S \to S$ analytic (ie. $f(z)$ and $f(\frac{1}{z})$ both meromorphic) $\Rightarrow f$ is a rational function.

McMullen §11 ④ **Local behavior: maximum principle, open mapping principle.**

Ahlfors 4.3.4, 4.6.1 lec. 29

- Cauchy $\Rightarrow$ mean value formula $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})\, d\theta$ if f analytic on $\overline{B(z,r)}$
- <u>maximum principle</u>: f analytic on U, nonconstant $\Rightarrow |f|$, $\mathrm{Re}(f)$ don't achieve max. anywhere in U. If f continuous on $\overline{U}$ compact, then max achieved at ∂U.

Lec. 30

- <u>Schwarz lemma</u>: f analytic on $D = \{|z|<1\}$, $|f(z)| < 1\ \forall z \in D$, $f(0) = 0$
  $\Rightarrow |f'(0)| \le 1$ and $|f(z)| \le |z|\ \forall z \in D$. Equality implies $f(z) = cz$ for some $c \in S^1$.
- $f = u + iv$ is analytic $\Rightarrow u = \mathrm{Re} f$, $v = \mathrm{Im} f$ are <u>harmonic</u> ie. $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
- conversely, u harmonic on simply-conn^d open $U \subset \mathbb{C} \Rightarrow \exists$ analytic $f: U \to \mathbb{C}$ st. $u = \mathrm{Re} f$.
- hence: C^2 harmonic functions are C^∞, satisfy mean value formula and max. principle.
- (Riemann mapping thm: $U \subset \mathbb{C}$ nonempty simply conn^d open, $U \ne \mathbb{C} \Rightarrow \exists$ bihol. $\varphi: U \xrightarrow{\sim} D = \{|z|<1\}$ ie. analytic bijection w/ analytic inverse.)
- <u>open mapping principle</u>: a nonconstant analytic f^n is an open mapping, ie. U open $\Rightarrow f(U)$ open

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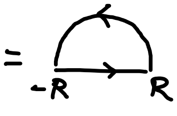
McMullen §13 ⑤ **Residue calculus** (= applications of Cauchy to calculate integrals)

Ahlfors 4.5 lec. 31

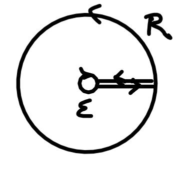
- <u>argument principle</u>: $f: U \to \mathbb{C}$ analytic, $U \supset \overline{D}$ bounded domain, $\partial D = \gamma$
  assume $f \ne 0$ on γ, then #zeros of f in D (with multiplicity) $= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz$
  Similarly, $c \notin f(\gamma) \Rightarrow \# f^{-1}(c)$ in D (with mult.) $= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - c}\, dz$
  (loc. constant f^n of c: open mapping principle) = <u>winding number</u> of $f(\gamma)$ around $c \in \mathbb{C}$.
- if f is meromorphic, winding$(f \circ \gamma)$ = # zeros − # poles in D (w/ multiplicities).
- Rouché's thm: f, g analytic in $U \supset \overline{D}$, $|f(z) - g(z)| < |f(z)|\ \forall z \in \gamma \Rightarrow \# f^{-1}(0) = \# g^{-1}(0)$ (w/ multiplicities)
- The <u>residue</u> of f at p (isolated singularity): $\mathrm{Res}_p(f) = \frac{1}{2\pi i}\int_{S^1(p,\varepsilon)} f(z)\, dz$ = coeff^t of $(z-p)^{-1}$ in Laurent series
  (if simple pole: $= \lim_{z \to p} (z-p) f(z)$).
- <u>Residue thm</u>: f analytic on $U \supset \overline{D} - \{p_i\}$ isolated, $\partial D = \gamma \Rightarrow \frac{1}{2\pi i}\int_\gamma f(z)\, dz = \sum_{p_i} \mathrm{Res}_{p_i}(f)$.
  compact
- Definite integrals via residue:
  1) $\int_0^{2\pi} R(\sin\theta, \cos\theta)\, d\theta$ (rational function) $\Rightarrow$ set $z = e^{i\theta}$ to get $\int_{S^1}$, via $\cos\theta = \frac{z + z^{-1}}{2}$, $d\theta = \frac{dz}{iz}$, ...
  + apply residue thm to unit disc.
  2) 3) $\int_{-\infty}^{\infty} R(x)\, dx$ (rational function), $\int_{-\infty}^{\infty} R(z) e^{iz}\, dz \Rightarrow$ close path to $\int_{C_R}$, $C_R =$ 

Lec. 32 (This requires bounds on integrand to check $\int$ on semicircle $\to 0$ as $R \to \infty$)

$\rightsquigarrow$ sum over residues at poles in $\{\operatorname{Im} z > 0\}$.

3') if there's a pole on the contour of integration, make a detour and estimate $\int$ on small semicircle $\left(\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \right)$.

4) branch behavior eg. $\int_0^\infty x^\alpha R(x)\, dx$: "keyhole" contour + use the multivalued nature of the integrand $\Rightarrow$ $\rightleftarrows$ don't cancel !



Ahlfors 5.1–5.2

⑥ Sum and product expansions:

Lec. 33

- partial fractions: $R(z)$ rational function $\Rightarrow$ $R(z) = \sum_j P_j\left(\frac{1}{z-b_j}\right) + S(z)$ $\quad$ (b_j = poles)

  where $P_j\left(\frac{1}{z-b_j}\right) = \frac{a_{-m}}{(z-b_j)^m} + \dots + \frac{a_{-1}}{z-b_j}$ polar part at $z = b_j$; $\quad S(z)$ polynomial.

- for $f(z)$ meromorphic with (isolated) poles b_j, with polar parts $P_j\left(\frac{1}{z-b_j}\right)$:

  $\sum P_j\left(\frac{1}{z-b_j}\right)$ might not converge, but $\exists$ polynomials $q_j(z)$ = truncated Taylor series of $P_j\left(\frac{1}{z-b_j}\right)$

  st. $\sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z) \right)$ converges (absolutely, uniformly on compact sets).

  Then we get $f(z) = \sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z) \right) + g(z)$, $\;g$ entire analytic function.

- Ex: $\dfrac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$, $\quad \pi \cot(\pi z) = \dfrac{1}{z} + \sum_{n \neq 0} \left(\dfrac{1}{z-n} + \dfrac{1}{n} \right)$.

Lec. 34

- infinite products: $\prod_{i=1}^{\infty} p_i$ converges if
  - at most finitely many terms are zero.
  - $\prod_{p_i \neq 0} p_i$ converges to a nonzero limit

  This forces $p_i \to 1$, and convergence amounts to that of $\sum \log(p_i)$.

  (Unif.) convergent products of analytic functions are analytic, orders of zeros = sum of orders of the factors that equal zero.

- for $f(z)$ entire function with (isolated) zeros at b_j with order m_j

  $\prod_j \left(1 - \frac{z}{b_j}\right)^{m_j}$ might not converge, but $\exists\, q_j(z) = \frac{z}{b_j} + \frac{1}{2}\left(\frac{z}{b_j}\right)^2 + \dots + \frac{1}{d}\left(\frac{z}{b_j}\right)^d$ polynomial

  (truncated Taylor series of $-\log\left(1 - \frac{z}{b_j}\right)$ st. $\prod_j \left[\left(1 - \frac{z}{b_j}\right) e^{q_j(z)} \right]^{m_j}$ converges (absolutely, unif. on compact sets).

  Then we get $f(z) = z^{m_0} \prod_j \left[\left(1 - \frac{z}{b_j}\right) e^{z/b_j + \dots + (z/b_j)^d/d} \right]^{m_j} e^{g(z)}$, $\;g(z)$ entire f$^{\underline{n}}$.

- Ex: $\sin(\pi z) = \pi z \prod_{n \neq 0} \left(\left(1 - \frac{z}{n}\right) e^{z/n} \right)$.

- in sum & product expressions, find the unknown term $g(z)$ by comparing (log.) derivatives and/or by showing g is bounded (hence constant), etc.