Homework 2

Math 55b
Due Tuesday, 10 Feb 2009.

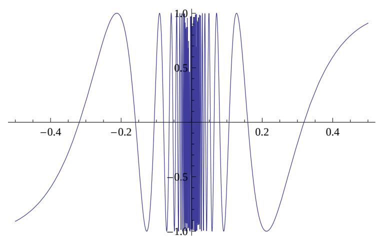

Figure 1. A connected set (?)

1. Let $B' = X - B$. Give an example of set $A \subset \mathbb{R}$ such that, by repeatedly taking closures and complements (i.e. by forming A, A', $\overline{A}$, $\overline{A}'$, $\overline{A'}, \ldots$), you obtain a total of 14 distinct subsets of $\mathbb{R}$.

2. Prove that for any subset E of a metric space (X, d), we have $\partial E = \overline{E} - \operatorname{int}(E)$. (Recall $x \in \partial E$ iff every neighborhood of x meets both E and $X - E$.)

3. Let E_1 denote the set of limit points of E, and E_{n+1} the set of limit points of E_n. For each $n > 0$, given an example of a set $E \subset [0, 1]$ such that $E_n \neq \emptyset$ but $E_{n+1} = \emptyset$.

   Prove or disprove: for any $E \subset [0, 1]$, $F = \bigcap E_n$ is perfect (it satisfies $F_1 = F$).

4. Prove that any open subset $U \subset \mathbb{R}$ can be expressed as a union $U = \bigcup_{i \in I}(a_i, b_i)$ of disjoint open intervals (we allow $\pm\infty$ as endpoints). Prove that the number of intervals $|I|$ appearing in this union is at most countable.

   Now assume U is bounded. Does $\partial U = \bigcup\{a_i, b_i\}$?

5. A collections of open sets $\mathcal{B}$ forms a *base* for a metric space (X, d) if for every open set $U \subset X$ and $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.

(i) Prove that $\mathbb{R}$ has a countable base.

(ii) Prove that if X has a countable base, then any open cover of X has a countable subcover.

(iii) Suppose every infinite subset of X has a limit point. Prove (directly from the definitions) that X has a countable base.

6. (See Figure.) Prove that $[0, 1]$ is connected. Is the locus

$$X = (0 \times [-1, 1]) \cup \{(x, y) \; : \; x \neq 0, y = \sin(1/x)\}$$

in $\mathbb{R}^2$ also connected?