Solutions to Homework 4

Math 55b

1. *Let $U_n \subset \mathbb{R}$ be a sequence of open sets containing $\mathbb{Q}$. Prove that $\bigcap U_n$ contains an irrational number.*

   Label the rationals $q_1, q_2, \ldots$, and construct inductively a nested sequence of closed intervals $I_1 \supset I_2 \supset \cdots$ with the property that $I_n \subset U_1 \cap \cdots \cap U_n$ and $q_n \notin I_n$. Start by taking I_1 an interval containing q_1 and contained in U_1 (this is made possible by the assumption that U_1 is an open neighborhood of q_1); having constructed I_n, take an open interval J contained in I_n and not containing q_n, and take for I_{n+1} any closed interval contained in the open set $U_1 \cap \cdots \cap U_{n+1} \cap J$, which is nonempty since $U_1 \cap \cdots \cap U_{n+1}$ is an open neighborhood of $\mathbb{Q}$, and $\mathbb{Q} \cap J \neq \emptyset$.

   It is manifest that $\bigcap I_n \subset \bigcap U_n$; and that $\bigcap I_n \neq \emptyset$ for a descending chain of (closed) intervals follows from the completeness axioms of $\mathbb{R}$; alternatively, from the finite intersection property characterization of compactness. Finally, $\bigcap I_n$ contains no rational number, since by construction $q_n \notin I_n$. The conclusion follows. ∎

   Remark. Some of you noted (a special case of the) **Baire category theorem**: if X is a complete metric space and U_n a sequence of dense open subsets, then $\bigcup U_n$ is dense. The way this applies is by taking $X := \mathbb{R}$ and the sequence $U_n - \{q_n\}$ of dense open subsets: the intersection of this sequence must be dense and in particular nonempty, but it contains no rational number, hence the conclusion. The proof of the general result is exactly the same. ∎

2. *Let $f : [0,1] \to \mathbb{R}$ be a function, and $\Gamma(f) \subset [0,1] \times \mathbb{R}$ its graph. Prove or disprove each of the four implications:*

   $$f \textit{ is continuous } \Leftrightarrow \Gamma(f) \textit{ is compact } \Leftrightarrow \Gamma(f) \textit{ is connected.}$$

   The only implication that does not hold is that connectedness of the graph does not imply compactness, as demonstrated by the example of the topologist's sine curve (whose connectedness was shown on the second homework):

   $$f(x) := \begin{cases} \sin(1/x), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

The other three implications hold. Note, first of all, that $\Gamma(f)$ is, by definition, the image of the compact connected interval $[0,1]$ under the map $\mathrm{id} \times f : [0,1] \to [0,1] \times \mathbb{R}$; this map is continuous if and only if f is. Since the continuous image of $[0,1]$ is compact and connected, it follows that continuity of f implies compactness and connectedness of $\Gamma(f)$.

It remains to show that compactness of $\Gamma(f)$ implies continuity of f. For *any* function $f : [0,1] \to \mathbb{R}$, the projection map $\Gamma(f) \to [0,1]$ onto the x-axis is a continuous bijection. Recall a very important fact shown in class: a continuous bijection $g : X \to Y$ from a compact space X is a homeomorphism, i.e. $g^{-1} : Y \to X$ is also continuous (Proof: continuity of g^{-1} means that g is a closed map. To show this, note that compactness of X implies that every closed subset $F \subset X$ is compact; since the continuous image of a compact space is compact, it follows that g maps any closed subset of X onto a compact, hence closed, subset of Y, hence g is a closed map). In our situation, we conclude that compactness of $\Gamma(f)$ implies that the projection map $\Gamma(f) \to [0,1]$ onto the x-axis is a homeomorphism; and this is equivalent to the continuity of f. $\blacksquare$

3. *Prove that if $a_n \geq 0$ and $\sum_n a_n$ converges, then so does $\sum_n \sqrt{a_n}/n$.*

   This is an application of the Cauchy-Schwartz inequality: $\left(\sum_{n \leq N} \sqrt{a_n}/n\right)^2 \leq \left(\sum_{n \geq N} a_n\right)\left(\sum_{n \leq N} \frac{1}{n^2}\right) < \infty$, since $\sum_{n \leq N} \frac{1}{n^2} < 1 + \sum_{n<N} \frac{1}{n(n+1)} = 1 + \sum_{n<N}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2 - \frac{1}{N} < 2$ for all n. $\blacksquare$

4. *Let $a_1 = 1$ and $a_{n+1} = (a_n + 2/a_n)/2$. Let $\epsilon_n := a_n - \sqrt{2}$. (i) Show that $\epsilon_n \to 0$ (ii) Compute $\lim_n \log\log(1/\epsilon_n)$.*

   Let $\eta_n := a_n + \sqrt{2}$ (whenever you see $a_n - \sqrt{2}$ and a recurrence with rational coefficients, always think about $a_n + \sqrt{2}$!), and note that the recurrence $a_{n+1} = (a_n + 2/a_n)/2$ may be rewritten as $\frac{\epsilon_{n+1}}{\eta_{n+1}} = \frac{\epsilon_n^2}{\eta_n^2}$. This shows $\frac{\epsilon_{n+1}}{\eta_{n+1}} = \frac{\epsilon_1^{2^n}}{\eta_1^{2^n}} = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{2^n}$. Since the right-hand side converges to 0 and $\eta_n > \sqrt{2}$ for all n, it follows that $\epsilon_n \to 0$. Moreover, as it then follows that η_n is bounded both below and above, the last formula implies $\lim \log\log(1/\epsilon_n) = \log 2$. $\blacksquare$

5. *Let $P(t) = t^d + a_1 t^{d-1} + \cdots + a_d \in \mathbb{Z}[t]$, and suppose $x \in \mathbb{R}$ satisfies $P(x) = 0$. Show that there exists a constant $C > 0$ such that every rational*

number $p/q \in \mathbb{Q} - \{x\}$ satisfies $\left|x - \frac{p}{q}\right| > \frac{C}{q^d}$. Use this to show that the number $\sum_n 10^{-n!}$ is transcendental.

The idea is to estimate $|P(p/q)|$ from above by the mean value theorem, and from below by the basic arithmetic estimate $|P(p/q)| \geq 1/q^d$, which is simply a reflection of the fact that $P(p/q)$ is a nonzero rational number of denominator dividing q^d. For the upper bound, note that the mean value theorem provides some $\alpha = \alpha(p/q)$ between x and p/q such that $|P(p/q)| = |P(p/q) - P(x)| = \left|x - \frac{p}{q}\right| \cdot P'(\alpha)$. Comparing the two bounds, we may take $C := \min\left(1, M^{-1}\right)$, where $M := \sup_{t\in[x-1,x+1]} |P'(t)| < \infty$.

This result is called the **Liouville inequality**, and yields a criterion of transcendence: if $\beta \in \mathbb{R}$ is such that for every d there exists a rational number $p/q \in \mathbb{Q} - \{\beta\}$ with $|\beta - p/q| < q^{-d}$, then β is transcendental; such numbers are called **Liouville numbers**. An example is the number $\alpha := \sum_n 10^{-n!}$ and its approximation by the partial sums $q_N := \sum_{n\leq N} 10^{-n!}$, which are rational numbers of denominators $10^{-N!}$: since $|\alpha - q_N| = \sum_{n>N} 10^{-n!} < 2 \cdot 10^{-(N+1)!} < 10^{-N!d}$ for $N \gg_d 0$, it follows that α is a Liouville, hence transcendental, number. ■

Remark. The Liouville inequality is only sharp for $d \leq 2$. For $d > 2$, this basic result is superseded by a deep theorem of **Klaus Roth**, which states that the exponent d can be replaced by any number > 2. More precisely, for every $\kappa > 2$ and $\alpha \in \overline{\mathbb{Q}}$ algebraic, the set $\left\{p/q \in \mathbb{Q} \mid |\alpha - p/q| < q^{-\kappa}\right\}$ is finite. ■

6. *Prove that the function $S(t) := \sup_N \sum_{n=0}^{N} \sin(nt)$ is finite for all $t \in \mathbb{R}$. Is it bounded?*

There is an exact formula for the sum: $\sum_{n=1}^{N} \sin(nt) = \frac{\sin(Nt/2)\sin((N+1)t/2)}{\sin(t/2)}$. This follows upon successively setting $y := nt + t/2$, $x := nt - t/2$ in the formula $\cos x - \cos y = 2\sin\frac{y+x}{2}\sin\frac{y-x}{2}$, and forming the resulting telescoping sum of identites for $n = 1, 2, \ldots, N$. The finiteness of $S(t)$ follows from this expression: it is bounded above in absolute value by $1/|\sin(t/2)|$, when $t \in 2\pi\mathbb{Z}$, and it is identically 0 for $t \in 2\pi\mathbb{Z}$. It also follows from this closed form expression that $S(t)$ is not bounded in t, as seen upon taking $t := \pi/q$ for $q = 1, 2, 3, \ldots$. ■

Remark. In fact, it is not difficult to find the exact value of $S(t)$ from the given closed form evaluation:

$$S(t) = \begin{cases} \frac{1}{|\sin(t/2)|}, & \text{if } t/\pi \notin \mathbb{Q}, \\ \frac{\cos(\pi/q)}{|\sin(t/2)|}, & \text{if } t/\pi \text{ is rational with denominator } q > 1, \\ 0, & \text{if } t/\pi \in \mathbb{Z}. \end{cases}$$

■

7. *Prove that if if $\sum a_n = S$, then $\lim_{r \to 1^-} a_n r^n = S$.*

As noted by most of you, the argument appears in Rudin, Theorem 8.2. Let $s_N := \sum_{n \le N} a_n$ be the partial sums, and denote $a_{-1} := 0$. We have $\sum_{n \le N} a_n r^n = \sum_{n \le N} (s_n - s_{n-1}) r^n = s_N r^N + (1-r) \sum_{n \le N-1} s_n r^n$, showing for $r < 1$ that the series $\sum_n a_n r^n$ converges to the series $(1-r) \sum_n s_n r^n$. Letting $\varepsilon > 0$ arbitrary, there is an $M > 0$ such that $n > M$ yields $|s_n - S| < \varepsilon/2$. Then $\left| \sum_n a_n r^n - S \right| = (1-r) \left| \sum_n (s_n - S) r^n \right| \le (1 - r) \sum_{n \le M} |s_n - S| r^n + \varepsilon/2$. For $r > 1 - \varepsilon / \big(2M \max_0^M |s_n - M|\big)$, the sum in the right hand-side is $< \varepsilon/2$, and the inequality becomes $\left| \sum_n a_n r^n - S \right| < \varepsilon$. Letting $\varepsilon \to 0$ proves the required limit $\lim_{r \to 1^-} a_n r^n = S$. ■

8. *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f'(x)$ is continuous. Prove that $f_n(x) := n\big(f(x + 1/n) - f(x)\big)$ converges to $f'(x)$ uniformly on compact intervals $[a, b]$. Give an example where the convergence is not uniform on $\mathbb{R}$.*

On a compact metric space, a continuous function is uniformly continuous; thus f' converges uniformly on compact intervals, and hence, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $x, y \in [a, b+1]$ and $|x - y| < \delta$ imply $|f'(x) - f'(y)| < \varepsilon$. For $x \in [a, b]$, the mean value theorem implies the existence of $\xi \in (x, x + 1/n)$ with $n\big(f(x + 1/n) - f(x)\big) = f'(\xi)$; for $n > 1/\delta$, this ξ satisfies $|\xi - x| < \delta$, and the uniform continuity implies $|f'(\xi) - f'(x)| < \varepsilon$; that is, $|f_n(x) - f'(x)| < \varepsilon$ for all $x \in [a, b]$ and all $n > 1/\delta$. This shows the uniform convergence of f_n to f' on the compact intervals $[a, b]$.

The convergence need not be uniform on $\mathbb{R}$; the simplest examples are $f(x) = x^3$ and $f(x) = e^x$. ■