We know how to express analytic functions as power series over discs, or as Laurent series over annuli, but this isn't always the best representation. For rational functions, the best way to convey information about the poles is partial fractions! Of course, the amount of info. in a rational function is finite...

- Product expressions: (⇒ zeros & poles) $\quad R(z) = \dfrac{P(z)}{Q(z)} \Rightarrow$ can factor $R(z) = c\, \dfrac{\prod_{i=1}^{k} (z-a_i)^{n_i}}{\prod_{i=1}^{\ell} (z-b_i)^{m_i}}$
- Sums (partial fractions): if the poles are all simple, can write

$$R(z) = \frac{c_1}{z-b_1} + \dots + \frac{c_\ell}{z-b_\ell} + S(z) \quad \text{where } c_i \in \mathbb{C},\ S(z) \text{ polynomial.}$$

↳ useful: residues!

or in general, $R(z) = \dfrac{C_1(z)}{(z-b_1)^{m_1}} + \dots + \dfrac{C_\ell(z)}{(z-b_\ell)^{m_\ell}} + S(z)$

where $C_1, \dots, C_\ell, S$ are polynomials, $\deg(C_i) \le m_i - 1$.

We'll learn how to find similar (infinite) sum or product expansions for general meromorphic functions.

* Starting point: if $f(z)$ is meromorphic with a pole of order m at $b \in \mathbb{C}$, then we can write $f(z) = \dfrac{g(z)}{(z-b)^m}$ with $g(z)$ analytic in a nbd. of b.

Or, expressing $g(z)$ as a power series in $(z-b)$, $g(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$ we have a Laurent series for f with finite negative part, as already noticed:

$$f(z) = \underbrace{\left[\frac{a_0}{(z-b)^m} + \frac{a_1}{(z-b)^{m-1}} + \dots + \frac{a_{m-1}}{z-b}\right]}_{\text{THE POLAR PART OF } f \text{ at } z=b} + h(z), \quad h(z) = \sum_{n=0}^{\infty} a_{m+n} (z-b)^n$$

analytic near b.

This looks a lot like partial fractions, and in fact, for rational functions, it is partial fractions: if f is meromorphic with finitely many poles $b_1 \dots b_\ell$, by induction on #poles (observe: remainder $h(z)$ has one fewer pole than f), we get $f(z) = \dfrac{C_1(z)}{(z-b_1)^{m_1}} + \dots + \dfrac{C_\ell(z)}{(z-b_\ell)^{m_\ell}} + g(z)$, $C_i(z)$ polynomials of degree $< m_i$, where $g(z)$ is now analytic everywhere. What if there's ∞ many poles?

* Given $f(z)$ meromorphic on all of $\mathbb{C}$, with infinitely many (isolated) poles $b_1, b_2, \dots$ we have near each b_j the polar part (=(finite) negative part) of the Laurent expansion,

$$P_j\left(\frac{1}{z-b_j}\right) = \frac{a_{-m}}{(z-b_j)^m} + \dots + \frac{a_{-1}}{z-b_j}$$

(a polynomial without constant term in the variable $\frac{1}{z-b_j}$).

and we hope to be able to write $f(z) = \sum_{j=1}^{\infty} p_j\left(\frac{1}{z-b_j}\right) + g(z)$

where $g(z)$ no longer has any poles hence is an entire function.

Questions: → when do these kinds of sums converge? uniformly?
→ what meromorphic functions can be represented in such a way?
→ existence: given a discrete set of poles b_j and orders m_j, does there exist a meromorphic function with exactly those poles? can we prescribe the polar parts $p_j\left(\frac{1}{z-b_j}\right)$ arbitrarily?

(Apparent problem: expressions like $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$ don't seem to make sense?)

Example: let's consider the function $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$, with poles (of order 2) exactly at the integers.

The polar part at 0 can be found by expanding

$$\sin \pi z = \pi z - \frac{\pi^3}{6} z^3 + \dots \Rightarrow \sin^2(\pi z) = \pi^2 z^2 - \frac{\pi^4}{3} z^4 + \dots$$
$$= \pi^2 z^2 \left(1 - \frac{\pi^2}{3} z^2 + \dots\right)$$

so $\frac{\pi^2}{\sin^2(\pi z)} = \frac{1}{z^2}\left(1 + \frac{\pi^2}{3} z^2 + \dots\right)$ ⇒ the polar part at 0 is just $\frac{1}{z^2}$.

Since f is periodic $(f(z+1) = f(z))$, the polar part at $z = n \in \mathbb{Z}$ is $\frac{1}{(z-n)^2}$.

Observe: the sum $h(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ is convergent $\forall z \in \mathbb{C} - \mathbb{Z}$

and the convergence is uniform over compact subsets of $\mathbb{C}-\mathbb{Z}$ (prove it!)

(key observation: $K \subset \mathbb{C}-\mathbb{Z}$ compact $\Rightarrow K \subset B(0,R)$, so for $|n|$ large $\gg R$ the terms are bounded by $\sum_{|n|>n_0} \frac{1}{(|n|-R)^2}$, which converges. Apply M-test.)

so the sum is an analytic function on $\mathbb{C}-\mathbb{Z}$, easily checked to have the correct behavior (pole of order 2 with polar part $\frac{1}{(z-n)^2}$) at each $n \in \mathbb{Z}$:

indeed $h(z) - \frac{1}{(z-n)^2} = \sum_{m\neq n} \frac{1}{(z-m)^2}$ converges uniformly near $z=n$ hence analytic at n.

Hence: $g(z) = f(z) - h(z)$ is an entire analytic function (the polar parts cancel at each $z=n$)

→ can write $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} + g(z)$

where $g(z)$ is an entire function, periodic: $g(z+1) = g(z)$. What is g?

Observe: for $\mathrm{Im}(z) \to +\infty$, $|e^{i\pi z}| = e^{-\pi \mathrm{Im} z} \ll e^{\pi \mathrm{Im} z} = |e^{-i\pi z}|$, so $|f(z)| \approx \frac{4\pi^2}{e^{2\pi \mathrm{Im} z}} \to 0$.
& similarly for $\mathrm{Im}\, z \to -\infty$.

Meanwhile, for $h(z)$: if $z = x+iy$, $y \to +\infty$, $x \in [0,1]$ wlog by periodicity,

then $\left|\frac{1}{(z-n)^2}\right| = \frac{1}{|z-n|^2} = \frac{1}{(n-x)^2+y^2}$ $\Rightarrow$ terms with $|n|<y$ are $\leq 1/y^2$, $|n|>y$ are $\leq 1/(n-1)^2$

$$\Rightarrow |h(z)| \leq 2y \cdot \frac{1}{y^2} + 2\sum_{n\geq y} \frac{1}{n^2} \leq \frac{C}{y}.$$

Similarly for $y \to -\infty$.

<u>So</u>: $g(z)$ is an entire function, $g(z+1) = g(z)$, $|g(z)| \to 0$ as $|\text{Im } z| \to \infty$ (uniformly $\forall \text{Re } z$)

$\to$ g is bounded on $\mathbb{C}$, hence constant !! and since $g \to 0$ as $y \to \infty$, the constant is 0.

<u>Conclusion</u>: $$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

<u>Q</u>: what about <u>simple poles</u>? can we find $f(z)$ with simple poles at all integers, and residue 1 at each? and can we express it as a partial fraction type of sum?

In terms of partial fractions, the natural guess would be $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$ ... but this series doesn't converge!

<u>Solution</u>: add to each term an analytic function of z to cancel the divergence.

In this case: just subtract from each term its value at 0, ie. $-1/n$:

$$f(z) = \frac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{z-n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0} \frac{z}{n(z-n)}$$

This series now converges $\forall z \in \mathbb{C} - \mathbb{Z}$, uniformly over compact subsets, and has the desired polar part at each integer point.

Can we use a similar trick to build meromorphic functions with arbitrary poles and polar parts at each pole? Answer: yes, but we may need to add more complicated counter-terms to achieve convergence.

<u>Thm</u>: Let $\{b_j\}$ be an arbitrary set of complex numbers with no limit points, and for each j, P_j an arbitrary polynomial without constant term. Then there exists a meromorphic function $f(z)$ on all of $\mathbb{C}$, analytic on $\mathbb{C} - \{b_j\}$, and whose polar part at b_j is $P_j\left(\frac{1}{z-b_j}\right)$ $\forall j$.

<u>Pf</u>: The proof uses the same idea as above, except to achieve convergence we subtract from each $P_j\left(\frac{1}{z-b_j}\right)$ (for $b_j \neq 0$) a polynomial in z: given $m_j \geq 0$ integer, let $q_j(z)$ = sum of the terms of degree $\leq m_j$ of the Taylor series of $P_j\left(1/(z-b_j)\right)$ at $z=0$. The point (see Ahlfors §5.2.1) is that we can choose the m_j's so that the series $f(z) = \sum_j \left(P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right)$ converges on $\mathbb{C} - \{b_j\}$.

How does one show this? First observe: $\{b_j\}$ no limit points $\Rightarrow$ only finitely many inside any compact subset of $\mathbb{C}$ $\Rightarrow$ $|b_j| \to \infty$. Next, we need explicit bounds on the remainder $P_j\left(\frac{1}{z-b_j}\right) - q_j(z)$ of the Taylor series of $P_j\left(\frac{1}{z-b_j}\right)$. Base case:

$$\frac{1}{z-b_j} = -\frac{1}{b_j}\,\frac{1}{1-\frac{z}{b_j}} = -\frac{1}{b_j}\left(1+\frac{z}{b_j}+\left(\frac{z}{b_j}\right)^2+\cdots\right) \text{ with remainder } \left(\frac{z}{b_j}\right)^{m_j+1}\frac{1}{z-b_j},$$

$$\Longrightarrow \left|P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right| \leq C_j \left(\frac{|z|}{|b_j|}\right)^{m_j+1} \text{ whenever } |z| \leq \frac{|b_j|}{2},$$

(after more work...) where C_j depends on $P_j(\dots)$ but not on m_j.

Now, pick m_j's suff$^{\text{tly}}$ large, eg. so $\frac{C_j}{2^{m_j+1}} \leq \frac{1}{j^2}$. Then $|z| \leq \frac{|b_j|}{2} \Rightarrow \left|P_j\left(\frac{1}{z-b_j}\right) - q_j(z)\right| \leq \frac{1}{j^2}$.

Since $|b_j| \to \infty$, this implies uniform convergence over compact subsets of $\mathbb{C}$.
(since all but finitely many terms of the series are bounded by $\sum \frac{1}{j^2}$) $\square$

- Back to our function with simple poles at all integers,

$$f(z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right) = \frac{1}{z} + \sum_{n\neq 0}\frac{z}{n(z-n)}$$

Since convergence is uniform on compact subsets of $\mathbb{C}-\mathbb{Z}$, using analyticity, we can differentiate term by term. (recall: f_n analytic, $f_n \to f$ uniformly $\Rightarrow f_n' \to f'$ uniformly on compacts).

We find $f'(z) = -\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \frac{-\pi^2}{\sin^2(\pi z)}$!

Recall: $\cot(t) = \frac{\cos(t)}{\sin(t)}$ has derivative $\cot'(t) = \frac{\sin\cdot\cos' - \cos\cdot\sin'}{\sin^2(t)} = \frac{-1}{\sin^2 t}$

$\Rightarrow$ hence: $f(z) = \pi\cot(\pi z) + C$.

Since both sides are odd functions of z $(f(-z) = -f(z))$, necess. $C=0$.

Hence: $$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

Remark: there's another way to achieve convergence in this case, instead of the general method of polynomial counter-terms: combining the terms for $\pm n$,

$\frac{1}{z-n} + \frac{1}{z+n} = \frac{2z}{z^2-n^2}$ which form a convergent series. (while $+\frac{1}{n} - \frac{1}{n}$ cancel).

Hence: $$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\geq 1}\frac{2z}{z^2-n^2}.$$