Math 55a: Honors Abstract Algebra

Homework Assignment #6 (8 October 2010):
Linear Algebra VI

Это не только отрицательная величина, но отрицательная величина возведенная в квадрат![1]

Some more results about bilinear pairings and inner products:

1. [Strange life in rational normed spaces.] Recall that two norms $\| \cdot \|$, $[[\cdot]]$ on the same vector space V are said to be "equivalent" if there exist constants C, C' such that $\|v\| \leq C[[v]]$ and $[[v]] \leq C'\|v\|$ for all $v \in V$. (Convince yourself that this is indeed an equivalence relation.) Give an example of a norm $[[\cdot]] : \mathbf{Q}^2 \to \mathbf{R}$ that is *not* equivalent with the standard Euclidean norm $|(x,y)| = (x^2 + y^2)^{1/2}$ on the $\mathbf{Q}$-vector space $\mathbf{Q}^2$.
   (Next term we shall see that all norms F^n are equivalent for $F = \mathbf{R}$ or $F = \mathbf{C}$.)

2. [Strange life in infinite-dimensional inner product spaces.] Let V be the space of continuous functions from $[0,1]$ to $F(= \mathbf{R}$ or $\mathbf{C})$, with the usual inner product $\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\, dx$. Let U be the subspace consisting of all functions such that $f(0) = 0$. Show that U is *not* closed in V (i.e. there exist $u_n \in U$ and $v \in V$ such that $|u_n - v| \to 0$ but $v \notin U$), and $U^\perp = \{0\}$ (whence $V \neq U \oplus U^\perp$). Is there a *closed* subspace $W \subset V$ such that $W \neq V$ but $W^\perp = \{0\}$? (Hint: show that $U^\perp$ is closed for any subspace $U \subset V$. We'll see next term that if $\mathcal{H}$ is any *complete* inner product space [a.k.a. Hilbert space] then $\mathcal{H} = U \oplus U^\perp$ for any closed subspace U.)

3. [Field involutions.] An *involution* is a bijection ι such that $\iota^2 = \mathrm{id}$; usually (as here) we require $\iota \neq \mathrm{id}$. Let K be a field and $\iota : K \to K$ a field involution (i.e. an involution that respects all the field operations), and set $F = \{x \in K : \iota x = x\}$. Show that F is a field and ι is an F-linear transformation of K. Then prove that this linear transformation has the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with respect to some basis of K, depending on whether $2 \neq 0$ or $2 = 0$ in K (and thus in F). In particular $\dim_F(K) = 2$ regardless of the characteristic. [The case $2 \neq 0$ has the familiar example $F = \mathbf{R}$, $K = \mathbf{C}$, with ι being complex conjugation; in this case we can use the standard basis $(1, i)$ to get the desired matrix.]

4. [Semidefinite pairings.] A symmetric or Hermitian pairing $\langle \cdot, \cdot \rangle$ on an $\mathbf{R}$- or $\mathbf{C}$-vector space V is said to be *positive semidefinite* if $\langle v, v \rangle$ is a nonnegative real number for all $v \in V$. Prove that $\langle v, v \rangle = 0$ if and only if v is in the kernel of the pairing, i.e., if and only if $\langle v, w \rangle = 0$ for all $w \in V$. In particular, the set of such v is a vector subspace V_0 of V. Show that $\langle \cdot, \cdot \rangle$ yields a well-defined inner product on the quotient space V/V_0.

About normal operators:

5.–6. Solve Exercises 1, 4, 6, 7 from Chapter 7 of the textbook (page 158; here V must be an inner-product space of finite dimension). Recall that in Axler's notation the equations in Exercises 6 and 7 mean what we would write as $T(V) = T^*(V)$, $\ker T^k = \ker T$, and $T^k(V) = T(V)$.

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[1] Attributed to J. V. Stalin in the article "On Sums of Squares and on Elliptic Curves over Function Fields" (*Journal of Number Theory* **3** (1971), 125–149) by J.W.S. Cassels, W.J. Ellison, and A. Pfister. I'm told that this quote translates to "Not only is this a negative quantity — it is a negative quantity squared!" I surmise that "squared" has a colloquial use in Russian comparable to the English "to the n-th degree".

A foretaste of Fourier analysis:[2]

7. Let V be an infinite-dimensional real or complex inner product space. Then V has at least a countably infinite orthonormal set $\{v_n\}_{n=1}^{\infty}$ (why?). Prove that for every $v \in V$ we have $\langle v, v_n \rangle \to 0$ as $n \to \infty$. Conclude that in particular for every continuous function $f : [0,1] \to \mathbf{C}$ we have

$$\int_0^1 f(x) \, e^{2\pi i n x} \, dx \to 0$$

as $n \to \pm\infty$.

8. Let V be the space of functions $f : \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable (a.k.a. "smooth": $d^n f / dx^n$ exists for $n = 1, 2, 3, \ldots$) and $\mathbf{Z}$-periodic (satisfy $f(x+m) = f(x)$ for all $x \in \mathbf{R}$ and $m \in \mathbf{Z}$). We make V into an inner-product space by defining

$$\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} \, dx.$$

Let $T : V \to V$ be the operator taking any $f \in V$ to its derivative df/dx (which again is in V).

i) Prove that T is a skew-Hermitian operator on V; that is, $T^* = -T$. What does this tell you *a priori* about the eigenvalues and eigenvectors of T?

ii) Determine the eigenvalues and eigenvectors of T. Make sure that eigenvectors with different eigenvalues are orthogonal, as we know they must be.
[Eigenvectors of an operator on a function space such as V are often called "eigenfunctions" in the literature.]

This problem set is due *Monday*, 18 October, at the beginning of class.

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[2]This requires a bit more of integral and differential calculus than is in the official prerequisites, but I'm sure everybody in Math 55a has seen the relevant elementary facts about integrals and differential equations.