Putnam Prep Session – Sept 25 2017

1. Show that for each $n = 1, 2, 3, \ldots$, the sum of the first n odd integers is n^2.

2. Suppose P_1, P_2, P_3, P_4, P_5 are five points inside a square of side length 1 (but not on the edges of the square),. Show that the closest pair among these five points is no further than $1/\sqrt{2}$ apart.

3. Let P be a polynomial with integer coefficients of degree at most n. Suppose $|P(x)| < n$ for all integers x with $|x| < n^2$. Show that P is constant.

4. Suppose A and B are 2×2 matrices with integer coefficients, such that each of the matrices $A, A + B, A + 2B, A + 3B$, and $A + 4B$ has an inverse with integer coefficients. Show that $A + 5B$ has an inverse with integer coefficients, too.

5. Let $f(x) = \sqrt{x^2 - 1}$ (for $x > 1$). Show that $f^{(n)}(2) > 0$ when n is odd and $f^{(n)}(2) < 0$ when n is even.

6. Consider the following two sequences of integers:

$$\begin{array}{llll} a_1 = 4, & a_2 = 484, & a_3 = 48484, & \ldots \\ b_1 = 8, & b_2 = 848, & b_3 = 84848, & \ldots \end{array}$$

Show that for each i, $4b_i - 7a_i = 4$. Also show that for each i, $b_i^2 - a_i^2 = c_i$ where c_i is the concatenation of a_i and b_i; for example $c_2 = 484848$.

7. Show that for every $n > 1$ the expansion of $(1 + x + x^2)^n$ contains at least one even coefficient.

8. Suppose r is a real number such that $r + (1/r)$ is an integer. Show that for each n, $r^n + (1/r^n)$ is also an integer.

9. McDonalds' Chicken McNuggets are sold in packages of 6, 9, and 20. A *McNugget Number* is a number n for which it is possible to buy n McNuggets as some combination of these packages. Describe the set of all McNugget Numbers.

10. Fifty-one different integers are chosen between 1 and 100, inclusive. Show that one of them divides another.