Here are some of the answers that were presented after we worked on the problems.

1. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

ANSWER: Prove this by induction. If n is even, write $n/2$ as such a sum

$$\sum_i 2^{r_i} 3^{s_i}$$

(The condition on divisibility is that for every $i \neq j$, if $r_i \leq r_j$ then $s_i > s_j$.) Then n itself may be written as

$$n = \sum_i 2^{r_i+1} 3^{s_i}$$

which also satisfies the divisibility condition. If instead n is odd, let 3^k be the largest power of 3 which is less than or equal to n, so that $3^k \leq n < 3^{k+1}$. Then $n - 3^k$ is even and as above we write

$$(n - 3^k)/2 = \sum_i 2^{r_i} 3^{s_i}$$

for some exponents r_i and s_i, i.e.

$$n = 3^k + \sum_i 2^{r_i+1} 3^{s_i}$$

Now, as above most of the pairs of terms satisfy the divisibility condition by induction, and none of the even terms divide 3^k, obviously. We must also check that 3^k does not divide any of the even terms, i.e. that each s_i is less than k. But if $s_i \geq k$, then $(n - 3^k)/2 \geq 3^{s_i} \geq 3^k$ would imply $n \geq 3^{k+1}$, which contradicts the choice of k. So the divisibility condition is satisfied for odd n, too, and the inductive step is complete.

2. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

ANSWER: It suffices to prove the given statement for positive integers n, which we do by induction. Suppose every $k < n$ has such an expression $E(k)$ which is a quotient of products of factorials of primes. Then if n is composite write $n = ab$ with $a, b < n$; then $n = E(a) \cdot E(b)$ is such an expression $E(n)$. If on the other hand n is prime, note that

$$n = \frac{n!}{(n-1)!} = \frac{n!}{\prod_{k<n} E(k)}$$

is a satisfactory expression $E(n)$ for n.

5. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

ANSWER: Since

$$1 = 1^{m+n} = \left(\frac{m}{m+n} + \frac{n}{m+n}\right)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} \left(\frac{m}{m+n}\right)^i \left(\frac{n}{m+n}\right)^{m+n-i}$$

we know that each of the summands is less than 1. When $i = m$ this tells us

$$\frac{(m+n)!}{m!n!} \left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n < 1$$

Rearranging the terms gives the desired inequality.

REMARK: By Stirling's approximation, $k!/k^k$ is approximately equal to $e^{-k}\sqrt{2\pi k}$ for all k, in the sense that the ratio of the two expressions tends to 1 as $k \to \infty$. So for large m and n the right side of the inequality in the statement of the problem is larger than the left side by a factor of about $\sqrt{2\pi(\frac{mn}{m+n})}$ which is larger than 1. But this is far too weak a result to use to prove the desired inequality.