We are given several pieces of information about four functions y_1, y_2, y_3 and f:

$$\begin{gathered} y_1''' + p(x)y_1'' + q(x)y_1' + r(x)y_1 = 0 \\ y_2''' + p(x)y_2'' + q(x)y_2' + r(x)y_2 = 0 \\ y_3''' + p(x)y_3'' + q(x)y_3' + r(x)y_3 = 0 \\ y_1^2 + y_2^2 + y_3^2 = 1 \\ y_1'^2 + y_2'^2 + y_3'^2 = f \end{gathered}$$

My plan is simply to take these equations and differentiation some of them a few times so that I have enough equations that I can make some some substitutions and so on to get an equation which has no y_i in it at all; that will be the final equation, from which I will read off the constants A and B.

The first three equations will allow us to get rid of any third derivatives of the y_i. There seems to be nothing to gain from differentiating these to learn about the higher derivatives of the y_i. I will differentiate the last equation only once because there seems to be no advantage to getting f'' into the picture. The middle equation I will differentiate three times only – by the third time I will be including $y_i'''(x)$ so I will be able to make use of the top equations, but there will be nothing additional learned by going further.

In other words, I will enlarge the original set of equations to get a larger set:

$$\begin{gathered} y_1''' + p(x)y_1'' + q(x)y_1' + r(x)y_1 = 0 \\ y_2''' + p(x)y_2'' + q(x)y_2' + r(x)y_2 = 0 \\ y_3''' + p(x)y_3'' + q(x)y_3' + r(x)y_3 = 0 \\ y_1^2 + y_2^2 + y_3^2 = 1 \\ y_1y_1' + y_2y_2' + y_3y_3' = 0 \\ y_1y_1'' + (y_1')^2 + y_2y_2'' + (y_2')^2 + y_3y_3'' + (y_3')^2 = 0 \\ y_1y_1''' + 3y_1'y_1'' + y_2y_2''' + 3y_2'y_2'' + y_3y_3''' + 3y_3'y_3'' = 0 \\ y_1'^2 + y_2'^2 + y_3'^2 = f \\ 2y_1'y_1'' + 2y_2'y_2'' + 2y_3'y_3'' = f' \end{gathered}$$

Calling these equations e_1 through e_9, we simply compute

$$-y_1e_1 - y_2e_2 - y_3e_3 + r(x)e_4 + q(x)e_5 + p(x)e_6 + e_7 - p(x)e_8 - (3/2)e_9,$$

a sum which makes the left side vanish and makes the right side equal $r(x) - p(x)f(x) - (3/2)f'(x)$. Solving for f' gives

$$f'(x) + (2/3)p(x)f(x) = (2/3)r(x)$$

i.e. the solution is $A = 2/3,\ B = 2/3$.

I suppose you could simply assume the question-posers are correct when they assert that such constants A and B exist that are independent of the functions p, q, r and the y_i; in that case, you can take any y_i you like whose squares sum to 1 (that is, any parameterization of a path on the unit sphere at the origin), compute the corresponding p, q, r as the solution to a system of three linear equations e_1, e_2, e_3 in three unknowns, compute

f as shown, and then the expression $E = f'(x) + Ap(x)f(x) - Br(x)$; this should vanish identically for the right A and B. You could check that by evaluating E at any two points $x = x_1, x = x_2$ to obtain two linear equations to solve for the two unknowns A, B; or if E is a rational function you could pick two powers of x in the numerator and choose A, B to make their coefficients vanish. I actually carried this out using

$$y_1(x) = (x^4 + x^2 - 1)/s(x), \quad y_2(x) = 2x/s(x), \quad y_3(x) = 2x^2/s(x)$$

where $s(x) = x^4 + x^2 + 1$ (or more precisely, I knew the sequence of commands to type into my computer to find three functions whose squares sum to 1, and from there to compute p, q, r, then f, then $f' + Apf - Br$, and the numerator of the latter was a polynomial whose coefficients were linear combinations of A and B which all vanished precisely when $A = B = 2/3$.) But I doubt the problem-graders would be happy with a solution that does not establish that A, B are independent of the y_i that I chose!