This game is interesting! Allow me to tell you how I analyzed it, before I give my solution.

I had Maple analyze the game in the following way: the state of the game at any moment can be represented as a 4-tuple $[a, b, c, d]$ with $a \leq 3, b \leq 4, c \leq 5, d \leq 6$, with none of the coordinates equal to 1. By rearranging the piles we may also assume that $0 \leq a \leq b \leq c \leq d$. I found that there are 90 such states, including the starting state [3,4,5,6] and the final state [0,0,0,0].

These 90 states become vertices in a directed tree: every state has up to 5 successors — players can move from one state to any of its successor in one turn. The play of the game will progress along the branches from the initial state to the final state.

Now assume that both players have analyzed the game tree and will play optimally. Then they will both know which vertices are states from which they can win the game no matter how well their opponents play, and which vertices are states from which they might lose the game if their opponent is careful. For example, obviously if it is your turn to play, and you see that the other player has left you with state [0,0,0,0], you have lost. Let us gather together the states which will mean a loss to you (if you face them at the start of your turn) in a set L. The other vertices will form the set W – these are the states which (if you see them at the start of your turn) will enable you to force a win.

The defining properties are these:

(1) if all successors of a state v lie in W, then v lies in L.

(2) if at least one successor of a state v lies in L, then v lies in W.

These properties allow us to work up the directed graph, starting with

$$L = \{[0, 0, 0, 0]\}, \qquad W = \{\ \}$$

and successively placing more and more vertices into L and W. I get $|W| = 59,\ |L| = 31$.

In the end, the computer claims that [3,4,5,6] lies in W : if you are looking at the piles in their initial configuration and it is your turn, you can force a win. That is, first player has a winning strategy. Well, great. But how exactly do you win? This is not enlightening.

Having discovered that [3,4,5,6] lay in W, I looked to find a successor of it that lay in L, that is, I wanted to find a move that would leave my opponent in a condition of not having any way to win. It turns out that [2,4,5,6] is the unique good opening move. Then for each of my opponent's (four) possible responses, I looked again for a counter-response that returned the state to something in L. And so on. In this way, I found a game tree that indicated a set of moves I could use that would bring me a win in every case (always taking 11 turns total).

But now how to write this up as a proof? I decided that there must be some easily-remembered strategy; otherwise they would not pose this as a Putnam problem. (*Warning:* the same conclusion does not hold for games in general! The fact that player 1 can force a win does not guarantee that there is an easily-communicated "strategy"!) In our case I thought it would be instructive to note the states that I returned the game to after each move. These were:

$$0000, 0004, 0006, 0044, 0046, 0055, 0444, 0256, 0446, 0455, 2456$$

This is an EXCELLENT time to stop and see if you can now devise a strategy...

Armed with some more computer power, I tested this conjecture and found a perfect dichotomy between W and L, which suggested this Theorem, which I could then prove:

Theorem. *A player can force a win from any state in which there are either an odd number of 3's, or and odd number of 2's and 5's altogether. Conversely, a player can be forced to lose from any state in which the number of 3's is even, and also the sum of the numbers of 2's and 5's is even. In particular, the first player has a winning strategy – to force the play to alternate between these states.*

Proof: Let W be the set of states meeting the first condition, and L be the set of states meeting the second condition. We will show that for all states v,

(a) $v \in L \Rightarrow$ for every successor w of v, $v \in W$

(b) $v \in W \Rightarrow$ there exists a successor w of v with $v \in L$

So if the game starts in a state in W, then the first player can always move the state to L, and the second player will have to restore it to W. This continues until state 0000, which is in L, that is, it will be player 2 who faces it, and then loses.

For any state v, and for each $i = 0, 1, \ldots, 6$, let c_i be the number of piles with i stones. (Note that $c_1 = 0$ for every state in our game.) Then $v \in L$ iff both c_3 and $c_2 + c_5$ are even.

Suppose a player is to take his turn, and the current state is indeed in L. What can this player do? If he removes a stone from a pile with 6 stones, then c_6 decreases by 1 and c_5 increases by 1; so now $c_2 + c_5$ is odd. Removing one from a pile of five will decrease $c_2 + c_5$ by one, again making this sum odd. Removing one from a pile of 4 will increase c_3 by one, making it odd, and removing one from a pile of 3 will decrease c_3 by one, again making it odd. Removing a pile of 3 also decreases c_3, and removing a pile of 2 decreases $c_2 + c_5$. So any legal move will indeed make one of c_3 or $c_2 + c_5$ odd, giving a state in W.

Now suppose v is in W. We will find a legal move that gives a state in L.

If c_3 is odd (and thus nonzero) and $c_2 + c_5$ is also odd, remove one stone from a pile of size 3, decreasing c_3 and increasing c_2, making both indicators even.

If c_3 is odd but $c_2 + c_5$ is even, remove a pile of size 3, making c_3 even.

If c_3 is even but $c_2 + c_5$ is odd, then at least one of c_2 and c_5 is nonzero. If $c_2 > 0$, we remove a pile of size 2; if $c_5 > 0$ we can remove a stone from a pile of size 5. Either way, $c_2 + c_5$ is decreased by 1 (while c_3 is unchanged).

In all cases, we have found a move which leaves a state in L. QED

Note that this theorem allows us to analyze any similar game with any number of piles and any number of starting stones (except 1) in each pile: you should go first unless both c_3 and $c_2 + c_5 + c_7 + c_9 + \ldots$ are even. Obviously one can adjust the rules to create similar games in which there are related strategies based on the parities of sums of various c_i. The interested reader is invited to create such games to tick off his friends. :-)