U.T. PUTNAM PRACTICE 2019 — week 3

You asked for some problems that are, or appear to be, "just calculus". Here are some – including some that have appeared on actual Putnam exams in the past!

1. Compute the minimum value of the function

$$f(u, v) = \left(u - v \right)^2 + \left((3 - u) - \left(\frac{5}{v} \right) \right)^2$$

on the region where $v > 0$.

2. Evaluate

$$\Sigma_{n=2}^{\infty} \log \left(\frac{n^3 - 1}{n^3 + 1} \right)$$

3. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$ given that the x_i are positive real numbers and $\sum_{j=0}^{\infty} x_j = A$?

4. Show that this improper integral converges:

$$\int_0^{\infty} \sin(x) \sin(x^2) \, dx$$

5. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2x \, f(x)$ for all x. Show that $x = 0$ for $x \in [-1, 1]$.

6. Let $p(x)$ be a polynomial that is non-negative for all real x. Prove that p is a sum of squares, that is, for some integer k there are polynomials $f_1(x), f_2(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2$$