Here is a solution to problem A4 of the the 2018 Putnam exam.

We are asked to show that two elements of a group commute. It is helpful to repeatedly use two key lemmas:

Lemma 1: if $x, y \in G$ and $x \in \langle y \rangle$ then x and y commute. Here $\langle y \rangle$ is the subgroup of G generated by y — the collection of all powers of y and their inverses.

Lemma 2: if x commutes with xy or yx then x commutes with y. (Simply multiply the premise equation by x^{-1}.)

We will prove this result by induction on M =max(m,n). When $m = n = 1$ the problem is trivial since the starting premise is that $gh = e$, i.e. that g and h are inverses of each other, and hence commute.

If $M > 1$ then m and n are distinct since they are given to be coprime. The argument is a little different depending on which of the two is larger.

Suppose first that $m > n$. Let $m' = m - n$ and note that the exponents associated to (m', n) will be $a'_k = \lfloor m'k/n \rfloor - \lfloor m'(k-1)/n \rfloor = (\lfloor mk/n \rfloor - k) - (\lfloor m(k-1)/n \rfloor - (k-1)) = a_k - 1$. So the starting premise that pertains to (m, n) may be written

$$ghh^{a'_1}ghh^{a'_2}ghh^{a'_3}\dots ghh^{a'_n} = e$$

That is, the group elements $g' = gh$ and h satisfy precisely the starting premise that pertains to (m', n). Since $\max(m', n) < \max(m, n)$, we know by induction that $g' = gh$ commutes with h. Then by Lemma 2, g commutes with h too.

Now suppose that $m < n$. In that case, the consecutive values of the expression mk/n increase by less than 1 as k increases, so the exponents a_k are all either 0 or 1: the starting premise is just a string of gs and hs (no exponents on the hs). Now, the sum of all the a_k (for $k = 1, 2, \dots, n$) telescopes to $(mn/n) - (m0/n) = m$, that is, there are m of these hs altogether, interspersed among the n gs, including the very last h. ($a_n = 1$ whenever $m < n$.)

Collecting together the intervening gs, this starting premise may be written in the form

$$g^{c_1}hg^{c_2}h\dots g^{c_m}h = e$$

for some integer exponents c_i. Specifically, the rth h in our premise equation occurs only when $mk/n \geq r$, i.e. when $k = \lceil nr/m \rceil = n - \lfloor n(m-r)/m \rfloor$ of the gs have been passed; so this value of k will equal $c_1 + c_2 + \dots + c_r$. So we now know the exponents in this starting premise when $m < n$:

$$c_1 = n - \lfloor n(m-1)/m \rfloor; c_2 = \lfloor n(m-1)/m \rfloor - \lfloor n(m-2)/m \rfloor; \dots; c_m = \lfloor n/m \rfloor$$

So now assume that elements g, h satisfy this equation $g^{c_1}hg^{c_2}h\dots g^{c_m}h = e$. Take inverses of both sides to obtain $h^{-1}g^{-c_m}h^{-1}g^{-c_{m-1}}\dots h^{-1}g^{-c_1} = e$. From our characterization of the exponents c_i we see that this is precisely the starting premise pertaining to (n, m), with g replaced by h^{-1} and h replaced by g^{-1}. Since $n > m$, our previous inductive argument shows that *this* equation implies that g^{-1} commutes with h^{-1}. But then of course g commutes with h as well, and we are done.

Let me illustrate this idea by working out the computations when $n = 5$ and $m = 3$. The exponents a_k are all 0 or 1 and the starting equation is $gghgghgh = e$. View this as $(ggh)(ggh)(gh) = e$; it tells us gh is the inverse of the square of ggh so $gh \in \langle ggh \rangle$ and so gh will commute with ggh by Lemma 1. Then use Lemma 2 once to show gh commutes with g, and use it again to conclude h commutes with g.

Note that the starting premise when $m = 5$ and $n = 3$ is $ghgh^2gh^2 = e$, which has the predicted relationship with the one for $n = 5, m = 3$.

Allow me to share with you a distracting idea that may be helpful in a future Putnam exam. Let $b_k = \lfloor (mk/n) \rfloor$, so that $a_k = b_k - b_{k-1}$, and note that $b_0 = 0$. Then the given equation may be written

$$g\, h^{b_1} g h^{-b_1}\, h^{b_2} g h^{-b_2}\, h^{b_3} g h^{-b_3}\, \ldots\, h^{b_{n-1}} g h^{-b_{n-1}}\, h^{b_n} \;=\; e$$

Note that $b_n = m$, so if we multiply both sides of this equation on the right by h^{-m} we get

$$g\, g^{h^{b_1}}\, g^{h^{b_n}}\, \ldots g^{b_{n-1}} = h^{-m}$$

where I am using a standard group-theoretic convention: if a and b are elements of a group then a^b denotes the *conjugate of a by b*, i.e. $a^b = bab^{-1}$. This notion of conjugation is very natural and important; you should check that it has properties like $a^{bc} = (a^b)^c$, $(ab)^c = a^c v b^c$, etc. Nonetheless, I didn't see any reasonable way to use the fact that a power of h was a product of conjugates of g.