On Oct 25, I outlined one approach to problem #1 of the Oct 18 handout. Allow me to flesh this out a bit. It's not necessarily the easiest or most natural way to solve this problem, but it does lead to more cool math!

Problem: Suppose the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y, 2x + 2y + 1, 3x + 3y + 1$ are all perfect squares.

First, I wanted to put the equation into a form more familiar to me. Suppose (x, y) is a solution to the main equation. Multiply the equation by 48 (!) and complete squares to see that

$$(12y + 2)^2 - 6(4x + 1)^2 = -2$$

In other words, the integers $A = 12y+2$ and $B = 4x+1$ satisfy a form of what is sometimes called "Pell's Equation",

$$A^2 - 6B^2 = -2.$$

This is a well-studied Diophantine equation, which I encourage you to look up some time because it admits multiple analyses, some simple, some deep. I like to analyze it this way: any such solution (A, B) gives us a factorization of -2 as a product

$$(A + B\sqrt{6})(A - B\sqrt{6}) = -2,$$

where all the terms are elements of the ring $\mathbf{Z}[\sqrt{6}] = \{a + b\sqrt{6}; \quad a, b \in \mathbf{Z}\}$. One such factorization comes from the simple solution $(A, B) = (2, 1)$, i.e. $(2 + \sqrt{6})(2 - \sqrt{6}) = -2$.

Fortunately, there are some true-but-not-obvious facts that will help us here: (1) The ring $\mathbf{Z}[\sqrt{6}]$ has unique factorization, so that there is only one decomposition of any given ring element into irreducible factors (up to rearranging the factors and multiplying them by units) (2) the elements $(2 \pm \sqrt{6})$ are both irreducible in this ring. (3) The only units in this ring are the (positive or negative) powers of $5 + 2\sqrt{6}$, and their negatives. (4) The two elements $(2 \pm \sqrt{6})$ differ by a unit. Specifically we have

$$(2 + \sqrt{6}) = (2 - \sqrt{6}) \cdot (-1) \cdot (5 + 2\sqrt{6})$$

Putting these facts together shows us that if (A, B) is a solution to this Pell equation, that is, if

$$(A + B\sqrt{6})(A - B\sqrt{6}) = (2 + \sqrt{6})(2 - \sqrt{6}),$$

then $A + B\sqrt{6} = \pm(2 + \sqrt{6}) \cdot (5 + 2\sqrt{6})^n$ for some integer n. You might want to expand this out to generate a few solutions: $(A, B) =$

$$(2, 1), \quad (22, 9), \quad (218, 89), \quad (2158, 881), \ldots$$

(These correspond to $n = 0, 1, 2, 3, \ldots$; taking $n = -1, -2, -3, \ldots$ gives the negative conjugates of these. In our setting it is clear that we are only interested in the values of $|A|$ and $|B|$, so effectively we have computed *all* solutions of the Pell equation.)

Since our problem involves *positive* x and y, we are interested only in those solutions with $A, B > 0$ and with $A \equiv 2 \pmod{12}$. Those arise precisely by taking even n in our formula above. (The first few solutions are then $(x, y) =$

$$(0,0), \quad (22,18), \quad (2180,1780), \ldots$$

And indeed we have a formula to express all such solutions: for non-negative values of k we compute A, B as above, and separate out the rational and irrational components. For example

$$A = \frac{(2+\sqrt{6})\cdot(5+2\sqrt{6})^{2k} - (2+\sqrt{6})\cdot(5-2\sqrt{6})^{2k+1}}{2}$$

and there is a very similar formula for B. We may then compute more expressions of the same type for x and y and then also for $x - y$, $2x + 2y + 1$, and $3x + 3y + 1$; each of them is of the form $\alpha u^{2k} + \beta v^{2k} + \gamma$ where $u, v = 5 \pm 2\sqrt{6}$ (so that in particular $u \cdot v = 1$).

At this point it's clear what has to happen: the three linear expressions in x and y were chosen precisely so that it would be possible to compute two more ring elements δ, ϵ so that

$$\alpha u^{2k} + \beta v^{2k} + \gamma = (\delta u + \epsilon v)^2$$

and so that in addition δ and ϵ are conjugates of each other, which would make $\delta u + \epsilon v$ be rational (and thus integral). This then demonstrates that each of the chosen linear combinations of x and y is indeed a perfect square.

In fact, I compute

$$x - y = (u^k - v^k)^2/24 = (\frac{1}{2\sqrt{6}}u^k - \frac{1}{2\sqrt{6}}v^k)^2$$

$$2x + 2y + 1 = ((2+\sqrt{6})u^k - (2-\sqrt{6})v^k)^2/24$$

$$3x + 3y + 1 = (\frac{2+\sqrt{6}}{4}u^k + \frac{2-\sqrt{6}}{4}v^k)^2$$

So this completes the proof that the given expressions are squares. But note that the form of the answer suggests that other answers are possible. For example, what seems clear is that the things that e.g. $x - y$ are squares OF will themselves be solutions to similar Pellian equations. I have not pursued this but I'm sure there must be such solutions available on the web!