Here is a solution to problem B4 of the the 2018 Putnam exam.

The first few terms may be computed directly:

$$1, a, a, 2a^2 - 1, 4a^3 - 3a, 16a^5 - 20a^3 + 5a, \ldots$$

This may not mean much to you but when I saw the $2a^2 - 1$ I realized this was using the double-angle formula for cosine. The $2x_n x_{n-1}$ was also suggestive and the $4a^3 - 3a$ confirmed it for me.

So let us pick numbers y_n with $x_n = \cos(y_n)$; writing θ for y_1 we can select $y_0 = 0$ and $y_2 = \theta$. Then $x_3 = 2a^2 - 1 = 2\cos(\theta)^2 - 1 = \cos(2\theta)$ so we may pick $y_3 = 2\theta$. This now suggests a pattern: what if $y_n = k_n \theta$ for some sequence of integers k_n? We already have $k_0 = 0, k_1 = 1, k_2 = 1, k_3 = 2$. Such a pattern matches our recurrence relation if

$$\begin{aligned}
\cos(k_{n+1}\theta) = x_{n+1} &= 2x_n x_{n-1} - x_{n-2} \\
&= 2\cos(k_n\theta)\cos(k_{n-1}\theta) - \cos(k_{n-2}\theta) \\
&= \cos((k_n + k_{n-1})\theta) + \cos((k_n - k_{n-1})\theta) - \cos(k_{n-2}\theta)
\end{aligned}$$

and this equation will indeed hold if we match the cosines in pairs, that is if

$$k_{n+1}\theta = (k_n + k_{n-1})\theta \qquad \text{and} \qquad k_{n-2}\theta = (k_n - k_{n-1})\theta$$

Happily, both these equations will hold for all n if the k_n follow the Fibonacci recurrence, and that is consistent with the first few terms we found above.

Therefore we have a proof by induction that

$$x_n = \cos(F_n\theta) \qquad \text{where } \theta \text{ is chosen so } \cos(\theta) = a$$

You might object that this assumes $-1 \le a \le +1$ and no such assumption is given in the problem. However, it is not necessary that a lie in this interval since the calculations above are perfectly valid if θ is imaginary! Here I am simply defining the cosine function by the formula $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$, and this function takes on all real values as θ runs along *both* axes in the complex plane.

Now, the premise of the problem is that for some m we have $x_m = 0$, which requires $F_m\theta$ to be an odd multiple of $\pi/2$, say $\theta = (k/F_m)(\pi/2)$. (In particular, θ will be real, forcing $|a| \le 1$ after all!) In this setting it is helpful to know that when we compute the Fibonacci numbers modulo any modulus M, the sequence is periodic. The proof is not hard: consider all the *pairs* of consecutive numbers in the sequence. There are at most M^2 different values of this pair modulo M, and each pair uniquely determines the next pair, so after at most M^2 terms the pairs must start to repeat, and a fortiori the underlying sequence repeats. Applying this idea in the case that $M = 4F_m$, we see that there is a number $N < M^2$ such that $F_{n+N} \equiv F_n \pmod{4F_m}$ for all n, that is, $F_{n+N} = F_n + 4t\,F_m$ for some integer t (varying with n). Then

$$x_{n+N} = \cos((F_n + 4tF_m)\,\frac{k}{F_m}\,\frac{\pi}{2}) = \cos(F_n\,\frac{k}{F_m}\,\frac{\pi}{2} + tk(2\pi)) = \cos(F_n\,\frac{k}{F_m}\,\frac{\pi}{2}) = x_n$$

and the sequence is periodic.