Week1, problem 1

There was some grumbling about this problem...

L'Hopital's Rule certainly applies but you will need SEVEN iterations of it to get an answer! The seventh derivative in the denominator is simply 7!, but in the numerator it's of the form $P(x)\sin(u) + Q(x)\cos(u)$ where $u = x + \frac{x^3}{6} + \frac{3x^5}{40}$ and where P and Q are polynomials of degree 28 involving fractions with 7-digit denominators. Ugh!

Much simpler is to use the Taylor Series for $\sin(u)$,

$$\sin(u) = u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \ldots$$

and simply substitute in our u. This does require that you expand a few powers of this trinomial u, but you know you can ignore all powers of x beyond x^7. Indeed, the Taylor series of the numerator turns out to be

$$\sin(u) - x = \frac{-5x^7}{112} - \frac{65x^9}{8064} - \ldots$$

so the desired limit is $-5/112$.

In fact I chose this u to be the initial part of the Taylor series of $\arcsin(x)$ so that $\sin(u)$ is nearly equal to x itself. More terms are here:

$$\arcsin(x) = x + \frac{x}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \ldots$$