M93 Prof. Bernoff
Fall, 2013 Prof. Omar

Putnam TNG #5 – Algebra and Group Theory

These problems will be discussed on Tuesday 10/15/13 at 6PM in TLB B450.

The Putnam TNG seminar assumes that you have tried to work the problems in advance of the seminar and have solved at least one problem that you are willing to present in class.

E1: Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$ we have $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove $a * (b * a) = b$ for all $a, b \in S$. (Putnam, 2001)

E2: Let H be the group generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Prove H is the trivial group. (Omar)

E3: Suppose R is a ring and for every $a \in R$ we have $a^2 = a$. Prove R is commutative. (Hint: First prove that for any $a, b \in R$, $ab = -ba$.) (Omar)

E4: Let A be a subset of a finite group G, and A contains more than one-half of the elements of G. Prove that each element of G is the product of two elements of A. (Putnam, 1968)

E5: Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible polynomial over F of the form $x^2 + bx + c$ such that $b, c \in F$. For how many elements k in F is $p(x) + k$ irreducible over F? (Putnam, 1979)

E6: Let p be a prime number. Let J be the set of all 2-by-2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ whose trace and determinant are 1 and 0 (in $\mathbb{Z}/p\mathbb{Z}$) respectively. Determine how many members J has. (Putnam, 1968)

E7: Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$). (Putnam, 1997)

E8: Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, ...\}$ is finite. Must S be a group? (Putnam, 1989)