Putnam practice – Sept 27 2012
By popular request, we will spend a little time today on Number Theory.

1. How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{40} and 20^{30}?

2. Do there exist one million consecutive integers, each of which is divisible by a perfect square (larger than 1)?

3. Show that if p is an odd prime, and $x = ((p-1)/2)!$, then $x^2 \equiv \pm 1$ modulo p.

4. For each integer ≥ 0, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

5. Suppose n is a positive integer. How many ordered pairs (x, y) of positive integers are there with
$$\frac{xy}{x+y} = n$$