MORE LINEAR-ALGEBRA PUTNAM PROBLEMS

(Handed out Oct 25 2012. Let's see some solutions on Nov 1 !)

1. (99B5) For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{j,k})$ has entries $a_{j,k} = \cos(j\theta + k\theta)$ for all j, k.

2. (10B1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

3. (95A5) Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable f which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\cdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

4. (95A6) Suppose that each of n people writes down the numbers 1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

5. (01A4) Triangle ABC has area 1. Points E, F, G lie, respectively, on the sides BC, CA, and AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of triangle RST.

Now flip over for some additional practice with Axiomatic Mathematics!

BONUS ROUND! A vector space may be defined as a set V on which two binary operations called $+$ and $\cdot$ are defined (respectively as functions $V \times V \to V$ and $\mathbf{R} \times V \to V$) subject to a set of axioms. We may express these axioms in the following way:

VS_1. For all $u, v, w \in V$ we have $u + (v + w) = (u + v) + w$

VS_2. For all $u, v \in V$ we have $u + v = v + u$

VS_3. There is a vector $u \in V$ so that for all $v \in V$ we have $u + v = v$

VS_4. For all $u, v, w \in V$, if $u + w = v + w$ then $u = v$; likewise if $w + u = w + v$ then $u = v$.

VS_5. For all $u, v \in V$ and all $a \in \mathbf{R}$ we have $a \cdot (u + v) = a \cdot u + a \cdot v$

VS_6. For all $u \in V$ and all $a, b \in \mathbf{R}$ we have $(a + b) \cdot u = a \cdot u + b \cdot u$

VS_7. For all $u \in V$ and all $a, b \in \mathbf{R}$ we have $(ab) \cdot u = a \cdot (b \cdot u)$

VS_8. For all $u \in V$ we have $1 \cdot u = u$

For each of these axioms, give an example of an object which satisfies all the axioms EXCEPT the given one, that is, a non-vector space that satisfies the other seven axioms.

Here's an example. Take the set V to be the set of real numbers; define "vector addition" on $V4$ to be ordinary addition of real numbers; and define "scalar multiplication" by

$$c \cdot v = 0 \qquad \text{for all scalars } c \text{ and vectors } v$$

Then axioms VS_1 through VS_7 are satisfied but axiom VS_8 is not. Your job is to construct other examples (saying exactly what V, "+", and "$\cdot$" are) where seven of the axioms are satisfied but the remaining one is not. (I'm looking for one example where VS_1 is violated, another where VS_2 is violated, etc.)

This can be done for seven of the axioms, but one of these axioms is actually redundant — it automatically follows from the other seven axioms. Which of the eight axioms is redundant?