Problems for 2018 University of Texas Putnam Prep Session, week 1 (Sept 20)

1. The equation $y^3 + x^2y + 2x^3 - 3x^2 + 1 = 0$ defines a curve in the plane. We view this curve as the graph of a function $y = f(x)$. Find all the critical points of this function and classify them as local maxima or local minima.

2. Show that $I = \displaystyle\int_1^2 \frac{1}{4+x^4}\,dx$ lies between $\frac{1}{20}$ and $\frac{7}{24}$. Five points extra credit goes to the contestant who finds the smallest such interval containing the value of I.

3. Does the following series converge? (Why or why not?)

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)\\ =&3\cdot\frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3\cdot\frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3\cdot\frac{\ln 10}{10} - \cdots\end{aligned}$$

4. Compute the limit or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} (\cos(x+y))^{\cot(x^2-xy+y^2)}$$

5. Compute

$$\int_{y=0}^{2}\left(\int_{x=0}^{3}\frac{x-y}{(x+y)^3}\,dx\right)dy \qquad \text{and} \qquad \int_{x=0}^{3}\left(\int_{y=0}^{2}\frac{x-y}{(x+y)^3}\,dy\right)dx$$

6. Compute (with explanation) the following limit, or show that it does not exist:

$$\lim_{x\to 0}\frac{x^2\sin(\frac{1}{x})}{\sin(x)}$$

7. Compute the derivative of $f(x) = x^{x^x}$.

8. Compute $\displaystyle\int \frac{\sin(t)+\cos(t)}{\sqrt{2\sin(t)\cos(t)}}\,dt$. (*Hint*: if $u = \sin(t) - \cos(t)$, what is u^2?)

Extra Credit: Use this idea to evaluate $\displaystyle\int \sqrt{\tan(t)}\,dt$ by first computing

$$\int \sqrt{\tan(t)} + \sqrt{\cot(t)}\,dt \quad \text{and} \quad \int \sqrt{\tan(t)} - \sqrt{\cot(t)}\,dt$$

9. Do these series converge or diverge? Explain.

$$(A)\ \sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^{-n} \qquad\qquad (B)\ \sum_{n=1}^{\infty}(-1)^n\frac{2+\cos(\pi n)}{n}$$

10. Find the volume of the intersection of the solid bounded by the cylinders $x^2 + z^2 = R^2$ and $y^2 + z^2 = R^2$