Here an answer for problem A6 of the the 2018 Putnam exam. There may be some algebra errors.

*Caution:* I will use the letters $A, B, C, D$ to mean something different from the problem statement.

Since the two triangles have a common edge, we treat that edge as the base of the triangles; then the ratio of the areas is the same as the ratio of the heights (the distances from the common edge to the two top points).

Let us place the figure on a coordinate plane, with the shared edge stretching from (0,0) to $(a, 0)$, and let the other points of the two triangles be $(x_1, y_1)$ and $(x_2, y_2)$. Let $b$ and $d$ be the distances from these top points to the origin, and let $c$ and $e$ be the distances from the top points to $(a, 0)$. Finally let $f$ be the distance between the two top points. For convenience we will write $A$ for $a^2$, etc.; we are told that $A, \ldots, F$ are all rational.

The number we wish to prove rational is the ratio of the heights of the triangles, which is $y_2/y_1$.

Let us compute the coordinates $(x_i, y_i)$. From the distance formulas we know

$$x_1^2 + y_1^2 = B, \qquad (x_1 - a)^2 + y_1^2 = C$$

Subtract to see $x_1 = (A + B - C)/(2a)$, so that

$$y_1^2 = B - (A + B - C)^2/(4A) = (\,2(AB + BC + CA) - (A^2 + B^2 + C^2)\,)/(4A)$$

Similarly $x_2 = (A + D - E)/(2a)$, $y_2^2 = (\,2(AD + DE + EA) - (A^2 + D^2 + E^2)\,)/(4A)$. Note that the $x_i^2$ and $y_i^2$, as well as the product $x_1 x_2$, may be written as rational functions of $A, B, C, D, E, F$ and hence by the statement of the problem are all rational numbers.

Now, the last edge length may computed as

$$F = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1^2 + x_2^2 + y_1^2 + y_2^2) - 2x_1x_2 - 2y_1y_2$$

All the other terms in this expression being rational, we conclude $y_1 y_2$ is rational as well. Since $y_1^2$ is too, we know the desired ratio $y_2/y_1 = (y_1y_2)/(y_1^2)$ is rational.

An alternative approach uses the formula which gives the volume of a tetrahedron from the lengths of its six edges, akin to Heron's formula for the area of a triangle.

Heron's formula itself can be used first to find 16 times the square of the area of the triangles:

$$A^2 + B^2 + C^2 - 2(AB + BC + CA) \qquad \text{and} \qquad A^2 + D^2 + E^2 - 2(AD + DE + EA)$$

We are asked to prove that the ratio of the square roots of these is rational, i.e. that these numbers differ by a rational-square factor. It will suffice to show that their product is a rational-square instead, i.e. that

$$P = A^4 - 2(B + C + D + E)A^3 + \ldots + (B - C)^2(D - E)^2$$

is the square of a rational number.

From the terms shown, one might guess that this expression is the square of $A^2 - (B+C+D+E)A + (B-C)(D-E)$, but this is incorrect — the coefficients of $A^2$ and $A^1$ don't match. The difference turns out to be $(4A)(B^2E + BE^2 + C^2D + CD^2 + \ldots)$, where I list just enough terms to suggest the next move.

The following "Cayley-Menger determinant" equals 288 times the square of the volume (where $A, B$, etc. are the squares of the lengths of the same edges as labeled above):

$$\begin{vmatrix} 0 & A & B & D & 1 \\ A & 0 & C & E & 1 \\ B & C & 0 & F & 1 \\ D & E & F & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

The planarity of our figure means this determinant must vanish, giving a polynomial relation among these six rational numbers $A = a^2$ etc. Expand, divide by 2, and rearrange terms to get a polynomial in 6 variables that evaluates to zero for any planar tetrahedron:

$$R = (ABF+BFE+FEA+EAB)+(ADF+DFC+FCA+CAD)+(BDE+DEC+ECB+CBD)$$
$$-(ABC+ADE+BDF+CEF) \quad -(A^2F+AF^2)-(B^2E+BE^2)-(D^2C+DC^2)=0$$

Notice the terms $(B^2E + BE^2) + (D^2C + DC^2)\ldots$

So now we have a hope: we don't need the *polynomial* $P$ to be the square of another *polynomial* $Q$; we just need to know that the *values* of $P$ are rational squares whenever the tetrahedron is planar, i.e. whenever $R$ evaluates to zero. This will be true as long as $P - Q^2 = LR$ where $L$ is any other polynomial (in this case necessarily of degree 1). Taking our cue from the previous calculation of $P - Q^2$ suggests that we use $L = 4A$, and indeed $P^2 - 4AR$ may be calculated to be

$$A^4 - 2(B + C + D + E \; - 2F\,)A^3 + \ldots + (B-C)^2(D-E)^2$$

and again from the terms shown we guess that perhaps this might be the square of $Q = A^2 - (B + C + D + E - 2F)A + (B - C)(D - E)$, and this time the conjecture proves to be correct!

To summarize, then, if $\Delta_1$ and $\Delta_2$ are the areas of two faces of a tetrahedron which share an edge of length $a$, and if $V$ is the volume of this tetrahedron, then

$$(4\Delta_1)^2(4\Delta_2)^2 - (4a^2)(144V^2) = Q^2$$

When the tetrahedron is planar, $V = 0$ and so $(16\Delta_1\Delta_2)^2 = Q^2$, so $\Delta_1\Delta_2 = Q/16$ is rational. Since by Heron's formula $\Delta_1^2$ is rational too, we may divide to prove $\Delta_2/\Delta_1$ is rational and we are done.

I don't know if there is a geometric interpretation of $Q$ that would allow us to prove the last displayed equation by consideration of certain hypervolumes in $\mathbf{R}^8$. ( ! )