UT Putnam Prep Problems, Nov 9, 2016
GEOMETRY PUTNAM PROBLEMS

1. Given a convex polygon S of area A and perimeter p, what is the area of the set of point which lie within a distance of 1 from S?

2. Let A be the region in the first quadrant bounded by the line $y = x/2$, the x-axis, and the ellipse $x^2/9 + y^2 = 1$, and let B be the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the same ellipse $x^2/9 + y^2 = 1$. For what positive number m do the regions A and B have the same area?

3. Find a parameterization of the (entire) curve $y^2 = x^3 + x^2$.

4. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exists $i < j < k \leq 12$ for which d_i, d_j, d_k are the lengths of the sides of an acute triangle.

5. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

6. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?

7. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a non-negative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

8. Here is a "Tangram puzzle". Suppose D and S are a diamond and a square of equal area. (That is, they are two congruent squares whose sides lie on lines making 45-degree angles with each other.) Find a decomposition of D and S into polygons D_i and S_i respectively such that each D_i is a translation of the corresponding S_i.

9. Show that there exist tetrahedra of arbitrarily large volume whose vertices lie at integer points and which do not contain any other lattice points (neither on their boundaries nor in their interiors). (Thus there is no "Pick's Theorem in 3 dimensions.)

10. There was a time when national leaders were clear-thinking and clever. Prove this theorem which is attributed to Napoleon: Given a triangle, erect equilateral triangles on all its edges. Show that the centers of the three equilateral triangles form themselves the vertices of an equilateral triangle.