Problem 4

We are given a sequence $S_1 = \log(a)$, and $S_n = \sum_{i=1}^{n-1} \log(a - S_i)$. A simple manipulation reveals that in fact, $S_{n+1} = S_n + \log(a - S_n)$. Now define a sequence E_n by $E_n = S_n - (a-1)$, so that $E_{n+1} = E_n + \log(1 - E_n)$. We want to show that E_n tends to 0.

We first observe that $E_n \leq 0$ for $n > 1$, since

$$E_{n+1} = E_n + \log(1 - E_n) \leq E_n + (1 - E_n) - 1 = 0.$$

(Here we used the tangent-line bound, $\log(x) \leq x - 1$.)

Also, if $E_n < 0$, we have that $E_{n+1} > E_n$. To see this, rewrite the inequality as $E_n + \log(1 - E_n) > E_n$ and note that $\log(1 - E_n)$ is positive as long as E_n is negative.

This shows that the sequence $\{E_n\}$ is an increasing sequence of non-positive numbers. Therefore, E_n has a limit L. This value must satisfy $L = L + \log(1 - L)$, so we conclude that $L = 0$ as desired.