Here is a solution to problem A3 of the the 2018 Putnam exam.

We will use the trig identities

$$\cos(3x) = 4\cos(x)^3 - 3\cos(x) \qquad \sin(3x) = \sin(x)\,\left(4\cos^2(x) - 1\right)$$

According to taste, we can analyze this problem either using the trigonometric format in which the problem was given, or letting $y_i = \cos(x_i)$ and then maximizing the sum of $4y_i^3 - 3y_i$, subject to the constraints that each $y_i \in [-1, 1]$ and the y_i sum to zero.

We first seek the local maxima. Observe that at a local maximum, the function would decrease (locally) if any two variables are changed (subject to the constraints), leaving the others constant. This observation allows us to work with just two variables at a time, asking for example to maximize $\cos(3x_1) + \cos(3x_2)$, subject to keeping $\cos(x_1) + \cos(x_k) = C$, a constant. This is a classic Lagrange Multipliers question; the condition for a critical point is that the gradients be parallel, i.e. that $(\sin(3x_1), \sin(3x_2))$ be parallel to $(\sin(x_1), \sin(x_2))$. This happens iff either x_i is a multiple of π or if $\sin(3x_1)/\sin(x_1) = \sin(3x_2)/\sin(x_2)$; in the latter case we use the trig identity and solve to conclude $\cos(x_1) = \pm\cos(x_2)$ (and of course in the former case instead we conclude $\cos(x_i) = \pm 1$). Observe that this is equivalent to the conclusion that either one $y_i = \pm 1$ or that $y_1 = \pm y_2$.

Applying this argument to all the pairs of variables we find that at a critical point all the cosines that are not equal to 1 or -1 are equal to or negative of each other. That is, at any critical point there will be a number $a \in [0, 1)$ and natural numbers k_1, k_2, k_3, k_4 adding up to 10 such that:

$$\begin{aligned} &k_1 \text{ of the cosines } \cos(x_i) = y_i \text{ are equal to } +1 \\ &k_2 \text{ of the cosines } \cos(x_i) = y_i \text{ are equal to } -1 \\ &k_3 \text{ of the cosines } \cos(x_i) = y_i \text{ are equal to } +a \\ &k_4 \text{ of the cosines } \cos(x_i) = y_i \text{ are equal to } -a \end{aligned}$$

The constraints of the problem require that $k_1 - k_2 + (k_3 - k_4)a = 0$, so that $a = (k_1 - k_2)/(k_4 - k_3)$.

Applying the triple-angle formula shows the value of the objective function will be $k_1(+1) + k_2(-1) + k_3(4a^3 - 3a) + k_4(-4a^3 + 3a) = (k_1 - k_2) + (k_3 - k_4)a(4a^2 - 3) = 4(k_1 - k_2)(1 - a^2)$. Thus at a maximal point we will have $k_1 > k_2$, and so also $k_3 > k_4$.

Since $(k_1 - k_2) + (k_4 - k_3) = (k_1 + k_2 + k_3 + k_4) - 2(k_2 + k_3)$, the numerator and denominator of a are now distinct positive integers which sum to an even number no larger than 10. The possible combinations are

$$(1, 3), (1, 5), (1, 7), (1, 9), (2, 4), (2, 6), (2, 8), (3, 5), (3, 7), (4, 6)$$

(Observe that for the first six pairs there will be multiple combinations of the k_i which will yield these same numerator and denominator.)

With a brute-force computation we find that among these ten pairs, $(n, d) = (3, 7)$ gives the largest value of $4n(1 - (n/d)^2)$. (Obviously for each n we want to maximize d so we only check four cases.)

This combination requires $k_1 - k_2 = 3, k_4 - k_3 = 7$ and thus $k_2 = 0, k_3 = 0, k_1 = 3, k_4 = 7$, that is: three y_i equal $+1$ and seven of them equal $-3/7$. Then three values of $\cos(3x)$ are equal to $+1$ and seven are equal to $+333/347$, giving the objective function the value of $3 + 333/49$, a bit less than 10.