Here are some comments about problem A5 of the the 2018 Putnam exam.

Let us consider some features of a hypothetical counterexample f, i.e. a function defined and infinitely differentiable on the entire real line, for which both f and all its derivaties are everywhere non-negative, with $f(1)$ strictly positive but $f(0) = 0$.

If $a < 0$ then by the Mean Value Theorem there is a point c between a and 0 where $f'(c) = (f(0) - f(a))/(0 - a) = f(a)/a$; since f' is everywhere non-negative and a is negative this implies $f(a) \leq 0$. But we are told $f(a) \geq 0$ too, so we see $f(a) = 0$ for all $a < 0$.

Similarly if $a \in [0, 1]$ then the MVT assures us that $0 \leq f(a) \leq f(1) = 1$

I suspect that the way to get a contradiction would be to use the analogous theorem for higher-order derivatives, which is Lagrange's Theorem for Taylor series. From a putative counterexample f I would construct the Taylor series at 1, i.e. the formal power series

$$F(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

(Incidentally the Taylor series at 0 would just be $0 + 0x + 0x^2 + \ldots$!) This Taylor series does not necessarily converge at *any* x other than $x = 1$, but Lagrange helps us estimate the error: if $T_n(x)$ is the nth partial sum of this series, i.e. the degree-n Taylor Polynomial, then Lagrange shows there exists a number c between 1 and x where $F(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1}$. In particular, if we suppose that all the derivatives of f are everywhere non-negative, this shows that $F(x) \geq T_n(x)$ for all n and all $x \geq 1$, as well as all $x < 1$ whenever n is odd, although $F(x) \leq T_n(x)$ for all $x < 1$ whenever n is even. This does tell us that all the derivatives of f are strictly positive at $x = 1$: if any of them were equal to zero then two consecutive Taylor polynomials would be identical, contradicting $T_{2k}(x) \leq 0 \leq T_{2k\pm1}(x)$ for all $x < 0$.

So we have a sequence of polynomials alternately greater than and less than $F(x)$ on $(0, 1)$; if we knew these polynomials converged pointwise to F here, I could get a contradiction, I think. But I don't know how to make any progress without knowing the Taylor series converges somewhere beside $x = 1$.

It is important to note that there ARE functions of this sort: they are infinitely differentiable, but not represented by power series. The classic example is

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

One has to show that $f^{(n)}(0)$ exists for every n, but it does, and each of these derivatives is zero, which means that the Taylor series of f at the origin is $0 + 0x + 0x^2 + \ldots$, which clearly does not converge to $f(x)$ for all x in a neighborhood of 0 (no matter how small)!

Of course there are others: the antiderivative $F(x) = \int_{-\infty}^{x} f(t)\,dt$ of such a function is another such function, and grows even faster than f. You can repeat this to get examples which grow faster than any quadratic polynomial, or faster than any cubic, or $\ldots$.

All these functions have one of their higher-order derivatives equal to that first example $f(x) = e^{-1/x}$, and a glance at the graph of this function shows it must have an inflection point, where $f''(c) = 0$. So none of these functions are counterexamples to problem A5, but they illustrate how close one may come to such a counterexample.

Also note that $g(x) = f(x)f(1-x)$ is positive for $x \in (0,1)$ but positive everywhere else, and still infinitely differentially; the graph is a smooth "bump" over the real line. Such functions are extremely useful in Differential Geometry. We sometimes also use functions like $G(x) = \int_{-\infty}^{x} g(t)\,dt$ to create infinitely differentiable functions that transition from one constant value on $(-\infty, 0]$ to another on $[1, \infty)$. It's not hard to show that the Taylor series of this G at $x = 1$ is just $G(1) + 0(x-1) + 0(x-1)^2 + \ldots$, which does not converge to $G(x)$ for any $x < 1$, illustrating my frustration with my proposed method of proof.

(Functions of the form $G(ax+b)$ are similar but transition at other points besides $x = 0$ and $x = 1$.)

You might also like the bump function $h(x) = 1 - e^{-1/x^2}$ which is infinitely differentiable and everywhere positive and whose Taylor series at 0 again vanishes identically, but $h(x) \neq 0$ for every $x \neq 0$, so this Taylor series does not converges to $h(x)$ for *any* x except $x = 0$!