Putnam 2017 — some solutions (D.Rusin, 2017-12-03)

I thought this year's problems were fun and more of them were doable. Here are some thoughts on the solutions, incorporating some comments from conversations with other participants.

A1. None of the three rules forces a multiple of 5 to lie in S unless some other multiple of 5 is already in S. Therefore, a minimal solution S includes no multiples of 5.

On the other hand, it contains every non-multiple of 5, except the number 1. To see this, note that since $2 \in S$, rule (c) forces $7^2 = 49$ and then $54^2 = 2916$ to be in S. Note that $2916 \equiv 1 \pmod 5$. Since rules (b) and (c) together imply that $n + 5 \in S$ whenever $n \in S$, it follows that all sufficiently large number which are $\equiv 1 \pmod 5$ will also lie in S.

So now given $n \not\equiv 0 \pmod 5$, we know $(n^2)^2 = n^4 \equiv 1$ by Euler's Theorem. Square a few more times to get an integer which is still $\equiv 1$ but now large enough to know for sure that it lies in S. Then use rule (b) repeatedly to deduce (eventually) that $n^4 \in S$, and thus that $n^2 \in S$, and thus $n \in S$.

A2. Seeing that the Q_n would satisfy a determinant-like relationship,

$$Q_{n+1}Q_{n-1} - Q_n^2 = -1$$

just as the Fibonacci numbers do, I thought there might be a simple two-term recurrence relationship, and a review of the first four Q_k indeed suggested one:

$$Q_{k+1} = xQ_k - Q_{k-1}$$

This relationship is easily shown to hold for the first few k. If it has already been shown to hold for all $k < n$ then we compute

$$(xQ_n - Q_{n-1})Q_{n-1} - Q_n^2 = Q_n(xQ_{n-1} - Q_n) - Q_{n-1}^2 = Q_nQ_{n-2} - Q_{n-1}^2 = -1$$

and so $(xQ_n - Q_{n-1})$ satisfies the equation that is supposed to define Q_{n+1}, which means $(xQ_n - Q_{n-1}) = Q_{n+1}$, completing the inductive step.

From THIS recurrence relation, then, it is obvious that each Q_n is an integer polynomial.

A3 Let F be the portion of the interval where $f(x) > g(x)$ and likewise let G be the portion where g exceeds f.

Then $\int_F (f-g)(f/g)^{n+1}\,dx > \int_F (f-g)(f/g)^n$ since $f - g > 0$ and $f/g > 1$ there. But by the same reasoning, $\int_G (f-g)(f/g)^{n+1}\,dx > \int_G (f-g)(f/g)^n$ (both integrals are negative). Of course on the complement of F and G , $f - g = 0$ and so both integrals there are zero too. Adding up these inequalities we find $\int (f-g)(f/g)^{n+1}\,dx > \int (f-g)(f/g)^n\,dx$ when integrating over the whole interval. In particular, all these integrals are larger than the one for $n = 0$, which is just $\int f\,dx - \int g\,dx = 0$, so these integrals are all positive.

But $\int (f-g)(f/g)^n\,dx = \int f^{n+1}/g^n\,dx - \int f^n/g^{n-1}\,dx = I_n - I_{n-1}$, so the positivity of each integral shows that the I_n form an increasing sequence. The fact that the gaps themselves are increasing shows moreover that $I_n \geq nI_1$ for all n, so the I_n increase to ∞.

A4 is incomplete and surely not ideal.

We are told there is a set of scored student papers, with say a_i of them receiving a score of i (for $0 \le i \le 10$). The total number of papers $\sum a_i$ is given to be an even number $2N$ and the average score is exactly 7.4, that is, $\sum i a_i = 74(N/5)$. (It follows that N is a multiple of 5 but that is not relevant.) We are also told that each $a_i > 0$. Note that reading the score equation mod 2 shows $a_1 + a_3 + a_5 + a_7 + a_9$ to be even (that is, an even number of these a_i are odd); since the sum of *all* a_i is even, then likewise there is an even number of odd a_i among $a_0, a_2, a_4, a_6, a_8, a_{10}$.

We are asked to split the student papers into two subsets of equal cardinality and equal averages. Let the piles contain respectively b_i and c_i papers with a score of i. My thinking is that the most natural way to create the two piles would be to give each pile about half of the papers with each score; that is, for each i we let $b_i = (a_i - \epsilon_i)/2, c_i = (a_i + \epsilon_i)/2$ for some integers ϵ_i. Integrality requires $\epsilon_i \equiv a_i \pmod 2$. Non-negativity requires $|\epsilon_i| \le a_i$. Equality of cardinality requires $\sum \epsilon_i = 0$. Equality of averages requires $\sum i\epsilon_i = 0$.

So for example if each a_i is even we simply take each $\epsilon_i = 0$. The difficulty is accounting for other combinations of even and odd a_i. The possibilities are limited by what was noted above: exactly 0, 2, or 4 among $\{a_1, a_3, a_5, a_7, a_9\}$ are odd, and exactly 0, 2, 4, or all 6 among $\{a_0, a_2, a_4, a_6, a_8, a_{10}\}$ are odd. Also note that since $a_i \ge 1$ for every i, we may always elect to use $\epsilon_i = 0, 1$, or -1; in addition, whenever a_i is even, we may use $\epsilon_i = \pm 2$.

I haven't yet noticed a way to do this systematically. For example, it's not obvious what to do when the a_i are the sequence $1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 14, 10$ (40 papers, 296 total points). I have patterns that cover most of the combinations and I am still working on the others. I don't think this is the Putnam Way, though!

Suppose for example that a_0, a_2, a_4, a_6 are odd but a_8, a_{10} are even. In that case we use $\epsilon_0 = +1, \epsilon_2 = -1, \epsilon_4 = -1, \epsilon_6 = +1, \epsilon_8 = 0, \epsilon_{10} = 0$. The parity and two sum conditions are met, as well of course as the inequalities. Exactly the same pattern of ϵ_i works when the parities of these a_i are $(0, 1, 1, 1, 1, 0)$ or $(0, 0, 1, 1, 1, 1)$ or $(1, 1, 0, 0, 1, 1)$. (We will also use this solution for some of the cases in which four among a_1, a_3, a_5, a_9 are odd.) When the parities are $(1, 1, 1, 0, 1, 0)$ we may use these ϵ's: $(+1, -1, +1, -2, +1, 0)$. A similar solution applies to the translates and reflections of this "11101" pattern. When the parities are 111001 (or its reflection) we can use epsilons $(-1, +1, +1, -2, +2, -1)$. So we have found collections of ϵ_i to use for every combinations of parities of the a_{2i} that involve precisely four odds and two evens, except for the case that the odds are a_4 and a_8 (or, symmetrically, when the odds are a_2 and a_6.)

If a_{2i-2} and a_{2i+2} are the only odd ones, we can use $\epsilon_{2i-2} = +1, \epsilon_{2i} = -2, \epsilon_{2i+2} = +1$. A similar solution holds if only a_{2i-4} and a_{2i+4} are odd. (Of course we will have similar cases to consider with the odd-subscripted a_i and can use similar solutions.)

Up to reflection and translation, then, the only cases we must consider more carefully among the even-subscripted a_{2i} are those with parity strings 111111, 110101, 110000, 100100, 100001. (Among the odd-subscripted a_i the only problem parity strings are 110000 and 10010 and their translates.)

A5 Let $v_n = (a_n, b_n, c_n)$ be the vector of probabilities that the three players win a game of n cards. Then $v_1 = (1, 0, 0)$ and $v_2 = (1/2, 1/2, 0)$. For larger n look at the game tree: with probability $1/n$ player A selects card 1 and wins; and for each $k > 1$, she selects card k with probability $1/n$, and then play continues, with player B beginning a game that has $k-1$ cards, at which point players A, B, C will win with probabilities $c_{k-1}, a_{k-1}, b_{k-1}$ respectively. Thus

$$v_n = (1/n)v_1 + \sum_{i=1}^{n-1}(1/n)(c_i, a_i, b_i)$$

Subtracting two such equations from each other gives us a recursion

$$(n+1)v_{n+1} - nv_n = (c_n, a_n, b_n) = Rv_n$$

where R is a 120-degree rotation around the line $x = y = z$. With some experimentation (or intuition) it becomes clear that the v_n quickly approach the vector $(1/3, 1/3, 1/3)$ and of course each v_n lies in the plane $x+y+z = 1$; the only question is to decide which third of space contains this vector: whether $\max(x, y, z) = x$ or y or z.

For algebraic or geometric convenience let us scale and project to the plane $x+y+z = 0$: let $w_n = n!(v_n - (1/3, 1/3, 1/3))$; then $w_{n+1} = nw_n + Rw_n$. These w_n spiral (slowly) away from the origin; we only need to determine which third of the plane they are in, i.e. to determine (approximately) the θ of their polar coordinates.

Let's find out how much θ moves each time, that is, let us determine the angle between two consecutive w's. We compute dot products: since Rw_n has the same length as w_n but is a third of a turn away, $\langle w_{n+1}, w_n\rangle = n\langle w_n, w_n\rangle + n\cdot(-1/2)\langle w_n, w_n\rangle = (n-1/2)\langle w_n, w_n\rangle$ and similarly $\langle w_{n+1}, w_{n+1}\rangle = n^2\langle w_n, w_n\rangle + \langle w_n, w_n\rangle + 2n\cdot(-1/2)\langle w_n, w_n\rangle = (n^2-n+1)\langle w_n, w_n\rangle$ So the cosine of the angle θ_n between them is $(n-1/2)/\sqrt{n^2-n+1}$ and thus

$$\sin^2(\theta_n) = 1 - (n-1/2)^2/(n^2-n+1) = (3/4)/(n^2-n+1)$$

From the Taylor series for sine (or arcsine) we deduce that $\theta_n = \sqrt{3}/(2n) + O(1/n^2)$. Adding these changes in direction we deduce that the direction in which vector w_N points is of the form $(\sqrt{3}/2)\log(N) + O(1)$, and in particular, rotates infinitely often among the three sectors. (It takes exponentially longer to make each change in the lead, however.)

A6. First note that there are three types of colored triangles: besides the Putnam type, a triangle may be either monochromatic or tricolor. But if we label the colors by the elements $\{0, 1, 2\}$ of the set R of integers mod 3, and then assign to each face the sum of its edges, then we discover the Putnam triangles are exactly the ones for which the face-sum is nonzero.

Next consider any 2-simplex, that is, a collection of V vertices, some pairs of which are joined to make E edges, and some triples of which are connected to give F faces. An edge-coloring of the simplex simply assigns to each of the E edges a number in R, that is, it is an E-tuple in R^E. Any such coloring determines a face-sum for every face and hence an F-tuple of elements in R; that would be an element of R^F. The function which computes this F-tuple from the E-tuple is a linear map $L : R^E \to R^F$; in fact, the matrix that represents it is just the incidence matrix showing which edges are in which faces.

If this linear map has a kernel of dimension k then it follows that the number of edge-colorings which produce a given vector v of face-sums is either 0 (if v is not in the image of L) or $|R|^k = 3^k$ (if it is).

I will show that in our context L is onto. Since our $L : R^{30} \to R^{20}$, a surjection will have dimension 10. The vectors v of interest to us are the 20-tuples of non-zero elements of R, of which there are 2^{20}. Thus the number of colorings will be $3^{10} \cdot 2^{20} = 12^{10}$.

So let us show that indeed for any pre-assigned collection of face-sums for the icosahedron, there is an edge-coloring that will match it. This is where we have to look closely at how the icosahedron is assembled. Call one vertex the "North Pole"; then in the obvious way we get the "Arctic Circle" of 5 vertices and the 5 vertices that join them, the five "North polar faces", the five "North Tropical faces" that share an edge with them, and then the corresponding faces in the southern hemisphere, as well as the polar and equitorial edges. Very well then, here is the observation: given any edge-coloring, add the face-sums of the North polar faces and the South tropical faces, and then subtract those of the South polar faces and North tropical faces. This alternating sum will cancel the contributions of most edges, leaving only the double of the sum of the North polar edges minus that of the South polar edges.

Now, conversely, let us show how to reconstruct an edge-coloring from any face-sum. Pick colors for 9 of the ten polar edges at will; then choose the color of the last polar edge so that the consistency condition of the previous paragrph is met. Now the Arctic circle can be colored, so as to yield the desired face-sums of the North polar faces; likewise the Antarctic circle. Color one of the equatorial edges at will; then the desired face-sums of nine tropical faces will determine the colorings of the other nine equatorial edges in turn. The edges are now all colored, and the face-sums for 19 of the faces match the initial data. But the face sum of the last face is also forced to match because we have chosen in advance to have the consistency condition satisfied.

To recap: any sequence of face-sums can be achieved by an edge-coloring, thus L is onto, and thus the pre-image of the set of 2^{20} everywhere-nonzero vectors has cardinality $2^{20} \cdot 3^{10}$.

In exactly the same way we can conclude that there are 144 Putnam colorings of the tetrahedron: in this case any alternating sum (across adjacent triangles) will identify the difference of the colors of an appropriate pair of non-intersecting colors. Pick the color of one of those and compute the color of the other; color another edge at will and then compute the colors of the remaining three edges. This shows $L : R^6 \to R^4$ is onto, giving $3^2 \cdot 2^4 = 144$ Putnam colorings.

You might want to look up "simplicial homology" to see where these ideas are put to use in Topology.

B1. First note that two rays PA_1 and PA_2 are real multiples of each other iff the points P, A_1, and A_2 are collinear. So in the problem we are asked to consider the following statement about two lines L_1 and L_2:

Condition C: Given any point P lying in neither line and given any real $\lambda \neq 0$ there is a third line L_3 whose points of intersection A_i with lines L_i make $PA_2 = \lambda PA_1$.

If two lines in $\mathbf{R}^2$ do not intersect, they are parallel (and distinct). In that case condition (C) does not hold: let P be a point separated from L_1 by L_2; then for every L_3 the points of intersection A_i of L_i with L_3 will make the vectors PA_1 and PA_2 parallel, i.e. $PA_2 = \lambda PA_1$ for some λ, but we will always have $\lambda > 1$.

Conversely let us suppose L_1 and L_2 do intersect, and suppose P is given. Drop a perpendicular from P to L_1 and call that the x-axis; the perpendicular to that line passing through P will be the y-axis. Let the unit of measure be the distance from P to L_1. Then in this coordinate system we have $P = (0, 0)$, L_1 is the line $x = 1$, and L_2 is not vertical and hence may be described as $y = mx + b$ for some m and b. Any third line L_3 passing through P and meeting L_1 is of the form $y = kx$ for some k. The intersection A_1 with L_1 is clearly $(1, k)$ and the intersection with L_2 comes out to be $A_2 = (b/(k-m), kb/(k-m))$, so that the vector PA_2 is $b/(k-m)$ times the vector PA_1. Every non-zero multiplier λ may be achieved for the appropriate k, namely $k = m + b/\lambda$.

Let me remark that some people consider it to be "unaestehtic" to use coordinates to solve a geometry problem. I may be one of them! I certainly know that if you wish to do so, you should choose your coordinate system carefully, e.g. as in this case rotating the data so that one line is vertical.

B2. An integer N is the sum of $k > 1$ consecutive positive integers starting with a iff $2N = k(k + (2a - 1))$. Note that the latter factor is strictly larger than the first, i.e. $k < \sqrt{2N}$, and of the opposite parity. So there is a one-to-one correspondence between these trapezoidal representations of N and the factorizations of $2N$ (other than $2N = 1 \cdot 2N$) in which the two factors have opposite parity; k is the lesser factor in every case. (Perfect squares do not present an additional option because of the required parity difference.)

It is uncommon for N to have such a representation for only one $k > 1$. The parity difference condition means N cannot be a power of 2. If N is divisible by two distinct odd primes p, q, then $2N = p \cdot (2N/p) = q \cdot (2N/q)$ are two distinct factorizations of N into factors of opposite parity. And if N is divisible by p^2 then $2N = p \cdot (2N/p) = p^2 \cdot (2N/p^2)$ provide two.

Thus we see $N = 2^n p$ for some $n \geq 0$ and some odd prime p; the lone factorization of $2N$ is as $2^{n+1} \cdot p$, where k is the lesser of the two factors and $2a - 1$ is their difference, $|p - 2^{n+1}|$.

In order for an integer N to have only one trapezoidal representation *and* have it use $k = 2017$, it must be true that $p = 2017$; asking further to minimize a (or equivalently to minimize $2a - 1$) requires that 2^{n+1} be the power of two closest to 2017; clearly that would be $2^{11} = 2048$, so that $n = 10, N = 2^{10} \cdot 2017, 2a - 1 = 31$, and so $a = 16$.

B3. Suppose $g(x) = \sum c_i x^i$ is a power series with all coefficients $c_i = 0$ or 1. Then $g(1/2)$ is a power-series representation of a number but it is also the dyadic (i.e. base -2) representation of that number, which is unique except for the possible substitution of $\sum_{i>N} x^i$ for a final x^N in a terminating series (a polynomial). In particular, since every rational number has an eventually-repeating dyadic expansion, $g(1/2)$ will be rational iff $g(x) = P(x) + x^N Q(x)/(1 - x^M)$ for some polynomials P and Q of degrees less than N and M, respectively, each with coefficients of 0 and 1 only.

If, for this same power series, $g(r) = 1$ for some rational number r, then r is a root of the polynomial $R(x) = (P(x) - 1)(x^M - 1) + x^N Q(x)$. We can draw conclusions in several cases, but in particular if the constant term of g is 0 then the constant term of R is $+1$. By the Rational Root Theorem then numerator of r must be ± 1. (And the denominator is also absolutely bounded.)

If we apply this to $g(x) = xf(x)$ and $r = 2/3$ we obtain a contradiction: we cannot have $f(2/3) = 3/2$ and $f(1/2)$ rational.

B4 – no idea. Writing $4k+i$ as $(4k)(1+i/4k)$ and using properties of the logarithm, the kth summand is $\log(4k) \cdot A(x) + B(x)$, where both $A(x)$ and $B(x)$ equal $3/8k^2 + O(1/k^3)$. So the series converges (fairly rapidly).

B5 – no real idea. I misread the problem, thinking I wanted to slice off a triangle of half the original area (correct) and half the original perimeter (incorrect). FWIW, that incorrect problem leads to some nasty coordinate calculations, althoug they are not too bad if the coordinate system is set up so the original triangle's vertices are given coordinates $(0,0), (s,1), (t,1)$ and the equalizer line cuts off a new triangle that includes $(0,0)$ as a vertex. I solved for the y-intercept of the equalizer, in terms of its slope, and then found the slope must satisfy a quartic polynomial; so nominally there could be as many as four such lines crossing each pair of points! (Surely the number of REAL roots to those quartics is not always 4, though.)

B6 – No real idea. I did not notice until now that 2016 is a triangular number as in B2: $1 + 2 + 3 + ... + 63 = 2016$.