UT Putnam Prep Problems, Nov 2, 2016 (Go, Cubbies, Go!)
NUMBER-THEORY PUTNAM PROBLEMS

1. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A *perfect square* is the square of an integer; that is, a member of the set $\{0, 1, 4, 9, 16, \ldots\}$. We say "$a$ is within n of b" if $b - n \leq a \leq b + n$.)

2. Find the smallest positive integer n such that for every integer m, with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}$$

3. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

4. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, and z positive integers.

5. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

6. Prove that, for any integers a, b, c there exists a positive integer n such that

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

7. Let N be the positive integer with 2016 decimal digits, all of them being 1, that is, $N = 111 \ldots 111$ (2016 digits). Find the thousandth digit after the decimal point of $\sqrt{N}$.

8. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that either $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exists $s, t \in S$ such that $\gcd(s, t)$ is prime. [Here $\gcd(a, b)$ denotes the greatest common divisor of a and b.]

9. Show that no four consecutive binomial coefficients can lie in an arithmetic progression:

$$\binom{n}{r}, \quad \binom{n}{r+1}, \quad \binom{n}{r+2}, \quad \binom{n}{r+3}$$