We are asked to prove that the improper integral

$$\int_0^\infty \sin(x)\sin(x^2)\,dx$$

converges. We must investigate the integrals over finite intervals $[0, B]$.

Using the angle-addition formula we may rewrite the product

$$\sin(x)\sin(x^2) = \frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x))$$

to convert the integral over $[0, B]$ into half the difference of two integrals. Using the substitutions $u = x - \frac{1}{2}$ and $u = x + \frac{1}{2}$ respectively, these will *both* become integrals of $\cos(u^2 - \frac{1}{4})$, the first one over the interval $[-\frac{1}{2}, B - \frac{1}{2}]$ and the second over the interval $[+\frac{1}{2}, B + \frac{1}{2}]$. Subtracting, we may cancel the common parts of these intervals and write our original integral as

$$\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(u^2 - \frac{1}{4})\,du - \frac{1}{2}\int_{B-\frac{1}{2}}^{B+\frac{1}{2}} \cos(u^2 - \frac{1}{4})\,du$$

The first integral is just some constant (my computer tells me it's about 0.49) so we need only investigate the behaviour of the other integral as $B \to \infty$.

Viewing the integrand as $(1/u)(u\cos(u^2 - \frac{1}{4}))$, we apply integration by parts to obtain

$$(1/u)\frac{1}{2}\sin(u^2 - \frac{1}{4}) + \frac{1}{2}\int \sin(u^2 - \frac{1}{4})/u^2\,du$$

for the antiderivative, and thus the definite integral is

$$\sin(B^2 + B)/(2B + 1) - \sin(B^2 - B)/(2B - 1) + \frac{1}{2}\int_{B-\frac{1}{2}}^{B+\frac{1}{2}} \sin(u^2 - \frac{1}{4})/u^2\,du$$

Now simply observe everywhere that $|\sin(\theta)| \leq 1$, so that the magnitude of the above is bounded by

$$1/(2B + 1) + 1/(2B - 1) + \frac{1}{2}\int_{B-\frac{1}{2}}^{B+\frac{1}{2}} 1/u^2\,du$$

which is roughly $1/B$ and anyway clearly drops to zero as $B \to \infty$.

Remark: Using the substitutions I indicated, you can explicitly compute an antiderivative of this function in terms of the Fresnel functions, available e.g. in Maple, where they are defined as

$$\mathrm{FresnelC}(x) = \int_{t=0}^{t=x} \cos(\frac{\pi}{2}t^2)\,dt$$

$$\mathrm{FresnelS}(x) = \int_{t=0}^{t=x} \sin(\frac{\pi}{2}t^2)\,dt\ .$$

Our integral may be expressed as

$$\sqrt{\frac{\pi}{2}}\left\{\cos(\frac{1}{4})\,\mathrm{FresnelC}\left(\frac{1}{\sqrt{2\pi}}\right) + \sin(\frac{1}{4})\,\mathrm{FresnelS}\left(\frac{1}{\sqrt{2\pi}}\right)\right\}$$