A-1. Supposing that an integer n is the sum of two triangular numbers,

$$n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2},$$

write $4n + 1$ as the sum of two squares, $4n + 1 = x^2 + y^2$, and show how x and y can be expressed in terms of a and b. Show that, conversely, if $4n + 1 = x^2 + y^2$, then n is the sum of two triangular numbers. [Of course, a, b, x, y are understood to be integers.]

A-2. (a) For which ordered pairs of real numbers b and c do both roots of the quadratic equation

$$z^2 + az + b$$

lie inside the unit disk $\{|z| < 1\}$ in the complex plane?

(b) Draw a reasonably accurate graph of the region in the real bc-plane for which the above condition holds. Identify precisely the boundary curves of this region.

A-3. Let a, b and c be constants with $0 < a < b < c$. At what points of the set

$$\{x^b + y^b + z^b = 1, x \geq 0, y \geq 0, z \geq 0\}$$

in three-dimensional space R^3 does the function $f(x, y, z) = x^a + y^b + z^c$ assume its maximum and minimum values?

A-4. Let $n = 2m$, where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1 - \theta)^{-1}$ explicitly as a polynomial in θ,

$$a_k\theta^k + a_{k-1}\theta^{k-1} + \cdots + a_1\theta + a_0,$$

with integer coefficients a_i. [Note that θ is a primitive n-th root of unity, and thus it satisfies all of the identities which hold for such roots.]

A-5. On some interval I of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation

$$y'' = f(x)y,$$

wherre $f(x)$ is a continuous real-valued function. Suppose that $y_1(x) > 0$ and $y_2(x) > 0$ on I. Show that there exists a positive constant c such that, on I, the function

$$z(x) = c\sqrt{y_1(x)y_2(x)}$$

satisfies the equation

$$z'' + \frac{1}{z^3} = f(x)z.$$

State clearly the manner in which c depends on $y_1(x)$ and $y_2(x)$.

A-6. Let P_1, P_2, and P_3 be vertices of an acute-angled triangle situated in three-dimensional space. Show that it is always possible to locate two additional points P_4 and P_5 in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

B-1. In the additive group of ordered pairs of integers (m, n) [with addition defined componentwise: $(m, n) + (m', n') = (m + m', n + n')$] consider the subgroup H generated by the three elements

$$(3, 8), \qquad (4, -1), \qquad (5, 4).$$

Then H has another set of generators of the form

$$(1, b), \qquad (0, a)$$

for some integers a and b with $a > 0$. Find a.

B-2. In three-dimensional Euclidean space, define a *slab* to be the open set of points lying between two parallel planes. The distance between the planes is called the *thickness* of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$ respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

B-3. Let $s_k(a_1, \ldots, a_n)$ denote the k-th elementary symmetric function of $a_1, \ldots, a_n$. With k held fixed, find the supremum (or least upper bound) M_k of

$$s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k$$

for arbitrary $n \geq k$ and arbitrary n-tuples $a_1, \ldots, a_n$ of positive real numbers.

B-4. Does there exist a subset B of the unit circle $x^2 + y^2 = 1$ such that

(1) B is topologically closed, and

(2) B contains exactly one point from each pair of diametrically opposite points on the circle?

B-5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = x f_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e$$

B-6. Show that if

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

then

(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$ and

(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.