Here is a solution to problem B5 of the the 2018 Putnam exam.

The positivity of the partials means that each of the two component functions f_1 and f_2 will increase as we move in any direction north or east (" in the first quadrant") from any point. Therefore, if we look at any level curve $\{(x,y)\,|\,f_1(x,y)=c\}$, it will stretch vaguely southeast and northwest. (Equivalently, by the Implicit Function Theorem, the level curve is at every point locally the graph of a function with negative derivative.) If we parameterize this path with a parameterization $(x,t)=p(t)$ heading southeast, then $f_1(p(t))$ will stay constant for all t. Computing the derivative then shows $\nabla f_1(p(t))\cdot p'(t) = 0$ for all t: our velocity vector $p'(t)$ is always perpendicular to the level curve.

On the other hand, $z=f_2(p(t))$ will definitely vary, and indeed will vary monotomically. The derivative $dz/dt = \nabla f_2(p(t))\cdot p'(t)$ can *never* be zero, because otherwise that would imply $p'(t)$ is also perpendicular to $\nabla f_2(p(t))$ at some point P, and thus the two gradients, of f_1 and f_2, would be parallel at P. Since both components of both gradients were given to be positive at each point, there would be a (positive) constant λ such that $\nabla f_2(P) = \lambda \nabla f_1(P)$. But then at the point P we would have

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2}+\frac{\partial f_2}{\partial x_1}\right)^2 &= \lambda\frac{\partial f_1}{\partial x_1}\frac{\partial f_1}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2}+\lambda\frac{\partial f_1}{\partial x_1}\right)^2 \\
&= -\frac{1}{4}\left(\frac{\partial f_1}{\partial x_2}^2 - 2\lambda\frac{\partial f_1}{\partial x_2}\frac{\partial f_1}{\partial x_1} + \lambda^2\frac{\partial f_1}{\partial x_1}^2\right) \\
&= -\frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} - \lambda\frac{\partial f_1}{\partial x_1}\right)^2 \le 0,
\end{aligned}$$

a contradiction.

Therefore, as we move along all the points where $f_1=c$, the other component f_2 must be monotonically increasing or decreasing, and we can never return to a later point where both f_1 and f_2 have the same values as at an earlier point.

To complete this proof we must show that the level curves of f_1 are connected. Indeed suppose, say, $f_1(x,y)=xy$; then the set of points where $f_1=1$ is a hyperbola which is a set of 2 disjoint curves. If we travel along either one of them, then the preceding proof shows f_2 will monotonically increase or decrease and so f will be one-to-one there. But I have not proved that there cannot be two points on the two separate branches of the hyperbola where f_2 takes on the same values.

So let's analyze the shape of a level set $C = f_1^{-1}(c)$. Let K be one of its path components, and K_x the projection of K to the x-axis. Then this is also necessarily path-connected, hence it is an interval. If that interval is bounded on the left, let $a=\inf(K_x)$. I claim a is not in K_x. If it were, then there would be a point (a,y) in K. Because the partials of f_1 at (a,y) are positive, by the Implicit Function Theorem there is an neighborhood of this point whose intersection with K is the graph of a function, and in particular K includes points path-connected to (a,y) but having smaller x-coordinates, contradicting the definition of a. So the interval K_x is open on the left, and similarly will be open on the right. Now, we have already noted that f_1 increases as we go north, so that in particular C will pass the "vertical line test", so that for each $x \in K_x$ there is a unique point (x, y) in K, i.e. K is (globally) the graph of a function g defined on K_x (and the IFT makes it smooth and decreasing). If $L = \lim_{x \to a^+} g(x)$ existed, then by the continuity of f_1 we would have $(a, L) \in K$ and thus $a \in K_x$ after all. So we must have $L = \infty$, that is, the connected component "goes off the top of the plane" (i.e. g has a vertical asymptote at $x = a$). To recap, then, this path-connected component of $f_1^{-1}(c)$ is the graph of a smooth, decreasing function on an interval, and either this interval extends left to $-\infty$ or the graph of g rises to $+\infty$ as we approach the left end of the interval.

Of course the same remarks apply to the right side of K_x as well, and we could equally well analyze the projection K_y of K to the y-axis. Thus we conclude that each connected component of a level set of f_1 has one of a few shapes: vaguely resembling the graphs of $y = -\tan(x)$ on $(-\pi/2, \pi/2)$ or $y = -\arctan(x)$ on $(-\infty, \infty)$; or of $y = 1/x$ or $y = -\log(x)$ on $(0, \infty)$; or of $y = 1/x$ or $y = \log(-x)$ on $(-\infty, 0)$; or of course simply something like the curve $x + y = 0$, depending on whether K_x and K_y are bounded on the left or right

In most of those cases, either K_x or K_y (or both) is the whole real line. Since C passes the "horizontal line test" as well as the vertical line test, that means this one component K must be the entirety of C, and our earlier proof suffices to show f is one-to-one on C.

The only other possibility is that C contains two components, one with K_x and K_y bounded above and the other with K_x and K_y bounded below — something like the whole hyperbola $xy = 1$. But this is impossible as well. If (x_1, y_1) is a point in the first component and (x_2, y_2) in the second, then $x_1 < x_2$ and $y_1 < y_2$. But then the positivity of both partial derivatives makes $c = f_1(x_1, y_1) < f_1(x_2, y_1) < f_1(x_2, y_2) = c$, which is a contradiction.

So indeed the level sets $f_1^{-1}(c)$ are all path-connected, and the main part of the proof applies.

Remarks: The test of whether a function is *locally* one to one is simply that the determinant of the derivative matrix not vanish, that is,

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_2}\frac{\partial f_1}{\partial x_1} \neq 0$$

at all points in the plane. But that's not quite the same as being *globally* one-to-one. The classic example is the exponential function $f : \mathbf{C} \to \mathbf{C}$ given by $f(z) = e^z$. Viewing $\mathbf{C}$ as $\mathbf{R}^2$ we can write this as $f_1(x, y) = e^x \cos(y), f_2(x, y) = e^x \sin(y)$. Then $f(x, y) = f(x, y + 2\pi)$ so f is not one-to-one.

Also of interest is the *Jacobian Conjecture*, which asks whether the non-vanishing of that determinant makes f globally one-to-one when f is "algebraic". I invite you to research this topic for more details.