Problem set six (Nov 9 2010) — Number Theory!

1. Suppose n is a positive integer. How many ordered pairs (x, y) of positive integers are there with

$$\frac{xy}{x+y} = n$$

2. Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.

3. Do there exist one million consecutive integers, each of which is divisible by a perfect square (larger than 1)?

4. Prove that there are no positive integers x and y for which

$$x^2 + 3xy - 2y^2 = 122.$$

Show on the other hand that the equation $x^2 - y^2 = a^3$ has integer solutions for every a.

5. For p an odd prime, let F be the function defined by

$$F(n) = 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}$$

Show that if a and b are distinct integers in $\{0, 1, 2, .., p-1\}$ then $F(a)$ and $F(b)$ are not congruent mod p.

6. Define integers a_k by $a_1 = 2$ and then, for $k > 1$, let $a_{k+1} = 2^{a_k}$. Show that for every integer $n > 1$, $a_n \equiv a_{n-1} \pmod{n}$.

7. Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.