Putnam practice problems from Linear Algebra – Oct 9 2017 – **ANSWERS**

1. Suppose $A$ is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that $A$ has a positive determinant.

If $v$ is any eigenvector of $A$, corresponding to the eigenvalue $\lambda$, then $A^3 v = \lambda^3 v$ and $(A+I)v = (\lambda+1)v$. Since these two are equal (and $v$ is not the zero vector), it follows that $\lambda^3 = \lambda + 1$, i.e. that $\lambda$ is a root of $X^3 - X - 1$. A glance at the graph of this polynomial shows that it has one real root $\lambda_0$ (which is positive), and a conjugate pair of complex roots $\lambda_1, \lambda_2$ (and the product of any nonzero complex number and its conjugate is also positive).

Now, the determinant of a matrix is the product of its eigenvalues, each taken with its algebraic multiplicity (i.e. as a root of the characteristic polynomial). Since the matrix is real, so is its characteristic polynomial, which means each non-real root appears with the same multiplicity as its complex conjugate. Hence $\det(A) = \lambda_0^r (\lambda_1 \lambda_2)^s$ for some positive integers $r$ and $s$, and this product is positive.

2. Suppose $A, B \in M_2(\mathbf{C})$ satisfy $AB = BA$. Assume that $A$ is not of the form $aI_2$ for any complex number $a$. Show that $B = yA + zI$ for some complex numbers $x, y$.

Many, many problem in Linear Algebra can be treated like this: first you conjugate (i.e. switch to a more convenient basis), then do your work, and then conjugate back. The computations can be much simpler that way. Let me clarify.

Every complex $2 \times 2$ matrix is diagonalizable or nearly so: there will always be complex matrices $P$ and $Q$ with $PQ = QP = I$ such that $PAQ = A'$ where $A'$ is either diagonal or of the form

$$\begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix}$$

So the strategy is to conjugate the whole problem by $P$ and $Q$. That is, let $B' = PBQ$ and then note $A'$ and $B'$ commute too:

$$A'B' = (PAQ)(PBQ) = PAIBQ = P(AB)Q = P(BA)Q = PBIAQ = (PBQ)(PAQ) = B'A'$$

Also if $A'$ were equal to $aI$ for some scalar $a$, then we would have $A = QA'P = Q(aI)P = a(QIP) = aI$, a contradiction. And finally if $B' = yA' + zI$ for some $y$ and $z$ then $B = QB'P = Q(yA' + zI)P = y(QA'P) + z(QIP) = yA + zI$. In other words, it will be sufficient to prove the desired result for $A'$ and $B'$.

So we need only solve the problem in the special case that $A$ is of one of these two forms:

$$\begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix} \qquad \mathit{or} \qquad \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$$

where $e \neq f$.

Very well then. Suppose $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then in the diagonal case

$$AB = \begin{pmatrix} ea & eb \\ fc & fd \end{pmatrix} \qquad \text{while} \qquad BA = \begin{pmatrix} ea & fb \\ ec & fd \end{pmatrix}$$

so for the two to be equal we need $b = c = 0$. This makes $B$ also into a diagonal matrix, and we will indeed have $B = yA + zI$ as long as $a = ye + z, d = yf + z$; two such complex numbers $y$ and $z$ do exist since $\det \begin{pmatrix} e & 1 \\ f & 1 \end{pmatrix} \neq 0$.

Similarly in the non-diagonal case we expand out $AB - BA$ and observe that this matrix will be zero iff $a = d$ and $c = 0$. We easily solve a system of equations to discover that if $y = b$ and $z = a - be$ then $B = yA + zI$.

So the result is true for both types of matrices $A$, and we are done.

Let me remark that every matrix $A$ commutes with every polynomial in $A$, that is, with every $B = zI + yA + xA^2 + wA^3 + \ldots$. For *most* matrices, there is nothing else that commutes with $A$; only when the characteristic polynomial of $A$ has repeated roots can there be any other commuting matrices $B$. It's also true that every polynomial in $A$ can also be expressed as a polynomial in $A$ *of degree less than* $n$ (the size of the matrix). This problem essentially asked you to prove all this when $n = 2$.

3.(a) Find two real matrices $A, B$ with

$$A^2 + B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

(b) Show that if $A, B$ are real matrices with

$$A^2 + B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

then $AB \neq BA$

For (a) I instinctively began with $A = c\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ where $c = \sqrt{3}$. Then $B^2$ would have to be $-I$, which is possible: $-I$ represents a $180°$ rotation of the plane, which can be thought of as the composite of two $90°$ rotations, which are represented by $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Alternatively, Dennis noted that $\begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and that adding this matrix to its transpose gives the desired sum. So using $A = \begin{pmatrix} 1 & 3/2 \\ 0 & 1 \end{pmatrix}$ and $B = A^T$ will do the job.

4. Suppose $A$ is a $3 \times 3$ matrix with rational entries, for which $A^8 = I$. Show that in fact $A^4 = I$.

This is actually a number-theory question!

If $\lambda$ is any eigenvalue of $A$, let $v$ be an eigenvector with $Av = \lambda v$. Then $v = Iv = A^8 v = \lambda^8 v$, so $\lambda^8 = 1$. (There are 8 complex numbers $\lambda$ which satisfy this equation.) Now, each of these eigenvalues is a root of the characteristic polynomial, which is a cubic polynomial with rational coefficients. There are indeed such polynomials which have 1, -1, $i$, and $-i$ as roots. But the other four roots have degree 4 over the rationals, that is, the smallest-degree rational polynomials which have such numbers as roots are of degree 4 (such as $X^4 + 1 = 0$). So those numbers are not among the eigenvalues of $A$.

Each of the other roots of $X^8 - 1$ is already a root of one of its factors, namely $X^4 - 1$. So since $A$ is diagonalizable and $\lambda^4 = 1$ for each of its eigenvalue, it follows that $A^4 = I$ too.

5. Suppose $A, B$ are real $3 \times 3$ matrices. Prove that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq 3$$

iff there is an invertible matrix $X$ with $AXB = O_3$

6. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer $m$?

No there is not. Let $s_m = \sum a_i^m$. Then $s_2^2 - s_4 = (\sum a_i^2)^2 - \sum a_i^4 = 2 \sum_{i<j} (a_i)^2 (a_j)^2$ which is clearly positive (as long as there are at least two nonzero numbers $a_i$.) That is, we will surely have $s_2^2 > s_4$ (and thus if $s_2 = 2$ then $s_4 < 4$) unless all but one $a_i$ is zero; but in that case the equations $s_1 = 1$ and $s_2 = 2$ are contradictory.

Thus was a Putnam question in 2010.

7. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $f$ which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} > 0$. Suppose that for all $i$, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

8. Alan and Barbara play a game in which they take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player chooses a real number

and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

9. If $A$ and $B$ are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

No. Here is a counterexample:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Geometrically the idea is this: $AB$ annihilates a plane and sends all of space into that plane, so $(AB)^2 = 0$. On the other hand $BA = A$ only annihilates a line and so it is geometrically impossible for $(BA)^2$ to be zero. (dim(ker($BA$))=1 implies dim(ker(($BA)^2$))$\leq 2$.)