Hey gang,

Since I spoke at length about the virtue of finding alternative ways to compute — or at least estimate — numerical integrals, I set myself the challenge of estimating more accurately the value of the first integral in yesterday's Putnam set. I found the value "exactly"; perhaps you will enjoy this as much as I did.

Recall that the problem was to compute

$$I = \int_0^\pi a^{\sin^2(x)}\,dx$$

where $a = e = 2.71828\ldots$ I will show that we can estimate this integral for any positive number a quite easily.

We can rewrite the integral using symmetry, u-substitution, the half-angle formulas, and symmetry again:

$$\begin{aligned} I &= 2\int_0^{\pi/2} a^{\sin^2(x)}\,dx \\ &= \int_0^\pi a^{\sin^2(u/2)}\,du \\ &= \int_0^\pi a^{\frac{1}{2}(1-\cos(u))}\,du \\ &= \sqrt{a}\int_0^\pi a^{-\cos(u)/2}\,du \\ &= \sqrt{a}\int_0^{\pi/2}\left(a^{-\cos(u)/2} + a^{-\cos(\pi-u)/2}\right)du \\ &= \sqrt{a}\int_0^{\pi/2}\left(a^{-\cos(u)/2} + a^{+\cos(u)/2}\right)du \end{aligned}$$

This is enough to answer the Putnam question, since for any positive number X we have $X+X^{-1} \geq 2$, and thus the integral is at least equal to 2 times the width of that interval, showing the integral is at least equal to $\sqrt{a}\,\pi$. Since $e > (3/2)^2$, we are done — the Putnam integral is at least $\frac{3}{2}\pi$.

But let's try to compute the integral more accurately. You can write the integrand using cosh but I never found that function to be very helpful. On the other hand this seems like a good time to start using Taylor series (which is easy when $a = e$, which we assume now). The Taylor series is $e^x = 1 + x + x^2/2 + x^3/3! + \ldots$ so the integrand is

$$2\left(1 + \frac{\cos^2(u)}{8} + \frac{\cos^4(u)}{2^4\cdot 4!} + \ldots\right)$$

It happens to be true that

$$\int_0^{\pi/2} \cos^{2k}(u)\,du = \frac{\binom{2k}{k}\pi}{2^{2k+1}}$$

(ask me about that some time!) so our integral may be written as an infinite series

$$I = 2\sqrt{e}\sum_{k\geq 0}\left(\frac{\binom{2k}{k}\pi}{(2k)!\cdot 2^{4k+1}}\right)$$

or simply

$$I = \pi\sqrt{e} \sum_{k \geq 0} \left(\frac{1}{(k!)^2 \cdot 16^k} \right)$$

Those denominators there in the final sum get really large, really fast (each one is several digits longer than the one before it, the pace quickening as k increases!) so the sequence converges very rapidly, and monotonically.

The original problem asked us for an approximation which was a rational multiple of π; this analysis shows they should probably have asked for a rational multiple of $\pi\sqrt{e}$; the correct value of the integral and first few the partial sums are shown here:

$$\begin{aligned}
\pi\sqrt{e} &= 5.179610631848751409866699511466031864914 \\
\tfrac{17}{16}\pi\sqrt{e} &= 5.503336296339298372983368230932658856470 \\
\tfrac{1089}{1024}\pi\sqrt{e} &= 5.508394509846963169282066179674324903213 \\
\tfrac{156817}{147456}\pi\sqrt{e} &= 5.508429636329655285923029359873919806318 \\
\tfrac{40145153}{37748736}\pi\sqrt{e} &= 5.508429773542478302003658122296574473906 \\
\tfrac{5352687067}{5033164800}\pi\sqrt{e} &= 5.508429773885510359543859694202631110575 \\
\tfrac{9249443251777}{8697308774400}\pi\sqrt{e} &= 5.508429773886105901310422544153856903348 \\
\tfrac{7251563509393169}{6818690079129600}\pi\sqrt{e} &= 5.508429773886106660930022751870631426041 \\
\tfrac{2475200344539535019}{2327446213676236800}\pi\sqrt{e} &= 5.508429773886106661671838767698479838662 \\
\tfrac{9623578939569712153873}{9049110878773208678400}\pi\sqrt{e} &= 5.508429773886106661672411156599581573550 \\
\tfrac{15397726303311539446196801}{14478577406037133885440000}\pi\sqrt{e} &= 5.508429773886106661672411514342644762134 \\
I &= 5.508429773886106661672411514527509633546
\end{aligned}$$

For what it's worth, the infinite series itself has a closed-form expression; it's a value ("$I_0(1/2)$") of one of the *Bessel functions*, which are solutions of certain simple second-order differential equations. But as far as I know there is no way to describe the number in terms of anything more elementary.