UT Putnam Prep Problems, Oct 19 2016

I was asked to provide some questions about Combinatorics (a.k.a. Advanced Counting).

1. Determine (with proof) the number of ordered triples (A_1, A_2, A_3) of sets which satisfy
   (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and
   (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$
where $\emptyset$ denotes the empty set. Express that answer in the form $2^a 3^b 5^c 7^d$ where a, b, c, and d are nonnegative integers.

2. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$ what is the smallest possible number of elements in A_S?

3. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi_0(x) = \pi_0(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

4. Call a set *selfish* if it has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, ..., n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

5. Suppose S is a set of triangles, no two of which are congruent to each other. If every triangle in S has sides of integer length, how many triangles in S can have a perimeter of 15?

6. For each positive integer k let $f(k) = k!/k^k$. Show that for all positive integers m, n we have $f(m + n) < f(m)f(n)$.

7. Given any five points in the interior of a square of side 1, show that there must be two of them closer together than a distance of $k = 1/\sqrt{2}$. Is the result true for a smaller number k?

8. Let $a(n)$ be the number of representations of positive integer n as a sum of 1's and 2's taking order into account. Let $b(n)$ be the number of representations of n as a sum of integers greater than one. For example $a(4) = b(6) = 5$ because

$$4 = (1 + 1 + 1 + 1) = (1 + 2 + 1) = (1 + 1 + 2) = (2 + 1 + 1) = (2 + 2) \quad \text{and}$$
$$6 = (3 + 3) = (2 + 2 + 2) = (4 + 2) = (2 + 4) = (6)$$

Prove that for every positive integer n, $a(n) = b(n + 2)$

9. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

10. How many polynomials P with coefficients $0, 1, 2$ or 3 have $P(2) = n$, where n is a given positive integer?