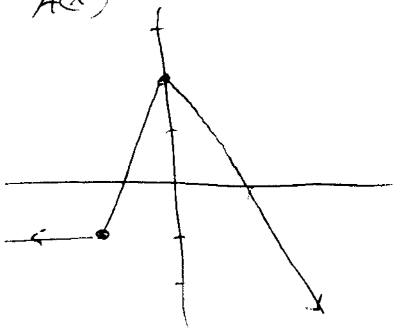
#4 from 3rd set

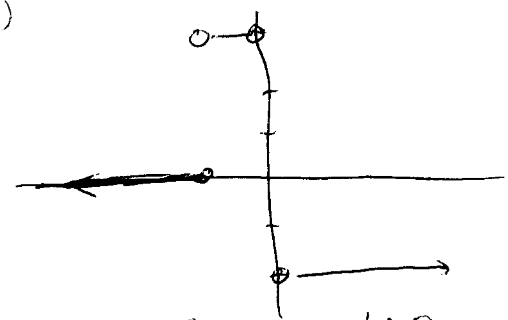
find polynomials $f(x)$ $g(x)$ and $h(x)$ so that $\forall x \in \mathbb{R}$

$$A(x) = |f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{for } x < -1 \\ 3x+2 & \text{for } -1 \le x \le 0 \\ -2x+2 & \text{for } x > 0 \end{cases}$$

graph of $A(x)$

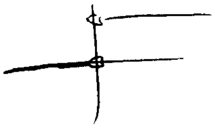


graph of $A'(x)$



let $u(t)$ be the heaviside step function $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \\ \text{undef} & \text{else} \end{cases}$

graph $u(t)$



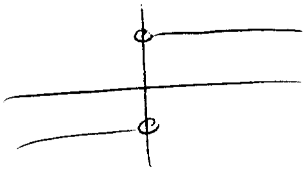
In terms of step functions, ~~A~~

$$A'(x) = 3u(x+1) - 5u(x)$$

observing that $\frac{d}{dx}|x| = \frac{|x|}{x}$ and $\frac{d}{dx}|f(x)| = f'(x)\frac{|f(x)|}{f(x)}$

we can define $\frac{d}{dx}|x|$ in terms of the unit step function

graph of $\frac{|x|}{x}$



thus $\frac{|x|}{x} = [u(x)-1] + u(x)$

$= 2u(x) - 1$

$\rightarrow$ $u(x) = \left(\frac{|x|}{x} + 1\right)\frac{1}{2}$

so ~~substituting~~ substituting into $A'(x)$ this equation for $u(t)$ we see $A'(x) = \frac{3}{2}\frac{|x+1|}{x+1} + \frac{3}{2} - \frac{5}{2}\frac{|x|}{x} - \frac{5}{2} = \frac{3}{2}\frac{|x+1|}{x+1} - \frac{5}{2}\frac{|x|}{x} - 1$

setting $\frac{3}{2}\frac{|x+1|}{x+1} - \frac{5}{2}\frac{|x|}{x} - 1 = f'(x)\frac{|f(x)|}{f(x)} - g'(x)\frac{|g(x)|}{g(x)} + h'(x)$

we can "guess" that $f'(x)\frac{|f(x)|}{f(x)} = \frac{3}{2}\frac{|x+1|}{x+1}$ $\quad g'(x)\frac{|g(x)|}{g(x)} = \frac{5}{2}\frac{|x|}{x}$

and $h'(x) = -1$. This implies $|f(x)| = \left|\frac{3}{2}x + \frac{3}{2}\right| + C_1$ $|g(x)| = \left|\frac{5}{2}x\right| + C_2$

and $h(x) = -x + C_3$. putting these into $A(x)$ we get . . on ~~next~~ next page

for some constants C_1, C_2, C_3