UT Putnam Prep Problems, Nov 9, 2016
GEOMETRY PUTNAM PROBLEMS

1. Given a convex polygon $S$ of area $A$ and perimeter $p$, what is the area of the set of points which lie within a distance of 1 from $S$?

**ANSWER:** The problem is meant to imply that $S$ is a 2-dimensional set, i.e. the polygon and its interior; we will address below the more complicated problem of taking $S$ to mean the boundary of such a set. In the simpler setting, the set in question is just the union of rectangles of width 1 resting along the edges of $S$, together with sectors of a circle of radius 1, one at each of the vertices. If the angle of the sector at vertex $i$ is $\theta_i$ then we know $\sum \theta_i = \pi$ as the direction vectors of the sides of $S$ wrap half-way around the circle. Hence the areas $1^2 \cdot \theta$ of the sectors also add to $\pi$, while the areas of the rectangles sum to $1 \cdot p$, making the total area of the set in question equal to $A + p + \pi$.

When $S$ is a (hollow) polygon, it is mostly sufficient to add together the areas of rectangles that surround the edges and have width 2, and have a combined area of $2p$. At each vertex there is some overlap of the rectangles on the inside of $S$; that intersection is a kite with two right angles on the sides joined by line segments of length 1 to the vertex, at some angle $\theta$. The rectangles also leave a gap on the outside of $S$, which we will fill with a sector of a circle of radius 1 and central angle $\theta$ (as above). So the vertex region adds a net area of $\theta/2 - \sin(\theta/2)$ to the total area. Adding this up over the vertices makes the total area be

$$2p + \pi/2 - \sum \sin(\theta/2)$$

Note that in the limit as the polygon approaches a circular shape, the angles approach zero and the sum of the sines approaches $\pi/2$ so that the area approaches $2p$, which is indeed the area of the annulus between circles of radius $r+1$ and $r-1$ (since $p = 2\pi r$ in that case).

2. Let $A$ be the region in the first quadrant bounded by the line $y = x/2$, the $x$-axis, and the ellipse $x^2/9 + y^2 = 1$, and let $B$ be the region in the first quadrant bounded by the line $y = mx$, the $y$-axis, and the same ellipse $x^2/9 + y^2 = 1$. For what positive number $m$ do the regions $A$ and $B$ have the same area?

**ANSWER:** Dylan's proof is cleaner than mine because it avoids actually computing the areas. Here's his idea: Using the transformation $(x, y) \to (x', y) = (x/3, y)$ we transform the ellipse to the unit circle, transform the lines to the lines $y = 3x'/2$ and $y = 3mx'$ respectively, and transform all areas by a factor of 3; in particular the transformed regions $A'$ and $B'$ must still have equal area. But by symmetry (interchanging the roles of $x'$ and $y$) the regions will have equal area if the slopes are reciprocals, which is to say, if $m = 2/9$.

The transformation is also useful if you actually want to compute the areas; it shows the area of $B$ is $(3/2)\arctan(1/3m)$, for example, working simply with the area of a sector of a circle rather than a hard intergral.

3. Find a parameterization of the (entire) curve $y^2 = x^3 + x^2$.

**ANSWER:** Every point may be connected to the origin with a line, and each nonvertical line through the origin meets the curve in exactly one other point: if $y = mx$ then $(mx)^2 = x^3 + x^2$ so (either $x = y = 0$ or) $m^2 = x + 1$. So we parameterize the curve by the slopes of these lines: $m \to (x, y) = (m^2 - 1, m^3 - m)$.

This is very similar to the parameterization of a circle or indeed any quadratic curve: pair off each point on the curve with the slopes of the line joining that point to one fixed point (e.g. $(-1, 0)$ in the case of the unit circle). This parameterization of the circle can be used to transform any integral of trig functions into an integral of rational functions! (It's essentially the "half-angle substitution".)

4. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exists $i < j < k \leq 12$ for which $d_i, d_j, d_k$ are the lengths of the sides of an acute triangle.

**ANSWER:** We may assume the $d_i$ are labelled by size, so $1 < d_1 < d_2 < \ldots d_{12} < 12$. Then if there are no acute triangles, then in particular each triangle $\{d_{i-1}, d_i, d_{i+1}\}$ is not acute, which requires $d_{i-1}^2 + d_i^2 \geq d_{i+1}^2$. Since $d_2^2 \geq d_1^2 > 1$, it follows by induction that each $d_i^2$ is larger than the $i$th Fibonacci number

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

In particular, $d_{12}^2 > 144$, which violates the assumption that $d_{12} < 12$.

5. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

**ANSWER:** Yes! Taking the circle to be centered at $(0, 1)$, i.e. the circle $x^2 + y^2 = 2y$, we can show that a very narrow parabola $\varepsilon y = x^2$ is longer than a trip down the diameter and back. The parabola meets the circle where $\varepsilon y + y^2 = 2y$, i.e. at the bottom $(x = y = 0)$ and near the top ($y = 2 - \varepsilon$, with $x = \pm\sqrt{2\varepsilon - \varepsilon^2}$) The length of either half of the parabola may be computed by the integral

$$\int_0^{\sqrt{2\varepsilon-\varepsilon^2}} \sqrt{1 + (2x/\varepsilon)^2}\, dx$$

As $\varepsilon \to 0$ this integral does indeed approach 2 but is a little larger than 2 for some small $\varepsilon$. I don't remember if there is an elegant way to compute or at least to estimate this integral, so I cheated and evaluated it with Maple. The actual length seems to be

$$\frac{\varepsilon}{4} \log\left(\frac{\sqrt{8 - 4\varepsilon} + \sqrt{8 - 3\varepsilon}}{\sqrt{\varepsilon}}\right) + \frac{1}{2}\sqrt{16 - 14\varepsilon + 3\varepsilon^2}$$

This is larger than 2 when $0 < \varepsilon < 0.02879$; it takes a maximum value of about 2.001335 near $\varepsilon = 0.010628$. Thus the effect exists on $y = 35x^2$ but is most noticeable at $y = 94x^2$ or so.

6. A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle $ABC$ has $H$ as the intersection of the altitudes, $O$ the center of the circumscribed circle, $M$ the midpoint of $BC$, and $F$ the foot of the altitude from $A$. What is the length of $BC$?

**ANSWER:** They say it's heresy to use coordinates on a problem like this but I'm going to do it anyway. Draw axes through $O$ so that we can coordinatize the points $O = (0, 0)$, $M = (0, -5)$, $H = (-11, 0)$. "Rectangle" makes $F = (-11, -5)$. "Foot" and "midpoint" makes the line $FM$ the same as the line $BC$, so $B = (x_1, -5)$ and $C = (x_2, -5)$ for some $x_i$; "midpoint" also makes $x_1 = -x_2$. "Altitude" makes $A = (-11, y)$ for some $y$. Since the orthocenter is equidistant from the three vertices, $x_1^2 + 25 = 121 + y^2$. Finally since the altitude from $B$ passes through $H$, the vector $BH$ is perpendicular to the edge $AC$, i.e.

$$0 = (x_1 + 11, -5) \cdot (-11 + x_1, y + 5) = x_1^2 - 121 - 5(y + 5)$$

Subtracting the previous equation then yields $y^2 - 5y - 50 = 0$ and since $y = -5$ leads to a degenerate triangle, we must have $y = 10$, hence $x_1 = \pm 14$, hence $BC = 28$.

7. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a non-negative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

**ANSWER:** Let $S$ be the set of pairs $(f, v)$ consisting of a face and a vertex from that face. Note that $|S| = 60$ because each face has three vertices in it. Let $n(f, v)$ be the number on the face $f$. Then the sum of $n(f, v)$ over all the pairs in $S$ is $3 \cdot 39 = 117$ since every face is counted three times. Now, if there were *not* two faces sharing a vertex with the same integer on them, then when we sum over the five summands in this sum having a common $v$, we would have to have $n(f_1, v) + n(f_2, v) + n(f_3, v) + n(f_4, v) + n(f_5, v) \geq 0 + 1 + 2 + 3 + 4 = 10$ and hence the sum over all elements of $S$ would have to be at least $20 \cdot 10 = 120$, a contradiction.

8. Here is a "Tangram puzzle". Suppose $D$ and $S$ are a diamond and a square of equal area. (That is, they are two congruent squares whose sides lie on lines making 45-degree angles with each other.) Find a decomposition of $D$ and $S$ into polygons $D_i$ and $S_i$ respectively such that each $D_i$ is a translation of the corresponding $S_i$.

**ANSWER:** I think this is too hard to be a Putnam question but I like it anyway because it makes us think and because it reminds us of other famous problems (Banach-Tarski, Hilbert Problems, etc.)

Let us write $A \sim B$ if there is a decomposition of $A$ and $B$ into two parts each, and the corresponding parts are translation-equivalent. Thus for example, if $A$ is a $1 \times 2N$ rectangle and $B$ is a $2 \times N$ rectangle then $A \sim B$, as is obvious by simply splitting $A$ vertically down the middle. Herewith are the steps I would use to show $D \sim S$:

1) $D \sim$ a parallelogram $P$ of area 1 and sides of length 1 and $\sqrt{2}$. (Slice $D$ horizontally.)

2) $P \sim$ a $(1/\sqrt{2}) \times \sqrt{2}$ rectangle $R$. (Slice a triangle off the end of $P$.)

3) $R \sim S$. Here I'll make two cuts. First mark off a distance of 1 from the bottom-left corner, along the bottom edge. Cut from there to the top-left corner of $R$, making a line with slope $1/\sqrt{2}$. Also mark off a distance of 1 from the top-right corner, along the top edge and cut from there straight down to the previous cut. That leaves a small triangle

under the left portion of the top edge; remove that triangle, slide the other top piece up and to the left, and then tuck the removed triangle into the space on the bottom right. You can check that all the lengths and angles match.

In fact in this way we can show all unit squares, regardless of orientation, are equivalent!

9. Show that there exist tetrahedra of arbitrarily large volume whose vertices lie at integer points and which do not contain any other lattice points (neither on their boundaries nor in their interiors). (Thus there is no "Pick's Theorem in 3 dimensions.)

**ANSWER:** I should have said "neither on their boundaries nor in their interiors, *except for the four vertices*", of course.

Simply consider the tetrahedron (i.e. simplex)

$$\{x_0P_0 + x_1P_1 + x_2P_2 + x_3P_3;\ \text{each } x_i \geq 0 \text{ and } x_0 + x_1 + x_2 + x_3 = 1\}$$

which is the convex hull of the points $P_0 = (0,0,0), P_1 = (1,0,0), P_2 = (0,1,0)$, and $P_3 = (1,1,n)$. The volume is base$\times$height$/3 = n/6$ and yet the points in it are $(x_1+x_3, x_2+x_3, nx_3)$ for some $x_i \in [0,1]$; in order to be a lattice point we must have $x_1 + x_3 = 0$ or 1, but then either $x_1 = x_3 = 0$ or $x_2 = x_4 = 0$. Similarly either $x_2 = x_3 = 0$ or $x_1 = x_4 = 0$. In each of the four combinations, three $x_i$ must vanish leaving the remaining $x_i = 1$, i.e. we are at one of the vertices of the tetrahedron.

10. There was a time when national leaders were clear-thinking and clever. Prove this theorem which is attributed to Napoleon: Given a triangle, erect equilateral triangles on all its edges. Show that the centers of the three equilateral triangles form themselves the vertices of an equilateral triangle.

**ANSWER:** View the plane as a Euclidean vector space: three points $X, Y, Z$ form an equilateral triangle iff the vectors $XY, YZ$, and $ZX$ are 120-degree rotations of each other. Then use William's proof, in which this plane is identified with the complex numbers, so that 120-degree rotations are accomplished by multiplying by cube roots of unity (or their negatives). Let $\omega = (-1 + i\sqrt{3})/2$, a cube root of unity; then a triangle $XYZ$ (with the vertices listed in counterclockwise order) is equilateral iff $YZ = \omega XY$ and $ZX = \omega YZ = \omega^2 XY$. So if we erect equilateral triangles on the sides of $ABC$ we construct new points $D, E, F$ with

$$\begin{array}{lll} AD = \omega BA, & BE = \omega CB, & CF = \omega AC \\ DB = \omega^2 BA, & EC = \omega^2 CB, & FA = \omega^2 AC \end{array}$$

Of course we can recover points from vectors, identifying $A$ with $OA$, etc. Then $D = A + AD = A + \omega BA$ and likewise for $E$ and $F$. Then the center of the first equilateral triangle is $(A + B + D)/3 = A + ((-1 + \omega)/3)BA = ((2 + \omega)/3)A + ((1 - \omega)/3)B$ and likewise for the other two equilateral triangles. From this we compute the vector joining the first two triangles' centers as

$$\frac{-2-\omega}{3}A + \frac{1+2\omega}{3}B + \frac{1-\omega}{3}C$$

When we multiply this by $\omega$, and use the identity $\omega^2 = -1 - \omega$, we get

$$\frac{1-\omega}{3}A + \frac{-2-\omega}{3}B + \frac{1+2\omega}{3}C$$

which by the preceding analysis is also the vector joining the second and third triangles' centers. In the same way the vector joining the centers of the 3rd and 1st equilateral triangles is $\omega$ times this last displayed vector. So the vectors joining the three centers do indeed form an equilateral triangle.

I don't know Napolean's proof but it surely was not this one. Incidentally I have been told that US President Garfield came up with his own proof of the Pythagorean Theorem.