I have a few comments about this week's problems.

1. Find the value of b for which

$$\lim_{x\to 0}\left(\frac{1/(3x+b)-2}{x}\right)$$

exists; then, find the value of this limit.

ANSWER: This is exactly the limit that shows up when computing the derivative of $f(t) = 2/(6t+1)$ at $t = 0$, so you may compute that derivative and substitute in $t = 0$.

2. Evaluate this integral:

$$\int_{x=0}^{x=1} \frac{x^4(1-x)^4}{1+x^2}\,dx$$

ANSWER: I mentioned that the long division can be made to go a little faster using the fact that $x^4 = (x^2+1)(x^2-1)+1$; for what remains you could also use the fact that $(1-x)^2 = (1+x^2)+2x$. However you do it, you discover the integral is $22/7-\pi$, which is especially cool since it is obvious that the integrand is positive, so we learn that $22/7 > \pi$. The next "great" rational approximation to π (found in the Continued Fractions expansion) is $355/113$, so I challenge you to find a sum of squares of rational functions whose integral equals $\pi - 355/113$!

3. Show that this improper integral converges:

$$\int_0^\infty \sin(x)\,\sin(x^2)\,dx$$

ANSWER: My solutions to this problem involve several ingredients: directly rewriting the integral using substitutions and Integration By Parts; using trig identities; estimating values of oscillating integrals carefully; and looking for cancellations of similar terms. I will illustrate with a couple of proofs; you can mix and match the ideas to get somewhat different proofs.

The trig identity I have in mind is writes products of trig functions as sums of them. For example, we may rewrite the original integrand as

$$(\tfrac{1}{2})\cos(x^2-x) - (\tfrac{1}{2})\cos(x^2+x)$$

Actually I prefer to complete the square and write this as

$$\begin{gathered}(\tfrac{1}{2})\cos((x-\tfrac{1}{2})^2-\tfrac{1}{4}) - (\tfrac{1}{2})\cos((x+\tfrac{1}{2})^2-\tfrac{1}{4}) = \\ a\cos((x-\tfrac{1}{2})^2) - b\sin((x-\tfrac{1}{2})^2) - a\cos((x+\tfrac{1}{2})^2) + b\sin((x+\tfrac{1}{2})^2),\end{gathered}$$

where $a = \frac{1}{2}\cos(\frac{1}{4}),\ b = \frac{1}{2}\sin(\frac{1}{4})$. So integrating our integrand over an interval really only requires integrating $\cos(y^2)$ and $\sin(y^2)$ over two (different) intervals. In particular the integral over $[1, T]$ will be

$$\begin{aligned} &a\int_{\frac{1}{2}}^{T-\frac{1}{2}} \cos(y^2)\,dy - b\int_{\frac{1}{2}}^{T-\frac{1}{2}} \sin(y^2)\,dy - a\int_{3/2}^{T+\frac{1}{2}} \cos(y^2)\,dy + b\int_{3/2}^{T+\frac{1}{2}} \sin(y^2)\,dy \\ &= a\int_{\frac{1}{2}}^{3/2} \cos(y^2)\,dy - b\int_{\frac{1}{2}}^{3/2} \sin(y^2)\,dy - a\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \cos(y^2)\,dy - b\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \sin(y^2)\,dy, \end{aligned}$$

that is, there's a lot of cancellation except near the ends of the interval. Since the integral over $[1, \infty)$ is said to converge iff the integrals over $[1, T]$ converge as $T \to \infty$, we see we need only show that the integrals of $\cos(y^2)$ and $\sin(y^2)$ over intervals of length 1 have definite limits as the interval moves further to the right. Indeed we will see that their limit is zero, so that the original improper integral converges to

$$a\int_{1/2}^{3/2} \cos(y^2) - b\int_{1/2}^{3/2} \sin(y^2),$$

a fact which can be verified numerically.

So now we need to estimate these integrals over short windows. I think this is best done with a change of variables. Let $y = \sqrt{u}$; then

$$\int_A^B \cos(y^2)\,dy = \frac{1}{2}\int_{A^2}^{B^2} \cos(u)/\sqrt{u}\,du$$

and the integral of $\sin(y^2)$ will be similar. In the cases of interest to us, the length of the left interval is only 1 but then the length of the right interval is $A + B$, which will be large, including many periods of the cosine. If the cosine crosses from positive to negative at a point p in this interval, then it is positive on $[p - \pi, p]$, and the integrand there lies between $\cos(u)/\sqrt{p}$ and $\cos(u)/\sqrt{p - \pi}$, so that its integral lies between $2/\sqrt{p}$ and $2/\sqrt{p - \pi}$. Likewise the integral over $[p, p + \pi]$ lies between $-2/\sqrt{p}$ and $-2/\sqrt{p + \pi}$. Thus the integral over the entire period is positive but less than $2/\sqrt{p - \pi} - 2/\sqrt{p + \pi}$. The sum over multiple consecutive periods inside $[A^2, B^2]$ then telescopes and is no larger than $2/A - 2/B$. We have already seen that the integrals over the incomplete periods at the ends of the interval are themselves no larger in magnitude than $2/A$, and thus the entire integral $\int_{A^2}^{B^2} \cos(u)du/\sqrt{u}$ is no larger than $6/A$. Hence as the interval moves to the right, the value of the integral tends to 0. (Note that the telescoping sums here also show directly that $\int_A^\infty \cos(y^2)\,dy$ converges, if you wish to construct a proof along those lines.)

Here is a different approach to the problem. First verify this identity:

$$\begin{aligned} \sin(x)\sin(x^2) = &\frac{d}{dx}\left(-\frac{\sin(x)\cos(x^2)}{2x} + \frac{\cos(x)\sin(x^2)}{4x^2}\right) \\ &-\frac{\sin(x)\cos(x^2)}{2x^2} + \frac{\sin(x)\sin(x^2)}{4x^2} + \frac{\cos(x)\sin(x^2)}{2x^3} \end{aligned}$$

The Fundamental Theorem of Calculus then allows us to compute $\int_1^T \sin(x)\sin(x^2)\,dx$ as a sum of a couple of constants, two functions of T which are of magnitudes $O(1/T)$ and $O(1/T^2)$ respectively, and three integrals bounded by multiples of $\int_1^T dx/x^r$ with $r = 2$ or $r = 3$; as $T \to \infty$ all these converges, as then does our original improper integral.

Of course the hard part here is to discover this identity in the first place but it can be discovered naturally as one attempts to evaluate the antiderivative. First use the substitution $x = \sqrt{u}$ as in the other proof to rewrite this as

$$\frac{1}{2}\int_1^{T^2} \frac{\sin(\sqrt{u})}{\sqrt{u}} \sin(u)\,du$$

which we wish to evaluate by the Fundamental Theorem of Calculus. We find an antiderivative by using integration by parts twice. Without worrying about explicit coefficients we can handle more general integrals: let V be the vector space of functions spanned by $\sin(x)$ and $\cos(x)$. Then we observe that for any f and g in V and any $r > 0$ we have

$$\int \frac{f(\sqrt{u})}{u^r} g(u)\,du = \frac{f(\sqrt{u})}{u^r} h_1(u) + \int \frac{f(\sqrt{u})}{u^{r+1}} h_2(u)\,du + \int \frac{h_3(\sqrt{u})}{u^{r+\frac{1}{2}}} h_4(u)\,du$$

for some other functions $h_i \in V$. So starting with an integrand of this type with $r = 1/2$, we use this expansion twice to get several such integrals having $r = 3/2$ and $r = 2$, together with terms of the form

$$\frac{h_5(\sqrt{u})}{u^{\frac{1}{2}}} h_6(u) + \frac{h_7(\sqrt{u})}{u} h_8(u)$$

I merely repackaged the resulting calculations in terms of x and the Product Rule to obtain the mysterious identity.

4. Evaluate

$$\sum_{n=2}^{\infty} \log\left(\frac{n^3 - 1}{n^3 + 1}\right)$$

ANSWER: The trick is that $n^3+1 = (n+1)(n^2-n+1)$ and $n^3-1 = (n-1)(n^2+n+1) = (n-1)((n+1)^2-(n+1)+1)$ so the properties of logarithm produce two telescoping sums.

Please note that (IMHO) a Putnam-level solution requires discussing the partial sums and then their limit, rather than simply writing a bunch of series that end with "$+\ldots$". (If you're unclear why that would be insufficient come talk to me.)

5. Evaluate

$$\int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x}\,dx$$

ANSWER: More generally

$$\int_0^\infty \frac{\arctan(Kx) - \arctan(x)}{x}\,dx$$

$$\begin{aligned}
&= \int_0^\infty \int_1^K \frac{1}{1+(xy)^2}\,dy\,dx \\
&= \int_1^K \int_0^\infty \frac{1}{1+(xy)^2}\,dx\,dy \\
&= \int_1^K \frac{1}{y} \int_0^\infty \frac{1}{1+z^2}\,dz\,dy \\
&= (\pi/2)\log(K).
\end{aligned}$$

It is appropriate to study the nature of the convergence of the improper integral to justify the interchange of integrals. For small x, $\arctan(x) \approx x$, so the integrand is roughly $(K-1)$. For large x, use the substitution $u = 1/x$ and recall that $\arctan(x) + \arctan(1/x) = \pi/2$.

6. Suppose that f is differentiable and that $f'(x)$ is strictly increasing on $[0, \infty)$. Suppose further that $f(0) = 0$. Prove that $g(x) = f(x)/x$ is strictly increasing on $(0, \infty)$

7. Let f be a three times differentiable function (defined on $\mathbf{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

ANSWER: The function $g(x) = e^{x/2}f(x)$ is also thrice-differentiable and has 5 zeros, so its k-th derivative has $5-k$ zeros ($k = 0, 1, 2, 3, 4$), and so $8e^{(-x/2)}g'''(x)$ has two zeros. That comes out to $f + 6f' + 12f'' + 8f'''$.

8. Find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

among positive numbers x, y, z subject to $x + y + z = 1$.

ANSWER: I didn't do anything slick: simply think of this as a function of x and y on the open triangle $x > 0, y > 0, x + y < 1$. The only point where both partials vanish is at $(1/3, 1/3)$, where the second derivative is positive-definite, so that point is a local minimum. The fact that it's a global minimum can be proved with a compactness argument (and by noting that the function tends to infinite all along the boundary of the triangle).

9. Prove that $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$

ANSWER: Try a proof by induction: if you already know $n! < \frac{(n+1)^n}{2^n}$, then

$$(n+1)! < \frac{(n+1)^{n+1}}{2^n} = \frac{(n+2)^{n+1}}{2^{n+1}} \cdot 2/\left(\frac{n+2}{n+1}\right)^{n+1}.$$

But that denominator is $X = \left(1 + \frac{1}{n+1}\right)^{n+1}$ which increases to e as $n \to \infty$ and since it's more than 2 already when $n = 1$, the fraction $2/X$ is less than one, giving us the inductive step.

Incidentally, in a problem like this when you need to understand the size of $n!$, it can be helpful to recall Stirling's approximation, most accurately stated as

$$\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1$$

10. Show that

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$$

for all positive real numbers a, b, c

ANSWER: