Practice Proof for Dr. Rusin's Fall 2017 Putnam Practice

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For a sequence x_i of positive real numbers, let $f(x_i) = \sum_{n=1}^{\infty} x_n$ and let $g(x_i) = \sum_{n=1}^{\infty}(x_n)^2$. If $f(x_i) = A$ for some positive real number A, then what are the possible values of $g(x_i)$?

Answer: $0 < g(x_i) < A^2$.

1 0 Is Greatest Lower Bound

$g(x_i)$ is a sum of the squares of positive numbers, so clearly $g(x_i) > 0$ for any x_i. To show that 0 is the greatest lower bound, consider the following observations.

$$\frac{x}{2} + \frac{x}{2} = x$$

$$(\frac{x}{2})^2 + (\frac{x}{2})^2 = \frac{x^2}{2}$$

Let $g(x_i) = T$. For any sequence x_i with $f(x_i) = M$, let $x_i' = \frac{x_1}{2}, \frac{x_1}{2}, \frac{x_2}{2}, \frac{x_2}{2}, \frac{x_3}{2}, \ldots$. Then

$$f(y_i) = f(x_i) = A$$

$$g(x_i') = (\frac{x_1}{2})^2 + (\frac{x_1}{2})^2 + (\frac{x_2}{2})^2 + (\frac{x_2}{2})^2 + \ldots$$

$$g(x_i') = \sum_{n=1}^{\infty}(\frac{x_n}{2})^2 + (\frac{x_n}{2})^2 = \sum_{n=1}^{\infty}\frac{x_n^2}{2} = T/2$$

Similarly, $g(x_i'') = \frac{T}{4}$ and $g(x_i''') = \frac{T}{8}$.

In this way, given a sequence x_i with $g(x_i) = T$, one can construct a sequence y_i with $g(y_i) = \frac{T}{2^n}$ for arbitrary $n \in \mathbb{N}$. The greatest lower bound of the sequence $\frac{T}{2^i}$ is 0, completing the proof of the claim.

2 A^2 Is an Upper Bound

$$f(x_i) = A$$

$$g(x_i) = (x_1^2 + x_2^2 + x_3^2 + \ldots) < (x_1 + x_2 + x_3 + \ldots)^2 = A^2$$
$$g(x_i) < A^2$$

3 A^2 Is Least Upper Bound

Let $\epsilon \in \mathbb{R}, \epsilon > 0$. Let $f(q_i) = \epsilon, g(q_i) = Q$. Let $y_i = (A\epsilon), q_1, q_2, q_3, \dots$. Then

$$f(y_i) = (A - \epsilon) + f(q_i) = (a - \epsilon) + \epsilon = A$$

$$\begin{aligned} g(y_i) &= (A - \epsilon)^2 + g(q_i) \\ &= A^2 - 2\epsilon + \epsilon^2 + Q \end{aligned}$$

ϵ^2 and Q are both positive, so $|A^2 - g(y_i)| < 2\epsilon$.

So, choosing an arbitraily small ϵ causes $g(y_i)$ to be arbitrarily close to A^2, i.e. A^2 is the least upper bound.

4 Continuity

So far, bounds on $g(x_i)$ have been proven, but we have not shown that $g(x_i)$ can take *any* value in the interval $(0, A^2)$. Let $f(x_i) = f(y_i) = A$. Without loss of generality, let $g(x_i) > g(y_i)$. Now, we define two useful functions. Let

$$h(t) = g(tx_1+(1-t)y_1, tx_2+(1-t)y_2+\dots) = (tx_1+(1-t)y_1)^2+(tx_2+(1-t)y_2)^2+\dots$$

$$j(t) = f(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2 + \dots)$$

These functions have some nice properties. First,
$j(t) = t*f(x_i)+(1-t)*f(y_i) = t*A+(t-1)*A = A(t+(1-t)) = A$ for all t. Second, $h(t)$ is continuous (because it is a sum of continuos functions of t) and it has a maximum of $g(x_i)$ and a minimum of $g(y_i)$. Furthermore, each value of t with $0 < t < 1$ corresponds to the sequence $\{tx_1 + (1-t)y_1, tx_2 + (1-t)y_2 + \dots\}$.

So, for any r with $g(y_i) < r < g(x_i)$ there must exist a t such that $h(t) = r$, and that value of t corresponds to a sequence z_i such that $f(z_i) = A$ and $g(z_i) = r$. This proves that $g(x_i)$ can take on any value within its bounds.