Putnam Exam: meet Linear Algebra!

1. Do there exist square matrices A and B with $AB - BA = I$?

2. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$,

3. Suppose A, B, C, D are $n \times n$ matrices, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Prove that $A^tD - C^tB = I$.

4. Suppose A is an $n \times n$ matrix for which

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|$$

for all $i = 1, 2, \ldots n$. Prove that A is invertible.

5. An $n \times n$ matrix M has the feature that $M_{ij} = M_{kl}$ whenever $i - j \equiv k - l \pmod{n}$. Compute the eigenvalues of M (in terms of the entries in the first row of M: $M_{11}, M_{12}, \ldots, M_{1n}$.)

6. Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all $+1$. Show that $ab \leq n$.

7. Let $M_3(\mathbf{C})$ denote the collection of 3×3 matrices whose entries are complex numbers. Suppose $A, B \in M_3(\mathbf{C})$ with $B \neq 0$ and $AB = 0$. Prove that there exists a nonzero $D \in M_3(\mathbf{C})$ such that

$$AD = DA = 0$$

(Here 0 means the 3×3 matrix filled with zeros.)

8. Let S be a set of 2×2 matrices with complex entries, and let T be the subset of S consisting of those matrices in S whose eigenvalues are ± 1 (i.e. the eigenvalues of each such matrix are either $\{1, 1\}$, $\{-1, -1\}$, or $\{1, -1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A and B (possibly equal) in S such that AB is not in S.

9. Let Z be the set of points in $\mathbf{R}^n$ whose coordinates are all 0's and 1's. (That is, Z is the set of vertices of the unit hypercube in $\mathbf{R}^n$. For a fixed integer $k = 1, 2, \ldots, n$, what is the largest number $N(k)$ of elements of Z which can be found in a subspace of $\mathbf{R}^n$ of dimension k?

10. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Since we worked on Number Theory problems last week, I invite you to work on the following

11. Let $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, ... be the Fibonacci sequence. Compute the determinant of the matrix

$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

12. Suppose n is a positive integer. How many ordered pairs (x, y) of positive integers are there with

$$\frac{xy}{x+y} = n \ ?$$

13. Prove that there are no positive integers x and y for which $x^2 + 3xy - 2y^2 = 122$.

14. Define integers a_k by $a_1 = 2$ and then, for $k > 1$, let $a_{k+1} = 2^{a_k}$. Show that for every integer $n > 1$, $a_n \equiv a_{n-1} \pmod{n}$.