Putnam practice – Oct 11, 2012

1. Suppose the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y, 2x + 2y + 1, 3x + 3y + 1$ are all perfect squares.

2. Suppose $A \in M_n(\mathbf{C})$ has rank r, where $1 \le r \le n-1$ and $n > 1$. Show that there exist matrices $B \in M_{n,r}(\mathbf{C})$ and $C \in M_{r,n}(\mathbf{C})$ with $A = BC$.

3. Suppose $A, B \in M_4(\mathbf{R})$ commute, and $\det(A^2 + AB + B^2) = 0$. Prove that

$$\det(A + B) + 3\det(A - B) = 6\det(A) + 6\det(B).$$

4. (Problem 2009-A-3). Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos(1), \cos(2), \ldots, \cos(n^2)$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

(The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

5. (Problem 2008-A-2). Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?