I solved problem B-1 on the 2010 Putnam exam with the following observation:

$$0 \leq \sum(-3a_i + 2a_i^2)^2 = \sum(9a_i^2 - 12a_i^3 + 4a_i^4) = 9 \cdot 2 - 12 \cdot 3 + 4 \cdot 4 = -2$$

This contradiction shows we cannot even have $\sum a_i^n = n$ for $n = 2, 3$ and 4.

Let me show where I got this funny contradiction; that's more educational. For any finite collection of numbers a_i we can make a *Vandermonde matrix* $M_{ij} = a_j^i$. There's a lot you can say about them; for example, the determinant is the product of all the a_i and their differences! But for our purposes the key thing is that when you compute the matrix $M\,M^t$, then entries are

$$(M\,M^t)_{ij} = \sum_k a_k^i a_k^j = \sum_k a_k^{i+j}$$

so that $M\,M^t$ is "striped" by the sums of the different powers of the a_k.

Now, for any matrix M, the product $M\,M^t$ is special — it's "non-negative semidefinite", which means that whenever you multiply by a column vector v on the right and its transpose v^t on the left, you get something non-negative. Indeed, $v^t\,M\,M^t v = (M^t v)^t\,(M^t v)$ is the square of the magnitude of $M^t v$. On the other hand, our "striped" matrix doesn't have this property. (A matrix will only have it if the determinants of all square submatrices nestled along the main diagonal are positive, and for example the determinant of $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ is negative.) In fact the way I obtained my starting proof is by using the negativity of this determinant to conclude the quadratic form is negative at the vector $v^t = (-3, 2)$, and then just expanded everything in sight.

So a bit more mathematical muscle has revealed more than the non-existence of the presumed sequence of real numbers. It's also shown what more generally can be said about the relationships between the sums of the different powers of the a_i : those sums have to give a "striped" matrix which is non-negative semidefinite. For example if $\sum a_i^2 = 2$ and $\sum a_i^3 = 3$ then we would need $\sum a_i^4$ to be at least 4.5. (But at that point, solutions do exist, for example $1.458193848, 0.7915868512, 0.02642730622$ fit the bill.)

Dean gave a nice Cauchy-Schwarz proof of B1: let $u = (a_1, a_2, a_3, \ldots)$ and $v = (a_1^2, a_2^2, a_3^2, \ldots)$. Then since $|u \cdot v|^2 \leq ||u||^2 \cdot ||v||^2$ we conclude $(\sum a_i^3)^2 \leq (\sum a_i^2)(\sum a_i^4)$, which in the Putnam problem would lead to the contradiction $9 \leq 2 \cdot 4$.

Dylan noted we can get a contradiction from less information: $(\sum a_i^2)^2 = (\sum a_i^4) + 2\sum_{i<j}(a_i a_j)^2$ would contradict $\sum a_i^2 = 2$ and $\sum a_i^4 = 4$ unless all the pairwise products are zero, which to say at most one of the a_i is nonzero (in which case that one must equal ± 2, but that contradicts *any* of the other given sums!)