1. A rectangle $HOMF$ has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of its altitudes, O as the center of its circumscribed circle, M as the midpoints of BC, and F as the foot of the altitude from A. What is the length of BC?

OK, I will commit the heresy of using coordinates. Draw coordinates around any triangle ABC so that $B = (0, 0)$, $C = (1, 0)$, and $A = (a, b)$ for some $a, b \in \mathbf{R}$. Then clearly $M = (1/2, 0)$ and $F = (a, 0)$, and of course $BC = 1$ in these units. The circumscribed circle is equidistant from all of A, B, C; in particular it must lie on the line $x = 1/2$ in order to be equidistant from B and C. Its y coordinate then satisfies

$$(1/2)^2 + y^2 = (1/2 - a)^2 + (y - b)^2$$

i.e. $y = (a^2 + b^2 - a)/(2b)$. Finally we can locate F by dropping a perpendicular from C to AB. The line AB has slope b/a so the perpendicular through C will have equation $y = (-a/b)(x-1)$; this line meets the other altitude $x = a$ at the point $H = (a, (a-a^2)/b)$.

So now we draw an important conclusion: the quadrilateral $HOMF$ was given to be a rectangle, and MF is horizontal in this picture; thus HO is as well, meaning $(a - a^2)/b = (a^2 + b^2 - a)/(2b)$. This forces $b^2 = 3(a - a^2)$, which in turn rewrites the coordinates of $O = (1/2, \sqrt{(a - a^2)/3})$ and $H = (a, \sqrt{(a - a^2)/3})$. The lengths of the sides of the rectangle are, in these units, $HO = |a - (1/2)|$ and $OM = \sqrt{(a - a^2)/3}$, so we can finally deduce the value of a (and then b):

$$(11/5)^2 = (HO/OM)^2 = (a-1/2)^2/((a-a^2)/3) = 3((a^2-a)+(1/4))/(a-a^2) = (3/4)/(a-a^2)-3$$

Thus $a - a^2 = (3/4)/(196/25) = 75/784$ and so $a = 3/28$ (or $25/28$). In that case $HO = |a - 1/2| = 11/28$, which is to say our units are off by a factor of precisely 28 from the original units. Since in our picture $BC = 1$, this means $BC = 28$ in the original diagram. Indeed, we can now locate all the points and lines in the original picture:

$$A = (3, 15), B = (0, 0), C = (28, 0), F = (3, 0), M = (14, 0), O = (14, 5), H = (3, 5)$$

5. The longest arc of a parabola which fits inside the unit circle has length approximately 4.00167 (and is certainly longer than 4).

Taking the circle to be the set of points $x^2 + y^2 = 1$ and the parabola to be the points where $y = x^2/K - 1$, (for some $K > 0$ to be determined) we see that the parabola touches the circle at $(0, -1)$ and two points near the top. For K near zero this parabola is very tall and thin and has length nearly equal to 4. Specifically we can find the points of intersection to be at $(\pm\sqrt{2K - K^2}, 1 - K)$ and so we compute the length of the parabolic arc as in Calculus classes: it's

$$L = \int_{x=-\sqrt{2K-K^2}}^{x=+\sqrt{2K-K^2}} \sqrt{1 + (2x/K)^2}\, dx$$

It's perhaps easier to rewrite the integral using $u = x/\sqrt{K}$:

$$L = \int_{u=\sqrt{2-K}}^{u=\sqrt{2-K}} \sqrt{K + (4u^2)}\, du$$

Our task is then to estimate the values of this integral L as K decreases to zero.

It is actually possible to give a closed-form expression for this integral using Calculus techniques; the cleanest form I know is

$$\sqrt{(2-K)(8-3K)} + (K/2)\log(\sqrt{8-3K} + \sqrt{8-4K}) - (K/4)\log(K)$$

It's not hard to derive the expected result that the limit as $K \to 0^+$ is 4. To see *how* the function approaches this limit, note that the two main terms have a well-defined positive slope at $K = 0$, but the last term has infinite derivative at $K = 0$, that is, for small positive increases in K away from $K = 0$ the values of $-K\log(K)$ increase faster than any linear function of K. Thus in particular this increase overwhelms the (negative) first derivative of the other terms in the expression for the integral, meaning that for small values of K, this expression increases. (Numerically I find the maximum to occur when K is near 0.01063.)

With the precise result in hand it's probably easy to find approximations to the length-integral that are sufficient to prove the integral can exceed 4.0 but I haven't worked hard on that.