Problem 6 of week 6 is (I think) fairly simple in idea but a little subtle in execution, so I will take the liberty of providing an answer for this one.

I think it pays to prove something a little stronger because then the induction is easier. We shall prove the following, which implies the desired result.

Theorem. *For every positive integer n there exists an integer $k = k(n)$ for which*

$$a_k \equiv a_{k+1} \equiv a_{k+2} \ldots \ (\ mod \ n \ )$$

The smallest such value of k satisfies $k(n) < n$.

The proof is by "strong" induction on n, that is, we will prove this result for one value of n, assuming it is true for *all* lower values of n. Obviously the result holds for $n = 1$.

If $n = 2^e$ then as long as $a_k > e$ we have $a_{k+1} = 2^{a_k}$ divisible by $2^e = n$; that is, all terms of the sequence are congruent to zero. This is certainly true for $k \geq e$, and in particular holds for all $k \geq n$. (Indeed you will probably never see a sloppier inequality than one which concludes e.g. for $n = 2^3$ that $2^{2^{2^{2^{2^{2^{2^{2}}}}}}} > 3$!)

If n is odd, then Euler's Theorem asserts that $2^N - 1$ is a multiple of n if N is a multiple of $\varphi(n)$. Since $\varphi(n) < n$ we may assume by induction that a bound $k(\varphi(n)) < \varphi(n)$ has been found, past which each $a_k - a_{k-1}$ is indeed a multiple of $\varphi(n)$. Thus $2^{a_k - a_{k-1}} - 1$ is a multiple of n, and if we multiply by $2^{a_{k-1}}$ we see $a_{k+1} - a_k = 2^{a_k} - 2^{a_{k-1}}$ is also a multiple of n. That is, we have $k(n) \leq k(\varphi(n)) + 1$.

Finally, note that if $n = ab$ where a and b are coprime, then the desired congruences hold modulo n iff they hold modulo a and b; thus we have $k(ab) = \max(k(a), k(b)) < \max(a, b) < n$. In particular we can use this where a and b are respectively the 2-primary part of n and the odd part of n.