Problems for 2018 University of Texas Putnam Prep Session, week 3 (Oct 4)
Some problems from Geometry

1. Suppose we have a triangle ABC and two more points E, F on sides AC and AB respectively. We are given angle measures in degrees: $ABC = BCA = 80$, $ACF = 30$, and $ABE = 20$. Compute the angle measure x of angle BEF.

2. There is a circle of grass with radius r. We want to let a sheep eat the grass from that circle by attaching the sheep's leash on the edge of the circle. What must be the length of the leash for the sheep to eat exactly half of the grass?

3. What is the maximum number of rational points that can be on a circle in R^2 whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

4. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

5. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

6. For what point P inside a convex quadrilateral $ABCD$ is $PA + PB + PC + PD$ minimized?

7. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

8. Show that there exist tetrahedra of arbitrarily large volume whose vertices lie at integer points and which do not contain any other lattice points (neither on their boundaries nor in their interiors).

9. Let R and S be two squares in the plane of equal side length. Find a decomposition of R into pieces which can be moved by *translation alone* to give a decomposition of S. Try to find a decomposition with as few pieces as possible.

SPECIAL BONUS IRRITANT:

The points A, B and C are collinear (and in that order). $AB = 2$ and $BC = 7$. Choose a point D such that angle ABD measures exactly $11\pi/30$ (66°) and angle ACD measures $\pi/9$ (20°). Prove that angle ADB meaures $\pi/5$ (36°).