A new month of Putnam prep problems – Nov 2 2010

1 (Geometry) Show that four points on the parabola $y = x^2$, say $(a, a^2), \dots (d, d^2)$ (with a, b, c, d distinct) are concylic if and only if $a + b + c + d = 0$.

2. The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

3. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

4. Express $1/[(x + 1)(x + 2)(x + 3)]$ in the form $A/(x + 1) + B/(x + 2) + C/(x + 3)$.

5. If $P(x)$ is a polynomial of degree n such that $P(k) = k/(k + 1)$ for $k = 0, \dots n$, determine $P(n + 1)$.

6. Show that each number in the sequence 49, 4489, 444889, 44448889, ... is a perfect square.

7. Prove that $(2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}$ is rational.