Problems for 2018 University of Texas Putnam Prep Session, week 8 (Nov 8)
Functional Equations

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ satisfies the equation

$$f(x+1) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

for all x. Prove that f is periodic.

2. Let G be a set of functions $f : \mathbf{R} \to \mathbf{R}$ of the form $f(x) = ax + b$ where $a \neq 0$ and b are real numbers, such that
(a) if $f, g \in G$ then $f \circ g \in G$
(b) If $f \in G$ then $f^{-1} \in G$
(c) For each $f \in G$ there is a point $x_f \in \mathbf{R}$ where $f(x_f) = x_f$.
Prove that there exists a point $x_* \in \mathbf{R}$ with $f(x_*) = x_*$ for all $f \in G$.

3. Let $P_1(x) = x^2 - 2$ and for $j > 1$ let $P_j(x) = P_1(P_{j-1}(x))$. Show that for each positive integer n, the solutions of the equation $P_n(x) = x$ are real and distinct.

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is $(n+1)$-times differentiable and that $f^{(k)}(a) = f^{(k)}(b) = 0$ for every non-negative $k < n$. Prove that there is a point $c \in [a, b]$ where $f^{(n+1)}(c) = f(c)$.

5. (Putnam 1959) Find all functions $f : \mathbf{C} \to \mathbf{C}$ for which

$$f(z) + zf(1-z) = 1 + z$$

for all points $z \in \mathbf{C}$.

6. Suppose $f(m, n)$ is a real-valued function defined for all pairs of natural numbers, such that
(a) $f(0, n) = n + 1$
(b) $f(m+1, 0) = f(m, 1)$
(c) $f(m+1, n+1) = f(m,\ f(m+1, n))$
for all $m, n \geq 0$. Compute $f(4, 2018)$.

7. Find all differentiable functions $f : \mathbf{R} \to \mathbf{R}$ for which

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

for every $x, y \in \mathbf{R}$.

8. Find all analytic functions $f : [0, 1) \to \mathbf{R}$ for which

$$f(x) + f(x^2) = x$$

for every $x \in [0, 1)$. What if the domain of the function must include the point 1 as well?

9. (Putnam 1972) The series

$$\sum_{k\geq 0} \frac{x^k(x-1)^{2k}}{k!}$$

may be expanded into a power series $\sum_{i\geq 0} a_i x^i$. Show that for no value of i do we have $a_{i-1} = a_i = a_{i+1} = 0$.

10. Show that there is no function $f : \mathbf{R} \to \mathbf{R}$ which satisfies

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

for all real x, y.

11. (Putnam 1988) Prove that there is a unique function $f : [0,\infty) \to [0,\infty)$ such that $f(f(x)) = 6x - f(x)$ for all $x > 0$.

12. (Putnam 1997) Suppose $g(x) : \mathbf{R} \to [0,\infty)$ is given. Show that any solution to the differential equation

$$f''(x) + f(x) = -x\, g(x)\, f'(x)$$

is bounded.

13. (Putnam 1979) First, find a solution (not identically zero) of the homogeneous linear differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0.$$

(Intelligent guessing of the form of a solution may be helpful.) Then let $y = f(x)$ be the solution of the *inhomogeneous* differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 6(6x + 1)$$

which has $f(0) = 1$ and $(f(-1) - 2)(f(1) - 6) = 1$. Find integers a, b, c such that

$$(f(-2) - a)(f(2) - b) = c$$

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