UT Putnam Prep 2017-11-20 — "All of the above"

1. Show that if a and b are positive then for every positive integer n,

$$(n-1)a^n + b^n \geq na^{n-1}b$$

2. Prove that in any group of 6 people there are either 3 mutual friends or 3 mutual strangers.

3. Do there exist 2017 consecutive integers each of which is divisible by a square (other than 1)?

4. Find all positive rational solutions of $x^{x+y} = (x+y)^y$.

5. Suppose F is a polynomial with integer coefficients such that $F(x) = 5$ for four distinct integers $x = x_1, x_2, x_3, x_4$. Show that $F(x) \neq 8$ for any integer x.

6. Suppose G is a group containing two elements a, b for which

$$ab = ba^{-1} \qquad \text{and} \qquad ba = ab^{-1}$$

Show that $a^4 = b^4 = e$ (the identity element of G.

7. Sum the infinite series

$$\frac{3}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \frac{9}{4 \times 5 \times 6} + \cdots$$

8. Does $\sum \frac{\sin(n)}{n}$ converge? (Of course this refers to the sine of n radians, and of course you must prove your answer.)

9. Suppose $f : R \to R$ is continuous and satisfies $f(a)f(b) = f(c)$ whenever $a^2 + b^2 = c^2$. Prove that $f(x) = A^{x^2}$ for some real number A.

10. Compute $\lim_{x \to \infty} x \int_0^x e^{t^2 - x^2}\, dt$.

11. If $a, b, c > 0$ and $(1+a)(1+b)(1+c) = 8$, prove $abc \leq 1$.

12. Find the area of the convex octagon that is inscribed in a circle and has four consecutive sides of length 3 and four consecutive sides of length 2. Your answer should be of the form $r + s\sqrt{t}$ where r, s, t are integers.

13. If two altitudes of a tetrahedon are coplanar, the edge joining the two vertices from which these altitudes issue is orthogonal to the opposite edge of the tetrahedron.