Here is a solution to problem A2 of the the 2018 Putnam exam.

Let M_n be the matrix described in the problem. We will show that for all $n > 0$,

$$\det(M_{n+1}) = -\det(M_n)^2$$

Fix an ordering of $Pow(n)\backslash\emptyset$ as indicated in the problem. Then order the elements of $Pow(n+1)\backslash\emptyset$ as follows:

$$S_1, S_2, \ldots, S_{2^n-1}, N = \{n+1\}, S_1 \cup N, S_2 \cup N, \ldots S_{2^n-1} \cup N$$

Then the matrix M_{n+1} may be described in blocks as

$$\begin{pmatrix} M & O_c & M \\ O_r & 1 & J_r \\ M & J_c & JJ \end{pmatrix}$$

where O_r is a row of zeros, J_r is a row of 1s, O_c and J_c are columns of 0s and 1s, and JJ is a square matrix filled with 1s.

Subtract the first $2^n - 1$ rows from the last $2^n - 1$ rows in the obvious pairs to zero out the lower-left corner. The matrix JJ will be replaced by the complement of M.

Then subtract the middle row from all the rows below it; J_c will be replaced by O_c, and the lower right corner will be replaced by $-M$.

Now the matrix is in block-diagonal form, so its determinant is $\det(M) \cdot 1 \cdot \det(-M)$. Since $2^n - 1$ is odd, $\det(-M) = -\det(M)$ and we are done.

In the case $n = 1$, M is just the 1×1 matrix $M = (1)$ whose determinant is 1. By the preceding we then see by induction that $\det(M) = -1$ for all $n > 1$.