UT Putnam Prep Problems, Oct 19 2016

I was very pleased that, between the whole gang of you, you solved almost every problem this week! Let me add a few comments here.

1. Determine (with proof) the number of ordered triples (A_1, A_2, A_3) of sets which satisfy
   (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and
   (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$

where $\emptyset$ denotes the empty set. Express that answer in the form $2^a 3^b 5^c 7^d$ where a, b, c, and d are nonnegative integers.

ANSWER: In this and other problems, it makes sense to practice presenting your answers with good, slick notation. I might phrase the answer that was presented tonight something like this.

For every positive integer n, I will let J_n denote the set $\{1, 2, \ldots, n\}$ (and I will retain this notation in the other problems below). Also let T_n denote the collection of triples (A_1, A_2, A_3) satisfying the conditions $A_1 \cup A_2 \cup A_3 = J_n$ and $A_1 \cap A_2 \cap A_3 = \emptyset$. We will show $|T_n| = 6^n$ for every n, so that in particular $|T_{10}| = 2^{10}3^{10}5^0 7^0$, giving the a, b, c, d that were requested.

In order to count the elements in T_n, consider the function $F : T_n \to T_{n-1}$ defined by $F(A, B, C) = (A', B', C') \overset{def}{:=} (A \cap J_{n-1}, B \cap J_{n-1}, C \cap J_{n-1})$ Note that $A' \cap B' \cap C' = (A \cap B \cap C) \cap J_{n-1} = \emptyset$ and $A' \cup B' \cup C' = (A \cup B \cup C) \cap J_{n-1} = J_{n-1}$, showing that F does indeed map into T_{n-1}. This F is actually onto, and indeed we can describe all the pre-images (A, B, C) of a given (A', B', C'): we must have $A = A'$ or $A = A' \cup \{n\}$ and likewise for B and C: that gives 8 candidate triples. Of these we must exclude only the combination $(A, B, C) = (A', B', C')$ (because it violates condition (i)) and the combination $(A, B, C) = (A' \cup \{n\}, B' \cup \{n\}, C' \cup \{n\})$ (because it violates condition (ii)). The other 6 combinations each describe an element of T_n mapping onto any given $(A', B', C') \in T_{n-1}$, showing that the map F is precisely 6-to-1, and thus $|T_n| = 6|T_{n-1}|$. Since T_0 contains only the single element $(\emptyset, \emptyset, \emptyset)$, it follows by induction that $|T_n| = 6^n$ for every n.

Obviously there are many ways to phrase this idea according to your personal style, and the Putnam scorers would surely give full marks for any of them that conveys the idea completely and clearly. But do try to pick good notation that lets you avoid "..." or a "proof by example", which would not be scored as highly.

2. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$ what is the smallest possible number of elements in A_S?

ANSWER: On the one hand the example $S = J_n$ from the previous problem shows that A_S can have as few as $2n - 3$ elements (which is pretty small compared to the *maximum* size, which can be $|A_S| = n(n-1)/2$!). On the other hand, for any set S, $x_1 + x_n$ must be greater than each of $x_1 + x_{n-1} > \ldots > x_1 + x_3 > x_1 + x_2$ and less than each of

$x_2+x_n < x_3+x_n < \ldots < x_{n-1}+x_n$, from which we see A_S cannot have fewer than $2n-3$ elements.

By the way, on exam day please don't do what I did just now: don't refer to your solution to another problem, because the problems will be graded by different people.

3. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x)=\pi(y)$ and $\pi'(x)=\pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

4. Call a set *selfish* if it has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1,2,...,n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

ANSWER: Let's write S_n for the collection of selfish subsets of J_n, that is,

$$S_n = \{U \subseteq J_n; |U| \in U\}$$

and write M_n for the minimal elements of S_n, that is,

$$M_n = \{U \in S_n;\ \text{ for all } T \subsetneq U, T \notin S_n\}$$

I claim that M_n can be characterized more simply as

$$M_n = \{U \subseteq J_n; |U| \in U \text{ and } |U| = min(U)\}$$

That is, the minimal-selfish subsets of J_n are those whose *least* element is their cardinality. Indeed, suppose S is minimal-selfish; if $j \in S$ and $j < k$, then S will contain the smaller selfish set consisting of j and any other $j-1$ elements of S. Conversely if S is selfish of cardinality k and k really is its least element, then S clearly can contain no smaller selfish set, i.e. S is minimal-selfish.

So what are the minimal selfish subsets of J_n? It is clear that $M_{n-1} \subseteq M_n$; in fact M_{n-1} is precisely that portion of M_n consisting of subsets whose largest element is less than n. As for the rest of M_n, consider the injective function $G: M_{n-2} \to M_n$ defined by

$$G(\{k, x_2, x_3, \ldots, x_k\}) = \{k+1, x_2+1, \ldots, x_k+1, n\}$$

that is, we increase all the elements by 1 and adjoin n. This gives a new set whose minimal element is its cardinality, so it is minimal-selfish, and this new set is not in M_{n-1} because of the presence of the largest element. (We also observe that this new set really does have $k+1$ *distinct* elements because $x_k+1 < n$ since the set lies in J_{n-2}.)

In this way we see $M_n = M_{n-1} \cup G(M_{n-2})$, a disjoint union, and thus $|M_n| = |M_{n-1}| + |M_{n-2}|$, giving a recurrence that we will use to compute cardinalities. Now, M_1 and M_2 consist only of $\{1\}$, so by induction we conclude the $|M_n|$ is the nth Fibonacci number.

You might (either for your own benefit or to illustrate your proof) illustrate this inductive step for some small n. For example you can compute that $M_3 = \{ \{1\}, \{2, 3\} \}$, so the inductive step uses the observation that $M_4 = M_3 \cup G(M_2) = \{ \{1\}, \{2, 3\} \} \cup \{ \{2, 4\} \}$.

5. Suppose S is a set of triangles, no two of which are congruent to each other. If every triangle in S has sides of integer length, how many triangles in S can have a perimeter of 15?

ANSWER: Thanks to the "side-side-side" theorem from high-school geometry, the triangles are uniquely specified by their side-lengths, which I will write in increasing order. That is, the set of (congruence classes of) triangles is in one-to-one correspondence with the set of triples (a, b, c) with $0 < a \le b \le c < a+b$, that last inequality being the Triangle Inequality.

So let us generalize the problem and count the elements of

$$T_n = \{(a, b, c);\ 0 < a \le b \le c < a + b \quad \text{and} \quad a + b + c = n\ \}$$

for all integers n, where we will restrict our attention to triangles with integer side lengths. We want $|T_{15}|$.

Since a, b, c are integers, the constraints on a triple (a, b, c) to be a member of T_n are that $w = a - 1, x = b - a, y = c - b$, and $z = (a + b) - c - 1$ must all be non-negative. (The first of these is redundant, as $w = y + z$.) Observe that x, y, z are also integers, and that

$$(a, b, c) = (1, 1, 1) + (y + z, x + y + z, x + 2y + z) = (1, 1, 1) + x(0, 1, 1) + y(1, 1, 2) + z(1, 1, 1)$$

Taking all non-negative integral combinations of x, y, z will produce all the integer triangles. (In effect, every integral triangle is obtained from the smallest (equilateral) one, $(1, 1, 1)$, by a unique combination of repeated transformations that replace a given triangle (a, b, c) with, respectively, $(a, b + 1, b + 1)$, $(a + 1, b + 1, c + 2)$, or $(a + 1, b + 1, c + 1)$.)

We compute the perimeter of the triangle (a, b, c) above as $3 + 2x + 4y + 3z$, so we conclude that $|T_n|$ is exactly the number of solutions (x, y, z) in non-negative integers to the Diophantine equation $2x + 4y + 3z = n - 3$. For example, when $n = 15$, the 7 solutions are $(x, y, z) = (6, 0, 0), (4, 1, 0), (2, 2, 0), (0, 3, 0), (3, 0, 2), (1, 1, 2), (0, 0, 4)$, which correspond respectively to the triangles $(a, b, c) = (1, 7, 7), (2, 6, 7), (3, 5, 7)$, $(4, 4, 7)$, $(3, 6, 6)$, $(4, 5, 6)$, $(5, 5, 5)$.

Whenever $n - 3$ is a multiple of 12, say $n - 3 = 12k$, we may argue thus: one solution is $(x, y, z) = (6k, 0, 0)$; other solutions are obtained by exchanging 2 xs for a y or 3 xs for 2 zs, that is, by adding non-negative multiples of $(-2, 1, 0)$ or $(-3, 0, 2)$ to that first solution, as long as all three components are non-negative. Thus the solution set may be written

$$\{(x, y, z); 2x+4y+3z = n-3\} = \{(6k, 0, 0)+u(-2, 1, 0)+v(-3, 0, 2); 0 \le u, v; 2u+3v \le 6k\}$$

so that there is one solution (x, y, z) for each lattice point in the triangle bounded by $(u, v) = (0, 0)$, $(3k, 0)$, and $(0, 2k)$. There are exactly $3k^2+3k+1$ such points (u, v), so there are exactly that many non-negative solutions (x, y, z) to the equation $2x + 4y + 3z = 12k$, and so

$$|T_{12k+3}| = 3k^2 + 3k + 1$$

(The seven solutions for $n = 15$ correspond in this way to the points $(u, v) = (0, 0), (1, 0)$, $(2, 0)$, $(3, 0)$, $(0, 1)$, $(1, 1)$, and $(0, 2)$.)

We can obtain similar expressions for $|T_n|$ when n is in a different congruence class modulo 12. For example whenever $n-3$ is odd, every solution to $2x+4y+3z = n-3$ must have $z \geq 1$ (in fact z must be odd), so these solutions are in one-to-one correspondence with the solutions to $2x + 4y + 3z = n - 6$. Here are all the values for the cardinalities:

$$\begin{aligned}
|T_{12k+3}| &= |T_{12k}| = 3k^2 + 3k + 1 \\
|T_{12k+1}| &= |T_{12k-2}| = 3k^2 + 2k \\
|T_{12k-1}| &= |T_{12k-4}| = 3k^2 + k \\
|T_{12k-3}| &= |T_{12k-6}| = 3k^2 \\
|T_{12k-5}| &= |T_{12k-8}| = 3k^2 - k \\
|T_{12k-7}| &= |T_{12k-10}| = 3k^2 - 2k
\end{aligned}$$

6. For each positive integer k let $f(k) = k!/k^k$. Show that for all positive integers m, n we have $f(m + n) < f(m)f(n)$.

ANSWER: Expand $(m + n)^N$ using the Binomial Theorem, with $N = m + n$. Each of the many terms will be $\binom{N}{k} m^k n^{N-k}$ for some k. In the special case $k = m$ this term is $\binom{m+n}{m} m^m n^n$. Since all the other terms are positive we conclude

$$\binom{m+n}{m} m^m n^n < (m + n)^{m+n}$$

Express the binomial coefficient in terms of factorials and rearrange terms to get the desired inequality.

7. Given any five points in the interior of a square of side 1, show that there must be two of them closer together than a distance of $k = 1/\sqrt{2}$. Is the result true for a smaller number k?

ANSWER: Use the Pigeonhole Principle on the four natural quadrants of the square.

8. Let $a(n)$ be the number of representations of positive integer n as a sum of 1's and 2's taking order into account. Let $b(n)$ be the number of representations of n as a sum of integers greater than one. For example $a(4) = b(6) = 5$ because

$$4 = (1 + 1 + 1 + 1) = (1 + 2 + 1) = (1 + 1 + 2) = (2 + 1 + 1) = (2 + 2) \quad \text{and}$$
$$6 = (3 + 3) = (2 + 2 + 2) = (4 + 2) = (2 + 4) = (6)$$

Prove that for every positive integer n, $a(n) = b(n + 2)$

9. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

10. How many polynomials P with coefficients $0, 1, 2$ or 3 have $P(2) = n$, where n is a given positive integer?

ANSWER: Let k_n be the number of such polynomials representing n.

When n is even, we must necessarily have the constant term $p_0 = P(0)$ also be even (i.e. equals 0 or 2). If $p_0 = 0$, our polynomial P is of the form $xQ(x)$ where Q is a polynomial (with the same set of allowed coefficients) which represents $n/2$; if $p_0 = 2$ then $P(x) = 2 + xQ(x)$ where Q represents $(n-2)/2$. Conversely the polynomials representing $n/2$ or $(n/2) - 1$ each lead to distinct polynomials P in this way. So we find $k_n = k_{n/2} + k_{n/2-1}$.

Similarly when n is odd we note $p_0 = 1$ or 3. It follows that $P(x) - 1$ is an acceptable polynomial representing $n - 1$, and thus $k_n = k_{n-1}$ in this case.

In either case, it is then an easy matter to show $k_n = 1 + floor(n/2)$.