Since no one else volunteered to answer question 1, I will do it here. But if that's because y'all feel weak with geometry (a common problem), then please force yourself to try some geometric problems! As I will demonstrate here, it is often quite possible to reduce the problem to a bunch of algebra – purists will object that that's not the "right" way to solve the problem, but at least it works!

So let's suppose the four points (a_i, a_i^2) are known. They are concyclic iff there is a point (h, k) for which the distances to (h, k) are equal; this happens iff the three equations

$$(a_i - h)^2 + (a_i^2 - k)^2 = (a_{i+1} - h)^2 + (a_{i+1}^2 - k)^2$$

are satisfied for $i = 1, 2, 3$. Note that, once expanded, these equations are seen to be linear in the unknowns h and k:

$$a_i^2 - 2a_ih + a_i^4 - 2a_i^2k = a_{i+1}^2 - 2a_{i+1}h + a_{i+1}^4 - 2a_{i+1}k$$

Very well then, what we must answer is the question, under what conditions can three linear equations $A_ih + B_ik + C_i = 0$ be satisfied simultaneously for some h and k? Geometrically all we want is for there to be a vector $(h, k, 1)$ which is simultaneously parallel to the three vectors (A_i, B_i, C_i) (which is true iff those three vectors are coplanar) with the added condition that the last coordinate be 1 (which can be achieved by scaling as long as the last coordinate is not 0).

So in order for the points to be concyclic, the three vectors

$$(2(a_i - a_{i+1}), 2(a_i^2 - a_{i+1}^2), -(a_i^2 - a_{i+1}^2) - (a_i^4 - a_{i+1}^4))$$

must be coplanar. It is equivalent for any nonzero scalar multiples of these vectors to be coplanar, so we will divide the ith vector here by $(a_i - a_{i+1})$ to get

$$(2, 2(a_i + a_{i+1}), -(a_i + a_{i+1}) - (a_i + a_{i+1})(a_i^2 + a_{i+1}^2))$$

which we will abbreviate as $(2, 2D_i, -D_i - D_iE_i)$.

These are coplanar iff the determinant with these are rows vanishes. We can check this by first dividing the first and second columns each by 2, themn by adding the second column to the third, then changing sign in the third column, to get the simpler matrix

$$\begin{pmatrix} 1 & D_1 & D_1E_1 \\ 1 & D_2 & D_2E_2 \\ 1 & D_3 & D_3E_3 \end{pmatrix}$$

This matrix's determinant can be expanded along the third column to get

$$D_1E_1(D_3 - D_2) + D_2E_2(D_1 - D_3) + D_3E_3(D_2 - D_1)$$

At this point I think I have to multiply out the whole thing longhand because the terms are not symmetrical: I get

$$(a_1^3+a_1^2a_2+a_1a_2^2+a_2^3)(a_4-a_2)+(a_2^3+a_2^2a_3+a_2a_3^2+a_3^3)(a_1+a_2-a_3-a_4)+(a_3^3+a_3^2a_4+a_3a_4^2+a_4^3)(a_3-a_1)$$

I advise patience when expanding this out! But we know that the determinant will surely vanish if say $a_1 = a_3$ because then $D_1 = D_2$ and $E_1 = E_2$ making rows 1 and 2 identical; likewise if $a_2 = a_4$. By symmetry, it must also vanish if $a_1 = a_4$ (but not necessarily for the other three pairs, since we have already divided by the three factors $a_i - a_{i+1}$). These observations are not necessary, but they do facilitate a check on the final determinant, which I make out to be

$$(a_1 - a_3)(a_2 - a_4)(a_4 - a_1)(a_1 + a_2 + a_3 + a_4)$$

Therefore the determinant vanishes iff the sum of the a_i does.

So the stated condition is equivalent to the existence of a nonzero vector perpendicular to each (A_i, B_i, C_i). This is almost what we want. We do need to address the extra condition that maybe all such vectors have last coordinate zero. That would happen iff the three vectors (A_i, B_i) are collinear. But as noted above these are (up to scaling) the vectors $(1, D_i)$, and these will be collinear iff they are equal, which happens iff $a_1 = a_3$ and $a_2 = a_4$, conditions violated by our assumptions.

I have to say I don't like the loss of symmetry in my solution. Probably it is better to say something like this: the determinant of the original 3×3 matrix will vanish if any two a_i are equal, making it a product of all the $a_i - a_j$. Comparing degrees, we see the entire determinant is a product of these and one more linear factor. By symmetry, that factor must be a scalar multiple of $a_1 + a_2 + a_3 + a_4$ and we're done. (We do have to make sure the determinant does not vanish identically.) Except for one thing: although the VANISHING of the determinant is a condition which will be symmetric in the a_i, I don't actually see that the VALUE of the determinant must be. It does turn out to be, though!