Solution to UT Putnam practice, 2017, week 3, problem 5

I don't think anyone presented this solution so here it is, FYI.

The problem is to give (part of) the Maclaurin series of

$$f(x) = \frac{5x-7}{(x-1)(x-2)}$$

The solution is to use Partial Fractions. This is a general tool for analyzing rational functions – it's not just for integration!

We know we may write $f(x) = A/(x-1) + B/(x-2)$ for some real numbers A and B. The right values will satisfy $5x - 7 = A(x-2) + B(x-1)$ for all x; taking in particular $x = 1$ and $x = 2$ shows $A = 2$ and $B = 3$.

Thus we see

$$f(x) = \frac{(-2)}{1-x} + \frac{(-3/2)}{1-\frac{x}{2}}$$

and I write it in this way because we recognize the sum of an infinite geometric sequence in each summand. That is,

$$f(x) = \sum_{n \geq 0} (-2)x^n + (-3/2)\left(\frac{x}{2}\right)^n$$

In particular the Maclaurin series begins

$$f(x) = \left(\frac{-7}{2}\right) + \left(\frac{-11}{4}\right)x + \left(\frac{-19}{8}\right)x^2 + \left(\frac{-35}{16}\right)x^3 + \cdots$$