Putnam practice – Oct 11, 2018

1. A square matrix X is said to be *idempotent* if $X^2 = X$. Note that in this case, $X(I - X) = 0$.

(a) Show that if X is idempotent, so is $I - X$.

(b) Find a matrix X which is not idempotent but satisfies $X^2(I - X) = 0$.

(c) Find a matrix X which is not idempotent but satisfies $X(I - X)^2 = 0$.

(d) Show however that if $X^2(I - X) = 0$ *and* $X(I - X)^2 = 0$ then X is idempotent.

2. Suppose $A \in M_n(\mathbf{C})$ has rank r, where $1 \le r \le n - 1$ and $n > 1$. Show that there exist matrices $B \in M_{n,r}(\mathbf{C})$ and $C \in M_{r,n}(\mathbf{C})$ with $A = BC$.

3. Suppose $A, B \in M_4(\mathbf{R})$ commute, and $\det(A^2 + AB + B^2) = 0$. Prove that

$$\det(A + B) + 3\det(A - B) = 6\det(A) + 6\det(B).$$

4. (Problem 2009-A-3). Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos(1), \cos(2), \ldots, \cos(n^2)$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

(The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

5. Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is also invertible and its inverse has integer entries.

6. The exponential function is defined for matrices via the usual power series:

$$e^A = \sum_{n\ge 0} \frac{1}{n!} A^n$$

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(a) Compute e^A where $A = \begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix}$.

(b) Show that if A is skew-symmetric then e^A is orthogonal.

If you don't know enough linear algebra to do these problems, here is a Calculus puzzler for you to work on: prove that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

Remark: This proves that π is NOT equal to 22/7 !