Putnam practice, 2013-11-14

1. Given a finite number of points in the Euclidean plane, show that there is a line which contains exactly two of the points, unless all the points are collinear.

2. (Putnam 1963 B1) If a is a real number for which the polynomial $x^2 - x + a$ divides $x^{13} + x + 90$, then give the value of a.

3. Evaluate in closed form

$$\sum_{k=0}^{n} \binom{n}{k} k^2.$$

4. Define a sequence of real numbers by

$$S_1 = \log a \qquad \text{and} \qquad S_n = \sum_{i=1}^{n-1} \log(a - S_i) \qquad \text{for } n > 1$$

Show that $\lim_{n\to\infty} S_n = a - 1$.

5. (Putnam 1999, B-4) Suppose $f : R \to R$ has a continous 3rd derivative, and suppose that for all $x \in R$, $f(x) > 0, f'(x) > 0$, and $f'''(x) < f(x)$. Show that for all $x \in R$, $f'(x) < 2f(x)$.

(WARNING: I consider that problem to be very difficult and too suggestive of many questions for research! So here is an alternative ODE question.)

6. (Putnam 1995, A-5) Let $x_1, x_2, \ldots x_n$ be differentiable real-valued functions of a single variable t, which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n \\
\ldots & \quad \ldots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{22}x_2 + \ldots a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots x_n$ necessarily linearly dependent?