Last week I left you with the following problem:

Let R and S be two squares in the plane of equal side length. Find a decomposition of R into pieces which can be moved by *translation alone* to give a decomposition of S. Try to find a decomposition with as few pieces as possible.

This was very entertaining but we didn't succeed in getting an answer! So I looked again at my source, which is a homework from what must be a very interesting course that I encourage you to investigate:

www.its.caltech.edu/~dannyc/courses/139/

I did a little more reading on the background. There is a family of questions that ask which kinds of geometric figures can be transformed into which other ones using cut-and-paste techniques. Specifically, given two subsets X and Y of Euclidean space, we want to know when there are decompositions

$$X = \bigcup X_i, \qquad Y = \bigcup Y_i$$

where for each i we can obtain Y_i by applying specific types of operations to X_i, typically Euclidean motions.

If the X_i are supposed to be disjoint but otherwise arbitrary, and likewise for the Y_i, and the allowable operations are all the proper Euclidean motions (translations and rotations, but not reflections) then the answer is more positive than one might think. For example, Banach and Tarski proved the paradoxical result that there is such a decomposition in which X is a sphere of radius 1 and Y is a disjoint union of *two* spheres of radius 1.

If the X_i are supposed to be measurable, then no such cutting and pasting is possible, and in fact it is clear that whenever X have different volumes (or areas or measures) then no such cutting and pasting is possible using measurable pieces.

For polygons in the plane, that's the only constraint; in fact, given any two polygons X and Y there are polygons X_i meeting only at their edges (and likewise the Y_i) with each X_i congruent to the corresponding Y_i. You have surely seen such a decomposition that shows that a parallelogram has the same area as certain rectangle.

For polyhedra X and Y in R^3 there is an additional invariant which must be matched in order to find a polyhedral decomposition of X and Y into congruent parts.

Other results have been obtained, wherein for example the operations used to transform X_i into Y_i must be motions in space which do not at any intermediate time cause the moved X_i to intersect any of the Y_i.

The problem at hand considers another restricted class of motions, namely we are allowed to use translations alone (no rotations or reflections) This was addressed in 1990 by Laczkovich, who proved that all polygons of equal area admit such a decomposition. (He also proved that there are translation-only equivalences between the circle and the square, but the construction uses the Axiom of Choice and constructs something like 10^{50} parts X_i !)

By reviewing some of the literature I see now how elementary results of this type can be proved. Let us consider what types of polygons are "equivalent" under this notion of scissors-decomposition into other polygons. Some helpful results:

Theorem 1. A $s \times (2t)$ rectangle is equivalent to a $(2s) \times t$ rectangle.
Proof: Cut it in half.

By iterating this process one may show that every rectangle is equivalent to a similarly-oriented rectangle whose aspect ratio (width/height) is between 1 and 4.

Theorem 2. The $1 \times s^2$ rectangle is equivalent to the $s \times s$ square.
Proof: By the previous result we may assume $1 < s < 2$. Set the rectangle in the plane with vertices at $(0,0), (0,1), (s^2, 0)$ and $(s^2, 1)$. Slice along the line segment from $(0,1)$ to $(s,0)$. Slice the upper piece again with the vertical line at $x = s^2 - s$, moving the left half down southeast and sliding the right half northwest. A square will result.

Combining this result with the previously-recalled rearrangement of a parallelogram we conclude

Theorem 3. Every parallelogram is equivalent to a square oriented with one pair of its sides.

Corollary: All unit squares (irrespective of orientation) are equivalent
Proof: Draw the unit-area parallelogram with one side matching a side of the first square and with a base parallel to a side of the other square. That parallelogram can be shown equivalent to either of the two squares!

I computed a very accurate picture of the six pieces needed to decompose a diamond into a square; it's in this directory too.