Problems for 2018 University of Texas Putnam Prep Session, week 2 (Sept 27)
You want Number Theory? I'll give you Number Theory!

1. Show that for every positive integer n, the fraction $\dfrac{21n+4}{14n+3}$ is in lowest terms.

2. Suppose $n > 1$ and $p = 2^n + n^2$ is prime. Show that $n \equiv 3 \bmod 6$.

3. Do there exist one million consecutive integers, each of which is divisible by a perfect square (larger than 1)?

4. Find all integral solutions to $x^2 + 3xy - 2y^2 = 122$.

5. Find a multiple of 37 whose base-10 representation consist of just 0s and 1s.

6. We learn in Calculus that the partial sums of the harmonic series are unbounded, that is, among the numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

we can find arbitrarily large values. Show, however, that these numbers are never integers for $n > 1$.

7. Show that for every natural number n, the alternating sum of binomial coefficients

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots$$

is either zero or $\pm 2^k$ for some k. Bonus: for which values of n is the sum positive? negative? zero? What is the power of k in each case?

8. Suppose that a, b, c are distinct integers and that $p(x)$ is a polynomial with integer coefficients. Show that it is not possible to have $p(a) = b, p(b) = c, p(c) = a$.

9. A triangular number is a positive integer of the form $n(n+1)/2$. Show that m is a sum of two triangular numbers iff $4m+1$ is a sum of two squares. (A-1, Putnam 1975)

10. Suppose N is an integer of at most 1000 decimal digits. Describe a way to compute the central binomial coefficient

$$\binom{2^{1000}}{2^{999}} \pmod{N}$$

using at most one billion arithmetic operations on integers of at most 1000 digits each.