Question #4 from Week #1. We are asked to evaluate

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\,dx$$

My proposal is to turn this into a double integral so that we can Fubinate. The numerator of the integrand is of the form $F(b) - F(a)$ which we know can be expressed as an integral $\int_a^b f(t)\,dt$ where f is the derivative of F. So I rewrite the original integral as

$$\int_{x=0}^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\,dx$$

$$= \int_{x=0}^\infty \frac{\int_{t=x}^{t=\pi x} \frac{dt}{1+t^2}}{x}\,dx$$

You can push the denominator x inside the inner integral since it is "a constant" (i.e. does not involve t). Then you can use Fubini's theorem to view this iterated integral as a "2-dimensional integral", the region of integration being the subset of the (t, x) plane where t lies between x and πx, and $x > 0$. That's a sector – the region between two rays at the origin. You can equally well describe it by saying that it's the region where $t > 0$ and x is between t/π and t. So, using Fubini's theorem again, the integral may be written

$$\int_{t=0}^\infty \int_{x=t/\pi}^{x=t} \frac{1}{x(1+t^2)}\,dxdt$$

The inner integral is obviously $\frac{\log(t)-\log(t/\pi)}{1+t^2}$ which is $\frac{\log(\pi)}{1+t^2}$. Integrating this now from 0 to ∞ gives $\log(\pi)\cdot\pi/2$.

Strictly speaking, this proof is inadequate: Fubini's theorem is for rectangles, not unbounded regions. We could write the original integral as the limit of integrals from $x = 0$ to $x = R$ and then let $R \to \infty$. For a fixed R we then have a double integral of the form

$$\int_{t=0}^{t=R} \int_{x=t/\pi}^{x=t} + \int_{t=R}^{t=\pi R} \int_{x=t/\pi}^{x=R}$$

Fortunately the second integral is easily bounded: the integrand is $O(1/R^3)$ and the area of the region of integration is only $O(R^2)$, so the integral is only about $O(1/R)$ so that the limiting value as $R \to \infty$ is zero; the other integral is $\log(\pi) \cdot \arctan(R)$ which approaches the value $\log(\pi)\pi/2$ claimed above.