Here is a proof that $P_n := (1+x+x^2)^n$ has at least one even coefficient for every integer $n > 1$. We will show that one of the coefficients is congruent to zero, modulo 2.

It is a consequence of the Binomial Theorem that for every prime p we have

$$(x+y)^p \equiv x^p + y^p \pmod{p}$$

as polynomials in two indeterminates. Setting $x = x_1$ and $y = x_2 + \cdots + x_n$, we prove by induction on n that

$$(\sum_{i=1}^{n} x_i)^p \equiv \sum_{i=1}^{n} (x_i)^p \pmod{p}$$

for any $n \geq 1$.

So suppose that for some n we expand $(1+x+x^2)^n = \sum a_i x^i$. Then by the observation above, $(1+x+x^2)^{2n} \equiv \sum a_i x^{2i} \pmod{2}$, and thus the coefficient of every odd power of x in $(1+x+x^2)^{2n}$ is even. (More generally, the coefficient of x^i in $(1+x+x^2)^N$ is even unless i is divisible by as many powers of 2 as N is.) Furthermore,

$$(1+x+x^2)^{2n+1} \equiv \sum a_i x^{2i}(1+x+x^2) \equiv \sum a_i x^{2i+1} + \sum (a_i + a_{i-1}) x^{2i} \pmod{2}$$

so that in particular the odd coefficients of P_{2n+1} are the same as those of P_n and so by induction we are done (except for the case

$$P_3 = 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$$

which we check manually).

If $N = \sum 2^{k_i}$ is the binary expansion of N then in the spirit of this proof we may express

$$P_N \equiv \prod \left(1 + x^{2^{k_i}} + x^{2^{k_i+1}}\right) \pmod{2}$$

which is especially easy if each $k_i \leq k_{i-1} + 2$ (i.e. if there are no consecutive 1s in the binary expansion of N). For example,

$$P_{10} \equiv (1+x^8+x^{16})(1+x^2+x^4) = 1 + x^2 + x^4 + x^8 + x^{10} + x^{12} + x^{16} + x^{18} + x^{20}$$

The integral expansion is actually

$$1 + 10x + 55x^2 + 210x^3 + 615x^4 + 1452x^5 + 2850x^6 + 4740x^7 + 6765x^8 + 8350x^9 + 8953x^{10}$$
$$+ 8350x^{11} + 6765x^{12} + 4740x^{13} + 2850x^{14} + 1452x^{15} + 615x^{16} + 210x^{17} + 55x^{18} + 10x^{19} + x^{20}$$