Here we go: one last bunch of problems of all kinds of types. You have TWO weeks to work on this set because we only meet after Thanksgiving, and Nov. 29 will be our last meeting before the Putnam two days later.

Good luck and enjoy the break!

1. (1952-A1) Let $f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$ be a polynomial of degree n with integral coefficients. If a_0, a_n, and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

2. (1956-B7) The polynomials $P(z)$ and $Q(z)$ have complex coefficients. $P(z)$ and $Q(z)$ have precisely the same sets of zeros, possibly with different multiplicities. $P(z) + 1$ and $Q(z) + 1$ likewise have the same zeros, possibly with different multiplicities. Prove that $P = Q$.

3. (1977-A2) Find all solutions in real numbers x, y, z, w to the system

$$x + y + z = w, \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}$$

4. (1983-B3) Assume that the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has solutions $y_1(x), y_2(x)$, and $y_3(x)$ defined on the whole real line such that

$$y_1^2(x) + y_2^2(x) + y_3^2(x) = 1$$

for all real x. Let

$$f(x) = (y_1'(x))^2 + (y_2'(x))^2 + (y_3'(x))^2.$$

Find constants A and B such that $f(x)$ is a solution to the differential equation

$$y' + Ap(x)y = Br(x)$$

5. (1991-A4) Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that (i) the c_i have no limit point in the finite plane, (ii) the sum of the areas of the D_i is finite, and (iii) every line in the plane intersects at least one of the D_i?

6. (2001-A2). You have coins $C1, C2, ..., C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

7. (2010-A5) Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbf{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbf{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.