The *Arithmetic-Geometric Mean Inequality* states that the arithmetic mean of several positive numbers is at least as large as the geometric mean. There are many interesting proofs. But you can use the sum-of-squares argument that I mentioned in today's Putnam Prep session to prove the AGM in the smallest cases.

For example, given two positive numbers a and b we want to prove that $(a+b)/2$ exceeds $\sqrt{ab}$. Squaring both sides, we see we want to prove $(a+b)^2/4 - ab$ is positive. But that's obvious: the difference is exactly $((a-b)/2)^2$.

The case of three numbers has nearly-identical logic but much trickier algebra: to prove $(a+b+c)/2 \geq (abc)^{1/3}$ it suffices to show $(a+b+c)^3 - 27abc$ is positive when a, b, c are. Well, you can check it yourself: $(a+b+c) \cdot ((a+b+c)^3 - 27abc)$ is exactly equal to

$$\begin{array}{rcc}
4ab & \cdot & (a-b)^2 \\
+4bc & \cdot & (b-c)^2 \\
+4ca & \cdot & (c-a)^2 \\
+2ab & \cdot & (a+b-2c)^2 \\
+2bc & \cdot & (b+c-2a)^2 \\
+2ca & \cdot & (c+a-2b)^2 \\
+1 & \cdot & ((a^2+b^2+c^2) - (ba+ca+bc))^2 \\
+1 & \cdot & (\frac{1}{2}ab + 2bc - \frac{5}{2}ca)^2 \\
+3 & \cdot & (\frac{3}{2}ab - bc - \frac{1}{2}ca)^2
\end{array}$$

Thus if a, b, c are positive we have a sum of positive multiples of squares, which shows $(a+b+c)^3 - 27abc$ is indeed positive. I would put this in the category of "quick but unenlightening proof" since I didn't give the tiniest clue as to where these squares come from!

You may enjoy reading about this topic:

`http://en.wikipedia.org/wiki/Hilbert's_seventeenth_problem`