Pizza and Linear Algebra and Vector Calculus (oh my!) 10/10/13

1. Let A and B be matrices of sizes 3×2 and 2×3. Suppose that their product in the order AB is given by

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$$

Show that the product in the order BA is given by

$$\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

2. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)b(y) + b(x)d(y)$$

holds identically? (2003 B1)

3. Let Z denote the set of points in R^n whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are vertices of a hypercube in R^n.) Given a vector subspace V of R^n, let $Z(V)$ denote the number of members of Z which lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq R^n$ of dimension k, of the number of points in $V \cap Z$. (2006 B4)

4. (2009 B4) Say that a polynomial with real coefficients in two variables, x, y, is "balanced" if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over R. Find the dimension of V.

5. (1998 A2). Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

6. (Putnam Exam 1984) Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a \geq 0, b \geq 0, c \geq 0$. Let B be the set of all points which are at distance at most 1 from some point of A (in particular, $A \subseteq B$). Express the volume of B as a polynomial in a, b, c.