#3 from 1st set

let $Y = \sum_{n=2}^{\infty} \log\left(\frac{n^3-1}{n^3+1}\right)$ then let $\{Y_N\}$ be the sequence of $Y_N = \sum_{n=2}^{N} \log\left(\frac{n^3-1}{n^3+1}\right)$.

letting $X_N = \exp(Y_N)$ exp is exponential function e^x

$$X_N = \exp\left(\log\left(\frac{2^3-1}{2^3+1}\right) + \log\left(\frac{3^3-1}{3^3+1}\right) + \dots + \log\left(\frac{N^3-1}{N^3+1}\right)\right)$$

$$X_N = \exp\left(\log\left(\frac{2^3-1}{2^3+1}\right)\right) \cdot \exp\left(\log\left(\frac{3^3-1}{3^3+1}\right)\right) \cdot \dots \cdot \exp\left(\log\left(\frac{N^3-1}{N^3+1}\right)\right)$$

that is, X_N is a product of each $\exp\left(\frac{i^3-1}{i^3+1}\right)$ for $2 \le i < N$

so $X_N = \prod_{n=2}^{N} \exp\left(\log\left(\frac{n^3-1}{n^3+1}\right)\right)$ but $\exp(\log x) = x \; (\forall x \in \mathbb{R})$

$\rightarrow X_N = \prod_{n=2}^{N} \left(\frac{n^3-1}{n^3+1}\right)$ factor $n^3-1 = (n-1)(n^2+n+1)$ and $n^3+1 = (n+1)(n^2-n+1)$

$X_N = \prod_{n=2}^{N} \left(\frac{n-1}{n+1}\right)\left(\frac{n^2+n+1}{n^2-n+1}\right)$ since multiplication is commutative and associative, we can take

$X_N = \left(\prod_{n=2}^{N} \left(\frac{n-1}{n+1}\right)\right)\left(\prod_{n=2}^{N}\left(\frac{n^2+n+1}{n^2-n+1}\right)\right)$ let $A_N = \prod_{n=2}^{N} \frac{n-1}{n+1}$, $B_N = \prod_{n=2}^{N} \frac{n^2+n+1}{n^2-n+1}$

then writing out terms of A_N we will see a pattern of terms "canceling" out: $A_N = \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{\cancel{3}}{5} \cdot \frac{\cancel{4}}{\cancel{6}} \cdot \frac{\cancel{5}}{\cancel{7}} \cdot \frac{\cancel{6}}{\cancel{8}} \cdots \frac{\cancel{N-2}}{N} \cdot \frac{\cancel{N-1}}{N+1}$

for $N \ge 3$ $\quad A_N = \frac{2}{N(N+1)}$

writing $B_N = \prod_{n=2}^{N} \frac{(n+\frac{1}{2})^2 + \frac{3}{4}}{(n-\frac{1}{2})^2 + \frac{3}{4}} = \frac{\cancel{(2+\frac{1}{2})^2 + \frac{3}{4}}}{(2-\frac{1}{2})^2 + \frac{3}{4}} \cdot \frac{(3+\frac{1}{2})^2 + \frac{3}{4}}{\cancel{(3-\frac{1}{2})^2 + \frac{3}{4}}} \cdots \frac{(N+\frac{1}{2})^2 + \frac{3}{4}}{\cancel{(N-\frac{1}{2})^2 + \frac{3}{4}}}$

again, we get the same canceling of the numerator of the previous term canceling the denominator of the next term so that for $N \ge 2$ $\quad B_N = \frac{(N+\frac{1}{2})^2 + \frac{3}{4}}{(2-\frac{1}{2})^2 + \frac{3}{4}} = \frac{N^2+N+1}{3}$

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