Answers for 2018 University of Texas Putnam Prep Session, week 2 (Sept 27)

1. Show that for every positive integer n, the fraction $\dfrac{21n+4}{14n+3}$ is in lowest terms.

Answer: Note that $3(14n+3) - 2(21n+4) = 1$, so any common factor of the numerator and denominator would also divide 1.

Lesson: Remember that the gcd of two integers is the smallest positive linear combination of those numbers.

2. Suppose $n > 1$ and $p = 2^n + n^2$ is prime. Show that $n \equiv 3 \bmod 6$.

Answer: If n is even, so is n^2 and thus (assuming $n > 0$) so is p. So if p is of this form and prime, then $n \equiv 1 \pmod 2$.

Note that this means $2^n \equiv 2 \pmod 3$. That's quickly proved by induction: $2^{k+2} = 2^k \cdot 4 \equiv 2 \cdot 1 = 2 \pmod 3$. Now, if $n \equiv \pm 1 \pmod 3$ then $n^2 \equiv 1$ so $p \equiv 2 + 1 \equiv 0 \pmod 3$, and the only prime divisible by 3 is 3 itself, and $p = 3$ will not occur since we are told $n > 1$. Thus p will not be prime if $n \equiv \pm 1 \pmod 3$. That leaves only $n \equiv 0 \pmod 3$.

Combining these two paragraphs we conclude $n \equiv 3 \pmod 6$.

Lesson: using the Chinese Remainder Theorem you never really have to work modulo a modulus that's divisible by more than one prime at a time.

3. Do there exist one million consecutive integers, each of which is divisible by a perfect square (larger than 1)?

Answer: Yes. By the Chinese Remainder Theorem we can find integers which are congruent to different things modulo any set of coprime moduli. So find an integer n which satisfies one million congruences

$$n \equiv 0 \pmod 4 \qquad n \equiv -1 \pmod 9 \qquad n \equiv -2 \pmod{25} \qquad \ldots$$

and so on: $n \equiv -k \pmod{p_k^2}$ for each $1 \le k \le 1000000$, where p_k denotes the kth prime.

Lesson: Chinese Remainder Theorem all the way.

4. Find all integral solutions to $x^2 + 3xy - 2y^2 = 122$.

Answer: there are none. If there were such a pair of integers (x, y) then we would also have

$$488 = 4x^2 + 12xy - 8y^2 = (2x+3y)^2 - 17y^2 \equiv (2x+37)^2 \pmod{17}$$

Well, square each of the 17 possible residues modulo 17: $(-8)^2, (-7)^2, \ldots, (+7)^2, (+8)^2$. I get these values: $13, 15, 2, 8, 16, 9, 4, 1, 0$, that is $\{0, \pm 1, \pm 2, \pm 4, \pm 8\}$. Since $488 \equiv 12 \equiv -5$ is not in this list, it is impossible to solve the equation in integers.

Lesson: any integral solution to a Diophantine problem is also a mod-p solution for every p (and is also a solution in real numbers); if any of these is impossible then so is the integral problem. (A little more subtle than solving a problem modulo p is solving it

"'p-adically". Amazingly, a QUADRATIC equation has integer solutions iff it has both real solutions and p-adic solutions for every prime p.)

There is a well-defined procedure for solving quadratic equations of two variables in integers. Look up "Pell's Equation".

5. Find a multiple of 37 whose base-10 representation consist of just 0s and 1s.

Answer. Oops: $3 \times 37 = 111$. Here's what I meant to say. If n is any integer which is coprime to 10, then for some integer k we will have $10^k \equiv 1 \pmod{n}$. There's a theorem that guarantees that: Euler proved this for $k = \phi(n)$ (but I never told you exactly what this function $\phi(n)$ is). However you can prove the existence of k by considering all the powers of 10 modulo n: at some point they must repeat, and you can prove that the first repetition must be of the form $10^k \equiv 10^0$, since 10 will have an inverse modulo n. Anyway, once you know $10^k \equiv 1$ then it follows also that $10^{rk} \equiv 1$ for $r = 1, 2, 3, \ldots$, and then $10^k + 10^{2k} + \ldots + 10^{nk} \equiv n \equiv 0 \bmod n$, i.e. $n \mid (10^k + \cdots + 10^{nk})$, and the latter is a number expressed with all 1s and 0s.

If n has factors of 2s and 5s then it will divide a similar expression with extra zeros at the end: simply apply the previous argument to $n/2^a 5^b$.

Lesson: Euler's Theorem (which generalizes Fermat's Theorem, the case when n is prime).

6. We learn in Calculus that the partial sums of the harmonic series are unbounded, that is, among the numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

we can find arbitrarily large values. Show, however, that these numbers are never integers for $n > 1$.

Answer: Assume that H_n is indeed an integer. Let M be the LEAST common multiple of the integers from 1 to n; then $M \cdot H_n$ is the sum of the integers $M + (M/2) + (M/3) + \ldots (M/n)$. Now, for $n > 1$ we must surely have M even, so that $M \cdot H_n$ is even too. And so are most of the integers M/k.

But consider the greatest power of 2 which is less than or equal to n, say $2^r \leq n < 2^{r+1}$. Our M must be a multiple of this denominator, so $2^r \mid M$. On the other hand no higher power of 2 appears in any of the denominators, so 2^r is the precise power of 2 which divides M. Now, what powers of 2 divide those denominators 1 through n? Obviously no $k < 2^r$ is divisible by 2^r — only by lower powers of 2 (or none at all), so that M/k is even in all those cases. But no $k > 2^r$ is divisible by 2^r either: the first such candidate would be $2 \cdot 2^r$, and that is already larger than n. So all the integers M/k for $k > 2^r$ are also even. Only $M/2^r$ itself is odd.

Thus the sum $(M/1) + (M/2) + \ldots + (M/n)$ is odd, but $M \cdot H_n$ is even, a contradiction.

This is kind of a classic question.

7. Show that for every natural number n, the alternating sum of binomial coefficients

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots$$

is either zero or $\pm 2^k$ for some k. Bonus: for which values of n is the sum positive? negative? zero? What is the power of k in each case?

Answer: Use DeMoivre's Theorem: $e^{i\theta} = \cos(\theta) + i\,\sin(\theta)$, first with $\theta = \pi/4$ and then with $\theta = n\pi/4$. That will give two expressions for $\left(e^{\pi/4\cdot i}\right)^n = e^{n\pi/4\cdot i}$, so that

$$\left(\frac{1+i}{\sqrt{2}}\right)^n = \cos(n\pi/4) + i\sin(n\pi/4)$$

Using the Binomial Theorem, the left side expands to

$$\frac{1}{2^{n/2}}\sum_{j=0}^{n}\binom{n}{j} i^j$$

Comparing real and imaginary parts this gives us

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = 2^{n/2}\cos(n\pi/4)$$

As you know the values of that cosine rotate values through $1, 1/\sqrt{2}, 0, -1/\sqrt{2}, -1, -1/\sqrt{2}, 0, 1/\sqrt{2}$ and back to 1, which allows you to evaluate the alternating sum given.

Lesson: remember the connection of the binomial coefficients to the Binomial Theorem. (And to Pascal's Triangle. And to factorials.)

8. Suppose that a, b, c are distinct integers and that $p(x)$ is a polynomial with integer coefficients. Show that it is not possible to have $p(a) = b, p(b) = c, p(c) = a$.

Answer: Consider the polynomial $f(x) = p(x) - b$. The fact that $f(a) = 0$ means that $(x - a)$ divides $f(x)$. In particular that means $b - a$ divides $c - b$. Similarly $c - b$ divides $a - c$ and $a - c$ divides $b - a$. By transitivity we conclude that each of these three integers divides both others. But two integers that divide each other are either equal or negatives of each other. In our case we cannot have e.g. $a - c = -(c - b)$ because that would mean $a = b$, while we are told the integers are different. So we conclude the three quantities are exactly equal. From say $a - b = b - c$ we deduce $b = (a + c)/2$, that is, b is the average of a and c. But likewise a is the average of b and c and c is the average of a and b. The only way each can be the average of the other two is iff all three are equal (think about their ordering!), and that contradicts the assumptions.

9. A triangular number is a positive integer of the form $n(n + 1)/2$. Show that m is a sum of two triangular numbers iff $4m+1$ is a sum of two squares. (A-1, Putnam 1975)

Answer: $m = x(x+1)/2+y(y+1)/2$ iff $4m+1 = 2x^2+2x+2y^2+2y+1 = (x+y+1)^2+(x-y)^2$ So clearly if m is a sum of triangular numbers then $4m+1$ is a sum of squares. Conversely if $4m + 1 = a^2 + b^2$ then solve the equations $a = x + y + 1, b = x - y$ for x and y to see that m is the sum of these trianglar *expression*.

We must pause here to show that x and y are integers other than 0 and -1 (which will then guarantee the triangular expressions are positive, as required). Indeed, the solutions are $x = (a-1+b)/2$ and $y = (a-1-b)/2$. Now, $4m+1$ is odd, so if $4m+1 = a^2+b^2$ then a and b have opposite parity, making $a-1+b$ and $a-1-b$ both even, so x and y are integers.

As for the positivity, I see the problem's claim is actually not true: for example $4 \cdot 3 + 1 = 2^2 + 3^2$ is a sum of squares but 3 is not a sum of "triangular numbers", unless you include 0 as a candidate. A number that is a sum of two square presents actually multiple candidates for the a and b since $a^2+b^2 = b^2+a^2 = (-a)^2+(-b)^2$ etc. Making different choices for the a and b allows the argument above to go through unless (a, b) lies on one of the four lines $\pm a \pm b = 1$. In those cases one of our two constructed "triangular numbers" will have to be zero. This situation occurs when m itself is a triangular. (Note however that sum triangular numbers may indeed be written as a sum of two other triangular numbers! e.g. 21 = 6 + 15.)

10. Suppose N is an integer of at most 1000 decimal digits. Describe a way to compute the central binomial coefficient

$$\binom{2^{1000}}{2^{999}} \pmod N$$

using at most one billion arithmetic operations on integers of at most 1000 digits each.

Answer: There is no known solution to this problem. I will explain in a separate sheet why I asked this question.