Pizza and Problem Solving 9/26/13

Pigeons and Holes

A1) Friendship on Facebook is a symmetric and non-reflexive relation. That is, if Alice is Bob's friend, then Bob is Alice's friend; and nobody can count themselves as a friend. Is it possible to have an isolated community of 30 people such that no two people have the same number of friends?

A2) (a) A 10×10 matrix A is made from 92 1's and 8 0's. What is the largest possible value of $|\det(A)|$?

(b) What are the possible values of $\det(A)$ if A is made from 91 1's and 9 0's?

A3) For this problem, assume that all years have 365 days, so that February 29 doesn't exist. Also, note that December 31 and January 1 are consecutive days. What's the largest set of people such that no two have the same birthday and

(a) no two birthdays fall on consecutive days?

(b) no two birthdays are exactly a week apart?

(c) no two birthdays are exactly 5 days apart?

A4) Let p be a prime integer. Show that $(p-1)! \equiv -1 \pmod{p}$.

A5) Let C be a bounded convex region of the plane. Show that there is a line that divides both the area and perimeter of C in half. Incidentally, the result can be extended to non-convex C, but you have to add conditions about the smoothness and finite length of the boundary.