Putnam Study Group 2015-1

A1) Friendship is a symmetric and non-reflexive relation. That is, if Alice is Bob's friend, then Bob is Alice's friend, and nobody can count themselves as a friend. Is it possible to have an isolated community of 30 people such that no two people have the same number of friends?

A2) (a) A 10×10 matrix A is made from 92 1's and 8 0's. What is the largest possible value of $\det(A)$?

(b) What are the possible values of $\det(A)$ if A is made from 91 1's and 9 0's?

A3) Given the positive numbers a and b and the natural number n, find the greatest among the $n + 1$ monomials in the binomial expansion of $(a + b)^n$.

A4) For this problem, assume that all years have 365 days, and that February 29 doesn't exist. Also, note that December 31 and January 1 are consecutive days. What's the largest set of people such that no two have the same birthday and

(a) no two birthdays fall on consecutive days?

(b) no two birthdays are exactly a week apart?

(c) no two birthdays are exactly 5 days apart?

A5) Charlotte writes a test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. Charlottes teacher announces that for every five consecutive questions on the test, the answers to exactly three of them are TRUE. Just before the test starts, the teacher whispers to Charlotte that the answers to the first and last questions are both FALSE.

(a) Determine the number of questions for which the correct answer is TRUE.

(b) What is the correct answer to the sixth question on the test?

(c) Explain how Charlotte can correctly answer all 100 questions on the test.

A6) Let C be a bounded convex region of the plane. Show that there is a line that divides both the area and perimeter of C in half.

A7) A function f satisfies the equation

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$$

for every real number x except for $x = 0$ and $x = 1$. Find a closed formula for f.

A8) Prove there is an integer k for which $k^3 - 36k^2 + 51k - 97$ is a multiple of 3^{2015}.