Question: A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans, respectively. Agamemnon and Brunhilde move alternately. The set M of legal moves consists of taking either

(a) one bean from a heap, provided at least two beans are left behind in that heap, or

(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Answer: Go first.

The game is a guaranteed win for Agamemnon, and it's all because at the beginning there is exactly one heap having exactly 3 beans in it — Agamemnon can force a win any time we start the game with any number of heaps with any numbers of beans in them, with the only restriction being that there be exactly one heap of cardinality 3.

It's pretty easy to tell how exactly long the game will last because, as we shall see, no one will ever (after the first move) choose to remove just one bean from a heap of 3. Thus each heap of size $h < 3$ will be cleared in one turn, and — assuming that indeed no one ever removes just one bean from a heap of 3 — all heaps of size $h \geq 3$ will take $h - 2$ turns to clear. So just sum over all the heaps and you'll know the number of turns the game will last. If it's even, Brunhilde will have the last turn and win; if odd, Agamemnon wins. Yawn. This way the game is completely determined and about as interesting as CandyLand.

So the only thing that makes the game interesting is that, when a heap has cardinality exactly 3, one can choose to have the clearing of that heap take two more turns instead of just one. So whenever a player has a heap of 3 in front of him, he has the opportunity to change the parity of the number of remaining moves, and thus change the projected winner.

And indeed this is the start of Agamemnon's strategy: on his first move he either clears the heap-of-3 or changes it to a heap-of-2 depending on whether or not the number of moves in the game will be odd or even.

Thereafter, A maintains control of the game as long as he alone gets to decide how each heap will finally be removed. It's easy for him to do this: if at the start of his turn A sees a heap of 3, he uses his turn to remove it. (Thus heaps-of-3 will never accumulate — A will never see more than one on any turn.) Otherwise he can take any move he wants, except to change a heap of 4 to a heap of 3, so that B will never see a 3 and thus never get a chance to change the parity of the number of remaining moves in the game. (It will never be true that A is boxed out, i.e. on A's turn the game will never consist only of several heaps of 4, because that would mean the parity of the remaining number of moves is even, and on his very first turn A arranged it that this not be the case, and he then maintained that condition throughout the game.)

In our case the heaps initially have cardinalities 3,4,5,6, which would ordinarily be cleared in $1 + 2 + 3 + 4 = 10 \equiv 0 (\text{mod } 2)$ moves, giving B the win. So Agamemnon will begin by taking one bean from the short heap, leaving cardinalities 2,4,5,6, thus requiring $1 + 2 + 3 + 4 = 10 \equiv 0$ more moves, giving him the win. His strategy at any later move can be summarized by:

Is there a 3? Clear that heap.
If no 3: is there a 2, 5, or 6? Reduce the size of one of those heaps.
If no 3 AND no 2,5, or 6, then only 4s remain. But this will never happen!

Note that in a similar way if A chooses to leave that initial heap of 3 untouched, then B obtains control of the game and can set the parity and maintain it thereafter in exactly the same way. In other words, if B is playing cleverly, then she will win if A does not make the correct opening move.

More generally A can win if there is an odd number of heaps of 3 shown initially. On his first term he will choose to change one of the threes to a 2 or a 0 to leave the desired number of remaining moves, also leaving an even number of 3's. If at any point B changes the number of 3s, A can counter to restore both the parity of the number of moves remaining AND the (even) parity of number of 3s remaining. For example if B increases the number of 3s (i.e. changes a 4 to a 3) then A can simply take that new 3 away: that heap has been removed in the expected number of moves (2) and there is an unchanged number of 3s left. If instead B lowers the number of 3s (either removing a 3 or changing it to a 2) then A takes the same action on another 3 (leaving two fewer 3s and an even number of remaining moves).

Similarly B can win if at any point *she* sees an odd number of 3s; and that means that if A sees an even number of 3s he has to leave that number unchanged, neither changing a 3 to a 2 or 0 nor changing a 4 to a 3. By symmetry, B will adopt the same strategy, that is, no one will touch any 3s or 4s if there is an even number of 3s out. In that case the winner of the game will be determined by the number of beans beyond the 3s and 4s: if a_i is the number of heaps of size i then the number of "safe" moves is

$$N = a_1 + a_2 + \sum_{i>3} (i-4)a_i$$

After N moves, one player will be forced to increase or decrease the value of a_3 to an odd value, in which case the other player can force a win as above.

For example if the initial configuration is 2,3,3,4,4,4,5,5,7 then there are six safe moves in total for the two players, leaving two 3s and six 4s after three moves by A and three by B. Now it is A's turn but he must either take a bean from a 3 (leaving B a single-3 game like the Putnam problem, which she knows she can win) or take a bean from a 4, leaving three 3s and five 4s. (But since three is odd, player B can now force a win by removing a single 3, then mirroring any later move that A makes on the heaps of 3, or removing any new 3s that A creates by shortening a 4.) Either way, this initial configuration allows player B to force a win.

To summarize, A can force a win if $(1 + a_3)(1 + a_1 + a_2 + a_5 + a_7 + a_9 + \ldots)$ is even; otherwise B can force a win. Our question had $a_3 = 1$ so A can win. The strategy is simply: always arrange to leave an even number of 3s and, when facing an odd number of 3's, also arrange to leave an even number of "expected" moves.