UT Putnam Prep 2017-11-20 — "All of the above" — some answers

1. Show that if a and b are positive then for every positive integer n,

$$(n-1)a^n + b^n \geq na^{n-1}b$$

ANSWER: When $n = 1$ this is just the tautology $b \geq b$. For $n = 2$ it is the assertion that $a^2 + b^2 - 2ab = (a-b)^2$ is non-negative, which is also obviously true.

Now to complete a proof by induction suppose we know that

$$(n-1)a^n - na^{n-1}b + b^n \geq 0$$

Multiply by $b \geq 0$ and add $na^{n+1} - 2na^n b + na^{n-1}b^2 = na^{n-1}(a-b)^2$, which is also non-negative, to obtain the next inequality,

$$na^{n+1} + (-n-1)a^n b + b^{n+1} \geq 0$$

This problem can also be solved by noting that $1 - nx^{n-1} + (n-1)x^n = (1-x)^2(1 + 2x + 3x^2 + \ldots + (n-1)x^{n-2})$ is positive for positive x.

I actual prefer Jeffrey's solution, which is to use the
Arithmetic Mean-Geometric Mean Inequality: If $x_1, x_2, \ldots, x_n$ are positive then their arithmetic mean $(x_1 + x_2 + \cdots + x_n)/n$ exceeds their geometric mean $(x_1 x_2 \ldots x_n)^{1/n}$ unless all the x_i are equal.
To answer question 1, simply use this inequality with $x_1 = b^n$ and all other x_i equal to a^n.

2. Prove that in any group of 6 people there are either 3 mutual friends or 3 mutual strangers.

ANSWER: Treat this as a graph-theory question. Let each person be a vertex in the complete graph on six vertices, and color each edge red or blue according to whether the two people are friends or strangers. We need to find a monochromatic triangle in this graph.

Look at the five edges connected to the first vertex. By the Pigeonhole Principle there are at least three among those edges that have the same color; without loss of generality assume that the edges joining v_1 to v_2, v_3, and v_4 are all red. Now consider the edges joining these last three vertices. If any of them (e.g. $e_{23} = v_2v_3$) is also red, then $\{v_1, v_2, v_3\}$ is a red triangle. Otherwise, $\{v_2, v_3, v_4\}$ is a blue triangle. Either way we have found the monochromatic triangle we seek.

The branch of mathematics studying such problems is called Ramsey Theory. If you increase the number of colors or the size of the desired subset the problem is solvable with a sufficiently large initial set of people. ("Complete chaos is impossible.") But determining the minimal number of people needed to guarantee a monochromatic k-tuple gets rapidly more difficult with increasing k.

3. Do there exist 2017 consecutive integers each of which is divisible by a square (other than 1)?

ANSWER: Let p_n be the nth prime number and consider the set of congruences

$$x \equiv -1 (\text{mod } p_1), \quad x \equiv -2 (\text{mod } p_2), \quad \ldots, \quad x \equiv -2017 (\text{mod } p_{2017})$$

Since the moduli are all pairwise coprime, the Chinese Remainder Theorem assures us that there exist integers x which satisfy all these congruences simultaneously. For such x, we have

$$(p_1)^2 | (x+1), \qquad (p_2)^2 | (x+2), \qquad \ldots, \quad (p_{2017})^2 | (x+2017)$$

4. Find all positive rational solutions of $x^{x+y} = (x+y)^y$.

ANSWER: Let's change coordinates: let $r = y/x$, so we are looking for positive pairs (r, x) making $x^{(x+rx)} = (x(1+r))^{rx}$. Dividing by x^{rx} we see the defining condition is that $x^x = (1+r)^{rx}$; taking xth roots, this becomes simply $x = (1+r)^r$. So the solution set is parameterized by r as the set of ordered pairs $(x, y) = (\,(1+r)^r, r(1+r)^r\,)$.

Generally speaking, equations to be solved in rational numbers (or more generally over a field) can be viewed geometrically: a set of k equations in n unknowns defines a $(n-k)$-dimensional subset of n-dimensional space. You should expect to define the solution set to the equations in terms of $n-k$ free parameters (here, that's just $\{r\}$). This is only a heuristic, but it can help organize your thoughts on the Putnam problems.

5. Suppose F is a polynomial with integer coefficients such that $F(x) = 5$ for four distinct integers $x = x_1, x_2, x_3, x_4$. Show that $F(x) \neq 8$ for any integer x.

ANSWER: We are given four distinct roots of the polynomial $G(x) = F(x) - 5$, so $G(x) = (x-x_1)(x-x_2)(x-x_3)(x-x_4)H(x)$ for some other polynomial; H is also seen to have integer coefficients by long division. So then if there is an integer x with $F(x) = 8$, that same integer makes $G(x) = 3$, i.e. 3 is the product of five integers $x-x_1$, $x-x_2$, $x-x_3$, $x-x_4$, and $H(x)$, the first four of which are all distinct. Since the integer divisors of 3 are only $1, -1, 3$, and -3, those must be precisely the first four factors, leaving $H(x) = 3/9$, which contradicts the fact that $H(x)$ is an integer.

6. Suppose G is a group containing two elements a, b for which

$$ab = ba^{-1} \qquad \text{and} \qquad ba = ab^{-1}$$

Show that $a^4 = b^4 = e$ (the identity element of G.

ANSWER: This is actually pretty standard Group Theory but I know many of you have not seen much of that. The product $b^{-1}ab$ is called the conjugate of a by b; conjugation by a single element b permutes the other elements of G around — in fact this permutation is a homomorphism from G to itself. The fact that conjugation by b inverts a (meaning $b^{-1}ab = a^{-1}$) means that a second conjugation by b will then return a to itself: $b^{-2}ab^2 = a$, i.e. a commutes with b^2. Since we also know a inverts b ($a^{-1}ba = b^{-1}$) it follows that a will also invert b^2 — simply square the previous equation. But then we have two separate conclusions:

$$b^2 = a^{-1}ab^2 = a^{-1}b^2a = b^{-2}$$

which together imply that $b^4 = e$. In exactly the same way we conclude $a^4 = e$.

(FWIW, the subgroup generated by a and b is then a quotient of the semi-direct product of $\mathbf{Z}_4$ with $\mathbf{Z}_4$, one of the many groups of order 16.)

7. Sum the infinite series

$$\frac{3}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{7}{3 \times 4 \times 5} + \frac{9}{4 \times 5 \times 6} + \cdots$$

ANSWER: Use Partial Fractions to write the nth term as

$$\frac{2n+1}{n(n+1)(n+2)} = \frac{1/2}{n} + \frac{-1}{n+1} + \frac{1/2}{n+2}$$

So for example the third partial sum is now written

$$\frac{1/2}{1} + \frac{-1/2}{2} + \frac{-1/2}{4} + \frac{1/2}{5}$$

after we collect terms with like denominators. Then by induction it is easy to prove that, similarly, the sum of the first N terms is

$$\frac{1/2}{1} + \frac{-1/2}{2} + \frac{-1/2}{N+1} + \frac{1/2}{N+2}$$

The infinite sum is then the limit of these: $1/4$.

8. Does $\sum \frac{\sin(n)}{n}$ converge? (Of course this refers to the sine of n radians, and of course you must prove your answer.)

ANSWER: Once again we need to look at the partial sums and decide whether they converge to anything. This is best done by "summation by parts", a kind of discrete analogue of integration by parts. Our sum may be written as $\sum a_n b_n$ where $a_n = \sin(n)$ and $b_n = 1/n$. Let $A_n = \sum_{k=1}^n a_k$ (the "integral" of the a's) ($A_0 = 0$ by definition) and let $B_n = b_{n+1} - b_n$ (the "derivative" of the b's). Then $a_n = A_n - A_{n-1}$ so

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} (A_n - A_{n-1}) b_n = A_N b_N - A_0 b_1 - \sum_{n=1}^{N-1} A_n B_n$$

In particular, since $A_0 = 0$ we have

$$\sum_{n=1}^{\infty} a_n b_n = \lim(A_N b_N) - \sum_{n=1}^{\infty} A_n B_n$$

in the sense that if one infinite sum converges then the other does too, and their sum are related as shown, as long as $\lim A_N b_N$ exists.

In our particular series, the a_n are the imaginary components of e^{in}, which are easily summed as a geometric series as $e^i(e^{iN}-1)/(e^i-1)$. These are clearly bounded as $N \to \infty$, so the A_n are as well, while the $b_n = 1/n$ converge to zero, so that $\lim A_n b_n = 0$. Since $B_n = -1/n(n+1)$ the sum $\sum_{n=1}^{\infty} A_n B_n$ is bounded by a multiple of $\sum |B_n| = 1$, so these series all converge.

This method can be used to numerically evaluate the infinite series quickly. Our original series only converges about as fast as the alternating harmonic series; the new one converges as fast as the p-series with $p = 2$. You can also repeat the trick, as the "integrals" of the A_n also stay bounded; the "derivatives" of the B_n converge to zero even faster than the B_n themselves.

9. Suppose $f : R \to R$ is continuous and satisfies $f(a)f(b) = f(c)$ whenever $a^2 + b^2 = c^2$. Prove that $f(x) = A^{x^2}$ for some real number A.

ANSWER: Let $A = f(1)$, and then let $S = \{x \in \mathbf{R}\,;\, f(x) = A^{x^2}\}$. Our job is to show S is the whole real line.

If $f(x) = 0$ for every x, we are done, using $A = 0$. (That is, we are done if 0^0 is interpreted to mean 0; otherwise, this function f is a counterexample to the claim we are asked to prove!) Otherwise, $f(c) \neq 0$ for some c; since $0^2 + c^2 = c^2$ we conclude $f(0) = 1$, so $0 \in S$. Also, since $0^2 + (-x)^2 = x^2$ we then see $f(-x) = f(x)$ for every x, and in particular if $s \in S$ then $-s \in S$ too.

Let us dispense with the squares: for $x > 0$ let $g(x) = f(\sqrt{x})$; then whenever $x = u+v$ we have $\sqrt{x}^2 = \sqrt{u}^2 + \sqrt{v}^2$ and so $f(\sqrt{x}) = f(\sqrt{u})f(\sqrt{v})$, i.e. $g(x) = g(u)g(v)$. We will show that for every $x > 0$, $g(x) = A^x$ and so $\sqrt{x} \in S$. This will mean S contains all positive real numbers, and hence as noted above $S = \mathbf{R}$ and we are done.

If $r > 0$ and n is a positive integer, then $(n+1)r = r + nr$, so $g((n+1)r) = g(r)g(nr)$; by induction on n it follows that $g(nr) = g(r)^n$ for $n = 1, 2, 3, \ldots$. Also, $r = (r/2) + (r/2)$ so $g(r) = g(r/2)^2$; this shows first of all that $g(r) \geq 0$ for every positive r and thus we may compute $g(r/2) = \sqrt{g(r)}$; in particular if $g(r) = A^r$ then $g(r/2) = A^{r/2}$.

So, starting with the fact that $g(1) = A^1$, we conclude first that $g(1/2^k) = A^{1/2^k}$ by induction on k, and then that $g(n/2^k) = A^{n/2^k}$ by induction on n. That is, $g(r) = A^r$ whenever r is a positive rational number whose denominator is a power of 2.

But such rational numbers are dense in the positive reals, so by continuity we conclude $g(r) = A^r$ for every positive r, and then as noted above, $f(x) = A^{x^2}$ for every real x.

Composing with a logarithm, this is essentially the proof that the only continuous homomorphisms $\phi : \mathbf{R} \to \mathbf{R}$ are of the form $\phi(x) = kx$ for some fixed k. But we have to take some care as above to deal with the possibilities that $f(x) \leq 0$.

10. Compute $\displaystyle\lim_{x\to\infty} x \int_0^x e^{t^2-x^2}\, dt$.

ANSWER: Dennis noted that this problem is much simpler than I thought. The key phrase is: L'Hopital's Rule. Indeed the function whose limit we wish to compute may be written

$$\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}/x}$$

Both numerator and denominator clearly grow without bound as x does, so we may apply L'HR. By the Fundamental Theorem of Calculus, the derivative of the numerator is exactly e^{x^2}, and the derivative in the denominator may be computed to be $e^{x^2}(-1/x^2) + 2xe^{x^2}(1/x) = e^{x^2}(2 - 1/x^2)$. Then the ratio of these derivatives is $x^2/(2x^2 - 1)$, which approaches $1/2$ as $x \to \infty$.

11. If $a, b, c > 0$ and $(1 + a)(1 + b)(1 + c) = 8$, prove $abc \leq 1$.

ANSWER: Once again we can use the Arithmetic Mean-Geometric Mean Inequality: it proves both

$$a + b + c \geq 3(abc)^{1/3} \qquad \text{and} \qquad ab + bc + ca \geq 3(abc)^{2/3}$$

Since the opening premise is that $1 + (a + b + c) + (ab + bc + ca) + (abc) = 8$, we thus get

$$\left(1 + (abc)^{1/3}\right)^3 = 1 + 3(abc)^{1/3} + 3(abc)^{2/3} + (abc) \leq 8$$

Take cube roots and subtract 1.

Less cleverly one could reduce this to a two-variable problem, say by eliminating $c = 8/(1 + a + b + ab) - 1$: we are given that $a > 0$, $b > 0$, and $1 + a + b + ab < 8$, and asked to prove that $ab(7 - a - b - ab) \leq 1 + a + b + ab$. This can be treated as a Lagrange-Multiplier optimization problem.

12. Find the area of the convex octagon that is inscribed in a circle and has four consecutive sides of length 3 and four consecutive sides of length 2. Your answer should be of the form $r + s\sqrt{t}$ where r, s, t are integers.

ANSWER: The consecutivity is not really relevant: we are simply looking at the combined area of 8 triangles, four with sides of lengths $a = 2$, r, and r, and the other four with sides of lengths $b = 3$, r, and r, where r is the radius of the circumscribed circle. Using Heron's formula for the area of a triangle, the combined area will be

$$A = a\sqrt{4r^2 - a^2} + b\sqrt{4r^2 - b^2}$$

We need only determine r so that these 8 triangles will nestle together in the circle, that is, so that the apex angles will sum to 2π. This requires that joining together one triangle of each type makes an angle of measure $\pi/2$.

If we draw the corresponding diagram in the first quadrant, we see we need a point $P = (x, y)$ whose distance from the origin is r, whose distance from $(r, 0)$ is a, and whose distance from $(0, r)$ is b. Compute the point with Cartesian or polar coordinates or by drawing a bunch of triangles; the point must be $(r - \frac{a^2}{2r}, r - \frac{b^2}{2r})$. But then the distance to the origin is only right if

$$(a^2 + b^2 - 2r^2)^2 = 2(ab)^2$$

so that $r^2 = (a^2 + b^2)/2 \pm ab/\sqrt{2} = 13/2 \pm 3\sqrt{2}$ and then the area is

$$a\sqrt{a^2 + 2b^2 + 2\sqrt{2}ab} + b\sqrt{2a^2 + b^2 + 2\sqrt{2}ab} = a(a + \sqrt{2}b) + b(b + \sqrt{2}a) = a^2 + b^2 + 2\sqrt{2}ab$$

In our case that's $13 + 12\sqrt{2}$.

Notice that I determined r because I had a quadrilateral with edges and diagonals whose lengths I knew (in terms of r), and that imposed a constraint. This is actually a very interesting sidebar! Given any four points in space we can connect them in pairs to form a tetrahedron with six edges; the lengths of the edges determine the tetrahedron up to congruence, and in particular they determine the volume V of that tetrahedron. There's a formula for V^2 as a polynomial in the six side lengths; it's best described as a certain determinant. What's most interesting about that formula (to me!) is that

(a) it's supposed to compute V^2 and in particular should be positive. For some combinations of the six variables the polynomial is negative and so there is no such tetrahedron. This positivity of this polynomial is then a "tetrahedral inequality", akin to the "triangle inequality" which limits the combinations of three edge lengths that can form an actual triangle.

(b) The four points are coplanar iff the volume of the tetrahedron is zero; thus we have a polynomial equation $V = 0$ which relates the four side lengths and the two diagonals in a (planar) quadrilateral. What I computed above is the special case where the sides have lengths r, r, a, b and the diagonals have lengths r and $\sqrt{2}r$.

13. If two altitudes of a tetrahedon are coplanar, the edge joining the two vertices from which these altitudes issue is orthogonal to the opposite edge of the tetrahedron.

ANSWER: vector geometry.