I limit myself to two instructional comments on this week's problems.

1. Let me expand upon the second solution I described for problem 1.

The "stars and bars" theorem we discussed can be stated this way: the number of ways to distribute k identical objects into n distinguishable boxes is $\binom{k+n-1}{k}$. For example there are $\binom{5}{3} = 10$ ways to put 3 items into 3 boxes, and here they are (viewing the bars as the interior walls separating the 3 boxes lined up in a row):

$$***||, **|*||, **||*, *|**||, *|*|*, *||**, |***|, |**|*, |*|**, ||***$$

For example, the first configuration $***||$ (or $|***|||$) shows the three items in the first box, with the other two boxes empty.

So imagine all the ways to choose 3 items from among 12 (say, 3 pockets into which we will place balls, chosen from a set of 12 pockets). View the set of 12 pockets as a grid of 3 columns of height 4:

$$\begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

When we go to pick 3 locations for the balls, we can do it by first deciding how many balls will be in each column. There are precisely 10 ways to do this, as shown in the previous paragraph. For each of these 10 choices, there are then many ways to actually pick the pockets in each column that will hold the balls. For example the choice $**|*|$ accounts for many of the $\binom{12}{3}$ ways to put 3 balls into the pockets, e.g.

$$\begin{pmatrix} * & * & \circ \\ * & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}, \begin{pmatrix} * & \circ & \circ \\ \circ & * & \circ \\ * & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}, \text{ etc.}$$

Here is the key observation: most of these partially-filled 4×3 grids come in sets of three that simply have the three columns rotated around. For example, the last grid gets paired with

$$\begin{pmatrix} \circ & \circ & * \\ * & \circ & \circ \\ \circ & \circ & * \\ \circ & \circ & \circ \end{pmatrix} \text{ and } \begin{pmatrix} \circ & * & \circ \\ \circ & \circ & * \\ \circ & * & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

So if all the arrangements come in threes, then $\binom{12}{3}$ is a multiple of three, right? Well, no, because there are arrangements that come in sets of 1 alone (all of them corresponding to the arrangement $*|*|*$ above) in which the three columns are identical, e.g.

$$\begin{pmatrix} \circ & \circ & \circ \\ * & * & * \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}$$

There are exactly 4 of these, so $\binom{12}{3}$ is 4 more than a multiple of 3, i.e.

$$\binom{12}{3} \equiv \binom{4}{1} \pmod 3$$

In exactly the same way we can put pb items into pa pockets (arranged in a columns of height p) but rotating the columns shows that these $\binom{pa}{pb}$ objects all come in sets of p except the $\binom{a}{b}$ arrangements that place b objects into a pockets in the first column, and then make the other $p-1$ columns all identical to the first. Thus $\binom{pa}{pb} \equiv \binom{a}{b} \pmod p$ for all a and b.

2. I made the statement that much of combinatorics consists of counting things in two ways, so let me clarify.

You remember Fubini's Theorem in Calculus, right? It's supposed to be about integration over rectangles, but in practice we use it like this. In order to integrate a function $F(x,y)$ over a region A in the x,y-plane, we let $\pi(A)$ be the projection of A to the x-axis, i.e. $\pi(A) = \{x; (x,y) \in A \text{ for some } y\}$, and for each $x \in \pi(A)$ we let $A_x = \{y; (x,y) \in A\}$. Then $\int\int_A F(x,y)\,dxdy = \int_{\pi(A)} \int_{A_x} F(x,y)\,dy\,dx$ (and the double integral can of course be written as an iterated integral the other way around as well).

As it turns out counting is a special kind of integration and Fubini's theorem can be used in that context too. If A is a finite set of pairs and $F : A \to \mathbf{R}$ is any function, then

$$\sum_{(x,y)\in A} F(x,y) = \sum_{x\in\pi(A)} \left(\sum_{y\in A_x} F(x,y) \right) = \sum_{y\in\pi'(A)} \left(\sum_{x\in A_y} F(x,y) \right)$$

where the notation for the last double sum is supposed to be clear from context.

So for example this is Dylan's solution to problem number 2. Let A be the set of pairs (v,f) where v is a vertex of an icosahedron, f is a face, and $v \in f$. Let $F(v,f)$ be the number shown on face f. Then for any face f, $\sum_{v\in A_f} F(v,f)$ will be 3 times the number shown on face f since every face f is incident with three vertices v. Thus the final double sum is 3 times the sum of the numbers on the 20 faces, i.e. $3 \times 39 = 117$.

On the other hand, for any vertex v, the numbers shown on the five faces f incident to v are supposed to be five different non-negative integers, so $\sum_{x\in A_y} F(x,y)$ will be at least $0+1+2+3+4 = 10$. Summing over all 12 vertices gives a total which is at least $120 > 117$, a contradiction.

You deduce all kinds of combinatorial structure for a Platonic solid by using this double-sum idea with $F(x,y)$ simply equal to 1 for each pair (x,y). Try it with (x,y) running over all incident pairs of vertices and edges, or of edges and faces, or of vertices and faces.