Linear Algebra and Differential Equations (Nov 16 2015)

1. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$.

2. Let A and B be 2×2 matrices such that for each $k = 0, 1, 2, 3, 4$, the matrix $A + kB$ has integer entries and has an inverse which also has integer entries. Show that the same is true when $k = 5$.

3. For any square matrix A we can define $\sin(A)$ by the usual power series

$$\sin(A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin(A) = \begin{pmatrix} 1 & 2015 \\ 0 & 1 \end{pmatrix}$$

4. Find all polynomials $p(x)$ with real coefficients satisfying the differential equation

$$7\frac{d}{dx}[xp(x)] = 3p(x) + 4p(x+1)$$

over the real line.

5. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

6. Let $f : (1, \infty) \to \mathbf{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)}$$

for all $x > 1$. Prove that

$$\lim_{x \to \infty} f(x) = \infty$$

7. Prove that there exists a unique function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that

$$f(f(x)) = 6x - f(x) \qquad \text{and} \qquad f(x) > 0 \qquad \text{for all } x > 0$$