UT Putnam Prep Problems, Nov 16, 2016

You asked for an assortment of problems of different types. All righty, then; here's a grab-bag of unrelated questions.

1. Find all solutions in real numbers x, y, z, w to the system

$$x + y + z = w, \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}$$

ANSWER: We need to find nonzero x, y, z (with $x + y + z$ also nonzero) such that

$$\begin{aligned} 0 &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{x+y+z} \\ &= \frac{(xy + yz + zx)(x + y + z) - (xyz)}{xyz(x + y + z)} \\ &= \frac{(x + y)(y + z)(x + z)}{xyz(x + y + z)} \end{aligned}$$

so obviously we must have two of x, y, z be negatives of each other (in which case w will equal the third). Admittedly you have to spot that factorization but it's clear by inspection that the solution set in $\mathbf{R}^3$ includes the planes $x + y = 0$, $y + z = 0$, $z + x = 0$ so we expect those as factors.

(You might want to learn more about *Algebraic Geometry*, the study of solution sets to systems of (polynomial) equations in multiple variables, thought of as subsets of Euclidean Space.)

2. Let M be a finite set of points in the plane such that for any two of them, there is a third point in M on the same line. Show that M is contained in a single straight line.

ANSWER: It was cruel of me to give this problem as an exercise but I thought maybe you could come up with something! This is known as the Sylvester-Gallai Theorem and it has a celebrated backstory which I invite you to read about. There are several proofs known; the shortest ones are kind of contrary to the spirit of the problem because there is no obvious reason we should have to use tools like Euclidean geometry to solve a problem stated with just incidence data, but, well, that's the best proof I know.

I will prove this theorem which I think you can see is equivalent to the original problem: if P is a finite set of points in the plane, not all collinear, then one of the lines joining two points of P contains no other element of P besides those two.

Let L be the set of all lines passing through two or more points of P. Note that L is a finite set because P is, but has more than one element by assumption. The set $P \times L$ of ordered pairs (p, ℓ) of points and lines contains the subset D of all such pairs where the point p is actually on the line ℓ; again, the non-collinearity assumption assures us that $P \times L$ contains other pairs besides those in D.

So let (p, ℓ) be an element in $(P \times L) \setminus D$ which minimizes the distance from the point p to the line ℓ. (There's only a finite set of candidate pairs and they all involve a positive distance.)

I claim ℓ is a line that contains only two elements of P. Indeed, if we drop the perpendicular from p to ℓ, that splits ℓ into two (closed) rays; if ℓ contains three or more points of P, then one of those two rays contains two of these points, call them q and r, with q closer to the endpoint of the ray than r is. Well, in that case, let $m \in L$ be the line joining p to r; a quick sketch shows that q is closer to m than p is to ℓ, which violates our assumed minimality.

3. Let $f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$ be a polynomial of degree n with integral coefficients. If a_0, a_n, and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

ANSWER: If $x = p/q$ is a rational root written in lowest terms, then $\sum_{i=0}^{i=n} a_i q^i p^{n-i} = f(p/q) \cdot q^n = 0$ and thus $\sum_{i=0}^{i=n} a_i q^i p^{n-i} \equiv 0 (\mathrm{mod}\ 2)$. Now, p and q cannot both be even. If p is but q is not, then this equation would force $a_n \equiv 0$, contrary to hypothesis. Likewise if q were even but p were not, then a_0 would be even. But then p and q are both odd, in which case the left side is congruent to $\sum a_i = f(1)$, which is odd, too. So no combination works, and therefore there can be no such rational root p/q.

4. Define a sequence of real numbers r_n by the recurrence

$$\begin{aligned} r_1 &= 1 \\ r_{n+1} &= 1 + \tfrac{n}{r_n} \end{aligned}$$

Show that for every n, $\sqrt{n} \leq r_n \leq 1 + \sqrt{n}$.

ANSWER: It's an interesting induction proof: we use the left inequality $n \leq r_n^2$ to prove the right inequality $(r_{n+1} - 1)^2 \leq n + 1$ and use the right inequality $(r_n - 1)^2 \leq n$ to prove the left one $n + 1 \leq r_{n+1}^2$.

The first is easy: $(r_{n+1} - 1)^2 = n(n/r_n^2) \leq n \cdot 1 < n + 1$. For the second, expand $r_{n+1}^2 - (n+1) = (1 + \frac{n}{r_n})^2 - (n+1) = n(n + 2r_n - r_n^2)/r_n^2 = n(1 + [n - (r_n - 1)^2])/r_n^2 > 0$.

Observe that life is simpler if we avoid square roots!

5. Let g be a continuous function mapping a real interval J into itself. Suppose that there is an integer $n \geq 2$ for which the n-fold composite $g \circ g \circ g \ldots \circ g = I$, the identity functions. (That is, $I(x) = x$ for all $x \in J$.) Prove that $g \circ g = I$.

ANSWER: Dylan did this one for us. (g increasing makes $g = I$ and g decreasing makes $g \circ g$ increasing.)

6. In a round-robin tournament with players $P_1, \ldots, P_n$, each player plays every other player exactly once, and there are no ties. Let w_r and ℓ_r denotes the numbers of games won and lost, respectively, by player r. Prove that

$$\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} \ell_r^2$$

ANSWER: The difference between the two sides is

$$\sum_{r=1}^{n} w_r^2 - \sum_{r=1}^{n} \ell_r^2 = \sum_{r=1}^{n} (w_r^2 - \ell_r^2) = \sum_{r=1}^{n} (w_r + \ell_r)(w_r - \ell_r) = (n-1)\left(\sum_{r=1}^{n} (w_r) - \sum_{r=1}^{n} (\ell_r)\right)$$

which is zero because every match played gives someone a win and someone a loss, increasing each summation by 1.

7. The polynomials $P(z)$ and $Q(z)$ have complex coefficients. $P(z)$ and $Q(z)$ have precisely the same sets of zeros, possibly with different multiplicities. $P(z)+1$ and $Q(z)+1$ likewise have the same zeros, possibly with different multiplicities. Prove that $P = Q$.

ANSWER: Here's something I never notieced before: a polynomial P can have a lot of repeated roots, or $P+1$ can, but not both! More precisely,
Lemma If P is a polynomial of degree D, and P has R distinct roots and $P+1$ has S distinct roots, then $R+S > D$.
Proof. Factor P as $c\prod(x-r_i)^{n_i}$. Then P' has $\prod(x-r_i)^{(n_i-1)}$ as a factor, of degree $\sum(n_i-1) = D-R$. Likewise $(P+1)'$ has a factor of degree $D-S$. Since $P' = (P+1)'$ has degree $D-1$, and the roots of P and $P+1$ are distinct, we must have $(D-R)+(D-S) \leq (D-1)$, proving the lemma.

Returning to our problem, we may assume Q has degree at most D. Then by assumption $P - Q = (P+1) - (Q+1)$ vanishes at the same R roots as P and at the same S roots as $P+1$, hence has more roots than its (nominal) degree, meaning it must be identically zero.

8. Suppose the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has three different solutions $y_1(x), y_2(x), y_3(x)$, each defined on the whole real line, such that

$$y_1(x)^2 + y_2(x)^2 + y_3(x)^2 = 1$$

for all real x. Let

$$f(x) = y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$$

Find constants A and B so that f is a solution of

$$y' + Ap(x)y = Br(x)$$

ANSWER: We will use the notation $S_{mn} = \sum \frac{d^m y_i}{dx^m}\frac{d^n y_i}{dx^n}$, so that $S_{m,n} = S_{n,m}$ and, by the product rule, $\frac{d}{dx}S_{m,n} = S_{m+1,n} + S_{m,n+1}$. We are given that $S_{0,0} = 1$ is constant, so by differentiating several times we conclude these are all zero: $S_{0,1}, S_{1,1} + S_{0,2}, 3S_{1,2} + S_{0,3}$. Now each y_i satisfies that third-order differential equation, so $3S_{1,2} = -S_{0,3} = pS_{0,2} + qS_{0,1} + rS_{0,0} = -pS_{1,1} + r$. Now, $f = S_{1,1}$ so $f' = 2S_{1,2} = (2/3)(-pS_{1,1} + r) = (-2/3)pf + (2/3)r$. Thus $A = B = 2/3$.