Week 2, number 4 – deducing the Taylor series of f from the data $f(1/n) = n^2/(n^2+1)$.

I don't want to give a solution here — I'm not sure I can! – but I do want to make a couple of points.

The first is that it is easy to find such a function f: we want $f(1/n) = n^2/(n^2+1)$ for all natural numbers n; with a little algebra this is found to mean $f(x)$ agrees with $g(x) = 1/(x^2+1)$ for every real number x of the form $x = 1/n$. Since the problem asks for the Taylor series of f and no other information is given which precludes the possibility that $f(x) = g(x)$, then the answer has to be that the Taylor series is that of this one particular function $g(x)$, that is, it's

$$f(x) = 1 - x^2 + x^4 - \ldots + (-1)^n x^{2n} \ldots$$

(Possibly they are allowing for the possibility that there are different series which could arise for various such functions f; in that case we only know that this is one possible answer.)

Well then for any function f meeting the conditions of the problem, let $h(x) = f(x) - g(x)$. Then we know $h(x) = 0$ for $x = 1, 1/2, 1/3, \ldots$. Is it necessarily true that the Taylor series of h is

$$h(x) = 0 + 0x + 0x^2 + \ldots\ ?$$

That is, can we conclude that all derivatives of h vanish at the origin, if all we know is that h itself vanishes at this collection of points near the origin? (Assuming of course that h HAS derivatives of all orders at 0.)

(Remark: this is an EXCELLENT strategy for Putnam problems: if you have to prove something about a function, or number, or matrix... and you haven't been told what X is, but you have a candidate example Y that might play this role, then make sure you can prove whatever you're supposed to prove for Y first; after that, see what you know about $X - Y$ and what you have to prove about it. This trick will peel away a lot of complicated information, and you'll find you're trying to prove instead that something that "feels like" zero really is zero!)

So this is the question: if $h(x)$ vanishes at all our special x's, is its Taylor series just a bunch of zeros?

Certainly if $h'(0)$ exists, then h is continuous at 0, in which case $h(0) = \lim h(1/n) = \lim 0 = 0$. So yes, the zero-th term of the Taylor series vanishes. Beyond that...?

This is not trivial! Among complex analytic functions, it is true that when h vanishes on any set E which includes a limit point, then h is identically zero (on any connected domain which includes the limit point). But this is not directly applicable to our situation – we could try to extend the problem to the complex domain, but I don't see how to do that unless we assume that the Taylor series of h converges to h on some open set, and that information was not given.

Staying among real-analytic functions, things are much murkier. Consider the function defined by

$$h(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ e^{-1/x^2} & \text{if } x < 0 \end{cases}$$

This is a classic example which I encourage you to work with: it's "infinitely smooth"; that is, the two pieces fit together to give a function which is differentiable to all orders, even at $x = 0$. This function DOES vanish at $x = 0$, and it DOES have a zero Taylor series, but it's NOT the zero function! You could add this to g to get a function which meets the condition of the problem but is not equal to $1/(1 + x^2)$

So I think what is intended in this problem is to be able to use the information given to deduce the derivatives of h at 0. I have already shown that $h(0) = 0$ above. We can then compute

$$h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x} = \lim_{x \to 0} \frac{h(x)}{x}$$

Since we are given that this limit exists, we may compute it by taking any sequence of x that converges to 0. Obviously we should choose $x = 1/n$ since we know $h(1/n) = 0$, making this limit equal to 0 as well. Thus we have $h'(0) = 0$: the zero-th and first terms of the Taylor series are now what we want them to be.

Beyond that I am not sure where to go. We should compute

$$h''(0) = \lim_{x \to 0} \frac{h'(x) - h'(0)}{x} = \lim_{x \to 0} \frac{h'(x)}{x}$$

but I don't have any information about $h'(x)$ at any other point besides $x = 0$! If for example h' is known to exist at all points of an interval $[0, \epsilon)$, we can use the Mean Value Theorem to assert that for each n there is a point $x_n \in (1/(n + 1), 1/n)$ where

$$h'(x_n) = \frac{h(1/n) - h(1/(n + 1))}{1/n - 1/(n + 1)} = 0$$

and then if $h''(0)$ exists, h' must be continuous at 0, so that $h'(0) = \lim_{n \to \infty} h'(x_n) = \lim 0 = 0$. Similar techniques could show $h^{(k)}(0) = 0$ for all k, but only if I know that $h^{(k)}$ is defined on some interval for every k. Perhaps I'm missing something, but that's not clear to me.