Here are a solution to problem B3 of the the 2018 Putnam exam.

First note that $n|2^n$ requires n to be a power of 2, say $n = 2^r$.

Next we prove a lemma: if a, b are positive integers and $(2^a - 1)|(2^b - 1)$ then $a|b$. Indeed, use the Division Algorithm to write $b = a \cdot q + s$ with $s < a$; then $2^b = 2^{aq+s} = (2^a)^q(2^s) \equiv 2^s$ modulo $(2^a - 1)$. On the other hand $(2^a - 1)|(2^b - 1)$ implies $2^b \equiv 1$. Thus we conclude $2^s \equiv 1$, i.e. $(2^s - 1)$ is a multiple of $2^a - 1$. But since $s < a$, $2^s - 1 < 2^a - 1$, and the only multiple of $2^a - 1$ that is smaller than it is zero. So $2^s - 1 = 0$, so $s = 0$, so $a|b$.

Very well then: if $(n-1)|(2^n - 1)$ and we have already shown $n = 2^r$ then by the lemma we have $r|n$. But since $n = 2^r$ this means r itself is a power of 2, say $r = 2^k$, and $n = 2^r = 2^{2^k}$.

Now, finally, when does $n-2 = 2(2^{r-1}-1)$ divide $2^n - 2 = 2(2^{n-1}-1)$? Cancelling the factors of 2, we may again apply the lemma to deduce that $(r-1)|(n-1)$, i.e. $(2^k-1)|(2^r-1)$. One more application of the lemma shows $k|r = 2^k$ so that k as well is a power of 2 ! Write $k = 2^m$; then $r = 2^{2^m}$ and so $n = 2^{2^{2^m}}$.

The first few examples of these numbers (m, k, r, n) and then $(0, 1, 2, 4)$, $(1, 2, 4, 16)$, $(2, 4, 16, 65536)$, and $(3, 8, 256, 2^{256})$. The next would be $(4, 16, 65536, 2^{65536})$, but already this is too big: since $10^3 = 1000 < 1024 = 2^{10}$, we see $10^{100} < (10^3)^{34} < 2^{340}$ and we are done.

So $n \in \{4, 16, 65536, 2^{65536}\}$.