Putnam Prep 10/24/13

1. (2010-A5) Let G be a group, with operation $*$. Suppose that

(i) G is a subset of R^3 (but $*$ need not be related to addition of vectors);

(ii) For each $a, b \in G$, either $a \times b = a * b$ or $a \times b = 0$ (or both), where $\times$ is the usual cross product in R^3.

Prove that $a \times b = 0$ for all $a, b \in G$.

2. Suppose $*$ is a binary operation on a set S, and satisfies the conditions

for all $x, y \in S$, we have $x * (x * y) = y$; and

for all $x, y \in S$, we have $(y * x) * x = y$

Then show this binary operation is commutative. Show also that it need not be associative.

3. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all real numbers x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0 \end{cases}$$

4. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

5. Suppose n is a positive integer. How many ordered pairs (x, y) of positive integers are there with

$$\frac{xy}{x+y} = n$$