Putnam practice problems from Linear Algebra – Oct 9 2017

1. Suppose A is a real $n \times n$ matrix which satisfies $A^3 = A + I_n$. Show that A has a positive determinant.

2. Suppose $A, B \in M_2(\mathbf{C})$ satisfy $AB = BA$. Assume that A is not of the form aI_2 for any complex number a. Show that $B = bA + cI$ for some complex numbers b, c.

3.(a) Find two real matrices A, B with

$$A^2 + B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

(b) Show that if A, B are real matrices with

$$A^2 + B^2 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

then $AB \neq BA$

4. Suppose A is a 3×3 matrix with rational entries, for which $A^8 = I$. Show that in fact $A^4 = I$.

5. Suppose A, B are real 3×3 matrices. Prove that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq 3$$

iff there is an invertible matrix X with $AXB = O_3$

6. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

7. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable f which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

8. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

9. If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?