UT Putnam Prep session 2017-10-23: Combinatorics

1. Prove that the number of ways of writing n as a sum of distinct positive integers is equal to the number of ways of writing n as a sum of odd positive integers.

2. Let A and B be any two sets. Find all sets X with the property that

$$A \cap X = B \cap X = A \cap B \qquad \text{and} \qquad A \cup B \cup X = A \cup B.$$

3. A number n of tennis players take part in a tournament in which each of them plays exactly one game with each of the others. Suppose that for each i, the ith play wins x_i of the matches and loses y_i of them. (There are no ties or draws in tennis.) Show that

$$x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2$$

4. Prove that every graph has two vertices that are endpoints of the same number of edges.

5. Determine the number of permutations $a_1, a_2, ..., a_{2004}$ of the numbers 1, 2,. . . , 2004 which move each number equally far, that is, for which

$$|a_1 - 1| = |a_2 - 2| = \ldots = |a_{2004} - 2004|$$

6. Given $2n - 1$ subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

7. Let A be a nonempty set and let $f : P(A) \to P(A)$ be an increasing function on the set $P(A)$ of subsets of A, meaning that $f(X) \subset f(Y)$ if $X \subset Y$. Prove that there exists a subset $T \subset A$ such that $f(T) = T$.

8. We play the following solitaire game with an equilateral triangle of $n(n+1)/2$ pennies (i.e. with n pennies on each side). Initially, all of the pennies are turned heads up. On each turn, we may turn over three pennies which are mutually adjacent; the goal is to make all of the pennies show tails. For which values of n can this be achieved?

9. There is an odd number, n, of students at a university. Some of those students join together to form several clubs (a student may belong to different clubs). Some of those clubs join together to form several societies (a club may belong to different societies). There are k societies altogether. Suppose that the following hold:

(i) each pair of students is in exactly one club,
(ii) for each student and each society, the student is in exactly one club of the society,
(iii) a club that is in $m > 0$ societies has exactly $2m + 1$ student members.

Find all possible values of k.

10. Suppose $R_1, R_2, \ldots, R_n$ and $B_1, B_2, \ldots, B_n$ are points in the plane, no three of which are collinear. Show that there is a permutation σ of the set $\{1, 2, \ldots, n\}$ so that the n line segments joining each R_i to $B_{\sigma(i)}$ are disjoint.