OK gang, one last set of Putnam-ish problems.

1. Show that if p is an odd prime, and $x = ((p-1)/2)!$, then $x^2 \equiv \pm 1$ modulo p.

2. Among all 10×10 matrices A whose entries consist of 92 1's and eight 0's, what is the largest possible value of $\det(A)$? What values of $\det(A)$ are possible if there are only 91 1's and nine 0's?

3. A line meets the graph of $y = 2x^4 + 7x^3 + 3x - 5$ in four distinct points (x_i, y_i). What is the value of $\frac{x_1+x_2+x_3+x_4}{4}$?

4. Let $p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6$, and for $k = 1, 2, 3, 4$ define

$$I_k = \int_0^\infty \frac{x^k}{p(x)}\, dx$$

For which of these k is I_k largest?

5. Suppose the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x) = 0$$

has three different solutions $y_1(x), y_2(x), y_3(x)$, each defined on the whole real line, such that

$$y_1(x)^2 + y_2(x)^2 + y_3(x)^2 = 1$$

for all real x. Let

$$f(x) = y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$$

Find constants A and B so that f is a solution of

$$y' + Ap(x)y = Br(x)$$