I mentioned a method for finding the largest root of a polynomial (hoping to provide an alternative method of proof for the last question of Oct 17). Let me explain that method by example.

Here is a random cubic:

$$P(X) = 9198X^3 + 9038X^2 - 80183X + 75946$$

If its roots are r, s, t then we can then expand the following product to get:

$$Q(Y) := P(X)P(-X) = 5767794916 - 5056513593Y + 1556731912Y^2 - 84603204Y^3$$

where I have written Y for X^2 since only even powers of X will appear (clearly $P(X)P(-X)$ is an even function of X!). But it is clear from the factorization that

$$Q(Y) = -9198^2(X-r)(X-s)(X-t)(X+r)(X+s)(X+t) = -9198(Y-r^2)(Y-s^2)(Y-t^2)$$

in other words the new polynomial Q has roots which are the squares of those of P.

Continuing in this way we may find polynomials R, S, T, ... whose roots are the 4^{th}, 8^{th}, 16^{th}, ... powers of the roots of P. But if the roots of P are of different magnitudes, this difference will be much magnified by successive squarings. In particular, the sum of all of the roots will differ very little (percentage-wise) from the largest root all by itself.

But the sum of the roots of a polynomial is easily computed as the ratio of the coefficients of two largest powers of the variable. Thus in our example

$$\begin{aligned} r + s + t &= -9038/9198 \\ r^2 + s^2 + t^2 &= 1556731912/84603204 \\ &\cdots \\ r^{16} + s^{16} + t^{16} &= a/b, \end{aligned}$$

where I compute the coefficients a and b in the last example to be

$$\begin{aligned} a &= 55230774788388100247464469662246602295892210749874446611186192837272704 00 \\ b &= 26247895226284664787222083880620023802522495365156948624278159 36 \end{aligned}$$

If we expect one root (say r) to be much larger than the others, then in particular the last example shows $|r| \approx (a/b)^{1/16} = 3.825541483$. In fact, the largest root of P is -3.825541546, so this worked very well!

In practice, the procedure works well as long as one root is "much" bigger than the any of the others (including the complex ones, judged according to their magnitudes). By contrast the system requires tinkering when the roots are of similar magnitude, especially when they occur in complex conjugate pairs or are multiple roots. But it's not bad for 18th century technology!