Show that the value of this integral exceeds $\frac{3\pi}{2}$:

$$\int_0^\pi e^{\sin^2 x}\,dx$$

Proof. We note that the Taylor series for e^x is convergent everywhere, so we have that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Longrightarrow e^{\sin^2 x} = \sum_{n=0}^{\infty} \frac{\sin^{2n} x}{n!}$$

$$\Longrightarrow \int_0^\pi e^{\sin^2 x}\,dx = \int_0^\pi \sum_{n=0}^{\infty} \frac{\sin^{2n} x}{n!}\,dx = \sum_{n=0}^{\infty} \int_0^\pi \frac{\sin^{2n} x}{n!}\,dx$$

(The last step here requires some justification, but is true.) And furthermore,

$$\forall n \geq 0, \frac{\sin^{2n} x}{n!} \geq 0 \Longrightarrow \int_0^\pi \frac{\sin^{2n} x}{n!}\,dx \geq 0$$

So we may consider the partial sum:

$$\sum_{n=0}^{2} \int_0^\pi \frac{\sin^{2n} x}{n!}\,dx = \int_0^\pi dx\, + \int_0^\pi \sin^2 x\,dx\, + \int_0^\pi \frac{sin^4 x}{2}\,dx = \pi + \frac{\pi}{2} + \frac{3\pi}{8} = \frac{13\pi}{8} > \frac{3\pi}{2}$$

It follows that the original integral exceeds $\frac{3\pi}{2}$. $\square$