U.T. Putnam preparation — Oct 16 2017 — Number Theory

1. Find all integers n for which $n(n+1)$ is a perfect square.

2. Prove that there is no integer n for which n^5 can be written as a product of six consecutive positive integers.

3. Let $n \geq 3$ be an odd integer. Prove that every positive integer less than n can be written as a sum or difference of two other integers, each of which is less than n and coprime to n.

4. Let p be a prime of the form $3k+2$. Suppose that there are integers a and b such that p divides a^2+ab+b^2. Prove that p already divides a and b.

5. Suppose p is prime. Show that there are infinitely many positive integers n such that p divides $2^n - n$.

6. Show that if k is odd then

$$(1+2+\cdots+n) \mid (1^k+2^k+\cdots+n^k)$$

for all positive integers n.

7. Prove that the sum of 3 consecutive integers is not a perfect square. What about the sum of 4 consecutuive integers?

8. Show that for all positive integers the number

$$S(m,n) = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n}$$

is not an integer.