Problem set 3, October 2, 2017, #8

8. Let s be any arc of the unit circle lying entirely inside the first quadrant. Let A be the area of the region lying below s and above the x-axis, and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

Proof:

Suppose s is the arc that runs from c radians to d radians along the unit circle, where $0 \leq c \leq d \leq \pi/2$. Then

$$A = \int_{\cos(d)}^{\cos(c)} \sqrt{1-x^2}dx \qquad \text{and} \qquad B = \int_{\sin(c)}^{\sin(d)} \sqrt{1-y^2}dy.$$

Focusing on the formula for A, since $x = \sqrt{1-y^2}$ on the unit circle in the first quadrant, we have $dx = -y \cdot dy/\sqrt{1-y^2}$. To change the limits of integration, since we also have $y = \sqrt{1-x^2}$, observe that $\sqrt{1-\cos^2(t)} = \sin(t)$ for $0 \leq t \leq \pi$. Thus we may rewrite

$$A = \int_{\sin(d)}^{\sin(c)} y \cdot \frac{-y}{\sqrt{1-y^2}}dy.$$

Switching the upper and lower limits of integration in our formula for A and rewriting B to create a common denominator, we have

$$A+B = \int_{\sin(c)}^{\sin(d)} \frac{y^2}{\sqrt{1-y^2}}dy + \int_{\sin(c)}^{\sin(d)} \frac{1-y^2}{\sqrt{1-y^2}}dy = \int_{\sin(c)}^{\sin(d)} \frac{1}{\sqrt{1-y^2}}dy = \arcsin(y)\Big|_{\sin(c)}^{\sin(d)} = d-c.$$

Since $d-c$ is the length of s, our claim holds. $\square$