U.T. PUTNAM PRACTICE 2019 — week 1

1. The equation $x^y = y^x$ describes a curve in the first quadrant of the plane containing the point $P = (4, 2)$. Compute the slope of the line that is tangent to this curve at P. Some extra credit will be given for a good sketch of the graph of this curve.

2. Determine whether this series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

3. Compute $\displaystyle\int_0^{\pi/4} \frac{1}{\cos(x) + \sin(x)}\, dx.$

4. A *wedding ring* is the three-dimensional solid that remains after drilling a cylindrical hole through the center of a sphere. Compute, with proof, the volume of metal in a metallic wedding ring that is 6mm tall when it rests on a table, as a function of the radius r of the hole that has been drilled.

5. The curve parameterized by $x(t) = \cos^3(t)$, $y(t) = \sin^3(t)$, $z(t) = \cos(2t)$ passes through the point $(1, 0, 1)$ when $t = 0$ and passes through the point $(0, 1, -1)$ when $t = \pi/2$, having traversed a path of length $5/2$. (You don't have to prove this.) What point will it pass through after having traversed a length of exactly 1 ?

6. For which real numbers r does this limit exist?

$$\lim_{x \to 0^+} x^r \ln(x)$$

7. Find an antiderivative of $\cos^4(x) - \sin^4(x)$.

8. Do these series converge or diverge? Explain.

$$(A)\ \sum_{n=1}^{\infty} \sin\left(\frac{\cos(n)}{n^2}\right) \qquad\qquad (B)\ \sum_{n=1}^{\infty} \cos\left(\frac{\sin(n)}{n^2}\right)$$

9. Compute $\dfrac{dy}{dx}$ where $y = \arcsin(2uv)$, $u = \cos(x)$, and $v = \sin(x)$. You may assume that $x \in [0, \pi/4]$.

10. A 1-meter-long rod is lying at the base of a 5-meter-tall streetlamp. The rod is oriented north-south. A runner raises the rod to a height of 2 meters and heads east at a rate of 4 meters per second, always keeping the rod perpendicular to his path, level to the ground, and at a height of 2 meters. The rod will then produce a moving shadow on the ground. How rapidly does the width of the rod's shadow increase as the runner moves eastward?

11. Find a polynomial $f(x)$ which has the same values as $g(x) = \frac{120}{x}$ for $x = 1, 2, 3, 4, 5$. (That is, we need $f(1) = 120$, $f(2) = 60$, etc.)

12. Suppose A and B are square matrices of the same size, and that $ABABA = I$.
(a) Explain why A is invertible.
(b) Show that $AB = BA$.

13. The exponential function is defined for square matrices A by the usual power series:

$$e^A = I + A + \frac{1}{2}A^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Compute e^A when $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

14. A linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ is called a *projection* if $L(L(v)) = L(v)$ for each $v \in \mathbf{R}^n$. For example the function $L(x, y, z) = (2y + 3z, y, z)$ is a projection in $\mathbf{R}^3$. Show that the only possible eigenvalues of a projection L are 0 and 1.

15. Find an invertible matrix P for which $PAP^{-1} = B$ where

$$A = \begin{pmatrix} 1 & 2018 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 41 \\ 0 & 1 \end{pmatrix}$$

16. If $f(x)$ is the function defined by

$$f'(x) = \frac{f(x)}{4f(x) + 3x - 3} \qquad \text{and} \qquad f(0) = 1\ ,$$

what is the value of $f(3)$? (Partial credit will be given for a numerical estimate of this value, with more credit for a closer approximation.)

17. For some functions $A(x)$ and $B(x)$, the set of solutions of the differential equation

$$y' = A(x)y + B(x)$$

includes both the tangent function $y = \tan(x)$ and the cosine function $y = \cos(x)$. What is the solution to the initial-value problem

$$y' = A(x)y + B(x), \qquad y(0) = \pi\ ?$$

18. Find a solution to the partial differential equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z$$

which is not a polynomial in x and y. For extra credit give the general solution.

19. Find a (nonzero) solution of the linear differential equation

$$5x^2y'' + x(1 + x)y' - y = 0$$

20. Does every solution of the differential equation $y'' + e^x y = 0$ stay bounded as $x \to \infty$?