## UT Putnam Prep 2017-11-13 — Geometry

The word "geometry" means many different things in mathematics, but most of them are not applicable to the Putnam exam. (I don't think I've ever seen a question which really used Differential Geometry, Finite Geometries, etc.) But it is helpful to know some ideas from classical plane and solid geometry — Pythagoras' Theorem, Heron's Theorem, the Platonic Solids, etc. — and to be comfortable with Cartesian geometry, Trigonometry, and the study of curves and surfaces in calculus. Occasionally some combinatorial topology is useful (e.g. Euler's formula for $V - E + F$.) It's also *very* helpful to keep a geometric mindset when pursuing other problems (e.g. to think of sets of equations as describing an algebraic variety, or to view matrices as representing geometric motions).

Undoubtedly a few nontrivial theorems in classical geometry are useful:

*Pythagoras' Theorem* If a right triangle has a hypotenuse of length $a$ and legs of lengths $b$ and $c$ then

$$a^2 = b^2 + c^2$$

This is generalized in the *Law of Cosines*: If angles $\alpha, \beta, \gamma$ lie, respectively, opposite sides $a, b, c$ in a triangle, then $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ (and similarly for the the two other choices of a length to appear on the left side).

*Heron's Theorem* If a planar triangle has sides of lengths $a$, $b$, and $c$ then its area is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$ (the "semi-perimeter"). This can be generalized to *Brahmagupta's Theorem* If a cyclic quadrilateral has sides of lengths $a$, $b$, $c$, and $d$ then its area is

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = (a+b+c+d)/2$.

*Ptolemy's Theorem* If a planar quadrilateral has sides of lengths $a$, $b$, $c$, and $d$ and diagonals of length $e$ and $f$, then $ef \leq ac + bd$, with equality iff the vertices all lie on a circle. (Here the products are to be of the lengths of opposite sides.)

*Law of Sines* If angles $\alpha, \beta, \gamma$ lie opposite sides $a, b, c$ in a triangle, respectively, then

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

There are many other useful formulas in trigonometry which may or may not be considered geometric. Apart from the ones that in essence define $\tan, \cot, \sec, \csc$ in terms of sin and cos, and the consequences of Pythagoras's theorem, the most useful ones are probably the *angle-addition formulas*

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta), \qquad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

and their consequences (the *double-angle formulas*, the *half-angle formulas*, etc. These addition formulas are in turn consequences of the matrix representations of rotations.

A personal favorite: *Pick's Theorem* The area of a lattice polygon is one less than the number of interior points plus half the number of boundary points.

1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. Suppose the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^3$ satisfy

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq \mathbf{0}$$

Show that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

3. A trapezoid is inscribed in a circle, having a diameter of the circle as its base. A triangle is also inscribed in the same circle, with each of its three sides parallel to a side of the trapezoid. Show that the triangle and the trapezoid have equal area.

4. Suppose four points $(x_i, y_i)$ lie on the hyperbola $xy = 1$, and are also concyclic (i.e. there is a circle containing all four of them). Show that $x_1 x_2 x_3 x_4 = 1$.

5. Find the maximum number of points on a sphere of radius 1 in $\mathbf{R}^n$ such that the distance between any two points is strictly larger than $\sqrt{2}$.

6. A rectangle $R$ is tiled by finitely many rectangles, each of which has at least one side of integral length. Prove that $R$ itself has at least one side of integral length.

7. Suppose $S_1$ and $S_2$ are squares contained in the closed unit disk, and each has sides of length 0.9. Show that $S_1 \cap S_2 \neq \emptyset$

8. Every unit square in $\mathbf{R}^2$, when projected onto the $x$-axis, projects onto a line segment of length at most $\sqrt{2}$. What is the largest possible area of a projection to the $x, y$ plane of a unit cube in $\mathbf{R}^3$?

9. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

10. Can an arc of a parabola inside a circle of radius 1 have length greater than 4?

11. Find the smallest volume bounded by the coordinate planes in $\mathbf{R}^3$ and by a plane which is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

12. For $i = 1, 2$ let $T_i$ be an acute triangle with side lengths $a_i, b_i, c_i$ and area $A_i$. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$. Does it follow that $A_1 \leq A_2$?

13. Show that there are no equilateral triangles in $\mathbf{R}^2$ whose vertices all have integer coordinates.

14. A regular pentagon has side of length 1. Compute the length of its diagonal.