Let me take the liberty of generalizing the lovely solution that Dean M. submitted for problem 6.

Let $P(x,y)$ be any rational function which is homogeneous of degree -4. (That is, it's a ratio of two (homogeneous) polynomials in the two variables and $P(kx,ky) = k^{-4}P(x,y)$ for any constant k.) Then we can find a closed-form antiderivative of $P(\sqrt{\sin(x)}, \sqrt{\cos(x)})$ as follows:

$$\begin{aligned}
\int P(\sqrt{\sin(x)}, \sqrt{\cos(x)})\, dx &= \int \sec^2(x) P(\sqrt{\tan(x)}, 1)\, dx \qquad \text{(by homogeneity)} \\
&= \int P(\sqrt{u}, 1)\, du \qquad \text{(letting } \tan(x) = u\text{)} \\
&= \int 2v\, P(v, 1)\, dv \qquad \text{(letting } u = v^2\text{)}
\end{aligned}$$

Since this last is now a rational function of v, we may compute a partial-fractions decomposition and then integrate.

(A similar argument works as long as P is homogeneous of any degree which is a *multiple* of 4; we simply have additional factors of $\sqrt{\cos(x)}^4 = 1/\sec^2(x) = 1/(u^2+1) = 1/(v^4+1)$. I'm not sure what larger family of integrands can be characterized as having an antiderivative which is a rational function of $\sqrt{\sin(x)}$ and $\sqrt{\cos(x)}$.)

In our example, $P(x,y) = (x+y)^{-4}$, so the rational function is $2v/(v+1)^4 = 2/(v+1)^3 - 2/(v+1)^4$ whose antiderivative is $-\frac{2}{2}(v+1)^{-2} + \frac{2}{3}(v+1)^{-3}$; replace v with $\sqrt{u} = \sqrt{\tan(x)}$ to obtain an antiderivative of the original function.

Since we are computing a definite integral, we may just as easily transform the endpoints when we change variables: $x = 0$ and $x = \pi/2$ correspond to $u = 0, u = \infty$, i.e. to $v = 0, v = \infty$, and thus our integral evaluates to $-2/2 + 2/3 = 1/3$.

Easy as pie, right? :-)