Problem 26

October 9, 2017

Problem

Let $A, B \in M_2(\mathbb{Z})$ with $AB = BA$ such that $det(A) = det(B) = 0$. Show that $det(A^3 + B^3)$ is the cube of an integer.

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. For any real number λ, I will compute the determinant of the matrix $A - \lambda B$. The determinant would be equal to

$$(a - \lambda e)(d - \lambda h) - (b - \lambda f)(c - \lambda g) = (ad - bc) - (ah + de - bg - cf)\lambda + (eh - fg)\lambda^2$$

Remember that the the determinants and A and B are precisely $ad - bc$ and $eh - fg$, respectively. We are given that those values equal 0. As a result, the determinant of $A - \lambda B$ is $-(ah + de - bg - cf)\lambda$ I will let $ah + de - bg - cf = z$ Because the terms of the matrices A and B are integers, we have that z is also an integer. Note that $det(A^3 + B^3) = det(A + B)det(A + \omega B)det(A + \omega^2 B)$. Plugging in $\lambda = -1, -\omega, -\omega^2$, the expression equals

$$(-z)(-1) * (-z)(-\omega) * (-z)(-\omega^2) = z^3 * w^3 = z^3$$

which is the cube of an integer.