Simulation-Based Learning: Theory and applications.

M2 MVA 2014-15.

Homework 1.

Exercice 1:

1)

(a) X is the number of time you have to run the algorithm. (X is N-valued)

Let us find $P(X=k)$ for $k \geq 1$.

$$P(X=k) = P\left(\sum_{i=1}^{k-1} p(i) < U \leq \sum_{i=1}^{k} p(i)\right)$$

$$= F_U\left(\sum_{i=1}^{k} p(i)\right) - F_U\left(\sum_{i=1}^{k-1} p(i)\right) = \sum_{i=1}^{k} p(i) - \sum_{i=1}^{k-1} p(i) = p(k).$$

(b) The average number of loops is:

$$E[X] = \sum_{i=1}^{+\infty} i \times p(i) \text{ with } p(i) = \frac{6}{\pi^2 i^2}$$

$$= \sum_{i=1}^{+\infty} \frac{6}{\pi^2 i^2} \times i = \frac{6}{\pi^2} \sum_{i=1}^{+\infty} \frac{1}{i} \text{ wich diverge.}$$

Rq: It's due to the construction of X for $p(i) = \frac{6}{\pi^2 i^2}$. In fact if $U = 0,99\ldots$ then it will take a infinite time (never reach) the stop condition.

2)

(a) Distribution of $[1/U]$.

$$P([1/U]=k) = P\left(k \leq \frac{1}{U} < k+1\right) = P\left(\frac{1}{k+1} < U \leq \frac{1}{k}\right) = F_U\left(\frac{1}{k}\right) - F_U\left(\frac{1}{k+1}\right)$$

$$= \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}.$$

(b) + (c): Because the samples are independent, the algorithm terminates at first succes in a sequence of independent trials with same probability of succes:

$$p = P(V \leq (X+1)/(2X))$$

$$= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{V \leq (X+1)/(2X)\}} | X]]$$

$$= \mathbb{E}\left[P\left(V \leq \frac{X+1}{2X}\right)\right] = \mathbb{E}\left[\frac{X+1}{2X}\right] = \sum_{k=1}^{+\infty} \frac{k+1}{2k} \times \frac{1}{(k+1)k}$$

$$= \frac{1}{2} \sum_{k \geq 1} \frac{1}{k^2} = \frac{\pi^2}{12}$$

The average time is thus $\frac{1}{p} = \frac{12}{\pi^2}$.

The algorithm draws a r.v X with distribution q and then a r.v V uniformly distributed on $[0,1]$ until:

$$Vq(X)M \leq p(X) \quad \text{with } q(X) = \frac{1}{k(k+1)} \text{ for } X = k.$$

Following the arguments that we have shown in lecture 1, we can show that the pair $(X, Vq(X)M)$ is uniformly distributed under the curve $\{(x,y) : 0 \leq y \leq q(x)M\}$; Then that the element X is distributed according to P.

Exercice 2:

1) At the end of the loop line 4:

$$P(E \leq x \mid E^2 \leq \frac{2E'}{t}) = \frac{P(E \leq x, \; E^2 \leq \frac{2E'}{t}) \,(*)}{P(E^2 \leq \frac{2E'}{t})}$$

$$(*)\; P(E \leq x, E^2 \leq \frac{2E'}{t}) = \mathbb{E}\left(\mathbb{E}\left[\mathbb{1}_{\{E \leq x\}} \mathbb{1}_{\{E^2 \leq \frac{2E'}{t}\}} \mid E\right]\right)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{E \leq x\}} P(E^2 \leq \frac{2E'}{t})\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{E \leq x\}} P(E' \geq \frac{t}{2}E^2)\right] ; \quad P(E' \geq \frac{t}{2}E^2) = 1 - 1 + e^{-\frac{tE^2}{2}} = e^{-\frac{tE^2}{2}}.$$

$$= \mathbb{E}\left[\mathbb{1}_{\{E \leq x\}} e^{-\frac{tE^2}{2}}\right] = \int_0^{+\infty} \mathbb{1}_{\{y \leq x\}} e^{-\frac{ty^2}{2}} e^{-y} \, dy.$$

$$= \begin{cases} \int_0^x e^{-\frac{1}{2}(\sqrt{t}y + \frac{1}{\sqrt{t}})^2 + \frac{1}{2t}} dy = e^{\frac{1}{2t}} \int_0^x e^{-\frac{1}{2}(\sqrt{t}y + \frac{1}{\sqrt{t}})^2} dy & \text{if } x \geq 0. \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{e^{1/2t}}{\sqrt{t}} e \int_{\frac{1}{\sqrt{t}}}^{\sqrt{t}x + \frac{1}{\sqrt{t}}} e^{-\frac{1}{2}u^2} du = \frac{e^{\frac{1}{2t}}}{\sqrt{t}} \left(\Phi(\sqrt{t}x + \frac{1}{\sqrt{t}}) - \Phi(\frac{1}{\sqrt{t}})\right) & \text{if } x \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

$\rightarrow$ with Φ the c.d.f of $\mathcal{N}(0,1)$

and $P(E^2 \leq 2E'/t) = \frac{e^{1/2t}}{\sqrt{t}}(1 - \Phi(\frac{1}{\sqrt{t}}))$ if $x \geq 0$

$$\text{So } P(E \leq x \mid E^2 \leq \frac{2E'}{t}) = \frac{\Phi(\sqrt{t}x + \frac{1}{\sqrt{t}}) - \Phi(\frac{1}{\sqrt{t}})}{1 - \Phi(\frac{1}{\sqrt{t}})} \mathbb{1}_{\{x \geq 0\}}$$

2) $X = \dfrac{t}{(1+tE)^2}$

$$P(X\leq y) = P\left(\frac{t}{(1+tE)^2}\leq y\right) = P\left(\frac{t}{y}\leq (1+tE)^2\right) = 1 - P\left((1+tE)^2 \leq t/y\right)$$

$$= 1 - P\left(-\frac{1}{\sqrt{ty}} - 1 \leq E \leq \frac{1}{\sqrt{ty}} - 1\right)$$

$$= 1 - \left(F_E\left(\frac{1}{\sqrt{ty}} - 1\right) - F_E\left(-\frac{1}{\sqrt{ty}} - 1\right)\right)$$

$$F_E\left(\frac{1}{\sqrt{ty}} - 1\right) = \frac{\Phi\left(\frac{1}{\sqrt{y}}\right) - \Phi\left(\frac{1}{\sqrt{t}}\right)}{1 - \Phi\left(\frac{1}{\sqrt{t}}\right)}$$

So $P(X\leq y) = 1 - \dfrac{2\Phi\left(\frac{1}{\sqrt{y}}\right) - 1}{1 - \Phi\left(\frac{1}{\sqrt{t}}\right)} \mathbb{1}_{\{y\geq 0\}}$.

densité de X: $\dfrac{\partial F_X(y)}{\partial y}$

$$\frac{\partial \Phi\left(\frac{1}{\sqrt{y}}\right)}{\partial y} = -\frac{1}{2} y^{-3/2} \varphi\left(\frac{1}{\sqrt{y}}\right) \text{ with } \varphi \text{ the density of } \mathcal{N}(0,1)$$

$$= -\frac{1}{2} y^{-3/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^{-1}} = -\frac{1}{2}\left(\frac{1}{2\pi y^3}\right)^{\frac{1}{2}} e^{-\frac{1}{2} y^{-1}}$$

So $\dfrac{\partial F_X(y)}{\partial y} = \dfrac{1}{1-\Phi\left(\frac{1}{\sqrt{t}}\right)} \left(\dfrac{1}{2\pi y^3}\right)^{1/2} e^{-\frac{1}{2} y^{-1}} \mathbb{1}_{\{y\geq 0\}}$.

3) $P(X\leq w \mid U\leq \alpha) = \dfrac{P(X\leq w, U\leq \alpha)\,(*)}{P(U\leq \alpha)}$

$(*)$ $P(X\leq w, \leq \alpha) = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{X\leq w\}} \mathbb{1}_{\{U\leq \alpha\}} \mid X\right]\right]$

$$= \mathbb{E}\Big[\mathbb{1}_{\{X\leq w\}} \underbrace{P(U\leq \alpha)}_{\alpha}\Big] = \mathbb{E}\left[\mathbb{1}_{\{X\leq w\}} \exp\left(-\frac{1}{2\varphi^2} X\right)\right]$$

$$= \int \mathbb{1}_{\{x \leq w\}} \exp\left(-\frac{1}{2\gamma^2} x\right) \times \frac{1}{1-\Phi\left(\frac{1}{\sqrt{t}}\right)} \left(\frac{1}{2\pi x^3}\right)^{1/2} e^{-\frac{1}{2}x^{-1}} \mathbb{1}_{\{x \geq 0\}} \, dx.$$

$$= \frac{1}{1-\Phi\left(\frac{1}{\sqrt{t}}\right)} \times \int \mathbb{1}_{\{x \leq w\}} \left(\frac{1}{2\pi x^3}\right)^{1/2} \underbrace{\exp\left(-\frac{1}{2\gamma^2 x} x^2 - \frac{1}{2\gamma^2 x}\gamma^2\right)}_{= \exp\left(-\frac{1}{2\gamma^2 x}\left[(x-\gamma)^2 + 2\gamma x\right]\right)} \mathbb{1}_{\{x \leq 0\}}$$

$$= \begin{cases} \dfrac{\exp(-1/2\gamma)}{1-\Phi\left(\frac{1}{\sqrt{t}}\right)} \displaystyle\int_{-\infty}^{w} \left(\frac{1}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{1}{2\gamma^2 x} \times (x-\gamma)^2\right) \mathbb{1}_{\{x \geq 0\}} & \text{if } w \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

Same as previously, $P(U \leq \alpha) = P(X < \infty, U \leq \alpha)$

$$= \frac{\exp(-1/2\gamma)}{1-\Phi\left(\frac{1}{\sqrt{t}}\right)} \underbrace{\int_{\mathbb{R}} \left(\frac{1}{2\pi x^3}\right) \exp\left(-\frac{(x-\gamma)^2}{2\gamma^2 x}\right) \mathbb{1}_{\{x \geq 0\}}}_{=1}$$

(inverse gaussian distribution with parameters $(\gamma, 1)$).

$$\text{So } P(X \leq w \mid U \leq \alpha) = \int_{-\infty}^{w} \left(\frac{1}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{1}{2\gamma^2} \frac{(x-\gamma)^2}{x}\right) \mathbb{1}_{\{x \geq 0\}}$$

$$= F_{IG}(w) \text{ with parameters } (\gamma, 1).$$

Exercice 3

1. (a).

Let f be a measurable function:

$$\mathbb{E}[f(Y)] = \mathbb{E}[f(\varphi(X))] = \int_0 f(\varphi(x))\, g(x)\,dx = \int_{0_1} f(\varphi_1(x))\, g(x)\,dx + \int_{0_2} f(\varphi_2(x))\, g(x)\,dx$$

$$\mathbb{E}[f(Y)] = \int_\Delta f(y)\, |(\varphi_1^{-1})'(y)|\, g(\varphi_1^{-1}(y))\,dy + \int_\Delta f(y)\, |(\varphi_2^{-1})'(y)|\, g(\varphi_2^{-1}(y))\,dy$$

$$\mathbb{E}[f(Y)] = \int_\Delta f(y)\left[|(\varphi_1^{-1})'(y)|\, g(\varphi_1^{-1}(y)) + |(\varphi_2^{-1})'(y)|\, g(\varphi_2^{-1}(y))\right] dy$$

This works for every f measurable:

$\Rightarrow$

$$\forall y \in \Delta,\ h(y) = |[\varphi_1^{-1}]'(y)|\cdot g(\varphi_1^{-1}(y)) + |[\varphi_2^{-1}]'(y)|\, g(\varphi_2^{-1}(y))$$

1. (b).

Let f be a measurable function and X the random variable obtained by Algorithm 2:

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X), X \in 0_1] + \mathbb{E}[f(X), X \in 0_2]$$

$$= \mathbb{E}[f(X), X = \varphi_1^{-1}(Y)] + \mathbb{E}[f(X), X = \varphi_2^{-1}(Y)]$$

$$= \int_\Delta f(\varphi_1^{-1}(y))\, h(y)\,dy\, \frac{|\varphi_1^{-1}{}'(y)|\, g(\varphi_1^{-1}(y))}{h(y)} + \int_\Delta f(\varphi_2^{-1}(y))\, h(y)\,dy\, \frac{|\varphi_2^{-1}{}'(y)|\, g(\varphi_2^{-1}(y))}{h(y)}$$

$$= \int_\Delta f(\varphi_1^{-1}(y))\, |\varphi_1^{-1}{}'(y)|\, g(\varphi_1^{-1}(y))\,dy + \int_\Delta f(\varphi_2^{-1}(y))\, |\varphi_2^{-1}{}'(y)|\, g(\varphi_2^{-1}(y))\,dy$$

$$= \int_{0_1} f(x)\, g(x)\,dx + \int_{0_2} f(x)\, g(x)\,dx$$

$$\mathbb{E}[f(X)] = \int_0 f(x)\, g(x)\,dx$$

So X's distribution is g.

2.(a) $\frac{(x-\mu)^2}{\mu^2 x} = y \iff x^2 - 2x\mu + \mu^2 - \mu^2 x y = 0$

$\iff x^2 - x(2\mu + \mu^2 y) + \mu^2 = 0$

There are two solutions for this second order equation: x_1 and x_2:

$$x_1 x_2 = \mu^2$$

and:

$$x_i = \frac{1}{2}\mu(2+\mu y) \pm \frac{1}{2}\sqrt{\mu^2(2+\mu y)^2 - 4\mu^2}, \quad i = 1,2$$

If we study the variations of $\psi(x)$, $\forall x \in \mathbb{R}_+$

$$\frac{d}{dx}\psi(x) = \frac{2x(x-\mu) - (x-\mu)^2}{\mu^2 x^2} = \frac{(\mu+x)}{\mu^2 x^2}(\mu - x)$$

So

$\psi(O_1) = \mathbb{R}_+$ and $\psi(O_2) = \mathbb{R}_+$

and knowing that $x_1 < x_2$

We get

$$\psi_1^{-1}(y) = \mu + \frac{1}{2}\mu^2 y - \frac{1}{2}\mu\sqrt{4\mu y + (\mu y)^2} \in [0, \mu]$$

and $\psi_2^{-1}(y) = \frac{\mu^2}{\psi_1^{-1}(y)} \in]\mu, +\infty[$

2.(b) if $Y \sim N(0,1)$ what is the distribution of $Y^2 \sim h_2(y)\,dy$

We define: $\phi : \mathbb{R} \longrightarrow \mathbb{R}_+$
$x \longmapsto x^2$

$\phi_+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$
$x \longmapsto x^2$

$\phi_- : \mathbb{R}_- \longrightarrow \mathbb{R}_+$
$x \longmapsto x^2$

ϕ_+ is a C^1-diffeomorphisme so is ϕ_-.

$$\phi_+^{-1} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \qquad \text{and} \qquad \phi_-^{-1} : \mathbb{R}_+ \longrightarrow \mathbb{R}_-$$
$$x \longmapsto \sqrt{x} \qquad\qquad x \longmapsto -\sqrt{x}$$

$$(\phi_+^{-1})'(x) = \frac{1}{2\sqrt{x}} = -(\phi_-^{-1})'(x)$$

We apply then the result of question 1.(a)

$$\forall y \in \mathbb{R}_+,\ h_2(y) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y\right) + \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y\right)$$

$$h_2(y) = \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{1}{2} y\right)$$

We now have to prove that steps 2 to 8 are equivalent to Algorithm 2:

Actually:

$$\psi_2^{-1}(y)\, \psi_1^{-1}(y) = \nu^2 \Rightarrow \psi_2^{-1\prime}(y)\, \psi_1^{-1}(y) + \psi_2^{-1}(y)\, (\psi_1^{-1})'(y) = 0$$

$$\Rightarrow \frac{[\psi_2^{-1}]'(y)}{[\psi_1^{-1}]'(y)} = -\frac{\psi_2^{-1}(y)}{\psi_1^{-1}(y)}$$

$$\Rightarrow \left| \frac{[\psi_2^{-1}]'(y)}{[\psi_1^{-1}]'(y)} \right| = \frac{\nu^2}{|\psi_1^{-1}(y)|^2} = \frac{\nu^2}{X^2}$$

$$g(\psi_2^{-1}(Y)) = \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\psi_2^{-1}(Y)}\right)^{3/2} \exp\left(-\lambda\, \psi(\psi_2^{-1}(Y))\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{1/2} \left(\frac{1}{\psi_2^{-1}(Y)}\right)^{3/2} \exp(-\lambda Y)$$

So:
$$\frac{g(\psi_2^{-1}(Y))}{g(\psi_1^{-1}(Y))} = \left(\frac{\psi_1^{-1}(Y)}{\psi_2^{-1}(Y)}\right)^{3/2} = \frac{X^3}{\nu^3}$$

This means that

$$\left[1+\left|\frac{[\Psi_2^{-1}]'(y)}{[\Psi_1^{-1}]'(y)}\right|\frac{g(\Psi_2^{-1}(y))}{g(\Psi_1^{-1}(y))}\right]^{-1}=\left(1+\frac{X}{\mu}\right)^{-1}=\frac{\mu}{\mu+X}$$

This means that steps 1 to 8 yields: $X \sim IG(1,\mu)$

Algorithm 3 repeats these steps until X is inferior to t

This means that Algorithm 3 yields an inverted gaussian with parameters: $(1,\mu)$

and truncated on $[0,t]$.

4. Problem

4.1. Rejection Method

1. $\Leftarrow$ This one is easy

$$\begin{cases}\exists q\geq 0 \text{ s.t } u\leq S_{2q+1}(x)\\ S_{2q+1}(x)<f_x(x)\end{cases}\Rightarrow u<f_x(x)$$

$\Rightarrow$ To prove this we need to prove, first, that $S_n(x)$ converges

Set $x\in\mathbb{R}_+^*$: $\forall n\geq 0$

$$n^2a_n(x)\,\cancel{\alpha}\,\left(n+\frac{1}{2}\right)n^2\left[\exp\left(-\frac{2(n+1/2)^2}{x}\right)\mathbb{1}_{0<x\leq t}+\exp\left[-\frac{(n+1/2)^2\pi^2x}{2}\right]\mathbb{1}_{x>t}\right]$$

$$\Rightarrow n^2a_n(x)=O\left[n^3\exp\left(-cst.\,n^2\right)\right],$$

$$\Rightarrow n^2a_n(x)\xrightarrow[n\infty]{}0$$

and thus $S_n(x)\xrightarrow[n\infty]{}S(x)$

Let's suppose now that

$\exists u \geq 0$ st.

$$u < f_{\alpha}(x) \text{ and } \forall q \geq 0 \; S_{2q+1}(x) < u$$

$$\Rightarrow \quad S_{2q+1} \xrightarrow[q\infty]{} S^{(1)}(x) < u < f_{\alpha}(x) \leq \lim_{q\infty} S_{2q}(x)$$

$$\Rightarrow \quad \limsup_{n\infty} S_n(x) \neq \liminf_{n\infty} S_n(x) \; (= S^{(1)}(x))$$

which is absurd

4.

Set: $V = \dfrac{U}{S_0(x)} \in [0,1]$ so $\quad V \sim U([0,1])$

We know that: $\exists K > 0$ s.t.: $g(x) = K\,S_0(x), \; \forall x$

when we stop at n odd (ie: $n = 2q+1$): we have:

$$U \leq S_{2q+1}(X) \Rightarrow \frac{U}{g(X)} = \frac{U}{K\,S_0(x)} \leq \frac{S_{2q+1}(X)}{g(X)}.$$

$$\Rightarrow \quad \frac{U}{K} \leq \frac{S_{2q+1}(X)}{g(X)}$$

This means that:

$$V \leq \frac{S_{2q+1}(X)}{g(X)} K \leq K \frac{f_{\alpha}(X)}{g(X)}$$

Algorithm 4 is then a readaptation of the rejection algorithm to sample $X \sim f_{\alpha}(x)\,dx$

4.2 Sampling the proposal distribution:

1. Back to the definition: $\forall u \in \mathbb{R}_+^*$

$$S_0(u) = \cosh(z)\exp(-u z^2/2)\cdot\frac{\pi}{2}\begin{cases}\left(\frac{2}{\pi u}\right)^{3/2}\exp\left(-\frac{1}{2u}\right), & u \in]0,t] \\ \exp\left(-\frac{\pi^2 u}{8}\right), & u > t\end{cases}$$

$$S_0(u) = \frac{\pi}{2}\cosh(z)\left(\frac{2}{\pi u}\right)^{3/2}\exp\left(-\frac{u^2z^2+1}{2u}\right)\mathbb{1}_{u\in]0,t]} + \frac{\pi}{2}\cosh(z)\exp\left(-\left(\frac{z^2}{2}+\frac{\pi^2}{8}\right)u\right)\mathbb{1}_{u\in]t,\infty[}$$

$$S_0(u) = \frac{\pi}{2}\cosh(z)\left(\frac{2}{\pi u}\right)^{3/2}\exp\left(-\frac{(uz-1)^2}{2u}-z\right)\mathbb{1}_{u\in]0,t]} + \frac{\pi}{2}\cosh(z)\exp\left(-\left(\frac{z^2}{2}+\frac{\pi^2}{8}\right)u\right)\mathbb{1}_{u\in]t,+\infty[}$$

$$S_0(u) = \frac{\pi}{2}\cosh(z)\exp(-z)\left(\frac{2}{\pi u}\right)^{3/2}\exp\left(-\frac{(u-\frac{1}{z})^2}{2u\left(\frac{1}{z}\right)^2}\right)\mathbb{1}_{u\in]0,t]} + \frac{\pi}{2}\cosh(z)\exp(-Ku)\mathbb{1}_{u\in]t,+\infty[}$$

where $K = \frac{z^2}{2} + \pi^2/8$

We define: $g_1(u)\,du \sim IG(\lambda,\mu)$

$g_2(u)\,du \sim \mathcal{E}(K)$

with $\lambda = 1$

$\mu = \frac{1}{z}$

$K = \frac{z^2}{2} + \pi^2/8$

Then clearly: $g(u) = (1-\alpha)\, g_1(u)\,\mathbb{1}_{[0,t]}(u) + \alpha\, g_2(u)\,\mathbb{1}_{]t,\infty[}(u)$

$$1 = \int_{\mathbb{R}_+} g(u)\,du = (Cst)^{-1}\int_{\mathbb{R}_+} S_0(u)\,du$$

$$\int_{\mathbb{R}_+} S_0(u)\,du = \left[\int_{[0,t]}\exp(-z)\left(\frac{2}{\pi}\right)^{3/2}\sqrt{2\pi}\; g_1(u)\,du + \int_t^{+\infty}\frac{1}{K}K\exp(-Ku)\,du\right]\frac{\pi}{2}\cosh(z)$$

$$Cst = \frac{\pi}{2}\cosh(z)\left[4/\pi\,\exp(-z)\,F_{IG}(t) + K^{-1}\exp(-Kt)\right]$$

$$\alpha = \int_t^{+\infty} g(u) = \int_t^{+\infty}(Cst)^{-1}S_0(u)\,du = \frac{\frac{\pi}{2}\cancel{\cosh(z)}\,K^{-1}\exp(-Kt)}{\left[K^{-1}\exp(-Kt) + F_{IG}(t)\,4\pi^{-1}\exp(-z)\right]\frac{\pi}{2}\cancel{\cosh(z)}}$$

2. Let $Y = t + X$ and $X \sim \mathcal{E}(\lambda)$, Set f measurable function:

$$\mathbb{E}[f(Y)] = \int_{\mathbb{R}} f(t+u)\, \lambda \exp(-\lambda u)\, \mathbb{1}_{u \in \mathbb{R}_+}\, du$$

$$= \int_{\mathbb{R}} f(y)\, \lambda \exp(-\lambda y) \exp(\lambda t)\, \mathbb{1}_{y \in [t, +\infty[}\, dy$$

We can observe that

$$e^{\lambda t} = \left[\int_t^{+\infty} \lambda e^{-\lambda u} du \right]^{-1}$$

So $t+X$ is just the truncated exponential with parameter λ ~~and~~ in the interval $[t, +\infty[$.

Algorithm 5: Sample G2

1. Draw $U \sim \mathcal{U}[0,1]$
2. $G2 \leftarrow \dfrac{-\log(U)}{K}$
3. $G2 \leftarrow G2 + t$
4. Return G2

Algorithm 6: Sample G1

1. If $t < \frac{1}{3}$
2. $G1 \leftarrow$ Algorithm 1
3. else
4. $G1 \leftarrow$ Algorithm 3
5. end.
6. Return G1

$$F_g(x) = \begin{cases} (1-\alpha) F_{IG}(x), & x \le t \\ (1-\alpha) F_{IG}(t) + \alpha F_{g_2}(x), & x > t \end{cases} \Rightarrow F_g^{-1}(u) = \begin{cases} F_{IG}^{-1}\left(\dfrac{u}{1-\alpha}\right), & u < (1-\alpha) F_{IG}(t) \\ F_{g_2}^{-1}\left(\dfrac{u - (1-\alpha) F_{IG}(t)}{\alpha}\right), & u > (1-\alpha) F_{IG}(t) \end{cases}$$

Algorithm 7: Sample G

1. Draw $U \sim \mathcal{U}([0,1])$
2. If $U < 1-\alpha$
3. $\quad X \leftarrow$ Sample G1
4. else
5. $\quad X \leftarrow$ Sample G2
6. end
7. return X

In fact, let f be a measurable function

$$\mathbb{E}[f(X)] = \mathbb{E}[f(X), U < 1-\alpha] + \mathbb{E}[f(X), U > 1-\alpha]$$

$$= \int f(x) \cdot g_1(x) \mathbb{1}_{x \in [0,t]} (1-\alpha)\, dx + \int f(x) \cdot g_2(x) \mathbb{1}_{x \in [t,+\infty[} [1-(1-\alpha)]\, dx$$

$$= \int_{\mathbb{R}_+} f(x)\, g(x)\, dx$$

4.3 Implemtation

1.

Algorithm 8: Home Walk

1. repeat
2. $n \leftarrow 0$;
3. $X \leftarrow$ Sample G ;
4. Draw $U \sim \mathcal{U}[0, S_0(X)]$;
5. Iteratively calculate $S_n(X)$, starting at $S_1(X)$ and until $U \leq S_n(X)$ for an odd n or $U > S_n(X)$ for an even n;
6. Until n is odd
7. return X

(i) c.f Figure 1

(ii)

$z = 3000$, $\mathbb{E}[T/4] = 2{,}7561 \cdot 10^5$

$z = 2000$, $\mathbb{E}[T/4] = 1{,}1904 \cdot 10^5$

$z = 1000$, $\mathbb{E}[T/4] = 3{,}155 \cdot 10^4$