# Homework 2

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## 1 EXERCISE 1.

1. Let $f$ be a borelian function and $Y$ the resulting random variable from the algorithm:

$$
\begin{aligned}
\mathbb{E}[f(Y)] &= \mathbb{E}_{\epsilon,B}[f(x\epsilon)\mathbb{1}_{B=1} + f(\frac{x}{\epsilon})\mathbb{1}_{B=0}] \\
&= \mathbb{E}_{\epsilon}[f(x\epsilon)\frac{1}{2} + f(\frac{x}{\epsilon})\frac{1}{2}] \\
&= \frac{1}{2}\int_{\mathbb{R}}[f(x\epsilon) + f(\frac{x}{\epsilon})]g(\epsilon)\mathbb{1}_{(-1,1)}(\epsilon)d\epsilon \\
&= \frac{1}{2}\int_{|u|<|x|} f(u)g(\frac{u}{x})\frac{du}{|x|} + \frac{1}{2}\int_{|u|>|x|} f(u)g(\frac{x}{u})\frac{du|x|}{u^2}
\end{aligned}
$$

So if we denote by $K(x,y)$ the p.d.f. of the proposal distribution:

$$K(x,y) = \frac{1}{2}g(\frac{y}{x})\frac{1}{|x|}\mathbb{1}_{|y|<|x|} + \frac{1}{2}g(\frac{x}{y})\frac{|x|}{y^2}\mathbb{1}_{|y|>|x|} \tag{1.1}$$

2. Let $\pi$ be the invariant distribution :

$$
\begin{aligned}
\alpha(x,y) &= min(1, \frac{\pi(y)}{\pi(x)}\frac{|y|}{|x|}\mathbb{1}_{|y|<|x|} + \frac{|y|}{|x|}\mathbb{1}_{|y|>|x|}) \\
&= min(1, \frac{\pi(y)}{\pi(x)}|\epsilon|)\mathbb{1}_{|y|<|x|} + min(1, \frac{\pi(y)}{\pi(x)}\frac{1}{|\epsilon|})\mathbb{1}_{|y|>|x|}
\end{aligned}
$$

$$\alpha(x,y) = min(1, \frac{\pi(y)}{\pi(x)}|\frac{y}{x}|) \tag{1.2}$$

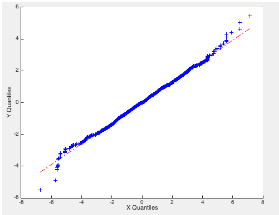



Figure 1.1: QQ-plot comparing the MCMC invariant sequence to the true one

3. We implement this scheme for two distributions:

   - The hyperbolic secant distribution:

$$f(x) = sech(\frac{\pi}{2}x)$$

   Its quantile function is:

$$Q(p) = -\frac{2}{\pi}\log\,(\,\tan\,(\frac{\pi}{2}))$$

   - The boltzman distribution of a quartic potential:

$$V(x) \propto e^{-\frac{x-\mu)^2}{2\sigma}(x-\mu)^2+\lambda(x-\mu)^4}$$

4. In the two cases, we estimate the mean with:

$$\hat{\mu}_n = \frac{1}{N}\sum_{n=0}^{N} X_n$$

We set:

$$X_0 := 2$$

$$N := 5000$$

$$\mu := 0$$

and the other parameters to 1.
We get for the:

   - The logistic distribution:

$$\hat{\mu}_n = -0.0308$$

   - The quartic potential well:

$$\hat{\mu}_n = 0.0131$$

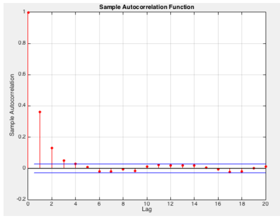



Figure 1.2: Auto correlation plot for the secant hyperbolic distribution MCMC sample.

5. We use here 5000 samples.
The two distrivutions concide well enough around the mean where we can sample with high praobability. However on the tails the two distributions differ.(c.f figure 1.1)

6. In figure 1.2, we can see that the autocorrelation coefficients decay very fast toward 0. We can say then taht the burn in period of this scheme is small.

# 2 Exercise 2.

1. When $f_0 \equiv 1$ , It is clear from step (b) that $X_n \in \{0,1\}, \forall n = 1, \dots, +\infty$ and the transition kernel is a bernouli distribution. We can see that this algorithm resembles the rejection algorithm. In fact, the step $(b)$ rejects $X_n$. The difference here is that the samples are not independent.

2. Let $h$ be a borelian function. Set $n \in \mathbb{N}$ :

$$\begin{aligned}
\mathbb{E}[h(X_{n+1})|X_n] &= \mathbb{E}_{U_{n+1}} \int_{\mathbb{R}} h(x) f_0(x) \prod_{j=1}^{l} \mathbb{1}_{U_{n+1}^j \le f_j(x)} dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^l} h(x) f_0(x) \prod_{j=1}^{l} \mathbb{1}_{u^j \le f_j(x)} \frac{1}{f_j(X_n)} \mathbb{1}_{[0,f_j(X_n)]} d^l u dx \\
&= \int_{\mathbb{R}} h(x) f_0(x) \prod_{j=1}^{l} \int_{\mathbb{R}} \frac{1}{f_j(X_n)} \mathbb{1}_{0 \le u^j \le min(f_j(x), f_j(X_n))} du^j dx \\
&= \int_{\mathbb{R}} h(x) f_0(x) \prod_{j=1}^{l} \frac{min(f_j(x), f_j(X_n))}{f_j(X_n)} dx
\end{aligned}$$

So the transition p.d.f. is:

$$K(x,y) = f_0(y) \prod_{j=1}^{l} \frac{min(f_j(y), f_j(x))}{f_j(x)} \tag{2.1}$$

3. If we prove that $\pi$ is symmetric w.r.t. $K$ then $\pi$ would be invariant.

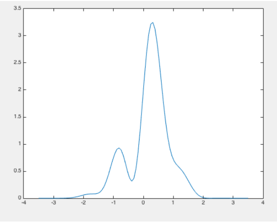
R 

Figure 3.1: The density shape.

$$
\begin{aligned}
\pi(x)K(x,y) &= f_0(x)\prod_{j=1}^{l} f_j(x) f_0(y) \prod_{j=1}^{l} \frac{min(f_j(y), f_j(x))}{f_j(x)} \\
&= f_0(x) f_0(y) \prod_{j=1}^{l} min(f_j(y), f_j(x)) \\
&= f_0(y)\prod_{j=1}^{l} f_j(y) f_0(x) \prod_{j=1}^{l} \frac{min(f_j(y), f_j(x))}{f_j(y)} \\
&= \pi(y)K(y,x)
\end{aligned}
$$

Let $\mathscr{A}$ be a borel set:

$$
\begin{aligned}
\int_{\mathbb{R}\times\mathscr{A}} \pi(x)K(x,y) &= \int_{\mathbb{R}\times\mathscr{A}} \pi(y)K(y,x)dxdy \\
&= \int_{\mathscr{A}} \pi(y)(\int_{\mathbb{R}} K(y,x)dx)dy \\
&= \int_{\mathscr{A}} \pi(y)dy
\end{aligned}
$$

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# 3 Exercise 3.

1. It is obviously not symmetric with two regions that are more probable and also separated. The problem that could rise then is that the chain stays too long in one region (possibly the right one) and thus we may not be aware of the local minima in the middle.(c.f. figure 3.1)

2. We used two proposals:

   (i) An independent proposition kernel:

$$K(x,y) = \mathscr{N}(x,0,1)$$

(ii) A random walk proposition:

$$Y_{n+1} = X_n + Z_{n+1}$$

where $Z_n \sim \mathcal{N}(0,1)$

3.
In this figure 3.2, we see that the independent proposition converges faster than the random walker. In average the acceptance ratio is 62% for the first case and around 51% in the second one.
4. For:

$$N = 100000$$

and

$$X_0 = -2$$

we run the indHM 10 times, we get:

$$\hat{\mu}_n \approx 0.19$$

$$\widehat{Var}_n \approx 0.46$$

and an estimate for the mode being $\approx 0.25$.

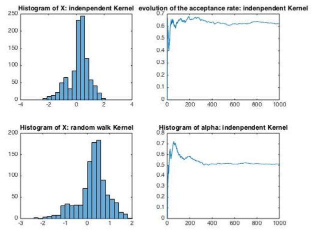


Figure 3.2: The comparison between the two samplers.