Department of Statistics and Applied Probability
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ST4234 Bayesian Statistics
Tutorial 4 Group 6

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3. Consider a geometric random variable with pdf

$$\boldsymbol{p(x|\theta) = \theta(1-\theta)^{x-1}, x = 1, 2, 3 \ldots,} \qquad \textbf{(1)}$$

where 0<$\boldsymbol{\theta}$<1. Suppose that there are 50 observations ($\boldsymbol{x_1, \ldots, x_{50}}$)of X and $\boldsymbol{\sum_{i=1}^{50} x_i = 200}$. Under the prior,

$$\boldsymbol{p(\theta) \propto \theta^{-1}(1-\theta)^{-\frac{1}{2}}, 0 < \theta < 1,}$$

(a) Find the posterior probabilities in order to distinguish between the two hypotheses, {$\boldsymbol{\theta < 0.2}$} and{$\boldsymbol{\theta > 0.25}$}

Solution:

The prior distribution is $p(\theta) \propto \theta^{-1}(1-\theta)^{-\frac{1}{2}}$,$0 < \theta < 1$
The likelihood distribution is $p(x|\theta) = \theta(1-\theta)^{x-1}, x = 1,2,3 \ldots,$
The posterior distribution is

$$\begin{aligned}
p(\theta|x) &\propto p(x|\theta)p(\theta) \\
&\propto \prod_{i=1}^{50} p(x_i|\theta)\, p(\theta) \\
&\propto \prod_{i=1}^{50} \theta(1-\theta)^{x_i-1}\theta^{-1}(1-\theta)^{-\frac{1}{2}} \\
&\propto \theta^{50}(1-\theta)^{\sum_{i=1}^{50} x_i - 50}\theta^{-1}(1-\theta)^{-\frac{1}{2}} \\
&\propto \theta^{50}(1-\theta)^{200-50}\theta^{-1}(1-\theta)^{-\frac{1}{2}} \\
&\propto \theta^{50}(1-\theta)^{150}\theta^{-1}(1-\theta)^{-\frac{1}{2}} \\
&\propto \theta^{49}(1-\theta)^{149.5}
\end{aligned}$$

The posterior distribution of θ is Beta (49+1, 149.5+1) = Beta (50, 150.5)

To test the two hypothesis:

$$H_0: \theta < 0.2 \qquad H_1: \theta > 0.25$$

$$p_0 = p(\theta < 0.2|x) = 0.04746901$$
$$p_1 = p(\theta > 0.25|x) = 0.4809932$$

R code

```
> (P0=pbeta(q=0.2,shape1=50,shape2 = 150.5))
[1] 0.04746901
> (p1=1-pbeta(q=0.25,shape1=50,shape2 = 150.5))
[1] 0.4809932
```

(b) Find the probability that a new observation $\boldsymbol{X_{51}}$ distributed from (1) is less than 3

The posterior prediction distribution of X_{n+1} is the conditional distribution of X_{n+1} given observed data X :

$$\begin{aligned}
\mathrm{p}(\mathrm{X}_{51}|\mathrm{x}) &= \int_0^1 \mathrm{p}(\mathrm{x}_{51}, \theta|\mathrm{x})d\theta \\
&= \int_0^1 \mathrm{p}(\mathrm{x}_{51}|\ \theta, \mathrm{x})\mathrm{p}(\theta|\mathrm{x})d\theta \\
&= \int_0^1 \frac{\theta(1-\theta)^{\mathrm{x}_{51}-1}\theta^{49}(1-\theta)^{149.5}}{\mathrm{Beta}(50, 150.5)}d\theta \\
&= \int_0^1 \frac{\theta(1-\theta)^{\mathrm{x}_{51}-1}\theta^{49}(1-\theta)^{149.5}}{\mathrm{Beta}(50, 150.5)}d\theta \\
&= \frac{\mathrm{Beta}(51, 149.5+\mathrm{x}_{51})}{\mathrm{Beta}(50, 150.5)}\int_0^1 \frac{\theta^{50}(1-\theta)^{149.5+\mathrm{x}_{51}-1}}{\mathrm{Beta}(51, 149.5+\mathrm{x}_{51})}d\theta \\
&= \frac{\mathrm{Beta}(51, 149.5+\mathrm{x}_{51})}{\mathrm{Beta}(50, 150.5)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(\mathrm{X}_{51} < 3|\mathrm{X}) &= \mathrm{P}(\mathrm{x}_{51} = 1\ |\mathrm{X}) + \mathrm{P}(\mathrm{x}_{51} = 2|\mathrm{X}) \\
&= \frac{\mathrm{Beta}(51,149.5+1)}{\mathrm{Beta}(50,150.5)} + \frac{\mathrm{Beta}(51,149.5+2)}{\mathrm{Beta}(50,150.5)} \\
&= \frac{\mathrm{Beta}(51,150.5)}{\mathrm{Beta}(50,150.5)} + \frac{\mathrm{Beta}(51,151.5)}{\mathrm{Beta}(50,150.5)} \\
&= 0.4356355
\end{aligned}$$

R code

```
> beta(51,150.5)/beta(50,150.5)+beta(51,151.5)/beta(50,150.5)
[1] 0.4356355
```

4. Suppose that the results of a certain test are known to be approximately distributed as $N(\mu, \frac{1}{\tau})$. Suppose that your prior belief about (μ, τ) is

$$\mu|\tau \sim N\left(75, \frac{3}{\tau}\right) \text{ and } \tau \sim Gamma\left(1, \frac{1}{2}\right)$$

We obtained 33 observations $(x_1,\ldots, x_{33})$ from the population with their sample mean $\tilde{x} = \frac{1}{33}\sum_{i=1}^{33} x_i = 80$ and sample variance $s^2 = \frac{1}{33}\sum_{i=1}^{33}(x_i - \tilde{x})^2 = 30$. Based on the posterior probabilities,

(a) Do you favour $\sigma^2 < 25$or $\sigma^2 > 35$? Recall that $\sigma^2 = \frac{1}{\tau}$

Solution:

In this case, likelihood distribution is known to be normally distributed with parameter μ and $\frac{1}{\tau}$. Though we know the distributions that μ and $\frac{1}{\tau}$ follow, they still remain as random variables which are unknown. Thus, we regard this case as an inference when mean and variance are unknown.

According to 3.8, when likelihood distribution is normal, with i.i.d observations, it can be expressed as an Inverse-gamma and a normal distribution. A conjugate prior can then be expressed as

$$p(\theta, \sigma^2) = p(\theta|\sigma^2)p(\sigma^2)$$

and where

$$\theta|\sigma^2 \sim \mathrm{N}\left(\mu_0, \frac{\sigma^2}{n_0}\right), \quad \sigma^2 \sim \text{Inv-Gamma}\left(\frac{\nu_0}{2}, \frac{S_0}{2}\right)$$

$Given\ \mu|\tau \sim N\left(75, \frac{3}{\tau}\right), \tau \sim Gamma\left(1, \frac{1}{2}\right)$, we get:

$$\mu_0 = 75 \quad \text{and} \quad 3\sigma^2 = \frac{\sigma^2}{n_0} \quad => \quad n_0 = 1/3$$

$$\frac{v_0}{2} = 1 \ => \ v_0 = 2$$

$$\frac{s_0}{2} = \frac{1}{2} \ => \ s_0 = 1$$

Using this conjugate prior and combining with the observations,

The posterior is of the form

$$p(\theta, \sigma^2|\boldsymbol{y}) = p(\theta|\sigma^2, \boldsymbol{y})p(\sigma^2|\boldsymbol{y}),$$

where

$$\theta|\sigma^2, \boldsymbol{y} \sim \mathrm{N}\left(\mu_1, \frac{\sigma^2}{n_1}\right), \qquad \sigma^2|\boldsymbol{y} \sim \text{Inv-Gamma}\left(\frac{\nu_1}{2}, \frac{S_1}{2}\right),$$

$$\mu_1 = \frac{n\bar{y} + n_0\mu_0}{n + n_0}, \qquad \nu_1 = \nu_0 + n,$$

$$n_1 = n + n_0, \qquad S_1 = S_0 + S + \frac{nn_0(\bar{y} - \mu_0)^2}{n + n_0}.$$

Giving that $\tilde{x}$ = 80, $s^2 = 30$, we get:

$$\mu_1 = \frac{33(80) + \frac{1}{3}(75)}{33 + \frac{1}{3}} = \frac{1599}{20}$$

$$n_1 = 33 + \frac{1}{3} = 33\frac{1}{3} \qquad v_1 = 33 + 2 = 35$$

$$s_1 = 1 + 30 * 32 + \frac{33\left(\frac{1}{3}\right)(80 - 75)^2}{33 + \frac{1}{3}} = \frac{3877}{4}$$

Thus we get the posterior distribution:

$$p(\mu, \sigma^2|x) = p(\mu\,|\sigma^2, x)p(\,\sigma^2|x)$$

where

$$\mu|\sigma^2, x \sim\ N\left(\frac{1599}{20}, \frac{\sigma^2}{\frac{100}{3}}\right) \quad , \quad \sigma^2|x \sim InverseGamma(\frac{35}{2}, \frac{3877}{8})$$

To test the hypothesis:

$$H_0: \sigma^2 < 25 \qquad H_1: \sigma^2 > 25$$

$$P(\sigma^2 < 25|x) = \int_0^{25} P(\sigma^2|x)\, d\sigma^2 = 0.3034621$$

$$P(\sigma^2 > 35|x) = 1 - \int_0^{35} P(\sigma^2|x)\, d\sigma^2 = 0.1944627$$

Hence, thes result is in favour of $\sigma^2 < 25$ because its posterior probability is larger.

R code

```
require(pscl)
> pigamma(25,35/2,3877/8)
[1] 0.3034621

> 1-pigamma(35,35/2,3877/8)
[1] 0.1944627
```

(b) Do you favour $\boldsymbol{\mu < 78\ or\ \mu > 81}$

The marginal posterior distribution of μ is:

$$P(\mu|x) = \int P(\mu, \sigma^2|x) d\sigma^2$$

$$let\ t = \sqrt{\frac{n_1 v_1}{S_1}}\,(\mu - \mu_1), \quad t \sim t_{v_1} = t_{35}$$

To test the hypothesis:

$$H_0: \mu < 78 \quad H_1: \mu > 81$$

Transforming marginal posterior distribution of μ into Student's t-distribution with degrees of freedom v_1, by letting $t = \sqrt{\frac{n_1 v_1}{S_1}}\,(\mu - \mu_1)$ and applying change-of-variable formula we get:

$$P(\mu < 78|x) = P\left(\sqrt{\frac{n_1 v_1}{S_1}}\,(\mu - \mu_1) < \sqrt{\frac{n_1 v_1}{S_1}}\,(78 - \mu_1)\middle|\, x\right)$$

$$= P\left(t < \sqrt{\frac{\frac{100}{3} \times 35}{\frac{3877}{4}}}\left(78 - \frac{1599}{20}\right)\middle|\, x\right) = P(t < -2.13939\,|x) = 0.0197$$

$$P(\mu > 81|x) = 1 - P(\mu < 81|x) = 1 - P\left(\sqrt{\frac{n_1 v_1}{S_1}}\,(\mu - \mu_1) < \sqrt{\frac{n_1 v_1}{S_1}}\,(81 - \mu_1)\middle|\, x\right)$$

$$= 1 - P\left(t < \sqrt{\frac{\frac{100}{3} \times 35}{\frac{3877}{4}}}\left(81 - \frac{1599}{20}\right)\middle|\, x\right) = 1 - P(t < 1.15198\,|x) = 0.129$$

Hence, the result is in favour of $\mu > 81$ because its posterior probability is larger.

R code

```
> pt(-2.139390689, 35, lower.tail = TRUE)
[1] 0.01972806
> pt(1.151979602, 35, lower.tail = FALSE)
[1] 0.1285688
```